DIFFERENTIALLY ALGEBRAIC GAPS

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Abstract. H-fields are ordered differential fields that capture some basic properties of Hardy fields and fields of transseries. Each H-field is equipped with a convex valuation, and solving first-order linear differential equations in H-field extensions is strongly affected by the presence of a "gap" in the value group. We construct a real closed H-field that solves every first-order linear differential equation, and that has a differentially algebraic H-field extension with a gap. This answers a question raised in [1]. The key is a combinatorial fact about the support of transseries obtained from iterated logarithms by algebraic operations, integration, and exponentiation.

Introduction

This paper is motivated by a basic problem about H-fields, the *gap problem,* as we explain later in this introduction. In this paper "differential field" means "ordinary differential field of characteristic 0"; H-fields are ordered differential fields whose ordering and derivation interact in a strong way. The category of H-fields was defined in [1] as a common algebraic framework for two points of view on the asymptotic behavior of one-variable real-valued functions at infinity: the theory of Hardy fields (see [9]), and the more recent theory of transseries fields, introduced by Dahn and Göring [3] as well as Écalle [4], and further developed in [15], [13], [14], [11]. We hope that the theory of H-fields will lead to a better (model-theoretic) understanding of Hardy fields, and of their relation to fields of transseries.

For this introduction, we assume that the reader has access to [1] and [2]; in particular, the notations and conventions in these papers remain in force. We just recall here that any H-field K (with constant field C) comes equipped with a *dominance relation* $\preccurlyeq$: for $f, g \in K$, we have

$$f \preccurlyeq g \iff |f| \leqslant c|g| \text{ for some } c \in C,$$

and we write $f \prec g$ if $f \preccurlyeq g$ and $g \not\preccurlyeq f$; we also write $g \succcurlyeq f$ instead of $f \preccurlyeq g$, and $g \succ f$ instead of $f \prec g$. (If $K \supseteq \mathbb{R}$ is a Hardy field, then K is an H-field and, in Landau's O-notation, $f \preccurlyeq g \Leftrightarrow f = O(g)$ and $f \prec g \Leftrightarrow f = o(g)$.) For some basic properties of these asymptotic relations we refer to [16] in the case of transseries fields, and [2] for H-fields in general.

Let K be an H-field. The set $K^{\preccurlyeq 1} = \{f \in K : f \preccurlyeq 1\}$ of *bounded* elements of K is a convex subring of K; we shall always denote the associated valuation by $v\colon K \to \Gamma \cup \{\infty\}$, with $\Gamma = v(K^\times)$, $K^\times := K \setminus \{0\}$. For $f, g \in K$ we write $f \asymp g$ if $v(f) = v(g)$, that is, $f \preccurlyeq g$ and $g \preccurlyeq f$. An element f of K is said to be *infinitesimal*

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if $f \prec 1$, equivalently, $|f| < c$ for all positive constants $c \in C$, and *infinite* if $f \succ 1$, equivalently, $|f| > C$.

An H-field K is *Liouville closed* if K is real closed, and any first-order linear differential equation $y' + fy = g$ with $f, g \in K$ has a solution in K. A *Liouville closure* of an H-field K is a Liouville closed H-field L extending K which is minimal with this property. Every H-field K has at least one, and at most two, Liouville closures, up to isomorphism over K. Given a differential field F, an element $f \in F^\times$ and an element y in some differential field extension of F we let $f^\dagger := f'/f$ denote the logarithmic derivative of f, and let $F\langle y\rangle := F(y, y', y'', \dots)$ be the differential field generated by y over F. A differential field F is said to be *closed under integration* if for each $g \in F$ there is $f \in F$ with $f' = g$.

Gaps in H-fields. In an H-field, asymptotic relations between elements of non-zero valuation may be differentiated: if $f, g \not\asymp 1$, then $f \prec g \Leftrightarrow f' \prec g'$. In particular, if f is infinitesimal and g is infinite, then $f' \prec g'$. Also, if ε and δ are non-zero infinitesimals, then $\varepsilon' \prec \delta^\dagger$. A *gap* in an H-field K is an element $\gamma = v(g)$, $g \in K^\times$, of its value group Γ such that $\varepsilon' \prec g \prec \delta^\dagger$ for all non-zero infinitesimals ε, δ. An H-field has at most one gap, and has no gap if it has a smallest comparability class or is Liouville closed. Further examples of H-fields without a gap can be obtained using the H-field of transseries of finite exponential and logarithmic depth with real coefficients, denoted by $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ in [14], and by $\mathbb{R}[[[x]]]$ in [15]: each ordered differential subfield of $\mathbb{R}[[[x]]]$ that contains $\mathbb{R}$ is an H-field without a gap.

If an H-field K has a gap $v(g)$ as above, then K has exactly two Liouville closures, up to isomorphism over K: one in which $g = \varepsilon'$ with infinitesimal ε, and one where $g = h'$ with infinite h. This "fork in the road" due to a gap causes much trouble. For a model-theoretic analysis of (existentially closed) H-fields, one needs to understand when a given H-field can have a differentially algebraic H-field extension with a gap. (An extension $L|K$ of differential fields is said to be *differentially algebraic* if every element of L is a zero of a non-constant differential polynomial over K).

The gap problem. The simplest type of differentially algebraic extensions are Liouville extensions. If K is a real closed H-field and $L = K(y)$ is an H-field extension with $y' \in K$, then L has a gap if and only if K does, by [1], [2]. However, [2] also has an example of a real closed H-field K without a gap, but such that some H-field extension $L = K(y) \supseteq K$ with $y \neq 0$, $y^\dagger \in K$, has a gap. It may even happen that an H-field K has no gap, but its real closure does. These examples raise the question (called the "gap problem" in [1]) whether the creation of gaps in differentially algebraic H-field extensions can be confined to Liouville extensions. More precisely, we asked the following:

Suppose L is a differentially algebraic H-field extension of a Liouville closed H-field K. Can L have a gap? (A negative answer would have been welcome.)

Our main result is an example where the answer is positive. This example is about as simple as possible, and may well be *generic* in some sense.

Outline of the example. No differentially algebraic H-field extension of $\mathbb{R}[[[x]]]$ can have a gap, by [2], Corollary 12.2, and this statement remains true when $\mathbb{R}[[[x]]]$

is replaced by any Liouville closed H-subfield. Our example will indeed live in a *larger* field $\mathbb{T}$ of transseries, as we shall indicate.

First, let $\mathfrak{L}$ denote the multiplicative ordered subgroup of $\mathbb{R}[[[x]]]^{>0}$ generated by the real powers of the iterated logarithms

$$\ell_0 := x,\ \ell_1 := \log x,\ \ell_2 := \log\log x, \dots, \ell_n := \log_n x, \dots$$

of x (the group of *logarithmic monomials,* see Section 2). This gives rise to

$$\mathbb{L} := \mathbb{R}[[\mathfrak{L}]] \quad \text{(the } \textit{field of logarithmic transseries}\text{)}.$$

In the beginning of Section 3 we equip $\mathbb{L}$ with a derivation making it an H-field with constant field $\mathbb{R}$. Let $\mathbb{T}$ be the field of *transseries of finite exponential depth and logarithmic depth at most* ω*,* with real coefficients (denoted by $\mathbb{R}^{\omega}_{<\omega}[[[x]]]$ in [15]). At this stage we only mention that $\mathbb{T}$ is obtained from $\mathbb{L}$ by an inductive procedure of closure under exponentiation. (Details of this procedure are in [15], Chapter 2, and are recalled at the beginning of Section 4.) As a result of its construction $\mathbb{T}$ comes equipped with a derivation that makes it a real closed H-field extension of $\mathbb{L}$ (with same constant field $\mathbb{R}$), and with an isomorphism exp of the ordered additive group of $\mathbb{T}$ onto its positive multiplicative group $\mathbb{T}^{>0}$, whose inverse is denoted by log, such that $\exp(f)' = f'\exp(f)$ for all $f \in \mathbb{T}$ and $\log \ell_n = \ell_{n+1}$ for all n.

Moreover, the sequence $\ell_0, \ell_1, \ell_2, \dots$ is coinitial in the set of positive infinite elements of $\mathbb{T}$ and hence $1/\ell_0, 1/\ell_1, 1/\ell_2, \dots$ is cofinal in the set of positive infinitesimals of $\mathbb{T}$. Also, $\mathbb{R}[[[x]]] \subseteq \mathbb{T}$, as H-fields and as exponential fields. Here is a diagram illustrating the various H-fields and their inclusions (indicated by arrows):

$$\begin{array}{ccc}
\mathbb{L} = \mathbb{R}[[\mathfrak{L}]] & \longrightarrow & \mathbb{T} \\
\uparrow & & \uparrow \\
\mathbb{R}(\mathfrak{L}) & \longrightarrow & \mathbb{R}[[[x]]]
\end{array}$$

Whereas the H-field $\mathbb{L}$ does not have a gap (see Section 3), the H-field $\mathbb{T}$ *does*. In particular, $\mathbb{T}$ is not Liouville closed. To see this, we set as in [4], Chapter 7:

$$\Lambda := \ell_1 + \ell_2 + \ell_3 + \cdots \in \mathbb{L}.$$

In $\mathbb{T}$ we have $(\ell_n)^\dagger = (\ell_{n+1})' = \exp\big(-(\ell_1 + \ell_2 + \cdots + \ell_{n+1})\big)$, and thus

$$(1/\ell_n)' \prec \exp(-\Lambda) \prec (1/\ell_n)^\dagger \qquad \text{for all } n.$$

(Intuitively, $\exp(-\Lambda)$ represents the infinitely long logarithmic monomial $\frac{1}{\ell_0\ell_1\ell_2\cdots}$.) Therefore $v(\exp(-\Lambda))$ is a gap in $\mathbb{T}$, and hence is a gap in each H-subfield of $\mathbb{T}$ that contains $\exp(\Lambda)$. So any Liouville closed H-subfield K of $\mathbb{T}$ with a differentially algebraic H-field extension $L \subseteq \mathbb{T}$ containing $\exp(\Lambda)$ is an example as claimed. Put

$$\lambda := \Lambda' = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \frac{1}{\ell_0\ell_1\ell_2} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} + \cdots \in \mathbb{L}.$$

Let $\varrho := 2\lambda' + \lambda^2 \in \mathbb{L}$. A computation shows that

$$\varrho = -\left(\frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2} + \cdots\right).$$

We shall prove (Corollary 5.13):

Theorem. *There exists a Liouville closed H-subfield $K \supseteq \mathbb{R}(\mathfrak{L})$ of $\mathbb{T}$ such that $\varrho \in K$.*

Given K as in the theorem, let $L := K\big(\exp(\Lambda), \lambda\big) \subseteq \mathbb{T}$. Since $\exp(\Lambda)^\dagger = \lambda$ and $\lambda' = \varrho - (1/2)\lambda^2$, L is an H-subfield of $\mathbb{T}$ and differentially algebraic over K; thus K and L are an example as claimed.

We shall construct a K as in the theorem by isolating a condition on transseries in $\mathbb{T}$, namely "to have decay > 1", a condition satisfied by ϱ, but not by λ. The main effort then goes into showing that this condition defines a Liouville closed H-subfield of $\mathbb{T}$ as in the Theorem.

Organization of the paper. After preliminaries in Section 1 on transseries, we introduce in Section 2 the property of subsets $\mathfrak{S}$ of $\mathfrak{L}$ to have decay > 1. In Section 3 we consider the subset $\mathbb{L}_1$ of $\mathbb{L}$ consisting of those series whose support has decay $>$ 1, and show that $\mathbb{L}_1$ is an H-subfield of $\mathbb{L}$ closed under integration and taking logarithms of positive elements. (By construction, $\varrho \in \mathbb{L}_1$, but $\lambda \notin \mathbb{L}_1$.) Section 4 is the most technical; it focuses on subgroups $\mathfrak{M}$ of the group $\mathfrak{T}$ of monomials of $\mathbb{T}$ and shows, under mild assumptions including $\exp(\Lambda) \notin \mathfrak{M}$, that then the transseries field $\mathbb{R}[[\mathfrak{M}]]$ is closed under a natural derivation on $\mathbb{R}[[\mathfrak{T}]]$ extending that of $\mathbb{T}$, and is also closed under integration. (Here we make essential use of the Implicit Function Theorem from [17].) In Section 5 we prove the main theorem by extending $\mathbb{L}_1$ to a Liouville closed H-subfield $\mathbb{T}_1$ of $\mathbb{T}$. We finish with comments on the transseries λ and ϱ.

1. Preliminaries

In our notations we mostly follow [17]. Throughout this paper we let m and n range over $\mathbb{N} := \{0, 1, 2, \dots\}$.

Strong linear algebra. Let $(\mathfrak{M}, \preccurlyeq)$ be an ordered set. (We do not assume that $\preccurlyeq$ is total, but we do follow the convention that ordered abelian groups and ordered fields are totally ordered.) A subset $\mathfrak{S}$ of $\mathfrak{M}$ is said to be *noetherian* if for every infinite sequence $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ in $\mathfrak{S}$ there exist indices $i < j$ such that $\mathfrak{m}_i \succcurlyeq \mathfrak{m}_j$. If the ordering $\preccurlyeq$ is total, then $\mathfrak{S} \subseteq \mathfrak{M}$ is noetherian if and only if $\mathfrak{S}$ is well-ordered for the reverse ordering $\succcurlyeq$, that is, there is no strictly increasing infinite sequence $\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \cdots$ in $\mathfrak{S}$. Let C be a field. Then

$$C[[\mathfrak{M}]] := \left\{ f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m} : \text{all } f_{\mathfrak{m}} \in C,\ \operatorname{supp} f \subseteq \mathfrak{M} \text{ is noetherian} \right\},$$

where $\operatorname{supp} f = \{\mathfrak{m} \in \mathfrak{M} : f_{\mathfrak{m}} \neq 0\}$ is the *support* of f, denotes the C-vector space of *transseries with coefficients in C and monomials from $\mathfrak{M}$*. We refer to [17] for terminology and basic results concerning "strong linear algebra" in $C[[\mathfrak{M}]]$. In particular, a family $(f_i)_{i \in I}$ in $C[[\mathfrak{M}]]$ is called *noetherian* if the set $\bigcup_{i \in I} \operatorname{supp} f_i \subseteq \mathfrak{M}$ is noetherian and for each $\mathfrak{m} \in \mathfrak{M}$ there exist only finitely many $i \in I$ such that $\mathfrak{m} \in \operatorname{supp} f_i$. In this case, we put

$$\sum_{i \in I} f_i := \sum_{\mathfrak{m} \in \mathfrak{M}} \left(\sum_{i \in I} f_{i,\mathfrak{m}} \right) \mathfrak{m},$$

an element of $C[[\mathfrak{M}]]$.

Let $(\mathfrak{N}, \leqslant)$ be a second ordered set. A C-multilinear map $\Phi \colon C[[\mathfrak{M}]]^n \to C[[\mathfrak{N}]]$ is called *strongly multilinear* if for all noetherian families

$$(f_{1,i_1})_{i_1 \in I_1}, \dots, (f_{n,i_n})_{i_n \in I_n}$$

in $C[[\mathfrak{M}]]$ the family

$$\big(\Phi(f_{1,i_1},\dots,f_{n,i_n})\big)_{(i_1,\dots,i_n)\in I_1\times\cdots\times I_n}$$

in $C[[\mathfrak{N}]]$ is noetherian and

$$\Phi\left(\sum_{i_1\in I_1} f_{1,i_1},\dots,\sum_{i_n\in I_n} f_{n,i_n}\right)=\sum_{(i_1,\dots,i_n)\in I_1\times\cdots\times I_n}\Phi(f_{1,i_1},\dots,f_{n,i_n}).$$

In the case $n=1$ we say that Φ is *strongly linear.* Clearly a strongly multilinear map $C[[\mathfrak{M}]]^n\to C[[\mathfrak{N}]]$ is strongly linear in each of its n variables.

A map $\varphi\colon \mathfrak{M}\to C[[\mathfrak{N}]]$ is said to be *noetherian* if for every noetherian subset $\mathfrak{S}\subseteq\mathfrak{M}$, the family $(\varphi(\mathfrak{m}))_{\mathfrak{m}\in\mathfrak{S}}$ in $C[[\mathfrak{N}]]$ is noetherian; equivalently, for every infinite sequence $\mathfrak{m}_1\succ\mathfrak{m}_2\succ\cdots$ of monomials in $\mathfrak{M}$ and $\mathfrak{n}_i\in\operatorname{supp}\varphi(\mathfrak{m}_i)$ for $i\geqslant 1$, there exist $i<j$ such that $\mathfrak{n}_i\succ\mathfrak{n}_j$. A noetherian map $\mathfrak{M}\to C[[\mathfrak{N}]]$ extends to a unique strongly linear map $C[[\mathfrak{M}]]\to C[[\mathfrak{N}]]$ (Proposition 3.5 in [17]), and every strongly linear map $C[[\mathfrak{M}]]\to C[[\mathfrak{N}]]$ restricts to a noetherian map $\mathfrak{M}\to C[[\mathfrak{N}]]$.

A map $\Phi\colon C[[\mathfrak{M}]]\to C[[\mathfrak{N}]]$ is called *noetherian* if there exists a family $(M_n)_{n\in\mathbb{N}}$ of strongly multilinear maps

$$M_n\colon C[[\mathfrak{M}]]^n\to C[[\mathfrak{N}]]$$

such that for every noetherian family $(f_k)_{k\in K}$ in $C[[\mathfrak{M}]]$ the family

$$\big(M_n(f_{k_1},\dots,f_{k_n})\big)_{n\in\mathbb{N},k_1,\dots,k_n\in K}$$

in $C[[\mathfrak{N}]]$ is noetherian and

$$\Phi\left(\sum_{k\in K} f_k\right)=\sum_{\substack{n\in\mathbb{N}\\ k_1,\dots,k_n\in K}} M_n(f_{k_1},\dots,f_{k_n}).$$

The family (M_n) is called a *multilinear decomposition* of Φ. If $\operatorname{char} C=0$, then the M_n may chosen to be symmetric, and in this case the sequence $(M_n)_{n\in\mathbb{N}}$ is uniquely determined by Φ ([17], Proposition 5.8). Every strongly linear map $\Phi\colon C[[\mathfrak{M}]]\to C[[\mathfrak{N}]]$ is noetherian, with multilinear decomposition (M_n) given by $M_1=\Phi$ and $M_n=0$ for $n\neq 1$. Conversely, if C is infinite, then every linear noetherian map is strongly linear, as we show next.

