DIFFERENTIALLY ALGEBRAIC GAPS

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Abstract. H-fields are ordered differential fields that capture some basic properties of Hardy fields and fields of transseries. Each H-field is equipped with a convex valuation, and solving first-order linear differential equations in H-field extensions is strongly affected by the presence of a “gap” in the value group. We construct a real closed H-field that solves every first-order linear differential equation, and that has a differentially algebraic H-field extension with a gap. This answers a question raised in [1]. The key is a combinatorial fact about the support of transseries obtained from iterated logarithms by algebraic operations, integration, and exponentiation.

Introduction

This paper is motivated by a basic problem about H-fields, the gap problem, as we explain later in this introduction. In this paper “differential field” means “ordinary differential field of characteristic 0”; H-fields are ordered differential fields whose ordering and derivation interact in a strong way. The category of H-fields was defined in [1] as a common algebraic framework for two points of view on the asymptotic behavior of one-variable real-valued functions at infinity: the theory of Hardy fields (see [9]), and the more recent theory of transseries fields, introduced by Dahn and Göring [3] as well as Écalle [4], and further developed in [15], [13], [14], [11]. We hope that the theory of H-fields will lead to a better (model-theoretic) understanding of Hardy fields, and of their relation to fields of transseries.

For this introduction, we assume that the reader has access to [1] and [2]; in particular, the notations and conventions in these papers remain in force. We just recall here that any H-field K (with constant field C) comes equipped with a dominance relation ≼: for f, g ∈ K, we have

\[ f ≼ g \iff |f| ≼ c |g| \quad \text{for some } c ∈ C, \]

and we write \( f ≢ g \) if \( f \not\sim g \) and \( g \not\sim f \); we also write \( g \gg f \) instead of \( f \not\sim g \), and \( g \gg f \) instead of \( f \not\sim g \). (If \( K \supseteq \mathbb{R} \) is a Hardy field, then \( K \) is an H-field and, in Landau’s O-notation, \( f ≼ g \iff f = O(g) \) and \( f \gg g \iff f = o(g) \).) For some basic properties of these asymptotic relations we refer to [16] in the case of transseries fields, and [2] for H-fields in general.

Let K be an H-field. The set \( K_× = \{ f ∈ K : f \not\sim 1 \} \) of bounded elements of K is a convex subring of K; we shall always denote the associated valuation by \( v: K \to \Gamma \cup \{ \infty \} \), with \( \Gamma = v(K_×), K_× := K \setminus \{ 0 \} \). For \( f, g ∈ K \) we write \( f \approx g \) if \( v(f) = v(g) \), that is, \( f \approx g \) and \( g \approx f \). An element \( f \) of K is said to be infinitesimal.

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if \( f \prec 1 \), equivalently, \( |f| < c \) for all positive constants \( c \in C \), and infinite if \( f \succ 1 \), equivalently, \( |f| > C \).

An \( H \)-field \( K \) is Liouville closed if \( K \) is real closed, and any first-order linear differential equation \( y' + f y = g \) with \( f, g \in K \) has a solution in \( K \). A Liouville closure of an \( H \)-field \( K \) is a Liouville closed \( H \)-field \( L \) extending \( K \) which is minimal with this property. Every \( H \)-field \( K \) has at least one, and at most two, Liouville closures, up to isomorphism over \( K \). Given a differential field \( F \), an element \( f \in F^\times \) and an element \( y \) in some differential field extension of \( F \) we let \( f' := f'/f \) denote the logarithmic derivative of \( f \), and let \( F(y) := F(y, y', y'', \ldots) \) be the differential field generated by \( y \) over \( F \). A differential field \( F \) is said to be closed under integration if for each \( g \in F \) there is \( f \in F \) with \( f' = g \).

**Gaps in \( H \)-fields.** In an \( H \)-field, asymptotic relations between elements of non-zero valuation may be differentiated: if \( f, g \neq 1 \), then \( f \prec g \iff f' \prec g' \). In particular, if \( f \) is infinitesimal and \( g \) is infinite, then \( f' \prec g' \). Also, if \( \varepsilon \) and \( \delta \) are non-zero infinitesimals, then \( \varepsilon' \prec \delta' \). A gap in an \( H \)-field \( K \) is an element \( \gamma = v(g) \), \( g \in K^\times \), of its value group \( \Gamma \) such that \( \varepsilon' \prec g \prec \delta' \) for all non-zero infinitesimals \( \varepsilon, \delta \). An \( H \)-field has at most one gap, and has no gap if it has a smallest comparability class or is Liouville closed. Further examples of \( H \)-fields without a gap can be obtained using the \( H \)-field of transseries of finite exponential and logarithmic depth with real coefficients, denoted by \( \mathbb{R}((x^{-1}))^{\mathcal{L}} \) in [14], and by \( \mathbb{R}[[[x]]] \) in [15]: each ordered differential subfield of \( \mathbb{R}[[[x]]] \) that contains \( \mathbb{R} \) is an \( H \)-field without a gap.

If an \( H \)-field \( K \) has a gap \( v(g) \) as above, then \( K \) has exactly two Liouville closures, up to isomorphism over \( K \): one in which \( g = \varepsilon' \) with infinitesimal \( \varepsilon \), and one where \( g = h' \) with infinite \( h \). This “fork in the road” due to a gap causes much trouble. For a model-theoretic analysis of (existentially closed) \( H \)-fields, one needs to understand when a given \( H \)-field can have a differentially algebraic \( H \)-field extension with a gap. (An extension \( L|K \) of differential fields is said to be differentially algebraic if every element of \( L \) is a zero of a non-constant differential polynomial over \( K \)).

**The gap problem.** The simplest type of differentially algebraic extensions are Liouville extensions. If \( K \) is a real closed \( H \)-field and \( L = K(y) \) is an \( H \)-field extension with \( y' \in K \), then \( L \) has a gap if and only if \( K \) does, by [1], [2]. However, [2] also has an example of a real closed \( H \)-field \( K \) without a gap, but such that some \( H \)-field extension \( L = K(y) \supseteq K \) with \( y \neq 0, y' \in K \) has a gap. It may even happen that an \( H \)-field \( K \) has no gap, but its real closure does. These examples raise the question (called the “gap problem” in [1]) whether the creation of gaps in differentially algebraic \( H \)-field extensions can be confined to Liouville extensions. More precisely, we asked the following

*Suppose \( L \) is a differentially algebraic \( H \)-field extension of a Liouville closed \( H \)-field \( K \). Can \( L \) have a gap?* (A negative answer would have been welcome.)

Our main result is an example where the answer is positive. This example is about as simple as possible, and may well be generic in some sense.

**Outline of the example.** No differentially algebraic \( H \)-field extension of \( \mathbb{R}[[[x]]] \) can have a gap, by [2], Corollary 12.2, and this statement remains true when \( \mathbb{R}[[[x]]] \)
is replaced by any Liouville closed $H$-subfield. Our example will indeed live in a larger field $T$ of transseries, as we shall indicate.

First, let $\mathcal{L}$ denote the multiplicative ordered subgroup of $\mathbb{R}[[x]]^{>0}$ generated by the real powers of the iterated logarithms

$$
\ell_0 := x, \ell_1 := \log x, \ell_2 := \log \log x, \ldots, \ell_n := \log^n x, \ldots
$$

of $x$ (the group of logarithmic monomials, see Section 2). This gives rise to

$$
L := \mathbb{R}[[\mathcal{L}]] \quad \text{(the field of logarithmic transseries)}.
$$

In the beginning of Section 3 we equip $L$ with a derivation making it an $H$-field with constant field $\mathbb{R}$. Let $T$ be the field of transseries of finite exponential depth and logarithmic depth at most $\omega$, with real coefficients (denoted by $\mathbb{R}_{\omega}^\omega[[x]]$ in [15]). At this stage we only mention that $T$ is obtained from $L$ by an inductive procedure of closure under exponentiation. (Details of this procedure are in [15], Chapter 2, and are recalled at the beginning of Section 4.) As a result of its construction $T$ comes equipped with a derivation that makes it a real closed $H$-field extension of $L$ (with same constant field $\mathbb{R}$), and with an isomorphism $\exp$ of the ordered additive group of $T$ onto its positive multiplicative group $T^{>0}$, whose inverse is denoted by $\log$, such that $\exp(f') = f' \exp(f)$ for all $f \in T$ and $\log \ell_n = \ell_{n+1}$ for all $n$.

Moreover, the sequence $\ell_0, \ell_1, \ell_2, \ldots$ is coinitial in the set of positive infinite elements of $T$ and hence $1/\ell_0, 1/\ell_1, 1/\ell_2, \ldots$ is cofinal in the set of positive infinitesimals of $T$. Also, $\mathbb{R}[[x]] \subseteq T$, as $H$-fields and as exponential fields. Here is a diagram illustrating the various $H$-fields and their inclusions (indicated by arrows):

$$
\begin{array}{ccc}
L &=& \mathbb{R}[[\mathcal{L}]] \\
\mathbb{R}[[\mathcal{L}]] &\longrightarrow& T \\
\mathbb{R}[[x]] &\longrightarrow& \mathbb{R}[[x]]
\end{array}
$$

Whereas the $H$-field $L$ does not have a gap (see Section 3), the $H$-field $T$ does. In particular, $T$ is not Liouville closed. To see this, we set as in [4], Chapter 7:

$$
\Lambda := \ell_1 + \ell_2 + \ell_3 + \cdots \in L
$$

In $T$ we have $(\ell_n)^\dagger = (\ell_{n+1})' = \exp(-(\ell_1 + \ell_2 + \cdots + \ell_{n+1}))$, and thus

$$
(1/\ell_n)^\dagger = \exp(-\Lambda) \cdot (1/\ell_n)^\dagger \quad \text{for all } n.
$$

(Intuitively, $\exp(-\Lambda)$ represents the infinitely long logarithmic monomial $1/\ell_0 \ell_1 \ell_2 \cdots$.) Therefore $v(\exp(-\Lambda))$ is a gap in $T$, and hence is a gap in each $H$-subfield of $T$ that contains $\exp(\Lambda)$, so any Liouville closed $H$-subfield $K$ of $T$ with a differentially algebraic $H$-field extension $L \subseteq T$ containing $\exp(\Lambda)$ is an example as claimed. Put

$$
\lambda := \Lambda' = \frac{1}{\ell_0} + \frac{1}{\ell_0 \ell_1} + \frac{1}{\ell_0 \ell_1 \ell_2} + \cdots + \frac{1}{\ell_0 \ell_1 \cdots \ell_n} + \cdots \in L.
$$

Let $\varrho := 2\lambda' + \lambda^2 \in L$. A computation shows that

$$
\varrho = -\left(\frac{1}{\ell_0^2} + \frac{1}{(\ell_0 \ell_1)^2} + \frac{1}{(\ell_0 \ell_1 \ell_2)^2} + \cdots + \frac{1}{(\ell_0 \ell_1 \cdots \ell_n)^2} + \cdots \right).
$$

We shall prove (Corollary 5.13):

**Theorem.** There exists a Liouville closed $H$-subfield $K \supseteq \mathbb{R}(\mathcal{L})$ of $T$ such that $\varrho \in K$. 

Given $K$ as in the theorem, let $L := K(\exp(\Lambda), \lambda) \subseteq \mathbb{T}$. Since $\exp(\Lambda)^{t} = \lambda$ and $\lambda' = \varrho - (1/2)\lambda^2$, $L$ is an $H$-subfield of $\mathbb{T}$ and differentially algebraic over $K$; thus $K$ and $L$ are an example as claimed.

We shall construct a $K$ as in the theorem by isolating a condition on transseries in $\mathbb{T}$, namely “to have decay $> 1$”, a condition satisfied by $\varrho$, but not by $\lambda$. The main effort then goes into showing that this condition defines a Liouville closed $H$-subfield of $\mathbb{T}$ as in the Theorem.

**Organization of the paper.** After preliminaries in Section 1 on transseries, we introduce in Section 2 the property of subsets $\mathcal{S}$ of $\mathcal{L}$ to have decay $> 1$. In Section 3 we consider the subset $L_1$ of $L$ consisting of those series whose support has decay $> 1$, and show that $L_1$ is an $H$-subfield of $L$ closed under integration and taking logarithms of positive elements. (By construction, $\varrho \in L_1$, but $\lambda \not\in L_1$.) Section 4 is the most technical; it focuses on subgroups $\mathcal{M}$ of the group $\mathbb{T}$ of monomials of $\mathbb{T}$ and shows, under mild assumptions including $\exp(\Lambda) \not\in \mathcal{M}$, that then the transseries field $\mathbb{R}[[\mathcal{M}]]$ is closed under a natural derivation on $\mathbb{R}[[\mathbb{T}]]$ extending that of $\mathbb{T}$, and is also closed under integration. (Here we make essential use of the Implicit Function Theorem from [17].) In Section 5 we prove the main theorem by extending $L_1$ to a Liouville closed $H$-subfield $\mathbb{T}_1$ of $\mathbb{T}$. We finish with comments on the transseries $\lambda$ and $\varrho$.

1. Preliminaries

In our notations we mostly follow [17]. Throughout this paper we let $m$ and $n$ range over $\mathbb{N} := \{0, 1, 2, \ldots\}$.

**Strong linear algebra.** Let $(\mathcal{M}, \preceq)$ be an ordered set. (We do not assume that $\preceq$ is total, but we do follow the convention that ordered abelian groups and ordered fields are totally ordered.) A subset $\mathcal{S}$ of $\mathcal{M}$ is said to be noetherian if for every infinite sequence $m_0, m_1, \ldots$ in $\mathcal{S}$ there exist indices $i < j$ such that $m_i \preceq m_j$. If the ordering $\preceq$ is total, then $\mathcal{S} \subseteq \mathcal{M}$ is noetherian if and only if $\mathcal{S}$ is well-ordered for the reverse ordering $\succeq$, that is, there is no strictly increasing infinite sequence $m_0 < m_1 < \cdots$ in $\mathcal{S}$. Let $C$ be a field. Then

$$C[[\mathcal{M}]] := \left\{ f = \sum_{m \in \mathcal{M}} f_m m : \text{all } f_m \in C, \text{ supp } f \subseteq \mathcal{M} \text{ is noetherian} \right\},$$

where $\text{supp } f = \{m \in \mathcal{M} : f_m \neq 0\}$ is the support of $f$, denotes the $C$-vector space of transseries with coefficients in $C$ and monomials from $\mathcal{M}$. We refer to [17] for terminology and basic results concerning "strong linear algebra" in $C[[\mathcal{M}]]$. In particular, a family $(f_i)_{i \in I}$ in $C[[\mathcal{M}]]$ is called noetherian if the set $\bigcup_{i \in I} \text{supp } f_i \subseteq \mathcal{M}$ is noetherian and for each $m \in \mathcal{M}$ there exist only finitely many $i \in I$ such that $m \in \text{supp } f_i$. In this case, we put

$$\sum_{i \in I} f_i := \sum_{m \in \mathcal{M}} \left( \sum_{i \in I} f_{i, m} \right) m,$$

an element of $C[[\mathcal{M}]]$.

Let $(\mathcal{M}, \preceq)$ be a second ordered set. A $C$-multilinear map $\Phi : C[[\mathcal{M}]]^n \to C[[\mathcal{M}]]$ is called strongly multilinear if for all noetherian families

$$(f_{1, i_1})_{i_1 \in I_1}, \ldots, (f_{n, i_n})_{i_n \in I_n}$$
in $C[[\mathcal{M}]]$ the family

$$\left( \Phi(f_{i_1, i_2, \ldots, i_n}) \right)_{(i_1, \ldots, i_n) \in I_1 \times \cdots \times I_n}$$

in $C[[\mathcal{N}]]$ is noetherian and

$$\Phi \left( \sum_{i_1 \in I_1} f_{1, i_1, \ldots, i_n} \right) = \sum_{(i_1, \ldots, i_n) \in I_1 \times \cdots \times I_n} \Phi(f_{i_1, i_2, \ldots, i_n}).$$

In the case $n = 1$ we say that $\Phi$ is \textit{strongly linear}. Clearly a strongly multilinear map $C[[\mathcal{M}]] \to C[[\mathcal{N}]]$ is strongly linear in each of its $n$ variables.

A map $\varphi : \mathcal{M} \to C[[\mathcal{N}]]$ is said to be \textit{noetherian} if for every noetherian subset $\mathbf{S} \subseteq \mathcal{M}$, the family $(\varphi(m))_{m \in \mathbf{S}}$ in $C[[\mathcal{N}]]$ is noetherian; equivalently, for every infinite sequence $m_1 \succ m_2 \succ \cdots$ of monomials in $\mathcal{M}$ and $n_i \in \text{supp} \varphi(m_i)$ for $i \geq 1$, there exist $i < j$ such that $n_i \succ n_j$. A noetherian map $\mathcal{M} \to C[[\mathcal{N}]]$ extends to a unique strongly linear map $C[[\mathcal{M}]] \to C[[\mathcal{N}]]$ (Proposition 3.5 in [17]), and every strongly linear map $C[[\mathcal{M}]] \to C[[\mathcal{N}]]$ restricts to a noetherian map $\mathcal{M} \to C[[\mathcal{N}]]$.

