Lesson 9 — H-fields

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Differential fields

Definition

*A **differential field** is a field* K *with a **derivation*** $\partial\colon K \to K$ *such that, for all* $a, b \in R$,

D1. $\partial(a+b) = \partial a + \partial b$.

D2. $\partial(ab) = (\partial a)b + a(\partial b)$.

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Example. $\mathbb{T}$ with $\delta := x\partial$ is isomorphic to $\mathbb{T} \circ \exp$ with ∂.

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Note. The set $\mathrm{Der}_{\prec}(K)$ of small derivations on K forms a $\mathcal{O}_K$-module.

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Note. For each of the above examples, the derivation is small.

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H1. If $y \in (K^{\mathrm{rc}})^{>C}$, then $y^n \sim u \in K^{>C}$ for some $n \geqslant 1$.
Then $(y^n)' \sim u'$, whence $y' \sim y u^\dagger / n > 0$.

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H2. Let $y \in K^{\mathrm{rc}}$ with $y \preccurlyeq 1$.
Then $P(y) = 0$ for some $P := D + E$ with $D \in C[Y]^{\neq 0}$ and $E \in K[Y]^{\prec 1}$.
Since $D(y) \prec 1$ and D splits over $C^{\mathrm{rc}} = C_{K^{\mathrm{rc}}}$, we have $y \sim c$ for some $c \in C_{K^{\mathrm{rc}}}$.

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Preservation of smallness. Exercise. □

Gaps

Proposition

There is a unique map $^\dagger\colon \Gamma^{\neq 0} \to \Gamma$ *with* $v(a)^\dagger = v(a^\dagger)$ *for all* $a \in K^{\not\asymp 1}$.

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Exactly one of the following situations occurs:

1. $(\Gamma^{\neq 0})^\dagger$ *has a largest element* (*we say that* K *is* ***grounded***).
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3. $\Gamma = (\Gamma^{\neq 0})^\dagger \cup \{\beta\} \cup (\Gamma^{>0})'$ *with* $(\Gamma^{\neq 0})^\dagger < \beta < (\Gamma^{>0})'$ (*we say that* K *has a* ***gap***)

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3. Example: $\mathbb{T}_1^{\mathrm{wb}}$, with $\beta := v(\gamma)$, $\gamma := \frac{1}{x\log x\log_2 x\cdots}$. We have $\varepsilon' \prec \gamma \prec \delta^\dagger$ for any $\varepsilon,\delta\prec 1$.

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Furthermore, $\mathbb{T}_0^{\mathrm{wb}}$ has asymptotic integration (whence no gap), but

$$\lambda \;:=\; -\gamma^\dagger \;=\; \frac{1}{x} + \frac{1}{x \log x} + \frac{1}{x \log x \log_2 x} + \cdots \;\in\; \mathbb{T}_0^{\mathrm{wb}}.$$

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Lemma

Let $L := K(y)$ with $y' = \ell^\dagger$. There is a unique ordering on L with $y > 0$ for which $L \supseteq K$ is an extension of H-fields. We have $C_L = C$, $\Gamma_L = \Gamma \oplus \mathbb{Z}\,v(y)$, $\Gamma^{<0} < \mathbb{Z}^{>0}v(y)$, $(\Gamma_L^{\neq 0})^\dagger \leqslant v(y^\dagger)$. Moreover, if $F \supseteq K$ is another H-field extension and $a \in F^{>0}$ satisfies $a' = \ell^\dagger$, then there exists a unique embedding of H-fields $\varphi\colon L \to F$ with $\varphi(y) = a$.

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Example. $K = \mathbb{E}$ with $\ell = x$. Then $y' = \ell^\dagger = \frac{1}{x}$, so $y \in \log x + C$, e.g. $y = \log x$.

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Proof. We may assume wlog that $\partial=\delta$, whence $\ell^\dagger=1$ and $v(a^\dagger)\leqslant 0$ for all $a\in K^{\neq 1}$.

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Proof. We may assume wlog that $\partial = \delta$, whence $\ell^\dagger = 1$ and $v(a^\dagger) \leqslant 0$ for all $a \in K^{\neq 1}$.

Since $y^\dagger = \ell^{-1} \prec 1$, we have $\mathbb{Z}\, v(y) \cap \Gamma = \emptyset$, so $\beta := v(y)$ lies in a cut over Γ.