Lemma 1.1. *Suppose the field C is infinite and $(f_i)_{i\in\mathbb{N}}$ is a noetherian family in $C[[\mathfrak{M}]]$. Let $\phi\colon C\to C[[\mathfrak{M}]]$ be given by $\phi(\lambda)=\sum_i\lambda^i f_i$, and suppose ϕ is C-linear. Then $f_i=0$ for all $i\neq 1$.*

Proof. Suppose $\mathfrak{m}\in\bigcup_i\operatorname{supp} f_i$; let $i_1<\cdots<i_n$ be the indices i such that $\mathfrak{m}\in\operatorname{supp} f_i$, and put $c_k:=(f_{i_k})_{\mathfrak{m}}\in C$ for $k=1,\dots,n$. With $\lambda\in C$ we have $\phi(\lambda)_{\mathfrak{m}}=\lambda\phi(1)_{\mathfrak{m}}$, that is,

$$\lambda^{i_1}c_1+\cdots+\lambda^{i_n}c_n=\lambda(c_1+\cdots+c_n).$$

Since C is infinite, this yields $n=1$ and $i_1=1$. $\square$

Corollary 1.2. *Suppose the field C is infinite, and the map $\Phi\colon C[[\mathfrak{M}]]\to C[[\mathfrak{N}]]$ is noetherian and C-linear. Then Φ is strongly linear.*

Proof. Let $(M_n)_{n\in\mathbb{N}}$ be a multilinear decomposition of Φ. Let $f \in C[[\mathfrak{M}]]$, and define $\phi\colon C \to C[[\mathfrak{N}]]$ by $\phi(\lambda) = \Phi(\lambda f)$. Then

$$\phi(\lambda) = \sum_i \lambda^i f_i \qquad \text{with } f_i := M_i(f,\dots,f),$$

and ϕ is C-linear. Hence $f_i = 0$ for all $i \neq 1$, by the previous lemma. It follows that $\Phi = M_1$. □

We equip the disjoint union $\mathfrak{M} \amalg \mathfrak{N}$ with the least ordering extending those of $\mathfrak{M}$ and $\mathfrak{N}$. The natural inclusions $i\colon \mathfrak{M} \to \mathfrak{M} \amalg \mathfrak{N}$ and $j\colon \mathfrak{N} \to \mathfrak{M} \amalg \mathfrak{N}$ extend uniquely to strongly linear maps $\widehat{i}\colon C[[\mathfrak{M}]] \to C[[\mathfrak{M} \amalg \mathfrak{N}]]$, and $\widehat{j}\colon C[[\mathfrak{N}]] \to C[[\mathfrak{M} \amalg \mathfrak{N}]]$. This yields a C-linear bijection

$$(f,g) \mapsto \widehat{i}(f) + \widehat{j}(g)\colon C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M} \amalg \mathfrak{N}]].$$

When convenient, we identify $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]]$ with $C[[\mathfrak{M} \amalg \mathfrak{N}]]$ by means of this bijection. For example, we say that a map $\Phi\colon C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ is strongly linear (respectively, noetherian) if Φ, considered as a map $C[[\mathfrak{M} \amalg \mathfrak{N}]] \to C[[\mathfrak{M}]]$, is strongly linear (respectively, noetherian). The following is the strongly linear case of Theorems 6.1 and 6.3 in [17] (Van der Hoeven's implicit function theorem):

Theorem 1.3. *Let the map $(f,g) \mapsto \Phi(f,g)\colon C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ be strongly linear such that* $\operatorname{supp} \Phi(\mathfrak{m},0) \prec \mathfrak{m}$ *for all $\mathfrak{m} \in \mathfrak{M}$. Then for each $g \in C[[\mathfrak{N}]]$ there is a unique $f = \Psi(g) \in C[[\mathfrak{M}]]$ such that $\Phi(f,g) = f$. For each $g \in C[[\mathfrak{N}]]$ the family $\big(\Psi_{n+1}(g) - \Psi_n(g)\big)_{n\in\mathbb{N}}$ in $C[[\mathfrak{M}]]$ with*

$$\Psi_0(g) = \Phi(0,g), \qquad \Psi_{n+1}(g) = \Phi\big(\Psi_n(g),g\big) \text{ for all } n$$

is noetherian with

$$\Psi(g) = \Psi_0(g) + \sum_{n\in\mathbb{N}} \big(\Psi_{n+1}(g) - \Psi_n(g)\big).$$

The map $g \mapsto \Psi(g)\colon C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$ is noetherian.

The following consequence for inverting strongly linear maps is important later:

Corollary 1.4. *Suppose that C is infinite. Let $\Phi\colon C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$ be a strongly linear map such that* $\operatorname{supp} \Phi(\mathfrak{m}) \prec \mathfrak{m}$ *for all $\mathfrak{m} \in \mathfrak{M}$. Then the strongly linear operator* $\operatorname{Id} + \Phi$ *on $C[[\mathfrak{M}]]$ is bijective with strongly linear inverse given by*

$$(\operatorname{Id} + \Phi)^{-1}(g) = \sum_{n=0}^{\infty} (-1)^n \Phi^n(g). \tag{1.1}$$

Proof. Let $\Phi_1\colon C[[\mathfrak{M}]] \times C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$ be given by $\Phi_1(f,g) = g - \Phi(f)$. Then Φ_1 is strongly linear and $\operatorname{supp} \Phi_1(\mathfrak{m},0) = \operatorname{supp} \Phi(\mathfrak{m}) \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}$. By the theorem above with Φ_1 in place of Φ we obtain a a noetherian $\Psi\colon C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$ such that $(\operatorname{Id} + \Phi) \circ \Psi = \operatorname{Id}$. By Corollary 1.2, Ψ is strongly linear.

The assumption on Φ yields that $\operatorname{Id} + \Phi$ has trivial kernel, so $\operatorname{Id} + \Phi$ is injective, and thus Ψ is even a two-sided inverse of $\operatorname{Id} + \Phi$. Moreover, in the notation of Theorem 1.3 we have

$$\Psi_0(g) = g,\ \Psi_1(g) = g - \Phi(g),\ \Psi_2(g) = g - \Phi(g) + \Phi^2(g),\ \dots$$

for every g, which yields (1.1). □

Transseries fields. *In the rest of this section, $(\mathfrak{M}, \preccurlyeq)$ is a multiplicative ordered abelian group.* (In particular the ordering $\preccurlyeq$ is total.) Then $C[[\mathfrak{M}]]$ is a field, called the *transseries field* with coefficients in C and monomials from $\mathfrak{M}$. If $\mathfrak{S}, \mathfrak{S}' \subseteq \mathfrak{M}$ are noetherian, so is $\mathfrak{S}\mathfrak{S}'$. For $\mathfrak{S} \subseteq \mathfrak{M}$, let $\mathfrak{S}^*$ be the multiplicative submonoid of $\mathfrak{M}$ generated by $\mathfrak{S}$; if $\mathfrak{S} \subseteq \mathfrak{M}$ is noetherian and $\mathfrak{S} \preccurlyeq 1$, then $\mathfrak{S}^*$ is noetherian.

For non-zero $f \in C[[\mathfrak{M}]]$ we put

$$\mathfrak{d}(f) := \max_{\preccurlyeq} \operatorname{supp} f \quad (\textit{dominant monomial} \text{ of } f)$$

and we call $f_{\mathfrak{d}(f)}\mathfrak{d}(f) \in C^\times \cdot \mathfrak{M}$ the *dominant term* of f. We extend the ordering $\preccurlyeq$ on $\mathfrak{M}$ to a dominance relation on $C[[\mathfrak{M}]]$: for series f and g in $C[[\mathfrak{M}]]$, we put

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad \big(f \neq 0, g \neq 0, \mathfrak{d}(f) \preccurlyeq \mathfrak{d}(g)\big), \text{ or } f = 0 \\ f \asymp g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \wedge g \preccurlyeq f, \end{aligned}$$

so for non-zero f and g: $f \asymp g \Longleftrightarrow \mathfrak{d}(f) = \mathfrak{d}(g)$. We have the *canonical decomposition* of $C[[\mathfrak{M}]]$ into C-linear subspaces:

$$C[[\mathfrak{M}]] = C[[\mathfrak{M}]]^{\uparrow} \oplus C \oplus C[[\mathfrak{M}]]^{\downarrow},$$

where

$$C[[\mathfrak{M}]]^{\uparrow} := \{f \in C[[\mathfrak{M}]] : \operatorname{supp} f \succ 1\} = C[[\mathfrak{M}^{\succ 1}]]$$

and

$$C[[\mathfrak{M}]]^{\downarrow} := \{f \in C[[\mathfrak{M}]] : \operatorname{supp} f \prec 1\} = C[[\mathfrak{M}]]^{\prec 1} = C[[\mathfrak{M}^{\prec 1}]],$$

the maximal ideal of the valuation ring $C[[\mathfrak{M}]]^{\preccurlyeq 1} = C \oplus C[[\mathfrak{M}]]^{\downarrow}$ of $C[[\mathfrak{M}]]$. Every $f \in C[[\mathfrak{M}]]$ can be uniquely written as

$$f = f^{\uparrow} + f^{=} + f^{\downarrow},$$

where $f^{\uparrow} \in C[[\mathfrak{M}]]^{\uparrow}$, $f^{=} \in C$, and $f^{\downarrow} \in C[[\mathfrak{M}]]^{\downarrow}$. If C is an ordered field, then we turn $C[[\mathfrak{M}]]$ into an ordered field as follows:

$$f > 0 \quad \Longleftrightarrow \quad f_{\mathfrak{d}(f)} > 0, \qquad \text{for } f \in C[[\mathfrak{M}]],\ f \neq 0. \tag{1.2}$$

In this case,

$$C[[\mathfrak{M}]]^{\uparrow} = \{f \in C[[\mathfrak{M}]] : |f| > C\}$$

and

$$C[[\mathfrak{M}]]^{\downarrow} = \{f \in C[[\mathfrak{M}]] : |f| < C^{>0}\},$$

and the valuation ring $C[[\mathfrak{M}]]^{\preccurlyeq 1}$ of $C[[\mathfrak{M}]]$ is a convex subring of $C[[\mathfrak{M}]]$. Given an ordered field C we shall refer to $C[[\mathfrak{M}]]$ as an *ordered transseries field over C* to indicate that $C[[\mathfrak{M}]]$ is equipped with the ordering defined by (1.2).

Example 1.5. Let $C = \mathbb{R}$ and $\mathfrak{M} = x^{\mathbb{R}}$, a multiplicative copy of the ordered additive group of real numbers, with isomorphism $r \mapsto x^r : \mathbb{R} \to x^{\mathbb{R}}$. Then we have

$$f^{\uparrow} = \sum_{r>0} a_r x^r, \quad f^{=} = a_0, \quad f^{\downarrow} = \sum_{r<0} a_r x^r$$

for $f = \sum_r a_r x^r \in \mathbb{R}[[x^{\mathbb{R}}]]$.

Let $X = (X_1, \dots, X_n)$ be a tuple of distinct indeterminates and

$$F(X) = \sum_{\nu} a_\nu X^\nu \in C[[X]]$$

a formal power series; here the sum ranges over all multiindices $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$, and $a_\nu \in C$, $X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n}$. For any n-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ of elements of $C[[\mathfrak{M}]]^{\downarrow}$, the family $(a_\nu \varepsilon^\nu)_\nu$ is noetherian [8], where $\varepsilon^\nu = \varepsilon_1^{\nu_1} \cdots \varepsilon_n^{\nu_n}$. Put

$$F(\varepsilon) := \sum_\nu a_\nu \varepsilon^\nu \in C[[\mathfrak{M}]]^{\preccurlyeq 1}.$$

The proof of the following lemma is similar to that of [12], Lemma 2.5.

Lemma 1.6. *Suppose that C is real closed and the group $\mathfrak{M}$ is divisible. Then any subfield $K \supseteq C[\mathfrak{M}]$ of $C[[\mathfrak{M}]]$ with the property that $F(\varepsilon) \in K$ for all $F \in C[[X]]$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_1, \dots, \varepsilon_n \in K^{\prec 1}$ is real closed.*

Differentiation. If $C[[\mathfrak{M}]]$ is an H-field with respect to a derivation $f \mapsto f'$ with constant field C and with respect to the ordering extending an ordering on C via (1.2), then the dominance relation $\preccurlyeq$ that $C[[\mathfrak{M}]]$ carries as a transseries field over C coincides with the dominance relation that it has as an H-field, and

$$\mathfrak{m} \preccurlyeq \mathfrak{n} \iff \mathfrak{m}' \preccurlyeq \mathfrak{n}', \qquad \text{for } \mathfrak{m}, \mathfrak{n} \in \mathfrak{M} \setminus \{1\}. \tag{1.3}$$

In the rest of this section we assume, more generally, that $C[[\mathfrak{M}]]$ is equipped with a derivation $f \mapsto f'$ with constant field C such that (1.3) *holds.*

Integration. A series $f \in C[[\mathfrak{M}]]$ is called the *distinguished integral* of $g \in C[[\mathfrak{M}]]$, written as $f = \int g$, if $f' = g$ and $f^{=} = 0$.

For every $\mathfrak{m} \in \mathfrak{M}$ there is at most one $\mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{n}' \asymp \mathfrak{m}$; we say that $C[[\mathfrak{M}]]$ is *closed under asymptotic integration* if for every $\mathfrak{m} \in \mathfrak{M}$ there exists such an $\mathfrak{n}$.

If the derivation on $C[[\mathfrak{M}]]$ is strongly linear and $C[[\mathfrak{M}]]$ is closed under integration, then it is closed under asymptotic integration: for $\mathfrak{m} \in \mathfrak{M}$ we have $\mathfrak{m} \asymp \mathfrak{n}'$ where $\mathfrak{n} := \mathfrak{d}(\int \mathfrak{m})$. The following converse is very useful:

Lemma 1.7. *Suppose that C is infinite, the derivation on $C[[\mathfrak{M}]]$ is strongly linear, and $C[[\mathfrak{M}]]$ is closed under asymptotic integration. Then each $g \in C[[\mathfrak{M}]]$ has a distinguished integral in $C[[\mathfrak{M}]]$, and the operator $g \mapsto \int g$ on $C[[\mathfrak{M}]]$ is strongly linear.*

Proof. Define $\mathrm{I}\colon \mathfrak{M} \to C[[\mathfrak{M}]]$ by $\mathrm{I}(\mathfrak{m}) = c\mathfrak{n}$ with $c \in C$, $\mathfrak{n} \in \mathfrak{M}$ such that $c\mathfrak{n}' - \mathfrak{m} \prec \mathfrak{m}$. Then by (1.3) the map I is noetherian, hence extends to a strongly linear operator on $C[[\mathfrak{M}]]$, which we also denote by I. Let D be the derivation on $C[[\mathfrak{M}]]$. The strongly linear operator $\Phi = D \circ \mathrm{I} - \mathrm{Id}$ satisfies $\operatorname{supp} \Phi(\mathfrak{m}) \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}$. Hence by Corollary 1.4 the strongly linear operator $D \circ \mathrm{I} = \mathrm{Id} + \Phi$ has a strongly linear two-sided inverse Ψ given by

$$\Psi(g) = (D \circ \mathrm{I})^{-1}(g) = g - \Phi(g) + \Phi^2(g) - \Phi^3(g) + \cdots.$$

Since $\mathrm{I}(\mathfrak{m})^{=} = 0$ for all $\mathfrak{m} \in \mathfrak{M}$, the strongly linear operator $\int := \mathrm{I} \circ \Psi$ assigns to each $g \in C[[\mathfrak{M}]]$ its distinguished integral. ☐

Exponentials and logarithms. Suppose now that $C = \mathbb{R}$. For $f \in \mathbb{R}[[\mathfrak{M}]]^{\preccurlyeq 1}$, write $f = c + \varepsilon$ with $c \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}[[\mathfrak{M}]]^{\downarrow}$, and put

$$\exp(f) = \exp(c + \varepsilon) := e^c \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!},$$

where $t \mapsto e^t$ is the usual exponential function on $\mathbb{R}$. Then exp is an *exponential* on $\mathbb{R}[[\mathfrak{M}]]^{\preccurlyeq 1}$: for $f, g \in \mathbb{R}[[\mathfrak{M}]]^{\preccurlyeq 1}$

$$\exp(f) \geqslant 1 \Leftrightarrow f \geqslant 0, \quad \exp(f) \geqslant f+1, \quad \text{and} \quad \exp(f+g) = \exp(f)\exp(g).$$

Thus exp is injective with image

$$\{g \in \mathbb{R}[[\mathfrak{M}]] : g > 0, \mathfrak{d}(g) = 1\}$$

and inverse

$$\log\colon \{g \in \mathbb{R}[[\mathfrak{M}]] : g > 0, \mathfrak{d}(g) = 1\} \to \mathbb{R}[[\mathfrak{M}]]^{\preccurlyeq 1}$$

given by

$$\log g := \log a + \log(1+\varepsilon)$$

for $g = a(1+\varepsilon)$, $a \in \mathbb{R}^{>0}$, $\varepsilon \prec 1$, where $\log a$ is the usual natural logarithm of the positive real number a and

$$\log(1+\varepsilon) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \varepsilon^n.$$

If $\mathbb{R}[[\mathfrak{M}]]$ is closed under integration, then the above logarithm extends to a function $\log\colon \mathbb{R}[[\mathfrak{M}]]^{>0} \to \mathbb{R}[[\mathfrak{M}]]$ by

$$\log g := \log a + \log \mathfrak{m} + \log(1+\varepsilon)$$

for $g = a\mathfrak{m}(1+\varepsilon)$ with $a \in \mathbb{R}^{>0}$, $\mathfrak{m} \in \mathfrak{M}$, and $\varepsilon \prec 1$, and $\log \mathfrak{m} := \int \mathfrak{m}^\dagger$. Note that $\log(fg) = \log f + \log g$ for $f, g \in \mathbb{R}[[\mathfrak{M}]]^{>0}$.

More notation. For non-zero $f, g \in C[[\mathfrak{M}]]$ we put

$$\begin{aligned} f \preceq\!\!\!\prec g \quad &:\Longleftrightarrow \quad f^\dagger \preccurlyeq g^\dagger, \\ f \prec\!\!\!\prec g \quad &:\Longleftrightarrow \quad f^\dagger \prec g^\dagger, \\ f \asymp\!\!\!\asymp g \quad &:\Longleftrightarrow \quad f^\dagger \asymp g^\dagger. \end{aligned}$$

Suppose $\mathbb{R}[[\mathfrak{M}]]$, with its ordering as an ordered transseries field over $C = \mathbb{R}$, is an H-field. Then by [2], Proposition 7.3, we have for $f, g \in \mathbb{R}[[\mathfrak{M}]]^{\succ 1}$:

$$\begin{aligned} f \preceq\!\!\!\prec g \qquad &\Longleftrightarrow \qquad |f| \leqslant |g|^n \text{ for some } n > 0, \\ f \prec\!\!\!\prec g \qquad &\Longleftrightarrow \qquad |f|^n < |g| \text{ for all } n > 0. \end{aligned}$$

2. Logarithmic Monomials

Let $\mathfrak{L}$ be the multiplicative subgroup of *logarithmic monomials* of $\mathbb{R}[[[x]]]^{>0}$ generated by the real powers of the iterated logarithms $\ell_0 := x, \ell_1 := \log x, \ell_2 := \log\log x, \dots, \ell_n := \log_n x, \dots$ of x; that is,

$$\mathfrak{L} = \{\ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n} : (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^n, n = 0, 1, 2, \dots\}.$$

Thus $\mathfrak{L}$ is a multiplicatively written ordered vector space over the ordered field $\mathbb{R}$, with basis $\ell_0, \ell_1, \ell_2, \dots$ satisfying

$$\ell_0 \succ\!\!\!\succ \ell_1 \succ\!\!\!\succ \ell_2 \succ\!\!\!\succ \cdots \succ\!\!\!\succ \ell_n \succ\!\!\!\succ \cdots.$$

We define the group of *continued logarithmic monomials* $\overline{\mathfrak{L}}$ by

$$\overline{\mathfrak{L}} := \{\ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n} \cdots : (\alpha_0, \alpha_1, \dots, \alpha_n, \dots) \in \mathbb{R}^{\mathbb{N}}\}$$

and by requiring that $(\alpha_0, \alpha_1, \ldots) \mapsto \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots : \mathbb{R}^{\mathbb{N}} \to \overline{\mathfrak{L}}$ is an isomorphism of the additive group $\mathbb{R}^{\mathbb{N}}$ onto the multiplicative group $\overline{\mathfrak{L}}$. We order $\overline{\mathfrak{L}}$ lexicographically: given $\mathfrak{m} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots$ and $\mathfrak{n} = \ell_0^{\beta_0} \ell_1^{\beta_1} \cdots$ with $(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots) \in \mathbb{R}^{\mathbb{N}}$, put

$$\mathfrak{m} \preccurlyeq \mathfrak{n} \quad :\Longleftrightarrow \quad (\alpha_0, \alpha_1, \ldots) \leqslant (\beta_0, \beta_1, \ldots) \text{ lexicographically.}$$

This ordering makes $\overline{\mathfrak{L}}$ into an ordered group, and extends the ordering $\preccurlyeq$ on $\mathfrak{L}$. We also extend the relation $\prec\!\!\!\prec$ ("flatter than") from $\mathfrak{L}$ to $\overline{\mathfrak{L}}$ in the natural way:

$$\mathfrak{m} \prec\!\!\!\prec \mathfrak{n} \quad :\Longleftrightarrow \quad l(\mathfrak{m}) > l(\mathfrak{n}),$$

where $l(\mathfrak{m}) := \min\{i : \alpha_i \neq 0\} \in \mathbb{N}$ if $\mathfrak{m} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \neq 1$, and $l(1) := \infty > \mathbb{N}$.

Definition 2.1. A sequence $(\mathfrak{m}_i)_{i \geqslant 1}$ in $\overline{\mathfrak{L}}$ is called a *monomial Cauchy sequence* if for each $k \in \mathbb{N}$ there is an index i_0 such that for all $i_2 > i_1 > i_0$ we have $\mathfrak{m}_{i_2}/\mathfrak{m}_{i_1} \prec\!\!\!\prec \ell_k$. A continued logarithmic monomial $\mathfrak{l} \in \overline{\mathfrak{L}}$ is a *monomial limit* of $(\mathfrak{m}_i)_{i \geqslant 1}$ if for all $k \in \mathbb{N}$ there is an i_0 such that for all $i > i_0$ we have $\mathfrak{m}_i/\mathfrak{l} \prec\!\!\!\prec \ell_k$.

