Effective asymptotic analysis for finance

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Abstract

It is known that an adaptation of Newton's method allows for the computation of functional inverses of formal power series. We show that it is possible to successfully use a similar algorithm in a fairly general analytical framework. This is well suited for functions that are *highly tangent to identity* and that can be expanded with respect to asymptotic scales of "exp-log functions". We next apply our algorithm to various well-known functions coming from the world of quantitative finance. In particular, we deduce asymptotic expansions for the inverses of the Gaussian and the Black–Scholes pricing functions.

Keywords: Asymptotic expansion, algorithm, pricing, Hardy field, exp-log function, Black–Scholes formula

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1 Introduction

One notoriously complex problem in finance is the pricing of derivative products that are frequently traded on financial markets. Practitioners have proposed various sophisticated models for the dynamics of financial assets. In particular, it has been necessary to account for the existence of U-shaped "volatility smiles" which play a central role in the pricing of so-called vanilla options. Some models seem more reasonable than others because they explain not only the volatility smile, but also have properties that are directly exploitable in practice, notably the existence of easily implementable pricing formulas involving mathematical parameters that are easy to calibrate.

Subsequently, the volatility smile has been studied in a fairly general way, with a minimum of hypotheses on the probabilistic distribution of the assets [3, 4, 8, 16, 25]. This has made it possible to isolate intrinsic behaviours that are shared by a large number of models in the study of volatility smiles.

The next step has been to study the volatility smile in a model-free setting. This ultimately leads to focusing not on the Black–Scholes formula itself but on its inverse [10, 14, 30, 36]. A notable advantage of this approach is that it simplifies pricing problems. Indeed, in the case of vanilla options, such problems usually do not admit closed form solutions (except in the Black–Scholes model), so we need to resort to approximate solutions. Different techniques have been proposed to this purpose: perturbation methods with partial or stochastic differential equations, Lie symmetry theory, Watanabe theory, heat kernel expansion theory and Minakshisundaran–Pleijel's formula, large deviation theory, etc. [7, 12, 17, 20, 26, 27]. Most of these techniques give the asymptotics of price for large or small values of certain parameters involved in the computation of option prices. The study of the inverse function of the Black–Scholes formula then transforms vanilla option price asymptotics into implicit volatility asymptotics, which is the quantity of interest.

The problem of inverting Black–Scholes formula is challenging because of its non-analytic boundary behaviour. In fact, since the Black–Scholes model (as any other stochastic model) uses Brownian motion, it is not surprising that the asymptotics of the Black-Scholes formula involves logarithms. More precisely, after a suitable change of variables, the relation between vanilla option price and volatility can be expressed *via* an asymptotic expansion

$$y \;\approx\; x+\phi_0+\frac{\phi_1}{x}+\frac{\phi_2}{x^2}+\cdots, \tag{1}$$

where $\phi_0, \phi_1, \ldots$ are polynomials in $\log x$ [10, 14]. In particular, this means that

$$y \;=\; x+\phi_0+\frac{\phi_1}{x}+\cdots+\frac{\phi_n}{x^n}+\mathscr{O}(x^{-n-1/2}), \tag{2}$$

for every $n\in\mathbb{N}$. We are interested in computing a similar expansion for x in terms of y.

In computer algebra, various techniques have been developed for asymptotic expansions in general asymptotic scales. For instance, several algorithms exist for the asymptotic expansion of "exp-log" functions [15, 22, 29, 34, 35]. Such functions are built up from the rationals and an infinitely large variable $x\to\infty$ using the field operations, exponentiation and logarithm. An example of an exp-log function is $\exp(x^2-x\log x)/\log\log(x^x+3)$. The theory of transseries [9, 21, 23] makes it possible to cover asymptotic expansions of an even wider class of functions comprising many formal solutions to non-linear differential equations.

Several algorithms also exist for the functional inversion of exp-log functions [32, 33]. However, the right-hand side $\phi(x):=x+\phi_0+\phi_1\,x^{-1}+\phi_2\,x^{-2}$ of (1) is usually not an exp-log function, so these algorithms cannot be applied directly. When considering $\phi(x)$ as a formal transseries, there are also methods for computing the formal inverse $\psi=y+\psi_0+\psi_1\,y^{-1}+\psi_2\,y^{-2}+\cdots$ of ϕ [21, 23]. However, *a priori*, the analytic meaning (2) is lost during such formal computations. In this paper, we will show how to invert asymptotic expansions of the form (1) from the analytic point of view.

For each $n\in\mathbb{N}$, let $\mathscr{G}^n$ be the ring of n-fold continuously differentiable functions at infinity ($x\to\infty$). Then $\mathscr{G}^\infty:=\bigcap_{n\in\mathbb{N}}\mathscr{G}^n$ is a differential ring. We recall that a *Hardy field* is a differential subfield K of $\mathscr{G}^\infty$. It is well-known that Hardy fields [5, 18, 19] provide a suitable setting for asymptotic analysis. In section 2, we will introduce the abstract notion of an "effective Hardy field", which formalizes what we need in order to make this asymptotic calculus fully effective. Typical examples of effective Hardy fields are generated by exp-log functions. For instance, in Sections 2.3 and 2.4, we will show that $\mathbb{Q}(\log x, x, \mathrm{e}^x, \mathrm{e}^{x^2})$ is effective Hardy field. Using the aforementioned work on expansions of exp-log functions, it is possible to construct various other effective Hardy fields.

Let K be a Hardy field. We say that $\xi\in K^>\setminus\mathbb{R}$ with $\xi=\mathscr{o}(1)$ is *steep* if for any $f\in K$, there exists a $c\in\mathbb{R}$ with $f=\mathscr{O}(\xi^c)$. An element $f\in K$ is said to be *highly tangent to identity* if there exists a $c>0$ with $(f-x)/x=\mathscr{O}(\xi^c)$. For instance, if $K=\mathbb{Q}(\log x,x)$, then $\xi=x^{-1}$ is steep and $x+\log x+3\log^2 x/x$ is highly tangent to identity, contrary to $x+x/\log x$. Now assume that K is an effective Hardy field. We say that a germ $f\in\mathscr{G}^\infty$ admits an effective asymptotic expansion over K if for every $n\in\mathbb{N}$ we can compute an element $\varphi_n\in K$ with $f-\varphi_n=\mathscr{O}(\xi^n)$. If φ_1 is highly tangent to identity and $f'=\mathscr{o}(1)$, then we will prove in Section 3 that f admits a functional inverse that also admits an effective asymptotic expansion over K. Applied to the case when $K=\mathbb{Q}(\log x,x)$, this gives an algorithm for inverting asymptotic expansions of the form (1). Our algorithm relies on two main ingredients: Taylor's formula for right composition with functions that are highly tangent to identity, and Newton's method for reducing functional inversion to functional composition.

For our application to mathematical finance, it would have sufficed to work with the particular effective Hardy field $K=\mathbb{Q}(\log x,x)$. There are several reasons why we have chosen to prove our main result for general effective Hardy fields. First of all, the more general result may be useful in other areas such as combinatorics [31]. Indeed, functional inverses frequently occur when analyzing asymptotic behavior using the saddle point method. Secondly, our general setup only requires a moderate "investment" in the terminology from Section 2. Finally, it is natural to prove the results from Section 3 in this setup; the proofs would not become substantially shorter in the special case when $K=\mathbb{Q}(\log x,x)$.

This paper contains three main contributions. As far as we are aware, the application of modern asymptotic expansion algorithms to mathematical finance is new. Secondly, we introduce the framework of effective Hardy fields which we believe to be of general interest for effective asymptotic analysis. One major advantage of this framework is that it separates the potentially difficult question of constructing a suitable effective Hardy field from its actual use. The existing literature on exp-log functions and transseries can be put to use for such constructions. But for various other problems, it suffices to assume the effective Hardy field to be given as a blackbox. The third contribution of this paper is to show that this is particularly the case for the inversion of asymptotic expansions that are "highly tangent to identity".

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2 Effective Hardy fields

2.1 Hardy fields

Consider the differential ring $\mathscr{G}^{\infty} := \bigcap_{n \in \mathbb{N}} \mathscr{G}^n$, where $\mathscr{G}^n$ denotes the ring of n-fold continuously differentiable functions at infinity ($x \to \infty$) for each n. We recall that a *Hardy field* is a differential subfield K of $\mathscr{G}^{\infty}$. Since any non zero element f of Hardy fields is invertible, the sign of $f(x)$ is ultimately constant for $x \to \infty$. We define $f > 0$ if $f(x)$ is ultimately positive. It can be shown that this gives K the structure of an ordered field.

The well-known asymptotic relations $\preccurlyeq$, $\prec$, $\asymp$ and $\sim$ can be defined in terms of the ordering on K: given $f, g \in K$, we write

$$\begin{array}{rclcl} f = \mathscr{O}(g) & \iff & f \preccurlyeq g & \iff & \exists B \in \mathbb{Q}^{>}, |f| \leqslant B\,|g| \\ f = \mathscr{o}(g) & \iff & f \prec g & \iff & \forall \varepsilon \in \mathbb{Q}^{>}, |f| < \varepsilon\,|g| \end{array}$$

and

$$\begin{array}{rcl} f \asymp g & \iff & f \preccurlyeq g \preccurlyeq f \\ f \sim g & \iff & f - g \prec g. \end{array}$$

The quasi-ordering $\preccurlyeq$ is total on $K^{\neq}$: given $f, g \in K^{\neq}$, we have $f \preccurlyeq g \Leftrightarrow g \not\prec f$.