A map $\Phi : C[[\mathcal{M}]] \to C[[\mathcal{N}]]$ is called \textit{noetherian} if there exists a family $(M_n)_{n \in \mathbb{N}}$ of strongly multilinear maps

$$M_n : C[[\mathcal{M}]]^n \to C[[\mathcal{N}]]$$

such that for every noetherian family $(f_k)_{k \in K}$ in $C[[\mathcal{M}]]$ the family

$$(M_n(f_{k_1}, \ldots, f_{k_n}))_{n \in \mathbb{N}, k_1, \ldots, k_n \in K}$$

in $C[[\mathcal{N}]]$ is noetherian and

$$\Phi \left( \sum_{k \in K} f_k \right) = \sum_{n \in \mathbb{N}, k_1, \ldots, k_n \in K} M_n(f_{k_1}, \ldots, f_{k_n}).$$

The family $(M_n)$ is called a \textit{multilinear decomposition} of $\Phi$. If char $C = 0$, then the $M_n$ may be chosen to be symmetric, and in this case the sequence $(M_n)_{n \in \mathbb{N}}$ is uniquely determined by $\Phi$ ([17], Proposition 5.8). Every strongly linear map $\Phi : C[[\mathcal{M}]] \to C[[\mathcal{N}]]$ is noetherian, with multilinear decomposition $(M_n)$ given by $M_1 = \Phi$ and $M_n = 0$ for $n \neq 1$. Conversely, if $C$ is infinite, then every linear noetherian map is strongly linear, as we show next.

\begin{lemma}
Suppose the field $C$ is infinite and $(f_i)_{i \in \mathbb{N}}$ is a noetherian family in $C[[\mathcal{M}]]$. Let $\phi : C \to C[[\mathcal{M}]]$ be given by $\phi(\lambda) = \sum \lambda^i f_i$, and suppose $\phi$ is $C$-linear. Then $f_i = 0$ for all $i \neq 1$.
\end{lemma}

\textit{Proof.} Suppose $m \in \bigcup \text{supp } f_i$; let $i_1 < \cdots < i_n$ be the indices $i$ such that $m \in \text{supp } f_i$, and put $c_k := (f_{i_k})_m \in C$ for $k = 1, \ldots, n$. With $\lambda \in C$ we have $\phi(\lambda)_m = \lambda \phi(1)_m$, that is,

$$\lambda^{i_1} c_1 + \cdots + \lambda^{i_n} c_n = \lambda (c_1 + \cdots + c_n).$$

Since $C$ is infinite, this yields $n = 1$ and $i_1 = 1$. \hfill \Box

\begin{corollary}
Suppose the field $C$ is infinite, and the map $\Phi : C[[\mathcal{M}]] \to C[[\mathcal{N}]]$ is noetherian and $C$-linear. Then $\Phi$ is strongly linear.
\end{corollary}
Proof. Let \((M_n)_{n \in \mathbb{N}}\) be a multilinear decomposition of \(\Phi\). Let \(f \in C[\mathcal{M}]\), and define \(\phi: C \to C[\mathcal{M}]\) by \(\phi(\lambda) = \Phi(\lambda f)\). Then

\[
\phi(\lambda) = \sum_{i} \lambda^i f_i \quad \text{with} \quad f_i := M_i(f, \ldots, f),
\]

and \(\phi\) is \(C\)-linear. Hence \(f_i = 0\) for all \(i \neq 1\), by the previous lemma. It follows that \(\Phi = M_1\).

We equip the disjoint union \(\mathcal{M} \amalg \mathcal{M}\) with the least ordering extending those of \(\mathcal{M}\) and \(\mathcal{N}\). The natural inclusions \(i: \mathcal{M} \to \mathcal{M} \amalg \mathcal{M}\) and \(j: \mathcal{N} \to \mathcal{M} \amalg \mathcal{M}\) extend uniquely to strongly linear maps \(\hat{i}: C[\mathcal{M}] \to C[\mathcal{M} \amalg \mathcal{M}]\), and \(\hat{j}: C[\mathcal{N}] \to C[\mathcal{M} \amalg \mathcal{M}]\). This yields a \(C\)-linear bijection

\[
(f, g) \mapsto \hat{i}(f) + \hat{j}(g): C[\mathcal{M}] \times C[\mathcal{N}] \to C[\mathcal{M} \amalg \mathcal{M}].
\]

When convenient, we identify \(C[\mathcal{M}] \times C[\mathcal{N}]\) with \(C[\mathcal{M} \amalg \mathcal{M}]\) by means of this bijection. For example, we say that a map \(\Phi: C[\mathcal{M}] \times C[\mathcal{N}] \to C[\mathcal{N}]\) is strongly linear (respectively, noetherian) if \(\Phi\), considered as a map \(C[\mathcal{M} \amalg \mathcal{M}] \to C[\mathcal{N}]\), is strongly linear (respectively, noetherian). The following is the strongly linear case of Theorems 6.1 and 6.3 in [17] (Van der Hoeven’s implicit function theorem):

**Theorem 1.3.** Let the map \((f, g) \mapsto \Phi(f, g): C[\mathcal{M}] \times C[\mathcal{N}] \to C[\mathcal{N}]\) be strongly linear such that \(\text{supp} \Phi(m, 0) \prec m\) for all \(m \in \mathcal{M}\). Then for each \(g \in C[\mathcal{N}]\) there is a unique \(f = \Psi(g) \in C[\mathcal{M}]\) such that \(\Phi(f, g) = f\). For each \(g \in C[\mathcal{N}]\) the family

\[\Psi_n(g) = \Phi(0, g), \quad \Psi_{n+1}(g) = \Phi(\Psi_n(g), g)\]

for all \(n \in \mathbb{N}\) is noetherian with

\[\Psi(g) = \Psi_0(g) + \sum_{n \in \mathbb{N}} (\Psi_{n+1}(g) - \Psi_n(g)).\]

The map \(g \mapsto \Psi(g): C[\mathcal{M}] \to C[\mathcal{N}]\) is noetherian.

The following consequence for inverting strongly linear maps is important later:

**Corollary 1.4.** Suppose that \(C\) is infinite. Let \(\Phi: C[\mathcal{M}] \to C[\mathcal{N}]\) be a strongly linear map such that \(\text{supp} \Phi(m) \prec m\) for all \(m \in \mathcal{M}\). Then the strongly linear operator \(\text{Id} + \Phi\) on \(C[\mathcal{M}]\) is bijective with strongly linear inverse given by

\[
(\text{Id} + \Phi)^{-1}(g) = \sum_{n=0}^{\infty} (-1)^n \Phi^n(g).
\]

**Proof.** Let \(\Phi_1: C[\mathcal{M}] \times C[\mathcal{M}] \to C[\mathcal{M}]\) be given by \(\Phi_1(f, g) = g - \Phi(f)\). Then \(\Phi_1\) is strongly linear and \(\text{supp} \Phi_1(m, 0) = \text{supp} \Phi(m) \prec m\) for all \(m \in \mathcal{M}\). By the theorem above with \(\Phi_1\) in place of \(\Phi\) we obtain a noetherian \(\Psi: C[\mathcal{M}] \to C[\mathcal{M}]\) such that \(\text{Id} + \Phi_1 \circ \Psi = \text{Id}\). By Corollary 1.2, \(\Psi\) is strongly linear.

The assumption on \(\Phi\) yields that \(\text{Id} + \Phi\) has trivial kernel, so \(\text{Id} + \Phi\) is injective, and thus \(\Psi\) is even a two-sided inverse of \(\text{Id} + \Phi\). Moreover, in the notation of Theorem 1.3 we have

\[\Psi_0(g) = g, \quad \Psi_1(g) = g - \Phi(g), \quad \Psi_2(g) = g - \Phi(g) + \Phi^2(g), \ldots\]

for every \(g\), which yields (1.1). \(\square\)
Transseries fields. In the rest of this section, $(\mathcal{M}, \preceq)$ is a multiplicative ordered abelian group. (In particular the ordering $\preceq$ is total.) Then $C[[\mathcal{M}]]$ is a field, called the transseries field with coefficients in $C$ and monomials from $\mathcal{M}$. If $\mathcal{S}, \mathcal{S}^+ \subseteq \mathcal{M}$ are noetherian, so is $\mathcal{S}\mathcal{S}^+$. For $\mathcal{S} \subseteq \mathcal{M}$, let $\mathcal{S}^+$ be the multiplicative submonoid of $\mathcal{M}$ generated by $\mathcal{S}$; if $\mathcal{S} \subseteq \mathcal{M}$ is noetherian and $\mathcal{S} \preceq 1$, then $\mathcal{S}^+$ is noetherian.

For non-zero $f \in C[[\mathcal{M}]]$ we put
\[\partial(f) := \max_{\preceq} \text{supp } f \quad (\text{dominant monomial of } f)\]
and we call $f_0(f)\partial(f) \in C^* \cdot \mathcal{M}$ the dominant term of $f$. We extend the ordering $\preceq$ on $\mathcal{M}$ to a dominance relation on $C[[\mathcal{M}]]$: for series $f$ and $g$ in $C[[\mathcal{M}]]$, we put
\[f \preceq g \iff f \neq 0, g \neq 0, \partial(f) \preceq \partial(g), \text{ or } f = 0\]
\[f \asymp g \iff f \preceq g \land g \preceq f,\]
so for non-zero $f$ and $g$: $f \asymp g \iff \partial(f) = \partial(g)$. We have the canonical decomposition of $C[[\mathcal{M}]]$ into $C$-linear subspaces:
\[C[[\mathcal{M}]] = C[[\mathcal{M}]^+] + C \simeq C[[\mathcal{M}^-]^+] + C[[\mathcal{M}^-]^-],\]
where
\[C[[\mathcal{M}]^+] := \{f \in C[[\mathcal{M}]] : \text{supp } f \neq 1\} = C[[\mathcal{M}^+]^+]\]
and
\[C[[\mathcal{M}^-]^+] := \{f \in C[[\mathcal{M}]] : \text{supp } f \neq 1\} = C[[\mathcal{M}^-]^+] = C[[\mathcal{M}^-]^1],\]
the maximal ideal of the valuation ring $C[[\mathcal{M}]]^1 = C \simeq C[[\mathcal{M}^-]^+]$ of $C[[\mathcal{M}]]$. Every $f \in C[[\mathcal{M}]]$ can be uniquely written as
\[f = f^+ + f^-, f^+ \in C, f^- \in C[[\mathcal{M}]^1].\]
If $C$ is an ordered field, then we turn $C[[\mathcal{M}]]$ into an ordered field as follows:
\[f > 0 \iff f_0(f) > 0, \quad \text{for } f \in C[[\mathcal{M}]], f \neq 0. \quad (1.2)\]
In this case,
\[C[[\mathcal{M}]]^+ = \{f \in C[[\mathcal{M}]] : |f| > C\}\]
and
\[C[[\mathcal{M}]^+] = \{f \in C[[\mathcal{M}]] : |f| < C^0\},\]
and the valuation ring $C[[\mathcal{M}]^1]$ of $C[[\mathcal{M}]]$ is a convex subring of $C[[\mathcal{M}]]$. Given an ordered field $C$ we shall refer to $C[[\mathcal{M}]]$ as an ordered transseries field over $C$ to indicate that $C[[\mathcal{M}]]$ is equipped with the ordering defined by (1.2).

Example 1.5. Let $C = \mathbb{R}$ and $\mathcal{M} = x^\mathbb{R}$, a multiplicative copy of the ordered additive group of real numbers, with isomorphism $r \mapsto x^r : \mathbb{R} \to x^\mathbb{R}$. Then we have
\[f^+ = \sum_{r>0} a_r x^r, \quad f^- = a_0, \quad f^- = \sum_{r<0} a_r x^r\]
for $f = \sum_r a_r x^r \in \mathbb{R}[x^\mathbb{R}]$.

Let $X = (X_1, \ldots, X_n)$ be a tuple of distinct indeterminates and
\[F(X) = \sum_{\nu} a_\nu X^\nu \in C[[X]].\]
a formal power series; here the sum ranges over all multiindices \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n \), and \( a_\nu \in C, X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n} \). For any \( n \)-tuple \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) of elements of \( C[[\mathfrak{M}]] \), the family \( (a_\nu \varepsilon^\nu) \) is noetherian [8], where \( \varepsilon^\nu = \varepsilon_1^{\nu_1} \cdots \varepsilon_n^{\nu_n} \). Put

\[
F(\varepsilon) := \sum_{\nu} a_\nu \varepsilon^\nu \in C[[\mathfrak{M}]]^{\leq 1}.
\]

The proof of the following lemma is similar to that of [12], Lemma 2.5.

**Lemma 1.6.** Suppose that \( C \) is real closed and the group \( \mathfrak{M} \) is divisible. Then any subfield \( K \supseteq C[[\mathfrak{M}]] \) of \( C[[\mathfrak{M}]] \) with the property that \( F(\varepsilon) \in K \) for all \( F \in C[[X]] \) and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) with \( \varepsilon_1, \ldots, \varepsilon_n \in K^{\leq 1} \) is real closed.

**Differentiation.** If \( C[[\mathfrak{M}]] \) is an \( H \)-field with respect to a derivation \( f \mapsto f' \) with constant field \( C \) and with respect to the ordering extending an ordering on \( C \) via \((1.2)\), then the dominance relation \( \preceq \) that \( C[[\mathfrak{M}]] \) carries as a transseries field over \( C \) coincides with the dominance relation that it has as an \( H \)-field, and

\[
m \preceq n \iff m' \preceq n', \quad \text{for } m, n \in \mathfrak{M} \setminus \{1\}. \tag{1.3}
\]

In the rest of this section we assume, more generally, that \( C[[\mathfrak{M}]] \) is equipped with a derivation \( f \mapsto f' \) with constant field \( C \) such that \((1.3)\) holds.

**Integration.** A series \( f \in C[[\mathfrak{M}]] \) is called the *distinguished integral* of \( g \in C[[\mathfrak{M}]] \), written as \( f = \int g \), if \( f' = g \) and \( f'' = 0 \).

For every \( m \in \mathfrak{M} \) there is at most one \( n \in \mathfrak{M} \) with \( n' \succeq m \); we say that \( C[[\mathfrak{M}]] \) is closed under asymptotic integration if for every \( m \in \mathfrak{M} \) there exists such an \( n \).

If the derivation on \( C[[\mathfrak{M}]] \) is strongly linear and \( C[[\mathfrak{M}]] \) is closed under integration, then it is closed under asymptotic integration: for \( m \in \mathfrak{M} \) we have \( m \asymp n' \) where \( n := \partial(fm) \). The following converse is very useful:

**Lemma 1.7.** Suppose that \( C \) is infinite, the derivation on \( C[[\mathfrak{M}]] \) is strongly linear, and \( C[[\mathfrak{M}]] \) is closed under asymptotic integration. Then each \( g \in C[[\mathfrak{M}]] \) has a distinguished integral in \( C[[\mathfrak{M}]] \), and the operator \( g \mapsto \int g \) on \( C[[\mathfrak{M}]] \) is strongly linear.

**Proof.** Define \( I : \mathfrak{M} \to C[[\mathfrak{M}]] \) by \( I(m) = cn \) with \( c \in C \), \( n \in \mathfrak{M} \) such that \( cn' \succeq m \). Then by \((1.3)\) the map \( I \) is noetherian, hence extends to a strongly linear operator on \( C[[\mathfrak{M}]] \), which we also denote by \( I \). Let \( D \) be the derivation on \( C[[\mathfrak{M}]] \). The strongly linear operator \( \Phi = D \circ I - \mathrm{Id} \) satisfies \( \supp(\Phi) \preceq m \) for all \( m \in \mathfrak{M} \). Hence by Corollary 1.4 the strongly linear operator \( D \circ I = \mathrm{Id} + \Phi \) has a strongly linear two-sided inverse \( \Psi \) given by

\[
\Psi(g) = (D \circ I)^{-1}(g) = g - \Phi(g) + \Phi^2(g) - \Phi^3(g) + \cdots.
\]

Since \( I(m)^\circ = 0 \) for all \( m \in \mathfrak{M} \), the strongly linear operator \( f := I \circ \Psi \) assigns to each \( g \in C[[\mathfrak{M}]] \) its distinguished integral. \( \square \)

**Exponentials and logarithms.** Suppose now that \( C = \mathbb{R} \). For \( f \in \mathbb{R}[[\mathfrak{M}]]^{\leq 1} \), write \( f = c + \varepsilon \) with \( c \in \mathbb{R} \) and \( \varepsilon \in \mathbb{R}[[\mathfrak{M}]]^{\leq 1} \), and put

\[
\exp(f) = \exp(c + \varepsilon) := e^c \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}.
\]
where \( t \mapsto e^t \) is the usual exponential function on \( \mathbb{R} \). Then \( \exp \) is an \textit{exponential} on \( \mathbb{R}[[\mathbb{M}]]^{\leq 1} \): for \( f, g \in \mathbb{R}[[\mathbb{M}]]^{\leq 1} \)

\[
\exp(f) \geq 1 \iff f \geq 0, \quad \exp(f) \geq f + 1, \quad \text{and} \quad \exp(f + g) = \exp(f) \exp(g).
\]

Thus \( \exp \) is injective with image

\[
\{ g \in \mathbb{R}[[\mathbb{M}]] : g > 0, \delta(g) = 1 \}
\]

and inverse

\[
\log : \{ g \in \mathbb{R}[[\mathbb{M}]] : g > 0, \delta(g) = 1 \} \to \mathbb{R}[[\mathbb{M}]]^{\leq 1}
\]

given by

\[
\log g := \log a + \log(1 + \varepsilon)
\]

for \( g = a(1 + \varepsilon), a \in \mathbb{R}_{\geq 0}, \varepsilon \ll 1 \), where \( \log a \) is the usual natural logarithm of the positive real number \( a \) and

\[
\log(1 + \varepsilon) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \varepsilon^n.
\]

If \( \mathbb{R}[[\mathbb{M}]] \) is closed under integration, then the above logarithm extends to a function

\[
\log : \mathbb{R}[[\mathbb{M}]]^{\geq 0} \to \mathbb{R}[[\mathbb{M}]]
\]

by

\[
\log g := \log a + \log m + \log(1 + \varepsilon)
\]

for \( g = am(1 + \varepsilon) \) with \( a \in \mathbb{R}_{\geq 0}, m \in \mathbb{M} \), and \( \varepsilon \ll 1 \), and \( \log m := \int m^t \). Note that \( \log(fg) = \log f + \log g \) for \( f, g \in \mathbb{R}[[\mathbb{M}]]^{\geq 0} \).