Given a continued logarithmic monomial $\mathfrak{m} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots$, let us write

$$e(\mathfrak{m}) := (\alpha_0, \alpha_1, \ldots) \in \mathbb{R}^{\mathbb{N}}$$

for its sequence of exponents. Then $e \colon \overline{\mathfrak{L}} \to \mathbb{R}^{\mathbb{N}}$ is an order-preserving isomorphism between the multiplicative ordered abelian group $\overline{\mathfrak{L}}$ and the additive group $\mathbb{R}^{\mathbb{N}}$, ordered lexicographically. With this notation, a sequence $(\mathfrak{m}_i)$ in $\overline{\mathfrak{L}}$ is a monomial Cauchy sequence if and only if $(e(\mathfrak{m}_i))$ is a *Cauchy sequence* in $\mathbb{R}^{\mathbb{N}}$, that is: for every $\varepsilon > 0$ in $\mathbb{R}^{\mathbb{N}}$ there exists an index i_0 such that $|e(\mathfrak{m}_{i_2}) - e(\mathfrak{m}_{i_1})| < \varepsilon$ for all $i_2 > i_1 > i_0$. Similarly, an element $\mathfrak{l} \in \overline{\mathfrak{L}}$ is a monomial limit of $(\mathfrak{m}_i)$ if and only if $e(\mathfrak{l})$ is a limit of the sequence $(e(\mathfrak{m}_i))$, in the usual sense: for every $\varepsilon > 0$ there exists i_0 such that $|e(\mathfrak{m}_i) - e(\mathfrak{l})| < \varepsilon$ for all $i > i_0$. If $(\mathfrak{m}_i)$ has a monomial limit in $\overline{\mathfrak{L}}$, then $(\mathfrak{m}_i)$ is a monomial Cauchy sequence. Conversely, every monomial Cauchy sequence $(\mathfrak{m}_i)$ in $\overline{\mathfrak{L}}$ has a unique monomial limit $\mathfrak{l}$ in $\overline{\mathfrak{L}}$, denoted by $\mathfrak{l} = \lim_{i \to \infty} \mathfrak{m}_i$. Moreover, every continued logarithmic monomial $\mathfrak{m} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n} \cdots \in \overline{\mathfrak{L}}$ is the monomial limit of some monomial Cauchy sequence in $\mathfrak{L}$:

$$\mathfrak{m} = \lim_{i \to \infty} \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \ell_i^{\alpha_i}.$$

(Thus, viewing $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ as topological groups in their interval topology, $\overline{\mathfrak{L}}$ is the completion of its subgroup $\mathfrak{L}$.) Given a subset $\mathfrak{S}$ of $\mathfrak{L}$, let $\overline{\mathfrak{S}}$ denote the set of all monomial limits of monomial Cauchy sequences in $\mathfrak{S}$ (so $\overline{\mathfrak{S}}$ is the closure of $\mathfrak{S}$ in $\overline{\mathfrak{L}}$), and $\widehat{\mathfrak{S}}$ the set of all monomial limits of *strictly decreasing* monomial Cauchy sequences $\mathfrak{m}_1 \succ \mathfrak{m}_2 \succ \cdots$ in $\mathfrak{S}$. Note that if $\mathfrak{S} \subseteq \mathfrak{L}$ is noetherian, then so is $\overline{\mathfrak{S}} \subseteq \overline{\mathfrak{L}}$, and $\overline{\mathfrak{S}} = \mathfrak{S} \cup \widehat{\mathfrak{S}}$.

Proposition 2.2. *Let $\mathfrak{S}, \mathfrak{S}' \subseteq \mathfrak{L}$ be noetherian. Then*

(1) *If $\mathfrak{S} \subseteq \mathfrak{S}'$, then $\widehat{\mathfrak{S}} \subseteq \widehat{\mathfrak{S}'}$ and $\overline{\mathfrak{S}} \subseteq \overline{\mathfrak{S}'}$.*
(2) *$\widehat{\mathfrak{S} \cup \mathfrak{S}'} = \widehat{\mathfrak{S}} \cup \widehat{\mathfrak{S}'}$ and $\overline{\mathfrak{S} \cup \mathfrak{S}'} = \overline{\mathfrak{S}} \cup \overline{\mathfrak{S}'}$.*
(3) *$\widehat{\mathfrak{S}\mathfrak{S}'} = \overline{\mathfrak{S}}\widehat{\mathfrak{S}'} \cup \widehat{\mathfrak{S}}\overline{\mathfrak{S}'}$ and $\overline{\mathfrak{S}\mathfrak{S}'} = \overline{\mathfrak{S}}\ \overline{\mathfrak{S}'}$.*
(4) *If $\mathfrak{S} \prec 1$, then $\widehat{\mathfrak{S}^*} \subseteq \mathfrak{S}^* \big(\widehat{\mathfrak{S}}\big)^*$ and $\overline{\mathfrak{S}^*} \subseteq \overline{\mathfrak{S}}^*$.*

Proof. Parts (1) and (2) are trivial.

For (3) consider a monomial limit $\mathfrak{l}$ of a sequence $\mathfrak{m}_1\mathfrak{n}_1 \succ \mathfrak{m}_2\mathfrak{n}_2 \succ \cdots$, where

$$(\mathfrak{m}_1, \mathfrak{n}_1), (\mathfrak{m}_2, \mathfrak{n}_2), \ldots$$

is a sequence in $\mathfrak{S} \times \mathfrak{S}'$. Since $\mathfrak{S}$ and $\mathfrak{S}'$ are noetherian, we may assume, after choosing a subsequence of $(\mathfrak{m}_1, \mathfrak{n}_1), (\mathfrak{m}_2, \mathfrak{n}_2), \ldots$, that $\mathfrak{m}_1 \succcurlyeq \mathfrak{m}_2 \succcurlyeq \cdots$ and $\mathfrak{n}_1 \succcurlyeq \mathfrak{n}_2 \succcurlyeq \cdots$. Because $(\mathfrak{m}_i \mathfrak{n}_i)$ is a monomial Cauchy sequence, both sequences $(\mathfrak{m}_i)$ and $(\mathfrak{n}_i)$ are monomial Cauchy sequences as well. The sequences $(\mathfrak{m}_i)$ and $(\mathfrak{n}_i)$ cannot both be ultimately constant. If one of these sequences is ultimately constant, say $\mathfrak{m}_i = \mathfrak{m}$ for all $i \geqslant i_0$, then

$$\mathfrak{l} = \lim_{i \to \infty} \mathfrak{m}_i \mathfrak{n}_i = \mathfrak{m} \lim_{i \to \infty} \mathfrak{n}_i \in \mathfrak{S}\widehat{\mathfrak{S}'}.$$

Otherwise, we have

$$\mathfrak{l} = \lim_{i \to \infty} \mathfrak{m}_i \mathfrak{n}_i = \lim_{i \to \infty} \mathfrak{m}_i \lim_{i \to \infty} \mathfrak{n}_i \in \widehat{\mathfrak{S}}\widehat{\mathfrak{S}'}.$$

Hence $\widehat{\mathfrak{S}\mathfrak{S}'} \subseteq \overline{\mathfrak{S}}\widehat{\mathfrak{S}'} \cup \widehat{\mathfrak{S}}\overline{\mathfrak{S}'}$. The other inclusions of (3) now follow easily.

As to (4), assume that $\mathfrak{S} \prec 1$ and let $\mathfrak{l}$ be a monomial limit of a sequence

$$\mathfrak{m}_1 = \mathfrak{m}_{1,1} \cdots \mathfrak{m}_{1,l_1} \succ \mathfrak{m}_2 = \mathfrak{m}_{2,1} \cdots \mathfrak{m}_{2,l_2} \succ \cdots,$$

where $(\mathfrak{m}_{1,1}, \ldots, \mathfrak{m}_{1,l_1}), (\mathfrak{m}_{2,1}, \ldots, \mathfrak{m}_{2,l_2}), \ldots$ is a sequence of tuples over $\mathfrak{S}$. Since the set of these tuples is noetherian for Higman's embeddability ordering [5], we may assume, after choosing a subsequence, that in this ordering

$$(\mathfrak{m}_{1,1}, \ldots, \mathfrak{m}_{1,l_1}) \succcurlyeq (\mathfrak{m}_{2,1}, \ldots, \mathfrak{m}_{2,l_2}) \succcurlyeq \cdots.$$

In particular, we have $l_1 \leqslant l_2 \leqslant \cdots$. We claim that the sequence (l_i) is ultimately constant. Assume the contrary. Then, after choosing a second subsequence, we may assume that $l_1 < l_2 < \cdots$. Let $1 \leqslant k_{i+1} \leqslant l_{i+1}$ be such that

$$(\mathfrak{m}_{i,1}, \ldots, \mathfrak{m}_{i,l_i}) \succcurlyeq (\mathfrak{m}_{i+1,1}, \ldots, \mathfrak{m}_{i+1,k_{i+1}-1}, \mathfrak{m}_{i+1,k_{i+1}+1}, \ldots, \mathfrak{m}_{i+1,l_{i+1}})$$

for all i, hence $\mathfrak{m}_i \succcurlyeq \mathfrak{m}_{i+1}/\mathfrak{m}_{i+1,k_{i+1}}$ for all i. Since $\mathfrak{S}$ is noetherian, the set $\{\mathfrak{m}_{2,k_2}, \mathfrak{m}_{3,k_3}, \ldots\}$ has a largest element $\mathfrak{v} \prec 1$. But then

$$\mathfrak{m}_{i+1}/\mathfrak{m}_i \preccurlyeq \mathfrak{m}_{i+1,k_{i+1}} \preccurlyeq \mathfrak{v}$$

for all i, which contradicts $(\mathfrak{m}_i)$ being a monomial Cauchy sequence. This proves our claim (l_i) is ultimately constant.

We now proceed as in (3) to finish the proof of (4). $\square$

Given $\mathfrak{S} \subseteq \mathfrak{L}$ we say that $\mathfrak{S}$ *has decay* > 1 if for each $\mathfrak{m} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \in \widehat{\mathfrak{S}}$ there exists $k_0 \in \mathbb{N}$ such that $\alpha_k < -1$ for all $k \geqslant k_0$. Each finite subset of $\mathfrak{L}$ has decay > 1.

Example 2.3. Fix $n \geqslant 1$, and define a sequence $(\mathfrak{m}_i)_{i \geqslant 0}$ in $\mathfrak{L}$ by

$$\mathfrak{m}_0 = \left(\frac{1}{\ell_0}\right)^n, \quad \mathfrak{m}_1 = \left(\frac{1}{\ell_0 \ell_1}\right)^n, \quad \ldots, \quad \mathfrak{m}_i := \left(\frac{1}{\ell_0 \ell_1 \cdots \ell_i}\right)^n \qquad (i \geqslant 0).$$

Then the continued logarithmic monomial

$$\mathfrak{l} = \left(\frac{1}{\ell_0 \ell_1 \cdots \ell_i \cdots}\right)^n \in \overline{\mathfrak{L}}$$

is the monomial limit of the sequence $\mathfrak{m}_0 \succ \mathfrak{m}_1 \succ \cdots$ in $\mathfrak{L}$. Hence the subset $\{\mathfrak{m}_i : i = 0, 1, 2, \ldots\}$ of $\mathfrak{L}$ has decay > 1 if $n > 1$, but not if $n = 1$.

Corollary 2.4. *If $\mathfrak{S}$ and $\mathfrak{S}'$ are noetherian subsets of $\mathfrak{L}$ of decay > 1, then $\mathfrak{S} \cup \mathfrak{S}'$ and $\mathfrak{S}\mathfrak{S}'$ are noetherian of decay > 1; if in addition $\mathfrak{S} \prec 1$, then $\mathfrak{S}^*$ is noetherian of decay > 1.* $\square$

3. Logarithmic Transseries of decay > 1

Consider the ordered field $\mathbb{L} := \mathbb{R}[[\mathfrak{L}]]$ of *logarithmic transseries*, and equip $\mathbb{L}$ with the strongly linear derivation $f \mapsto f'$ such that for each $\alpha \in \mathbb{R}$

$$(\ell_0^\alpha)' = \alpha \ell_0^{\alpha-1},\ (\ell_k^\alpha)' = \alpha \ell_k^{\alpha-1}(\ell_0\ell_1\cdots\ell_{k-1})^{-1} \quad \text{for } k > 0.$$

This makes $\mathbb{L}$ a real closed H-field with constant field $\mathbb{R}$, and $\mathbb{L}$ is closed under integration (see *example* at end of Section 11 in [2]). Hence by Lemma 1.7 the distinguished integration operator $\int$ on $\mathbb{L}$ is strongly linear.

A logarithmic transseries $f \in \mathbb{L}$ is said to have *decay* > 1 if its support $\operatorname{supp} f$ has decay > 1. By Corollary 2.4 above,

$$\mathbb{L}_1 := \{f \in \mathbb{L} : f \text{ has decay} > 1\}$$

is a subfield of $\mathbb{L}$ containing the subfield $\mathbb{R}(\mathfrak{L})$ of $\mathbb{L}$ generated by $\mathfrak{L}$ over $\mathbb{R}$. In addition $F(\varepsilon) \in \mathbb{L}_1$ for any formal power series $F(X) \in \mathbb{R}[[X]]$ and any n-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ of infinitesimals in $\mathbb{L}_1$, where $X = (X_1, \dots, X_n)$, $n \geqslant 1$. Hence by Lemma 1.6 the field $\mathbb{L}_1$ is real closed. Defining the logarithmic function on $\mathbb{L}^{>0}$ as in the subsection "Exponentials and logarithms" of Section 2, we obtain

$$\log(\ell_0^{\alpha_0}\ell_1^{\alpha_1}\cdots\ell_k^{\alpha_k}) = \alpha_0\ell_1 + \cdots + \alpha_k\ell_{k+1} \in \mathbb{L}_1$$

for $\alpha_0, \dots, \alpha_k \in \mathbb{R}$. It follows that $\log f \in \mathbb{L}_1$ for every positive $f \in \mathbb{L}_1$. Moreover:

Proposition 3.1. *The field $\mathbb{L}_1$ is closed under differentiation. (Thus $\mathbb{L}_1$ is an H-subfield of $\mathbb{L}$.)*

Proof. Let $\mathfrak{l} \in \overline{\mathfrak{L}}$ be a monomial limit of a strictly decreasing sequence in $\operatorname{supp} f'$, where $f \in \mathbb{L}_1$; hence $\mathfrak{l}$ is the monomial limit of a sequence

$$\mathfrak{m}_1\mathfrak{n}_1 \succ \mathfrak{m}_2\mathfrak{n}_2 \succ \cdots$$

where $\mathfrak{m}_i \in \operatorname{supp} f$ and $\mathfrak{n}_i \in \operatorname{supp} \mathfrak{m}_i^\dagger$ for all i. Note that $\mathfrak{n}_i \in \mathfrak{D}$, where

$$\mathfrak{D} = \left\{\frac{1}{\ell_0}, \frac{1}{\ell_0\ell_1}, \frac{1}{\ell_0\ell_1\ell_2}, \dots\right\}. \tag{3.1}$$

Since $\operatorname{supp} f$ and $\mathfrak{D}$ are noetherian, we may assume that

$$\mathfrak{m}_1 \succcurlyeq \mathfrak{m}_2 \succcurlyeq \cdots, \text{ and } \mathfrak{n}_1 \succcurlyeq \mathfrak{n}_2 \succcurlyeq \cdots$$

after choosing a subsequence. Therefore $(\mathfrak{m}_i)$ and $(\mathfrak{n}_i)$ are monomial Cauchy sequences. We claim that $(\mathfrak{m}_i)$ cannot be ultimately constant: if

$$\mathfrak{m}_i = \ell_0^{\alpha_0}\ell_1^{\alpha_1}\cdots\ell_k^{\alpha_k}$$

for all $i \geqslant i_0$, then

$$\mathfrak{n}_i \in \operatorname{supp} \mathfrak{m}_i^\dagger \subseteq \left\{\frac{1}{\ell_0}, \frac{1}{\ell_0\ell_1}, \dots, \frac{1}{\ell_0\ell_1\cdots\ell_k}\right\}$$

for all $i \geqslant i_0$, so $(\mathfrak{n}_i)$ and thus $(\mathfrak{m}_i\mathfrak{n}_i)$ would be ultimately constant. This contradiction proves our claim. If $(\mathfrak{n}_i)$ is ultimately constant, say $\mathfrak{n}_i = \mathfrak{n}$ for all $i \geqslant i_0$, then

$$\mathfrak{l} = \lim_{i\to\infty} \mathfrak{m}_i\mathfrak{n}_i = \left(\lim_{i\to\infty} \mathfrak{m}_i\right)\mathfrak{n}.$$

Otherwise

$$\lim_{i\to\infty} \mathfrak{n}_i = \frac{1}{\ell_0\ell_1\ell_2\cdots} \in \overline{\mathfrak{L}},$$

hence

$$\mathfrak{l} = \lim_{i\to\infty} \mathfrak{m}_i \mathfrak{n}_i = \left(\lim_{i\to\infty} \mathfrak{m}_i\right) \frac{1}{\ell_0 \ell_1 \ell_2 \cdots},$$

which proves our proposition. □

Example 3.2. We have $\mathbb{R}\langle\varrho\rangle = \mathbb{R}(\varrho, \varrho', \dots) \subseteq \mathbb{L}_1$ as differential fields. Clearly $\lambda \in \mathbb{L}$, but $\mathbb{L}_1$ does not contain any element of the form $\lambda + \varepsilon$, where $\varepsilon \in \mathbb{L}$ satisfies $\varepsilon \prec 1/(\ell_0 \ell_1 \cdots \ell_n)$ for all n. (See Example 2.3.) Note also that $\Lambda \notin \mathbb{L}_1$.

Next we want to show that the differential field $\mathbb{L}_1$ is closed under integration. For this we need the following two lemmas:

Lemma 3.3. *For any non-zero $\alpha \in \mathbb{R}$ and any $f \in \mathbb{L}$, the linear differential equation*

$$y' + \alpha y = f \tag{3.2}$$

has a unique solution $y = g \in \mathbb{L}$, and if $f \in \mathbb{L}_1$, then $g \in \mathbb{L}_1$.

Proof. Note that for each i, $\operatorname{supp} f^{(i)}$ is contained in the set $(\operatorname{supp} f)\mathfrak{D}^i$, where $\mathfrak{D}$ is as in (3.1). Since $\mathfrak{D}^* = \bigcup_i \mathfrak{D}^i$ is noetherian and each of its elements lies in $\mathfrak{D}^i$ for only finitely many i, the family $(f^{(i)})$ is noetherian. Hence we have an explicit formula for a solution g to (3.2):

$$g := \sum_{i=0}^{\infty} (-1)^i \frac{f^{(i)}}{\alpha^{i+1}}.$$

The solution $g \in \mathbb{L}$ is unique, since the homogeneous equation $y' + \alpha y = 0$ only has the solution $y = 0$ in $\mathbb{L}$. Now suppose $f \in \mathbb{L}_1$, and let $\mathfrak{l} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \in \overline{\mathfrak{L}}$ be a monomial limit of a sequence

$$\mathfrak{m}_1 \mathfrak{n}_1 \succ \mathfrak{m}_2 \mathfrak{n}_2 \succ \cdots$$

in $\operatorname{supp}(g)$ where $\mathfrak{m}_i \mathfrak{n}_i \in \operatorname{supp}(f^{k(i)})$, with $\mathfrak{m}_i \in \operatorname{supp}(f)$ and $\mathfrak{n}_i \in \mathfrak{D}^{k(i)}$. We can assume that $\mathfrak{m}_1 \succcurlyeq \mathfrak{m}_2 \succcurlyeq \cdots$ and $\mathfrak{n}_1 \succcurlyeq \mathfrak{n}_2 \succcurlyeq \cdots$. Hence $(\mathfrak{m}_i)$ and $(\mathfrak{n}_i)$ are monomial Cauchy sequences with limit $\mathfrak{m} \in \overline{\mathfrak{L}}$ and $\mathfrak{n} \in \overline{\mathfrak{L}}$, respectively, so that $\mathfrak{l} = \mathfrak{m}\mathfrak{n}$. The exponent of ℓ_0 in $\mathfrak{n}_i$ is $-k(i)$, and thus the sequence $(k(i))$ is bounded. Hence we can even assume that this sequence is constant. Then $\alpha_k < -1$ for all sufficiently large k, by Proposition 3.1. Hence $g \in \mathbb{L}_1$ as required. □

For $k \in \mathbb{N}$ we consider the embedding of ordered abelian groups

$$\mathfrak{m} = \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n} \mapsto \mathfrak{m} \circ \ell_k := \ell_k^{\alpha_0} \ell_{k+1}^{\alpha_1} \cdots \ell_{k+n}^{\alpha_n} \;:\; \mathfrak{L} \to \mathfrak{L}$$

and denote its unique extension to a strongly linear $\mathbb{R}$-algebra endomorphism of $\mathbb{L}$ by $f \mapsto f \circ \ell_k$. Note that $(f \circ \ell_k)' = (f' \circ \ell_k)\ell_k'$ for $f \in \mathbb{L}$, and if $f \in \mathbb{L}_1$, then $f \circ \ell_k \in \mathbb{L}_1$.