Example 1. The set $\mathscr{E}$ of *exp-log* germs at infinity is the smallest subset of $\mathscr{G}^{\infty}$ that contains $\mathbb{Q}$ and the identity function, and which is closed under $+$, $-$, $\times$, $/$, exp and log. For instance, $\exp(x^x - x \log x) / (x - 3) + \pi \log \log x \in \mathscr{E}$. In his founding work [18, 19], Hardy showed that $\mathscr{E}$ forms a Hardy field.

Example 2. More generally, given a Hardy field K, its Liouville closure K^{Li} is the smallest subset of $\mathscr{G}^{\infty}$ that contains K and that is stable under $+$, $-$, $\times$, $/$, exp, log and integration. It is well known that K^{Li} is again a Hardy field [5].

2.2 Basic properties

Let K be a Hardy field. Given $f, g \in K$, let us show that

$$f \preccurlyeq g \wedge g \not\asymp 1 \implies f' \preccurlyeq g' \tag{3}$$

$$f \prec g \wedge g \not\asymp 1 \implies f' \prec g'. \tag{4}$$

Let us first assume that $f' \not\prec g'$, whence $g' \preccurlyeq f'$, and let $x_0 \in \mathbb{R}$ and $A > 0$ be such that $|g'(x)| \leqslant A\,|f'(x)|$ for all $x \geqslant x_0$. Modulo a further increase of x_0, we may assume without loss of generality that the signs of $g'(x)$ and $f'(x)$ are constant for $x \geqslant x_0$. Then, for all $\sigma \geqslant x_0$, we have

$$\left| \int_{\sigma}^{x} g'(t)\,\mathrm{d}t \right| = \int_{\sigma}^{x} |g'(t)|\,\mathrm{d}t \leqslant A \int_{\sigma}^{x} |f'(t)|\,\mathrm{d}t = A \left| \int_{\sigma}^{x} f'(t)\,\mathrm{d}t \right|. \tag{5}$$

Consequently, $g + a \preccurlyeq f + b$ for suitable integration constants $a, b \in \mathbb{R}$. If $g \succ 1$, then this yields $g \preccurlyeq f$. If $f \prec 1$ and $g \prec 1$, then we may take $\sigma = \infty$ in (5), so that $a = b = 0$, and we again obtain $g \preccurlyeq f$. If $f \succcurlyeq 1$ and $g \prec 1$, then we clearly have $g \prec 1 \preccurlyeq f$. This proves that $f' \not\prec g' \Rightarrow f \not\prec g \vee g \asymp 1$, which implies (4). One proves $f' \not\preccurlyeq g' \Rightarrow f \not\preccurlyeq g \vee g \asymp 1$ and (3) in a similar way.

2.3 Effective Hardy fields

Let K be a Hardy field. We say that K is *effective* if its elements can be represented by instances of a concrete data structure and if we have algorithms for carrying out the basic operations $+, -, \times, /, \partial$, as well as effective tests for the relations $\leqslant$, $<$, $\preccurlyeq$ and $\prec$.

In particular, the effective inequality test for $\leqslant$ yields an equality test. Inversely, if we have an algorithm to compute signs of elements in K, then this yields effective inequality tests for both $\leqslant$ and $<$. Similarly, if, given $f \in K$, we have a way to test whether $f \preccurlyeq 1$ and $f \prec 1$, then this yields effective tests for the relations $\preccurlyeq$ and $\prec$. Indeed, given $f \in K$ and $g \in K^{\neq}$, we have $f \preccurlyeq g \Leftrightarrow f / g \preccurlyeq 1$ and $f \prec g \Leftrightarrow f / g \prec 1$.

Example 3. Let us show that $K = \mathbb{Q}(x)$ is an effective Hardy field. The basic operations $+$, $-$, $\times$, $/$ and ∂ can clearly be carried out by algorithm, and it is also clear how to do the equality test. Now consider $f = (P_p x^p + \cdots + P_0) / (Q_q x^q + \cdots + Q_0) \in K^{\neq}$ with $P_0, \ldots, P_p, Q_0, \ldots, Q_q \in \mathbb{Q}$ and $P_p \neq 0$, $Q_q \neq 0$. Then $f \sim (P_p / Q_q)\, x^{p-q}$. Consequently, $\operatorname{sign}(f) = \operatorname{sign}(P_p / Q_q)$ and $f \preccurlyeq 1 \Leftrightarrow p \leqslant q$ (resp. $f \prec 1 \Leftrightarrow p < q$).

Example 4. We claim that $K = \mathbb{Q}(\log x, x)$ is an effective Hardy field. As above, the basic operations $+$, $-$, $\times$, $/$, ∂ and the equality test are straightforward. Now any non zero element $f \in K^{\neq}$ can be written as a fraction $f = (P_p x^p + \cdots + P_0) / (Q_q x^q + \cdots + Q_0) \in K^{\neq}$ with $P_0, \ldots, P_p$, $Q_0, \ldots, Q_q \in \mathbb{Q}(\log x)$ and $P_p \neq 0$, $Q_q \neq 0$. Similarly, we may write $P_p / Q_q = (A_a (\log x)^a + \cdots + A_0) / (B_b (\log x)^b + \cdots + B_0) \in K^{\neq}$ with $A_0, \ldots, A_a, B_0, \ldots, B_b \in \mathbb{Q}$ and $A_a \neq 0$, $B_b \neq 0$. Then $f \sim (A_a / B_b)\, x^{p-q} (\log x)^{b-a}$. Consequently, $\operatorname{sign}(f) = \operatorname{sign}(A_a / B_b)$ and $f \preccurlyeq 1 \Leftrightarrow (p, a) \leqslant (q, b)$ (resp. $f \prec 1 \Leftrightarrow (p, a) < (q, b)$). Here we used the lexicographical ordering on pairs: $(p, a) \leqslant (q, b)$ if and only if $p < q$ or $p = q$ and $a \leqslant b$.

Example 5. Let K be an effective Hardy field and let $\varphi \in K$ be such that $\varphi > 0$ and $\varphi \succ 1$. Then $\varphi' > 0$, whence φ is ultimately strictly increasing and invertible for composition. Let $\psi = \varphi^{\mathrm{inv}}$ be the inverse of φ and assume that $\varphi' \circ \psi \in K$. Then $K \circ \varphi = \{f \circ \varphi : f \in K\}$ is again an effective Hardy field. Indeed, since right composition preserves the field operations and the ordering, $K \circ \varphi$ is effectively isomorphic to K as an ordered field. The derivation on $K \circ \varphi$ is given by $(f \circ \varphi)' = ((\varphi' \circ \psi) \cdot f') \circ \varphi$.

2.4 Adjunction of steep exponentials

Let $f, g \in K^{\neq}$ and let $\delta, \epsilon = \pm 1$ be such that $f^{\delta} \succcurlyeq 1$, $g^{\epsilon} \succcurlyeq 1$. We define the *flatness relations* $\preceq\!\!\!\prec$, $\prec\!\!\!\prec$ and $\asymp\!\!\!\asymp$ by

$$
\begin{aligned}
f \preceq\!\!\!\prec g &\iff \exists c \in \mathbb{Q}^{>}, |f|^{\delta} \preccurlyeq |g|^{\epsilon c} \\
f \prec\!\!\!\prec g &\iff \forall c \in \mathbb{Q}^{>}, |f|^{\delta} \prec |g|^{\epsilon c} \\
f \asymp\!\!\!\asymp g &\iff f \preceq\!\!\!\prec g \preceq\!\!\!\prec f.
\end{aligned}
$$

Let $f^{\dagger} = f' / f$ denote the logarithmic derivative of a function f. Taking logarithms, and using (3) and (4), we observe that

$$
\begin{aligned}
f \preceq\!\!\!\prec g &\iff \log |f| \preccurlyeq \log |g| \iff f^{\dagger} \preccurlyeq g^{\dagger} \\
f \prec\!\!\!\prec g &\iff \log |f| \prec \log |g| \iff f^{\dagger} \prec g^{\dagger} \\
f \asymp\!\!\!\asymp g &\iff \log |f| \asymp \log |g| \iff f^{\dagger} \asymp g^{\dagger},
\end{aligned}
$$

for all $f \in K^{\neq}$ and $g \in K^{\not\asymp} = \{h \in K : h \not\asymp 1\}$.

