The hyperserial field of surreal numbers*

Vincent Bagayoko[A], UMons, LIX

Joris van der Hoeven[B], CNRS, LIX

For any ordinal $\alpha > 0$, we show how to define a hyperexponential E_{ω^α} and a hyperlogarithm L_{ω^α} on the class $\mathbf{No}^{>,\succ}$ of positive infinitely large surreal numbers. Such functions are archetypes of extremely fast and slowly growing functions at infinity. We also show that the surreal numbers form a so-called hyperserial field for our definition.

1 Introduction

The ordered field **No** of surreal numbers was introduced by Conway in [11]. Conway originally used transfinite recursion to define both the surreal numbers (henceforth called *numbers*), the ordering on **No**, and the ring operations. For any two sets L and R of numbers with $L < R$ (i.e. $x < y$ for all $x \in L$ and $y \in R$), there exists a number $\{L \mid R\}$ with

$$L \; < \; \{L|R\} \; < \; R,$$

and all numbers can be obtained in this way. Given $x = \{x_L \mid x_R\}$ and $y = \{y_L \mid y_R\}$, we have

$$x + y \; := \; \{x_L + y, x + y_L \mid x_R + y, x + y_R\}$$

and similar recursive formulas exist for $-x$, $x y$ and for deciding whether $x = y$, $x \leqslant y$, and $x < y$. It is truly remarkable that **No** turns out to be a totally ordered real-closed field for such "simple" definitions [11]. The bracket $\{ \mid \}$ is called the *Conway bracket*. Using this bracket, we obtain a surreal number in any traditional Dedekind cut, which allows us to embed $\mathbb{R}$ into **No**. In addition, **No** contains all ordinal numbers

$$0 = \{ \mid \}, \quad 1 = \{0 \mid \}, \quad 2 = \{0, 1 \mid \}, \quad \ldots, \quad \omega = \{0, 1, 2, \ldots \mid \}, \quad \omega + 1 = \{0, 1, 2, \ldots, \omega \mid \}, \quad \ldots,$$

so **No** is actually a proper class.

An interesting question is which other operations from calculus can be extended to the surreal numbers. Gonshor has shown how to extend the real exponential function to the surreal numbers [19] and the resulting exponential field $(\mathbf{No}, \exp)$ turns out to be elementarily equivalent to $(\mathbb{R}, \exp)$ [13]. Berarducci and Mantova recently defined a derivation with respect to ω on the surreals [9], again with good model-theoretic properties [2]. In collaboration with Mantova, the authors constructed a surreal solution to the functional equation

$$E_\omega(x + 1) \; = \; \exp E_\omega x,$$

which is a bijection of $\mathbf{No}^{>,\succ} := \{x \in \mathbf{No} : x > \mathbb{R}\}$ onto itself [6]. We call E_ω a *hyperexponential* and its functional inverse L_ω a *hyperlogarithm*.

A. vincent.bagayoko@umons.ac.be

B. vdhoeven@lix.polytechnique.fr

*. This article has been written using GNU TeXmacs [27].

The first goal of this paper is to extend the results from [6] to the construction of hyperexponentials $E_{\omega^\alpha}: \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ of any ordinal *force* α, together with their functional inverses L_{ω^α}. If $\alpha = \beta + 1$ is a successor ordinal, then E_{ω^α} satisfies the functional equation

$$E_{\omega^{\beta+1}}(x+1) \;=\; E_{\omega^\beta}(E_{\omega^{\beta+1}}(x)).$$

Our second goal is to show that these hyperexponentials are "well-behaved" in the sense that they endow **No** with the structure of a *hyperserial field* in the sense of [5].

1.1 Motivation and background

Whereas it is natural to study surreal exponentiation and differentiation, it may seem more exotic to define and investigate the properties of surreal hyperexponentials and hyperlogarithms. In fact, the main motivation behind our work is a conjecture by the second author [26, p. 16] and a research program that was laid out in [1] for proving this conjecture. The ultimate goal is to expose the deep connections between two types of mathematical infinities: numerical infinities and growth rates at infinity. Let us briefly recall the rationale behind this connection.

Cantor's ordinal numbers provide us with a way to count beyond all natural numbers and to keep counting beyond the size of any set. However, ordinal arithmetic is rather poor in the sense that we have no subtraction or division and that addition and multiplication do not satisfy the usual laws of arithmetic, such as commutativity. We may regard Conway's surreal numbers as providing a calculus with Cantor's ordinal numbers which does extend the usual calculus with real numbers. In this sense, Conway managed to construct the ultimate framework for computations with numerical infinities.

Another source for computations with infinitely large quantities stems from the study of growth rates of real functions at infinity. The first major results towards a systematic asymptotic calculus of this kind are due to Hardy in [21, 22], based on earlier ideas by du Bois-Reymond [15, 16, 17]. Hardy defined an *L-function* to be a function constructed from x and the real numbers $\mathbb{R}$ using the field operations, exponentiation, and logarithms. He proved that the germs of L-functions at infinity form a totally ordered field. The framework of L-functions is suitable for asymptotic analysis since we have an ordering for comparing the growth at infinity of any two such functions. This is often rephrased by saying that L-functions have a *regular* growth at infinity.

Hardy also observed [21, p. 22] that "The only scales of infinity that are of any practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions." In other words, Hardy suggested that the framework of L-functions not only allows for the development of a systematic asymptotic calculus, but that this framework is also sufficient for all "practical" purposes. Alas, there are several "holes". First of all, the framework is not closed under various useful operations such as functional inversion and integration. Secondly, the framework does not contain any functions of extremely fast or slow growth at infinity, like E_ω and L_ω, although such functions naturally appear in the analysis of certain algorithms. For instance, the best known algorithm for multiplying two polynomials of degree n in $\mathbb{F}_2[x]$ runs in time $O(n \log n \, 4^{L_\omega n})$; see [23].

This raises the question how to construct a truly universal framework for computations with regular functions at infinity. Our next candidate is the class of transseries. A *transseries* is a formal object that is constructed from x (with $x \rightarrow \infty$) and the real numbers, using exponentiation, logarithms, and *infinite* sums. One example of a transseries is

$$\mathrm{e}^{\mathrm{e}^x + \mathrm{e}^{x/2} + \mathrm{e}^{x/3} + \cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^\pi + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + 24\,x^{-4} + \cdots + \mathrm{e}^{-x}.$$

Depending on conditions satisfied by their supports, there are different types of transseries. The first constructions of fields of transseries are due to Dahn and Göring [12] and Écalle [18]. More general constructions were proposed subsequently by the second author and his former student Schmeling [24, 25, 29]. Clearly, any L-function is a transseries, but the class of transseries is also closed under integration and functional inversion, contrary to the class of L-functions.

However, the class of transseries still does not contain any hyperexponential or hyperlogarithmic elements like $E_\omega x$ or $L_\omega x$. In our quest for a truly universal framework for asymptotic analysis, we are thus lead to look beyond: a *hyperseries* is a formal object that is constructed from x and the real numbers using exponentiation, logarithms, infinite sums, as well as hyperexponentials E_{ω^α} and hyperlogarithms L_{ω^α} of any force α. The hyperexponentials E_{ω^α} and the hyperlogarithms L_{ω^α} are required to satisfy functional equations

$$E_{\omega^{\alpha+1}} \circ T_1 = E_{\omega^\alpha} \circ E_{\omega^{\alpha+1}} \tag{1.1}$$

$$L_{\omega^{\alpha+1}} \circ L_{\omega^\alpha} = T_{-1} \circ L_{\omega^{\alpha+1}}, \tag{1.2}$$

where $T_s(u) \coloneqq u + s$. For $\gamma = \sum_{i=1}^p \omega^{\alpha_i} n_i$ in Cantor normal form with $\alpha_1 < \cdots < \alpha_p$, we also define

$$L_\gamma = L_{\omega^{\alpha_1}}^{\circ n_1} \circ \cdots \circ L_{\omega^{\alpha_p}}^{\circ n_p} \tag{1.3}$$

and we require that

$$L'_\gamma = \frac{1}{\prod_{\beta<\gamma} L_\beta}. \tag{1.4}$$

It is non-trivial to construct fields of hyperseries in which these and several other technical properties (see section 4 below) are satisfied. This was first accomplished by Schmeling for hyperexponentials E_{ω^n} and hyperlogarithms L_{ω^n} of finite force $n \in \mathbb{N}$. The general case was tackled in [14, 5].

