ON A DIFFERENTIAL INTERMEDIATE VALUE PROPERTY

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Abstract. Liouville closed H-fields are ordered differential fields whose ordering and derivation interact in a natural way and where every linear differential equation of order 1 has a nontrivial solution. (The introduction gives a precise definition.) For a Liouville closed H-field K with small derivation we show: K has the Intermediate Value Property for differential polynomials iff K is elementarily equivalent to the ordered differential field of transseries. We also indicate how this applies to Hardy fields.

Introduction

Throughout this introduction K is an ordered differential field, that is, an ordered field equipped with a derivation $\partial\colon K \to K$. (We usually write f' instead of ∂f, for $f \in K$.) Its constant field

$$C \ := \ \{f \in K : \ f' = 0\}$$

yields the (convex) valuation ring

$$\mathcal{O} \ := \ \big\{f \in K : \ |f| \leqslant c \text{ for some } c \in C\big\}$$

of K, with maximal ideal

$$\scriptstyle\mathcal{O} \ := \ \big\{f \in K : \ |f| < c \text{ for all } c > 0 \text{ in } C\big\}.$$

(It may help to think of the elements of K as germs of real valued functions and of $f \in \mathcal{O}g$ and $f \in {\scriptstyle\mathcal{O}}g$ as $f = O(g)$ and $f = o(g)$, respectively.) The above definitions exhibit C, $\mathcal{O}$, and ${\scriptstyle\mathcal{O}}$ as definable in K in the sense of model theory.

Key example: the ordered differential field $\mathbb{T}$ of **transseries**, which contains $\mathbb{R}$ as an ordered subfield, and where $C = \mathbb{R}$. We refer to [3] for the rather elaborate construction of $\mathbb{T}$ and for any fact about $\mathbb{T}$ that gets mentioned without proof.

Other important examples are Hardy fields. (Hardy [6] proved a striking theorem on logarithmic-exponential functions. Bourbaki [5] put this into the general setting of what they called Hardy fields.) Here we can give a definition from scratch that doesn't take much space. Notation: $\mathcal{C}$ is the ring of germs at $+\infty$ of continuous real-valued functions on halflines $(a, +\infty)$, $a \in \mathbb{R}$. For $r = 1, 2, \dots$, let $\mathcal{C}^r$ be the subring of $\mathcal{C}$ consisting of the germs at $+\infty$ of r-times continuously differentiable real-valued functions on such halflines. This yields the subring

$$\mathcal{C}^{<\infty} \ := \ \bigcap_{r \in \mathbb{N}^{\geqslant 1}} \mathcal{C}^r$$

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of $\mathcal{C}$, and $\mathcal{C}^{<\infty}$ is naturally a *differential* ring. For a germ $f \in \mathcal{C}$ we let f also denote any real valued function representing this germ, if this causes no ambiguity. A real number is identified with the germ of the corresponding constant function: $\mathbb{R} \subseteq \mathcal{C}$.

A **Hardy field** is by definition a differential subfield of $\mathcal{C}^{<\infty}$. *Examples*:

$$\mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{R}(x), \quad \mathbb{R}(x, \mathrm{e}^x), \quad \mathbb{R}(x, \mathrm{e}^x, \log x), \quad \mathbb{R}(\Gamma, \Gamma', \Gamma'', \dots),$$

where x denotes the germ at $+\infty$ of the identity function on $\mathbb{R}$. All these are actually *analytic* Hardy fields, that is, its elements are germs of real analytic functions.

Let H be a Hardy field. Then H is an *ordered* differential field: for $f \in H$, either $f(x) > 0$ eventually (in which case we set $f > 0$), or $f(x) = 0$, eventually, or $f(x) < 0$, eventually; this is because $f \neq 0$ in H implies f has a multiplicative inverse in H, so f cannot have arbitrarily large zeros. Also, if $f' < 0$, then f is eventually strictly decreasing; if $f' = 0$, then f is eventually constant; if $f' > 0$, then f is eventually strictly increasing.

In order to state the main result of this paper we need a bit more terminology: an **H-field** is a K (that is, an ordered differential field) such that:

- for all $f \in K$, if $f > C$, then $f' > 0$;
- $\mathcal{O} = C + \scriptstyle{\mathcal{O}}$ (so C maps isomorphically onto the residue field $\mathcal{O}/\scriptstyle{\mathcal{O}}$).