An element $\xi \in K^{>}$ is said to be *steep* if $f \preccurlyeq\!\!\prec \xi$ (whence $f^{\dagger} \preccurlyeq \xi^{\dagger}$) for all $f \in K^{\neq}$. If $\xi \prec 1$, then this allows us to define a valuation with respect to ξ: we set $v_{\xi}(f) = \lim\,(f^{\dagger} / \xi^{\dagger})$ for $f \in K^{\neq}$ and $v_{\xi}(0) = \infty$. Notice that the corresponding valuation group $\Gamma_{\xi} = \operatorname{im} v_{\xi}$ is a subgroup of $\mathbb{R}$. In particular, Γ_{ξ} is Archimedean. For $f \in K$ and $g \in K^{\neq}$, we notice that

$$f \preccurlyeq g \implies v_{\xi}(f) \geqslant v_{\xi}(g).$$

Indeed, since $f \preccurlyeq g \Leftrightarrow f / g \preccurlyeq 1$ and $v_{\xi}(f / g) = v_{\xi}(f) - v_{\xi}(g)$, it suffices to show this for $g = 1$. Now assume that $c := v_{\xi}(f) < 0$. Then $f^{\dagger} > \frac{c}{2}\, \xi^{\dagger}$, whence $\log |f| > \log \xi^{c/2} + C > \log \xi^{c/3}$ for some constant $C \in \mathbb{R}$. It follows that $f > \xi^{c/3} \succ 1$. If $\xi \succ\!\!\succ x$, then we also notice that $v_{\xi}(\xi^{\dagger}) = 0$. Indeed, $\xi \succ\!\!\succ x \Leftrightarrow \log \xi \succ \log x \Rightarrow \xi^{\dagger} = (\log \xi)' \succ (\log x)' \asymp x^{-1}$ and $1 / \xi^{\dagger} \prec x \Rightarrow (1 / \xi^{\dagger})' \asymp \xi^{\dagger\dagger} / \xi^{\dagger} \prec 1$, whence $v_{\xi}(\xi^{\dagger}) = \lim \xi^{\dagger\dagger} / \xi^{\dagger} = 0$.

Two examples of steep elements are x in $\mathbb{Q}(\log x, x)$ and e^{-x^2} in $\mathbb{Q}(\log x, x, \mathrm{e}^{x}, \mathrm{e}^{x^2})$. The aim of the remainder of this section is to generalize Example 4 and prove in particular that $\mathbb{Q}(\log x, x, \mathrm{e}^{x}, \mathrm{e}^{x^2})$ is indeed an effective Hardy field.

Let K be an effective Hardy field and let $\varphi \in K^{\succ} = \{h \in K : h \succ 1\}$ be such that $f^{\dagger} \prec \varphi'$ for all $f \in K$. By what precedes, this implies that $\psi := \mathrm{e}^{\varphi} \succ\!\!\succ f$ for all $f \in K$. We claim that $L := K(\psi)$ is again an effective Hardy field. Modulo the replacement of ψ by $|\psi^{-1}|$ (and φ by $-\varphi$), we may assume without loss of generality that $\psi > 0$ and $\psi \succ 1$. We clearly have algorithms for the field operations of L. Using the rule $\psi' = \varphi\, \psi$, it is also straightforward to compute derivatives of elements of L.

Now consider a polynomial $P(\psi) = P_p\, \psi^p + \cdots + P_0 \in K[\psi]$. If $P_p \neq 0$, then for each $i < p$, we have $P_i / P_p \prec\!\!\prec \psi$, so that $P_i\, \psi^i \prec P_p\, \psi^p$. Hence $P_p \neq 0$ implies $P(\psi) \sim P_p\, \psi^p$. This also shows that $P(\psi) = 0 \Leftrightarrow P_0 = \cdots = P_p = 0$, which provides us with an effective zero test for $K[\psi]$, as well as for L. Given a rational function $P(\psi) / Q(\psi) = (P_p\, \psi^p + \cdots + P_0) / (Q_q\, \psi^q + \cdots + Q_0) \in L$ with $P_p \neq 0$ and $Q_q \neq 0$, we also have $P(\psi) / Q(\psi) \sim (P_p / Q_q)\, \psi^{p-q}$. Consequently, $\operatorname{sign}(f) = \operatorname{sign}(P_p / Q_q)$ and $f \preccurlyeq 1$ if and only if $p < q$ or $p = q$ and $P_p \preccurlyeq Q_q$. Similarly, $f \prec 1$ if and only if $p < q$ or $p = q$ and $P_p \prec Q_q$.

Example 6. Starting with $K = \mathbb{Q}(\log x, x)$ as in Example 4, applying the above argument twice shows that both $K(\mathrm{e}^{x})$ and $K(\mathrm{e}^{x}, \mathrm{e}^{x^2}) = \mathbb{Q}(\log x, x, \mathrm{e}^{x}, \mathrm{e}^{x^2})$ are effective Hardy fields. Applying Example 5 for $\varphi = \log x$, we also obtain that $\mathbb{Q}(\log \log x, \log x, x, x^{\log x})$ is an effective Hardy field.

Remark 7. In order to compute with more general exp-log germs in $\mathscr{E}$, one also needs to show that fields such as $\mathbb{Q}(x, \mathrm{e}^{x}, \mathrm{e}^{\sqrt{2} x})$ form effective Hardy fields. One even more difficult problem is to provide an effective zero test for exp-log constants, i.e. constants formed from the rationals, using $+$, $-$, $\times$, $/$, exp and log. Provided that Schanuel's conjecture holds, such an algorithm was given by Richardson [28]. His algorithm always returns correct results, but might not terminate if one explicitly hits a counterexample to the conjecture. Given a zero-test for exp-log constants, it can be shown that $\mathscr{E}$ forms an effective Hardy field [22].

2.5 Limits and asymptotic scales

Let K be a Hardy field. Given $f \in K^{\preccurlyeq} = \{\varphi \in K : \varphi \preccurlyeq 1\}$, there exists a unique $\ell \in \mathbb{R}$ with $f - \ell \prec 1$, which is called the *limit* of f, and denoted by $\ell = \lim f$. We say that K is *closed under limits* if $\lim f \in K$ for all $f \in K$. If K is effective and $\lim : K^{\preccurlyeq} \to K$ is computable, then we say that K admits an *effective limit map*.

An *asymptotic scale* for K is a multiplicative subgroup $\mathfrak{M} \subseteq K^{>}$ such that $\mathfrak{M}$ is totally ordered for $\preccurlyeq$ and such that there exists a mapping $\mathfrak{d} : K^{\neq} \to \mathfrak{M}$ with $\mathfrak{d}(f) \asymp f$ for all $f \in K^{\neq}$. We call $\mathfrak{d}(f)$ the *dominant monomial* of f and notice that $\mathfrak{d}$ is necessarily a group homomorphism. If K is effective and $\mathfrak{d}$ is computable, then we call $\mathfrak{M}$ an effective asymptotic scale.

Assume that K is closed under limits and that K also admits an asymptotic scale $\mathfrak{M}$. Given $f \in K^{\neq}$, we call $\tau(f) = (\lim f / \mathfrak{d}(f))\, \mathfrak{d}(f)$ the *dominant term* of f, and notice that $f \sim \tau(f)$. If $\mathfrak{d}$ and lim are both computable, then the same clearly holds for τ.

Example 8. In Example 3, we have given a method for the explicit computation of an equivalent in $\mathbb{Q}^{\neq} x^{\mathbb{Z}} = \{c x^k : c \in \mathbb{Q}^{\neq}, k \in \mathbb{Z}\}$ for any $f \in \mathbb{Q}(x)^{\neq}$. This both shows that $\mathbb{Q}(x)$ admits an effective limit map and that it admits $x^{\mathbb{Z}}$ as an effective asymptotic scale. Similarly, Example 4 shows that the same holds for $\mathbb{Q}(\log x, x)$, in which case the asymptotic scale becomes $(\log x)^{\mathbb{Z}} x^{\mathbb{Z}}$.

More generally, let K be an effective Hardy field and let φ, ψ be as in Section 2.4. Assume that K admits an effective limit map and that $\mathfrak{M}$ is an effective asymptotic scale. For each $f \in K(\psi)^{\neq}$, we have shown how to compute an equivalent $f \sim g \psi^k \sim \tau(g) \psi^k$ with $g \in K^{\neq}$ and $k \in \mathbb{Z}$. Since $g \prec\!\!\prec \psi^k$ for any $g \in \mathfrak{M}$ and $k \neq 0$, the group $\mathfrak{M} \psi^{\mathbb{Z}}$ is totally ordered for $\preccurlyeq$. This shows that $K(\psi)$ admits both an effective limit map and an effective asymptotic scale $\mathfrak{M} \psi^{\mathbb{Z}}$.

Example 9. Let K be an effective Hardy field and let φ be as in Example 5. If K admits an effective limit map, then so does $K \circ \varphi$, since $\lim f \circ \varphi = \lim f$ for all $f \in K^{\neq}$. If K admits an effective asymptotic scale $\mathfrak{M}$, then $K \circ \varphi$ admits $\mathfrak{M} \circ \varphi$ as an effective asymptotic scale, with $\mathfrak{d}(f \circ \varphi) = \mathfrak{d}(f) \circ \varphi$ for all $f \in K^{\neq}$.

3 Composition and functional inversion

Let K be a Hardy field which contains the identity function x, as well as a steep element $\xi \in K^{>, \prec} = \{\varphi \in K^{>} : \varphi \prec 1\}$. If $\xi \asymp\!\!\asymp x$, then also assume that $\xi = x^{-1}$.

An element $f \in K$ is said to be *highly tangent to identity* if there exists a $c > 0$ with $(f - x)/x = \mathscr{O}(\xi^c)$. Equivalently, this means that f is of the form $f = x + \delta$ with $v_\xi(\delta) > v_\xi(x)$. If $\xi = x^{-1}$, then this is the case when $\delta \prec \xi^\alpha$ for some $\alpha > -1$. If $\xi \succ\!\!\succ x$, then we rather should have $\delta \prec \xi^\alpha$ for some $\alpha > 0$. In particular, in both cases we have $\delta' \prec 1$ and even $v_\xi(\delta') > 0$. We will denote by T the subset of K of all elements that are highly tangent to identity.

