Lesson 10 — H-closed H-fields

Joris van der Hoeven

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The $\mathscr{L}$-theory T^{nl} of H-closed H-fields is model complete.
It is the model companion of the $\mathscr{L}$-theory of H-fields.

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Lemma

Let E be an ω-free H-subfield of an H-closed H-field K and let $\varphi: E \to F$ be an embedding of E into a $|K|^+$-saturated H-closed H-field F. Then φ extends to an embedding $\varphi: K \to F$.

Obstruction to quantifier elimination

$$\mathrm{I}(K) := \{y' : y \in K^{\prec 1}\}$$

Proposition

Let K be an ω-free real closed H-field. Then $\mathrm{I}(K)$ is not qf-definable in the $\mathcal{L}_K$-structure K.

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Consider the immediate extensions $K\langle\lambda\rangle$ and $K\langle\lambda+\gamma\rangle$ of K with $\gamma := \ell^\dagger$, $\lambda := -\gamma^\dagger$.

$$\lambda = \frac{1}{x} + \frac{1}{x\log x} + \cdots, \qquad \lambda+\gamma = \frac{1}{x} + \frac{1}{x\log x} + \cdots + \frac{1}{x\log x\log_2 x\cdots}$$

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One verifies that they are isomorphic as H-fields over K via $\lambda \longmapsto \lambda + \gamma$.

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Let $f := (1/\ell)^\dagger = -\gamma \notin \mathrm{I}(K^*)$ and $g := (1/\ell)' = -\gamma/\ell \in \mathrm{I}(K^*)$ with $f^\dagger = -\lambda$, $g^\dagger = -(\lambda+\gamma)$.

$$f \;=\; \frac{-1}{x\log x \log_2 x \cdots}, \qquad g \;=\; \frac{-1}{x \log x \log_2 x \cdots \ell}$$

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Then $K\langle\lambda, f\rangle \not\ni \ell$ and $K\langle\lambda+\gamma, g\rangle \not\ni \ell$ are isomorphic via $\lambda \longmapsto \lambda+\gamma$ and $f \longmapsto g$.

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Now assume $\mathrm{I}(K) = \{y : \varphi(y)\}$, with φ quantifier-free in $\mathscr{L}_K$.
Then $K^* \models \neg\varphi(f)$ but $K^* \models \varphi(g)$ implies $K\langle\lambda,\gamma\rangle \models \neg\varphi(f)$ but $K\langle\lambda+\gamma, g\rangle \models \varphi(g)$.

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This violates the isomorphism between $K\langle\lambda, f\rangle$ and $K\langle\lambda+\gamma, g\rangle$. □

Quantifier elimination

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with the semantics

$$\iota(a) := a^{-1} \text{ if } a \neq 0, \qquad \iota(0) := 0$$

$$\Lambda(a) \Leftrightarrow (\exists y \prec 1) \quad a = -y^{\dagger\dagger}$$

$$\Omega(a) \Leftrightarrow (\exists y \neq 0) \quad 4y'' + ay = 0.$$

This yields a theory $T^{\mathrm{nl},\iota}_{\Lambda\Omega}$ that extends T^{nl}.

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The theory $T^{\mathrm{nl},\iota}_{\Lambda\Omega}$ eliminates quantifiers.

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Note. For model complete theories, obstruction to qf-elimination is a language issue.

Quantifier elimination — proof

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ΛΩ-field := H-field K with additional (ι, Λ, Ω)-structure.

Theorem

Let K and L be ω-free newtonian $\Lambda\Omega$-fields such that L is $|K|^+$-saturated. Let E be a substructure of K and let $\varphi\colon E \to L$ be an embedding. Then φ can be extended to an embedding $\hat{\varphi}\colon E \to L$.

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Proof ideas. Extension lemmas for various individual cases.

The relations Λ, Ω act as switchmen, dictating the direction to take at a fork. □

Completeness

$$T^{\mathrm{nl}}_{\mathrm{small}} := T^{\mathrm{nl}} + \text{small derivation} \qquad T^{\mathrm{nl},\iota}_{\Lambda\Omega,\mathrm{small}} := T^{\mathrm{nl},\iota}_{\Lambda\Omega} + \text{small derivation}$$

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The completions of T^{nl} are the two $\mathscr{L}$-theories $T^{\mathrm{nl}}_{\mathrm{small}}$ and $T^{\mathrm{nl}}_{\mathrm{large}}$.

The theories $T^{\mathrm{nl}}_{\mathrm{small}}$, $T^{\mathrm{nl}}_{\mathrm{large}}$, and T^{nl} are decidable.