More notation. For non-zero \( f, g \in C[[\mathbb{M}]] \) we put

\[
f \preceq g \iff f^t \preceq g^t,
\]

\[
f \prec g \iff f^t < g^t,
\]

\[
f \asymp g \iff f^t \asymp g^t.
\]

Suppose \( \mathbb{R}[[\mathbb{M}]] \), with its ordering as an ordered transseries field over \( C = \mathbb{R} \), is an \( H \)-field. Then by [2], Proposition 7.3, we have for \( f, g \in \mathbb{R}[[\mathbb{M}]]^{\leq 1} \):

\[
f \preceq g \iff |f| \leq |g|^n \text{ for some } n > 0,
\]

\[
f \prec g \iff |f|^n < |g| \text{ for all } n > 0.
\]

2. Logarithmic Monomials

Let \( \mathcal{L} \) be the multiplicative subgroup of \textit{logarithmic monomials} of \( \mathbb{R}[[[x]]]^{\geq 0} \) generated by the real powers of the iterated logarithms \( \ell_0 := x, \ell_1 := \log x, \ell_2 := \log \log x, \ldots, \ell_n := \log \log x, \ldots \) of \( x \); that is,

\[
\mathcal{L} = \{ \ell_0^{a_0} \ell_1^{a_1} \cdots \ell_n^{a_n} : (a_0, \ldots, a_n) \in \mathbb{N}^n, n = 0, 1, 2, \ldots \}.
\]

Thus \( \mathcal{L} \) is a multiplicatively written ordered vector space over the ordered field \( \mathbb{R} \), with basis \( \ell_0, \ell_1, \ell_2, \ldots \) satisfying

\[
\ell_0 \gg \ell_1 \gg \ell_2 \gg \cdots \gg \ell_n \gg \cdots
\]

We define the \textit{group of continued logarithmic monomials} \( \mathfrak{L} \) by

\[
\mathfrak{L} := \{ \ell_0^{a_0} \ell_1^{a_1} \cdots \ell_n^{a_n} : (a_0, a_1, \ldots, a_n, \ldots) \in \mathbb{R}^\infty \}
\]
and by requiring that \((a_0, a_1, \ldots) \mapsto \ell_0^{a_0} \ell_1^{a_1} \cdots : \mathbb{R}^N \to \mathbb{S}\) is an isomorphism of the additive group \(\mathbb{R}^N\) onto the multiplicative group \(\mathbb{S}\). We order \(\mathbb{S}\) lexicographically: given \(m = \ell_0^{a_0} \ell_1^{a_1} \cdots\) and \(n = \ell_0^{b_0} \ell_1^{b_1} \cdots\) with \((a_0, a_1, \ldots), (b_0, b_1, \ldots) \in \mathbb{R}^N\), put
\[
m \preceq n \iff (a_0, a_1, \ldots) \preceq (b_0, b_1, \ldots) \text{ lexicographically.}
\]
This ordering makes \(\mathbb{S}\) into an ordered group, and extends the ordering \(\preceq\) on \(\mathbb{S}\).
We also extend the relation \(\ll\) ("flatter than") from \(\mathbb{S}\) to \(\mathbb{S}\) in the natural way:
\[
m \ll n \iff l(m) > l(n),
\]
where \(l(m) := \min\{i : a_i \neq 0\} \in \mathbb{N}\) if \(m = \ell_0^{a_0} \ell_1^{a_1} \cdots \neq 1\), and \(l(1) := \infty > \mathbb{N}\).

**Definition 2.1.** A sequence \((m_i)_{i \geq 1}\) in \(\mathbb{S}\) is called a monomial Cauchy sequence if for each \(k \in \mathbb{N}\) there is an index \(i_0\) such that for all \(i_2 > i_1 > i_0\) we have \(m_{i_2}/m_{i_1} \ll \ell_k\). A continued logarithmic monomial \(l \in \mathbb{S}\) is a monomial limit of \((m_i)_{i \geq 1}\) if for all \(k \in \mathbb{N}\) there is an \(i_0\) such that for all \(i > i_0\) we have \(m_i/l \ll \ell_k\).

Given a continued logarithmic monomial \(m = \ell_0^{a_0} \ell_1^{a_1} \cdots\), let us write
\[
e(m) := (a_0, a_1, \ldots) \in \mathbb{R}^N
\]
for its sequence of exponents. Then \(\varepsilon : \mathbb{S} \to \mathbb{R}^N\) is an order-preserving isomorphism between the multiplicative ordered abelian group \(\mathbb{S}\) and the additive group \(\mathbb{R}^N\), ordered lexicographically. With this notation, a sequence \((m_i)\) in \(\mathbb{S}\) is a monomial Cauchy sequence if and only if \(\varepsilon(m_i)\) is a Cauchy sequence in \(\mathbb{R}^N\), that is: for every \(\varepsilon > 0\) in \(\mathbb{R}^N\) there exists an index \(i_0\) such that \(|\varepsilon(m_{i_2}) - \varepsilon(m_{i_1})| < \varepsilon\) for all \(i_2 > i_1 > i_0\). Similarly, an element \(l \in \mathbb{S}\) is a monomial limit of \((m_i)\) if and only if \(\varepsilon(l)\) is a limit of the sequence \(\varepsilon(m_i)\), in the usual sense: for every \(\varepsilon > 0\) there exists \(i_0\) such that \(|\varepsilon(m_i) - \varepsilon(l)| < \varepsilon\) for all \(i > i_0\). If \((m_i)\) has a monomial limit in \(\mathbb{S}\), then \((m_i)\) is a monomial Cauchy sequence. Conversely, every monomial Cauchy sequence \((m_i)\) in \(\mathbb{S}\) has a unique monomial limit \(l\) in \(\mathbb{S}\), denoted by \(l = \lim_{i \to \infty} m_i\).

Moreover, every continued logarithmic monomial \(m = \ell_0^{a_0} \ell_1^{a_1} \cdots \in \mathbb{S}\) is the monomial limit of some monomial Cauchy sequence in \(\mathbb{S}\):
\[
m = \lim_{i \to \infty} \ell_0^{a_0} \ell_1^{a_1} \cdots \ell_i^{a_i}.
\]
(Thus, viewing \(\mathbb{S}\) and \(\mathbb{S}\) as topological groups in their interval topology, \(\mathbb{S}\) is the completion of its subgroup \(\mathbb{S}\).) Given a subset \(\mathcal{S}\) of \(\mathbb{S}\), let \(\mathcal{S}\) denote the set of all monomial limits of monomial Cauchy sequences in \(\mathcal{S}\) (so \(\mathcal{S}\) is the closure of \(\mathcal{S}\) in \(\mathbb{S}\)), and \(\mathcal{S}\) the set of all monomial limits of strictly decreasing monomial Cauchy sequences \(m_1 \succ m_2 \succ \cdots\) in \(\mathcal{S}\). Note that if \(\mathcal{S} \subseteq \mathbb{S}\) is noetherian, then so is \(\mathcal{S} \subseteq \mathbb{S}\), and \(\mathcal{S} = \mathcal{S} \cup \mathcal{S}\).

**Proposition 2.2.** Let \(\mathcal{S}, \mathcal{S}' \subseteq \mathbb{S}\) be noetherian. Then

1. If \(\mathcal{S} \subseteq \mathcal{S}'\), then \(\mathcal{S} \subseteq \mathcal{S}'\) and \(\mathcal{S} \subseteq \mathcal{S}'\).
2. \(\mathcal{S} \cup \mathcal{S}' = \mathcal{S} \cup \mathcal{S}'\) and \(\mathcal{S} \cup \mathcal{S}' = \mathcal{S} \cup \mathcal{S}^*\).
3. \(\mathcal{S}^{**} = \mathcal{S}^{**} \cup \mathcal{S}^*\) and \(\mathcal{S}^{**} = \mathcal{S}^{**} \cup \mathcal{S}^*\).
4. If \(\mathcal{S} \ll 1\), then \(\mathcal{S}^{**} \subseteq \mathcal{S}^{**} (\mathcal{S})^*\) and \(\mathcal{S}^{**} \subseteq \mathcal{S}^{**}\).

**Proof.** Parts (1) and (2) are trivial.

For (3) consider a monomial limit \(l\) of a sequence \(m_1, m_2, \ldots\), where
\[
(m_1, m_1), (m_2, m_2), \ldots
\]
is a sequence in $\mathcal{S} \times \mathcal{S}'$. Since $\mathcal{S}$ and $\mathcal{S}'$ are noetherian, we may assume, after choosing a subsequence of $(m_1, n_1), (m_2, n_2), \ldots$, that $m_1 \succ m_2 \succ \cdots$ and $n_1 \succ n_2 \succ \cdots$. Because $(m_n)$ is a monomial Cauchy sequence, both sequences $(m_i)$ and $(n_i)$ are monomial Cauchy sequences as well. The sequences $(m_i)$ and $(n_i)$ cannot both be ultimately constant. If one of these sequences is ultimately constant, say $m_i = m$ for all $i \geq i_0$, then

$$l = \lim_{i \to \infty} m_i n_i = \lim_{i \to \infty} n_i \in \mathcal{S} \mathcal{S}' .$$

Otherwise, we have

$$l = \lim_{i \to \infty} m_i n_i = \lim_{i \to \infty} m_i \lim_{i \to \infty} n_i \in \mathcal{S} \mathcal{S}' .$$

Hence $\mathcal{S} \mathcal{S}' \subseteq \mathcal{S} \mathcal{S}' \cup \mathcal{S} \mathcal{S}'$. The other inclusions of (3) now follow easily.

As to (4), assume that $\mathcal{S} \prec 1$ and let $l$ be a monomial limit of a sequence

$$m_i = m_{i_1} \cdots m_{i_{l_i}} \succ m_2 = m_{2_1} \cdots m_{2_{l_2}} \succ \cdots ,$$

where $(m_{1_1}, \ldots, m_{1_{l_1}}), (m_{2_1}, \ldots, m_{2_{l_2}}), \ldots$ is a sequence of tuples over $\mathcal{S}$. Since the set of these tuples is noetherian for Higman's embeddability ordering [5], we may assume, after choosing a subsequence, that in this ordering

$$(m_{1_1}, \ldots, m_{1_{l_1}}) \succ (m_{2_1}, \ldots, m_{2_{l_2}}) \succ \cdots .$$

In particular, we have $l_1 \leq l_2 \leq \cdots$. We claim that the sequence $(l_i)$ is ultimately constant. Assume the contrary. Then, after choosing a second subsequence, we may assume that $l_1 < l_2 < \cdots$. Let $1 \leq k_{i+1} \leq l_{i+1}$ be such that

$$(m_{1_1}, \ldots, m_{1_{l_{i_1}}}) \succ (m_{1+1_1}, \ldots, m_{1+1_{k_{i+1}-1}}, m_{1+1, k_{i+1}+1}, \ldots, m_{1+1, l_{i+1}})$$

for all $i$, hence $m_{1_i} \succeq m_{1+1_i} / m_{1+1_i, k_{i+1}}$ for all $i$. Since $\mathcal{S}$ is noetherian, the set ${m_{2_{k_2}}, m_{3_{k_3}}, \ldots}$ has a largest element $v \prec 1$. But then

$$m_{1+1_i} / m_{1_i} \preceq m_{1+1_i, k_{i+1}} \preceq v$$

for all $i$, which contradicts $(m_{1_i})$ being a monomial Cauchy sequence. This proves our claim $(l_i)$ is ultimately constant.

We now proceed as in (3) to finish the proof of (4).

Given $\mathcal{S} \subseteq \mathcal{L}$ we say that $\mathcal{S}$ has decay $> 1$ if for each $m = l_0^{\alpha_0} l_1^{\alpha_1} \cdots \in \mathcal{S}$ there exists $k_0 \in \mathbb{N}$ such that $\alpha_k < -1$ for all $k \geq k_0$. Each finite subset of $\mathcal{L}$ has decay $> 1$.

**Example 2.3.** Fix $n \geq 1$, and define a sequence $(m_i)_{i \geq 0}$ in $\mathcal{L}$ by

$$m_0 = \left( \frac{1}{l_0} \right)^n, \quad m_1 = \left( \frac{1}{l_1 l_0} \right)^n, \quad \ldots, \quad m_i := \left( \frac{1}{l_i l_{i-1} \cdots l_0} \right)^n \quad (i \geq 0).$$

Then the continued logarithmic monomial

$$l = \left( \frac{1}{l_0 l_1 \cdots l_i \cdots} \right)^n \in \mathcal{L}$$

is the monomial limit of the sequence $m_0 \succ m_1 \succ \cdots$ in $\mathcal{L}$. Hence the subset ${m_i : i = 0, 1, 2, \ldots}$ of $\mathcal{L}$ has decay $> 1$ if $n > 1$, but not if $n = 1$.

**Corollary 2.4.** If $\mathcal{S}$ and $\mathcal{S}'$ are noetherian subsets of $\mathcal{L}$ of decay $> 1$, then $\mathcal{S} \cup \mathcal{S}'$ and $\mathcal{S} \mathcal{S}'$ are noetherian of decay $> 1$; if in addition $\mathcal{S} \prec 1$, then $\mathcal{S}'$ is noetherian of decay $> 1$. 

\[ \square \]
3. Logarithmic Transseries of decay > 1

Consider the ordered field $L := \mathbb{R}[[\mathcal{L}]]$ of logarithmic transseries, and equip $L$ with the strongly linear derivation $f \mapsto f'$ such that for each $\alpha \in \mathbb{R}$

$$(\ell^\alpha_0)' = \alpha \ell^\alpha_0, \quad (\ell^\alpha_k)' = \alpha \ell^\alpha_k \left( \ell_0 \ell_1 \cdots \ell_{k-1} \right)^{-1} \text{ for } k > 0.$$  

This makes $L$ a real closed $H$-field with constant field $\mathbb{R}$, and $L$ is closed under integration (see example at end of Section 11 in [2]). Hence by Lemma 1.7 the distinguished integration operator $\int$ on $L$ is strongly linear.

A logarithmic transseries $f \in L$ is said to have decay $> 1$ if its support $\text{supp } f$ has decay $> 1$. By Corollary 2.4 above,

$$L_1 := \{ f \in L : f \text{ has decay } 1 \}$$

is a subfield of $L$ containing the subfield $\mathbb{R}[[\mathcal{L}]]$ of $L$ generated by $\mathcal{L}$ over $\mathbb{R}$. In addition $F(\varepsilon) \in L_1$ for any formal power series $F(X) \in \mathbb{R}[[X]]$ and any $n$-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of infinitesimals in $L_1$, where $X = (X_1, \ldots, X_n)$, $n \geq 1$. Hence by Lemma 1.6 the field $L_1$ is real closed. Defining the logarithmic function on $L^{>0}$ as in the subsection “Exponentials and logarithms” of Section 2, we obtain

$$\log(\ell^{\alpha_0} \ell^{\alpha_1} \cdots \ell^{\alpha_k}) = \alpha_0 \ell_1 + \cdots + \alpha_k \ell_{k+1} \in L_1$$

for $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$. It follows that $\log f \in L_1$ for every positive $f \in L_1$. Moreover:

**Proposition 3.1.** The field $L_1$ is closed under differentiation. (Thus $L_1$ is an $H$-subfield of $L$.)

**Proof.** Let $l \in \mathcal{L}$ be a monomial limit of a strictly decreasing sequence in $\text{supp } f$, where $f \in L_1$; hence $l$ is the monomial limit of a sequence

$$m_1 n_1 \succ m_2 n_2 \succ \cdots$$

where $m_i \in \text{supp } f$ and $n_i \in \text{sup m}_i^+$ for all $i$. Note that $n_i \in \mathcal{D}$, where

$$\mathcal{D} = \left\{ \frac{1}{\ell_0}, \frac{1}{\ell_0 \ell_1}, \frac{1}{\ell_0 \ell_1 \ell_2}, \cdots \right\}. \quad (3.1)$$

Since $\text{supp } f$ and $\mathcal{D}$ are noetherian, we may assume that

$$m_1 \succ m_2 \succ \cdots, \quad \text{and } n_1 \succ n_2 \succ \cdots$$

after choosing a subsequence. Therefore $(m_i)$ and $(n_i)$ are monomial Cauchy sequences. We claim that $(m_i)$ cannot be ultimately constant: if

$$m_i = \ell^{\alpha_0} \ell^{\alpha_1} \cdots \ell^{\alpha_k}$$

for all $i \geq i_0$, then

$$n_i \in \text{sup m}_i^+ \subseteq \left\{ \frac{1}{\ell_0}, \frac{1}{\ell_0 \ell_1}, \cdots, \frac{1}{\ell_0 \ell_1 \cdots \ell_k} \right\}$$

for all $i \geq i_0$, so $(n_i)$ and thus $(m_i n_i)$ would be ultimately constant. This contradiction proves our claim. If $(n_i)$ is ultimately constant, say $n_i = n$ for all $i \geq i_0$, then

$$l = \lim_{i \to \infty} m_i n_i = \left( \lim_{i \to \infty} m_i \right) n.$$  

Otherwise

$$\lim_{i \to \infty} n_i = \frac{1}{\ell_0 \ell_1 \ell_2 \cdots} \in \mathcal{D},$$
hence
\[
l = \lim_{i \to \infty} m_i n_i = \left( \lim_{i \to \infty} m_i \right) \frac{1}{\ell_0 \ell_1 \ell_2 \cdots},
\]
which proves our proposition. \(\square\)

**Example 3.2.** We have \(\mathbb{R}(g) = \mathbb{R}(g, g', \ldots) \subseteq L_1\) as differential fields. Clearly \(\lambda \in L\), but \(L_4\) does not contain any element of the form \(\lambda + \varepsilon\), where \(\varepsilon \in L\) satisfies \(\varepsilon << 1/((\ell_0 \ell_1 \cdots \ell_n))\) for all \(n\). (See Example 2.3.) Note also that \(\Lambda \not\in L_1\).