Since Hardy fields are not necessarily closed under composition and functional inversion, the set T does not necessarily form a group. The main aim of this section is to show that a suitable completion of T does form a group (Theorem 20 below). Moreover, under suitable hypothesis, there are algorithms for computing asymptotic expansions of compositions and functional inverses.

3.1 First order functional inversion

Lemma 10. *Let $\delta \in T - x$. Then for any germ $\eta \in \mathscr{G}^\infty$ with $\eta \preccurlyeq \delta$ and $\eta' \prec 1$, we have*

$$(x + \eta)^{\mathrm{inv}} - x \;=\; \mathscr{O}(\delta).$$

Proof. Without loss of generality, we may assume that $\delta > 0$. For any $c \in \mathbb{R}$, we claim that $\delta \circ (x + c\,\delta) \sim \delta$. Indeed, given $\varepsilon > 0$, let x_0 be such that $\delta'(x)$ has constant sign and $|\delta'(x)| < \varepsilon$ for $x \geqslant x_0$. Assume also that $\delta(x + c\,\delta(x))$ is defined for $x \geqslant x_0$. Then

$$|\delta(x + c\,\delta(x)) - \delta(x)| \;\leqslant\; \left| \int_x^{x + c\delta(x)} \delta'(t)\,\mathrm{d}t \right| \;<\; \varepsilon\,|c|\,\delta(x),$$

for all $x \geqslant x_0$. We conclude that $\delta \circ (x + c\,\delta) - \delta \prec \delta$, by letting ε tend to zero.

The assumption that $\eta' \prec 1$ implies that $(x + \eta)' \sim 1$, whence $\varphi(x) := x + \eta(x)$ is strictly increasing for sufficiently large x. This shows that φ indeed admits an inverse function ψ at infinity. Let $A > 0$ be such that $|\eta(x)| \leqslant A\,\delta(x)$ for sufficiently large x. Setting $l(x) = x - 2\,A\,\delta(x)$ and $r(x) = x + 2\,A\,\delta(x)$, our claim implies

$$\begin{array}{rclclclcl} \varphi(l(x)) & = & l(x) + \eta(l(x)) & \leqslant & l(x) + A\,\delta(l(x)) & < & l(x) + 2\,A\,\delta(x) & = & x \\ \varphi(r(x)) & = & r(x) + \eta(r(x)) & \geqslant & r(x) - A\,\delta(r(x)) & > & r(x) - 2\,A\,\delta(x) & = & x, \end{array}$$

for sufficiently large x. Since φ is strictly increasing, it follows that $l(x) < \psi(x) < r(x)$. In other words, $|\psi(x) - x| \leqslant 2\, A\, \delta(x)$ for sufficiently large x. □

3.2 First order right composition

Lemma 11. *Let $f \in K$ and $g = x + \varepsilon \in T$. Then for any germs $\eta, \delta \in \mathscr{G}^\infty$ with $\eta \preccurlyeq f$ and $\delta \preccurlyeq \varepsilon$, we have*

$$\eta \circ (x + \delta) \;=\; \mathscr{O}(f).$$

Proof. Since ξ is a steep element, there exists a constant $A > 0$ with $|f^\dagger| \leqslant A\, |\xi^\dagger|$. We also notice that $\xi^\dagger \varepsilon \prec 1$. Indeed, this is immediate if $\xi = 1/x$. If $\xi \succ\!\!\succ x$, then $\varepsilon \prec \xi^c$ for some $c > 0$ and $\xi^\dagger \varepsilon \prec \xi^\dagger \xi^c \asymp (\xi^c)' \prec 1$, since $\xi^c \prec x$.

Let us first show that $f \circ g \sim f$, whenever $f \prec x$ and $g \geqslant x$. Since $f \prec x$ implies $f' \prec 1$, the function $|f'|$ is ultimately decreasing. For sufficiently large x, it follows that $|f'(t)| \leqslant |f'(x)|$ for $t \in [x, g(x)]$, whence

$$|f(g(x)) - f(x)| \;\leqslant\; \int_x^{g(x)} |f'(t)|\, \mathrm{d}t \;\leqslant\; |f'(x)|\, \varepsilon(x) \;\leqslant\; A\, |\xi^\dagger(x)|\, \varepsilon(x)\, |f(x)|.$$

Since $\xi^\dagger \varepsilon \prec 1$, this shows that $f \circ g \sim f$.

Let us next show that we also have $f \circ g \sim f$ in the case when $f \prec x$ and $g < x$ (so that $\varepsilon < 0$). Then Lemma 10 implies $g^{\mathrm{inv}} = x + \mathscr{O}(\varepsilon)$, whence $x < g^{\mathrm{inv}} < x - B\, \varepsilon$ for some $B \in \mathbb{R}^>$. Let $\lambda > 0$. By what precedes, there exists an x_0 with $|f(x - B\, \varepsilon(x)) - f(x)| \leqslant \lambda\, |f(x)|$ for all $x \geqslant x_0$. Modulo a further increase of x_0, we may also arrange that $f(x)$ is monotonic for $x \geqslant x_0$. It follows that $|f(g^{\mathrm{inv}}(x)) - f(x)| \leqslant \lambda\, |f(x)|$, whence $f \circ g^{\mathrm{inv}} \sim f$. Post-composing with g, we again obtain $f \circ g \sim f$.

Let us finally assume that $f \succcurlyeq x$. Then the above arguments prove that $(1/f) \circ g \sim (1/f)$. Consequently, $f \circ g = ((1/f) \circ g)^{-1} \sim (1/f)^{-1} = f$.

The above arguments conclude the proof in the case when $\eta = f$ and $\delta = \varepsilon$. Let us next consider the case when we still have $\eta = f$, but $\delta \preccurlyeq \varepsilon$ is general. Let $B > 0$ be such that $|\delta| \leqslant B\, |\varepsilon|$. For sufficiently large x, it follows that $f(x + \delta(x))$ is comprised between $f(x - B\, |\varepsilon(x)|)$ and $f(x + B\, |\varepsilon(x)|)$, which are both equivalent to $f(x)$. This shows that $f \circ (x + \delta) \sim f$.

As to the general case, let $C > 0$ be such that $|\eta| \leqslant C\, |f|$. By what precedes, we have $|\eta(x + \delta(x))| \leqslant C\, |f(x + \delta(x))| \leqslant 2\, C\, |f(x)|$ for all sufficiently large x. This shows that $\eta \circ (x + \delta) \preccurlyeq f$. □

3.3 General composition

Lemma 12. *Let $f \in K$ and $g \in T$. Let $\varphi, \psi \in K$ and $n \in \mathbb{N}$ be such that $x + \psi \in T$ and $f^{(n)} \cdot (g - x)^n \preccurlyeq \varphi$. Then for any $\delta, \varepsilon \in \mathscr{G}^\infty$ with $\delta \preccurlyeq \varphi$ and $\varepsilon \preccurlyeq \psi$, we have*

$$(f + \delta) \circ (g + \varepsilon) \;=\; f + f' \cdot (g - x) + \cdots + \tfrac{1}{(n-1)!}\, f^{(n-1)} \cdot (g - x)^{n-1} + \mathscr{O}(\max(|\varphi|, |f'\, \psi|)).$$

Proof. Let us first consider the case when $\delta = \varepsilon = 0$ and consider

$$\begin{aligned} \eta &= f + f'\, (g - x) + \cdots + \tfrac{1}{(n-1)!}\, f^{(n-1)}\, (g - x)^{n-1} \\ R &= f \circ g - \eta \end{aligned}$$

For sufficiently large x, Taylor's formula with integral remainder yields

$$R(x) \;=\; \int_x^{g(x)} \frac{1}{(n-1)!}\, f^{(n)}(t)\, (g(x) - t)^{n-1}\, \mathrm{d}t.$$

For sufficiently large x, the function $f^{(n)}$ is also monotonic, whence

$$|R(x)| \;\leqslant\; \frac{1}{n!} \max(|f^{(n)}(x)|, |f^{(n)}(g(x))|)\, |g(x) - x|^n.$$

By Lemma 11, we have $f^{(n)} \circ g \preccurlyeq f^{(n)}$, whence $R \preccurlyeq f^{(n)}\,(g-x)^n \preccurlyeq \varphi$. This completes the proof in the case when $\delta = \varepsilon = 0$.