In the statement of the next lemma we use the multiindex notation $\ell^\alpha := \ell_0^{\alpha_0} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n}$, for an $(n+1)$-tuple $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}$.

Lemma 3.4. *Let $n \in \mathbb{N}$ and suppose $(g_\alpha)_{\alpha \in \mathbb{R}^{n+1}}$ is a family in $\mathbb{L}_1$ such that the family $\big(\ell^\alpha \cdot (g_\alpha \circ \ell_{n+1})\big)_\alpha$ in $\mathbb{L}$ is noetherian. Then*

$$\sum_\alpha \ell^\alpha \cdot (g_\alpha \circ \ell_{n+1}) \in \mathbb{L}_1.$$

Proof. Let $\mathfrak{l} \in \overline{\mathfrak{L}}$ be a monomial limit of a sequence $\ell^{\alpha_1}\mathfrak{n}_1 \succ \ell^{\alpha_2}\mathfrak{n}_2 \succ \cdots$ where $\alpha_i \in \mathbb{R}^{n+1}$ and $\mathfrak{n}_i \in \operatorname{supp}(g_{\alpha_i} \circ \ell_{n+1})$ for all i. Then there exists an index i_0 such that $\alpha_{i_0} = \alpha_{i_0+1} = \cdots$, and hence $\mathfrak{n}_{i_0} \succ \mathfrak{n}_{i_0+1} \succ \cdots$ is a sequence in $\operatorname{supp}(g_{\alpha_{i_0}} \circ \ell_{n+1})$ with monomial limit $\mathfrak{l}/\ell^{\alpha_{i_0}}$. Since $g_{\alpha_{i_0}} \circ \ell_{n+1} \in \mathbb{L}_1$, the lemma follows. $\square$

Proposition 3.5. *The H-field $\mathbb{L}_1$ is closed under integration.*

Proof. Let $f \in \mathbb{L}_1$. Since $\frac{1}{\ell_0\ell_1\ell_2\cdots}$ is not a monomial limit of a sequence in $\operatorname{supp} f$, there exists $k \in \mathbb{N}$ such that

$$l(\mathfrak{m} \cdot \ell_0\ell_1\ell_2\cdots) \leqslant k \qquad \text{for all } \mathfrak{m} \in \operatorname{supp} f.$$

Take k minimal with this property. We proceed by induction on k. Write

$$f = \sum_{\alpha \in \mathbb{R}} x^{\alpha-1}(F_\alpha \circ \ell_1)$$

where $F_\alpha \in \mathbb{L}_1$ for each $\alpha \in \mathbb{R}$, and for $0 \neq \alpha \in \mathbb{R}$, let $g_\alpha \in \mathbb{L}_1$ be the unique solution to the linear differential equation $y' + \alpha y = F_\alpha$, by Lemma 3.3. Then

$$\int x^{\alpha-1}(F_\alpha \circ \ell_1) = x^\alpha(g_\alpha \circ \ell_1) \in \mathbb{L}_1,$$

for $\alpha \neq 0$. Since distinguished integration on $\mathbb{L}$ is strongly linear, we have

$$\int f = (g_0 \circ l_1) + \sum_{\alpha \neq 0} x^\alpha(g_\alpha \circ \ell_1) \in \mathbb{L},$$

where $g_0 := \int F_0$, and thus $\int f \in \mathbb{L}_1$ if $g_0 \in \mathbb{L}_1$ (by Lemma 3.4). If $k = 0$, then $F_0 = 0$, hence $g_0 = 0 \in \mathbb{L}_1$. If $k > 0$, then

$$l(\mathfrak{m} \cdot \ell_0\ell_1\ell_2\cdots) \leqslant k - 1 \qquad \text{for all } \mathfrak{m} \in \operatorname{supp} F_0,$$

hence g_0 in $\mathbb{L}_1$, by the induction hypothesis. We conclude that $\int f \in \mathbb{L}_1$. $\square$

4. Strong Differentiation, Strong Integration, and Flattening

For the convenience of the reader and to fix notations, we first state some facts about the field of transseries $\mathbb{T}$ in addition to those mentioned in the Introduction. For proofs, we refer to [15], where $\mathbb{T}$ is defined as exponential H-field, and to [11] for more details; see [6] for an independent construction of $\mathbb{T}$ as exponential field.

Facts about $\mathbb{T}$. As an ordered field, $\mathbb{T}$ is the union of an increasing sequence

$$\mathbb{L} = \mathbb{R}[[\mathfrak{T}_0]] \subseteq \mathbb{R}[[\mathfrak{T}_1]] \subseteq \cdots \subseteq \mathbb{R}[[\mathfrak{T}_n]] \subseteq \cdots$$

of ordered transseries subfields over $\mathbb{R}$, with $\mathfrak{T}_0 = \mathfrak{L}$, and where each inclusion $\mathbb{R}[[\mathfrak{T}_n]] \subseteq \mathbb{R}[[\mathfrak{T}_{n+1}]]$ comes from a corresponding inclusion $\mathfrak{T}_n \subseteq \mathfrak{T}_{n+1}$ of multiplicative ordered abelian groups. The exponential operation exp on $\mathbb{T}$ maps the ordered additive group $\mathbb{R}[[\mathfrak{T}_n]]^\uparrow$ isomorphically onto the ordered group $\mathfrak{T}_{n+1}$. Hence $\log \mathfrak{m} \in \mathbb{R}[[\mathfrak{T}_n]]^\uparrow$ for $\mathfrak{m} \in \mathfrak{T}_{n+1}$, where $\log\colon \mathbb{T}^{>0} \to \mathbb{T}$ is the inverse of exp. Also

$$\log(1+\varepsilon) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}\varepsilon^i \in \mathbb{R}[[\mathfrak{T}_n]] \tag{4.1}$$

for $1 \succ \varepsilon \in \mathbb{R}[[\mathfrak{T}_n]]$. For $f \in \mathbb{T}^{>0}$ and $r \in \mathbb{R}$ we put $f^r := \exp(r \log f) \in \mathbb{T}$; one checks easily that $f^r \geqslant 1$ if $f \geqslant 1$ and $r \geqslant 0$, and that this operation of raising to real powers makes $\mathbb{T}^{>0}$ into a multiplicative vector space over $\mathbb{R}$ containing each $\mathfrak{T}_n$ as a multiplicative $\mathbb{R}$-subspace.

We put $\mathfrak{T} := \bigcup_n \mathfrak{T}_n$ (an ordered subgroup of $\mathbb{T}^{>0}$), so the ordered transseries field $\mathbb{R}[[\mathfrak{T}]]$ over $\mathbb{R}$ contains $\mathbb{T}$ as an ordered subfield. The ordered field $\mathbb{R}[[\mathfrak{T}]]$ comes equipped with two strongly linear automorphisms $f \mapsto f{\uparrow}$ (*upward shift*) and $f \mapsto f{\downarrow}$ (*downward shift*), that are mutually inverse and map $\mathbb{T}$ to itself. The downward shift extends the map $f \mapsto f \circ \ell_1$ on $\mathbb{L}$ used in the last section, and also the composition operation $f \mapsto f \circ \log x$ on $\mathbb{R}[[[x]]]$. (See [15], Chapter 2.) We have $\exp(f){\uparrow} = \exp(f{\uparrow})$ for $f \in \mathbb{T}$, and hence $\log(f){\uparrow} = \log(f{\uparrow})$ and $(f^r){\uparrow} = (f{\uparrow})^r$ for $f \in \mathbb{T}^{>0}$, $r \in \mathbb{R}$. From these properties one obtains by induction that $\mathfrak{T}_n{\uparrow} \subseteq \mathfrak{T}_{n+1}$ and $\mathfrak{T}_n{\downarrow} \subseteq \mathfrak{T}_n$. (Hence $\mathfrak{m} \mapsto \mathfrak{m}{\uparrow}$ is an automorphism of the ordered group $\mathfrak{T}$.) We denote the n-fold functional composition of $f \mapsto f{\downarrow}$ by $f \mapsto f{\downarrow}^n$, and similarly we write $f \mapsto f{\uparrow}^n$ for the n-fold composition of $f \mapsto f{\uparrow}$.

The derivation on $\mathbb{T}$ restricts to a strongly linear derivation on each subfield $\mathbb{R}[[\mathfrak{T}_n]]$, and extends uniquely to a strongly linear derivation $D\colon f \mapsto f'$ on $\mathbb{R}[[\mathfrak{T}]]$. With this derivation, $\mathbb{R}[[\mathfrak{T}]]$ is a real closed H-field with constant field $\mathbb{R}$. We have

$$(f{\uparrow})' = e^x \cdot (f'){\uparrow}, \quad (f{\downarrow})' = \frac{1}{x} \cdot (f'){\downarrow} \qquad (f \in \mathbb{R}[[\mathfrak{T}]]).$$

Note that $v\big(\exp(-\Lambda)\big)$ remains a gap in $\mathbb{R}[[\mathfrak{T}]]$, so $\mathbb{R}[[\mathfrak{T}]]$ is not closed under asymptotic integration. There is also no natural extension of the exponential operation on $\mathbb{T}$ to one on $\mathbb{R}[[\mathfrak{T}]]$. Nevertheless, using (4.1) one easily checks that the function $\log\colon \mathbb{T}^{>0} \to \mathbb{T}$ extends to an embedding log of the ordered multiplicative group $\mathbb{R}[[\mathfrak{T}]]^{>0}$ into the ordered additive group $\mathbb{R}[[\mathfrak{T}]]^{>0}$, by setting

$$\log g := \log a\mathfrak{m} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \varepsilon^n$$

for $g = a\mathfrak{m}(1+\varepsilon)$, $a \in \mathbb{R}^{>0}$, $\mathfrak{m} \in \mathfrak{T}$, and $1 \succ \varepsilon \in \mathbb{R}[[\mathfrak{T}]]$.

Monomial subgroups of $\mathfrak{T}$. In the next section we construct a Liouville closed H-subfield of $\mathbb{T}$ containing $\mathbb{L}_1$; this will involve subgroups $\mathfrak{M}$ of $\mathfrak{T}$ such that the subfield $\mathbb{R}[[\mathfrak{M}]]$ of $\mathbb{R}[[\mathfrak{T}]]$ is closed under differentiation and integration. In the rest of this section, $\mathfrak{M}_n$ denotes an ordered subgroup of $\mathfrak{T}_n$, for every n, with the following properties:

- (M1) $\mathfrak{M}_0 = \mathfrak{L}$;
- (M2) $A_n := \log \mathfrak{M}_{n+1}$ is an $\mathbb{R}$-linear subspace of $\mathbb{R}[[\mathfrak{M}_n]]^{\uparrow}$ and is closed under truncation;
- (M3) $\mathfrak{M}_n \subseteq \mathfrak{M}_{n+1}$.

Here a set $A \subseteq \mathbb{R}[[\mathfrak{T}]]$ is said to be *closed under truncation* if for each $f = \sum_{\mathfrak{m}\in\mathfrak{T}} f_{\mathfrak{m}}\mathfrak{m} \in A$ and each final segment $\mathfrak{F}$ of $\mathfrak{T}$ we have $f|_{\mathfrak{F}} := \sum_{\mathfrak{m}\in\mathfrak{F}} f_{\mathfrak{m}}\mathfrak{m} \in A$.

We put $\mathfrak{M} := \bigcup_n \mathfrak{M}_n$, a subgroup of $\mathfrak{T}$. When needed we shall also impose:

- (M4) $\mathfrak{M}{\uparrow} \subseteq \mathfrak{M}$.

Example 4.1. Let $\mathfrak{M}_n := \mathfrak{T}_n$. Then the $\mathfrak{M}_n$ satisfy (M1)–(M4), with $A_n = \mathbb{R}[[\mathfrak{T}_n]]^{\uparrow}$ and $\mathfrak{M} = \mathfrak{T}$.

By (M1), the set $\log \mathfrak{M}_0$ is also an $\mathbb{R}$-linear subspace of $\mathbb{R}[[\mathfrak{M}_0]]$ closed under truncation. By (M1) and (M2), each $\mathfrak{M}_n$ is *closed under $\mathbb{R}$-powers:* if $\mathfrak{m} \in \mathfrak{M}_n$ and $r \in \mathbb{R}$, then $\mathfrak{m}^r \in \mathfrak{M}_n$. Also by (M1) and (M2), each subfield $\mathbb{R}[[\mathfrak{M}_n]]$ of $\mathbb{T}$ is closed under taking logarithms of positive elements, and so is the subfield $\mathbb{R}[[\mathfrak{M}]]$ of $\mathbb{R}[[\mathfrak{T}]]$. Moreover, each subfield $\mathbb{R}[[\mathfrak{M}_n]]$ of $\mathbb{T}$ is closed under differentiation, hence is an H-subfield of $\mathbb{T}$. (This follows by an easy induction on n: use (M1) for $n = 0$, and (M2) for the induction step.) It follows that the subfield $\mathbb{R}[[\mathfrak{M}]]$ of $\mathbb{R}[[\mathfrak{T}]]$ is closed under differentiation, hence is an H-subfield of $\mathbb{R}[[\mathfrak{T}]]$.

Lemma 4.2. *The H-field $\mathbb{R}[[\mathfrak{M}]]$ is closed under asymptotic integration if and only if $\exp(\Lambda) \notin \mathfrak{M}$. In this case, $\mathbb{R}[[\mathfrak{M}]]$ is closed under integration, and the map $f \mapsto \int f \colon \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]]$ is strongly linear.*

Proof. The H-field $\mathbb{R}[[\mathfrak{M}]]$ is closed under asymptotic integration if and only if it does not have a gap ([1], Section 2). The valuation of $\mathbb{R}[[\mathfrak{T}]]$ maps $\mathfrak{T}$ bijectively and order-reversingly onto the value group of $\mathbb{R}[[\mathfrak{T}]]$, and also $\mathfrak{M}$ onto the value group of $\mathbb{R}[[\mathfrak{M}]]$. The element $\exp(-\Lambda)$ of $\mathfrak{T}$ satisfies $(1/\ell_n)' \prec \exp(-\Lambda) \prec (1/\ell_n)^\dagger$ for all n. Because the sequence $1/\ell_0, 1/\ell_1, \dots$ is coinitial in $\mathfrak{M}^{\prec 1}$, this yields the first part of the lemma. The rest now follows from Lemma 1.7. □

Put $\mathfrak{M}'_n := \mathfrak{M}_n \cap \mathfrak{M}\!\uparrow$ and $\mathfrak{M}' := \bigcup_n \mathfrak{M}'_n$. The next easy lemma is left as an exercise to the reader.

Lemma 4.3. *The family $(\mathfrak{M}'_n)$ satisfies the following analogues of* (M1)–(M3): *$\mathfrak{M}'_0 = \mathfrak{L}$; $\log \mathfrak{M}'_{n+1}$ is an $\mathbb{R}$-linear subspace of $\mathbb{R}[[\mathfrak{M}'_n]]^\uparrow$ closed under truncation; $\mathfrak{M}'_n \subseteq \mathfrak{M}'_{n+1}$. If* (M4) *holds, then $\mathfrak{M}' = \mathfrak{M}\!\uparrow$ and $\mathfrak{M}'\!\uparrow \subseteq \mathfrak{M}'$.*

In the rest of this section $\mathfrak{N}$ denotes a convex subgroup of $\mathfrak{M}$, equivalently, a subgroup such that for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$

$$\mathfrak{m} \preccurlyeq \mathfrak{n} \in \mathfrak{N} \Longrightarrow \mathfrak{m} \in \mathfrak{N}.$$

Note that then $\mathfrak{N}$ is closed under $\mathbb{R}$-powers, and that $\mathfrak{N}\!\uparrow$ is a convex subgroup of $\mathfrak{M}\!\uparrow$. To $\mathfrak{N}$ we associate the set

$$I := \left\{\mathfrak{m} \in \mathfrak{M}^{\succ 1} : \exp \mathfrak{m} \preccurlyeq \mathfrak{n} \text{ for some } \mathfrak{n} \in \mathfrak{N}\right\} \subseteq \mathfrak{N}.$$

Then I is an initial segment of $\mathfrak{M}^{\succ 1}$ (with $I = \emptyset$ if $\mathfrak{N} = \{1\}$). Consequently, the complement $F = \mathfrak{M}^{\succ 1} \setminus I$ of I is a final segment of $\mathfrak{M}^{\succ 1}$, and

$$\mathfrak{R} := \{\mathfrak{r} \in \mathfrak{M} : \log \mathfrak{r} \in \mathbb{R}[[F]]\}$$

is also a subgroup of $\mathfrak{M}$ closed under $\mathbb{R}$-powers.

Lemma 4.4. *For all $\mathfrak{m} \in \mathfrak{M}$ we have:*

$$\mathfrak{m} \in \mathfrak{N} \quad \Longleftrightarrow \quad \log \mathfrak{m} \in \mathbb{R}[[I]].$$

Proof. The lemma holds trivially if $\mathfrak{N} = \{1\}$. Assume that $\mathfrak{N} \neq \{1\}$; hence $\ell_k \in \mathfrak{N}$ from some $k \in \mathbb{N}$. Let $\mathfrak{m} \in \mathfrak{M}_n$. We prove the desired equivalence by distinguishing the cases $n = 0$ and $n > 0$. If $n = 0$, then we take $k \in \mathbb{N}$ minimal such that $\ell_k \in \mathfrak{N}$, so

$$\mathfrak{N} \cap \mathfrak{L} = \left\{\ell_0^{\beta_0} \ell_1^{\beta_1} \cdots \in \mathfrak{L} : \beta_i = 0 \text{ for all } i < k\right\},$$

which easily yields the desired equivalence.

Suppose that $n > 0$. Then $\log \mathfrak{m} \in A_{n-1}$. Since A_{n-1} is closed under truncation we have $\log \mathfrak{m} = \varphi + \psi$ with $\varphi \in A_{n-1} \cap \mathbb{R}[[I]]$ and $\psi \in A_{n-1} \cap \mathbb{R}[[F]]$. Hence $e^\varphi, e^\psi \in \mathfrak{M}$. In fact $e^\varphi \in \mathfrak{N}$, because if $\varphi \neq 0$, then $\mathfrak{d}(\varphi) \in I$, so $e^\varphi \asymp e^{\mathfrak{d}(\varphi)} \preccurlyeq \mathfrak{n}$ for some $\mathfrak{n} \in \mathfrak{N}$. Similarly, if $\psi \neq 0$, then $e^\psi \notin \mathfrak{N}$. The desired equivalence now follows from $\mathfrak{m} = e^\varphi \cdot e^\psi$. □

With $\mathfrak{N}_n := \mathfrak{N} \cap \mathfrak{M}_n$ and $\mathfrak{R}_n := \mathfrak{R} \cap \mathfrak{M}_n$ we have:

Corollary 4.5. $\mathfrak{N} \cap \mathfrak{R} = \{1\}$ *and* $\mathfrak{M}_n = \mathfrak{N}_n \cdot \mathfrak{R}_n$.

It follows that $\mathfrak{M} = \mathfrak{N} \cdot \mathfrak{R}$, and the products $\mathfrak{n}\mathfrak{r}$ with $\mathfrak{n} \in \mathfrak{N}$ and $\mathfrak{r} \in \mathfrak{R}$ are ordered antilexicographically: $\mathfrak{n}\mathfrak{r} \succ 1$ if and only if $\mathfrak{r} \succ 1$, or $\mathfrak{r} = 1$ and $\mathfrak{n} \succ 1$. We think of the monomials in the convex subgroup $\mathfrak{N}$ as being *flat*. Accordingly we call $\mathfrak{R}$ the *steep supplement* of $\mathfrak{N}$.