We also say that K **has small derivation** if for all $f \in \scriptstyle{\mathcal{O}}$ we have $f' \in \scriptstyle{\mathcal{O}}$. Hardy fields have small derivation, and any Hardy field containing $\mathbb{R}$ is an H-field.

An H-field K is said to be **Liouville closed** if it is real closed and for every $f \in K$ there are $g, h \in K^\times$ such that $f = g' = h'/h$. The ordered differential field $\mathbb{T}$ is a Liouville closed H-field with small derivation. Any Hardy field $H \supseteq \mathbb{R}$ has a smallest (with respect to inclusion) Liouville closed Hardy field extension $\operatorname{Li}(H)$. (The notions of "H-field" and "Liouville closed H-field" are introduced in [1]. The capital H is in honor of Hardy, Hausdorff, and Hahn, who pioneered various aspects of our topic about a century ago, as did Du Bois-Reymond and Borel even earlier.)

Now a very strong property: we say K **has DIVP** (the Differential Intermediate Value Property) if for every polynomial $P \in K[Y_0, \dots, Y_r]$ and all $f < g$ in K with

$$P(f, f', \dots, f^{(r)}) \;<\; 0 \;<\; P(g, g', \dots, g^{(r)})$$

there exists $y \in K$ such that $f < y < g$ and $P(y, y', \dots, y^{(r)}) = 0$. (Existentially closed ordered differential fields have DIVP by [9] and [10, Proposition 1.5]; this has limited interest for us since the ordering and derivation in those structures do not interact.) Actually, DIVP is a bit of an afterthought: in [3] we considered instead two robust but rather technical properties, ω-freeness and newtonianity, and proved that $\mathbb{T}$ is ω-free and newtonian. (One can think of newtonianity as a variant of differential-henselianity.) Afterwards we saw that "ω-free + newtonian" is equivalent to DIVP, for Liouville closed H-fields. Our aim is to establish this equivalence: Theorem 2.7, the main result of this short paper.

We did not consider DIVP in [3], but it is surely an appealing property and easier to grasp than the more fundamental notions of ω-freeness and newtonianity. (The latter make sense in a wider setting of valued differential fields where the valuation does not necessarily arise from an ordering, as is the case for H-fields.)

Besides [3] we shall rely on [7], which focuses on a particular ordered differential subfield of $\mathbb{T}$, namely $\mathbb{T}_{\mathrm{g}}$, consisting of the so-called *grid-based* transseries; see also [3, Appendix A]. We summarize what we need from [7] as follows:

$\mathbb{T}_g$ *is a newtonian* ω*-free Liouville closed* H*-field with small derivation, and* $\mathbb{T}_g$ *has* DIVP. We alert the reader that the terms *newtonian* and ω*-free* do not occur in [7], and that $\mathbb{T}_g$ there is denoted by $\mathbb{T}$.

We call attention to the fact that K is a Liouville closed H-field iff $K \models \mathrm{LiH}$ for a set LiH (independent of K) of sentences in the language of ordered differential fields. Also, for H-fields, "ω-free" is expressible by a single sentence in the language of ordered differential fields, and "newtonian" as well as "DIVP" by a set of sentences in this language. The reason that "ω-free + newtonian" is central in [3] are various theorems proved there, which are also relevant here. To state these theorems, we consider an H-field K below as an $\mathcal{L}$-structure, where

$$\mathcal{L} \;:=\; \{\,0,\,1,\,+,\,-,\,\times,\,\partial,\,<,\,\preccurlyeq\,\}$$

is the language of ordered valued differential fields. The symbols $0, 1, +, -, \times, \partial, <$ name the usual primitives of K, and $\preccurlyeq$ encodes its valuation: for $a, b \in K$,

$$a \preccurlyeq b \quad :\Longleftrightarrow \quad a \in \mathcal{O}b.$$

We can now summarize what we need from [3, Chapters 15, 16] as follows:

The theory of newtonian ω*-free Liouville closed* H*-fields is model complete, and is the model companion of the theory of* H*-fields. The theory of newtonian* ω*-free Liouville closed* H*-fields whose derivation is small is complete and has* $\mathbb{T}$ *as a model.*

For an H-field K its valuation ring $\mathcal{O}$ and so the binary relation $\preccurlyeq$ on K can be defined in terms of the other primitives by an *existential* formula independent of K. However, by [3, Corollary 16.2.6] this cannot be done by a universal such formula and so for the model completeness above we cannot drop $\preccurlyeq$ from the language $\mathcal{L}$.