Next we want to show that the differential field \(L_1\) is closed under integration. For this we need the following two lemmas:

**Lemma 3.3.** For any non-zero \(\alpha \in \mathbb{R}\) and any \(f \in L\), the linear differential equation
\[
y' + \alpha y = f
\]
has a unique solution \(y = g \in L\), and if \(f \in L_1\), then \(g \in L_1\).

**Proof.** Note that for each \(i\), \(\text{supp} f^{(i)}\) is contained in the set \((\text{supp} f)\mathcal{D}^i\), where \(\mathcal{D}\) is as in (3.1). Since \(\mathcal{D}^* = \bigcup_i \mathcal{D}^i\) is noetherian and each of its elements lies in \(\mathcal{D}^i\) for only finitely many \(i\), the \((f^{(i)})\) is noetherian. Hence we have an explicit formula for a solution \(g\) to (3.2):
\[
g := \sum_{i=0}^{\infty} (-1)^i \frac{f^{(i)}}{i!}
\]
The solution \(g \in L\) is unique, since the homogeneous equation \(y' + \alpha y = 0\) only has the solution \(y = 0\) in \(L\). Now suppose \(f \in L_1\), and let \(l = \ell_0 \ell_1 \ell_2 \cdots \in \mathcal{D}\) be a monomial limit of a sequence
\[
m_1 n_1 \succ m_2 n_2 \succ \cdots
\]
in \(\text{supp}(g)\) where \(m_i n_i \in \text{supp}(f^{(i)})\), with \(m_i \in \text{supp}(f)\) and \(n_i \in \mathcal{D}^{k(i)}\). We can assume that \(m_1 \succ m_2 \succ \cdots\) and \(m_1 \succ n_1 \succ \cdots\). Hence \((m_i)\) and \((n_i)\) are monomial Cauchy sequences with limit \(m \in \mathcal{D}\) and \(n \in \mathcal{D}\), respectively, so that \(l = mn\). The exponent of \(\ell_0\) in \(n_i\) is \(-k(i)\), and thus the sequence \((k(i))\) is bounded. Hence we can even assume that this sequence is constant. Then \(\alpha_k \prec \alpha\) for all sufficiently large \(k\), by Proposition 3.1. Hence \(g \in L_1\) as required. \(\square\)

For \(k \in \mathbb{N}\) we consider the embedding of ordered abelian groups
\[
m = \ell_0^{a_0} \ell_1^{a_1} \cdots \ell_n^{a_n} \mapsto m \circ \ell_k := \ell_0^{a_0} \ell_{k+1}^{a_1} \cdots \ell_{k+n}^{a_n} : L \to L
\]
and denote its unique extension to a strongly linear \(\mathbb{R}\)-algebra endomorphism of \(L\) by \(f \mapsto f \circ \ell_k\). Note that \((f \circ \ell_k)' = (f' \circ \ell_k)\ell_k'\) for \(f \in L\), and if \(f \in L_4\), then \(f \circ \ell_k \in L_4\).

In the statement of the next lemma we use the multiindex notation \(\ell^\alpha := \ell_0^{a_0} \ell_1^{a_1} \cdots \ell_n^{a_n}\), for an \((n+1)\)-tuple \(\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}\).

**Lemma 3.4.** Let \(n \in \mathbb{N}\) and suppose \((g_\alpha)_{\alpha \in \mathbb{R}^{n+1}}\) is a family in \(L_4\) such that the family \(\ell^\alpha : (g_\alpha \circ \ell_{n+1})\) in \(L\) is noetherian. Then
\[
\sum_{\alpha} \ell^\alpha : (g_\alpha \circ \ell_{n+1}) \in L_1.
\]
Proof. Let \( l \in \mathfrak{g} \) be a monomial limit of a sequence \( \ell^{n_1} \succ \ell^{n_2} \succ \cdots \) where \( \alpha_i \in \mathbb{R}^{n+1} \) and \( n_i \in \text{supp}(g_n \circ \ell_{n+1}) \) for all \( i \). Then there exists an index \( i_0 \) such that \( \alpha_{i_0} = \alpha_{i_0+1} = \cdots \), and hence \( \ell_{i_0} \succ \ell_{i_0+1} \succ \cdots \) is a sequence in \( \text{supp}(g_{\alpha_{i_0}} \circ \ell_{n+1}) \) with monomial limit \( l/\ell^{\alpha_{i_0}} \). Since \( g_{\alpha_{i_0}} \circ \ell_{n+1} \in L_1 \), the lemma follows. \( \square \)

**Proposition 3.5.** The \( H \)-field \( L_1 \) is closed under integration.

Proof. Let \( f \in L_1 \). Since \( \frac{1}{l \ell_1 l_2 \cdots} \) is not a monomial limit of a sequence in \( \text{supp} f \), there exists \( k \in \mathbb{N} \) such that

\[
l(m \cdot l \ell_1 l_2 \cdots) \leq k \quad \text{for all } m \in \text{supp} f.
\]

Take \( k \) minimal with this property. We proceed by induction on \( k \). Write

\[
f = \sum_{\alpha \in \mathbb{R}} x^{\alpha-1} (F_\alpha \circ \ell_1)
\]

where \( F_\alpha \in L_1 \) for each \( \alpha \in \mathbb{R} \), and for \( 0 \neq \alpha \in \mathbb{R} \), let \( g_\alpha \in L_1 \) be the unique solution to the linear differential equation \( y' + \alpha y = F_\alpha \), by Lemma 3.3. Then

\[
\int x^{\alpha-1} (F_\alpha \circ \ell_1) = x^\alpha (g_\alpha \circ \ell_1) \in L_1,
\]

for \( \alpha \neq 0 \). Since distinguished integration on \( L \) is strongly linear, we have

\[
\int f = (g_0 \circ \ell_1) + \sum_{\alpha \neq 0} x^\alpha (g_\alpha \circ \ell_1) \in L,
\]

where \( g_0 := \int F_0 \), and thus \( \int f \in L_1 \) if \( g_0 \in L_1 \) (by Lemma 3.4). If \( k = 0 \), then \( F_0 = 0 \), hence \( g_0 = 0 \in L_1 \). If \( k > 0 \), then

\[
l(m \cdot l \ell_1 l_2 \cdots) \leq k - 1 \quad \text{for all } m \in \text{supp} F_0,
\]

hence \( g_0 \) in \( L_1 \), by the induction hypothesis. We conclude that \( \int f \in L_1 \). \( \square \)

4. **Strong Differentiation, Strong Integration, and Flattening**

For the convenience of the reader and to fix notations, we first state some facts about the field of transseries \( T \) in addition to those mentioned in the Introduction. For proofs, we refer to [15], where \( T \) is defined as exponential \( H \)-field, and to [11] for more details; see [6] for an independent construction of \( T \) as exponential field.

**Facts about \( T \).** As an ordered field, \( T \) is the union of an increasing sequence

\[
L = \mathbb{R}[[\mathbb{T}_0]] \subseteq \mathbb{R}[[\mathbb{T}_1]] \subseteq \cdots \subseteq \mathbb{R}[[\mathbb{T}_n]] \subseteq \cdots
\]

of ordered transseries subfields over \( \mathbb{R} \), with \( \mathbb{T}_0 = \mathbb{L} \), and where each inclusion \( \mathbb{R}[[\mathbb{T}_n]] \subseteq \mathbb{R}[[\mathbb{T}_{n+1}]] \) comes from a corresponding inclusion \( \mathbb{T}_n \subseteq \mathbb{T}_{n+1} \) of multiplicative ordered abelian groups. The exponential operation \( \exp \) on \( T \) maps the ordered additive group \( \mathbb{R}[[\mathbb{T}_n]] \) isomorphically onto the ordered group \( \mathbb{T}_{n+1} \). Hence \( \log m \in \mathbb{R}[[\mathbb{T}_n]] \) for \( m \in \mathbb{T}_{n+1} \), where \( \log : T^{>0} \rightarrow T \) is the inverse of \( \exp \). Also

\[
\log(1 + \varepsilon) = \sum_{i=1}^{\infty} (-1)^{i+1}/i \varepsilon^i \in \mathbb{R}[[\mathbb{T}_n]] \tag{4.1}
\]

for \( 1 \succ \varepsilon \in \mathbb{R}[[\mathbb{T}_n]] \). For \( f \in T^{>0} \) and \( r \in \mathbb{R} \) we put \( f^r := \exp(r \log f) \in T \); one checks easily that \( f^r \succ 1 \) if \( f \succ 1 \) and \( r \geq 0 \), and that this operation of raising to real powers makes \( T^{>0} \) into a multiplicative vector space over \( \mathbb{R} \) containing each \( \mathbb{T}_n \) as a multiplicative \( \mathbb{R} \)-subspace.
We put $\mathcal{T} := \bigcup_n \mathcal{T}_n$ (an ordered subgroup of $T^{>0}$), so the ordered transseries field $\mathbb{R}[\mathcal{T}]$ over $\mathbb{R}$ contains $T$ as an ordered subfield. The ordered field $\mathbb{R}[\mathcal{T}]$ comes equipped with two strongly linear automorphisms $f \mapsto f^\uparrow$ (upward shift) and $f \mapsto f^\downarrow$ (downward shift), that are mutually inverse and map $T$ to itself. The downward shift extends the map $f \mapsto f \circ \ell_1$ on $T$ used in the last section, and also the composition operation $f \mapsto f \circ \log x$ on $\mathbb{R}[\mathcal{I}]$. (See [15], Chapter 2.) We have $\exp(f)^\uparrow = \exp(f^\uparrow)$ for $f \in T$, and hence $\log(f)^\uparrow = \log(f^\uparrow)$ and $(f^r)^\uparrow = (f^\uparrow)^r$ for $f \in T^{>0}$, $r \in \mathbb{R}$. From these properties one obtains by induction that $\mathcal{T}_n^\uparrow \subseteq \mathcal{T}_{n+1}$ and $\mathcal{T}_n^\downarrow \subseteq \mathcal{T}_n$. (Hence $m \mapsto m^\uparrow$ is an automorphism of the ordered group $\mathcal{T}$.) We denote the $n$-fold functional composition of $f \mapsto f^\downarrow$ by $f \mapsto f^\downarrow_n$, and similarly we write $f \mapsto f^\uparrow_n$ for the $n$-fold composition of $f \mapsto f^\uparrow$.

The derivation on $T$ restricts to a strongly linear derivation on each subfield $\mathbb{R}[\mathcal{T}_n][x]$, and extends uniquely to a strongly linear derivation $D: f \mapsto f^\prime$ on $\mathbb{R}[\mathcal{T}]$. With this derivation, $\mathbb{R}[\mathcal{T}]$ is a real closed $H$-field with constant field $\mathbb{R}$. We have

$$(f^r)^\prime = e^r \cdot (f^r)^\uparrow, \quad (f^\downarrow)^\prime = \frac{1}{x} \cdot (f^\downarrow)^\downarrow \quad (f \in \mathbb{R}[\mathcal{T}]).$$

Note that $\nu(\exp(-Ax))$ remains a gap in $\mathbb{R}[\mathcal{T}]$, so $\mathbb{R}[\mathcal{T}]$ is not closed under asymptotic integration. There is also no natural extension of the exponential operation on $T$ to one on $\mathbb{R}[\mathcal{T}]$. Nevertheless, using (4.1) one easily checks that the function $\log: T^{>0} \to T$ extends to an embedding $\log$ of the ordered multiplicative group $\mathbb{R}[\mathcal{T}]^{>0}$ into the ordered additive group $\mathbb{R}[\mathcal{T}]^{>0}$, by setting

$$\log g := \log a + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^n$$

for $g = am(1 + \varepsilon)$, $a \in \mathbb{R}^{>0}$, $m \in \mathcal{T}$, and $1 > \varepsilon \in \mathbb{R}[\mathcal{I}]$.

**Monomial subgroups of $\mathcal{T}$**. In the next section we construct a Liouville closed $H$-subfield of $T$ containing $\mathcal{L}_1$; this will involve subgroups $\mathcal{M}$ of $\mathcal{T}$ such that the subfield $\mathbb{R}[\mathcal{M}]$ of $\mathbb{R}[\mathcal{T}]$ is closed under differentiation and integration. In the rest of this section, $\mathcal{M}_n$ denotes an ordered subgroup of $\mathcal{T}_n$, for every $n$, with the following properties:

$(M1)$ $\mathcal{M}_0 = \mathcal{L}$;

$(M2)$ $A_n := \log \mathcal{M}_{n+1}$ is an $H$-linear subspace of $\mathbb{R}[\mathcal{M}_n]^\uparrow$ and is closed under truncation;

$(M3)$ $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$.

Here a set $A \subseteq \mathbb{R}[\mathcal{T}]$ is said to be closed under truncation if for each $f = \sum_{m \in A} f_n m \in A$ and each final segment $\mathcal{S}$ of $\mathcal{T}$ we have $f|_{\mathcal{S}} := \sum_{m \in \mathcal{S}} f_n m \in A$.

We put $\mathcal{M} := \bigcup_n \mathcal{M}_n$, a subgroup of $\mathcal{T}$. When needed we shall also impose:

$(M4)$ $\mathcal{M} \subseteq \mathcal{M}$.

**Example 4.1.** Let $\mathcal{M}_n := \mathcal{T}_n$. Then the $\mathcal{M}_n$ satisfy (M1)-(M4), with $A_n = \mathbb{R}[\mathcal{T}_n]^\uparrow$ and $\mathcal{M} = \mathcal{T}$.

By (M1), the set $\log \mathcal{M}_0$ is also an $H$-linear subspace of $\mathbb{R}[\mathcal{M}_0]$ closed under truncation. By (M1) and (M2), each $\mathcal{M}_n$ is closed under $\mathbb{R}$-powers: if $m \in \mathcal{M}_n$ and $r \in \mathbb{R}$, then $m^r \in \mathcal{M}_n$. Also by (M1) and (M2), each subfield $\mathbb{R}[\mathcal{M}_n]$ of $T$ is closed under taking logarithms of positive elements, and so is the subfield $\mathbb{R}[\mathcal{M}]$ of $\mathbb{R}[\mathcal{T}]$. Moreover, each subfield $\mathbb{R}[\mathcal{M}_n]$ of $T$ is closed under differentiation, hence
is an $H$-subfield of $T$. (This follows by an easy induction on $n$: use (M1) for $n = 0$, and (M2) for the induction step.) It follows that the subfield $\mathbb{R}[[\mathcal{M}]]$ of $\mathbb{R}[\mathcal{X}]$ is closed under differentiation, hence is an $H$-subfield of $\mathbb{R}[\mathcal{X}]$.

**Lemma 4.2.** The $H$-field $\mathbb{R}[[\mathcal{M}]]$ is closed under asymptotic integration if and only if $\exp(\lambda) \notin \mathcal{M}$. In this case, $\mathbb{R}[[\mathcal{M}]]$ is closed under integration, and the map $f \mapsto \int f : \mathbb{R}[[\mathcal{M}]] \to \mathbb{R}[[\mathcal{M}]]$ is strongly linear.

**Proof.** The $H$-field $\mathbb{R}[[\mathcal{M}]]$ is closed under asymptotic integration if and only if it does not have a gap ([1], Section 2). The valuation of $\mathbb{R}[\mathcal{X}]$ maps $\mathcal{X}$ bijectively and order-reversingly onto the value group of $\mathbb{R}[\mathcal{X}]$, and also $\mathcal{M}$ onto the value group of $\mathbb{R}[[\mathcal{X}]]$. The element $\exp(-\lambda)$ of $\mathcal{X}$ satisfies $1/\ell_n \prec \exp(-\lambda) \prec (1/\ell_n)^+$ for all $n$. Because the sequence $1/\ell_{k_0}, 1/\ell_1, \ldots$ is coinitial in $\mathcal{M}^{-1}$, this yields the first part of the lemma. The rest now follows from Lemma 1.7. \[\Box\]

Put $\mathcal{M}_n := \mathcal{M}_n \cap \mathcal{M}^\uparrow$ and $\mathcal{M} := \bigcup_n \mathcal{M}_n$. The next easy lemma is left as an exercise to the reader.

**Lemma 4.3.** The family $(\mathcal{M}_n)$ satisfies the following analogues of (M1)-(M3):

1. $\mathcal{M}_0 = \mathbb{L}$; $\log \mathcal{M}_{n+1}$ is an $\mathbb{R}$-linear subspace of $\mathbb{R}[[\mathcal{M}_n]]$ closed under truncation;
2. $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$. If (M4) holds, then $\mathcal{M} = \mathcal{M}^\uparrow$ and $\mathcal{M}^\uparrow \subseteq \mathcal{M}$.