Proof. It is clear from the previous lemma that $\mathfrak{N} \cap \mathfrak{R} = \{1\}$. We now show $\mathfrak{M}_n = \mathfrak{N}_n \cdot \mathfrak{R}_n$. Let $\mathfrak{m} \in \mathfrak{M}_n$. Then $\log \mathfrak{m} \in \mathbb{R}[[\mathfrak{M}]]^{\uparrow}$, so $\log \mathfrak{m} = \varphi + \psi$ with $\varphi \in \mathbb{R}[[I]]$, $\psi \in \mathbb{R}[[F]]$. Since $\log \mathfrak{M}_n$ is truncation closed, we have $\varphi, \psi \in \log \mathfrak{M}_n$, so $\mathfrak{m} = \mathfrak{n}\mathfrak{r}$ with $\mathfrak{n} := e^{\varphi} \in \mathfrak{M}_n \cap \mathfrak{N} = \mathfrak{N}_n$ and $\mathfrak{r} := e^{\psi} \in \mathfrak{M}_n \cap \mathfrak{R} = \mathfrak{R}_n$, using the previous lemma. □

Corollary 4.6. *Suppose that* $x \in \mathfrak{N}$. *Then the following analogues of* (M1)–(M3) *hold:*

(N1) $\mathfrak{N}_0 = \mathfrak{L}$;
(N2) $\log \mathfrak{N}_{n+1}$ *is an* $\mathbb{R}$*-linear subspace of* $\mathbb{R}[[\mathfrak{N}_n]]^{\uparrow}$ *and is closed under truncation;*
(N3) $\mathfrak{N}_n \subseteq \mathfrak{N}_{n+1}$.

In particular, the subfield $\mathbb{R}[[\mathfrak{N}]]$ *of* $\mathbb{R}[[\mathfrak{M}]]$ *is closed under differentiation, and if* $e^{\Lambda} \notin \mathfrak{N}$, *then* $\mathbb{R}[[\mathfrak{N}]]$ *is also closed under integration.*

Remark 4.7. If we drop the assumption $x \in \mathfrak{N}$, then $\mathbb{R}[[\mathfrak{N}]]$ may fail to be closed under differentiation. To see this, take $\mathfrak{N} = \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m} \prec\!\!\prec x\}$ and $\mathfrak{m} = \log x \in \mathfrak{N}$; then $\mathfrak{m}' = 1/x \asymp\!\!\asymp x$, so $\mathfrak{m}' \notin \mathfrak{N}$.

Property (N2) of Corollary 4.6 follows easily from Lemma 4.4 and its proof (without assuming $x \in \mathfrak{N}$). The rest of the corollary is then obvious.

Lemma 4.8. *Suppose that* $x \in \mathfrak{N}$, *and that* $\mathfrak{m} \prec\!\!\prec \mathfrak{r}$, *where* $\mathfrak{m}, \mathfrak{r} \in \mathfrak{M}$, $\mathfrak{r} \notin \mathfrak{N}$. *Then* $\operatorname{supp} \mathfrak{m}' \prec\!\!\prec \mathfrak{r}$.

Proof. By induction on n such that $\mathfrak{m} \in \mathfrak{M}_n$. The claim is trivial for $n = 0$ since $\mathfrak{M}_0 = \mathfrak{N}_0 = \mathfrak{L}$ and $\mathfrak{m}' \in \mathbb{R}[[\mathfrak{L}]]$. Suppose $n > 0$ and write $\mathfrak{m} = e^{\varphi}$ with $\varphi \in A_{n-1}$. Since $\operatorname{supp} \varphi \prec\!\!\prec \mathfrak{m}$ we obtain $\operatorname{supp} \varphi' \prec\!\!\prec \mathfrak{r}$, by inductive hypothesis. Any $\mathfrak{u} \in \operatorname{supp} \mathfrak{m}'$ is of the form $\mathfrak{u} = \mathfrak{v} \cdot \mathfrak{m}$ with $\mathfrak{v} \in \operatorname{supp} \varphi'$, hence $\mathfrak{u} \prec\!\!\prec \mathfrak{r}$ as required. □

Flattening. We "flatten" the dominance relations $\prec$ and $\preccurlyeq$ on $\mathbb{R}[[\mathfrak{M}]]$ by the convex subgroup $\mathfrak{N}$ of $\mathfrak{M}$ as follows:

$$\begin{aligned} f \prec_{\mathfrak{N}} g \quad &:\Longleftrightarrow \quad (\forall \varphi \in \mathfrak{N} : \varphi f \prec g), \\ f \preccurlyeq_{\mathfrak{N}} g \quad &:\Longleftrightarrow \quad (\exists \varphi \in \mathfrak{N} : f \preccurlyeq \varphi g), \end{aligned}$$

for $f, g \in \mathbb{R}[[\mathfrak{M}]]$. We also define, for $f, g \in \mathbb{R}[[\mathfrak{M}]]$:

$$f \asymp_{\mathfrak{N}} g \quad :\Longleftrightarrow \quad f \preccurlyeq_{\mathfrak{N}} g \wedge g \preccurlyeq_{\mathfrak{N}} f,$$

hence $\mathfrak{N} = \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m} \asymp_{\mathfrak{N}} 1\}$. Flattening corresponds to coarsening the valuation: The value group $v(\mathfrak{M})$ of the natural valuation v on $\mathbb{R}[[\mathfrak{M}]]$ has convex subgroup $v(\mathfrak{N})$, so gives rise to the coarsened valuation $v_{\mathfrak{N}}$ on $\mathbb{R}[[\mathfrak{M}]]$ with (ordered) value group $v(\mathfrak{M})/v(\mathfrak{N})$ given by $v_{\mathfrak{N}}(f) := v(f) + v(\mathfrak{N})$ for $f \in \mathbb{R}[[\mathfrak{M}]]^{\times}$. Then we have the equivalences

$$\begin{aligned} f \prec_{\mathfrak{N}} g \quad &\Longleftrightarrow \quad v_{\mathfrak{N}}(f) > v_{\mathfrak{N}}(g) \quad \text{and} \\ f \preccurlyeq_{\mathfrak{N}} g \quad &\Longleftrightarrow \quad v_{\mathfrak{N}}(f) \geqslant v_{\mathfrak{N}}(g) \end{aligned}$$

for $f, g \in \mathbb{R}[[\mathfrak{M}]]$. (See also Section 14 of [2].) The restriction of $\preccurlyeq_{\mathfrak{N}}$ to $\mathfrak{M}$ is a quasi-ordering, i.e., reflexive and transitive; it is anti-symmetric (i.e., an ordering) if and only if $\mathfrak{N} = \{1\}$. The restriction of $\preccurlyeq_{\mathfrak{N}}$ to $\mathfrak{R}$ is the already given ordering on $\mathfrak{R}$. The following rules are valid for $f, g \in \mathbb{R}[[\mathfrak{M}]]$:

$$\text{the equivalence} \quad f \prec_{\mathfrak{N}} g \Longleftrightarrow f' \prec_{\mathfrak{N}} g' \quad \text{holds, provided } f, g \not\asymp_{\mathfrak{N}} 1;$$
$$1 \prec_{\mathfrak{N}} f \preccurlyeq_{\mathfrak{N}} g \Longrightarrow f^{\dagger} \preccurlyeq_{\mathfrak{N}} g^{\dagger};$$
$$f \preccurlyeq g \Longrightarrow f \preccurlyeq_{\mathfrak{N}} g, \text{ and hence } f \prec_{\mathfrak{N}} g \Longrightarrow f \prec g.$$

In our proofs below, we often reduce to the case that $x \in \mathfrak{N}$ by upward shift. Here are a few remarks about this case. If $x \in \mathfrak{N}$, then $\mathfrak{L} \subseteq \mathfrak{N}$, and for all $f \in \mathbb{R}[[\mathfrak{M}]]$:

$$\text{the equivalence} \quad f \asymp_{\mathfrak{N}} 1 \iff f' \asymp_{\mathfrak{N}} 1 \quad \text{holds, provided } f \not\asymp 1;$$
$$f \succ_{\mathfrak{N}} 1 \iff f' \succ_{\mathfrak{N}} 1. \tag{4.2}$$

(See [2], Lemma 13.4.) Moreover:

Lemma 4.9. *Suppose that $x \in \mathfrak{N}$. Then the following conditions on $\mathfrak{m} \in \mathfrak{M}$ are equivalent:*

(1) $\log \mathfrak{m} \preccurlyeq_{\mathfrak{N}} 1$,
(2) $\log \mathfrak{m} \in \mathbb{R}[[\mathfrak{N}]]$,
(3) $\mathfrak{m}^{\dagger} \in \mathbb{R}[[\mathfrak{N}]]$,
(4) $\mathfrak{m}^{\dagger} \preccurlyeq_{\mathfrak{N}} 1$.

Proof. From $\operatorname{supp}(\log \mathfrak{m}) \subseteq \mathfrak{M}^{\succ 1}$ we obtain (1) ⇒ (2). The implication (2) ⇒ (3) follows from Corollary 4.6, (3) ⇒ (4) is trivial, and (4) ⇒ (1) follows from (4.2). □

Flattened canonical decomposition. We have an isomorphism

$$\mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{N}]][[\mathfrak{R}]]$$

of $\mathbb{R}[[\mathfrak{N}]]$-algebras given by

$$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m} \mapsto \sum_{\mathfrak{r} \in \mathfrak{R}} \left(\sum_{\mathfrak{n} \in \mathfrak{N}} f_{\mathfrak{n}\mathfrak{r}} \mathfrak{n} \right) \mathfrak{r}.$$

In $\mathbb{R}[[\mathfrak{M}]]$ we have in fact

$$f = \sum_{\mathfrak{r} \in \mathfrak{R}} \left(\sum_{\mathfrak{n} \in \mathfrak{n}} f_{\mathfrak{n}\mathfrak{r}} \mathfrak{n} \right) \mathfrak{r},$$

where the sums are interpreted as in Section 1. We shall identify the (real closed, ordered) field $\mathbb{R}[[\mathfrak{M}]]$ with the (real closed, ordered) field $\mathbb{R}[[\mathfrak{N}]][[\mathfrak{R}]]$ by means of this isomorphism. For $f \in \mathbb{R}[[\mathfrak{M}]]$ we put

$$f_{\mathfrak{N},\mathfrak{r}} := \sum_{\mathfrak{n} \in \mathfrak{N}} f_{\mathfrak{n}\mathfrak{r}} \mathfrak{n} \in \mathbb{R}[[\mathfrak{N}]], \quad (\mathfrak{r} \in \mathfrak{R}), \text{ and}$$

$$\operatorname{supp}_{\mathfrak{N}} f := \{ \mathfrak{r} \in \mathfrak{R} : f_{\mathfrak{N},\mathfrak{r}} \neq 0 \}.$$

We have the *flattened canonical decomposition* of the $\mathbb{R}$-vector space $\mathbb{R}[[\mathfrak{M}]]$ (relative to $\mathfrak{N}$)

$$\mathbb{R}[[\mathfrak{M}]] = \mathbb{R}[[\mathfrak{M}]]^{\Uparrow} \oplus \mathbb{R}[[\mathfrak{M}]]^{\equiv} \oplus \mathbb{R}[[\mathfrak{M}]]^{\Downarrow},$$

where

$$\begin{aligned}\mathbb{R}[[\mathfrak{M}]]^{\Uparrow} &= \mathbb{R}[[\mathfrak{N}]][[\mathfrak{R}^{\succ 1}]];\\ \mathbb{R}[[\mathfrak{M}]]^{\equiv} &= \mathbb{R}[[\mathfrak{N}]];\\ \mathbb{R}[[\mathfrak{M}]]^{\Downarrow} &= \mathbb{R}[[\mathfrak{N}]][[\mathfrak{R}^{\prec 1}]].\end{aligned}$$

Accordingly, given a transseries $f \in \mathbb{R}[[\mathfrak{M}]]$, we write

$$f = f^{\Uparrow} + f^{\equiv} + f^{\Downarrow}$$

where

$$\begin{aligned}f^{\Uparrow} &= \sum_{1 \prec \mathfrak{m} \in \mathfrak{M} \setminus \mathfrak{N}} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathfrak{M}]]^{\Uparrow};\\ f^{\equiv} &= \sum_{\mathfrak{m} \in \mathfrak{N}} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathfrak{M}]]^{\equiv};\\ f^{\Downarrow} &= \sum_{1 \succ \mathfrak{m} \in \mathfrak{M} \setminus \mathfrak{N}} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathfrak{M}]]^{\Downarrow}.\end{aligned}$$

Example 4.10. Let $\mathfrak{w} \in \mathfrak{M}$, $\mathfrak{w} \not\asymp 1$, and consider the convex subgroup

$$\mathfrak{N} := \{\mathfrak{n} \in \mathfrak{M} : \mathfrak{n} \prec\!\!\prec \mathfrak{w}\}$$

of $\mathfrak{M}$. Suppose that $\exp(\mathfrak{M}^{\succ 1}) \subseteq \mathfrak{M}$. Then

$$I = \{\mathfrak{m} \in \mathfrak{M}^{\succ 1} : \exp \mathfrak{m} \prec\!\!\prec \mathfrak{w}\}$$

and thus

$$\mathfrak{R} = \{\mathfrak{r} \in \mathfrak{M} : \operatorname{supp} \log \mathfrak{r} \succcurlyeq \mathfrak{d}(\log \mathfrak{w})\}.$$

In this case we write $\operatorname{supp}_{\mathfrak{w}} f$ instead of $\operatorname{supp}_{\mathfrak{N}} f$, $\preccurlyeq_{\mathfrak{w}}$ instead of $\preccurlyeq_{\mathfrak{N}}$, and likewise for the other asymptotic relations. In the next section we take $\mathfrak{w} = e^x$.

Flatly noetherian families. Let $(f_i)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I$. The family (f_i) is said to be *flatly noetherian* (with respect to $\mathfrak{N}$) if (f_i) is noetherian as a family of elements in $C[[\mathfrak{R}]]$, where $C = \mathbb{R}[[\mathfrak{N}]]$. If (f_i) is flatly noetherian, then (f_i) is noetherian as a family of elements of $\mathbb{R}[[\mathfrak{M}]]$, and its sum $\sum_{i \in I} f_i \in C[[\mathfrak{R}]]$ as a flatly noetherian family equals its sum $\sum_{i \in I} f_i \in \mathbb{R}[[\mathfrak{M}]]$ as a noetherian family of elements of $\mathbb{R}[[\mathfrak{M}]]$. For any monomial $\mathfrak{m} \in \mathfrak{M}$, (f_i) is flatly noetherian if and only if $(\mathfrak{m} f_i)$ is flatly noetherian.

Note that if $\mathfrak{n}_1 \succ \mathfrak{n}_2 \succ \cdots$ is an infinite sequence of monomials in $\mathfrak{N}$, then $(\mathfrak{n}_i)_{i \geqslant 1}$ is a noetherian family which is not flatly noetherian.

A map $\Phi \colon \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]]$ is called *flatly strongly linear* (with respect to $\mathfrak{N}$) if Φ considered as a map $C[[\mathfrak{R}]] \longrightarrow C[[\mathfrak{R}]]$ is strongly linear, where $C = \mathbb{R}[[\mathfrak{N}]]$.

Lemma 4.11. *Suppose that $x \in \mathfrak{N}$. The map $\mathfrak{R} \to C[[\mathfrak{R}]] \colon \mathfrak{r} \mapsto \mathfrak{r}'$ is noetherian, where $C = \mathbb{R}[[\mathfrak{N}]]$, and thus extends uniquely to a flatly strongly linear map*

$$\varphi \colon \mathbb{R}[[\mathfrak{M}]] \longrightarrow \mathbb{R}[[\mathfrak{M}]].$$

Proof. Let $\mathfrak{r}_1 \succ_{\mathfrak{N}} \mathfrak{r}_2 \succ_{\mathfrak{N}} \cdots$ be elements of $\mathfrak{R}$ and $\mathfrak{u}_i \in \operatorname{supp} \mathfrak{r}_i'$ for each i. It suffices to show that then there exist indices $i < j$ such that $\mathfrak{u}_i \succ_{\mathfrak{N}} \mathfrak{u}_j$. Since differentiation on $\mathbb{R}[[\mathfrak{M}]]$ is strongly linear, we may assume, after passing to a subsequence, that $\mathfrak{u}_i \succ \mathfrak{u}_j$ for all $i < j$. If there exist $i < j$ such that $\mathfrak{u}_i \asymp_{\mathfrak{N}} \mathfrak{r}_i$ and $\mathfrak{u}_j \asymp_{\mathfrak{N}} \mathfrak{r}_j$, we are already done. So we may assume that $\mathfrak{u}_i \not\asymp_{\mathfrak{N}} \mathfrak{r}_i$ for all i, and also that $\mathfrak{r}_i \not\asymp_{\mathfrak{N}} \mathfrak{u}_1$ for all i. Write each $\mathfrak{u}_i$ as $\mathfrak{u}_i = \mathfrak{r}_i \mathfrak{m}_i$, with $\mathfrak{m}_i \in \operatorname{supp} \mathfrak{r}_i^\dagger$, $\mathfrak{m}_i \notin \mathfrak{N}$. We distinguish two cases:

(1) For all $i > 1$ there exists a $\mathfrak{v}_i \in \operatorname{supp} \log \mathfrak{u}_1$ such that $\mathfrak{m}_i \in \operatorname{supp} \mathfrak{v}_i'$. Since $\operatorname{supp} \log \mathfrak{u}_1$ is noetherian we may assume, after passing to a subsequence, that $\mathfrak{v}_i \succcurlyeq \mathfrak{v}_j$ for $1 < i < j$. Since differentiation on $\mathbb{R}[[\mathfrak{M}]]$ is strongly linear, we then find $i < j$ with $\mathfrak{m}_i \succcurlyeq \mathfrak{m}_j$. Hence $\mathfrak{m}_i \succcurlyeq_{\mathfrak{N}} \mathfrak{m}_j$, so $\mathfrak{u}_i \succ_{\mathfrak{N}} \mathfrak{u}_j$.

(2) There exists an $i > 1$ such that for all $\mathfrak{v} \in \operatorname{supp} \log \mathfrak{u}_1$ we have $\mathfrak{m}_i \notin \operatorname{supp} \mathfrak{v}'$. Take such i and choose $\mathfrak{v} \in \operatorname{supp} \log \mathfrak{r}_i$ such that $\mathfrak{m}_i \in \operatorname{supp} \mathfrak{v}'$. Then

$$\mathfrak{v} \in (\operatorname{supp} \log \mathfrak{r}_i) \setminus (\operatorname{supp} \log \mathfrak{u}_1) \subseteq \operatorname{supp} \log(\mathfrak{r}_i/\mathfrak{u}_1) \subseteq \mathfrak{M}^{\succ 1}$$

and hence $\mathfrak{v} \preccurlyeq\!\!\!\prec \log(\mathfrak{u}_1/\mathfrak{r}_i)$. Since $\log \mathfrak{m} \prec\!\!\!\prec \mathfrak{m}$ for $\mathfrak{m} \in \mathfrak{M} \setminus \{1\}$, this yields $\mathfrak{v} \prec\!\!\!\prec \mathfrak{u}_1/\mathfrak{r}_i$. By Lemma 4.8 we get $\mathfrak{m}_i \prec\!\!\!\prec \mathfrak{u}_1/\mathfrak{r}_i$. Hence if $\mathfrak{n} := \mathfrak{u}_1/\mathfrak{u}_i \in \mathfrak{N}$, then $\mathfrak{m}_i \prec\!\!\!\prec \mathfrak{u}_1/\mathfrak{r}_i = \mathfrak{m}_i \mathfrak{n}$, contradicting $\mathfrak{m}_i \notin \mathfrak{N}$. Therefore $\mathfrak{u}_1 \succ_{\mathfrak{N}} \mathfrak{u}_i$.

□

In the rest of this section we assume (M4).

In particular, our previous results apply to $\mathfrak{M}{\uparrow}^k$ instead of $\mathfrak{M}$ for $k = 1, 2, \ldots$, by Lemma 4.3. In this connection, the following fact will be useful.

Remark 4.12. A family $(f_i)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I$ is flatly noetherian with respect to $\mathfrak{N}$ if and only if the family $(f_i{\uparrow})_{i \in I} \in \mathbb{R}[[\mathfrak{M}{\uparrow}]]^I$ is flatly noetherian with respect to $\mathfrak{N}{\uparrow}$.