Corollary 0.1. *Every newtonian* ω*-free Liouville closed* H*-field has* DIVP.

Proof. Let K be a newtonian ω-free Liouville closed H-field. If the derivation of K is small, then DIVP follows from the results from [7] quoted earlier and the above completeness result from [3]. Suppose the derivation of K is not small. Replacing the derivation ∂ of K by a multiple $\phi^{-1}\partial$ with $\phi > 0$ in K transforms K into its so-called compositional conjugate K^{ϕ}, which is still a newtonian ω-free Liouville closed H-field, and K has DIVP iff K^{ϕ} does. By 4.4.7 and 9.1.5 in [3] we can choose $\phi > 0$ in K such that the derivation $\phi^{-1}\partial$ of K^{ϕ} is small. ☐

This gives one direction of Theorem 2.7. In the rest of this paper we prove a strong version, Corollary 2.6, of the other direction, without using [7] but relying heavily on various parts of [3] with detailed references. Theorem 2.7 and the results quoted above from [3] yield the result stated in the abstract: a Liouville closed H-field with small derivation is elementarily equivalent to $\mathbb{T}$ iff it has DIVP.

Connection to Hardy fields. Every Hardy field H extends to a Hardy field $H(\mathbb{R}) \supseteq \mathbb{R}$, and $H(\mathbb{R})$ is in particular an H-field. We refer to [4] for a discussion of the conjecture that *any Hardy field containing* $\mathbb{R}$ *extends to a newtonian* ω*-free Hardy field.* At the end of 2019 we finished the proof of this conjecture by considerably refining material in [3] and [8]; this amounts to a rather complete extension theory of Hardy fields. Note that every Hardy field extends to a maximal Hardy field, by Zorn, and so having established this conjecture we now know that all maximal Hardy fields are elementarily equivalent to $\mathbb{T}$, as ordered differential fields. Since $\mathcal{C}$ has the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, there are at most $2^{\mathfrak{c}}$

many maximal Hardy fields, and we also have a proof that there are exactly that many. (We thank Ilijas Farah for a useful hint on this point.) These remarks on Hardy fields serve as an announcement. A rather voluminous work containing the proof of the conjecture is currently being prepared for publication. We also hope to include there a proof of DIVP for newtonian ω-free H-fields that does not depend as in the present paper on it being true for $\mathbb{T}_{\mathrm{g}}$, whose proof in [7] uses the particular nature of $\mathbb{T}_{\mathrm{g}}$.

We have a second conjecture about Hardy fields in [4], whose proof is not yet finished at this time (May 2021): *for any maximal Hardy field H and countable subsets $A < B$ in H there exists $y \in H$ such that $A < y < B$.* This means that the underlying ordered set of a maximal Hardy field is an η_1-set in the sense of Hausdorff. Together with the (now established) first conjecture and results from [3] it implies: *all maximal Hardy fields are back-and-forth equivalent as ordered differential fields, and thus isomorphic assuming* CH*, the Continuum Hypothesis.*

1. Preliminaries

In order to make free use of the valuation-theoretic tools from [3] and to make this paper self-contained modulo references to specific results from the literature we provide more background in this section before returning to DIVP.

Notation and terminology. Throughout, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Given an additively written abelian group A we let $A^{\neq} := A \setminus \{0\}$. Rings are commutative with identity 1, and for a ring R we let $R^{\times}$ be the multiplicative group of units (consisting of the $a \in R$ such that $ab = 1$ for some $b \in R$). A *differential ring* will be a ring R containing (an isomorphic copy of) $\mathbb{Q}$ as a subring and equipped with a derivation $\partial\colon R \to R$; note that then $C_R := \{a \in R : \partial(a) = 0\}$ is a subring of R, called the ring of constants of R, and that $\mathbb{Q} \subseteq C_R$. If R is a field, then so is C_R. An ordered differential field is in particular a differential ring.