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Moreover, if $F \supseteq K$ *is another H-field extension and* $a \in F^{>0}$ *satisfies* $a' = \ell^\dagger$, *then there exists a unique embedding of H-fields* $\varphi: L \to F$ *with* $\varphi(y) = a$.

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Since $y^\dagger = \ell^{-1} \prec 1$, we have $\mathbb{Z} v(y) \cap \Gamma = \emptyset$, so $\beta := v(y)$ lies in a cut over Γ.

By Lemma TR-VAL from Lesson 8, L has a unique valuation with $\Gamma_L = \Gamma \oplus \mathbb{Z}\beta$ and $v(y) = \beta$. Moreover, $\boldsymbol{k}_L = \boldsymbol{k}_K$, and for any valued field extension $F \supseteq K$ and $a \in F^{\neq 0}$ with $v(a)$ in the same cut as β over K, there exists a unique valued field embedding $\varphi: L \to F$ with $\varphi(y) = a$.

Adjoining new logarithms

Lemma

Let $L := K(y)$ with $y' = \ell^\dagger$. There is a unique ordering on L with $y > 0$ for which $L \supseteq K$ is an extension of H-fields. We have $C_L = C$, $\Gamma_L = \Gamma \oplus \mathbb{Z}\, v(y)$, $\Gamma^{<0} < \mathbb{Z}^{>0} v(y) < 0$, $(\Gamma_L^{\neq 0})^\dagger \leqslant v(y^\dagger)$.

Moreover, if $F \supseteq K$ is another H-field extension and $a \in F^{>0}$ satisfies $a' = \ell^\dagger$, then there exists a unique embedding of H-fields $\varphi: L \to F$ with $\varphi(y) = a$.

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To do. Verify that v comes from an ordering that satisfies **H1** and **H2**.

Adjoining new logarithms — continued proof

Given $f \in K(y)^{\neq 0}$, there exist $u \in K^{\neq 0}$ and $n \in \mathbb{Z}$ with $v(f) = v(u) + n\beta$, whence $f \sim u y^n$.

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H2. If $v(f) = 0$, then $n = 0$ and $v(u) = 0$, so $u \in C + \mathcal{O}$ and $f \in C + \mathcal{O}_L$.

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φ preserves valuation $\Rightarrow$ φ preserves $\sim$ $\Rightarrow$ $\varphi(y) \sim u a^n > 0$. $\square$

Adjoining immediate integrals

Let K be a real closed H-field with $\Gamma := \Gamma_K$ and let $C := C_K$.
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Lemma

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Moreover, if $F \supseteq K$ is another H-field extension and $a \in F$ satisfies $a' = g$, then there exists a unique embedding of H-fields $\varphi: L \to F$ with $\varphi(y) = a$.

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Proof. We construct a pc-sequence (y_ρ) that approximates y:

- $y_0 := 0$.
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Conclude by Lemma TR-IMM + “routine verifications”. □

A fork in the road

Let K be a real closed H-field with $\Gamma := \Gamma_K$ and let $C := C_K$.
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Let $\epsilon \in \{1, -1\}$.

Let $L := K(y)$ with $y' = \gamma$. There is a unique ordering on L with $\epsilon y^\epsilon > C$ for which $L \supseteq K$ is an extension of H-fields. We have $C_L = C$, $\Gamma_L = \Gamma \oplus \mathbb{Z} v(y)$, $\Gamma^{<0} < \mathbb{Z} v(y)$, $(\Gamma_L^{\neq 0})^\dagger \leqslant v(y^\dagger)$.

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Example. $K = \mathbb{T}_1^{\mathrm{wb}}$, $\gamma := \dfrac{1}{x \log x \log_2 x \cdots}$.

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$\epsilon = -1$. Then $-y_{\mathrm{nat}}^{-1} \prec 1$ satisfies $(-y_{\mathrm{nat}}^{-1})' = \gamma / y_{\mathrm{nat}}^2$.

This "explains" why we may also impose $\int \gamma \prec 1$.

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Proof. Similar as for immediate integration. This time (y_ρ) is as follows:

- $y_0 := 1$.
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But there may be elements of $K/\approx$ that are not in $(K^{\neq 0})^\dagger/\approx$.

Lemma

Assume that Γ is divisible. Let $s \in K^{\neq 0}$ be such that $s - a^\dagger > \mathcal{O}'_K$ for all $a \in K^{\neq 0}$. Consider the differential field $L := K(y)$ with $y^\dagger = s$.