As to the general case, we have

$$|f(g(x)+\varepsilon(x)) - f(g(x))| \;\leqslant\; \int_{g(x)}^{g(x)+\varepsilon(x)} |f'(t)|\,\mathrm{d}t \;\leqslant\; \max(|f'(g(x))|, |f'(g(x)+\varepsilon(x))|)\,|\varepsilon(x)|,$$

for all sufficiently large x. Now Lemmas 10 and 11 imply $\varphi \circ (g+\varepsilon) = \varphi \circ (x + \varepsilon \circ g^{\mathrm{inv}}) \circ g \preccurlyeq \varphi \circ g \preccurlyeq \varphi$ and similarly $f' \circ (g+\varepsilon) \preccurlyeq f'$. Consequently,

$$\begin{aligned} |(f+\delta)\circ(g+\varepsilon) - \eta| &\leqslant |\delta\circ(g+\varepsilon)| + |f\circ(g+\varepsilon) - f\circ g| + |f\circ g - \eta| \\ &\preccurlyeq \max(|\varphi\circ(g+\varepsilon)|, |f'\,\varepsilon|, |\varphi|) \\ &\preccurlyeq \max(|f'\,\varepsilon|, |\varphi|). \end{aligned}$$

This concludes the proof in the general case. □

Lemma 13. *For any $f \in K$, $g \in T$ and $\varphi \in K^{\neq}$, there exists an $n \in \mathbb{N}$ with $f^{(n)} \cdot (g-x)^n \preccurlyeq \varphi$.*

Proof. Let us first consider the case when $\xi = x^{-1}$, so that $v_\xi(g-x) > -1$. For any $f \in K^{\neq}$, we have $f' = f^\dagger f \preccurlyeq \xi^\dagger f \asymp f/x$, whence $v_\xi(f') \geqslant v_\xi(f) + 1$. Consequently,

$$v_\xi(f^{(n)}\,(g-x)^n) \;\geqslant\; v_\xi(f) + n + n\,v_\xi(g-x).$$

It thus suffices to take $n > (v_\xi(\varphi) - v_\xi(f))/(v_\xi(g-x)+1)$ in order to ensure that $v_\xi(f^{(n)}\,(g-x)^n) > v_\xi(\varphi)$ and therefore $f^{(n)}\,(g-x)^n \preccurlyeq \varphi$.

Assume next that $\xi \succ\!\!\succ x$, so that $v_\xi(g-x) > 0$. We again have $f' \preccurlyeq \xi^\dagger f$ for all $f \in K^{\neq}$, but this time, we rather obtain $v_\xi(f') \geqslant v_\xi(f)$, since $v_\xi(\xi^\dagger) = 0$. Therefore,

$$v_\xi(f^{(n)}\,(g-x)^n) \;\geqslant\; v_\xi(f) + n\,v_\xi(g-x).$$

Taking $n > (v_\xi(\varphi) - v_\xi(f))/v_\xi(g-x)$, we again obtain the desired result. □

If K is an effective Hardy field, then the above lemmas lead to the following algorithm for approximate composition:

Algorithm compose(f, g, φ)
INPUT: $f \in K$, $g \in T$ and $\varphi \in K^{\neq}$ with $v_\xi(\varphi) > v_\xi(x\,f')$
OUTPUT: $h \in K$ with $f \circ g = h + \mathscr{O}(\varphi)$
Moreover, for all $\delta, \varepsilon \in \mathscr{G}^\infty$ with $\delta \preccurlyeq \varphi$, $\varepsilon\,f' \preccurlyeq \varphi$ and $v_\xi(\varepsilon/x) > 0$, we have $(f+\delta)\circ(g+\varepsilon) = h + \mathscr{O}(\varphi)$

Let $n \in \mathbb{N}$ be minimal with $f^{(n)} \cdot (g-x)^n \preccurlyeq \varphi$
Return $f + \cdots + \frac{1}{(n-1)!} f^{(n-1)} \cdot (g-x)^{n-1}$

Theorem 14. *The algorithm* **compose** *is correct.*

Proof. The existence of n is ensured by Lemma 13. Since K is effective, we have an algorithm for doing the test $f^{(n)}\,(g-x)^n \preccurlyeq \varphi$, which enables us to compute n. Setting $\psi = \varphi / f'$, our assumption that $v_\xi(\varphi) > v_\xi(x\,f')$ ensures that $x + \psi \in T$. The result now follows from Lemma 12. □

Remark 15. In addition, by considering both cases $\xi = x^{-1}$ and $\xi \succ\!\!\succ x$, it can be verified that $v_\xi(h) = v_\xi(f)$, that $f \in T$ implies $h \in T$, and that $v_\xi(f-1) > 0$ implies $v_\xi(h-1) > 0$.

3.4 General functional inversion

A well-known way to solve functional equations of the form $f \circ g = x$ is Newton's method [6]. We will now show that this method indeed yields a quadratic convergence in our setting.

Lemma 16. *Let $f, g \in T$ and $\varepsilon \in K$ be such that $f \circ g - x = \mathscr{O}(x\,\varepsilon)$ and $v_\xi(\varepsilon) > 0$. Let $\tilde{g} \in T$ be such that*

$$\tilde{g} \;=\; g - \frac{f \circ g - x}{f' \circ g} + \mathscr{O}(x\,\varepsilon^2).$$

Then $f \circ \tilde{g} - x = \mathscr{O}(x\,\varepsilon^2)$.

Proof. Since $f \sim x$, we notice that $f' \sim 1$ and $f' \circ g \sim 1$. Let $\delta = g - \tilde{g} = \frac{f \circ g - x}{f' \circ g} + \mathscr{O}(x\,\varepsilon^2) = \mathscr{O}(x\,\varepsilon)$. For all sufficiently large x, we have

$$f(\tilde{g}(x)) \;=\; f(g(x)) - f'(g(x))\,\delta(x) + \int_{g(x)}^{g(x)-\delta(x)} f''(t)\,(g(x) - \delta(x) - t)\,\mathrm{d}t,$$

whence, using the ultimate monotonicity of f'' on $[g(x), \tilde{g}(x)]$,

$$f \circ \tilde{g} \;=\; f \circ g - (f' \circ g)\,\delta + \mathscr{O}(\max(|f'' \circ g|, |f'' \circ \tilde{g}|)\,\delta^2).$$

Using Lemma 11, we also have $f'' \circ g \preccurlyeq f''$ and $f'' \circ \tilde{g} \preccurlyeq f''$, whence

$$f \circ \tilde{g} \;=\; f \circ g - (f' \circ g)\,\delta + \mathscr{O}(f''\,\delta^2).$$

Consequently,

$$\begin{aligned} f \circ \tilde{g} - x \;&=\; (f \circ g - x) - (f' \circ g)\,\delta + \mathscr{O}(f''\,\delta^2) \\ &=\; (f' \circ g)\,\delta + \mathscr{O}(x\,\varepsilon^2) - (f' \circ g)\,\delta + \mathscr{O}(f''\,\delta^2) \\ &=\; \mathscr{O}(x\,\varepsilon^2) + \mathscr{O}(f''\,\delta^2). \end{aligned}$$

Now $f - x \prec x$ implies $(f - x)' \prec 1 \prec \log x$ and $f'' = (f - x)'' \prec x^{-1}$. Consequently,

$$f \circ \tilde{g} - x = \mathscr{O}(x\,\varepsilon^2) + \mathscr{O}(f''\,\delta^2) = \mathscr{O}(x\,\varepsilon^2) + \mathscr{O}(\delta^2 / x) = \mathscr{O}(x\,\varepsilon^2).$$

This completes the proof. □

If K is an effective Hardy field, then this lemma leads to the following algorithm for the computation of approximate functional inverses:

Algorithm invert(f, ε)
INPUT: $f \in T$ and $\varepsilon \in K^{\neq}$ with $v_\xi(\varepsilon) > 0$
OUTPUT: $g \in T$ with $f^{\mathrm{inv}} = g + \mathscr{O}(x\,\varepsilon)$
Moreover, for any $\delta \in \mathscr{G}^\infty$ with $\delta \preccurlyeq x\,\varepsilon$ and $\delta' \prec 1$, we have $(f + \delta)^{\mathrm{inv}} = g + \mathscr{O}(x\,\varepsilon)$

Let $g := x$
repeat
Let $h :=$ **compose**$(f, g, x\,\varepsilon^2)$
If $h - x \preccurlyeq x\,\varepsilon$ then return g
Let $d :=$ **compose**$(f', g, x\,\varepsilon^2)$
Let $g := g - (h - x) / d$

Theorem 17. *Let $\eta = (f - x)/x$. The algorithm* **invert** *is correct and terminates after at most $\lfloor \log(v_\xi(\varepsilon)/v_\xi(\eta))/\log 2 \rfloor + 1$ iterations of the main loop.*

Proof. Let us first show that $g \in T$ throughout the algorithm. This is clear at the start. At each iteration $g := g - (h - x)/d$, Remark 15 implies $v_\xi(h - x) > v_\xi(x)$ and $v_\xi(d - 1) > v_\xi(1)$, whence $v_\xi((h - x)/d) > v_\xi(x)$, so that $g - (h - x)/d \in T$.

On termination, we have $h = f \circ g + \mathscr{O}(x\,\varepsilon^2)$ and $h = x + \mathscr{O}(x\,\varepsilon)$, whence $f \circ g - x = \mathscr{O}(x\,\varepsilon)$. Applying Lemma 10 with $x\,\varepsilon$ and $f \circ g - x$ in the roles of δ and η, we obtain $g^{\mathrm{inv}} \circ f^{\mathrm{inv}} - x = (f \circ g)^{\mathrm{inv}} - x = \mathscr{O}(x\,\varepsilon)$. Consequently, $f^{\mathrm{inv}} = g \circ (g^{\mathrm{inv}} \circ f^{\mathrm{inv}}) = g + \mathscr{O}(g' \cdot (g^{\mathrm{inv}} \circ f^{\mathrm{inv}} - x)) = g + \mathscr{O}(x\,\varepsilon)$. Furthermore, $(f + \delta)^{\mathrm{inv}} = [(x + \delta \circ f^{\mathrm{inv}}) \circ f]^{\mathrm{inv}} = f^{\mathrm{inv}} \circ (x + \delta \circ f^{\mathrm{inv}})^{\mathrm{inv}} = f^{\mathrm{inv}} \circ (x + \mathscr{O}(\delta))^{\mathrm{inv}} = f^{\mathrm{inv}} \circ (x + \mathscr{O}(x\,\varepsilon))^{\mathrm{inv}} = f^{\mathrm{inv}} \circ (x + \mathscr{O}(x\,\varepsilon)) = f^{\mathrm{inv}} + \mathscr{O}((f^{\mathrm{inv}})'\,x\,\varepsilon) = g + \mathscr{O}(x\,\varepsilon)$.