The construction of general hyperseries relies on the definition of an abstract notion of *hyperserial fields*. Whereas the hyperseries that we are really after should actually be hyperseries *in* an infinitely large variable x, abstract hyperserial fields potentially contain hyperseries that can not be written as infinite expressions in x. In the present paper, we define hyperexponentials E_{ω^α} and hyperlogarithms L_{ω^α} on **No** for all ordinals α and show that this provides **No** with the structure of an abstract hyperserial field. Moreover, any hyperseries f in x can naturally be evaluated at $x = \omega$ to produce a surreal number $f(\omega)$. The conjecture from [26, p. 16] states that, for a sufficiently general notion of "hyperseries in x", *all* surreal numbers can actually be obtained in this way. We plan to prove this and the conjecture in a follow-up paper.

1.2 General overview and summary of our new contributions

Our main goal is to define hyperexponentials $E_{\omega^\alpha}: \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ for any ordinal $\alpha > 1$ and to show that **No** is a hyperserial field for these hyperexponentials. Since our construction builds on quite some previous work, the paper starts with three sections of reminders.

In section 2, we recall basic facts about well-based series and surreal numbers. In particular, we recall that any surreal number $x \in \mathbf{No}$ can be regarded as a well-based series

$$x = \sum_{\mathfrak{m} \in \mathbf{Mo}} x_{\mathfrak{m}} \mathfrak{m}$$

with real coefficients $x_{\mathfrak{m}} \in \mathbb{R}$. The corresponding group of monomials **Mo** consists of those positive numbers $\mathfrak{m} \in \mathbf{No}^>$ that are of the form $\mathfrak{m} = \{\mathbb{R}^> L \mid \mathbb{R}^> R\}$ for certain subsets L and R of **No** with $\mathbb{R}^> L < \mathbb{R}^> R$.

Section 3 is devoted to the theory of surreal substructures from [4]. One distinctive feature of the class of surreal numbers is that it comes with a partial, well-founded order $\sqsubseteq$, which is called the *simplicity* relation. The Conway bracket can then be characterized by the fact that, for any *sets* L and R of surreal numbers with $L<R$, there exists a unique $\sqsubseteq$-minimal number $\{L \mid R\}$ with $L<\{L \mid R\}<R$. For many interesting subclasses $\mathbf{S}$ of $\mathbf{No}$, it turns out that the restrictions of $\leqslant$ and $\sqsubseteq$ to $\mathbf{S}$ give rise to a structure $(\mathbf{S}, \leqslant, \sqsubseteq)$ that is isomorphic to $(\mathbf{No}, \leqslant, \sqsubseteq)$. Such classes $\mathbf{S}$ are called *surreal substructures* of $\mathbf{No}$ and they come with their own Conway bracket $\{\mid\}_{\mathbf{S}}$.

In section 4, we recall the definition of hyperserial fields from [5] and the main results on how to construct such fields. One major fact from [5] on which we heavily rely is that the construction of hyperserial fields can be reduced to the construction of *hyperserial skeletons*. In the context of the present paper, this means that it suffices to define the hyperlogarithms L_{ω^α} only for very special, so called *$L_{<\omega^\alpha}$-atomic elements*.

In the case when $\alpha=0$, the $L_{<1}$-atomic elements are simply the monomials in $\mathbf{Mo}$ and the definition of the general logarithm on $\mathbf{No}^>$ indeed reduces to the definition of the logarithm on $\mathbf{Mo}$: given $x\in\mathbf{No}^>$, we write $x=c\,\mathfrak{m}\,(1+\varepsilon)$, where $c\in\mathbb{R}$, $\mathfrak{m}\in\mathbf{Mo}$ and ε is infinitesimal, and we take $\log x := \log \mathfrak{m} + \log c + \varepsilon - \varepsilon^2/2 + \varepsilon^3/3 + \cdots$. This very special case will be considered in more detail in section 5.

In the case when $\alpha=1$, the $L_{<\omega}$-atomic elements of $\mathbf{No}^{>,\succ}$ are those elements $\mathfrak{a}\in\mathbf{No}^{>,\succ}$ such that $L_n\,\mathfrak{a}$ is a monomial for every $n\in\mathbb{N}$. The construction of L_ω on $\mathbf{No}^{>,\succ}$ then reduces to the construction of L_ω on the class $\mathbf{Mo}_\omega$ of $L_{<\omega}$-atomic numbers. This particular case was first dealt with in [6] and this paper can be used as an introduction to the more general results in the present paper.

For general ordinals α, we say that $\mathfrak{a}\in\mathbf{No}^{>,\succ}$ is $L_{<\omega^\alpha}$-atomic if $L_\beta\,\mathfrak{a}$ is a monomial for every $\beta<\alpha$. The advantage of restricting ourselves to such numbers $\mathfrak{a}$ when defining hyperlogarithms is that $L_\alpha\,\mathfrak{a}$ only needs to verify few requirements with respect to the ordering. This makes it possible to define $L_\alpha\,\mathfrak{a}$ using the fairly simple recursive formula

$$L_\alpha\,\mathfrak{a} \;:=\; \{\mathbb{R}, L_\alpha\,\mathfrak{a}' + (L_{<\alpha}\,\mathfrak{a}')^{-1} \mid L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a})^{-1}, L_{<\alpha}\,\mathfrak{a}\}, \tag{1.5}$$

where $\mathfrak{a}', \mathfrak{a}''$ range over $L_{<\alpha}$-atomic numbers with $\mathfrak{a}', \mathfrak{a}''\sqsubseteq\mathfrak{a}$ and $\mathfrak{a}'<\mathfrak{a}<\mathfrak{a}''$; see also (7.1).

In section 6, we prove that this definition is warranted and that the resulting functions L_α satisfy the axioms of hyperserial skeletons from [5, Section 3]. Our proof proceeds by induction on α and also relies on the fact that the class $\mathbf{Mo}_\alpha$ of $L_{<\omega^\alpha}$-atomic numbers actually forms a surreal substructure of $\mathbf{No}$. Our main result is the following theorem:

Theorem 1.1. *The definition* (1.5) *gives* $\mathbf{No}$ *the structure of a confluent hyperserial skeleton in the sense of [5]. Consequently, we may uniquely extend L_{ω^μ} to* $\mathbf{No}^{>,\succ}$ *in a way that gives* $\mathbf{No}$ *the structure of a confluent hyperserial field. Moreover, for each ordinal μ, the extended function $L_{\omega^\mu}\colon\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}^{>,\succ}$ is bijective.*

Our final section 7 is devoted to further identities that illustrate the interplay between the hyperexponential and hyperlogarithmic functions and the simplicity relation $\sqsubseteq$ on $\mathbf{No}$. We also prove the following more symmetric variant of (1.5):

$$L_\alpha\,\mathfrak{a} \;=\; \{\mathbb{R}, L_\alpha\,\mathfrak{a}' + (L_{<\alpha}\,\mathfrak{a}')^{-1} \mid L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a}'')^{-1}, L_{<\alpha}\,\mathfrak{a}\}, \tag{1.6}$$

where $\mathfrak{a}', \mathfrak{a}''$ again range over the $L_{<\alpha}$-atomic numbers with $\mathfrak{a}', \mathfrak{a}''\sqsubseteq\mathfrak{a}$ and $\mathfrak{a}'<\mathfrak{a}<\mathfrak{a}''$. An interesting open question is whether there exists an easy argument that would allow us to use (1.6) instead of (1.5) as a definition of $L_\alpha\,\mathfrak{a}$.

2 Basic notions

2.1 Ordered fields of well-based series

2.1.1 Well-based series

Let $(\mathfrak{M}, \times, 1, \prec)$ be a (possibly class-sized) linearly ordered abelian group. We write $\mathbb{S} := \mathbb{R}[[\mathfrak{M}]]$ for the class of functions $f: \mathfrak{M} \longrightarrow \mathbb{R}$ whose support

$$\operatorname{supp} f \;:=\; \{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\}$$

is a *well-based* set, i.e. a set which is well-ordered with respect to the reverse order $(\mathfrak{M}, \succ)$.