We now arrive at the main results of this section:

Theorem 4.13. *If $(f_i)_{i \in I}$ is a flatly noetherian family in $\mathbb{R}[[\mathfrak{M}]]$, then so is $(f_i')_{i \in I}$.*

Proof. Since the case $\mathfrak{N} = \{1\}$ is trivial, we may assume $\mathfrak{N} \neq \{1\}$. Then $x \in \mathfrak{N}{\uparrow}^k$ for sufficiently large $k \in \mathbb{N}$. Since $(f{\uparrow})' = e^x \cdot (f'){\uparrow}$ for $f \in \mathbb{R}[[\mathfrak{M}]]$, Remark 4.12 allows us to reduce to the case that $x \in \mathfrak{N}$. Then $\mathbb{R}[[\mathfrak{N}]]$ is closed under differentiation by Corollary 4.6. Now consider a flatly noetherian family $(f_i)_{i \in I} \in \mathbb{R}[[\mathfrak{M}]]^I$. Then (f_i) is noetherian, hence (f_i') is noetherian by strong linearity of differentiation. By the lemma above, the family (g_i) defined by

$$g_i := \sum_{\mathfrak{r} \in \mathfrak{R}} f_{i, \mathfrak{N}, \mathfrak{r}} \mathfrak{r}'$$

is flatly noetherian. Put

$$h_i := f_i' - g_i = \sum_{\mathfrak{r} \in \mathfrak{R}} (f_{i, \mathfrak{N}, \mathfrak{r}})' \mathfrak{r}.$$

We have $\operatorname{supp}_{\mathfrak{N}} h_i \subseteq \operatorname{supp}_{\mathfrak{N}} f_i$ for $i \in I$, since $\mathbb{R}[[\mathfrak{N}]]$ is closed under differentiation. It follows that (h_i) is flatly noetherian. Hence the family (f_i') is flatly noetherian since it is the componentwise sum of two flatly noetherian families. □

Theorem 4.14. *Suppose that* $\exp(\Lambda) \notin \mathfrak{M}$. *Then* $\mathbb{R}[[\mathfrak{M}]]$ *is closed under integration, and if $(f_i)_{i \in I}$ is a flatly noetherian family in $\mathbb{R}[[\mathfrak{M}]]$, then $\left(\int f_i\right)_{i \in I}$ is flatly noetherian.*

Before we begin the proof, we make some remarks about the summation of flatly noetherian families in $\mathbb{R}[[\mathfrak{M}]]$. Choose a basis $\mathfrak{B}$ for the $\mathbb{R}$-vector space $\mathbb{R}[[\mathfrak{N}]]$. We define a (partial) ordering $\preccurlyeq^*$ on $\mathfrak{B} \times \mathfrak{R}$ as follows:

$$(\mathfrak{b}, \mathfrak{r}) \preccurlyeq^* (\mathfrak{c}, \mathfrak{s}) \qquad \Longleftrightarrow \qquad \mathfrak{r} \prec_{\mathfrak{N}} \mathfrak{s}, \text{ or } \mathfrak{r} = \mathfrak{s} \text{ and } \mathfrak{b} = \mathfrak{c}, \tag{4.3}$$

for all $(\mathfrak{b},\mathfrak{r}),(\mathfrak{c},\mathfrak{s})\in\mathfrak{B}\times\mathfrak{R}$. Consider the $\mathbb{R}$-vector space $\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]$ of transsseries

$$f=\sum_{(\mathfrak{b},\mathfrak{r})\in\mathfrak{B}\times\mathfrak{R}} f_{(\mathfrak{b},\mathfrak{r})}(\mathfrak{b},\mathfrak{r})$$

with real coefficients $f_{(\mathfrak{b},\mathfrak{c})}$, whose support $\operatorname{supp} f:=\{(\mathfrak{b},\mathfrak{r}):f_{(\mathfrak{b},\mathfrak{c})}\neq 0\}$ is noetherian for $\preccurlyeq^*$; see Section 1. We have:

Lemma 4.15. *There exists a unique isomorphism* $\varphi\colon\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]\to\mathbb{R}[[\mathfrak{M}]]$ *of* $\mathbb{R}$*-vector spaces such that*

(1) $\varphi(\mathfrak{b},\mathfrak{r})=\mathfrak{b}\cdot\mathfrak{r}$ *for* $\mathfrak{b}\in\mathfrak{B}$, $\mathfrak{r}\in\mathfrak{R}$,
(2) *a family* $(f_i)_{i\in I}\in\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]^I$ *is noetherian if and only if* $\big(\varphi(f_i)\big)_{i\in I}$ *is flatly noetherian,*
(3) *if* $(f_i)_{i\in I}\in\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]^I$ *is noetherian, then* $\varphi\left(\sum_{i\in I}f_i\right)=\sum_{i\in I}\varphi(f_i)$.

Proof. Of course, there is at most one such φ. For existence, first note that the projection map $\pi\colon\mathfrak{B}\times\mathfrak{R}\to\mathfrak{R}$ is strictly increasing, and that a set $\mathfrak{S}\subseteq\mathfrak{B}\times\mathfrak{R}$ is noetherian if and only if $\pi(\mathfrak{S})\subseteq\mathfrak{R}$ is noetherian and each fiber $\pi^{-1}(\mathfrak{r})$, $(\mathfrak{r}\in\mathfrak{R})$ is finite. Applying this remark to $\mathfrak{S}:=\bigcup_{i\in I}\operatorname{supp} f_i$, where $(f_i)_{i\in I}$ is a noetherian family in $\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]$, it follows that the subset

$$\pi(\mathfrak{S})=\bigcup_{i\in I,\mathfrak{b}\in\mathfrak{B},\mathfrak{r}\in\mathfrak{R}}\operatorname{supp}_{\mathfrak{N}}\big(f_{i,(\mathfrak{b},\mathfrak{r})}\mathfrak{b}\cdot\mathfrak{r}\big)$$

of $\mathfrak{R}$ is noetherian, and that for each $\mathfrak{r}\in\mathfrak{R}$ there are only finitely many $(i,\mathfrak{b})\in I\times\mathfrak{B}$ with $\mathfrak{r}\in\operatorname{supp}_{\mathfrak{N}}\big(f_{i,(\mathfrak{b},\mathfrak{r})}\mathfrak{b}\cdot\mathfrak{r}\big)$. Therefore the family $\big(f_{i,(\mathfrak{b},\mathfrak{r})}\mathfrak{b}\cdot\mathfrak{r}\big)_{(i,\mathfrak{b},\mathfrak{r})\in I\times\mathfrak{B}\times\mathfrak{R}}$ of elements of $\mathbb{R}[[\mathfrak{M}]]$ is flatly noetherian. Thus, by setting

$$\varphi(f):=\sum_{\mathfrak{r}\in\mathfrak{R}}\left(\sum_{\mathfrak{b}\in\mathfrak{B}}f_{(\mathfrak{b},\mathfrak{r})}\mathfrak{b}\right)\mathfrak{r}\qquad\text{for } f\in\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]],$$

we obtain an $\mathbb{R}$-linear bijection $\varphi\colon\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]\to\mathbb{R}[[\mathfrak{M}]]$ such that for every noetherian family $(f_i)\in\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]^I$, the family $\big(\varphi(f_i)\big)$ is flatly noetherian and $\varphi\left(\sum_i f_i\right)=\sum_i\varphi(f_i)$. (See proof of Proposition 3.5 in [17].) If $(f_i)\in\mathbb{R}[[\mathfrak{B}\times\mathfrak{R}]]^I$ and $\big(\varphi(f_i)\big)$ is flatly noetherian, then, with $\mathfrak{S}:=\bigcup_i\operatorname{supp} f_i$,

$$\pi(\mathfrak{S})=\bigcup_{i\in I}\operatorname{supp}_{\mathfrak{N}}\varphi(f_i)$$

is noetherian and $\pi|\mathfrak{S}$ has finite fibers, so (f_i) is noetherian. □

We now begin the proof of Theorem 4.14. Using upward shifting and $\int(f\!\uparrow)=\big(\int(f\cdot x^{-1})\big)\!\uparrow$ for $f\in\mathbb{R}[[\mathfrak{M}]]$, we first reduce to the case that $e^x\in\mathfrak{N}$. In particular $x\in\mathfrak{N}$, so $\mathbb{R}[[\mathfrak{N}]]$ is closed under differentiation and integration, by Corollary 4.6. Partition $\mathfrak{M}=\mathfrak{V}\amalg\mathfrak{W}$ (disjoint union), where

$$\mathfrak{V}=\{\mathfrak{m}\in\mathfrak{M}:\mathfrak{m}^\dagger\preccurlyeq_{\mathfrak{N}}1\}$$

and

$$\mathfrak{W}=\{\mathfrak{m}\in\mathfrak{M}:\mathfrak{m}^\dagger\succ_{\mathfrak{N}}1\}.$$

Then $\mathfrak{V}$ is a convex subgroup of $\mathfrak{M}$ containing $\mathfrak{N}$ which is closed under $\mathbb{R}$-powers, and $\mathbb{R}[[\mathfrak{M}]]=\mathbb{R}[[\mathfrak{V}]]\oplus\mathbb{R}[[\mathfrak{W}]]$ as $\mathbb{R}$-vector spaces. Note that if $\mathfrak{n}\in\mathfrak{N}$, $\mathfrak{r}\in\mathfrak{R}$, then $\mathfrak{n}\cdot\mathfrak{r}\in\mathfrak{W}$ if and only if $\mathfrak{r}\in\mathfrak{W}$. It follows that $\mathfrak{W}=\mathfrak{N}\cdot\mathfrak{S}$, where $\mathfrak{S}:=\mathfrak{W}\cap\mathfrak{R}$. Since $x\in\mathfrak{V}$, the subfield $\mathbb{R}[[\mathfrak{V}]]$ of $\mathbb{R}[[\mathfrak{M}]]$ is closed under differentiation and integration, by Corollary 4.6. Moreover:

Lemma 4.16. *The $\mathbb{R}$-linear subspace $\mathbb{R}[[\mathfrak{W}]]$ of $\mathbb{R}[[\mathfrak{M}]]$ is closed under the operators $f \mapsto f'$ and $g \mapsto \int g$ on $\mathbb{R}[[\mathfrak{M}]]$.*

Proof. If $\mathbb{R}[[\mathfrak{W}]]$ is closed under $f \mapsto f'$, then it is also closed under $g \mapsto \int g$, because $\mathbb{R}[[\mathfrak{V}]]$ is closed under differentiation and $\mathbb{R}[[\mathfrak{M}]]$ is closed under integration. So let $\mathfrak{w} \in \mathfrak{W}$; it is enough to show that then $\operatorname{supp} \mathfrak{w}' \subseteq \mathfrak{W}$. Take $n > 0$ with $\mathfrak{w} \in \mathfrak{W} \cap \mathfrak{M}_n$, and write $\mathfrak{w} = e^{\varphi}$ with $\varphi \in A_{n-1}$. By Lemma 4.8 we have $\operatorname{supp} \varphi' \prec\!\!\prec \mathfrak{w}$. Hence $\mathfrak{m}^\dagger \asymp \mathfrak{w}^\dagger \succ_{\mathfrak{N}} 1$ and thus $\mathfrak{m} \in \mathfrak{W}$, for every $\mathfrak{m} \in \operatorname{supp} \mathfrak{w}'$. $\square$

Lemma 4.17. *For all $h \in \mathbb{R}[[\mathfrak{V}]]$, we have $\operatorname{supp}_{\mathfrak{N}} \int h \subseteq \operatorname{supp}_{\mathfrak{N}} h$.*

Proof. It is enough to prove the lemma for h of the form $h = f\mathfrak{r}$, where $f \in \mathbb{R}[[\mathfrak{N}]]$, $f \neq 0$, and $\mathfrak{r} \in \mathfrak{V} \cap \mathfrak{R}$, so $\mathfrak{r} = e^{\varphi}$ with $\varphi' = \mathfrak{r}^\dagger \preccurlyeq_{\mathfrak{N}} 1$. By Lemma 4.9, we have $\varphi' \in \mathbb{R}[[\mathfrak{N}]]$. We may assume $\varphi \neq 0$. Then $e^{\varphi} = \mathfrak{r} \succ\!\!\succ \mathfrak{N}$, so $\varphi' = \mathfrak{r}^\dagger \succ \mathfrak{n}^\dagger$ for all $\mathfrak{n} \in \mathfrak{N}$. Thus the strongly linear map

$$\Phi\colon \mathbb{R}[[\mathfrak{N}]] \to \mathbb{R}[[\mathfrak{N}]],\ \ g \mapsto g'/\varphi'$$

satisfies $\Phi(\mathfrak{n}) \prec \mathfrak{n}$ for all $\mathfrak{n} \in \mathfrak{N}$. Hence by Corollary 1.4 the strongly linear operator $\mathrm{Id} + \Phi$ on $\mathbb{R}[[\mathfrak{N}]]$ is bijective. We let $g := (\mathrm{Id} + \Phi)^{-1}(f/\varphi') \in \mathbb{R}[[\mathfrak{N}]]$. Then $g' + \varphi' g = f$ and thus $\int f\mathfrak{r} = g\mathfrak{r}$. $\square$

If (f_i) is a flatly noetherian family of elements of $\mathbb{R}[[\mathfrak{V}]]$, then by the previous lemma $\left(\int f_i\right)$ is flatly noetherian. To complete the proof of Theorem 4.14 it therefore remains to show:

Lemma 4.18. *If (f_i) is a flatly noetherian family of elements of $\mathbb{R}[[\mathfrak{W}]]$, then $\left(\int f_i\right)$ is flatly noetherian.*

Proof. Let $C = \mathbb{R}[[\mathfrak{N}]]$, let $\mathfrak{B}$ be a basis for C as $\mathbb{R}$-vector space, and let $\mathbb{R}[[\mathfrak{B} \times \mathfrak{R}]]$ and $\varphi\colon \mathbb{R}[[\mathfrak{B} \times \mathfrak{R}]] \to \mathbb{R}[[\mathfrak{M}]]$ be as in Lemma 4.15. Put $\mathfrak{S} := \mathfrak{W} \cap \mathfrak{R}$ as before. Then $\varphi(\mathfrak{B} \times \mathfrak{S}) = \mathfrak{B} \cdot \mathfrak{S} \subseteq \mathbb{R}[[\mathfrak{W}]]$, so φ restricts to an $\mathbb{R}$-linear map

$$\varphi_1\colon \mathbb{R}[[\mathfrak{B} \times \mathfrak{S}]] \to \mathbb{R}[[\mathfrak{W}]].$$

Clearly φ_1 is bijective, since $\mathfrak{W} = \mathfrak{N} \cdot \mathfrak{S}$. Consider the strongly linear operators $D\colon \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]]$ given by $f \mapsto f'$ and $\int\colon \mathbb{R}[[\mathfrak{M}]] \to \mathbb{R}[[\mathfrak{M}]]$ given by $f \mapsto \int f$. We have $D(f), \int f \in \mathbb{R}[[\mathfrak{W}]]$ for $f \in \mathbb{R}[[\mathfrak{W}]]$, by Lemma 4.16. By Theorem 4.13 and Lemma 4.15, the operator $D_1 := \varphi_1^{-1} \circ D_{\mathfrak{W}} \circ \varphi_1$ on $\mathbb{R}[[\mathfrak{B} \times \mathfrak{S}]]$ is strongly linear, where $D_{\mathfrak{W}} := D|_{\mathbb{R}[[\mathfrak{W}]]}\colon \mathbb{R}[[\mathfrak{W}]] \to \mathbb{R}[[\mathfrak{W}]]$. By Lemma 4.15 it suffices to prove that the operator $\int_1 := \varphi_1^{-1} \circ \int_{\mathfrak{W}} \circ \varphi_1$ on $\mathbb{R}[[\mathfrak{B} \times \mathfrak{S}]]$ is strongly linear, where $\int_{\mathfrak{W}} := \int|_{\mathbb{R}[[\mathfrak{W}]]}\colon \mathbb{R}[[\mathfrak{W}]] \to \mathbb{R}[[\mathfrak{W}]]$. Since $1 \notin \mathfrak{W}$, the operators $D_{\mathfrak{W}}$ and $\int_{\mathfrak{W}}$ on $\mathbb{R}[[\mathfrak{W}]]$ are mutually inverse, and hence the operators D_1 and $\int_1$ on $\mathbb{R}[[\mathfrak{B} \times \mathfrak{S}]]$ are mutually inverse.

For $t \in C^\times \cdot \mathfrak{S}$, let Δt and $\mathrm{I}t$ be the dominant term of the series t' and $\int t$ in $C[[\mathfrak{R}]]$, respectively, so $\Delta t, \mathrm{I}t \in C^\times \cdot \mathfrak{S}$ by Lemma 4.16. By the rules on $\succ_{\mathfrak{N}}$ listed earlier, if $t_1, t_2 \in C^\times \cdot \mathfrak{S}$ satisfy $t_1 \succ_{\mathfrak{N}} t_2$, then $\Delta t_1 \succ_{\mathfrak{N}} \Delta t_2$ and $\mathrm{I}t_1 \succ_{\mathfrak{N}} \mathrm{I}t_2$. Moreover, the maps $\mathrm{I}\colon C^\times \cdot \mathfrak{S} \to C^\times \cdot \mathfrak{S}$ and $\Delta\colon C^\times \cdot \mathfrak{S} \to C^\times \cdot \mathfrak{S}$ are mutually inverse, and $\varphi_1(\mathfrak{B} \times \mathfrak{S}) \subseteq C^\times \cdot \mathfrak{S} \subseteq \mathbb{R}[[\mathfrak{W}]]$. Now let

$$\begin{aligned} \Delta_1 &:= \varphi_1^{-1} \circ \Delta \circ \left(\varphi_1|_{\mathfrak{B}\times\mathfrak{S}}\right) \ :\ \mathfrak{B} \times \mathfrak{S} \to \mathbb{R}[[\mathfrak{B} \times \mathfrak{S}]], \\ \mathrm{I}_1 &:= \varphi_1^{-1} \circ \mathrm{I} \circ \left(\varphi_1|_{\mathfrak{B}\times\mathfrak{S}}\right) \ :\ \mathfrak{B} \times \mathfrak{S} \to \mathbb{R}[[\mathfrak{B} \times \mathfrak{S}]]. \end{aligned}$$

Then for $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathfrak{B} \times \mathfrak{S}$ we have

$$\mathfrak{v}_1 \succ^* \mathfrak{v}_2 \quad \Longrightarrow \quad \operatorname{supp} \Delta_1 \mathfrak{v}_1 \succ^* \operatorname{supp} \Delta_1 \mathfrak{v}_2,\ \operatorname{supp} \mathrm{I}_1 \mathfrak{v}_1 \succ^* \operatorname{supp} \mathrm{I}_1 \mathfrak{v}_2.$$

Hence the maps Δ_1, I_1 are noetherian, so they extend uniquely to strongly linear operators on $\mathbb{R}[[\mathfrak{B}\times\mathfrak{S}]]$. These extensions, again denoted by Δ_1 and I_1, respectively, are mutually inverse by [17], Proposition 3.10, because Δ and I are.

Now consider the strongly linear operator

$$\Phi := (D_1 - \Delta_1)\circ \mathrm{I}_1 = D_1\mathrm{I}_1 - \mathrm{Id}$$

on $\mathbb{R}[[\mathfrak{B}\times\mathfrak{S}]]$. Using

$$D_1\mathrm{I}_1|_{\mathfrak{B}\times\mathfrak{S}} \;=\; \varphi_1^{-1}\circ(D_{\mathfrak{W}}\circ \mathrm{I})\circ(\varphi_1|_{\mathfrak{B}\times\mathfrak{S}})$$

we obtain $\operatorname{supp}\Phi(\mathfrak{v}) \prec^* \mathfrak{v}$ for $\mathfrak{v}\in\mathfrak{B}\times\mathfrak{S}$. Hence by Corollary 1.4, the operator $\mathrm{Id}+\Phi = D_1\mathrm{I}_1$ on $\mathbb{R}[[\mathfrak{B}\times\mathfrak{S}]]$ is bijective with strongly linear inverse. Thus the operator $\mathrm{I}_1\circ(\mathrm{Id}+\Phi)^{-1}$ on $\mathbb{R}[[\mathfrak{B}\times\mathfrak{S}]]$ is strongly linear. Finally, note that

$$D_1\circ \mathrm{I}_1\circ(\mathrm{Id}+\Phi)^{-1} = D_1\circ \mathrm{I}_1\circ(D_1I_1)^{-1} = \mathrm{Id},$$

so $\int_1 = D_1^{-1} = \mathrm{I}_1\circ(\mathrm{Id}+\Phi)^{-1}$, and thus $\int_1$ is strongly linear. □

5. Transseries of decay > 1

In this section we extend $\mathbb{L}_1$ to a Liouville closed H-subfield $\mathbb{T}_1$ of $\mathbb{R}[[\mathfrak{T}]]$ by first extending $\mathbb{L}_1$ to a real closed H-subfield $\mathbb{S}$ of $\mathbb{R}[[\mathfrak{T}]]$ that is closed under taking logarithms of positive elements, and then closing off $\mathbb{S}$ under downward shifts. The H-field $\mathbb{T}_1$ will satisfy the requirements on K in the Theorem stated in the introduction.