Let R be a differential ring and $a \in R$. When its derivation ∂ is clear from the context we denote $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$ by $a', a'', \dots, a^{(n)}, \dots$, and if $a \in R^{\times}$, then $a^{\dagger}$ denotes a'/a, so $(ab)^{\dagger} = a^{\dagger} + b^{\dagger}$ for $a, b \in R^{\times}$. In Section 2 we need to consider the function $\omega = \omega_R\colon R \to R$ given by $\omega(z) = -2z' - z^2$, and the function $\sigma = \sigma_R\colon R^{\times} \to R$ given by $\sigma(y) = \omega(z) + y^2$ for $z := -y^{\dagger}$.

We have the differential ring $R\{Y\} = R[Y, Y', Y'', \dots]$ of differential polynomials in an indeterminate Y over R. We say that $P = P(Y) \in R\{Y\}$ has order at most $r \in \mathbb{N}$ if $P \in R[Y, Y', \dots, Y^{(r)}]$.

For $\phi \in R^{\times}$ we let R^{ϕ} be the *compositional conjugate of R by ϕ*: the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ instead of ∂. We then have an R-algebra isomorphism

$$P \mapsto P^{\phi} \;:\; R\{Y\} \to R^{\phi}\{Y\}$$

with $P^{\phi}(y) = P(y)$ for all $y \in R$; see [3, Section 5.7].

For a field K we have $K^{\times} = K^{\neq}$, and a (Krull) valuation on K is a surjective map $v\colon K^{\times} \to \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v\colon K \to \Gamma_{\infty} := \Gamma \cup \{\infty\}$ by $v(0) := \infty$, where the ordering on Γ is extended to a total ordering on Γ_{∞} by $\gamma < \infty$ for all $\gamma \in \Gamma$.

Let K be a *valued field*: a field (also denoted by K) together with a valuation ring $\mathcal{O}$ of that field. This yields a valuation $v\colon K^{\times} \to \Gamma$ on the underlying field such that $\mathcal{O} = \{a \in K : va \geqslant 0\}$ as explained in [3, Section 3.1]. We introduce various binary relations on the set K by defining for $a, b \in K$:

$$a \asymp b \;:\Leftrightarrow\; va = vb, \qquad a \preccurlyeq b \;:\Leftrightarrow\; va \geqslant vb, \qquad a \prec b \;:\Leftrightarrow\; va > vb,$$
$$a \succcurlyeq b \;:\Leftrightarrow\; b \preccurlyeq a, \qquad a \succ b \;:\Leftrightarrow\; b \prec a, \qquad a \sim b \;:\Leftrightarrow\; a - b \prec a.$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$, and that $\sim$ is an equivalence relation on $K^\times$. We also let $\scriptstyle\mathcal{O} = \{a \in K : va > 0\}$ be the maximal ideal of $\mathcal{O}$, so $\mathcal{O}/\scriptstyle\mathcal{O}$ is the residue field of the valued field K. A convex subgroup Δ of the value group Γ of v gives rise to the Δ-*coarsening* of the valued field K; see [ADH, 3.4].

H-fields and pre-H-fields. As in [3], a *valued differential field* is a valued field K with residue field of characteristic zero and equipped with a derivation $\partial\colon K \to K$. An *ordered valued differential field* is a valued differential field K equipped with an ordering on K making K an ordered field. We consider any H-field K as an ordered valued differential field whose valuation ring is the convex hull in K of its constant field C, in accordance with construing it as an $\mathcal{L}$-structure as specified in the introduction.