We see elements f of $\mathbb{S}$ as formal *well-based series* $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}}\, \mathfrak{m}$, where $f_{\mathfrak{m}}$ denotes the coefficient $f(\mathfrak{m}) \in \mathbb{R}$ of $\mathfrak{m}$ in f, for each $\mathfrak{m} \in \mathfrak{M}$. If $\operatorname{supp} f \neq \emptyset$, then we define $\mathfrak{d}_f := \max \operatorname{supp} f \in \mathfrak{M}$ to be the *dominant monomial* of f. For $\mathfrak{m} \in \mathfrak{M}$, we let $f_{\succ \mathfrak{m}} := \sum_{\mathfrak{n} \succ \mathfrak{m}} f_{\mathfrak{n}}\, \mathfrak{n}$ and we write $f_{\succ} := f_{\succ 1}$. We say that a series $g \in \mathbb{S}$ is a *truncation* of f and we write $g \trianglelefteq f$ if $\operatorname{supp}(f - g) \succ g$. The relation $\trianglelefteq$ is a well-founded partial order on $\mathbb{S}$ with minimum 0.

By [20], the class $\mathbb{S}$ is an ordered field under the pointwise sum

$$(f + g) \;:=\; \sum_{\mathfrak{m}} (f_{\mathfrak{m}} + g_{\mathfrak{m}})\, \mathfrak{m},$$

the Cauchy product

$$f g \;:=\; \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{u}\mathfrak{v} = \mathfrak{m}} f_{\mathfrak{u}}\, g_{\mathfrak{v}} \right) \mathfrak{m},$$

(where each sum $\sum_{\mathfrak{u}\mathfrak{v} = \mathfrak{m}} f_{\mathfrak{u}}\, g_{\mathfrak{v}}$ has finite support), and where the positive cone $\mathbb{S}^{>} = \{f \in \mathbb{S} : f > 0\}$ is given by

$$\mathbb{S}^{>} \;:=\; \{f \in \mathbb{S} : f \neq 0 \wedge f_{\mathfrak{d}_f} > 0\}.$$

The identification of $\mathfrak{m} \in \mathfrak{M}$ with the formal series $\sum_{\mathfrak{n} = \mathfrak{m}} 1 \cdot \mathfrak{n} \in \mathbb{S}$ induces an ordered group embedding $(\mathfrak{M}, \times, \prec) \longrightarrow (\mathbb{S}^{>}, \times, <)$.

We next define the following asymptotic relations on $\mathbb{S}$:

$$\begin{aligned} f \prec g \;&\Longleftrightarrow\; \mathbb{R}^{>} |f| < |g| \\ f \preccurlyeq g \;&\Longleftrightarrow\; \exists r \in \mathbb{R}^{>}, |f| \leqslant r |g| \\ f \asymp g \;&\Longleftrightarrow\; f \preccurlyeq g \preccurlyeq f. \end{aligned}$$

The relation $\prec$ extends the ordering on $\mathfrak{M}$. For non-zero $f, g \in \mathbb{S}$ we actually have $f \prec g$ (resp. $f \preccurlyeq g$, resp. $f \asymp g$) if and only if $\mathfrak{d}_f \prec \mathfrak{d}_g$ (resp. $\mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$, resp. $\mathfrak{d}_f = \mathfrak{d}_g$). We finally define

$$\begin{aligned} \mathbb{S}_{\succ} \;&:=\; \{f \in \mathbb{S} : \operatorname{supp} f \subseteq \mathfrak{M}^{\succ}\} \\ \mathbb{S}^{\prec} \;&:=\; \{f \in \mathbb{S} : \operatorname{supp} f \subseteq \mathfrak{M}^{\prec}\} \;=\; \{f \in \mathbb{S} : f \prec 1\} \\ \mathbb{S}^{>,\succ} \;&:=\; \{f \in \mathbb{S} : f > \mathbb{R}\} = \{f \in \mathbb{S} : f \geqslant 0 \wedge f \succ 1\}. \end{aligned}$$

Series in $\mathbb{S}_{\succ}$, $\mathbb{S}^{\prec}$ and $\mathbb{S}^{>,\succ}$ are respectively called *purely large*, *infinitesimal*, and *positive infinite*.

2.1.2 Well-based families

Let $(f_i)_{i \in I}$ be a family in $\mathbb{S}$, We say that $(f_i)_{i \in I}$ is *well-based* if

i. $\bigcup_{i \in I} \operatorname{supp} f_i$ is well-based, and

ii. $\{i \in I : \mathfrak{m} \in \operatorname{supp} f_i\}$ is finite for all $\mathfrak{m} \in \mathfrak{M}$.

In that case, we may define the sum $\sum_{i \in I} f_i$ of $(f_i)_{i \in I}$ by

$$\sum_{i \in I} f_i \;:=\; \sum_{\mathfrak{m}} \left(\sum_{i \in I} (f_i)_{\mathfrak{m}} \right) \mathfrak{m}.$$

If $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ is another field of well-based series and $\Psi: \mathbb{S} \longrightarrow \mathbb{U}$ is $\mathbb{R}$-linear, then we say that Ψ is *strongly linear* if for every well-based family $(f_i)_{i \in I}$ in $\mathbb{S}$, the family $(\Psi(f_i))_{i \in I}$ is well-based, with

$$\Psi\left(\sum_{i \in I} f_i\right) = \sum_{i \in I} \Psi(f_i).$$

2.2 Surreal numbers

2.2.1 Surreal numbers and simplicity

We denote by **On** the class of ordinal numbers. Following [19], we define **No** to be the class of *sign sequences*

$$a = (a[\beta])_{\beta < \alpha} \in \{-1, +1\}^{\alpha}$$

of ordinal *length* $\alpha \in \mathbf{On}$. The terms $a[\beta] \in \{-1, +1\}$ are called the *signs* of a and we write l_a for the length of a. Given two numbers $a, b \in \mathbf{No}$, we define

$$a \sqsubseteq b \iff l_a \leqslant l_b \wedge (\forall \beta < l_a, a[\beta] = b[\beta]).$$

We call $\sqsubseteq$ the *simplicity relation* on **No** and note that $(\mathbf{No}, \sqsubseteq)$ is well-founded. See [4, Section 2] for more details about the interaction between $\sqsubseteq$ and the ordered field structure of **No**.

Recall that the Conway bracket is characterized by the fact that, for any *sets* L and R of surreal numbers with $L < R$, there exists a unique $\sqsubseteq$-minimal number $\{L \mid R\}$ with $L < \{L \mid R\} < R$. Conversely, given a number $a \in \mathbf{No}$, we define

$$\begin{aligned} a_L &:= \{x \in \mathbf{No} : x \sqsubset a, x < a\} \\ a_R &:= \{x \in \mathbf{No} : x \sqsupset a, x > a\}. \end{aligned}$$

Then a can canonically be written as

$$a = \{a_L \mid a_R\}.$$

2.2.2 Ordinals as surreal numbers

The structure $(\mathbf{No}, \sqsubseteq)$ contains an isomorphic copy of $(\mathbf{On}, <)$ by identifying each ordinal α with the constant sequence $(+1)_{\beta < \alpha}$ of length α. We will write $\boldsymbol{\nu} \leqslant \mathbf{On}$ to state that $\boldsymbol{\nu}$ is either an ordinal or the class of ordinals.

For $\gamma \in \mathbf{On}$, we write ω^{γ} for the ordinal exponentiation of ω to the power γ and we define

$$\omega^{\mathbf{On}} := \{\omega^{\gamma} : \gamma \in \mathbf{On}\}.$$

If $\mu \in \mathbf{On}$ is a successor ordinal, then we define μ_- to be the unique ordinal with $\mu = \mu_- + 1$. We also define $\mu_- := \mu$ if μ is a limit ordinal. Similarly, if $\alpha = \omega^{\mu}$, then we set $\alpha_{/\omega} := \omega^{\mu_-}$. Recall that every ordinal γ has a unique Cantor normal form

$$\gamma = \omega^{\eta_1} n_1 + \cdots + \omega^{\eta_r} n_r,$$

where $r \in \mathbb{N}$, $n_1, \ldots, n_r \in \mathbb{N}^{>0}$ and $\eta_1, \ldots, \eta_r \in \mathbf{On}$ with $\eta_1 > \cdots > \eta_r$.