Construction of $\mathbb{S}$. The convex subgroup

$$\mathfrak{T}^\flat = \{\mathfrak{n}\in\mathfrak{T} : \mathfrak{n} \prec\!\!\!\prec e^x\}$$

of the ordered group $\mathfrak{T}$ is closed under $\mathbb{R}$-powers. Note that $\mathfrak{L}\subseteq\mathfrak{T}^\flat$. We call $\mathfrak{T}^\flat$ the *flat part of* $\mathfrak{T}$. Its steep supplement (as defined in the previous section) is the subgroup

$$\mathfrak{T}^\sharp = \{g\in\mathfrak{T} : \operatorname{supp}\log g \succcurlyeq x\}$$

of $\mathfrak{T}$, called the *steep part of* $\mathfrak{T}$. (See Examples 4.1 and 4.10.) We apply here Section 4 to $\mathfrak{M}=\mathfrak{T}$, and accordingly identify $\mathbb{R}[[\mathfrak{T}]]$ and $\mathbb{R}[[\mathfrak{T}^\flat]][[\mathfrak{T}^\sharp]]$. Every

$$f = \sum_{\mathfrak{m}\in\mathfrak{T}} f_{\mathfrak{m}}\mathfrak{m} \in \mathbb{R}[[\mathfrak{T}]]$$

can be written as

$$f = \sum_{\mathfrak{r}\in\mathfrak{T}^\sharp} f^\flat_{\mathfrak{r}}\mathfrak{r},$$

where the coefficients

$$f^\flat_{\mathfrak{r}} := \sum_{\mathfrak{n}\in\mathfrak{T},\, \mathfrak{n}\prec\!\!\!\prec e^x} f_{\mathfrak{n}\mathfrak{r}}\mathfrak{n}$$

are series in $\mathbb{R}[[\mathfrak{T}^\flat]]$. (In the notation of Section 4, we have $f^\flat_{\mathfrak{r}} = f_{\mathfrak{T}^\flat,\mathfrak{r}}$.) We may also decompose f as

$$f = f^{\Uparrow} + f^{\equiv} + f^{\Downarrow}, \tag{5.1}$$

where, with $\mathfrak{m}$ ranging over $\mathfrak{T}$,

$$\begin{aligned} f^{\Uparrow} &:= \sum_{\mathfrak{m}\succ 1,\, \mathfrak{m}\not\succcurlyeq e^x} f_{\mathfrak{m}}\mathfrak{m}; \\ f^{\equiv} &:= \sum_{\mathfrak{m}\prec\!\!\prec e^x} f_{\mathfrak{m}}\mathfrak{m}; \\ f^{\Downarrow} &:= \sum_{\mathfrak{m}\prec 1,\, \mathfrak{m}\not\succcurlyeq e^x} f_{\mathfrak{m}}\mathfrak{m}. \end{aligned}$$

Put $\mathbb{S}_0 := \mathbb{L}_1$, the latter as defined in Section 3. So $\mathbb{S}_0 \subseteq \mathbb{R}[[\mathfrak{T}_0]] \subseteq \mathbb{R}[[\mathfrak{T}^\flat]]$. Inductively, given the subfield $\mathbb{S}_n$ of $\mathbb{R}[[\mathfrak{T}_n]]$, we let $\mathbb{S}_{n+1}$ be the subfield of $\mathbb{R}[[\mathfrak{T}_{n+1}]]$ consisting of those $f \in \mathbb{R}[[\mathfrak{T}]]$ such that $f_{\mathfrak{r}}^\flat \in \mathbb{L}_1$ and $\log \mathfrak{r} \in \mathbb{S}_n^\uparrow$ for all $\mathfrak{r} \in \operatorname{supp}_{e^x} f$, that is, with $C := \mathbb{R}[[\mathfrak{T}^\flat]]$:

$$\mathbb{S}_{n+1} = \mathbb{L}_1[[\mathfrak{U}_{n+1}]] \subseteq C[[\mathfrak{T}^\sharp]]$$

where

$$\mathfrak{U}_{n+1} := \mathfrak{T}^\sharp \cap \exp(\mathbb{S}_n^\uparrow) = \exp\bigl(\mathbb{S}_n \cap \mathbb{R}[[\mathfrak{T}_n^{\succ x}]]\bigr),$$

a subgroup of $\mathfrak{T}^\sharp \cap \mathfrak{T}_{n+1}$ closed under $\mathbb{R}$-powers. It follows that $\mathbb{S}_{n+1} \subseteq \mathbb{R}[[\mathfrak{T}_{n+1}]]$. It is convenient to define $\mathfrak{R}_0 := \{1\} \subseteq \mathfrak{T}_0$.

Example 5.1. We have $\mathfrak{U}_1 = \exp\bigl(\mathbb{L}_1 \cap \mathbb{R}[[\mathfrak{L}^{\succ x}]]\bigr)$. Therefore $e^{x^2} \in \mathbb{S}_1$, but $e^{x^2}\!\downarrow = e^{(\log x)^2} \notin \mathbb{S}_1$.

Lemma 5.2. *Each $\mathbb{S}_n$ is a real closed subfield of $\mathbb{T}$, and $\mathfrak{U}_n \subseteq \mathfrak{U}_{n+1}$ for all n. (Hence $\mathbb{S}_n \subseteq \mathbb{S}_{n+1}$ for all n.)*

Proof. The first statement follows from the remarks at the beginning of Section 3 and Lemma 1.6. We show the other statement by induction on n. The case $n = 0$ being clear, suppose that $\mathfrak{U}_n \subseteq \mathfrak{U}_{n+1}$. Then

$$\mathbb{S}_n = \mathbb{L}_1[[\mathfrak{U}_n]] \subseteq \mathbb{L}_1[[\mathfrak{U}_{n+1}]] = \mathbb{S}_{n+1}$$

and thus

$$\mathfrak{U}_{n+1} = \mathfrak{T}^\sharp \cap \exp(\mathbb{S}_n^\uparrow) \subseteq \mathfrak{T}^\sharp \cap \exp(\mathbb{S}_{n+1}^\uparrow) = \mathfrak{U}_{n+2}$$

as required. ☐

We let $\mathbb{S}$ be the union of the increasing sequence $\mathbb{S}_0 \subseteq \mathbb{S}_1 \subseteq \cdots$ of real closed subfields of $\mathbb{T}$. Then $\mathbb{S}$ is a real closed subfield of $\mathbb{T}$. Moreover:

Lemma 5.3. $\log(\mathbb{S}_n^{>0}) \subseteq \mathbb{S}_n$ *for every n. (Hence* $\log(\mathbb{S}^{>0}) \subseteq \mathbb{S}$.)

Proof. The case $n = 0$ is discussed at the beginning of Section 3. Suppose $n > 0$. Every positive $f \in \mathbb{S}_n$ may be written in the form

$$f = g \cdot \mathfrak{u} \cdot (1 + \varepsilon)$$

where $0 < g \in \mathbb{L}_1$, $\mathfrak{u} \in \mathfrak{U}_n \subseteq \exp(\mathbb{S}_{n-1}^\uparrow)$, and $\varepsilon \prec_{e^x} 1$. We get

$$\log f = \log g + \log \mathfrak{u} + \log(1 + \varepsilon).$$

We have $\log g \in \mathbb{L}_1$ and (since $\varepsilon \prec 1$)

$$\log(1+\varepsilon) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \varepsilon^k \in \mathbb{S}_n.$$

Moreover $\log \mathfrak{u} \in \mathbb{S}_{n-1}$, thus $\log \mathfrak{u} \in \mathbb{S}_n$ by Lemma 5.2. Hence $\log f \in \mathbb{S}_n$. ☐

We now put $A_n := \mathbb{S}_n^\uparrow$, $\mathfrak{M}_{n+1} := \exp(A_n)$ for every n, and $\mathfrak{M}_0 := \mathfrak{L}$. Each A_n is an $\mathbb{R}$-linear subspace of $\mathbb{R}[[\mathfrak{T}_n]]$, and $\mathfrak{M}_n$ is a subgroup of $\mathfrak{T}_n$ closed under $\mathbb{R}$-powers. Here are some more properties of $\mathbb{S}_n$, A_n and $\mathfrak{M}_n$. A subset A of $\mathbb{R}[[\mathfrak{T}]]$ is said to be *closed under subseries* if for every $f = \sum_{\mathfrak{m}\in\mathfrak{T}} f_\mathfrak{m}\mathfrak{m} \in A$ the subseries $f|_\mathfrak{S} := \sum_{\mathfrak{m}\in\mathfrak{S}} f_\mathfrak{m}\mathfrak{m}$ is in A, for any subset $\mathfrak{S}$ of $\mathfrak{T}$.

Lemma 5.4. *For every n we have:*

1. $\mathbb{S}_n \subseteq \mathbb{R}[[\mathfrak{M}_n]]$. (*Hence* $A_n \subseteq \mathbb{R}[[\mathfrak{M}_n]]^\uparrow$.)
2. $\mathbb{S}_n$ *is closed under subseries.* (*Hence* A_n *is closed under subseries.*)
3. $\log \mathfrak{M}_n \subseteq A_n$. (*Hence* $\mathfrak{M}_n \subseteq \mathfrak{M}_{n+1}$.)
4. $\mathbb{S}_n\!\uparrow \subseteq \mathbb{S}_{n+1}$. (*Hence* $\mathfrak{M}_n\!\uparrow \subseteq \mathfrak{M}_{n+1}$.)

Proof. Parts (1)–(3) are obvious for $n = 0$. For the case $n = 0$ of (4) note first that $\mathfrak{L}\!\uparrow \subseteq \mathfrak{L}\cdot(\exp x)^\mathbb{R}$ with $\mathfrak{L}\cap(\exp x)^\mathbb{R} = \{1\}$. Moreover, if a subset $\mathfrak{S}$ of $\mathfrak{L}$ has decay > 1 and $\mathfrak{S}\!\uparrow \subseteq \mathfrak{L}\cdot(\exp x)^\beta$ with $\beta\in\mathbb{R}$, then $\pi(\mathfrak{S}\!\uparrow)$ has decay > 1, where $\pi\colon \mathfrak{L}\cdot(\exp x)^\mathbb{R} \to \mathfrak{L}$ is given by $\mathfrak{l}\cdot(\exp x)^\alpha \mapsto \mathfrak{l}$ for $\mathfrak{l}\in\mathfrak{L}$, $\alpha\in\mathbb{R}$. Hence $\mathbb{L}_1\!\uparrow \subseteq \mathbb{L}_1[[(\exp x)^\mathbb{R}]] \subseteq \mathbb{S}_1$ as required.

Let now $n > 0$. For (1) note that

$$\mathfrak{L} = \exp\log\mathfrak{L} \subseteq \exp(\mathbb{L}_1^\uparrow) \subseteq \exp(\mathbb{S}_{n-1}^\uparrow), \qquad \mathfrak{U}_n \subseteq \exp(\mathbb{S}_{n-1}^\uparrow),$$

hence

$$\mathbb{S}_n = \mathbb{L}_1[[\mathfrak{U}_n]] \subseteq \mathbb{R}[[\mathfrak{L}\cdot\mathfrak{U}_n]] \subseteq \mathbb{R}[[\exp(\mathbb{S}_{n-1}^\uparrow)]] = \mathbb{R}[[\mathfrak{M}_n]].$$

For (2) let $f = \sum_{\mathfrak{u}\in\mathfrak{U}_n} f_\mathfrak{u}^\flat\mathfrak{u} \in \mathbb{S}_n$, so $f_\mathfrak{u}^\flat \in \mathbb{L}_1$ for all $\mathfrak{u}$. Then for any subset $\mathfrak{S}$ of $\mathfrak{T}$ we have

$$f|_\mathfrak{S} = \sum_{\mathfrak{u}\in\mathfrak{U}_n} (f_\mathfrak{u}^\flat)|_{\mathfrak{S}_\mathfrak{u}}\mathfrak{u} \in \mathbb{S}_n,$$

where $\mathfrak{S}_\mathfrak{u} := \{\mathfrak{n}\in\mathfrak{T}^\flat : \mathfrak{n}\mathfrak{u}\in\mathfrak{S}\}$ for $\mathfrak{u}\in\mathfrak{U}_n$. For part (3) we have, by Lemma 5.2,

$$\log\mathfrak{M}_n = A_{n-1} = \mathbb{S}_{n-1}^\uparrow \subseteq \mathbb{S}_n^\uparrow = A_n$$

as required. For (4), we may assume inductively that $\mathbb{S}_{n-1}\!\uparrow \subseteq \mathbb{S}_n$. Since $\mathfrak{T}_{n-1}\!\uparrow \subseteq \mathfrak{T}_n$ we get

$$\mathfrak{U}_n\!\uparrow = \exp\left(\mathbb{S}_{n-1}\cap\mathbb{R}[[\mathfrak{T}_{n-1}^{\succ x}]]\right)\uparrow \subseteq \exp\left(\mathbb{S}_n\cap\mathbb{R}[[\mathfrak{T}_n^{\succ \exp x}]]\right) \subseteq \mathfrak{U}_{n+1}.$$

Together with $\mathbb{L}_1\!\uparrow \subseteq \mathbb{L}_1[[(\exp x)^\mathbb{R}]]$ this yields $\mathbb{S}_n\!\uparrow = (\mathbb{L}_1\!\uparrow)[[\mathfrak{U}_n\!\uparrow]] \subseteq \mathbb{S}_{n+1}$. □

We let $\mathfrak{M}$ be the union of the increasing sequence $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \cdots$ of ordered subgroups of $\mathfrak{T}$. Then $\mathfrak{M}$ is an ordered subgroup of $\mathfrak{T}$, and $\mathbb{S}$ is an ordered subfield of $\mathbb{R}[[\mathfrak{M}]]$. Note that the $\mathfrak{M}_n$ satisfy conditions (M1)–(M4) of the previous section. We have $\mathbb{S}\cap\mathbb{L} = \mathbb{L}_1$, hence $\exp(\Lambda)\notin\mathfrak{M}$, by part (3) of Lemma 5.4 and Example 3.2.

Proposition 5.5. *For every n, the field $\mathbb{S}_n$ is closed under differentiation.*

Proof. We proceed by induction on n. We have already dealt with the case $n = 0$ in Proposition 3.1. Let $f = \sum_{\mathfrak{u}\in\mathfrak{U}_{n+1}} f_\mathfrak{u}^\flat\mathfrak{u} \in \mathbb{S}_{n+1}$. By Theorem 4.13, the family $\left((f_\mathfrak{u}^\flat\mathfrak{u})'\right)_{\mathfrak{u}\in\mathfrak{U}_{n+1}}$ in $\mathbb{R}[[\mathfrak{T}_{n+1}]]$ is flatly noetherian. Hence for any $\mathfrak{s}\in\mathfrak{T}_{n+1}^\sharp$ the sum

$$\sum_{\mathfrak{u}\in\mathfrak{U}_{n+1}} \left[\left((f_\mathfrak{u}^\flat)' + f_\mathfrak{u}^\flat\mathfrak{u}^\dagger\right)\mathfrak{u}\right]_\mathfrak{s}^\flat$$

has only finitely many non-zero terms and equals $(f')_\mathfrak{s}^\flat$. Let $\mathfrak{u}\in\mathfrak{U}_{n+1}$ and $\mathfrak{s}\in\mathfrak{T}_{n+1}^\sharp$. By the induction hypothesis we have $\mathfrak{u}^\dagger\in\mathbb{S}_n$, hence $(\mathfrak{u}^\dagger)_{\mathfrak{s}/\mathfrak{u}}^\flat \in \mathbb{L}_1$. By

Proposition 3.1 we get $(f_{\mathfrak{u}}^{\flat})' \in \mathbb{L}_1$. Therefore $(f')_{\mathfrak{s}}^{\flat} \in \mathbb{L}_1$. It follows that $f \in \mathbb{S}_{n+1}$ as required. $\square$

Construction of $\mathbb{T}_1$. We have $\mathbb{S}{\downarrow}^k = (\mathbb{S}{\uparrow}){\downarrow}^{k+1} \subseteq \mathbb{S}{\downarrow}^{k+1}$ for every $k \in \mathbb{N}$, by Lemma 5.4, (4). We let $\mathbb{T}_1$ be the union of the increasing sequence

$$\mathbb{S} \subseteq \mathbb{S}{\downarrow} \subseteq \mathbb{S}{\downarrow}^2 \subseteq \cdots \subseteq \mathbb{S}{\downarrow}^k \subseteq \cdots$$

of real closed subfields of $\mathbb{T}$. The elements of the real closed subfield $\mathbb{T}_1$ of $\mathbb{T}$ are called *transseries of decay* > 1. The field $\mathbb{T}_1$ is closed under upward and downward shift: if $f \in \mathbb{T}_1$, then $f{\uparrow}, f{\downarrow} \in \mathbb{T}_1$. We have $\mathbb{L}_1 \subseteq \mathbb{T}_1$; in fact:

Lemma 5.6. $\mathbb{L}_1 = \mathbb{T}_1 \cap \mathbb{L}$.

Proof. Suppose $f \in \mathbb{T}_1 \cap \mathbb{L}$; so $f{\uparrow}^k \in \mathbb{S}_n$ where $k, n \in \mathbb{N}$; we claim that $f \in \mathbb{L}_1$. The case $k = 0$ being trivial, we may assume $k > 0$. Then

$$f{\uparrow}^k \in \mathbb{L}[[(\exp x)^{\mathbb{R}} \cdots (\exp_k x)^{\mathbb{R}}]] \cap \mathbb{S}_n \subseteq \mathbb{L}_1[[(\exp x)^{\mathbb{R}} \cdots (\exp_k x)^{\mathbb{R}}]],$$

where $\exp_m x = x{\uparrow}^m$ for all m. Hence f can be written in the form

$$f = \sum_{\alpha \in \mathbb{R}^k} \ell^{\alpha} \cdot (g_{\alpha} \circ \ell_k),$$

where $g_{\alpha} \in \mathbb{L}_1$ and $\ell^{\alpha} = \ell_0^{\alpha_0} \cdots \ell_{k-1}^{\alpha_{k-1}}$ for $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in \mathbb{R}^k$. By Lemma 3.4, we get $f \in \mathbb{L}_1$ as desired. $\square$

If A is a subset of $\mathbb{R}[[\mathfrak{T}]]$ which is closed under subseries, then so is $A{\downarrow}$, since $(f{\downarrow})|_{\mathfrak{S}} = (f|_{\mathfrak{S}\uparrow}){\downarrow}$, for any $f \in A$ and $\mathfrak{S} \subseteq \mathfrak{T}$. By induction on k it follows that each subfield $\mathbb{S}{\downarrow}^k$ of $\mathbb{R}[[\mathfrak{T}]]$ is closed under subseries. Hence $\mathbb{T}_1$ is closed under subseries.

Proof of the main theorem. In the remainder of this section, we show that $K = \mathbb{T}_1$ has the properties of the main theorem in the introduction.

Proposition 5.7. *The subfield $\mathbb{T}_1$ of $\mathbb{T}$ is closed under exponentiation and taking logarithms of positive elements.*

Proof. Since

$$\log(f{\downarrow}^m) = (\log f){\downarrow}^m \quad \text{for all } m \text{ and all } f \in \mathbb{S}^{>0},$$

Lemma 5.3 yields that $\mathbb{T}_1$ is closed under taking logarithms. Similarly,

$$\exp(f{\downarrow}^m) = (\exp f){\downarrow}^m \quad \text{for all } m \text{ and all } f \in \mathbb{S}.$$

Hence as to exponentiation, it suffices to prove that $\exp f \in \mathbb{T}_1$ for all $f \in \mathbb{S}$. Let $f \in \mathbb{S}_n$, and decompose f as in (5.1): $f = f^{\Uparrow} + f^{\equiv} + f^{\Downarrow}$, so

$$\exp f = (\exp f^{\Uparrow}) \cdot (\exp f^{\equiv}) \cdot (\exp f^{\Downarrow}).$$

Since $f^{\Downarrow} \in \mathbb{T}^{\prec 1}$ we get

$$\exp f^{\Downarrow} = \sum_{n=0}^{\infty} \frac{(f^{\Downarrow})^n}{n!} \in \mathbb{S}_n.$$

We have

$$f^{\Uparrow} = \sum_{\mathfrak{m} \succ 1, \mathfrak{m} \not\succcurlyeq e^x} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{S}_n \cap \mathbb{R}[[\mathfrak{T}_n^{\not\succcurlyeq x}]],$$

hence $\exp f^{\Uparrow} \in \mathfrak{U}_{n+1} \subseteq \mathbb{S}_{n+1}$. It remains to prove that $\exp f \in \mathbb{T}_1$ for all $f \in \mathbb{L}_1$. So let $f \in \mathbb{L}_1$. From $1 \notin \widehat{\operatorname{supp} f} \subseteq \mathfrak{L}$ we obtain $k \in \mathbb{N}$ such that $\ell_k \preccurlyeq\!\!\!\prec \mathfrak{m}$ for all

$\mathfrak{m} \in \operatorname{supp} f \backslash\{1\}$. Then $g^{\equiv} \in \mathbb{R}$ for $g = f{\uparrow}^{k+1}$, hence $\exp g \in \mathbb{S}$ by what we have shown above. We conclude that $\exp f = (\exp g){\downarrow}^{k+1} \in \mathbb{T}_1$. $\square$

Since $(f{\downarrow})' = (f'{\downarrow}) \cdot x^{-1}$ for all $f \in \mathbb{T}$, Proposition 5.5 yields:

Corollary 5.8. *The subfield $\mathbb{T}_1$ of $\mathbb{T}$ is closed under differentiation.* (*Hence $\mathbb{T}_1$ is an H-subfield of $\mathbb{T}$.*) $\square$

To prove that $\mathbb{T}_1$ is closed under integration, we first establish some auxiliary facts. Recall that $\mathbb{R}[[\mathfrak{M}]]$ is closed under differentiation and that $\exp(\Lambda) \notin \mathfrak{M}$. Hence $\mathbb{R}[[\mathfrak{M}]]$ is closed under integration.