A *pre-H-field* is by definition an ordered valued differential subfield of an H-field. By [3, Sections 10.1, 10.3, 10.5], an ordered valued differential field K is a pre-H-field iff the valuation ring $\mathcal{O}$ of K is convex in K, $f' > 0$ for all $f > \mathcal{O}$ in K, and $f' \prec g^\dagger$ for all $f, g \in K^\times$ with $f \preccurlyeq 1$ and $g \prec 1$. Any Hardy field H is construed as a pre-H-field by taking the convex hull of $\mathbb{Q}$ in H as its valuation ring, giving rise to the so-called "natural valuation" on H as an ordered field. At the end of Section 9.1 in [3] we give $\mathbb{Q}(\sqrt{2 + x^{-1}})$ as an example of a Hardy field that is not an H-field. Any ordered differential field K with the trivial valuation ring $\mathcal{O} = K$ is a pre-H-field (so the valuation ring of a pre-H-field K is not always the convex hull in K of its constant field, in contrast to Hardy fields and H-fields). If K is a pre-H-field whose valuation ring is nontrivial, then the valuation topology on K equals its order topology, by [3, Lemma 2.4.1].

Let K be a pre-H-field. Then the derivation of K and its valuation $v\colon K^\times \to \Gamma$ induce an operation $\psi\colon \Gamma^{\neq} \to \Gamma$, given by $\psi(vf) = v(f^\dagger)$ for $f \not\asymp 1$ in $K^\times$; the pair (Γ, ψ) is called the H-asymptotic couple of K; see [3, Section 9.1]. Below we assume some familiarity with (Γ, ψ), and properties of K based on it, such as K having *asymptotic integration* and K having a *gap* [3, Sections 9.1, 9.2]. The *flattening* of K is the $\Gamma^\flat$-coarsening of K where $\Gamma^\flat = \{vf : f \in K^\times,\ f' \prec f\}$, with associated binary relations $\asymp^\flat$, $\preccurlyeq^\flat$ etc.; see [ADH, 9.4].

2. DIVP

In this section K is a pre-H-field. We let $\mathcal{O}$ be its valuation ring, with maximal ideal $\scriptstyle\mathcal{O}$, and corresponding valuation $v\colon K^\times \to \Gamma = v(K^\times)$. Let (Γ, ψ) be its H-asymptotic couple, and $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^{\neq}\}$. Recall that "$K$ has DIVP" means: for all $P(Y) \in K\{Y\}$ and $f < g$ in K with $P(f) < 0 < P(g)$ there is a $y \in K$ such that $f < y < g$ and $P(y) = 0$. Restricting this to P of order $\leqslant r$, where $r \in \mathbb{N}$, gives the notion of r-DIVP. Thus K having 0-DIVP is equivalent to K being real closed as an ordered field. In particular, if K has 0-DIVP, then $\Gamma = v(K^\times)$ is divisible. From [3, Section 2.4] recall our convention that $K^> = \{a \in K : a > 0\}$, and similarly with $<$ replacing $>$.

Lemma 2.1. *Suppose* $\Gamma \neq \{0\}$ *and* K *has* 1-DIVP. *Then* $\partial K = K$, $(K^{>})^{\dagger} = (K^{<})^{\dagger}$ *is a convex subgroup of* K, Ψ *has no largest element, and* Ψ *is convex in* Γ.

Proof. We have $y' = 0$ for $y = 0$, and y' takes arbitrarily large positive values in K as y ranges over $K^{>\mathcal{O}} = \{a \in K : a > \mathcal{O}\}$, since by [3, Lemma 9.2.6] the set $(\Gamma^{<})'$ is coinitial in Γ. Hence y' takes all positive values on $K^{>}$, and therefore also all negative values on $K^{<}$. Thus $\partial K = K$. Next, let $a, b \in K^{>}$, and suppose $s \in K$ lies strictly between $a^{\dagger}$ and $b^{\dagger}$. Then $s = y^{\dagger}$ for some $y \in K^{>}$ strictly between a and b; this follows by noting that for $y = a$ and $y = b$ the signs of $sy - y'$ are opposite.

Let $\beta \in \Psi$ and take $a \in K$ with $v(a') = \beta$. Then $a \succ 1$, since $a \preccurlyeq 1$ would give $v(a') > \Psi$. Hence for $\alpha = va < 0$ we have $\alpha + \alpha^{\dagger} = \beta$, so $\alpha^{\dagger} > \beta$. Thus Ψ has no largest element. Therefore the set Ψ is convex in Γ. □

Thus the ordered differential field $\mathbb{T}_{\log}$ of logarithmic transseries [3, Appendix A] does not have 1-DIVP, although it is a newtonian $\boldsymbol{\omega}$-free H-field.