As to the termination, consider the quantity

$$\nu := \tilde{v}_\xi\!\left(\frac{f \circ g - x}{x}\right) := \sup\left\{\alpha \in \mathbb{R} : \frac{f \circ g - x}{x} = \mathscr{O}(\xi^\alpha)\right\}.$$

At the very start, we have $\nu = \tilde{v}_\xi(\eta) = v_\xi(\eta) > 0$. At every iteration $\tilde{g} := g - (h - x)/d$, we have $\tilde{g} = g - \frac{f \circ g - x}{f' \circ g} + \mathscr{O}(x\,\varepsilon^2)$. Lemma 16 therefore ensures that ν doubles at least, whereas the algorithm terminates as soon as $\nu > v_\xi(\varepsilon)$. This happens after at most $\lfloor \log(v_\xi(\varepsilon)/v_\xi(\eta))/\log 2 \rfloor + 1$ iterations. □

3.5 Effective asymptotic expansions

We now extend the definition of high tangency to identity to all germs. We say that a germ $f \in \mathscr{G}^\infty$ is *highly tangent to identity* if there exists a $c > 0$ with $f - x = \mathscr{O}(x\,\xi^c)$ and $f' = 1 + \mathscr{o}(1)$. We denote by $\mathscr{T}^\infty$ the set of such germs. We say that f admits an *asymptotic expansion* over K if for every $n \in \mathbb{N}$, there exists an element $\varphi_n \in K$ with $f - \varphi_n = \mathscr{O}(\xi^n)$. If we have an algorithm for computing φ_n as a function of n, then we say that f admits an *effective asymptotic expansion* over K.

Proposition 18. *Assume that $f \in \mathscr{G}^\infty$ and $g \in \mathscr{T}^\infty$ admit effective asymptotic expansions over K. Then so does $f \circ g$. If $f \in \mathscr{T}^\infty$, then $f \circ g \in \mathscr{T}^\infty$.*

Proof. Given $n \geqslant 1$, we may compute $\varphi_n \in K$ and $\psi_n \in T$ with $\delta := f - \varphi_n \preccurlyeq \xi^n$ and $\varepsilon := g - \psi_n \preccurlyeq \xi^n$. Assume that there exists an $n_0 \in \mathbb{N}$ with $f \not\preccurlyeq \xi^{n_0}$. Then for all $n > n_0$, we must have $\varphi_n \succ \xi^{n_0}$ and $v_\xi(x\,\varphi_n') \leqslant n_0 < n$. Consequently, we may compute $\chi_n = \mathbf{compose}(\varphi_n, \psi_n, \xi^n)$, and $f \circ g = (\varphi_n + \delta) \circ (\psi_n + \varepsilon) = \chi_n + \mathscr{O}(\xi^n)$. If $f \preccurlyeq \xi^n$ for all $n \in \mathbb{N}$, then we also have $f \circ g \preccurlyeq \xi^n$ for all $n \in \mathbb{N}$.

If $f \in \mathscr{T}^\infty$, then we also get $\varphi_n, \psi_n, \chi_n \in T$, whence $v_\xi(f \circ g - x) = v_\xi(\chi_n - x + \mathscr{O}(\xi^n)) \geqslant \min(v_\xi(\chi_n - x), n) > v_\xi(x)$. Moreover, $(f \circ g)' = g' \cdot f' \circ g = (1 + \mathscr{o}(1))\,(1 + \mathscr{o}(1) \circ g) = 1 + \mathscr{o}(1)$, whence $f \circ g \in \mathscr{T}^\infty$. □

Proposition 19. *Assume that $f \in \mathscr{T}^\infty$ admits an effective asymptotic expansion over K. Then so does f^{inv} and $f^{\mathrm{inv}} \in \mathscr{T}^\infty$.*

Proof. Given $n \geqslant 1$, we may compute $\varphi_n \in T$ with $\delta := f - \varphi_n \preccurlyeq \xi^n$. Let $\psi_n = \mathbf{invert}(\varphi_n, \xi^n/x)$. Then $f^{\mathrm{inv}} = (\varphi_n + \delta)^{\mathrm{inv}} = \psi_n + \mathscr{O}(\xi^n)$. Moreover, $(f^{\mathrm{inv}})' = (f' \circ f^{\mathrm{inv}})^{-1} = ((1 + \mathscr{o}(1)) \circ f^{\mathrm{inv}}) = 1 + \mathscr{o}(1)$, whence $f^{\mathrm{inv}} \in \mathscr{T}^\infty$. □

Combining these two propositions, we have shown the following:

Theorem 20. *The set of germs in $\mathscr{T}^\infty$ that admit effective asymptotic expansions over K forms a group for functional composition.* □

4 Examples and applications to finance

4.1 Lambert W function

The Lambert W function is defined to be the inverse function of $x \mapsto x\,\mathrm{e}^x$. Using our algorithm, we can compute the asymptotic expansion of the inverse function $W(\mathrm{e}^x)$ of $x \mapsto x + \log x$. This also yields the asymptotic expansion of $W(x)$ for large x.

4.2 Gaussian function

Let $(\Psi_n)_{n>0}$ be defined formally by

$$\log\left(1 + \sum_{n>0} a_n\, X^n\right) \;=\; \sum_{n>0} \Psi_n\,(a_1, \ldots, a_n)\, X^n$$

and let Φ be the Gaussian function:

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t, \qquad \varphi(t) = \frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{t^2}{2}}.$$

For any $n \in \mathbb{N}$, we have the well-known relation

$$1 - \Phi(x) \;=\; \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\,\pi}\,x}\left(1 + \sum_{i=1}^{n} \frac{(-1)^i\,(2\,i-1)!!}{x^{2i}} + \mathscr{O}\left(\frac{1}{x^n}\right)\right), \tag{6}$$

where we used the notation

$$(2\,i-1)!! \;:=\; \prod_{k=1}^{i}\,(2\,k-1).$$

The relation (6) shows that

$$f(x) \;=\; x + \log x + \varphi_0 + \sum_{i=1}^{n} \frac{\varphi_i}{x^i} + \mathscr{O}\left(\frac{1}{x^n}\right), \tag{7}$$

with $x > 0$,

$$\begin{aligned} f(x) \;&:=\; 2\log\,(1 - \Phi(\sqrt{x})) \\ \varphi_0 \;&:=\; 2\log\,(\sqrt{2\,\pi}) \\ \varphi_i \;&:=\; -\Psi_i(-1, \ldots, (-1)^i\,(2\,i-1)!!) \qquad (i>0) \end{aligned}$$

Our algorithm now allows us to compute the asymptotic expansion of the inverse function of Gaussian law at $+\infty$. This is potentially of great interest in finance when it comes to calculate risk measures. The formula (7) gives itself an asymptotic expansion of a Gaussian Value-at-Risk VaR_α in terms of its confidence level α.

4.3 Gaussian expected shortfall

The expected shortfall of a portfolio with confidence level $\alpha \in (0,1)$ is the expected loss conditional that the loss is greater than the α-th percentile of the loss distribution. When the return of the portfolio is Gaussian with mean μ and volatility σ, the expected shortfall is

$$\mathrm{ES}_\alpha = \mu + \sigma\,\frac{\varphi \circ \Phi^{-1}(\alpha)}{1-\alpha}.$$

With $x = \Phi^{-1}(\alpha)$, the relation (6) yields

$$\frac{\varphi(x)}{1-\Phi(x)} = \frac{x}{1+\sum_{i=1}^{n}\frac{(-1)^i\,(2\,i-1)!!}{x^{2i}}+\mathscr{O}\big(\frac{1}{x^n}\big)}.$$

So, for some constants $u_i \in \mathbb{R}$, we have

$$\frac{\mathrm{ES}_\alpha-\mu}{\sigma} = x+\sum_{i=0}^{n}\frac{u_i}{x^{2i+1}}+\mathscr{O}\bigg(\frac{1}{x^{2n+1}}\bigg).$$

By inverting (7), we get an asymptotic expansion of ES_α in terms of $\alpha \to 1$.