In the next lemma we fix $n > 0$. We have the following inclusions:

$$\mathfrak{L} \cdot \mathfrak{U}_n \subseteq \mathfrak{M}_n \subseteq \mathbb{S}_n \subseteq \mathbb{L}[[\mathfrak{U}_n]] = \mathbb{R}[[\mathfrak{L} \cdot \mathfrak{U}_n]] \subseteq \mathbb{R}[[\mathfrak{M}_n]].$$

The subfield $\mathbb{L}[[\mathfrak{U}_n]]$ of $\mathbb{R}[[\mathfrak{M}]]$ is closed under differentiation by Proposition 5.5, and closed under integration by the argument used to prove Lemma 4.2. Note that $\log \mathfrak{s} \in \mathbb{S}_{n-1} \subseteq \mathbb{L}[[\mathfrak{U}_n]]$ for all $\mathfrak{s} \in \mathfrak{U}_n$. In the next lemma we also fix a monomial $\mathfrak{u} \in \mathfrak{U}_n \setminus \{1\}$ and put

$$\mathfrak{S} := \left\{ \mathfrak{s} \in \mathfrak{U}_n : \mathfrak{s}^\dagger \prec_{e^x} \mathfrak{u}^\dagger \right\}, \tag{5.2}$$

a convex subgroup of $\mathfrak{U}_n$ closed under $\mathbb{R}$-powers.

Lemma 5.9. *The subfield $\mathbb{L}[[\mathfrak{S}]]$ of $\mathbb{L}[[\mathfrak{U}_n]]$ is closed under differentiation. Also, if $\mathfrak{u}^\dagger \succ_{e^x} 1$, then $\mathfrak{u}^\dagger \in \mathbb{L}[[\mathfrak{S}]]$.*

Proof. The first part will follow if $\mathfrak{s}' \in \mathbb{L}[[\mathfrak{S}]]$ for all $\mathfrak{s} \in \mathfrak{S}$. So let $\mathfrak{s} \in \mathfrak{S}$; we distinguish two cases:

(1) $\mathfrak{s}^\dagger \succ_{e^x} 1$. Then $\mathfrak{s} \notin \mathfrak{T}^\flat$, hence $\mathfrak{s} = e^\varphi$ with $\operatorname{supp} \varphi' \prec\!\!\prec \mathfrak{s}$ (by Lemma 4.8 applied to $\mathfrak{m} \in \operatorname{supp} \varphi$). Using $\varphi' = \mathfrak{s}^\dagger$, this yields $\mathfrak{m}^\dagger \asymp \mathfrak{s}^\dagger$ for every $\mathfrak{m} \in \operatorname{supp} \mathfrak{s}'$. Let $\mathfrak{v} \in (\operatorname{supp}_{e^x} \mathfrak{s}') \setminus \{1\}$, so $\mathfrak{v} \asymp_{e^x} \mathfrak{m}$ with $\mathfrak{m} \in \operatorname{supp} \mathfrak{s}'$. Then $\mathfrak{v}^\dagger \asymp_{e^x} \mathfrak{m}^\dagger \asymp \mathfrak{s}^\dagger \prec_{e^x} \mathfrak{u}^\dagger$, hence $\mathfrak{v} \in \mathfrak{S}$, as desired.

(2) $\mathfrak{s}^\dagger \preccurlyeq_{e^x} 1$. Then $\log \mathfrak{s} \in \mathbb{L}[[\mathfrak{U}_n]] \cap \mathbb{R}[[\mathfrak{T}^\flat]] = \mathbb{L}$ (by Lemma 4.9) and thus $\mathfrak{s}' = (\log \mathfrak{s})' \cdot \mathfrak{s} \in \mathbb{L}[[\mathfrak{S}]]$.

Suppose that $\mathfrak{u}^\dagger \succ_{e^x} 1$. Then $\log \mathfrak{u} \succ_{e^x} 1$ by Lemma 4.9, hence

$$(\log \mathfrak{u})^\dagger = \frac{\mathfrak{u}^\dagger}{\log \mathfrak{u}} \prec_{e^x} \mathfrak{u}^\dagger.$$

Therefore, if $\mathfrak{v} \in \operatorname{supp}_{e^x} \log \mathfrak{u}$, then $\mathfrak{v}^\dagger \preccurlyeq_{e^x} (\log \mathfrak{u})^\dagger \prec_{e^x} \mathfrak{u}^\dagger$, hence $\mathfrak{v} \in \mathfrak{S}$. Thus $\log \mathfrak{u} \in \mathbb{L}[[\mathfrak{S}]]$, and since $\mathbb{L}[[\mathfrak{S}]]$ is closed under differentiation, we get $\mathfrak{u}^\dagger \in \mathbb{L}[[\mathfrak{S}]]$. $\square$

Lemma 5.10. *Let $f \in \mathbb{S}$ with $\mathfrak{u}^\dagger \succ_{e^x} 1$ for all $\mathfrak{u} \in (\operatorname{supp}_{e^x} f) \backslash \{1\}$. Then $\int f \in \mathbb{S}$.*

Proof. We already know that $\mathbb{S}_0 = \mathbb{L}_1$ is closed under distinguished integration, by Proposition 3.5. So we may assume that $1 \notin \operatorname{supp}_{e^x} f$ by passing from f to $f - f_1^\flat$. Take $n > 0$ such that $f \in \mathbb{S}_n$. We shall prove that $\int f \in \mathbb{S}_n$. We have

$$f = \sum_{u \in \mathfrak{U}_n} f_\mathfrak{u}^\flat \mathfrak{u} \in \mathbb{L}_1[[\mathfrak{U}_n]] = \mathbb{S}_n.$$

Put $\mathfrak{N} := \mathfrak{M} \cap \mathfrak{T}^\flat$, a convex subgroup of $\mathfrak{M}$; note that $\mathbb{L} \subseteq \mathbb{R}[[\mathfrak{N}]]$. Let $\mathfrak{R}$ be the steep supplement of $\mathfrak{N}$ in $\mathfrak{M}$. The definitions of $\mathfrak{T}^\sharp$ and $\mathfrak{R}$ easily yield that $\mathfrak{M} \cap \mathfrak{T}^\sharp \subseteq \mathfrak{R}$; hence $\mathfrak{U}_n \subseteq \mathfrak{R}$. Therefore, the family $(f_\mathfrak{u}^\flat \mathfrak{u})_{\mathfrak{u} \in \mathfrak{U}_n}$ in $\mathbb{R}[[\mathfrak{M}]]$ is flatly noetherian with respect to $\mathfrak{N}$, with sum f. Thus by Theorem 4.14, the family $\left(\int f_\mathfrak{u}^\flat \mathfrak{u}\right)_{\mathfrak{u} \in \mathfrak{U}_n}$ in $\mathbb{R}[[\mathfrak{M}]]$

is also flatly noetherian, with sum $\int f$. Fix any $g \in \mathbb{L}_1$ and $\mathfrak{u} \in \mathfrak{U}_n$ with $\mathfrak{u}^\dagger \succ_{e^x} 1$; it suffices to show that then $\int g\mathfrak{u} \in \mathbb{S}_n = \mathbb{L}_1[[\mathfrak{U}_n]]$. Put $h := \frac{1}{\mathfrak{u}} \int g\mathfrak{u} \in \mathbb{L}[[\mathfrak{U}_n]]$; it remains to show that $h \in \mathbb{L}_1[[\mathfrak{U}_n]]$. Note that

$$h + (h'/\mathfrak{u}^\dagger) = g/\mathfrak{u}^\dagger.$$

Let $\mathfrak{S}$ be as in (5.2). Take a basis $\mathfrak{C}$ for the $\mathbb{R}$-vector space $\mathbb{L}$; extend $\mathfrak{C}$ to a basis $\mathfrak{B}$ for $\mathbb{R}[[\mathfrak{N}]]$, and let $\preccurlyeq^*$ be as in (4.3) and $\varphi\colon \mathbb{R}[[\mathfrak{B} \times \mathfrak{R}]] \to \mathbb{R}[[\mathfrak{M}]]$ as defined in Lemma 4.15. The map φ restricts to an $\mathbb{R}$-linear bijection

$$\varphi_1 : \mathbb{R}[[\mathfrak{C} \times \mathfrak{S}]] \to \mathbb{R}[[\mathfrak{L} \cdot \mathfrak{S}]] = \mathbb{L}[[\mathfrak{S}]].$$

By the previous lemma, the subfield $\mathbb{L}[[\mathfrak{S}]]$ of $\mathbb{L}[[\mathfrak{U}_n]]$ is closed under differentiation and contains $\mathfrak{u}^\dagger$. Hence the operator

$$\Phi\colon \mathbb{L}[[\mathfrak{U}_n]] \to \mathbb{L}[[\mathfrak{U}_n]], \quad y \mapsto y'/\mathfrak{u}^\dagger$$

maps $\mathbb{L}[[\mathfrak{S}]]$ to itself, and $(\mathrm{Id} + \Phi)(h) = g/\mathfrak{u}^\dagger$. By Theorem 4.13 the operator $\Phi_1 := \varphi_1^{-1} \circ \Phi \circ \varphi_1$ on $\mathbb{R}[[\mathfrak{C} \times \mathfrak{S}]]$ is strongly linear, and $\operatorname{supp} \Phi_1(\mathfrak{c}, \mathfrak{s}) \prec^* (\mathfrak{c}, \mathfrak{s})$ for all $(\mathfrak{c}, \mathfrak{s}) \in \mathfrak{C} \times \mathfrak{S}$. We now apply Corollary 1.4 with $\mathfrak{C} \times \mathfrak{S}$ in place of $\mathfrak{M}$, ordered by the restriction of $\preccurlyeq^*$ to $\mathfrak{C} \times \mathfrak{S}$, and Φ_1 in place of Φ. It follows that the family

$$\big((-1)^i \Phi^i(g/\mathfrak{u}^\dagger)\big)_{i \in \mathbb{N}}$$

in $\mathbb{L}[[\mathfrak{S}]]$ is flatly noetherian as a family in $\mathbb{R}[[\mathfrak{M}]]$, and that

$$h_1 := \sum_{i=0}^{\infty} (-1)^i \Phi^i(g/\mathfrak{u}^\dagger) \in \mathbb{L}[[\mathfrak{S}]]$$

satisfies

$$h_1 + (h_1'/\mathfrak{u}^\dagger) = g/\mathfrak{u}^\dagger = h + (h'/\mathfrak{u}^\dagger).$$

Hence $h = h_1 + c\mathfrak{u}^{-1}$ for some $c \in \mathbb{R}$. From $\Phi(\mathbb{L}_1[[\mathfrak{U}_n]]) \subseteq \mathbb{L}_1[[\mathfrak{U}_n]]$ we obtain that $\Phi^i(g/\mathfrak{u}^\dagger) \in \mathbb{L}_1[[\mathfrak{U}_n]]$ for all i. Hence $h_1 \in \mathbb{L}_1[[\mathfrak{U}_n]]$, and thus $h \in \mathbb{L}_1[[\mathfrak{U}_n]]$. □

Next we show that for suitable f the hypothesis in the last lemma is satisfied after a single upward shift:

Lemma 5.11. *For every $f \in \mathbb{S}$ with $f_1^\flat = 0$ and $\mathfrak{u} \in \operatorname{supp}_{e^x} f\!\uparrow$ we have $\mathfrak{u}^\dagger \succ_{e^x} 1$.*

Proof. Suppose $f \in \mathbb{S}_n$, $f_1^\flat = 0$, $n > 0$. Then

$$f\!\uparrow = \sum_{1 \neq \mathfrak{s} \in \mathfrak{U}_n} (f_\mathfrak{s}^\flat)\!\uparrow \cdot \mathfrak{s}\!\uparrow$$

with $\operatorname{supp}_{e^x} (f_\mathfrak{s}^\flat)\!\uparrow \subseteq (\exp x)^{\mathbb{R}}$ for $1 \neq \mathfrak{s} \in \mathfrak{U}_n$. So it suffices to show for such $\mathfrak{s}$ that $(\mathfrak{s}\!\uparrow)^\dagger \succ_{e^x} 1$. Write $\mathfrak{s} = e^\varphi$ with $0 \neq \varphi \in \mathbb{S}_{n-1} \cap \mathbb{R}[[\mathfrak{T}_{n-1}^{\succcurlyeq x}]]$. Then $\mathfrak{d}(\varphi) \succcurlyeq x$ and hence $\mathfrak{d}(\varphi\!\uparrow) = \mathfrak{d}(\varphi)\!\uparrow \succcurlyeq e^x$. Therefore $\mathfrak{d}(\varphi\!\uparrow)' \succcurlyeq (e^x)' = e^x \succ_{e^x} 1$, so $(\mathfrak{s}\!\uparrow)^\dagger = (\varphi\!\uparrow)' \asymp \mathfrak{d}(\varphi\!\uparrow)' \succ_{e^x} 1$ as required. □

Proposition 5.12. *The H-subfield $\mathbb{T}_1$ of $\mathbb{T}$ is closed under integration.*

Proof. We claim that for each $k \in \mathbb{N}$ and $g \in \mathbb{S}\!\downarrow^k$ there is $f \in \mathbb{S}\!\downarrow^{k+1}$ such that $f' = g$. We proceed by induction on k. First, let $g \in \mathbb{S}$. By Proposition 3.5 we may assume that $g_1^\flat = 0$. Consider $G = (g\!\uparrow) \cdot e^x \in \mathbb{S}$. By the previous lemma, all $\mathfrak{u} \in (\operatorname{supp}_{e^x} G) \setminus \{1\}$ satisfy $\mathfrak{u}^\dagger \succ_{e^x} 1$. By Lemma 5.10, we get $\int G \in \mathbb{S}$ and hence $\int g = (\int G)\!\downarrow \in \mathbb{S}\!\downarrow$. This proves the case $k = 0$ of our claim.

For the induction step we consider an element of $\mathbb{S}{\downarrow}^{k+1}$, and write it as $g{\downarrow}$ with $g \in \mathbb{S}{\downarrow}^{k}$. Then $g \cdot e^x \in \mathbb{S}{\downarrow}^{k}$, so inductively we have an $f \in \mathbb{S}{\downarrow}^{k+1}$ with $f' = g \cdot e^x$. Then $(f{\downarrow})' = g{\downarrow}$, and $f{\downarrow} \in \mathbb{S}{\downarrow}^{k+2}$. □

We now have the main theorem from the introduction, with $K = \mathbb{T}_1$:

Corollary 5.13. *The H-subfield $\mathbb{T}_1$ of $\mathbb{T}$ is Liouville closed, and $\varrho \in \mathbb{T}_1$.*

Proof. Propositions 5.7 and 5.12 yield that $\mathbb{T}_1$ is Liouville closed; the second part follows from $\varrho \in \mathbb{L}_1 \subseteq \mathbb{T}_1$. □

6. Final Remarks

The differential polynomial $2Z'+Z^2$ (the "Schwarzian" in [4]) has a close connection to the second-order linear differential equation $Y'' = fY$ where f is an element of some H-field: whenever y is a non-zero solution to $Y'' = fY$, then $z = 2y^\dagger$ satisfies $2z' + z^2 = f$. The cut in $\mathbb{R}[[[x]]] = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$ determined by $\varrho := 2\lambda' + \lambda^2 \in \mathbb{L}$ can be used to describe for which $f \in \mathbb{R}[[[x]]]$ the linear differential equation $Y'' = fY$ has a non-zero solution in $\mathbb{R}[[[x]]]$; see [14]. (Likewise for the existence of solutions in finite-rank Hardy fields, [10].) See also [7] for some observations about the role of gaps in Hardy fields, and of the transseries Λ, in the theory of ordinary differential equations over o-minimal expansions of the real exponential field.

The transseries ϱ makes another appearance in Écalle [4]: *Lemme 7.4* says that for any non-constant differential polynomial $P(Z, Z', \dots, Z^{(n)}) \in \mathbb{R}\{Z\}$, the series $P(\lambda, \lambda', \dots, \lambda^{(n)}) \in \mathbb{L}$ has infinite support, and the sum of its first ω terms, after possibly discarding finitely many initial terms, either has the form

$$c\ell_0^{-e_0}\ell_1^{-e_1}\cdots\ell_{k-1}^{-e_{k-1}}\left(\lambda{\downarrow}^k\right) \quad \text{with } e_0 \geqslant e_1 \geqslant \cdots \geqslant e_{k-1} > 1$$

or

$$c\ell_0^{-e_0}\ell_1^{-e_1}\cdots\ell_{k-1}^{-e_{k-1}}\left(\varrho{\downarrow}^k\right) \quad \text{with } e_0 \geqslant e_1 \geqslant \cdots \geqslant e_{k-1} > 2,$$

where $c \in \mathbb{R}^\times$, $k \in \mathbb{N}$, and the e_i are integers.

Given a real number $r \geqslant 0$, we say that a subset $\mathfrak{S}$ of $\mathfrak{L}$ has decay $> r$ if for every $\mathfrak{m} = \ell_0^{\alpha_0}\ell_1^{\alpha_1}\cdots$ in $\widehat{\mathfrak{S}}$ (with $\alpha_k \in \mathbb{R}$ for all k) there exists k_0 such that $\alpha_k < -r$ for all $k \geqslant k_0$. Let $\mathbb{L}_r$ be the set of all $f \in \mathbb{L}$ such that $\operatorname{supp} f$ has decay $> r$. (So $\mathbb{L}_r \subseteq \mathbb{L}_s$ for $0 \leqslant s \leqslant r$.) We have $\lambda \in \mathbb{L}_r \setminus \mathbb{L}_1$ for all $0 \leqslant r < 1$ and $\varrho \in \mathbb{L}_s \setminus \mathbb{L}_2$ for $0 \leqslant s < 2$. As with $\mathbb{L}_1$, one can show that $\mathbb{L}_r$ is a differential subfield of $\mathbb{L}$, which is closed under integration if and only if $r \geqslant 1$. (For $0 \leqslant r < 1$ we have $\lambda \in \mathbb{L}_r$, but $\int \lambda = \Lambda \notin \mathbb{L}_r$.) For $r \geqslant 1$, carrying out the construction of $\mathbb{T}_1$ with $\mathbb{L}_r$ in place of $\mathbb{L}_1$ yields a Liouville closed H-subfield $\mathbb{T}_r$ of $\mathbb{T}$ which doesn't contain an element of the form $\lambda + \varepsilon$, where $\varepsilon \in \mathbb{R}[[\mathfrak{T}]]$ satisfies $\varepsilon \prec 1/(\ell_0\ell_1\cdots\ell_n)$ for all n.

By the above result of Écalle, λ does not satisfy any differential equation of the form $P(\lambda, \lambda', \dots, \lambda^{(n)}) = f$, where $P(Z, Z', \dots, Z^{(n)}) \in \mathbb{R}\{Z\}$ is non-constant and $f \in \mathbb{T}_r$ with $r > 1$. (We suspect that λ is differentially transcendental over $\mathbb{L}_r$, and hence over $\mathbb{T}_r$, for any $r > 1$.) In particular, our construction of a differentially algebraic, non-Liouvillian gap could not have been carried out with $\mathbb{T}_1$ replaced by $\mathbb{T}_r$ for any $r > 1$, even if we replace $2Z' + Z^2$ by another non-constant differential polynomial $P(Z, Z', \dots, Z^{(n)}) \in \mathbb{R}\{Z\}$.

Finally, let us mention that the Newton polygon method of [15] can be used to obtain Hardy field examples of the various possibilities for the appearance of gaps exhibited in this paper. We shall leave the details for another occasion.

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