2.2.3 Surreal numbers as well-based series

We define **Mo** to be the class of positive numbers $\mathfrak{m} \in \mathbf{No}^{>}$ of the form $\mathfrak{m} = \{\mathbb{R}^{>} L \mid \mathbb{R}^{>} R\}$ for certain subsets L and R of **No** with $\mathbb{R}^{>} L < \mathbb{R}^{>} R$. Numbers in **Mo** are called *monomials*. It turns out [11, Theorem 21] that the monomials form a subgroup of $(\mathbf{No}^{>}, \times, <)$ and that there is a natural isomorphism between **No** and the ordered field $\mathbb{R}[[\mathbf{Mo}]]$. We will identify those two fields and thus see **No** as a field of well-based series. The ordinal ω, seen as a surreal number, is the simplest element, or $\sqsubseteq$-minimum, of the class $\mathbf{No}^{>,\succ}$.

3 Surreal substructures

3.1 Surreal substructures

In [4], we introduced the notion of *surreal substructures*. A surreal substructure is a subclass $\mathbf{S}$ of $\mathbf{No}$ such that $(\mathbf{No}, \leqslant, \sqsubseteq)$ and $(\mathbf{S}, \leqslant, \sqsubseteq)$ are isomorphic. The isomorphism $\mathbf{No} \longrightarrow \mathbf{S}$ is unique and denoted by $\Xi_{\mathbf{S}}$. Many important subclasses of $\mathbf{No}$ that are relevant to the study of hyperserial properties of $\mathbf{No}$ are surreal substructures. In particular, it is known that the following classes are surreal substructures:

- The classes $\mathbf{No}^{>}$, $\mathbf{No}^{>,\succ}$ and $\mathbf{No}^{\prec}$ of positive, positive infinite and infinitesimal numbers.
- The classes $\mathbf{Mo}$ and $\mathbf{Mo}^{\succ}$ of monomials and infinite monomials.
- The classes $\mathbf{No}_{\succ}$ and $\mathbf{No}_{\succ}^{>}$ of purely infinite and positive purely infinite numbers.
- The class $\mathbf{Mo}_{\omega}$ of log-atomic numbers.

If $\mathbf{U}, \mathbf{V}$ are surreal substructures, then the class $\mathbf{U} \prec \mathbf{V} \coloneqq \Xi_{\mathbf{U}} \mathbf{V}$ is a surreal substructure with $\Xi_{\mathbf{U} \prec \mathbf{V}} = \Xi_{\mathbf{U}} \circ \Xi_{\mathbf{V}}$.

3.2 Cuts

Given a subclass $\mathbf{X}$ of $\mathbf{No}$ and $a \in \mathbf{X}$, we will write

$$a_L^{\mathbf{X}} \coloneqq \{b \in \mathbf{X} : b < a \wedge b \sqsubseteq a\} \text{ and } a_R^{\mathbf{X}} \coloneqq \{b \in \mathbf{X} : b > a \wedge b \sqsubseteq a\},$$

so that $a_L \coloneqq a_L^{\mathbf{No}}$ and $a_R \coloneqq a_R^{\mathbf{No}}$. We also write $a_{\sqsubset}^{\mathbf{X}} \coloneqq a_L^{\mathbf{X}} \cup a_R^{\mathbf{X}}$ and $a_{\sqsubset} \coloneqq a_{\sqsubset}^{\mathbf{No}}$.

If $\mathbf{X}$ is a subclass of $\mathbf{No}$ and L, R are subsets of $\mathbf{X}$ with $L < S$, then the class

$$(L \mid R)_{\mathbf{X}} \;\coloneqq\; \{a \in \mathbf{X} : (\forall l \in L, l < a) \wedge (\forall r \in R, a < r)\}$$

is called a *cut* in $\mathbf{X}$. If $(L \mid R)_{\mathbf{X}}$ contains a unique simplest element, then we denote this element by $\{L \mid R\}_{\mathbf{X}}$ and say that (L, R) is a *cut representation* (of $\{L \mid R\}_{\mathbf{X}}$) in $\mathbf{X}$. These notations naturally extend to the case when $\mathbf{L}$ and $\mathbf{R}$ are subclasses of $\mathbf{X}$ with $\mathbf{L} < \mathbf{R}$.

A surreal substructure $\mathbf{S}$ may be characterized as a subclass of $\mathbf{No}$ such that for all cut representations (L, R) in $\mathbf{S}$, the cut $(L \mid R)_{\mathbf{S}}$ has a unique simplest element [4, Proposition 4.7].

Let $\mathbf{S}$ be a surreal substructure. Note that we have $a = \{a_L^{\mathbf{S}} \mid a_R^{\mathbf{S}}\}$ for all $a \in \mathbf{S}$. Let $a \in \mathbf{S}$ and let (L, R) be a cut representation of a in $\mathbf{S}$. Then (L, R) is *cofinal with respect to* $(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$ in the sense that L has no strict upper bound in $a_L^{\mathbf{S}}$ and R has no strict lower bound in $a_R^{\mathbf{S}}$ [4, Proposition 4.11(b)].

Given numbers $a, b \in \mathbf{No}$ with $a \leqslant b$, the number $c \coloneqq \{a_L \mid b_R\}$ is the unique $\sqsubseteq$-maximal number with $c \sqsubseteq a, b$. We have $a \leqslant c \leqslant b$. Let $\mathbf{S}$ be a surreal substructure. Considering the isomorphism $\Xi_{\mathbf{S}} : (\mathbf{No}, \leqslant, \sqsubseteq) \longrightarrow (\mathbf{S}, \leqslant, \sqsubseteq)$, we see that for all $a, b \in \mathbf{S}$ with $a \leqslant b$, there is a unique $\sqsubseteq$-maximal element c of $\mathbf{S}$ with $c \sqsubseteq a, b$, and we have $a \leqslant c \leqslant b$. In what follows, we will use this basic fact several times without further mention.

3.3 Cut equations

Let $\mathbf{X} \subseteq \mathbf{No}$ be a subclass, let $\mathbf{T}$ be a surreal substructure and $F : \mathbf{X} \longrightarrow \mathbf{T}$ be a function. Let λ, ρ be functions defined for cut representations in $\mathbf{X}$ and such that $\lambda(L, R), \rho(L, R)$ are subsets of $\mathbf{T}$ whenever (L, R) is a cut representation in $\mathbf{X}$. We say that (λ, ρ) is a *cut equation* for F if for all $a \in \mathbf{X}$, we have

$$\lambda(a_L^{\mathbf{X}}, a_R^{\mathbf{X}}) < \rho(a_L^{\mathbf{X}}, a_R^{\mathbf{X}}), \qquad F(a) = \{\lambda(a_L^{\mathbf{X}}, a_R^{\mathbf{X}}) \mid \rho(a_L^{\mathbf{X}}, a_R^{\mathbf{X}})\}_{\mathbf{T}}.$$

Elements in $\lambda(a_L^{\mathbf{X}}, a_R^{\mathbf{X}})$ (resp. $\rho(a_L^{\mathbf{X}}, a_R^{\mathbf{X}})$) are called *left* (resp. *right*) *options* of this cut equation at a.