In the rest of this section $\mathcal{M}$ denotes a convex subgroup of $\mathcal{M}$, equivalently, a subgroup such that for all $m, n \in \mathcal{M}$

$$m \preceq n \in \mathcal{M} \implies m \in \mathcal{M}.$$ 

Note that then $\mathcal{M}$ is closed under $\mathbb{R}$-powers, and that $\mathcal{M}^\uparrow$ is a convex subgroup of $\mathcal{M}^\uparrow$. To $\mathcal{M}$ we associate the set

$$I := \{m \in \mathcal{M}^\uparrow : \exp m \preceq n \text{ for some } n \in \mathcal{M} \} \subseteq \mathcal{M}.$$

Then $I$ is an initial segment of $\mathcal{M}^\uparrow$ (with $I = \emptyset$ if $\mathcal{M} = \{1\}$). Consequently, the complement $F = \mathcal{M}^\uparrow \setminus I$ of $I$ is a final segment of $\mathcal{M}^\uparrow$, and

$$\mathcal{N} := \{r \in \mathcal{M} : \log r \in \mathbb{R}[[F]]\}$$

is also a subgroup of $\mathcal{M}$ closed under $\mathbb{R}$-powers.

**Lemma 4.4.** For all $m \in \mathcal{M}$ we have:

$$m \in \mathcal{M} \iff \log m \in \mathbb{R}[[I]].$$

**Proof.** The lemma holds trivially if $\mathcal{M} = \{1\}$. Assume that $\mathcal{M} \neq \{1\}$; hence $\ell_k \in \mathcal{M}$ from some $k \in \mathbb{N}$. Let $m \in \mathcal{M}_n$. We prove the desired equivalence by distinguishing the cases $n = 0$ and $n > 0$. If $n = 0$, then we take $k \in \mathbb{N}$ minimal such that $\ell_k \in \mathcal{M}$, so

$$\mathcal{N} \cap \mathcal{L} = \{e^{\beta_0}e^{\beta_1}\cdots e^{\beta_k} : \beta_i = 0 \text{ for all } i < k\},$$

which easily yields the desired equivalence.

Suppose that $n > 0$. Then $\log m \in A_{n-1}$. Since $A_{n-1}$ is closed under truncation we have $\log m = \varphi + \psi$ with $\varphi \in A_{n-1} \cap \mathbb{R}[[F]]$ and $\psi \in A_{n-1} \cap \mathbb{R}[[F]]$. Hence $e^\varphi, e^\psi \in \mathcal{M}$. In fact $e^\varphi \in \mathcal{N}$, because if $\varphi \neq 0$, then $\varphi(\varphi) \in I$, so $e^\varphi \preceq e^{\varphi(\varphi)} \preceq n$ for some $n \in \mathcal{M}$. Similarly, if $\psi \neq 0$, then $e^\psi \in \mathcal{N}$. The desired equivalence now follows from $m = e^\varphi \cdot e^\psi$. \[\Box\]

With $\mathcal{M}_n := \mathcal{M} \cap \mathcal{M}_n$ and $\mathcal{N}_n := \mathcal{N} \cap \mathcal{M}_n$ we have:
Corollary 4.5. \( \mathfrak{M} \cap \mathfrak{N} = \{1\} \) and \( \mathfrak{M}_n = \mathfrak{N}_n \cdot \mathfrak{N}_n \).

It follows that \( \mathfrak{M} = \mathfrak{N} \cdot \mathfrak{M} \), and the products \( \mathfrak{m} \mathfrak{n} \) with \( \mathfrak{n} \in \mathfrak{N} \) and \( \mathfrak{r} \in \mathfrak{M} \) are ordered antilexicographically: \( \mathfrak{m} \mathfrak{n} > 1 \) if and only if \( \mathfrak{r} > 1 \), or \( \mathfrak{r} = 1 \) and \( \mathfrak{n} > 1 \). We think of the monomials in the convex subgroup \( \mathfrak{M} \) as being flat. Accordingly we call \( \mathfrak{M} \) the steep supplement of \( \mathfrak{N} \).

Proof. It is clear from the previous lemma that \( \mathfrak{M} \cap \mathfrak{N} = \{1\} \). We now show \( \mathfrak{M}_n = \mathfrak{M}_n \cdot \mathfrak{N}_n \). Let \( \mathfrak{m} \in \mathfrak{M}_n \). Then log \( \mathfrak{m} \in \mathbb{R}[\mathfrak{M}]^\times \), so log \( \mathfrak{m} = \varphi + \psi \) with \( \varphi \in \mathbb{R}[F] \), \( \psi \in \mathbb{R}[F] \). Since log \( \mathfrak{M}_n \) is truncation closed, we have \( \varphi, \psi \in \log \mathfrak{M}_n \), so \( \mathfrak{m} = \mathfrak{m}_n \) with \( \mathfrak{n} := e^\varphi \in \mathfrak{M}_n \cap \mathfrak{N} = \mathfrak{M}_n \) and \( \mathfrak{r} := e^\psi \in \mathfrak{N}_n \cap \mathfrak{N} = \mathfrak{N}_n \), using the previous lemma. \( \square \)

Corollary 4.6. Suppose that \( x \in \mathfrak{M} \). Then the following analogues of (M1)-(M3) hold:

1. (N1) \( \mathfrak{N}_0 = \mathbb{L} \);
2. (N2) \( \log \mathfrak{M}_n + 1 \) is an \( \mathbb{R} \)-linear subspace of \( \mathbb{R}[\mathfrak{M}]^\times \) and is closed under truncation;
3. (N3) \( \mathfrak{N}_n \subseteq \mathfrak{M}_n + 1 \).

In particular, the subfield \( \mathbb{R}[\mathfrak{M}] \) of \( \mathbb{R}[\mathfrak{M}]^\times \) is closed under differentiation, and if \( x \not\in \mathfrak{N} \), then \( \mathbb{R}[\mathfrak{M}] \) is also closed under integration.

Remark 4.7. If we drop the assumption \( x \in \mathfrak{N} \), then \( \mathbb{R}[\mathfrak{M}] \) may fail to be closed under differentiation. To see this, take \( \mathfrak{N} = \{ m \in \mathfrak{M} : m \not\prec x \} \) and \( m = \log x \in \mathfrak{N} \); then \( m' = 1/x = x \), so \( m' \not\in \mathfrak{N} \).

Property (N2) of Corollary 4.6 follows easily from Lemma 4.4 and its proof (without assuming \( x \in \mathfrak{N} \)). The rest of the corollary is then obvious.

Lemma 4.8. Suppose that \( x \in \mathfrak{M} \), and that \( m \not\prec r \), where \( m, r \in \mathfrak{M} \), \( r \not\in \mathfrak{N} \). Then \( \text{supp} m' \not\prec r \).

Proof. By induction on \( n \) such that \( m \in \mathfrak{M}_n \). The claim is trivial for \( n = 0 \) since \( \mathfrak{N}_0 = \mathfrak{N}_0 = \mathbb{L} \) and \( m' \in \mathbb{R}[\mathfrak{N}] \). Suppose \( n > 0 \) and write \( m = e^\varphi \) with \( \varphi \in \mathfrak{A}_n \). Since \( \text{supp} \varphi \not\prec m \) we obtain \( \text{supp} \varphi' \not\prec r \), by inductive hypothesis. Any \( u \in \text{supp} m' \) is of the form \( u = v \cdot m \) with \( v \in \text{supp} \varphi' \), hence \( u \not\prec r \) as required. \( \square \)

Flattening. We "flatten" the dominance relations \( \prec \) and \( \preceq \) on \( \mathbb{R}[\mathfrak{M}]^\times \) by the convex subgroup \( \mathfrak{M} \) of \( \mathfrak{M} \) as follows:

\[
\begin{align*}
f \prec_{\mathfrak{M}} g & \iff (\forall \varphi \in \mathfrak{M} : \varphi f < g), \\
f \preceq_{\mathfrak{M}} g & \iff (\exists \varphi \in \mathfrak{M} : f \leq \varphi g),
\end{align*}
\]

for \( f, g \in \mathbb{R}[\mathfrak{M}]^\times \). We also define, for \( f, g \in \mathbb{R}[\mathfrak{M}]^\times \):

\[
\begin{align*}
f \ll_{\mathfrak{M}} g & \iff f \prec_{\mathfrak{M}} g \wedge g \preceq_{\mathfrak{M}} f,
\end{align*}
\]

hence \( \mathfrak{M} = \{ m \in \mathfrak{M} : m \ll_{\mathfrak{M}} 1 \} \). Flattening corresponds to coarsening the valuation: The value group \( v(\mathfrak{M}) \) of the natural valuation \( v \) on \( \mathbb{R}[\mathfrak{M}]^\times \) has convex subgroup \( v(\mathfrak{M}) \), so gives rise to the coarsened valuation \( v_{\mathfrak{M}} \) on \( \mathbb{R}[\mathfrak{M}]^\times \) with (ordered) value group \( v(M)/v(\mathfrak{M}) \) given by \( v_{\mathfrak{M}}(f) := v(f) + v(\mathfrak{M}) \) for \( f \in \mathbb{R}[\mathfrak{M}]^\times \). Then we have the equivalences:

\[
\begin{align*}
f \ll_{\mathfrak{M}} g & \iff v_{\mathfrak{M}}(f) > v_{\mathfrak{M}}(g) \quad \text{and} \\
f \preceq_{\mathfrak{M}} g & \iff v_{\mathfrak{M}}(f) \geq v_{\mathfrak{M}}(g)
\end{align*}
\]
for $f, g \in \mathbb{R}[[\mathfrak{M}]]$. (See also Section 14 of [2].) The restriction of $\preceq_{\mathfrak{M}}$ to $\mathfrak{M}$ is a quasi-ordering, i.e., reflexive and transitive; it is anti-symmetric (i.e., an ordering) if and only if $\mathfrak{M} = \{1\}$. The restriction of $\preceq_{\mathfrak{M}}$ to $\mathfrak{M}$ is the already given ordering on $\mathfrak{M}$. The following rules are valid for $f, g \in \mathbb{R}[[\mathfrak{M}]]$:

-the equivalence $f \preceq_{\mathfrak{M}} g \iff f' \preceq_{\mathfrak{M}} g'$ holds, provided $f, g \neq 1$;

- $1 \preceq_{\mathfrak{M}} f \preceq_{\mathfrak{M}} g \iff f' \preceq_{\mathfrak{M}} g'$;

- $f \preceq_{\mathfrak{M}} g \implies f \preceq_{\mathfrak{M}} g$, and hence $f \preceq_{\mathfrak{M}} g \implies f \preceq_{\mathfrak{M}} g$.

In our proofs below, we often reduce to the case that $x \in \mathfrak{M}$ by upward shift. Here are a few remarks about this case. If $x \in \mathfrak{M}$, then $\mathbb{L} \subseteq \mathfrak{M}$, and for all $f \in \mathbb{R}[[\mathfrak{M}]]$:

-the equivalence $f \preceq_{\mathfrak{M}} 1 \iff f' \preceq_{\mathfrak{M}} 1$ holds, provided $f \neq 1$;

- $f \succ_{\mathfrak{M}} 1 \iff f' \succ_{\mathfrak{M}} 1$. (4.2)

(See [2], Lemma 13.4.) Moreover:

**Lemma 4.9.** Suppose that $x \in \mathfrak{M}$. Then the following conditions on $m \in \mathfrak{M}$ are equivalent:

1. $\log m \preceq_{\mathfrak{M}} 1$,
2. $\log m \in \mathbb{R}[[\mathfrak{M}]]$,
3. $m^l \in \mathbb{R}[[\mathfrak{M}]]$,
4. $m^l \preceq_{\mathfrak{M}} 1$.

**Proof.** From $\text{supp}(\log m) \subseteq \mathfrak{M}^{<1}$ we obtain (1) $\implies$ (2). The implication (2) $\implies$ (3) follows from Corollary 4.6, (3) $\implies$ (4) is trivial, and (4) $\implies$ (1) follows from (4.2). □

**Flattened canonical decomposition.** We have an isomorphism

$$\mathbb{R}[[\mathfrak{M}]] \rightarrow \mathbb{R}[[\mathfrak{M}]][[\mathfrak{M}]]$$

of $\mathbb{R}[[\mathfrak{M}]]$-algebras given by

$$f = \sum_{m \in \mathfrak{M}} f_m m \mapsto \sum_{r \in \mathfrak{M}} \left( \sum_{n \in \mathfrak{N}} f_{rn} n \right) r.$$ 

In $\mathbb{R}[[\mathfrak{M}]]$ we have in fact

$$f = \sum_{r \in \mathfrak{M}} \left( \sum_{n \in \mathfrak{N}} f_{rn} n \right) r,$$

where the sums are interpreted as in Section 1. We shall identify the (real closed, ordered) field $\mathbb{R}[[\mathfrak{M}]]$ with the (real closed, ordered) field $\mathbb{R}[[\mathfrak{M}]][[\mathfrak{M}]]$ by means of this isomorphism. For $f \in \mathbb{R}[[\mathfrak{M}]]$ we put

$$f_{\mathfrak{M},r} := \sum_{n \in \mathfrak{N}} f_{rn} n \in \mathbb{R}[[\mathfrak{M}]], \quad (r \in \mathfrak{M}), \text{ and}$$

$$\text{supp}_{\mathfrak{M}} f := \{ r \in \mathfrak{M} : f_{\mathfrak{M},r} \neq 0 \}.$$ 

We have the flattened canonical decomposition of the $\mathbb{R}$-vector space $\mathbb{R}[[\mathfrak{M}]]$ (relative to $\mathfrak{M}$)

$$\mathbb{R}[[\mathfrak{M}]] = \mathbb{R}[[\mathfrak{M}]]^0 \oplus \mathbb{R}[[\mathfrak{M}]]^\mathfrak{M} \oplus \mathbb{R}[[\mathfrak{M}]]^\mathfrak{M},$$
where
\[ \mathbb{R}[[\mathfrak{M}]]^1 = \mathbb{R}[[\mathfrak{M}]][[\mathfrak{M}^*]]; \]
\[ \mathbb{R}[[\mathfrak{M}]]^\ast = \mathbb{R}[[\mathfrak{M}]]; \]
\[ \mathbb{R}[[\mathfrak{M}]]^0 = \mathbb{R}[[\mathfrak{M}]][[\mathfrak{M}^*]]. \]

Accordingly, given a transseries \( f \in \mathbb{R}[[\mathfrak{M}]], \) we write
\[ f = f^0 + f^\ast + f^0 \]
where
\[ f^0 = \sum_{1 \cdot m \in \mathfrak{M} \setminus \mathfrak{N}} f_m m \in \mathbb{R}[[\mathfrak{M}]]^0; \]
\[ f^\ast = \sum_{m \in \mathfrak{N}} f_m m \in \mathbb{R}[[\mathfrak{M}]]^\ast; \]
\[ f^0 = \sum_{1 \cdot m \in \mathfrak{M} \setminus \mathfrak{N}} f_m m \in \mathbb{R}[[\mathfrak{M}]]^0. \]

**Example 4.10.** Let \( \mathfrak{w} \in \mathfrak{M}, \mathfrak{w} \neq 1, \) and consider the convex subgroup
\[ \mathfrak{N} := \{ n \in \mathfrak{M} : n \not\sim \mathfrak{w} \} \]
of \( \mathfrak{M}. \) Suppose that \( \exp(\mathfrak{M}^\ast) \subseteq \mathfrak{M}. \) Then
\[ I = \{ m \in \mathfrak{M}^\ast : \exp m \not\sim \mathfrak{w} \} \]
and thus
\[ \mathcal{R} = \{ x \in \mathfrak{M} : \text{supp}_m f \supseteq \delta(\log \mathfrak{w}) \}. \]

In this case we write \( \text{supp}_m f \) instead of \( \text{supp}_\mathfrak{M} f, \) \( \not\sim_\mathfrak{w} \) instead of \( \not\sim_\mathfrak{N}, \) and likewise for the other asymptotic relations. In the next section we take \( \mathfrak{w} = e^x. \)

**Flatly noetherian families.** Let \((f_i)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I.\) The family \((f_i)\) is said to be flatly noetherian (with respect to \( \mathfrak{M} \)) if \((f_i)\) is noetherian as a family of elements in \( C[[\mathfrak{M}]], \) where \( C = \mathbb{R}[[\mathfrak{M}]]. \) If \((f_i)\) is flatly noetherian, then \((f_i)\) is noetherian as a family of elements of \( \mathbb{R}[[\mathfrak{M}]], \) and its sum \( \sum_{i \in I} f_i \in C[[\mathfrak{M}]] \) as a flatly noetherian family equals its sum \( \sum_{i \in I} f_i \in \mathbb{R}[[\mathfrak{M}]] \) as a noetherian family of elements of \( \mathbb{R}[[\mathfrak{M}]]. \)

For any monomial \( m \in \mathfrak{M}, \) \((f_i)\) is flatly noetherian if and only if \((m f_i)\) is flatly noetherian.

Note that if \( n_1 \succ n_2 \succ \cdots \) is an infinite sequence of monomials in \( \mathfrak{M}, \) then \((n_i)_{i \geq 1}\) is a noetherian family which is not flatly noetherian.