There exists a unique ordering on L for which $L \supseteq K$ is an extension of H-fields with $y > 0$. We have $\boldsymbol{k}_L = \boldsymbol{k}_K$, $\Gamma_L = \Gamma \oplus \mathbb{Z} v(y)$, and ∂_L is small whenever ∂_K is small.

Moreover, if $F \supseteq K$ is another H-field extension and $a \in F^{>0}$ satisfies $a^\dagger = s$, then there exists a unique embedding of H-fields $\varphi: L \to F$ with $\varphi(y) = a$.

Liouville closure

Let K be an H-field and let $\Gamma := \Gamma_K$, $C := C_K$.

Theorem

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$$\lambda' \approx \left(\frac{1}{x} + \frac{1}{x \log x} + \frac{1}{x \log x \log_2 x} + \cdots\right)' = -\frac{1}{x^2} - \frac{1}{x^2 \log x} - \frac{1}{x \log x \log_2 x} - \cdots - \frac{1}{x^2 \log^2 x} - \cdots \approx \frac{-\lambda}{x}.$$

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- Similarly, λ cannot be introduced through exponentiation or real closure. □

The ω-cut

$$\lambda = x^{\dagger} + (\log x)^{\dagger} + (\log_2 x)^{\dagger} + \cdots = \frac{1}{x} + \frac{1}{x \log x} + \frac{1}{x \log x \log_2 x} + \cdots$$

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Theorem (Écalle, ADH)

For any $P \in \mathbb{R}\{Y\} \setminus \mathbb{R}$, the first ω terms of $\alpha P(\lambda) + \beta$ coincide with λ or ω, for certain $\alpha, \beta \in \mathbb{R}(x, \log x, \ldots, \log_r x)$.

First order conditions

property of γ	$(\forall \varepsilon \prec 1) \quad \varepsilon' \prec \gamma \prec \varepsilon^{\dagger}$
property of λ	$(\forall \varepsilon \prec 1) \quad \lambda + \varepsilon^{\dagger\dagger} \prec \varepsilon^{\dagger}$
property of ω	$(\forall \varepsilon \prec 1) \quad \omega - 2(\varepsilon^{\dagger\dagger})' + (\varepsilon^{\dagger\dagger})^2 \prec (\varepsilon^{\dagger})^2$

γ-freeness	$(\forall s)\,(\exists \varepsilon \prec 1) \quad s \preccurlyeq \varepsilon' \vee s \succcurlyeq \varepsilon^{\dagger}$
λ-freeness	$(\forall s)\,(\exists \varepsilon \prec 1) \quad s + \varepsilon^{\dagger\dagger} \succcurlyeq \varepsilon^{\dagger}$
ω-freeness	$(\forall s)\,(\exists \varepsilon \prec 1) \quad s - 2(\varepsilon^{\dagger\dagger})' + (\varepsilon^{\dagger\dagger})^2 \succcurlyeq (\varepsilon^{\dagger})$

$$\omega\text{-freeness} \implies \lambda\text{-freeness} \implies \gamma\text{-freeness}$$

Differential Newton polygon method

We need to generalize:

- Differential Newton polynomials.
- Equalizers.
- Resolution of quasi-linear differential equations.
- Unravelling.

Compositional conjugation

Consider $\delta := \phi^{-1}\partial$ with $\phi \in K^{>0}$.

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Any $P \in K\{Y\}$ can be rewritten as a polynomial $P^{\phi} \in K^{\phi}\{Y\} = K[Y, \delta Y, \delta^2 Y, \ldots]$:

$$\begin{aligned}
\partial &= \phi \delta \\
\partial^2 &= \phi^2 \delta^2 + \phi' \delta \\
\partial^3 &= \phi^3 \delta^3 + 3 \phi \phi' \delta^2 + \phi'' \delta \\
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We call P^{ϕ} the **compositional conjugate** of P by ϕ.