We say that the cut equation is *uniform* if we have

$$\lambda(L,R) < \rho(L,R), \quad F(\{L \mid R\}_{\mathbf{X}}) = \{\lambda(L,R) \mid \rho(L,R)\}_{\mathbf{T}}$$

whenever (L,R) is a cut representation in $\mathbf{X}$. For instance, given $r \in \mathbb{R}$, consider the translation $T_r: \mathbf{No} \longrightarrow \mathbf{No}; a \longmapsto a + r$ on $\mathbf{No}$. By [19, Theorem 3.2], we have the following uniform cut equation for T_r on $\mathbf{No}$:

$$\forall a \in \mathbf{No}, \quad a + r = \{a_L + r, a + r_L \mid a + r_R, a_R + r\}. \tag{3.1}$$

We will need the following result from [4]:

Proposition 3.1. [4, Proposition 4.36] *Let $\mathbf{S}, \mathbf{T}$ be surreal substructures. Let Λ be a function from $\mathbf{S}$ to the class of subsets of $\mathbf{T}$ such that for $x, y \in \mathbf{S}$ with $x < y$, the set $\Lambda(y)$ is cofinal with respect to $\Lambda(x)$. For $x \in \mathbf{S}$, let $\mathbf{\Lambda}[x]$ denote the class of elements u of $\mathbf{S}$ such that $\Lambda(x)$ and $\Lambda(u)$ are mutually cofinal. Let $\{\lambda \mid \rho\}_{\mathbf{T}}$ be a cut equation on $\mathbf{S}$ that is extensive in the sense that*

$$\forall x, y \in \mathbf{S}, \quad (x \sqsubseteq y \Longrightarrow (\lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \subseteq \lambda(y_L^{\mathbf{S}}, y_R^{\mathbf{S}}) \wedge \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \subseteq \rho(y_L^{\mathbf{S}}, y_R^{\mathbf{S}}))).$$

Let $F: \mathbf{S} \longrightarrow \mathbf{T}$ be strictly increasing with cut equation

$$\forall x \in \mathbf{S}, \quad F(x) = \{\Lambda(x), \lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \mid \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}})\}_{\mathbf{T}}.$$

Then F induces an embedding $(\mathbf{\Lambda}[x], \leqslant, \sqsubseteq) \longrightarrow (\mathbf{T}, \leqslant, \sqsubseteq)$ for each element x of $\mathbf{S}$.

3.4 Convex partitions

One natural way to obtain surreal substructures is *via* convex partitions. If $\mathbf{S}$ is a surreal substructure, then a *convex partition* of $\mathbf{S}$ is a partition $\mathbf{\Pi}$ of $\mathbf{S}$ whose members are convex subclasses of $\mathbf{S}$ for the order $\leqslant$. We may then consider the class $\mathbf{Smp}_{\mathbf{\Pi}}$ of simplest elements (i.e. $\sqsubseteq$-minima) in each member of $\mathbf{\Pi}$. Those elements are said $\mathbf{\Pi}$-*simple*. For $a \in \mathbf{S}$, we let $\mathbf{\Pi}[a]$ denote the unique member of $\mathbf{\Pi}$ containing a. By [4, Proposition 4.16], the class $\mathbf{\Pi}[a]$ contains a unique $\mathbf{\Pi}$-simple element, which we denote by $\pi_{\mathbf{\Pi}}(a)$. The function $\pi_{\mathbf{\Pi}}$ is a surjective non-decreasing function $\mathbf{S} \longrightarrow \mathbf{Smp}_{\mathbf{\Pi}}$ with $\pi_{\mathbf{\Pi}} \circ \pi_{\mathbf{\Pi}} = \pi_{\mathbf{\Pi}}$.

Given $a, b \in \mathbf{Smp}_{\mathbf{\Pi}}$, note that we have $a < b$ if and only if $\mathbf{\Pi}[a] < \mathbf{\Pi}[b]$. For $\mathbf{X} \subseteq \mathbf{No}$, we write $\mathbf{\Pi}[\mathbf{X}] = \bigcup_{a \in \mathbf{X}} \mathbf{\Pi}[a]$. We have the following criterion to characterize elements of $\mathbf{Smp}_{\mathbf{\Pi}}$.

Proposition 3.2. [4, Lemma 6.5] *An element a of $\mathbf{S}$ is $\mathbf{\Pi}$-simple if and only if there is a cut representation (L,R) of a in $\mathbf{S}$ with $\mathbf{\Pi}[L] < a < \mathbf{\Pi}[R]$. Equivalently $a \in \mathbf{S}$ is $\mathbf{\Pi}$-simple if and only if $\mathbf{\Pi}[a_L^{\mathbf{S}}] < a < \mathbf{\Pi}[a_R^{\mathbf{S}}]$.*

We say that $\mathbf{\Pi}$ is *thin* if each member of $\mathbf{\Pi}$ has a cofinal and coinitial subset. We then have:

Proposition 3.3. [4, Theorem 6.7 and Proposition 6.8] *If $\mathbf{\Pi}$ is thin, then the class $\mathbf{Smp}_{\mathbf{\Pi}}$ is a surreal substructure and $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}}$ has the following uniform cut equation:*

$$\forall z \in \mathbf{No}, \quad \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} z = \{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} z_L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} z_R]\}_{\mathbf{S}}.$$

3.5 Function groups

A special type of thin convex partitions is that of partitions induced by function groups acting on surreal substructures. A *function group* $\mathcal{G}$ on a surreal substructure $\mathbf{S}$ is a set-sized group of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$ under functional composition. We see elements f, g of $\mathcal{G}$ as actions on $\mathbf{S}$ and we sometimes write fg and fa instead of $f \circ g$ and $f(a)$, where $a \in \mathbf{S}$.

For such a function group $\mathcal{G}$, the collection $\mathbf{\Pi}_{\mathcal{G}}$ of classes

$$\mathcal{G}[a] \; := \; \{b \in \mathbf{S} : \exists f, g \in \mathcal{G}, fa \leqslant b \leqslant ga\}$$

with $a \in \mathbf{S}$ is a thin convex partition of $\mathbf{S}$. We write $\mathbf{Smp}_{\mathcal{G}} := \mathbf{Smp}_{\mathbf{\Pi}_{\mathcal{G}}}$. We have the uniform cut equation

$$\forall z \in \mathbf{No}, \quad \Xi_{\mathbf{Smp}_{\mathcal{G}}} z = \{\mathcal{G}\, \Xi_{\mathbf{Smp}_{\mathcal{G}}} z_L \mid \mathcal{G}\, \Xi_{\mathbf{Smp}_{\mathcal{G}}} z_R\}_{\mathbf{S}}. \tag{3.2}$$

Consider sets X, Y of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$, then we say that Y is *pointwise cofinal* with respect to X, and we write $X \trianglelefteq Y$, if we have $\forall f \in X, \forall a \in \mathbf{S}, \exists g \in Y, fa \leqslant ga$. We also define

$$\langle X \rangle \;:=\; \{f_0 \circ f_1 \circ \cdots \circ f_n : n \in \mathbb{N}, f_0, \ldots, f_n \in X \cup X^{-1}\}.$$

It is easy to see that $\langle X \rangle$ is a function group on $\mathbf{S}$ and that we have $\langle X \rangle \trianglelefteq \langle Y \rangle$ if $X \trianglelefteq Y$ or $X^{-1} \trianglelefteq Y^{-1}$. The relation $\langle X \rangle \trianglelefteq \langle Y \rangle$ trivially implies $\mathbf{Smp}_{\langle Y \rangle} \subseteq \mathbf{Smp}_{\langle X \rangle}$. If $X \trianglelefteq Y$ and $Y \trianglelefteq X$, then we say that X and Y are *mutually pointwise cofinal* and we write $X \trianglelefteq\!\!\!\!\!\!\ntriangleright Y$. We then have $\mathbf{Smp}_{\langle X \rangle} = \mathbf{Smp}_{\langle Y \rangle}$.

We write $X \leqslant Y$ (resp. $X < Y$) if we have $\forall a \in \mathbf{S}, \forall f \in X, \forall g \in Y, fa \leqslant ga$ (resp. $\forall a \in \mathbf{S}, \forall f \in X$, $\forall g \in Y, fa < ga$). We also write $f < Y$ and $X < g$ instead of $\{f\} < Y$ and $X < \{g\}$.