A map \( \Phi : \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]] \) is called flatly strongly linear (with respect to \( \mathfrak{M} \)) if \( \Phi \) considered as a map \( C[[\mathfrak{M}]] \to C[[\mathfrak{M}]] \) is strongly linear, where \( C = \mathbb{R}[[\mathfrak{M}]]. \)

**Lemma 4.11.** Suppose that \( x \in \mathfrak{M}. \) The map \( \mathfrak{R} \to C[[\mathfrak{M}]] : x \mapsto x' \) is noetherian, where \( C = \mathbb{R}[[\mathfrak{M}]], \) and thus extends uniquely to a flatly strongly linear map
\[ \varphi : \mathbb{R}[[\mathfrak{M}]] \longrightarrow \mathbb{R}[[\mathfrak{M}]]. \]

**Proof.** Let \( v_1 \prec_{\mathfrak{M}} v_2 \succ_{\mathfrak{M}} \cdots \) be elements of \( \mathfrak{M} \) and \( u_i \in \text{supp} x' \) for each \( i. \) It suffices to show that then there exist indices \( i < j \) such that \( u_i \prec_{\mathfrak{M}} u_j. \) Since differentiation on \( \mathbb{R}[[\mathfrak{M}]] \) is strongly linear, we may assume, after passing to a subsequence, that \( u_i \prec_{\mathfrak{M}} u_j \) for all \( i < j. \) If there exist \( i < j \) such that \( u_i \succ_{\mathfrak{M}} v_i \) and \( u_j =_{\mathfrak{M}} v_j, \) we are already done. So we may assume that \( u_i \not\prec_{\mathfrak{M}} v_i \) for all \( i, \) and also that \( v_i \not\succ_{\mathfrak{M}} u_i \) for
all \( i \). Write each \( u_i = r_i m_i \), with \( m_i \in \text{supp} r_i^1 \), \( m_i \not\in \mathfrak{M} \). We distinguish two cases:

1. For all \( i > 1 \) there exists a \( v_i \in \text{supp} \log u_i \) such that \( m_i \in \text{supp} v_i \). Since \( \text{supp} \log u_1 \) is noetherian we may assume, after passing to a subsequence, that \( v_i \gg v_j \) for \( 1 < i < j \). Since differentiation on \( \mathbb{R}[[\mathfrak{M}]] \) is strongly linear, we then find \( i < j \) with \( m_i \gg m_j \). Hence \( m_i \gg m_j \), so \( u_i \gg u_j \).

2. There exists an \( i > 1 \) such that for all \( v \in \text{supp} \log u_1 \) we have \( m_i \not\in \text{supp} v \). Take such \( i \) and choose \( v \in \text{supp} \log r_i \) such that \( m_i \in \text{supp} v \). Then

\[
v \in (\text{supp} \log r_i) \setminus (\text{supp} \log u_1) \subseteq \text{supp} \log (r_i/u_1) \subseteq \mathfrak{M}^{\rightarrow \cdot 1}
\]

and hence \( v \ll \log(u_1/r_i) \). Since \( \log m \ll m \) for \( m \in \mathfrak{M} \setminus \{1\} \), this yields \( v \ll u_1/r_i \). By Lemma 4.8 we get \( m_i \ll u_1/r_i \). Hence if \( n := u_1/r_i \in \mathfrak{M} \), then \( m_i \ll u_1/r_i = mn \), contradicting \( m_i \not\in \mathfrak{M} \). Therefore \( u_1 \gg u_i \).

\[
\Box
\]

In the rest of this section we assume \((\mathfrak{M})\).

In particular, our previous results apply to \( \mathfrak{M}^\dagger \) instead of \( \mathfrak{M} \) for \( k = 1, 2, \ldots \), by Lemma 4.3. In this connection, the following fact will be useful.

**Remark 4.12.** A family \( (f_i)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I \) is flatly noetherian with respect to \( \mathfrak{M} \) if and only if the family \( (f_i^\dagger)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I \) is flatly noetherian with respect to \( \mathfrak{M}^\dagger \).

We now arrive at the main results of this section:

**Theorem 4.13.** If \( (f_i)_{i \in I} \) is a flatly noetherian family in \( \mathbb{R}[[\mathfrak{M}]] \), then so is \( (f_i^\dagger)_{i \in I} \).

**Proof.** Since the case \( \mathfrak{M} = \{1\} \) is trivial, we may assume \( \mathfrak{M} \neq \{1\} \). Then \( x \in \mathfrak{M}^\dagger \) for sufficiently large \( k \in \mathbb{N} \). Since \( (f^\dagger)^k = e^{x \cdot (f^\dagger)^k} \) for \( f \in \mathbb{R}[[\mathfrak{M}]] \), Remark 4.12 allows us to reduce the case to the case that \( x \in \mathfrak{M} \). Then \( \mathbb{R}[[\mathfrak{M}]] \) is closed under differentiation by Corollary 4.6. Now consider a flatly noetherian family \( (f_i)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I \). Then \( (f_i) \) is noetherian, hence \( (f_i^\dagger) \) is noetherian by strong linearity of differentiation. By the lemma above, the family \( (g_i) \) defined by

\[
g_i := \sum_{r \in \mathfrak{M}} f_i g_i r^i
\]

is flatly noetherian. Put

\[
h_i := f_i^\dagger - g_i = \sum_{r \in \mathfrak{M}} (f_i g_i r^i)^\dagger r.
\]

We have \( \text{supp} n_i \subseteq \text{supp} f_i \) for \( i \in I \), since \( \mathbb{R}[[\mathfrak{M}]] \) is closed under differentiation. It follows that \( (h_i) \) is flatly noetherian. Hence the family \( (f_i^\dagger) \) is flatly noetherian since it is the componentwise sum of two flatly noetherian families.

**Theorem 4.14.** Suppose that \( \exp(\Lambda) \not\in \mathfrak{M} \). Then \( \mathbb{R}[[\mathfrak{M}]] \) is closed under integration, and if \( (f_i)_{i \in I} \) is a flatly noetherian family in \( \mathbb{R}[[\mathfrak{M}]] \), then \( \int f_i \) is flatly noetherian.

Before we begin the proof, we make some remarks about the summation of flatly noetherian families in \( \mathbb{R}[[\mathfrak{M}]] \). Choose a basis \( \mathcal{B} \) for the \( \mathbb{R} \)-vector space \( \mathbb{R}[[\mathfrak{M}]] \). We define a (partial) ordering \( \ll \) on \( \mathcal{B} \times \mathfrak{M} \) as follows:

\[
(b, r) \ll (c, s) \quad \iff \quad r \ll s \text{ or } r = s \text{ and } b = c,
\]

(4.3)
for all $(b,r), (c,s) \in B \times R$. Consider the $R$-vector space $R[[B \times R]]$ of transseries

$$f = \sum_{(b,r) \in B \times R} f_{(b,r)}(b,r)$$

with real coefficients $f_{(b,c)}$, whose support $\text{supp} f := \{(b,r) : f_{(b,c)} \neq 0\}$ is noetherian for $\s^*$; see Section 1. We have:

**Lemma 4.15.** There exists a unique isomorphism $\varphi : R[[B \times R]] \rightarrow R[[M]]$ of $R$-vector spaces such that

1. $\varphi(b,r) = b \cdot r$ for $b \in B$, $r \in R$,
2. a family $(f_i)_{i \in I} \in R[[B \times R]]^I$ is noetherian if and only if $(\varphi(f_i))_{i \in I}$ is flatly noetherian,
3. if $(f_i)_{i \in I} \in R[[B \times R]]^I$ is noetherian, then $\varphi(\sum_{i \in I} f_i) = \sum_{i \in I} \varphi(f_i)$.

**Proof.** Of course, there is at most one such $\varphi$. For existence, first note that the projection map $\pi : B \times R \rightarrow R$ is strictly increasing, and that a set $S \subseteq B \times R$ is noetherian if and only if $\pi(S) \subseteq R$ is noetherian and each fiber $\pi^{-1}(r)$, $r \in R$ is finite. Applying this remark to $S := \bigcup_{b \in B} \text{supp} f_b$, where $(f_b)_{b \in B}$ is a noetherian family in $R[[B \times R]]$, it follows that the subset

$$\pi(S) = \bigcup_{i \in I, b \in B, r \in R} \text{supp}_{R} (f_{i(b,r)}(b \cdot r))$$

of $R$ is noetherian, and that for each $r \in R$ there are only finitely many $(i, b) \in I \times B$ with $r \in \text{supp}_{R} (f_{i(b,r)}(b \cdot r))$. Therefore the family $(f_{i(b,r)}(b \cdot r))_{(i,b,r) \in I \times B \times R}$ of elements of $R[[M]]$ is flatly noetherian. Thus, by setting

$$\varphi(f) := \sum_{r \in R} \left(\sum_{b \in B} f_{(b,r)}(b)\right) r \quad \text{for } f \in R[[B \times R]],$$

we obtain an $R$-linear bijection $\varphi : R[[B \times R]] \rightarrow R[[M]]$ such that for every noetherian family $(f_i)_{i \in I} \in R[[B \times R]]^I$, the family $(\varphi(f_i))$ is flatly noetherian and $\varphi(\sum_{i} f_i) = \sum_{i} \varphi(f_i)$. (See proof of Proposition 3.5 in [17].) If $(f_i) \in R[[B \times R]]^I$ and $(\varphi(f_i))$ is flatly noetherian, then, with $S := \bigcup_i \text{supp} f_i$,

$$\pi(S) = \bigcup_{i \in I} \text{supp}_{R} (f_i)$$

is noetherian and $\pi|S$ has finite fibers, so $(f_i)$ is noetherian. \qed

We now begin the proof of Theorem 4.14. Using upward shifting and $\int(f \cdot x^{-1}) \uparrow f \in R[[M]]$, we first reduce to the case that $e^x \in R$. In particular $x \in R$, so $R[[R]]$ is closed under differentiation and integration, by Corollary 4.6. Partition $M = M \cup \bar{M}$ (disjoint union), where

$$M = \{m \in M : m^l \simeq_{R} 1\}$$

and

$$\bar{M} = \{m \in M : m^l >_{R} 1\}.$$ 

Then $M$ is a convex subgroup of $M$ containing $R$ which is closed under $R$-powers, and $R[[M]] = R[[K]] \oplus R[[\bar{M}]]$ as $R$-vector spaces. Note that if $n \in M$, $r \in R$, then $n \cdot r \in M$ if and only if $r \in \bar{M}$. It follows that $\bar{M} = R \cdot S$, where $S := M \cap R$. Since $x \in M$, the subfield $R[[M]]$ of $R[[R]]$ is closed under differentiation and integration, by Corollary 4.6. Moreover:
Lemma 4.16. The \(\mathbb{R}\)-linear subspace \(\mathbb{R}[[\mathcal{M}]]\) of \(\mathbb{R}[[\mathcal{N}]]\) is closed under the operators \(f \mapsto f'\) and \(g \mapsto \int f\) on \(\mathbb{R}[[\mathcal{N}]]\).

Proof. If \(\mathbb{R}[[\mathcal{M}]]\) is closed under \(f \mapsto f'\), then it is also closed under \(g \mapsto \int f\), because \(\mathbb{R}[[\mathcal{M}]]\) is closed under differentiation and \(\mathbb{R}[[\mathcal{N}]]\) is closed under integration. So let \(w \in \mathcal{W}\); it is enough to show that then \(\supp \nu w \subseteq \mathcal{W}\). Take \(n > 0\) with \(w \in \mathcal{W} \cap \mathcal{N}_n\), and write \(w = e^x\) with \(\nu \in \mathcal{N}_{n-1}\). By Lemma 4.8 we have \(\supp \nu' \preceq \nu\). Hence \(m^+ = \nu m^+ \succeq 1\) and thus \(m \in \mathcal{W}\), for every \(m \in \supp \nu w\). \(\square\)

Lemma 4.17. For all \(h \in \mathbb{R}[[\mathcal{N}]]\), we have \(\supp \int h \subseteq \supp \nu h\).

Proof. It is enough to prove the lemma for \(h\) of the form \(h = ft\), where \(f \in \mathbb{R}[[\mathcal{N}]]\), \(f \neq 0\), and \(t \in \mathcal{W} \cap \mathcal{N}\), so \(t = e^x\) with \(\nu' = t^+ \preceq 1\). By Lemma 4.9, we have \(\nu' \in \mathbb{R}[[\mathcal{N}]]\). We may assume \(\nu' \neq 0\). Then \(e^x = t \succ \mathcal{N}\), so \(\nu' = t^+ \succ n^+\) for all \(n \in \mathcal{N}\). Thus the strongly linear map

\[
\Phi : \mathbb{R}[[\mathcal{N}]] \to \mathbb{R}[[\mathcal{N}]], \quad g \mapsto g / \nu'
\]

satisfies \(\Phi(n) \prec n\) for all \(n \in \mathcal{N}\). Hence by Corollary 1.4 the strongly linear operator \(1d + \Phi\) on \(\mathbb{R}[[\mathcal{N}]]\) is bijective. We let \(g := (1d + \Phi)^{-1}(f / \nu') \in \mathbb{R}[[\mathcal{N}]]\). Then \(g + \nu' g = f\) and thus \(\int f t = g t\). \(\square\)

If \((f_i)\) is a flatly noetherian family of elements of \(\mathbb{R}[[\mathcal{N}]]\), then by the previous lemma \((f, f_i)\) is flatly noetherian. To complete the proof of Theorem 4.14 it therefore remains to show:

Lemma 4.18. If \((f_i)\) is a flatly noetherian family of elements of \(\mathbb{R}[[\mathcal{N}]]\), then \((f, f_i)\) is flatly noetherian.

Proof. Let \(C = \mathbb{R}[[\mathcal{N}]]\), let \(\mathcal{B}\) be a basis for \(C\) as \(\mathbb{R}\)-vector space, and let \(\mathbb{R}[[\mathcal{B} \times \mathcal{N}]]\) and \(\varphi : \mathbb{R}[[\mathcal{B} \times \mathcal{N}]] \to \mathbb{R}[[\mathcal{N}]]\) be as in Lemma 4.15. Put \(\mathcal{S} := \mathcal{W} \cap \mathcal{N}\) as before. Then \(\varphi(\mathcal{B} \times \mathcal{S}) = \mathcal{B} \cdot \mathcal{S} \subseteq \mathbb{R}[[\mathcal{N}]]\), so \(\varphi\) restricts to an \(\mathbb{R}\)-linear map

\[
\varphi_1 : \mathbb{R}[[\mathcal{B} \times \mathcal{S}]] \to \mathbb{R}[[\mathcal{N}]]
\]

Clearly \(\varphi_1\) is bijective, since \(\mathcal{W} = \mathcal{N} \cdot \mathcal{S}\). Consider the strongly linear operators \(D : \mathbb{R}[[\mathcal{N}]] \to \mathbb{R}[[\mathcal{N}]]\) given by \(f \mapsto f'\) and \(f : \mathbb{R}[[\mathcal{N}]] \to \mathbb{R}[[\mathcal{N}]]\) given by \(f \mapsto \int f\). We have \(D(f), f f \in \mathbb{R}[[\mathcal{N}]]\) for \(f \in \mathbb{R}[[\mathcal{N}]]\), by Lemma 4.16. By Theorem 4.13 and Lemma 4.15, the operator \(D_1 := \varphi_1^{-1} \circ D_{\mathcal{W}} \circ \varphi_1\) on \(\mathbb{R}[[\mathcal{B} \times \mathcal{S}]]\) is strongly linear, where \(D_{\mathcal{W}} := D_{[x, \mathcal{N}]}\) on \(\mathbb{R}[[\mathcal{N}]]\). By Lemma 4.15 it suffices to prove that the operator \(f_1 := \varphi_1^{-1} \circ f_{\mathcal{W}} \circ \varphi_1\) on \(\mathbb{R}[[\mathcal{B} \times \mathcal{S}]]\) is strongly linear, where \(f_{\mathcal{W}} := \int [x, \mathcal{N}]\) on \(\mathbb{R}[[\mathcal{N}]]\). Since \(1 \notin \mathcal{W}\), the operators \(D_{\mathcal{W}}\) and \(f_{\mathcal{W}}\) on \(\mathbb{R}[[\mathcal{N}]]\) are mutually inverse, and hence the operators \(D_1\) and \(f_1\) on \(\mathbb{R}[[\mathcal{B} \times \mathcal{S}]]\) are mutually inverse.

For \(t \in C^\times \cdot \mathcal{S}\), let \(\Delta t \) and \(\Delta t\) be the dominant term of the series \(t\) and \(\int t\) in \(C[[\mathcal{N}]]\), respectively, so \(\Delta, \Delta \in C^\times \cdot \mathcal{S}\) by Lemma 4.16. By the rules of \(\succ \mathcal{N}\) listed earlier, if \(t_1, t_2 \in C^\times \cdot \mathcal{S}\) satisfy \(t_1 \succ \mathcal{N} t_2\), then \(\Delta t_1 \succ \mathcal{N} \Delta t_2\) and \(\Delta t_1 \succ \mathcal{N} \Delta t_2\). Moreover, the maps \(I : C^\times \cdot \mathcal{S} \to C^\times \cdot \mathcal{S}\) and \(\Delta : C^\times \cdot \mathcal{S} \to C^\times \cdot \mathcal{S}\) are mutually inverse, and \(\varphi_1(\mathcal{B} \times \mathcal{S}) \subseteq C^\times \cdot \mathcal{S} \subseteq \mathbb{R}[[\mathcal{N}]]\). Now let

\[
\Delta_1 := \varphi_1^{-1} \circ \Delta \circ (\varphi_1|_{\mathcal{B} \times \mathcal{S}}) : \mathcal{B} \times \mathcal{S} \to \mathbb{R}[[\mathcal{B} \times \mathcal{S}]],
\]

\[
I_1 := \varphi_1^{-1} \circ I \circ (\varphi_1|_{\mathcal{B} \times \mathcal{S}}) : \mathcal{B} \times \mathcal{S} \to \mathbb{R}[[\mathcal{B} \times \mathcal{S}]].
\]

Then for \(v_1, v_2 \in \mathcal{B} \times \mathcal{S}\) we have

\[
v_1 \succ \mathcal{N} v_2 \implies \supp \Delta_1 v_1 \succ \supp \Delta_1 v_2, \quad \supp I_1 v_1 \succ \mathcal{N} \supp I_1 v_2.
\]
Hence the maps $\Delta_1, I_1$ are noetherian, so they extend uniquely to strongly linear operators on $\mathbb{R}[[\mathcal{B} \times \mathcal{G}]]$. These extensions, again denoted by $\Delta_1$ and $I_1$, respectively, are mutually inverse by [17], Proposition 3.10, because $\Delta$ and $I$ are.