Link with upward shifting

$$\uparrow \qquad \phi := \frac{1}{x} \qquad \partial = \phi\,\delta$$

$$\uparrow\uparrow \qquad \psi := \frac{1}{x\log x} \qquad \partial = \psi\,\theta$$

$$P = xYY'' - (Y')^2$$

$$P\uparrow = \frac{YY'' - YY'}{e^x} - \frac{(Y')^2}{e^{2x}}$$

$$P\uparrow\uparrow = \frac{YY'' - YY'}{e^{e^x+2x}} - \frac{YY'}{e^{e^x+x}} - \frac{(Y')^2}{e^{2e^x+2x}}$$

$$P = xY'' - (Y')^2$$

$$P^{\phi} = \frac{Y\delta^2 Y - Y\delta Y}{x} - \frac{(\delta Y)^2}{x^2}$$

$$P^{\psi} = \frac{Y\theta^2 Y - Y\theta Y}{x\log^2 x} - \frac{Y\theta Y}{x\log x} - \frac{(\theta Y)^2}{x^2\log^2 x}$$

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We say that ϕ is **active** if $\delta := \phi^{-1} \partial$ is small.

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$$\begin{aligned} P &= 2Y'Y''' - 3(Y'')^2 - \omega(Y')^2 \\ N(P) &= 2Y'Y''' - 3(Y'')^2 \end{aligned}$$

Applications of Newton polynomials

(Assuming that K is ω-free)

$$P(y) = 0, \qquad y \prec \mathfrak{v} \tag{$\star$}$$

$\mathfrak{m} \prec \mathfrak{v}$ starting monomial for $(\star)$ $\qquad N(P_{\times \mathfrak{m}}) \notin C Y^{\mathbb{N}}$

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	$\mathfrak{w} \prec \mathfrak{v} \implies \deg_{\prec \mathfrak{w}} P \leqslant \deg_{\prec \mathfrak{v}} P$

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K still ω-free and with a monomial group $\mathfrak{M} \subseteq K^{\neq 0}$.

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However, this is not good enough for convergence in arbitrary H-fields…

The equalizer theorem — continued proof

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For $k \geqslant 1$, let $d_k := \deg_{\prec \mathfrak{e}_k^\dagger - \mathfrak{e}_{k-1}^\dagger} R_{P,+\mathfrak{e}_k^\dagger}$ and $e_k := \deg_{\prec \mathfrak{e}_k^\dagger - \mathfrak{e}_{k-1}^\dagger} R_{P,+\mathfrak{e}_k^\dagger}$

We have $d_1 \geqslant d_2 \geqslant \cdots$ and $e_1 \geqslant e_2 \geqslant \cdots$

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We are done whenever $d_k = e_k = 0$

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For $k \geqslant 1$, let $d_k := \deg_{\prec \mathfrak{e}_k^\dagger - \mathfrak{e}_{k-1}^\dagger} R_{P,+\mathfrak{e}_k^\dagger}$ and $e_k := \deg_{\prec \mathfrak{e}_k^\dagger - \mathfrak{e}_{k-1}^\dagger} R_{P,+\mathfrak{e}_k^\dagger}$

We have $d_1 \geqslant d_2 \geqslant \cdots$ and $e_1 \geqslant e_2 \geqslant \cdots$

We are done whenever $d_k = e_k = 0$

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The equalizer theorem — continued proof

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Take $\mathfrak{e}_{k+2} := (\mathfrak{d}(R_{P,+\mathfrak{e}_{k+1}^{\dagger}}) / \mathfrak{d}(R_{Q,+\mathfrak{e}_{k+1}^{\dagger}}))^{1/(j-i)}$ instead of $\mathfrak{e}_{k+2} := \mathfrak{e}_{\text{approx}}(P_{\times \mathfrak{e}_{k+1}}, Q_{\times \mathfrak{e}_{k+1}})$

This ensures that $d_{k+2} < d_{k+1}$ or $e_{k+2} < e_{k+1}$. □

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Theorem

Let K be an ungrounded ω-free H-field with divisible Γ and real closed C. Then there exists a newtonian extension $K^n \supseteq K$ which embeds over K into any newtonian extension of K. This extension $K^n \supseteq K$ is immediate, differentially algebraic, and K^n is ω-free. We call it the ***newtonization*** *of K.*

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Corollary

*Let K be an ungrounded ω-free H-field with divisible Γ and real closed C. Then there exists a newtonian Liouville closed extension $K^{\mathrm{nl}} \supseteq K$ which embeds over K into any newtonian Liouville closed extension of K. This extension $K^{\mathrm{nl}} \supseteq K$ is differentially algebraic, ω-free, and we have $C_{K^{\mathrm{nl}}} = C$. We call K^{nl} the **Newton-Liouville closure** of K.*

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If K is newtonian, then K is asymptotically d-algebraically maximal.