Given a function group $\mathcal{G}$ on $\mathbf{S}$, the relation defined by $f < g \Longleftrightarrow \{f\} < \{g\}$ is a partial order on $\mathcal{G}$. We will frequently rely on the basic fact that $(\mathcal{G}, <)$ is *partially bi-ordered* in the sense that

$$\forall f, g, h \in \mathcal{G}, \quad \mathrm{id}_{\mathbf{S}} < g \Longleftrightarrow fh < fgh.$$

3.6 Remarkable function groups

Each of the examples of surreal substructures from Subsection 3.1 can be regarded as the classes $\mathbf{Smp}_{\mathcal{G}}$ for actions of the following function groups $\mathcal{G}$ acting on $\mathbf{No}$, $\mathbf{No}^{>}$ or $\mathbf{No}^{>,\succ}$. For $c \in \mathbb{R}$ and $r \in \mathbb{R}^{>}$, we define

$$\begin{array}{rcll} T_r & := & a \longmapsto a + c & \text{acting on } \mathbf{No} \text{ or } \mathbf{No}^{>,\succ}. \\ H_c & := & a \longmapsto r a & \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ}. \\ P_c & := & a \longmapsto a^r & \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ}. \end{array}$$

Now consider

$$\begin{array}{rcl} \mathcal{T} & := & \{T_c : c \in \mathbb{R}\}, \\ \mathcal{H} & := & \{H_r : r \in \mathbb{R}^{>}\}, \\ \mathcal{P} & := & \{P_r : r \in \mathbb{R}^{>}\}, \\ \mathcal{E}' & := & \langle E_n H_r L_n : n \in \mathbb{N}, r \in \mathbb{R}^{>} \rangle, \text{ and} \\ \mathcal{E}^* & := & \{E_n, L_n : n \in \mathbb{N}\}. \end{array}$$

Then we have the following list of correspondences $\mathcal{G} \longmapsto \mathbf{Smp}_{\mathcal{G}}$:

- The action of $\mathcal{T}$ on $\mathbf{No}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{No}_{\succ}$ (resp. $\mathbf{No}_{\succ}^{\gtrsim}$), e.g. $\mathbf{Smp}_{\mathcal{T}} = \mathbf{No}_{\succ}$.
- The action of $\mathcal{H}$ on $\mathbf{No}^{>}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{Mo}$ (resp. $\mathbf{Mo}^{\succ}$).
- The action of $\mathcal{P}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{Mo} \prec \mathbf{Mo} = E_1 \mathbf{Mo}^{\succ}$.
- The action of $\mathcal{E}'$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{Mo}_{\omega}$.
- The action of $\mathcal{E}^*$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{K} := \mathbf{Mo}_{\omega} \prec \mathbf{No}_{\succ}$ (which will coincide with $E_{\omega} \mathbf{No}_{\succ}^{\gtrsim}$).

Generalizations of those function groups will allow us to define certain surreal substructures related to the hyperlogarithms and hyperexponentials on $\mathbf{No}$.

4 Hyperserial fields

In this section, we briefly recall the definition of hyperserial fields from [5] and how to construct such fields from their hyperserial skeletons.

4.1 Logarithmic hyperseries

Let x be a formal, infinitely large indeterminate. The field $\mathbb{L}$ of *logarithmic hyperseries* of [14] is the smallest field of well-based series that contains all ordinal real power products of the hyperlogarithms $L_\alpha x$ with $\alpha \in \mathbf{On}$. It is naturally equipped with a derivation $\partial \colon \mathbb{L} \longrightarrow \mathbb{L}$ and composition law $\circ \colon \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$.

Definition Let α be an ordinal. For each $\gamma < \alpha$, we introduce the formal hyperlogarithm $\ell_\gamma := L_\gamma x$ and define $\mathfrak{L}_{<\alpha}$ to be the group of formal power products $\mathfrak{l} = \prod_{\gamma<\alpha} \ell_\gamma^{\mathfrak{l}_\gamma}$ with $\mathfrak{l}_\gamma \in \mathbb{R}$. This group comes with a monomial ordering $\succ$ that is defined by

$$\mathfrak{l} \succ 1 \Longleftrightarrow \mathfrak{l}_\beta > 0 \quad \text{for } \beta = \min \{\gamma < \alpha : \mathfrak{l}_\gamma \neq 0\}.$$

We define $\mathbb{L}_{<\alpha}$ to be the ordered field of well-based series $\mathbb{L}_{<\alpha} := \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$. If α, β are ordinals with $\beta < \alpha$, then we define $\mathfrak{L}_{[\beta,\alpha)}$ to be the subgroup of $\mathfrak{L}_{<\alpha}$ of monomials $\mathfrak{l}$ with $\mathfrak{l}_\gamma = 0$ whenever $\gamma < \beta$. As in [14], we write

$$\begin{aligned} \mathbb{L}_{[\beta,\alpha)} &:= \mathbb{R}[[\mathfrak{L}_{[\beta,\alpha)}]] \\ \mathfrak{L} &:= \bigcup_{\alpha \in \mathbf{On}} \mathfrak{L}_{<\alpha} \\ \mathbb{L} &:= \mathbb{R}[[\mathfrak{L}]]. \end{aligned}$$

We have natural inclusions $\mathfrak{L}_{[\beta,\alpha)} \subseteq \mathfrak{L}_{<\alpha} \subset \mathfrak{L}$, hence natural inclusions $\mathbb{L}_{[\beta,\alpha)} \subseteq \mathbb{L}_{<\alpha} \subset \mathbb{L}$.

Derivation on $\mathbb{L}_{<\alpha}$ The field $\mathbb{L}_{<\alpha}$ is equipped with a derivation $\partial \colon \mathbb{L}_{<\alpha} \longrightarrow \mathbb{L}_{<\alpha}$ which satisfies the Leibniz rule and which is strongly linear. Write $\ell_\gamma^\dagger := \prod_{\iota \leqslant \gamma} \ell_\iota^{-1} \in \mathfrak{L}_{<\alpha}$ for all $\gamma < \alpha$. The derivative of a logarithmic hypermonomial $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$ is defined by

$$\partial \mathfrak{l} := \left(\sum_{\gamma<\alpha} \mathfrak{l}_\gamma \ell_\gamma^\dagger \right) \mathfrak{l}.$$

So $\partial \ell_\gamma = \frac{1}{\prod_{\iota<\gamma} \ell_\iota}$ for all $\gamma < \alpha$. For $f \in \mathbb{L}_{<\alpha}$ and $k \in \mathbb{N}$, we will sometimes write $f^{(k)} := \partial^k f$.

Composition on $\mathbb{L}_{<\alpha}$ Assume that $\alpha = \omega^\nu$ for a certain ordinal ν. Then the field $\mathbb{L}_{<\alpha}$ is equipped with a composition $\circ \colon \mathbb{L}_{<\alpha} \times \mathbb{L}_{<\alpha}^{>,\succ} \longrightarrow \mathbb{L}_{<\alpha}$ that satisfies in particular:

- For $g \in \mathbb{L}_{<\alpha}^{>,\succ}$, the map $\mathbb{L}_{<\alpha} \longrightarrow \mathbb{L}_{<\alpha}; f \longmapsto f \circ g$ is a strongly linear embedding [14, Lemma 6.6].
- For $f \in \mathbb{L}_{<\alpha}$ and $g, h \in \mathbb{L}_{<\alpha}^{>,\succ}$, we have $g \circ h \in \mathbb{L}_{<\alpha}^{>,\succ}$ and $f \circ (g \circ h) = (f \circ g) \circ h$ [14, Proposition 7.14].
- For $g \in \mathbb{L}_{<\alpha}^{>,\succ}$ and successor ordinals $\mu < \nu$, we have $\ell_{\omega^\mu} \circ \ell_{\omega^{\mu_-}} = \ell_{\omega^\mu} - 1$ [14, Lemma 5.6].

The same properties hold for the composition $\circ \colon \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$ if α is replaced by $\mathbf{On}$. For $\gamma < \alpha$, the map $\mathbb{L}_{<\alpha} \longrightarrow \mathbb{L}_{<\alpha}; f \longmapsto f \circ \ell_\gamma$ is injective, with image $\mathbb{L}_{[\gamma,\alpha)}$ [14, Lemma 5.11]. For $g \in \mathbb{L}_{[\gamma,\alpha)}$, we define $g^{\uparrow\gamma}$ to be the unique series in $\mathbb{L}_{<\alpha}$ with $g^{\uparrow\gamma} \circ \ell_\gamma = g$.