Now consider the strongly linear operator

$$\Phi := (D_1 - \Delta_1) \circ I_1 = D_1 I_1 - \text{Id}$$

on $\mathbb{R}[[\mathcal{B} \times \mathcal{G}]]$. Using

$$D_1 I_1 |_{\mathcal{B} \times \mathcal{G}} = \varphi_1^{-1} \circ (D_{\mathcal{G}} \circ I) \circ (\varphi_1 |_{\mathcal{B} \times \mathcal{G}})$$

we obtain $\text{supp} \Phi(v) \prec v$ for $v \in \mathcal{B} \times \mathcal{G}$. Hence by Corollary 1.4, the operator $\text{Id} + \Phi = D_1 I_1$ on $\mathbb{R}[[\mathcal{B} \times \mathcal{G}]]$ is bijective with strongly linear inverse. Thus the operator $I_1 \circ (\text{Id} + \Phi)^{-1}$ on $\mathbb{R}[[\mathcal{B} \times \mathcal{G}]]$ is strongly linear. Finally, note that

$$D_1 \circ I_1 \circ (\text{Id} + \Phi)^{-1} = D_1 \circ I_1 \circ (D_1 I_1)^{-1} = \text{Id},$$

so $f_1 = D_1^{-1} = I_1 \circ (\text{Id} + \Phi)^{-1}$, and thus $f_1$ is strongly linear. \hfill \Box

5. Transseries of decay $> 1$

In this section we extend $L_4$ to a Liouville closed $H$-subfield $\mathcal{T}_1$ of $\mathbb{R}[[\mathcal{T}]]$ by first extending $L_4$ to a real closed $H$-subfield $\mathcal{S}$ of $\mathbb{R}[[\mathcal{T}]]$ that is closed under taking logarithms of positive elements, and then closing off $\mathcal{S}$ under downward shifts. The $H$-field $\mathcal{T}_1$ will satisfy the requirements on $K$ in the Theorem stated in the introduction.

Construction of $\mathcal{S}$. The convex subgroup

$$\mathcal{T}^0 = \{ n \in \mathcal{T} : n \prec e^x \}$$

of the ordered group $\mathcal{T}$ is closed under $\mathbb{R}$-powers. Note that $\mathcal{L} \subseteq \mathcal{T}^0$. We call $\mathcal{T}^0$ the flat part of $\mathcal{T}$. Its steep supplement (as defined in the previous section) is the subgroup

$$\mathcal{T}^1 = \{ g \in \mathcal{T} : \text{supp} \log g \succ x \}$$

of $\mathcal{T}$, called the steep part of $\mathcal{T}$. (See Examples 4.1 and 4.10.) We apply here Section 4 to $\mathcal{M} = \mathcal{T}$, and accordingly identify $\mathbb{R}[[\mathcal{T}]]$ and $\mathbb{R}[[\mathcal{T}^0]][[\mathcal{T}^1]]$. Every

$$f = \sum_{m \in \mathcal{T}} f_m m \in \mathbb{R}[[\mathcal{T}]]$$

can be written as

$$f = \sum_{r \in \mathcal{T}^1} f_r^r r,$$

where the coefficients

$$f_r^r := \sum_{n \in \mathcal{T}, e^r \prec n} f_{n} n$$

are series in $\mathbb{R}[[\mathcal{T}^0]]$. (In the notation of Section 4, we have $f_r^r = f_{\mathcal{T}^0, r}$.) We may also decompose $f$ as

$$f = f^0 + f^\infty + f^\flat,$$  \hspace{1cm} (5.1)
where, with $m$ ranging over $\mathcal{S}$,
\[
    f^0 := \sum_{m \not= 0, m \not= -e^x} f_m m;
\]
\[
    f^1 := \sum_{m \not= e^x} f_m m;
\]
\[
    f^0 := \sum_{m \not= 0, m \not= -e^x} f_m m.
\]

Put $S_0 := L_1$, the latter as defined in Section 3. So $S_0 \subseteq \mathbb{R}[L_0] \subseteq \mathbb{R}[L_1]$. Inductively, given the subfield $S_n$ of $\mathbb{R}[L_n]$, we let $S_{n+1}$ be the subfield of $\mathbb{R}[L_{n+1}]$ consisting of those $f \in \mathbb{R}[L]$ such that $f'_t \in L_1$ and $\log r \in S_1$ for all $r \in \text{supp}_t f$, that is, with $C := \mathbb{R}[L]$: $S_{n+1} = L_1 [L_{n+1}] \subseteq C[L_1],

where
\[
    \mathcal{U}_{n+1} := \mathcal{T}_1 \cap \exp(S_n^1) = \exp(S_n \cap \mathbb{R}[L^{\mathcal{T}_1}]),
\]
a subgroup of $\mathcal{T}_1 \cap \mathcal{T}_{n+1}$ closed under $\mathbb{R}$-powers. It follows that $S_{n+1} \subseteq \mathbb{R}[L_{n+1}]$. It is convenient to define $\mathcal{K}_0 := \{1\} \subseteq \mathcal{K}_0$.

Example 5.1. We have $\mathcal{U}_1 = \exp(L_1 \cap \mathbb{R}[L^{\mathcal{T}_1}])$. Therefore $e^{x^2} \in S_1$, but $e^{x^2} \downarrow = e^{(\log x)^2} \not\in \mathcal{S}_1$.

Lemma 5.2. Each $S_n$ is a real closed subfield of $T$, and $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ for all $n$. (Hence $S_n \subseteq S_{n+1}$ for all $n$.)

Proof. The first statement follows from the remarks at the beginning of Section 3 and Lemma 1.6. We show the other statement by induction on $n$. The case $n = 0$ being clear, suppose that $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$. Then
\[
    S_n = L_1 [L_1] \subseteq L_1 [L_{n+1}] = S_{n+1}
\]
and thus
\[
    \mathcal{U}_{n+1} = \mathcal{T}_1 \cap \exp(S_n^1) \subseteq \mathcal{T}_1 \cap \exp(S_{n+1}^1) = \mathcal{U}_{n+2}
\]
as required. \H

We let $S$ be the union of the increasing sequence $S_0 \subseteq S_1 \subseteq \cdots$ of real closed subfields of $T$. Then $S$ is a real closed subfield of $T$. Moreover:

Lemma 5.3. $\log(S_n^0) \subseteq S_n$ for every $n$. (Hence $\log(S^0) \subseteq S$.)

Proof. The case $n = 0$ is discussed at the beginning of Section 3. Suppose $n > 0$. Every positive $f \in S_n$ may be written in the form
\[
    f = g \cdot u \cdot (1 + \varepsilon)
\]
where $0 < g \in L_1$, $u \in \mathcal{U}_n \subseteq \exp(S_{n-1}^1)$, and $\varepsilon \prec 1$. We get
\[
    \log f = \log g + \log u + \log(1 + \varepsilon).
\]
We have $\log g \in L_1$ and (since $\varepsilon \prec 1$)
\[
    \log(1 + \varepsilon) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \varepsilon^k \in S_n.
\]
Moreover $\log u \in S_{n-1}$, thus $\log u \in S_n$ by Lemma 5.2. Hence $\log f \in S_n$. \H
We now put \( A_n := \mathcal{S}_n \cdot \mathcal{M}_{n+1} := \exp(A_n) \) for every \( n \), and \( \mathcal{M}_0 := \mathcal{L} \). Each \( A_n \) is an \( \mathbb{R} \)-linear subspace of \( \mathbb{R}[[\mathcal{T}_n]] \), and \( \mathcal{M}_n \) is a subgroup of \( \mathcal{T}_n \) closed under \( \mathbb{R} \)-powers. Here are some more properties of \( \mathcal{S}_n, A_n \) and \( \mathcal{M}_n \). A subset \( A \) of \( \mathbb{R}[[\mathcal{T}]] \) is said to be closed under subseries if for every \( f = \sum_{m \in \mathcal{T}} f_m m \in A \) the subseries \( f\big|_\mathcal{T} := \sum_{m \in \mathcal{T}} f_m m \) is in \( A \), for any subset \( \mathcal{T} \) of \( \mathcal{T} \).

**Lemma 5.4.** For every \( n \) we have:

1. \( \mathcal{S}_n \subseteq \mathbb{R}[[\mathcal{M}_n]] \). (Hence \( A_n \subseteq \mathbb{R}[[\mathcal{M}_n]]^\times \).)
2. \( \mathcal{S}_n \) is closed under subseries. (Hence \( A_n \) is closed under subseries.)
3. \( \log \mathcal{M}_n \subseteq A_n \). (Hence \( \mathcal{M}_n \subseteq \mathcal{M}_{n+1} \).)
4. \( \mathcal{S}_{n+1} \subseteq \mathcal{S}_{n+1} \). (Hence \( \mathcal{M}_{n+1} \subseteq \mathcal{M}_{n+1} \).)

*Proof.* Parts (1)–(3) are obvious for \( n = 0 \). For the case \( n = 0 \) of (4) note first that \( \mathcal{L} \subseteq \mathbb{R} \cdot (\exp x)^\mathbb{R} \) with \( \mathbb{R} \cap (\exp x)^\mathbb{R} = \{1\} \). Moreover, if a subset \( \mathcal{S} \) of \( \mathcal{L} \) has decay \( > 1 \) and \( x \subseteq \mathcal{L} \cdot (\exp x)^\mathbb{R} \), then \( \pi(x) \) has decay \( > 1 \), where \( \pi : \mathcal{L} \cdot (\exp x)^\mathbb{R} \to \mathcal{L} \) is given by \( l \cdot (\exp x)^\alpha \to l \) for \( l \in \mathcal{L} \), \( \alpha \in \mathbb{R} \). Hence \( L_1 \subseteq L_1 \left[ [\exp x]^\mathbb{R} \right] \subseteq S_1 \) as required.

Let now \( n > 0 \). For (1) note that

\[
\mathcal{L} = \exp \log \mathcal{L} \subseteq \exp (L_1^\mathbb{R}) = \exp (S_{n-1}^\mathbb{R}) = \exp (S_n^\mathbb{R}),
\]

hence

\[
\mathcal{S}_n = L_1 \left[ [\mathcal{U}_n] \right] \subseteq \mathbb{R}[[\mathcal{L} \cdot \mathcal{U}_n]] \subseteq \mathbb{R}[[\exp (S_{n-1}^\mathbb{R})]] = \mathbb{R}[[\mathcal{M}_n]].
\]

For (2) let \( f = \sum_{u \in \mathcal{U}_n} f_u u \in \mathcal{S}_n \), so \( f_u^\mathbb{R} \in L_1 \) for all \( u \). Then for any subset \( \mathcal{S} \) of \( \mathcal{T} \) we have

\[
f\big|_\mathcal{T} = \sum_{u \in \mathcal{U}_n} (f_u^\mathbb{R}) |_\mathcal{T} u \in S_n,
\]

where \( S_u := \{ n \in \mathcal{T} : n u \in \mathcal{S} \} \) for \( u \in \mathcal{U}_n \). For part (3) we have, by Lemma 5.2,

\[
\log \mathcal{M}_n = A_{n-1} = S_{n-1}^\mathbb{R} \subseteq S_n^\mathbb{R} = A_n
\]
as required. For (4), we may assume inductively that \( S_{n-1} \subseteq S_n \). Since \( \mathcal{T}_{n-1} \subseteq \mathcal{T}_n \) we get

\[
\mathcal{U}_n^\mathbb{R} = \exp \left( \mathcal{S}_{n-1} \cap \mathbb{R}[[\mathcal{T}_{n-1}^\mathbb{R}]] \right)^\mathbb{R} \subseteq \exp \left( \mathcal{S}_n \cap \mathbb{R}[[\mathcal{T}_n^\mathbb{R} \exp x]] \right) \subseteq \mathcal{U}_{n+1}.
\]

Together with \( L_1 \subseteq L_1 \left[ [\exp x]^\mathbb{R} \right] \) this yields \( S_n^\mathbb{R} \subseteq \mathcal{S}_n \subseteq \mathcal{S}_{n+1} \).

We let \( \mathcal{M} \) be the union of the increasing sequence \( \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \) of ordered subgroups of \( \mathcal{T} \). Then \( \mathcal{M} \) is an ordered subgroup of \( \mathcal{T} \), and \( \mathcal{S} \) is an ordered subfield of \( \mathbb{R}[[\mathcal{M}]] \). Note that the \( \mathcal{M}_n \) satisfy conditions (M1)–(M4) of the previous section. We have \( \mathcal{S} \cap \mathcal{L} = L_1 \), hence \( \exp(A) \notin \mathcal{M} \), by part (3) of Lemma 5.4 and Example 3.2.

**Proposition 5.5.** For every \( n \), the field \( \mathcal{S}_n \) is closed under differentiation.

*Proof.* We proceed by induction on \( n \). We have already dealt with the case \( n = 0 \) in Proposition 3.1. Let \( f = \sum_{u \in \mathcal{U}_{n+1}} f_u u \in \mathcal{S}_{n+1} \). By Theorem 4.13, the family \( (f_u u)^\mathbb{R} \) in \( \mathbb{R}[[\mathcal{T}_{n+1}]] \) is flatly noetherian. Hence for any \( s \in \mathcal{T}_{n+1}^\mathbb{R} \) the sum

\[
\sum_{u \in \mathcal{U}_{n+1}} \left[ ((f_u^\mathbb{R})') u \right]_s
\]

has only finitely many non-zero terms and equals \((f')_s\). Let \( u \in \mathcal{U}_{n+1} \) and \( s \in \mathcal{T}_{n+1}^\mathbb{R} \). By the induction hypothesis we have \( u^\mathbb{R} \in \mathcal{S}_n \), hence \( (u^\mathbb{R})' s/u \in L_1 \). By
Proposition 3.1 we get \((f'_u)^t \in L_1\). Therefore \((f')^t_s \in L_1\). It follows that \(f \in S_{n+1}\) as required. \(\square\)

**Construction of** \(T_1\). We have \(S^k = (S^1)^{k+1} \subseteq S^k \subseteq (S^1)^{k+1}\) for every \(k \in \mathbb{N}\), by Lemma 5.4, (4). We let \(T_1\) be the union of the increasing sequence
\[
S \subseteq S \subseteq S \subseteq \cdots \subseteq S^k \subseteq \cdots
\]
of real closed subfields of \(T\). The elements of the real closed subfield \(T_1\) of \(T\) are called *transseries of decay* \(\gamma > 1\). The field \(T_1\) is closed under upward and downward shift: if \(f \in T_1\), then \(f^\uparrow, f^\downarrow \in T_1\). We have \(L_1 \subseteq T_1\); in fact:

**Lemma 5.6.** \(L_1 = T_1 \cap L_1\).

*Proof.* Suppose \(f \in T_1 \cap L_1\); so \(f^\uparrow \in S_n\) where \(k, n \in \mathbb{N}\); we claim that \(f \in L_1\). The case \(k = 0\) being trivial, we may assume \(k > 0\). Then
\[
f^\uparrow \in L[[\exp x^\mathbb{R} \cdots \exp x^\mathbb{R}]] \cap S_n \subseteq L_1 \cap [\exp x^\mathbb{R} \cdots \exp x^\mathbb{R}]]\]
where \(\exp_m x = x^\uparrow_m\) for all \(m\). Hence \(f\) can be written in the form
\[
f = \sum_{\alpha \in \mathbb{R}^k} f^\alpha \circ l_k,
\]
where \(f^\alpha \in L_1\) and \(f^\alpha = l_0^{\alpha_0} \cdots l_{k-1}^{\alpha_{k-1}}\) for \(\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in \mathbb{R}^k\). By Lemma 3.4, we get \(f \in L_1\) as desired. \(\square\)

If \(A\) is a subset of \(\mathbb{R}[[\mathbb{R}]]\) which is closed under subseries, then so is \(A^\downarrow\), since \((f^\downarrow)^{\downarrow} = (f^\downarrow)^{\alpha}\), for any \(f \in A\) and \(\mathbb{R} \subseteq \mathbb{R}[[\mathbb{R}]]\). By induction on \(k\) it follows that each subfield \(S^k\) of \(\mathbb{R}[[\mathbb{R}]]\) is closed under subseries. Hence \(T_1\) is closed under subseries.

**Proof of the main theorem.** In the remainder of this section, we show that \(K = T_1\) has the properties of the main theorem in the introduction.

**Proposition 5.7.** The subfield \(T_1\) of \(T\) is closed under exponentiation and taking logarithms of positive elements.