Does DIVP imply that K is an H-field? No: take an $\aleph_0$-saturated elementary extension of $\mathbb{T}$ and let Δ be as in [3, Example 10.1.7]. Then the Δ-coarsening of K is a pre-H-field with DIVP and nontrivial value group, and has a gap, but it is not an H-field. On the other hand:

Lemma 2.2. *Suppose* K *has* 1-DIVP *and has no gap. Then* K *is an* H*-field.*

Proof. In [3, Section 11.8] we defined

$$\mathrm{I}(K) \ := \ \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\},$$

a convex $\mathcal{O}$-submodule of K. Since K has no gap, we have

$$\partial\smallo \ \subseteq \ \mathrm{I}(K) \ = \ \{y \in K : y \preccurlyeq f' \text{ for some } f \in \smallo\}.$$

Also $\Gamma \neq \{0\}$, and so (Γ, ψ) has asymptotic integration by Lemma 2.1. We show that K is an H-field by proving $\mathrm{I}(K) = \partial\smallo$, so let $g \in \mathrm{I}(K)$, $g < 0$. Since $(\Gamma^{>})'$ has no least element we can take positive $f \in \smallo$ such that $f' \succ g$. Since $f' < 0$, this gives $f' < g$. Since $(\Gamma^{>})'$ is cofinal in Γ we can also take positive $h \in \smallo$ such that $h' \prec g$, which in view of $h' < 0$ gives $g < h'$. Thus $f' < g < h'$, and so 1-DIVP yields $a \in \smallo$ with $g = a'$. □

We refer to Sections 11.6 and 14.2 of [3] for the definitions of $\boldsymbol{\lambda}$*-freeness* and r*-newtonianity* ($r \in \mathbb{N}$). From the introduction we recall that $\omega(z) := -2z' - z^2$. Below, compositionally conjugating an H-field K means replacing it by some K^{ϕ} with $\phi \in K^{>}$; this preserves most relevant properties like being an H-field, being $\boldsymbol{\lambda}$-free, r-DIVP, and r-newtonianity, and it replaces Ψ by $\Psi - v\phi$.

Lemma 2.3. *Suppose* K *is an* H*-field,* $\Gamma \neq \{0\}$*, and* K *has* 1-DIVP. *Then* K *is* $\boldsymbol{\lambda}$*-free and* 1*-newtonian, and the subset* $\omega(K)$ *of* K *is downward closed.*

Proof. Note that K has (asymptotic) integration, by Lemma 2.1. Assume towards a contradiction that K is not $\boldsymbol{\lambda}$-free. We arrange by compositional conjugation that K has small derivation, so K has an element $x \succ 1$ with $x' = 1$, hence $x > C$. A construction in the beginning of [3, Section 11.5] yields by [3, Lemma 11.5.2] a pseudocauchy sequence $(\boldsymbol{\lambda}_{\rho})$ in K with certain properties including $\boldsymbol{\lambda}_{\rho} \sim x^{-1}$ for all ρ. As K is not $\boldsymbol{\lambda}$-free, $(\boldsymbol{\lambda}_{\rho})$ has a pseudolimit $\boldsymbol{\lambda} \in K$ by [3, Corollary 11.6.1]. Then $s := -\boldsymbol{\lambda} \sim -x^{-1}$, and s creates a gap over K by [3, Lemma 11.5.14]. Now note that for $P := Y' + sY$ we have $P(0) = 0$ and $P(x^2) = 2x + sx^2 \sim x$, so by 1-DIVP we have $P(y) = 1$ for some $y \in K$, contradicting [3, Lemma 11.5.12].