4.2 Hyperserial fields

Let $\mathfrak{M}$ be an ordered group. A *real powering operation* on $\mathfrak{M}$ is a law

$$\mathbb{R} \times \mathfrak{M} \longrightarrow \mathfrak{M}; (r, \mathfrak{m}) \longmapsto \mathfrak{m}^r$$

of ordered $\mathbb{R}$-vector space on $\mathfrak{M}$. Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series with $\mathfrak{M} \neq 1$, let $\boldsymbol{\nu} \leqslant \mathbf{On}$, and let $\circ \colon \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ be a function. For $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$, we define $\mathfrak{M}_{\omega^{\boldsymbol{\mu}}}$ to be the class of series $s \in \mathbb{T}^{>,\succ}$ with $\forall \gamma < \omega^{\boldsymbol{\mu}}, \ell_\gamma \circ s \in \mathfrak{M}^{\succ}$. We say that $(\mathbb{T}, \circ)$ is a *hyperserial field* if

HF1. $\mathbb{L} \longrightarrow \mathbb{T}; f \longmapsto f \circ s$ is a strongly linear morphism of ordered rings for each $s \in \mathbb{T}^{>,\succ}$.

HF2. $f \circ (g \circ s) = (f \circ g) \circ s$ for all $f \in \mathbb{L}$, $g \in \mathbb{L}^{>,\succ}$, and $s \in \mathbb{T}^{>,\succ}$.

HF3. $f \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $f \in \mathbb{L}$, $t \in \mathbb{T}^{>,\succ}$, and $\delta \in \mathbb{T}$ with $\delta \prec t$.

HF4. $\ell_{\omega^\mu}^{\uparrow \gamma} \circ s < \ell_{\omega^\mu}^{\uparrow \gamma} \circ t$ for all ordinals μ, $\gamma < \omega^\mu$, and $s, t \in \mathbb{T}^{>,\succ}$ with $s < t$.

HF5. The map $\mathbb{R}^> \times \mathfrak{M}^\succ \to \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r := \ell_0^r \circ \mathfrak{m}$ extends to a real powering operation on $\mathfrak{M}$.

HF6. $\ell_1 \circ (s\,t) = \ell_1 \circ s + \ell_1 \circ t$ for all $s, t \in \mathbb{T}^{>,\succ}$.

HF7. $\operatorname{supp} \ell_1 \circ \mathfrak{m} \succ 1$ for all $\mathfrak{m} \in \mathfrak{M}^\succ$;
$\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ (\ell_\gamma \circ \mathfrak{a})^{-1}$ for all $1 \leqslant \mu < \boldsymbol{\nu}$, $\gamma < \omega^\mu$ and $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

For each $\mu \in \mathbf{On}$, we define the function $L_{\omega^\mu} \colon \mathfrak{M}_{\omega^\mu} \longrightarrow \mathbb{T}; \mathfrak{a} \longmapsto \ell_{\omega^\mu} \circ \mathfrak{a}$. The *skeleton* of $(\mathbb{T}, \circ)$ is defined to be the structure $(\mathbb{T}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ equipped with the real power operation from **HF5**.

We say that $(\mathbb{T}, \circ)$ is *confluent* if for all $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$, we have

$$\forall s \in \mathbb{T}^{>,\succ}, \exists \mathfrak{a} \in \mathfrak{M}_{\omega^\mu}, \exists \gamma < \omega^\mu, \quad \ell_\gamma \circ s \asymp \ell_\gamma \circ \mathfrak{a}.$$

In particular $(\mathbb{L}, \circ)$ is a confluent hyperserial field.

4.3 Hyperserial skeletons

It turns out that each hyperlogarithm L_{ω^μ} on a hyperserial field $\mathbb{T}$ can uniquely be reconstructed from its restriction to the subset of $L_{<\omega^\mu}$-atomic hyperseries (here we say that $f \in \mathbb{T}^{>,\succ}$ is $L_{<\omega^\mu}$-atomic if $L_\gamma f \in \mathfrak{M}$ for all $\gamma < \omega^\mu$). One of the main ideas behind [14] is to turn this fact into a way to *construct* hyperserial fields. This leads to the definition of a hyperserial skeleton as a field $\mathbb{T}$ with partially defined hyperlogarithms L_{ω^μ}, which satisfy suitable counterparts of the above axioms **HF1** until **HF7**.

More precisely, let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series and fix $\boldsymbol{\nu} \in \mathbf{On}^> \cup \{\mathbf{On}\}$. A *hyperserial skeleton* on $\mathbb{T}$ of *force* $\boldsymbol{\nu}$ consists of a family of partial functions L_{ω^μ} for $\mu < \boldsymbol{\nu}$, called *(hyper)logarithms*, which satisfy a list of axioms that we will describe now.

First of all, the domains $\mathfrak{M}_{\omega^\mu} := \operatorname{dom} L_{\omega^\mu}$ on which the partial functions L_{ω^μ} are defined should satisfy the following axioms:

Domains of definition:

DD$_\mathbf{0}$. $\operatorname{dom} L_1 = \mathfrak{M}^\succ$;

DD$_{\boldsymbol{\mu}}$. $\operatorname{dom} L_{\omega^\mu} = \bigcap_{\eta < \mu} \operatorname{dom} L_{\omega^\eta}$, if μ is a non-zero limit ordinal;

DD$_{\boldsymbol{\mu}}$. $\operatorname{dom} L_{\omega^\mu} = \{s \in \mathbb{T} : L_{\omega^{\mu_-}}^{\circ n}(s) \in \operatorname{dom} L_{\omega^{\mu_-}}$ for all $n\}$, if μ is a successor ordinal.

It will be convenient to also define the class $\mathfrak{M}_{\omega^{\boldsymbol{\nu}}}$ by

$$\begin{array}{lll} \mathfrak{M}_{\omega^{\boldsymbol{\nu}}} & := & \{s \in \mathbb{T} : L_{\omega^{\boldsymbol{\nu}_-}}^{\circ n}(s) \in \mathfrak{M}_{\omega^{\boldsymbol{\nu}_-}} \text{ for all } n\} \quad \text{if } \boldsymbol{\nu} \text{ is a successor ordinal} \\ \mathfrak{M}_{\omega^{\boldsymbol{\nu}}} & := & \bigcap_{\mu < \boldsymbol{\nu}} \mathfrak{M}_{\omega^\mu} \quad \text{if } \boldsymbol{\nu} \text{ is a non-zero limit ordinal.} \end{array}$$

Consider an ordinal $\gamma < \omega^{\boldsymbol{\nu}}$ written in Cantor normal form $\gamma = \sum_{i=1}^r \omega^{\eta_i} n_i$ where $\eta_1 > \eta_2 > \cdots > \eta_r$ and $n_1, \ldots, n_r < \omega$. We denote by L_γ the partial function

$$L_\gamma := L_{\omega^{\eta_1}}^{\circ n_1} \circ \cdots \circ L_{\omega^{\eta_r}}^{\circ n_r}. \tag{4.1}$$

It follows from the definition that for all $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$, the class $\mathfrak{M}_{\omega^\mu}$ consists of those series $s \in \mathbb{T}^{>,\succ}$ for which $s \in \operatorname{dom} L_\gamma$ and $L_\gamma s \in \mathfrak{M}^\succ$ for all $\gamma < \omega^{\boldsymbol{\mu}}$. We call such series *$L_{<\omega^\mu}$-atomic*.

Secondly, the hyperlogarithms L_{ω^μ} with $\mu<\boldsymbol{\nu}$ should satisfy the following axioms:

Axioms for the logarithm

Functional equation:

FE$_0$. $\forall \mathfrak{m},\mathfrak{n}\in\mathfrak{M}_1, L_1(\mathfrak{m}\,\mathfrak{n})=L_1\,\mathfrak{m}+L_1\,\mathfrak{n}$.

Asymptotics:

A$_0$. $\forall r\in\mathbb{R}^>, \forall \mathfrak{m}\in\mathfrak{M}_1, L_1\,\mathfrak{m}\prec\mathfrak{m}$.

Monotonicity:

M$_0$. $\forall \mathfrak{m},\mathfrak{n}\in\mathfrak{M}_1, \mathfrak{m}\prec\mathfrak{n}\Longrightarrow L_1\,\mathfrak{m}<L_1\,\mathfrak{n}$.