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- The axioms of $T^{\mathrm{nl}}_{\mathrm{small}}$, $T^{\mathrm{nl}}_{\mathrm{large}}$, and T^{nl} can effectively be enumerated. $\square$

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$T^{\mathrm{nl}}_{\mathrm{small}} := T^{\mathrm{nl}} +$ small derivation

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Note. $\mathbb{Q}(x)^{\mathrm{nl}}$ is a **prime model** of $T^{\mathrm{nl}}_{\mathrm{small}}$ (i.e. it embeds into any other model).

The intermediate value property

Theorem

Let K be an H-closed H-field. Then the differential intermediate value property (DIVP) holds in K: for any $P \in K\{Y\}$ and $f, g \in K$ with $f < g$ and $P(f)\,P(g) < 0$, there exists an $h \in K$ with $f < h < g$ and $P(h) = 0$.

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Proof. We may arrange the derivative on K to be small via $\partial \to \phi\partial$.

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Since $T^{\mathrm{nl}}_{\mathrm{small}}$ is complete, K satisfies the same theory as $\mathbb{T}$.

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The intermediate value property holds in $\mathbb{T}$. □

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The grid-based transseries $\mathbb{T}$ form a model of $T^{\mathrm{nl}}_{\mathrm{small}}$.
Since $T^{\mathrm{nl}}_{\mathrm{small}}$ is complete, K satisfies the same theory as $\mathbb{T}$.
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Corollary

Any $\mathscr{S}$-based field of transseries of finite logarithmic depth satisfies DIVP.

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Proof. We may arrange the derivative on K to be small via $\partial \to \phi\partial$.
The grid-based transseries $\mathbb{T}$ form a model of $T_{\text{small}}^{\text{nl}}$.
Since $T_{\text{small}}^{\text{nl}}$ is complete, K satisfies the same theory as $\mathbb{T}$.
The intermediate value property holds in $\mathbb{T}$. □

Corollary

Any $\mathscr{S}$-based field of transseries of finite logarithmic depth satisfies DIVP.

Theorem

Let K be a Liouville closed H-field. Then K is H-closed if and only if it satisfies DIVP.

H-closed Hardy fields

Theorem (vdH)

There is a Hardy field that is isomorphic as an H-field to the prime model $\mathbb{Q}(x)^{\mathrm{nl}}$ *of* $T^{\mathrm{nl}}_{\mathrm{small}}$.

H-closed Hardy fields

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There is a Hardy field that is isomorphic as an H-field to the prime model $\mathbb{Q}(x)^{\mathrm{nl}}$ *of* $T^{\mathrm{nl}}_{\mathrm{small}}$.

Note. We naturally have $\mathbb{Q}(x)^{\mathrm{nl}} \subseteq \mathbb{T}$. A Hardy field that is at the same time regarded as a subfield of $\mathbb{T}$ was called a **transserial Hardy field**.

H-closed Hardy fields

Theorem (vdH)

There is a Hardy field that is isomorphic as an H-field to the prime model $\mathbb{Q}(x)^{\mathrm{nl}}$ of $T^{\mathrm{nl}}_{\mathrm{small}}$.

Note. We naturally have $\mathbb{Q}(x)^{\mathrm{nl}} \subseteq \mathbb{T}$. A Hardy field that is at the same time regarded as a subfield of $\mathbb{T}$ was called a **transserial Hardy field**.

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Corollary

$\mathbb{R}(x)^{\mathrm{nl}}$, $\mathbb{T}$, *and all maximal Hardy fields are elementary equivalent.*

Proof ingredient I

We know that maximal Hardy fields are Liouville closed.

One may check that they are ω-free.

It remains to show that they are newtonian.

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Claim: $K\langle y \rangle$ is again a Hardy field for some root y of P with $y_\rho \rightsquigarrow y$.

Idea: further normalization of quasi-linear equations

$$e^{-2e^x} y'' y^3 + e^{-e^x} y^2 - y''' + e^x y'' - y' + e^x y - 2023\, e^{-e^x} \; = \; 0, \qquad y \; \prec \; 1$$

Proof ingredient II

Idea: further normalization of quasi-linear equations

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$$(\partial - e^x)(\partial + i)(\partial - i)\, y = e^{-2e^x} y'' y^3 + e^{-e^x} y^2 - 2023\, e^{-e^x}, \qquad y \prec 1$$

Proof ingredient III

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$$(\partial - e^{x})(\partial + i)(\partial - i)\, y \;=\; e^{-2e^{x}} y'' y^{3} + e^{-e^{x}} y^{2} - 2023\, e^{-e^{x}}, \qquad y \prec 1$$

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Subtlety

$$\mathrm{e}^{-\Phi(t)} \ll 1 \quad \rightsquigarrow \quad \int_\infty^x$$

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Done correctly, the process preserves realness and asymptotic properties…

In progress

Theorem… (ADH)

Let K be a maximal Hardy field.