Let $P \in K\{Y\}$ of order at most 1 have Newton degree 1; we have to show that P has a zero in $\mathcal{O}$. We know that K is $\boldsymbol{\lambda}$-free, so by [3, Proposition 13.3.6] we can pass to an elementary extension, compositionally conjugate, and divide by an element of $K^\times$ to arrange that K has small derivation and $P = D + R$ where $D = cY + d$ or $D = cY'$ with $c, d \in C$, $c \neq 0$, and where $R \prec^\flat 1$. Then $R(a) \prec^\flat 1$ for all $a \in \mathcal{O}$. If $D = cY + d$, then we can take $a, b \in C$ with $D(a) < 0$ and $D(b) > 0$, which in view of $R(a) \prec D(a)$ and $R(b) \prec D(b)$ gives $P(a) < 0$ and $P(b) > 0$, and so P has a zero strictly between a and b, and thus a zero in $\mathcal{O}$. Next, suppose $D = cY'$. Then we take $t \in o^{\neq}$ with $v(t^\dagger) = v(t)$, that is, $t' \asymp t^2$, so

$$P(t) \;=\; ct' + R(t), \quad P(-t) \;=\; -ct' + R(-t), \qquad R(t),\ R(-t) \;\prec\; t'.$$

Hence $P(t)$ and $P(-t)$ have opposite signs, so P has a zero strictly between t and $-t$, and thus P has a zero in $\mathcal{O}$.

From $\omega(z) = -z^2 - 2z'$ we see that $\omega(z) \to -\infty$ as $z \to +\infty$ and as $z \to -\infty$ in K, so $\omega(K)$ is downward closed by 1-IVP. □

For results involving r-DIVP for $r \geqslant 2$ we need a variant of [3, Lemma 11.8.31]. To state this variant we introduce as in [3, Section 11.8] the sets

$$\Gamma(K) \;:=\; \{a^\dagger : a \in K \setminus \mathcal{O}\} \;\subseteq\; K^>, \qquad \Lambda(K) \;:=\; -\Gamma(K)^\dagger \;\subseteq\; K.$$

The superscripts $\uparrow$, $\downarrow$ used in the statement of Lemma 2.4 below indicate upward, respectively downward, closure in the ordered set K, as in [3, Section 2.1].

Lemma 2.4. *Let K be an H-field with asymptotic integration. Then*

$$K^> \;=\; \mathrm{I}(K)^> \cup \Gamma(K)^\uparrow, \qquad \sigma\big(K^> \setminus \Gamma(K)^\uparrow\big) \;\subseteq\; \omega\big(\Lambda(K)\big)^\downarrow.$$

Proof. If $a \in K$, $a > \mathrm{I}(K)$, then $a \geqslant b^\dagger$ for some $b \in K^{\succ 1}$, and thus $a \in \Gamma(K)^\uparrow$. Next, let $s \in K^> \setminus \Gamma(K)^\uparrow$; we have to show $\sigma(s) \in \omega\big(\Lambda(K)\big)^\downarrow$. Note that $s \in \mathrm{I}(K)^>$ by what we just proved. From [3, 10.2.7 and 10.5.8] we obtain an immediate H-field extension L of K and $a \in L^{\succ 1}$ with $s = (1/a)'$. As in the proof of [3, 11.8.31] with L instead of K this gives $\sigma(s) \in \omega\big(\Lambda(L)\big)^\downarrow$, where $\downarrow$ indicates here the downward closure in L. It remains to note that ω is increasing on $\Lambda(L)$ by the remark preceding [3, 11.8.21], and that $\Lambda(K)$ is cofinal in $\Lambda(L)$ by [3, 11.8.14]. □

The concept of $\boldsymbol{\omega}$-*freeness* is introduced in [3, Section 11.7]. If K has asymptotic integration, then by [3, 11.8.30]: K is $\boldsymbol{\omega}$-free $\Leftrightarrow$ $K = \omega\big(\Lambda(K)\big)^\downarrow \cup \sigma\big(\Gamma(K)\big)^\uparrow$.

The next lemma also mentions the differential field extension $K[i]$ of K where $i^2 = -1$, as well as linear differential operators over differential fields like K and $K[i]$; for this we refer to [3, Sections 5.1, 5.2].