4.4 Incomplete Gamma function

Let $(u_k)_{k\geqslant 0}$ be defined by $u_0 = 1$ and, for $k > 0$, $u_k = (a-1)\,(a-2)...(a-k)$. A well-known relation for $\Gamma(a, x)$ tells that for $a \in \mathbb{R}$ and $x > 0$,

$$\Gamma(a,x) = x^{a-1}\,\mathrm{e}^{-x}\left(\sum_{k=0}^{n}\frac{u_k}{x^k}+\mathscr{O}\bigg(\frac{1}{x^n}\bigg)\right). \tag{8}$$

Taking logarithms, we get

$$-\log\Gamma(a,x) = x-(a-1)\log x+\sum_{k=1}^{n}\frac{\varphi_k}{x^k}+\mathscr{O}\bigg(\frac{1}{x^n}\bigg), \tag{9}$$

with $\varphi_k = -\Phi_k\,(u_1, ..., u_k)$. Considering the CEV process

$$\mathrm{d}\,F = \sigma\,F^\beta\,\mathrm{d}\,W,$$

where W is a Brownian motion and F denotes for instance a forward rate, the probability of absorption at zero before t-time is given by

$$\frac{1}{\Gamma(\nu)}\Gamma\left(\nu,\frac{2\,\nu^2\,F_0^{\frac{1}{\nu}}}{\sigma^2\,t}\right), \qquad \nu = \frac{1}{2\,(1-\beta)} \geqslant \frac{1}{2}.$$

Inverting (9) gives a confidence interval where the probability of absorption of F is lower than α, asymptotically, for $\alpha \ll 1$.

4.5 Black–Scholes formula

By definition, a call option is a contract which gives to the owner the value $\max\,(S_T-K, 0)$ at a future T-date (known today) called maturity of the contract, where S_T denotes the value at T-date (unknown today) of an asset (like a stock) whose initial value is S today, and K is a constant called strike (known today). The initial price of this contract is denoted by $C(S, K, T)$. In general, by no-arbitrage arguments, the option price $C(S, K, T)$ is always greater than the "intrinsic value" $(S-K)_+$ and lower than the spot value S:

$$(S-K)_+ < C(S,K,T) < S$$

In the Black–Scholes model, the dynamics of (S_t) is assumed to be log-normal:

$$\mathrm{dS}_t = \sigma\,S_t\,\mathrm{d}W_t$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and σ is a constant parameter called volatility. In this framework, the well known Black–Scholes formula gives the price of any call option. It can be shown that

$$C(S,K,T)=\mathrm{BS}(S,K,T,\sigma),$$

where

$$\mathrm{BS}(S,K,T,\sigma) \;=\; S\,\Phi(d_+)-K\,\Phi(d_-), \qquad d_\pm=\frac{\log\left(\frac{S}{K}\right)\pm\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}.$$

For simplicity, we have assumed that the interest rate is 0. If S,K,T are fixed, then it is easy to see that the function

$$\sigma \;\longmapsto\; \mathrm{BS}(S,K,T,\sigma) \tag{10}$$

is non-decreasing and one to one from $\mathbb{R}_+^*$ to $(\max(S-K,0),S)$. Therefore, in an *a priori* non Black–Scholes world and for a given call option price $C\in(\max(S-K,0),S)$ observed on the market, there is a unique solution $\sigma_{\mathrm{BS}}(K,T)$ (or simply σ_{BS}) of the equation

$$\mathrm{BS}\,(S,K,T,\sigma) \;=\; C$$

We call σ_{BS} the Black–Scholes implied volatility associated to K and T. For different reasons [10, 30], it is interesting to invert the Black–Scholes function BS in (10). For instance, using techniques from perturbation theory, sophisticated stochastic models (in a non Black–Scholes world) give only asymptotic expansions of an option price C in terms of the maturity T, whereas we really need a formula for the implied volatility [4, 20]. Indeed, call option prices are generally quoted in term of implied volatilities (and not as prices). This can be achieved in the following manner. In the Black–Scholes model and under the conditions that $T\ll 1$ and $K\neq S$, it can be proved [14] that the asymptotic expansion of the "time value" TV of the call price $\mathrm{BS}(S,K,T,\sigma)$, defined by

$$\mathrm{TV}(S,K,T,\sigma) \;:=\; \mathrm{BS}(S,K,T,\sigma)-(S-K)_+,$$

is given by

$$4\sqrt{\pi}\,\frac{\mathrm{e}^{-\frac{u}{2}}}{|u|}\left(\frac{\mathrm{TV}}{S}\right) \;=\; v^{3/2}\,\mathrm{e}^{-\frac{1}{v}}\sum_{k=0}^{n}\frac{(-1)^k}{2^k}\,a_k\!\left(\frac{u^2}{8}\right)v^k+\mathscr{O}\!\left(v^{2n+5}\,\mathrm{e}^{-\frac{1}{v}}\right), \tag{11}$$

with $n\in\mathbb{N}$ arbitrarily large,

$$u:=\log\left(\frac{K}{S}\right), \qquad v:=2\,\frac{\sigma^2 T}{u^2}$$

$$a_k(z):=(2\,k+1)!!, \qquad f_k(z):=\sum_{j=0}^{k}\frac{z^j}{j!\,(2\,j+1)!!}.$$

Therefore, setting

$$x:=\frac{1}{v}=\frac{\log^2\left(\frac{K}{S}\right)}{2\,\sigma^2\,T}, \qquad y:=-\log\!\left(\frac{\mathrm{TV}}{S}\right),$$

we have

$$y \;=\; x+\frac{3}{2}\log x+\varphi_0+\sum_{i=1}^{n}\frac{\varphi_i}{x^i}+\mathscr{O}\!\left(\frac{1}{x^n}\right), \tag{12}$$

for any integer n, where

$$\begin{aligned}\varphi_0 &:= -\log\left(\frac{|u|\,\mathrm{e}^{\frac{u}{2}}}{4\sqrt{\pi}}\right)\\ \varphi_i &:= -\Psi_i\left(-\frac{1}{2}a_1\left(\frac{u^2}{8}\right),\ldots,\frac{(-1)^i}{2^i}a_i\left(\frac{u^2}{8}\right)\right) \qquad (i>0)\end{aligned}$$

Formula (12) is nothing but another expression for Black-Scholes formula. Hence we get an asymptotic expansion for $\sigma^2 T$ in terms of $\log(\frac{\mathrm{TV}}{S})$. Note that (12) is an asymptotic expansion of a call price in terms of x for large x. So, it gives also an asymptotic expansion of a call price when $\left|\log\left(\frac{K}{S}\right)\right|$ is large i.e. small strike or large strike.

Notice also that there is another direct formula $\frac{C+S}{2S} = \Phi\left(\frac{\sigma\sqrt{T}}{2}\right)$ when $K=S$, which gives an asymptotic expansion of C in terms of $\sigma\sqrt{T}$.

At the limit when $T \gg 1$, the first author previously obtained a similar result [14]. Setting this time

$$\mathrm{CC} := S - \mathrm{BS}(S,K,T,\sigma)$$

and

$$x := \frac{\sigma^2 T}{8}, \qquad y := -\log\left(\frac{\mathrm{CC}}{S}\right),$$

we have

$$\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\frac{\mathrm{CC}}{S} = \frac{\mathrm{e}^{-x}}{\sqrt{x}}\sum_{k=0}^{n}\frac{(-1)^k}{2^k}c_k\left(\frac{u^2}{8}\right)\frac{1}{x^k} + \mathscr{O}\left(x^{-n-\frac{3}{2}}\,\mathrm{e}^{-x}\right),$$

where

$$c_k(z) := (2k-1)!!\,g_k(z), \qquad g_k(z) := \sum_{j=0}^{k}\frac{z^j}{j!\,(2j-1)!!}.$$

Therefore, we get

$$y = x + \frac{1}{2}\log x + \varphi_0 + \sum_{i=1}^{n}\frac{\varphi_i}{x^i} + \mathscr{O}\left(\frac{1}{x^n}\right), \tag{13}$$

where

$$\varphi_i := -\Psi_i\left(-\frac{1}{2}a_1\left(\frac{u^2}{8}\right),\ldots,\frac{(-1)^i}{2^i}c_i\left(\frac{u^2}{8}\right)\right) \qquad (i>0)$$

4.6 Bachelier's formula

In the Bachelier model, the dynamics of (S_t) is assumed to be normal:

$$\mathrm{d}S_t = \sigma\,\mathrm{d}W_t,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and σ is a constant parameter called normal volatility. In this framework, the price C of a call option with strike K and expiry T is given by Bachelier's formula

$$C = B(S,K,T,\sigma),$$

with

$$B(S,K,T,\sigma) = (S-K)\,\Phi\left(\frac{S-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\,\varphi\left(\frac{S-K}{\sigma\sqrt{T}}\right).$$

Denoting as before by

$$\mathrm{TV}(K,T) = C - (S-K)_+$$

the time-value of the call option, we have

$$\mathrm{TV}(T,K) = \frac{|S-K|}{4\sqrt{\pi}}\,\Gamma\!\left(-\frac{1}{2}, \frac{(S-K)^2}{2\,\sigma^2 T}\right)$$

for $S \neq K$. So, using (8), we deduce an asymptotic expansion of $\mathrm{TV}(T,K)$ in terms of $\frac{|S-K|}{\sigma\sqrt{T}}$ for large strike or small maturity in the Bachelier model. By inverting the Gamma function as before, this gives an asymptotic expansion of the normal volatility in terms of the time-value TV. Therefore, by comparing with (11), we obtain an equivalence between normal volatility and lognormal volatility in the cases when $T \ll 1$ or $K \gg S$. For example, when $T \ll 1$, the first terms of the expansion are given by

$$\sigma_{\mathrm{LN}} \;=\; \frac{\log m}{m-1}\,\tilde{\sigma}_N - \frac{(\log m)\cdot \log\!\left(\frac{\sqrt{m}\,\log m}{m-1}\right)}{(m-1)^3}\,\tilde{\sigma}_N^3\, T + O(T^2 \log(T)), \tag{14}$$

where m denotes the "moneyness" $m = \frac{K}{S}$ and $\tilde{\sigma}_N = \frac{\sigma_N}{S}$. Note that at the money ($K = S$), we have a closed form formula [13]:

$$\sigma_N \;=\; \sqrt{\frac{2\,\pi}{T}}\, S\left(2\,\Phi\!\left(\frac{\sigma_{\mathrm{LN}}\sqrt{T}}{2}\right) - 1\right).$$