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Corollary

All maximal Hardy fields are back-and-forth equivalent.

Under the continuum hypothesis, they are all isomorphic.

$$\tilde{f}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)!\, z^n$$

Resummation of divergent power series

$$\tilde{f}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)!\, z^n$$

Formal Borel $\tilde{\mathscr{B}}$ ↘

$$\hat{f}(\zeta) = \sum_{n=1}^{\infty} (-1)^{n-1} \zeta^{n-1} = \frac{1}{1+\zeta}$$

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Motivation compatability with composition $\longrightarrow$ o-minimality

Surreal numbers

Conway's recursive definition

- Given sets $L, R \subseteq \mathbf{No}$ with $L < R$, there exists a $\{L|R\} \in \mathbf{No}$ with $L < \{L|R\} < R$
- All numbers in $\mathbf{No}$ can be obtained in this way

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Definition using sign sequences

- **On**: class of ordinal numbers
- A surreal number x is a sequence $(x[\beta])_{\beta<\alpha} \in \{-,+\}^{\alpha}$ for some $\ell_x := \alpha \in \mathbf{On}$
- Lexicographical ordering on such sequences (modulo completion with zeros)

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Equivalence between (No, ⩽, {|}) and (No, ⩽, ⊑)

$$\{L|R\} := \min_{\sqsubseteq} \{a \in \mathbf{No} : L < a < R\}$$

Operations on No

Ring structure. For $x = \{x_L | x_R\}$ and $y = \{y_L | y_R\}$, we define

$$
\begin{aligned}
0 &:= \{\,|\,\} \\
1 &:= \{0|\,\} \\
-x &:= \{-x_R | -x_L\} \\
x + y &:= \{x_L + y, x + y_L | x_R + y, x + y_R\} \\
xy &:= \{x'y + xy' - x'y', x''y + xy'' - x''y'' | x'y + xy'' - x'y'', x''y + xy' - x''y'\}
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Surreal numbers as Hahn series

$$\mathbf{No} \cong \mathbb{R}[[\mathbf{Mo}]], \qquad \mathbf{Mo} := \omega^{\mathbf{No}}$$

Examples

$$
\begin{aligned}
0 &:= \{\,|\,\} \\
1 &:= \{0|\,\} \\
2 &:= \{0,1|\,\} \\
&\ \vdots \\
-1 &:= \{\,|0\} \\
-2 &:= \{\,|-1,0\} \\
&\ \vdots \\
{}^1\!/_2 &:= \{0|1\} \\
{}^1\!/_4 &:= \{0|{}^1\!/_2,1\} \\
{}^3\!/_8 &:= \{0,{}^1\!/_4|{}^1\!/_2,1\} \\
&\ \vdots \\
{}^1\!/_3 &:= \{0,{}^1\!/_4,{}^5\!/_{16},\ldots|\ldots,{}^3\!/_8,{}^1\!/_2,1\} \\
\pi &:= \{0,1,2,3,3\,{}^1\!/_{16},\ldots|\ldots,3\,{}^1\!/_4,3\,{}^1\!/_2,4\} \\
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\mathbb{R} &\subseteq \mathbf{No}
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\omega &:= \{0,1,2,\ldots|\,\} \\
\omega+1 &:= \{0,1,2,\ldots,\omega|\,\} \\
&\ \vdots \\
\omega 2 &:= \{0,1,2,\ldots,\omega,\omega+1,\ldots|\,\} \\
&\ \vdots \\
\omega^2 &:= \{0,1,2,\ldots,\omega,\ldots,\omega 2,\ldots|\,\} \\
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$$

$$
\begin{aligned}
\omega^{-1} &:= \{0|\ldots,{}^1\!/_4,{}^1\!/_2,1\} \\
\exp\omega &:= \{1,\omega,\omega^2,\omega^3,\ldots|\,\}
\end{aligned}
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Derivation on the surreal numbers

Theorem (Berarducci-Mantova)

There exists a strong exp-log derivation ∂_{BM} *on* **No** *with* $\partial_{\mathrm{BM}}\,\omega = 1$.

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- But ∂_{BM} is **not** the "right" derivation with respect to ω (see below.)
- Also: how to define a composition on **No**?