Lemma 2.5. *Suppose K is an H-field, $\Gamma \neq \{0\}$, $r \geqslant 2$, and K has* r-DIVP. *Then the following hold, with* (i), (ii), (iii) *using only the case $r = 2$:*

(i) *$K = \omega(K) \cup \sigma(K^>) = \omega\big(\Lambda(K)\big)^\downarrow \cup \sigma\big(\Gamma(K)\big)^\uparrow$;*
(ii) *K is $\boldsymbol{\omega}$-free and $\omega(K) = \omega\big(\Lambda(K)\big)^\downarrow$;*
(iii) *for all $a \in K$ the operator $\partial^2 - a$ splits over $K[i]$;*
(iv) *K is r-newtonian.*

Proof. For (i) we use the end of [3, Section 11.7] to replace K with a compositional conjugate so that $0 \in \Psi$. Then K has small derivation, and we have $a \in K^>$ such that $a \not\asymp 1$ and $a^\dagger \asymp 1$. Replacing a by a^{-1} if necessary this gives $a^\dagger = -\phi$ with $\phi \asymp 1$, $\phi > 0$, so $a \prec 1$. Then $\phi^{-1}a^\dagger = -1$; replacing K by K^ϕ and renaming the latter as K this means $a^\dagger = -1$. Let $f \in K$; to get $f \in \omega\big(\Lambda(K)\big)^{\downarrow} \cup \sigma\big(\Gamma(K)\big)^{\uparrow}$, note first that $1 = (1/a)^\dagger \in \Gamma(K)$, so $0 \in \Lambda(K)$. Also $\omega\big(\Lambda(K)\big)^{\downarrow} \subseteq \omega(K)$ by Lemma 2.3.

If $f \leqslant 0$, then $\omega(0) = 0$ gives $f \in \omega\big(\Lambda(K)\big)^{\downarrow}$. So assume $f > 0$; we first show that then $f \in \sigma(K^>)$. Now for $y \in K^>$, $f = \sigma(y)$ is equivalent (by multiplying with y^2) to $P(y) = 0$, where

$$P(Y) \ := \ 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in K\{Y\}.$$

See also [3, Section 13.7]. We have $P(0) = 0$ and $P(y) \to +\infty$ as $y \to +\infty$ (because of the term y^4). In view of 2-DIVP it will suffice to show that for some $y > 0$ in K we have $P(y) < 0$. Now with $y \in K^>$ and $z := -y^\dagger$ we have

$$\begin{aligned} P(y) \ &= \ y^2\big(\sigma(y) - f\big) \ = \ y^2\big(\omega(z) + y^2 - f\big), \text{ hence} \\ P(a) \ &= \ a^2\big(\omega(1) + a^2 - f\big) \ = \ a^2(-1 + a^2 - f) \ < \ 0, \end{aligned}$$

so $f \in \sigma(K^>)$. By the second inclusion of Lemma 2.4 this yields $f \in \omega\big(\Lambda(K)\big)^{\downarrow}$ or $f \in \sigma\big(\Gamma(K)^{\uparrow}\big)$. But we have $\sigma\big(\Gamma(K)^{\uparrow}\big) \subseteq \sigma\big(\Gamma(K)\big)^{\uparrow}$, because σ is increasing on $\Gamma(K)^{\uparrow}$ by the remark preceding [3, 11.8.30]. This concludes the proof of (i), and then (ii) follows, using for its second part also the fact stated just before [3, 11.8.29] that we have $\omega(K) < \sigma\big(\Gamma(K)\big)$.

Now (iii) follows from $K = \omega(K) \cup \sigma(K^>)$ by [3, Section 5.2, (5.2.1)]. As to (iv), let $P \in K\{Y\}$ of order at most r have Newton degree 1; we have to show that P has a zero in $\mathcal{O}$. For this we repeat the argument in the proof of Lemma 2.3 so that it applies to our P, using ω-freeness instead of λ-freeness, [3, Proposition 13.3.13] instead of [3, Proposition 13.3.6], and r-DIVP instead of 1-DIVP. □

Corollary 2.6. *If K is an H-field, $\Gamma \neq \{0\}$, and K has* DIVP*, then K is ω-free and newtonian.*

There are non-Liouville closed H-fields with nontrivial derivation that have DIVP; see [2, Section 14]. By Lemma 2.3 and Lemma 2.5(iii), Liouville closed H-fields having 2-DIVP are *Schwarz closed* as defined in [3, Section 11.8].

Theorem 2.7. *Let K be a Liouville closed H-field. Then*

$$K \text{ has DIVP} \iff K \text{ is } \omega\text{-free and newtonian.}$$

Proof. The forward direction is part of Corollary 2.6. The backward direction is Corollary 0.1. □

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