In particular, taking the derivative in (14) and then letting $K = S$ gives

$$\delta_{\mathrm{LN}} \;=\; \left(\delta_N - \frac{1}{2}\,\tilde{\sigma}_N\right)\left(1 + \frac{1}{8}\,\tilde{\sigma}_N^2\, T\right) + o(T), \tag{15}$$

where δ_{LN} and δ_N are the slopes of the smiles of volatility at the money:

$$\delta_{\mathrm{LN}} = \left(\frac{\partial\,\sigma_{\mathrm{LN}}}{\partial\,K}\right)_{K=S}, \qquad \delta_N = \left(\frac{\partial\,\tilde{\sigma}_N}{\partial\,K}\right)_{K=S}.$$

In particular when the volatility smile $K \mapsto \tilde{\sigma}_N$ is flat at the money ($\delta_N = 0$), we have:

$$\delta_{\mathrm{LN}} \;=\; -\frac{1}{2}\,\tilde{\sigma}_N\left(1 + \frac{1}{8}\,\tilde{\sigma}_N^2\, T\right) + o(T).$$

This is a refinement of a classical result $\delta_{\mathrm{LN}} \approx -\frac{1}{2}\,\tilde{\sigma}_N$. In general, (15) shows that if $\delta_N < \frac{1}{2}\,\tilde{\sigma}_N$, then the slope of the volatility smile for the lognormal volatility is negative and this condition holds up to order 1 in T [13].

.4.7 Implied volatility of a local volatility process

By inverting (12), we get the implied lognormal volatility σ_{LN} from the time-value TV of the call-option in a model-free setting. In general, thanks to Tanaka's theorem, this time-value can be calculated by integrating the density function of the stochastic process between 0 and the maturity T of the option. On the other hand, this density function can be obtained using Minakshisundaram–Pleijel expansion - modulo conditions of regularity [2]. So, in general this allows to translate an asymptotic expansion of p for $T \ll 1$ in an asymptotic expansion of σ_{LN}.

As an example, let us consider a local volatility model $\mathrm{d}S_t = f(S)\,\mathrm{d}W_t$. Then Tanaka's formula reads:

$$\mathrm{TV}(K,T) \;=\; \frac{1}{2}\,f(K)^2\int_0^T p(S,\Sigma,t)\,\mathrm{dt} \tag{16}$$

In this particular case, the Minakshisundaram–Pleijel expansion gives

$$p(S,\Sigma,t) \;=\; \frac{\mathrm{e}^{-\frac{2\,d(S,\Sigma)^2}{t}}}{\sqrt{2\,\pi\,t}\;f(\Sigma)}\left(\sum_{i=0}^{n} u_i(S,\Sigma)\,t^i + o(t^n)\right), \tag{17}$$

with $d(S,\Sigma) = \int_S^\Sigma \frac{\mathrm{d}u}{f(u)}$ and u_i is explictly given by induction [11, 37]. The asymptotic expansion (17) can be integrated by part and thus give an asymptotic expansion for $\mathrm{TV}(K,T)$ thanks to (16). Finally, by inverting (12), we get asymptotic expansion of the lognormal implied volatility.

4.8 Constant elasticity of variance model

The stochastic differential equation for the CEV diffusion model is

$$\mathrm{d}S(t) \;=\; \sigma\,S(t)^\beta\,\mathrm{d}W(t)$$

Let $p(S,\Sigma;T)$ be the probability density function to get state Σ at T-time starting from S at time 0. There is a closed form formula for p in terms of the modified Bessel function I:

$$p(S,\Sigma,T) \;=\; \frac{1}{2\,\sigma^2\,T}\,\mathrm{e}^{-\frac{S+\Sigma}{2\sigma^2T}}\left(\frac{S}{\Sigma}\right)^{-\theta/2} I_{|\theta|}\left(\frac{\sqrt{S\,\Sigma}}{\sigma^2\,T}\right) \tag{18}$$

with $\theta = \frac{1}{2(\beta-1)}$ (in the most popular cases we have $\beta\in(0,1)$) and

$$I_\nu(z) \;=\; \sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{19}$$

The following well known asymptotic expansion for $z\to\infty$ is due to Hankel:

$$I_\nu(z) \;\approx\; \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\left(1 - \frac{4\nu^2-1}{8z} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8z)^2} - \frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3!(8z)^3} + \cdots\right). \tag{20}$$

On the other hand, Tanaka's formula shows that the time-value

$$\mathrm{TV}(K,T) = C(S,K,T) - (S-K)_+$$

of the call-price $C(S,K,T)$ with strike K and maturity T is given by

$$\mathrm{TV}(K,T) \;=\; \frac{1}{2}\,\sigma^2\,K^{2\beta}\int_0^T p(S,\Sigma,u)\,\mathrm{d}u \tag{21}$$

For short maturities T, the combination of (18) and (20) yields an asymptotic expansion of $p(S,\Sigma,T)$ that can be integrated by parts to any order with respect to T. This leads to an asymptotic expansion of the time-value $\mathrm{TV}(K,T)$ of a call price. Using an asymptotic expansion of the inverse function of the Black-Scholes function seen in section 4.5, we deduce an asymptotic expansion of the implied lognormal volatility of the CEV model at any order in T.

4.9 Experimental Mathemagix implementation

We did an experimental implementation of our algorithm in the Mathemagix system [24]. Each of the above examples comes down to the computation of the functional inverse of a function $y(x)$ with an asymptotic expansion of the form

$$y \;=\; x+\alpha \log x+\varphi_0+\sum_{i=1}^{n} \frac{\varphi_i}{x^i}+\mathscr{O}\left(\frac{1}{x^n}\right).$$

For $n=3$, our algorithm yields:

$$\begin{aligned} x \;=\; & y-\alpha \log(y)-\varphi_0+(\alpha^2 \log(y)+\varphi_0\,\alpha-\varphi_1)\,\tfrac{1}{y}+\big(\tfrac{1}{2}\,\alpha^3 \log(y)^2+(\varphi_0\,\alpha^2-(\varphi_1+\alpha^2)\,\alpha)\log(y)+ \\ & \tfrac{1}{2}\,\varphi_0^2\,\alpha-(\varphi_0\,\alpha-\varphi_1)\,\alpha-\varphi_1\,\varphi_0-\varphi_2\big)\,\tfrac{1}{y^2}+\big(\tfrac{1}{3}\,\alpha^4 \log(y)^3+\big(\tfrac{-1}{3}\,(-(2\,\varphi_0\,\alpha-\alpha^2)\,\alpha-\varphi_0\,\alpha^2)\,\alpha- \\ & \varphi_1\,\alpha^2-\tfrac{7}{6}\,\alpha^4\big)\log(y)^2+\big(\tfrac{-1}{3}\,(-(\varphi_0^2-\varphi_0\,\alpha+\varphi_1)\,\alpha-(2\,\varphi_0\,\alpha-\alpha^2)\,\varphi_0)\,\alpha-\tfrac{1}{3}\,((2\,\varphi_0\,\alpha-2\,\varphi_1)\,\alpha+ \\ & 2\,\varphi_0\,\alpha^2)\,\alpha+(\varphi_1+\alpha^2)\,\alpha^2-2\,\varphi_1\,\varphi_0\,\alpha-\varphi_0\,\alpha^3+\varphi_1\,\alpha^2-2\,\varphi_2\,\alpha\big)\log(y)-(-(\varphi_0\,\alpha-\varphi_1)\,\alpha- \\ & \varphi_1\,\varphi_0-\varphi_2)\,\alpha+\tfrac{1}{3}\,(\varphi_0^2-\varphi_0\,\alpha+\varphi_1)\,\varphi_0\,\alpha-\tfrac{1}{3}\,(2\,\varphi_0\,\alpha-2\,\varphi_1)\,\varphi_0\,\alpha-(\varphi_1-\varphi_0\,\alpha)\,\varphi_1-\tfrac{1}{2}\,\varphi_0^2\,\alpha^2- \\ & \varphi_1\,\varphi_0^2-2\,\varphi_2\,\varphi_0-\varphi_3\big)\,\tfrac{1}{y^3}+\mathscr{O}\left(\tfrac{1}{y^3}\right). \end{aligned}$$

5 Conclusion

In this paper, we have presented an algorithm for calculating asymptotic expansions of functional inverses of functions that are highly tangent to identity. In particular, we obtained asymptotic expansions of the implied volatility of an option call price at any order in a model-free setting. hen, we have shown how this can be applied to the CEV model. It is envisaged to apply these techniques to more sophisticated models such as the SABR model [17]. More generally, it would be interesting to combine our approach with other algorithms for the calculation of heat kernel coefficients such as [1] to get automatic asymptotic expansion at any order of the implied volatility of option call prices for a large class of financial models [2].

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