*Proof.* Since
\[
\log (f^\downarrow) = (\log f)^\downarrow \text{ for all } m \text{ and all } f \in S^{>0},
\]
Lemma 5.3 yields that \(T_1\) is closed under taking logarithms. Similarly,
\[
\exp (f^\downarrow) = (\exp f)^\downarrow \text{ for all } m \text{ and all } f \in S.
\]
Hence as to exponentiation, it suffices to prove that \(\exp f \in T_1\) for all \(f \in S\). Let \(f \in S_n\), and decompose \(f\) as in (5.1): \(f = f^\uparrow + f^\uparrow + f^\downarrow\), so
\[
\exp f = (\exp f^\uparrow) \cdot (\exp f^\uparrow) \cdot (\exp f^\downarrow).
\]
Since \(f^\downarrow \in T^{\leq 1}\) we get
\[
\exp f^\downarrow = \sum_{n=0}^{\infty} (f^\downarrow)^n / n! \in S_n.
\]
We have
\[
f^\uparrow = \sum_{m \geq 0} f_m m \in S_n \cap \mathbb{R}[[\mathbb{R}^\mathbb{R}]],
\]
hence \(\exp f^\uparrow \in S_n \subseteq S_n\). It remains to prove that \(\exp f \in T_1\) for all \(f \in L_1\). So let \(f \in L_1\). From 1 \(\mathcal{L}\) \(\text{ supp } f \subseteq \mathcal{L}\) we obtain \(k \in \mathbb{N}\) such that \(\ell_k < m\) for all
m \in \text{supp } f \setminus \{1\}. Then \( g^m \in \mathbb{R} \) for \( g = f \downarrow^{k+1} \), hence \( \exp g \in S \) by what we have shown above. We conclude that \( \exp f = (\exp g)\downarrow^{k+1} \in T_1 \). \( \square \)

Since \( (f \downarrow)^\prime = (f \downarrow)^{-1} \cdot x^{-1} \) for all \( f \in T \), Proposition 5.5 yields:

**Corollary 5.8.** The subfield \( T_1 \) of \( T \) is closed under differentiation. (Hence \( T_1 \) is an \( H \)-subfield of \( T \).) \( \square \)

To prove that \( T_1 \) is closed under integration, we first establish some auxiliary facts. Recall that \( \mathbb{R}[[\mathcal{M}]] \) is closed under differentiation and that \( \exp(\Lambda) \not\in \mathcal{M} \). Hence \( \mathbb{R}[[\mathcal{M}]] \) is closed under integration.

In the next lemma we fix \( n > 0 \). We have the following inclusions:

\[ T \ni \mathcal{U}_n \subseteq \mathcal{M}_n \subseteq S_n \subseteq L[[\mathcal{U}_n]] = \mathbb{R}[[L \cdot \mathcal{U}_n]] \subseteq \mathbb{R}[[\mathcal{M}_n]]. \]

The subfield \( L[[\mathcal{U}_n]] \) of \( \mathbb{R}[[\mathcal{M}]] \) is closed under differentiation by Proposition 5.5, and closed under integration by the argument used to prove Lemma 4.2. Note that \( \log s \in S_{n-1} \subseteq L[[\mathcal{U}_n]] \) for all \( s \in \mathcal{U}_n \). In the next lemma we also fix a monomial \( u \in \mathcal{U}_n \setminus \{1\} \) and put

\[ \mathcal{G} := \{ s \in \mathcal{U}_n : s^t \prec_{\mathcal{E}} u^t \}, \tag{5.2} \]

a convex subgroup of \( \mathcal{U}_n \), closed under \( \mathbb{R} \)-powers.

**Lemma 5.9.** The subfield \( L[[\mathcal{G}]] \) of \( L[[\mathcal{U}_n]] \) is closed under differentiation. Also, if \( u^t \preceq_{\mathcal{E}} 1 \), then \( u^t \in L[[\mathcal{G}]] \).

**Proof.** The first part will follow if \( s' \in L[[\mathcal{G}]] \) for all \( s \in \mathcal{G} \). So let \( s \in \mathcal{G} \); we distinguish two cases:

1. \( s^t \succ_{\mathcal{E}} 1 \). Then \( s \not\in \mathcal{E}^t \), hence \( s = e^e \) with \( \text{supp } e^e \prec s \) (by Lemma 4.8 applied to \( m \in \text{supp } e^e \)). Using \( \varphi = s^t \), this yields \( m^t \prec_{\mathcal{E}} s^t \) for every \( m \in \text{supp } s' \). Let \( v \in (\text{supp } e^e \setminus \{1\} \), so \( v \succ_{\mathcal{E}} m \) with \( m \in \text{supp } s' \). Then \( v^t \prec_{\mathcal{E}} m^t \prec_{\mathcal{E}} s^t \) \( u^t \), hence \( v \in \mathcal{G} \), as desired.

2. \( s^t \preceq_{\mathcal{E}} 1 \). Then \( \log s \in L[[\mathcal{U}_n]] \cap \mathbb{R}[[\mathcal{E}^t]] = L \) (by Lemma 4.9) and thus \( s' = (\log s)^t \cdot s \in L[[\mathcal{G}]] \).

Suppose that \( u^t \succ_{\mathcal{E}} 1 \). Then \( \log u \succ_{\mathcal{E}} 1 \) by Lemma 4.9, hence

\[ (\log u)^t = \frac{u^t}{\log u} \prec_{\mathcal{E}} u^t. \]

Therefore, if \( v \in \text{supp } e^e \), then \( v^t \prec_{\mathcal{E}} (\log u)^t \prec_{\mathcal{E}} u^t \), hence \( v \in \mathcal{G} \). Thus \( \log u \in L[[\mathcal{G}]] \), and since \( L[[\mathcal{G}]] \) is closed under differentiation, we get \( u^t \in L[[\mathcal{G}]] \). \( \square \)

**Lemma 5.10.** Let \( f \in S \) with \( u^t \succ_{\mathcal{E}} 1 \) for all \( u \in (\text{supp}_{\mathcal{E}} f) \setminus \{1\} \). Then \( \int f \in S \).

**Proof.** We already know that \( S_0 = L_1 \) is closed under distinguished integration, by Proposition 3.5. So we may assume that \( f \not\in \text{supp}_{\mathcal{E}} f \) by passing from \( f \) to \( f - f_1 \). Take \( n > 0 \) such that \( f \in S_n \). We shall prove that \( \int f \in S_n \). We have

\[ f = \sum_{u \in \mathcal{U}_n} f_1^u u^t \in L_1[[\mathcal{U}_n]] = S_n. \]

Put \( \mathcal{R} := \mathcal{M} \cap \mathcal{E}^t \), a convex subgroup of \( \mathcal{M} \); note that \( L \subseteq \mathbb{R}[[\mathcal{M}]] \). Let \( \mathcal{R} \) be the steep supplement of \( \mathcal{R} \) in \( \mathcal{M} \). The definitions of \( \mathcal{E}^t \) and \( \mathcal{R} \) easily yield that \( \mathcal{M} \cap \mathcal{E}^t \subseteq \mathcal{R} \); hence \( \mathcal{U}_n \subseteq \mathcal{R} \). Therefore, the family \( (f_1^u u^t)_{u \in \mathcal{U}_n} \) in \( \mathbb{R}[[\mathcal{M}]] \) is flatly noetherian with respect to \( \mathcal{R} \), with sum \( f \). Thus by Theorem 4.14, the family \( (\int f_1^u u^t)_{u \in \mathcal{U}_n} \) in \( \mathbb{R}[[\mathcal{M}]] \)
is also flatly noetherian, with sum \( \int f \). Fix any \( g \in L_1 \) and \( u \in \mathcal{U}_n \) with \( u^\dagger \succ \varepsilon \cdot 1 \); it suffices to show that then \( \int g u \in S_n = L_1 [[\mathcal{U}_n]] \). Put \( h := \frac{1}{2} \int g u \in L_1 [[\mathcal{U}_n]] \); it remains to show that \( h \in L_1 [[\mathcal{U}_n]] \). Note that

\[
h + (h'/u^\dagger) = g/u^\dagger.
\]

Let \( \mathcal{S} \) be as in (5.2). Take a basis \( \mathcal{C} \) for the \( \mathbb{R} \)-vector space \( L \); extend \( \mathcal{C} \) to a basis \( \mathcal{B} \) for \( \mathbb{R}[[\mathcal{S}]] \), and let \( \varkappa^* \) be as in (4.3) and \( \varphi : \mathbb{R}[[\mathcal{B} \times \mathcal{R}]] \to \mathbb{R}[[\mathcal{S}]] \) as defined in Lemma 4.15. The map \( \varphi \) restricts to an \( \mathbb{R} \)-linear bijection

\[
\varphi_1 : \mathbb{R}[[\mathcal{C} \times \mathcal{S}]] \to \mathbb{R}[[\mathcal{S} : \mathcal{S}]] = L_1[[\mathcal{S}]].
\]

By the previous lemma, the subfield \( L_1[[\mathcal{S}]] \) of \( L_1[[\mathcal{U}_n]] \) is closed under differentiation and contains \( u^\dagger \). Hence the operator

\[
\Phi : L_1[[\mathcal{U}_n]] \to L_1[[\mathcal{U}_n]], \quad y \mapsto y'/u^\dagger
\]

maps \( L_1[[\mathcal{S}]] \) to itself, and \( (\text{Id} + \Phi)(h) = g/u^\dagger \). By Theorem 4.13 the operator \( \Phi_1 := \varphi_1^{-1} \circ \Phi \circ \varphi_1 \) on \( \mathbb{R}[[\mathcal{C} \times \mathcal{S}]] \) is strongly linear, and \( \text{supp} \Phi_1 (c, s) \succ^*(c, s) \) for all \( (c, s) \in \mathcal{C} \times \mathcal{S} \). We now apply Corollary 1.4 with \( \mathcal{C} \times \mathcal{S} \) in place of \( \mathcal{S} \), ordered by the restriction of \( \succ^* \) to \( \mathcal{C} \times \mathcal{S} \), and \( \Phi_1 \) in place of \( \Phi \). It follows that the family

\[
((-1)^i \Phi^i (g/u^\dagger))_{i \in \mathbb{N}}
\]

in \( L_1[[\mathcal{S}]] \) is flatly noetherian as a family in \( \mathbb{R}[[\mathcal{S}]] \), and that

\[
h_1 := \sum_{i=0}^{\infty} (-1)^i \Phi^i (g/u^\dagger) \in L_1[[\mathcal{S}]]
\]

satisfies

\[
h_1 + (h_1'/u^\dagger) = g/u^\dagger = h + (h'/u^\dagger).
\]

Hence \( h = h_1 + cu^\dagger \) for some \( c \in \mathbb{R} \). From \( \Phi(L_1[[\mathcal{U}_n]]) \subseteq L_1[[\mathcal{U}_n]] \) we obtain that \( \Phi^i (g/u^\dagger) \in L_1[[\mathcal{U}_n]] \) for all \( i \). Hence \( h_1 \in L_1[[\mathcal{U}_n]] \), and thus \( h \in L_1[[\mathcal{U}_n]] \). \( \square \)

Next we show that for suitable \( f \) the hypothesis in the last lemma is satisfied after a single upward shift:

**Lemma 5.11.** For every \( f \in S \) with \( f^\dagger = 0 \) and \( u \in \text{supp}_{\varepsilon \cdot 1} f \uparrow \) we have \( u^\dagger \succ \varepsilon \cdot 1 \).

**Proof.** Suppose \( f \in S_n, f^\dagger = 0, n > 0 \). Then

\[
f^\dagger = \sum_{1 \neq s \in \mathcal{U}_n} (f^s)^\uparrow \cdot s^\uparrow
\]

with \( \text{supp}_{\varepsilon \cdot 1} (f^s)^\uparrow \subseteq (\exp x)^\mathbb{R} \) for \( 1 \neq s \in \mathcal{U}_n \). So it suffices to show for such \( s \) that \( (s^\uparrow)^\dagger \succ \varepsilon \cdot 1 \). Write \( s = e^\varphi \) with \( 0 \neq \varphi \in S_{n-1} \cap \mathbb{R}[[\mathcal{S}_{n-1}]] \). Then \( \delta(\varphi) \succ x \) and hence \( \delta(\varphi^\dagger) \succ e^x \). Therefore \( \delta(\varphi^\dagger)^\dagger \succ (e^x)^\dagger = e^x \succ \varepsilon \cdot 1 \), so \( (s^\uparrow)^\dagger = (\varphi^\dagger)^\dagger \succ \varepsilon \cdot 1 \) as required. \( \square \)

**Proposition 5.12.** The \( H \)-subfield \( T_1 \) of \( T \) is closed under integration.

**Proof.** We claim that for each \( k \in \mathbb{N} \) and \( g \in \mathcal{S}_k^t \) there is \( f \in \mathcal{S}_k^{t+1} \) such that \( f^\dagger = g \). We proceed by induction on \( k \). Fix, let \( g \in S \). By Proposition 3.5 we may assume that \( g^\dagger = 0 \). Consider \( G = (g^\dagger) \cdot e^x \in S \). By the previous lemma, all \( u \in (\text{supp}_{\varepsilon \cdot 1} G) \setminus \{1\} \) satisfy \( u^\dagger \succ \varepsilon \cdot 1 \). By Lemma 5.10, we get \( \int G \in S \) and hence \( \int g = (\int G) \downarrow \in S \downarrow \). This proves the case \( k = 0 \) of our claim.
For the induction step we consider an element of $\mathbb{S}_k^{k+1}$, and write it as $g_\downarrow$ with $g \in \mathbb{S}_k^k$. Then $g \cdot e^x \in \mathbb{S}_k^k$, so inductively we have an $f \in \mathbb{S}_k^{k+1}$ with $f' = g \cdot e^x$. Then $(f')_\downarrow = g_\downarrow$, and $f_\downarrow \in \mathbb{S}_k^{k+2}$.

We now have the main theorem from the introduction, with $K = T_1$:

**Corollary 5.13.** The $H$-subfield $T_1$ of $T$ is Liouville closed, and $q \in T_1$.

**Proof.** Propositions 5.7 and 5.12 yield that $T_1$ is Liouville closed; the second part follows from $q \in L_1 \subseteq T_1$. \hfill $\square$

### 6. Final Remarks

The differential polynomial $2Z' + Z^2$ (the “Schwarzian” in [4]) has a close connection to the second-order linear differential equation $Y'' = fY'$ where $f$ is an element of some $H$-field: whenever $y$ is a non-zero solution to $Y'' = fY$, then $z = 2y'$ satisfies $2z' + z^2 = f$. The cut in $\mathbb{R}[[x]] = \mathbb{R}((x^{-1}))^{\text{LE}}$ determined by $q = 2L + \lambda^2 \in \mathbb{L}$ can be used to describe for which $f \in \mathbb{R}[[x]]$ the linear differential equation $Y'' = fY$ has a non-zero solution in $\mathbb{R}[[x]]$; see [14]. (Likewise for the existence of solutions in finite-rank Hardy fields, [10].) See also [7] for some observations about the role of gaps in Hardy fields, and of the transseries $\Lambda$, in the theory of ordinary differential equations over o-minimal expansions of the real exponential field.

The transseries $q$ makes another appearance in Écalle [4]: *Lemme 7.4* says that for any non-constant differential polynomial $P(Z, Z', \ldots, Z^{(n)}) \in \mathbb{R}(Z)$, the series $P(\lambda, \lambda', \ldots, \lambda^{(n)}) \in \mathbb{L}$ has infinite support, and the sum of its first $\omega$ terms, after possibly discarding finitely many initial terms, either has the form

$$c \Delta_0^{-c_0} \ell^{-c_1} \cdots \ell^{-c_{k-1}} (\lambda_1^k) \quad \text{with } c_0 \geq c_1 \geq \cdots \geq c_{k-1} > 1$$

or

$$c \Delta_0^{-c_0} \ell^{-c_1} \cdots \ell^{-c_{k-1}} (\lambda_1^k) \quad \text{with } c_0 \geq c_1 \geq \cdots \geq c_{k-1} > 2,$$

where $c \in \mathbb{R}$, $k \in \mathbb{N}$, and the $c_i$ are integers.

Given a real number $r \geq 0$, we say that a subset $\mathcal{S}$ of $\mathcal{L}$ has decay $r$ if for every $m$ in $\mathcal{L}$ there exists $k_0$ such that $a_k < -r$ for all $k \geq k_0$. Let $L_r$ be the set of all $f \in \mathcal{L}$ such that supp $f$ has decay $r$. (So $L_r \subseteq L_s$ for $0 \leq s \leq r$.) We have $\lambda \in L_r \setminus L_{r'}$ for all $r < r' < 1$ and $q \in L_s \setminus L_2$ for $0 \leq s < 2$. As with $L_1$, one can show that $L_r$ is a differential subfield of $L_1$, which is closed under integration if and only if $r \geq 1$. (For $0 \leq r < 1$ we have $\lambda \in L_r$, but $f \lambda = \lambda \not\in L_r$.) For $r \geq 1$, carrying out the construction of $T_1$ with $L_r$ in place of $L_1$ yields a Liouville closed $H$-subfield $T_r$ of $T$ which doesn't contain an element of the form $\lambda + \epsilon$, where $\epsilon \in \mathbb{R}[[x]]$ satisfies $\epsilon \ll 1/(\ell c_1 \cdots c_n)$ for all $n$.

By the above result of Écalle, $\lambda$ does not satisfy any differential equation of the form $P(\lambda, \lambda', \ldots, \lambda^{(n)}) = f$, where $P(Z, Z', \ldots, Z^{(n)}) \in \mathbb{R}(Z)$ is non-constant and $f \in T_r$ with $r > 1$. (We suspect that $\lambda$ is differentially transcendental over $L_r$, and hence over $T_r$, for any $r > 1$.) In particular, our construction of a differentially algebraic, non-Liouvilleian $\mathcal{S}$ could not have been carried out with $T_1$ replaced by $T_r$ for any $r > 1$, even if we replace $2Z' + Z^2$ by another non-constant differential polynomial $P(Z, Z', \ldots, Z^{(n)}) \in \mathbb{R}(Z)$.

Finally, let us mention that the Newton polygon method of [15] can be used to obtain Hardy field examples of the various possibilities for the appearance of gaps exhibited in this paper. We shall leave the details for another occasion.
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