Missing formal growth rates

Dubois–Reymond, Hardy, Kneser, …

There exist "regular" functions that grow faster than x, e^x, e^{e^x}, …

$$E_\omega(x+1) = e^{E_\omega(x)}$$

$\longrightarrow$ Écalle's "Grand Cantor"

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Hyperseries: generalization of transseries with

- Hyperexponentials and hyperlogarithm E_α, L_α of ordinal strength ($E_1 = \exp$)
- Nested hyperseries

Grand unification

Conjecture (vdH, 2006) ⟶ Theorem (Bagayoko-vdH, 2022)

The field $\mathbb{H}$ of hyperseries in $x \succ 1$ (for a suitable definition) is naturally isomorphic to **No***, via the map $\mathbb{H} \longrightarrow \mathbf{No}; f \longmapsto f(\omega)$ that evaluates a hyperseries f at ω.*

In particular, $\mathbb{H}$ is closed under all ***hyperexponentials*** *E_α and* ***hyperlogarithms*** *L_α for ordinal α, and $\mathbb{H}$ contains "****nested hyperseries****".*

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Problem with ∂_{BM}: $\partial_{\mathrm{BM}} E_\omega E_\omega \omega = E'_\omega E_\omega \omega \neq E'_\omega \omega \; E'_\omega E_\omega \omega$

Complex transseries

Start with logarithmic transseries at an arbitrary level $l \in \mathbb{Z}$:

$$\mathfrak{T}_0 := \mathfrak{L} \circ \exp_l z$$
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Take any ordering on $\mathbb{T}_k$ that

- is compatible with the $\mathbb{R}$-vector space structure;
- is such that $\mathfrak{m} \preccurlyeq \mathfrak{n} \Longleftrightarrow \log \mathfrak{m} \leqslant \log \mathfrak{n}$ for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{T}_k$.

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Example. We may impose $\mathrm{e}^{\mathrm{i}z} \succ 1$, $\mathrm{e}^{z^2} \prec 1$, and $\mathrm{e}^{\mathrm{i}\mathrm{e}^{\mathrm{i}z}} \succ 1$.

Complex transseries — continued

$$\begin{aligned} \mathfrak{T} &:= \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \cdots \\ \mathbb{T} &:= \mathbb{T}_0 \cup \mathbb{T}_1 \cup \cdots = \mathbb{C}[[\mathfrak{T}]]. \end{aligned}$$

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However. The isomorphism φ does not preserve $\exp$ or ∂. For instance, it might send $e^{iz} \succ_{\mathbb{T}} 1$ to $e^{-iz} \succ_{\tilde{\mathbb{T}}} 1$, but cannot send $\log(e^{iz}) = iz$ to $-iz$ or $(e^{iz})^{\dagger} = i$ to $-i$.

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However. ∂ is asymptotic, but not ordered: if $0 < \mathrm{e}^{\mathrm{i}z} \succ 1$, then $(\mathrm{e}^{\mathrm{i}z})'' = -\mathrm{e}^{\mathrm{i}z}$.

Closure results

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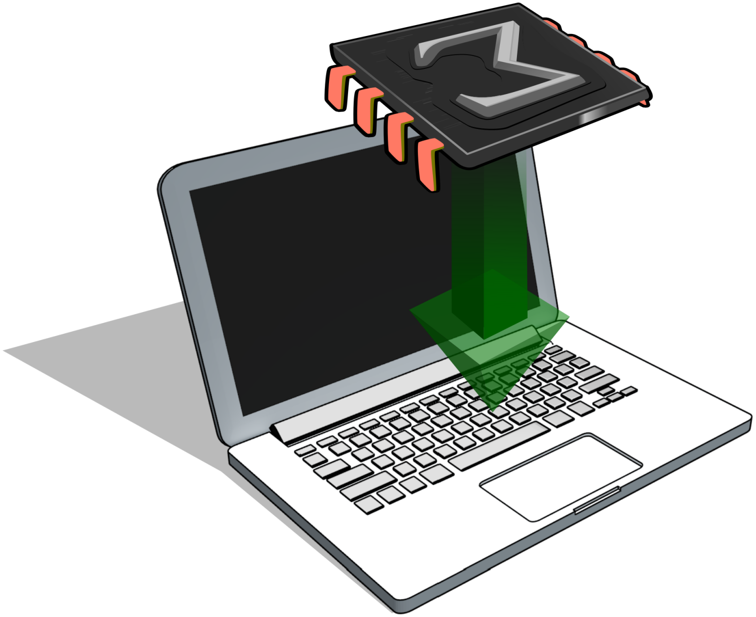
Question: what is the theory of fields of complex transseries?

Thank you !



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