Regularity:

R$_0$. $\forall \mathfrak{m}\in\mathfrak{M}_1, \operatorname{supp} L_1\,\mathfrak{m}\succ 1$.

Surjective logarithm:

SL. $\forall \varphi\in\mathbb{T}^>_\succ, \exists \mathfrak{m}\in\mathfrak{M}_1, \varphi=L_1\,\mathfrak{m}$.

Axioms for the hyperlogarithms (for each $\mu\in\mathbf{On}$ with $0<\mu<\boldsymbol{\nu}$ and $\beta:=\omega^\mu$)

Functional equation:

FE$_\mu$. $\forall \mathfrak{a}\in\mathfrak{M}_\beta, L_\beta L_{\beta/\omega}\,\mathfrak{a}=L_\beta\,\mathfrak{a}-1$ if μ is a successor ordinal.

Asymptotics:

A$_\mu$. $\forall \gamma<\beta, \forall \mathfrak{a}\in\mathfrak{M}_\beta, L_\beta\,\mathfrak{a}<L_\gamma\,\mathfrak{a}$.

Monotonicity:

M$_\mu$. $\forall \mathfrak{a},\mathfrak{b}\in\mathfrak{M}_\beta, \forall\gamma<\beta, \mathfrak{a}\prec\mathfrak{b}\Longrightarrow L_\beta\,\mathfrak{a}+(L_\gamma\,\mathfrak{a})^{-1}<L_\beta\,\mathfrak{b}-(L_\gamma\,\mathfrak{b})^{-1}$.

Regularity:

R$_\mu$. $\forall \mathfrak{a}\in\mathfrak{M}_\beta, \forall\gamma<\beta, \operatorname{supp} L_\beta\,\mathfrak{a}\succ(L_\gamma\,\mathfrak{a})^{-1}$.

Finally, for $\mu\leqslant\boldsymbol{\nu}$ with $\mu\in\mathbf{On}$, we also need the following axiom

Infinite products:

P$_\mu$. $\forall \mathfrak{a}\in\mathfrak{M}_\beta, \forall \mathfrak{l}\in\mathfrak{L}^>_{<\beta}, \sum_{\gamma<\beta}\mathfrak{l}_\gamma L_{\gamma+1}\,\mathfrak{a}\in L_1\,\mathfrak{M}^\succ$.

Note that **SL** and **R**$_0$ together imply $L_1\,\mathfrak{M}^\succ=\mathbb{T}^>_\succ$, whence **P**$_\mu$ automatically holds. This will in particular be the case for **No** (see Section 5).

In summary, we have:

Definition 4.1. [5, Definition 3.3] *Given $\boldsymbol{\nu}\in\mathbf{On}^>\cup\{\mathbf{On}\}$, we say that $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is a* **hyperserial skeleton** *of force $\boldsymbol{\nu}$ if it satisfies* **DD**$_\mu$, **FE**$_\mu$, **A**$_\mu$, **M**$_\mu$, *and* **R**$_\mu$ *for all $\mu<\boldsymbol{\nu}$, as well as* **P**$_\mu$ *for all ordinals $\mu\leqslant\boldsymbol{\nu}$.*

Assume that $\mathbb{T}$ is a hyperserial skeleton of force $\boldsymbol{\nu}$. The partial logarithm $L_1:\mathfrak{M}_1\longrightarrow\mathbb{T}$ extends naturally into a strictly increasing morphism $(\mathbb{T}^>,\times,<)\longrightarrow(\mathbb{T},+,<)$, which we call the *logarithm* and denote by L_1 or log [5, Section 4.1]. If $\mathbb{T}$ satisfies **SL**, then this extended logarithm is actually an isomorphism [29, Proposition 2.3.8]. In that case, for any $s\in\mathbb{T}^>$ and $r\in\mathbb{R}$, we define $s^r:=\exp(r\log s)\in\mathbb{T}^>$.

4.4 Confluence

Definition 4.2. [5, Definition 3.5] *Given a hyperserial skeleton $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ of force $\nu \in \mathbf{On}^{>}$ and $\mu<\nu$, we inductively define the notion of μ-**confluence** in conjunction with the definition of functions $\mathfrak{d}_{\omega^\mu}\colon \mathbb{T}^{>,\succ} \longrightarrow \mathfrak{M}_{\omega^\mu}$, as follows.*

- *The field $\mathbb{T}$ is said 0-confluent if $\mathfrak{M}$ is non-trivial. The function $\mathfrak{d}_1$ maps every positive infinite series $s\in\mathbb{T}^{>,\succ}$ onto its dominant monomial $\mathfrak{d}_s$. For each $s\in\mathbb{T}^{>,\succ}$, we write*

$$\mathcal{E}_1[s] \;:=\; \{t\in\mathbb{T}^{>,\succ}: t\asymp s\}.$$

Let $\mu\leqslant\boldsymbol{\nu}$ be such that $\mathbb{T}$ is η-confluent for all $\eta<\mu$ and let $s\in\mathbb{T}^{>,\succ}$.

- *If μ is a successor ordinal, then we write $\mathcal{E}_{\omega^\mu}[s]$ for the class of series t with*

$$(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(s) \;\asymp\; (L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(t)$$

  for a certain $n\in\mathbb{N}$.
- *If μ is a limit ordinal, then we write $\mathcal{E}_{\omega^\mu}[s]$ for the class of series t with*

$$L_{\omega^\eta}\mathfrak{d}_{\omega^\eta}(s) \;\asymp\; L_{\omega^\eta}\mathfrak{d}_{\omega^\eta}(t)$$

  for a certain $\eta<\mu$.

*We say that $\mathbb{T}$ is **μ-confluent** if each class $\mathcal{E}_{\omega^\mu}[s]$ contains a $L_{<\omega^\mu}$-atomic element; we then define $\mathfrak{d}_{\omega^\mu}(s)$ to be this element.*

This inductive definition is sound. Indeed, if $\mu\leqslant\nu+1$ and $\mathbb{T}$ is η-confluent for all $\eta<\mu$, then the functions $\mathfrak{d}_{\omega^\eta}\colon\mathbb{T}^{>,\succ}\longrightarrow\mathfrak{M}_{\omega^\eta}$ with $\eta<\mu$ are well-defined and non-decreasing. Thus, for $\eta<\mu$, the collection of $\mathcal{E}_{\omega^\eta}[s]$ with $s\in\mathbb{T}^{>,\succ}$ forms a partition of $\mathbb{T}^{>,\succ}$ into convex subclasses.

We say that $\mathbb{T}$ is *confluent* if it is ν-confluent. If $\mathbb{T}$ has force $\mathbf{On}$, then we say that $\mathbb{T}$ is $\mathbf{On}$-confluent, or *confluent*, if $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu})$ is μ-confluent for all $\mu\in\mathbf{On}$.

4.5 Correspondence between fields and skeletons

Proposition 4.3. [5, Theorem 1.1] *If $(\mathbb{T},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ is a confluent hyperserial skeleton, then there is a unique function $\circ\colon\mathbb{L}\times\mathbb{T}^{>,\succ}\longrightarrow\mathbb{T}$ with*

$$\forall\mu\in\mathbf{On},\forall\mathfrak{a}\in\mathfrak{M}_{\omega^\mu},\quad \ell_{\omega^\mu}\circ\mathfrak{a}=L_{\omega^\mu}\mathfrak{a}$$

such that $(\mathbb{T},\circ)$ is a confluent hyperserial field.

Assume now that $\mathbb{T}$ is only a hyperserial skeleton of force $\boldsymbol{\nu}\in\mathbf{On}^{>}\cup\{\mathbf{On}\}$ and that μ is an ordinal with $0<\mu<\boldsymbol{\nu}$ such that $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu})$ is μ-confluent. Let $\beta:=\omega^\mu$. By [5, Definition 4.11 and Lemma 4.12], the partial function L_β naturally extends into a function $\mathbb{T}^{>,\succ}\longrightarrow\mathbb{T}^{>,\succ}$ that we still denote by L_β. This extended function is strictly increasing, by‘ [5, Corollary 4.17]. If μ is a successor ordinal, then it satisfies the functional equation

$$\forall s\in\mathbb{T}^{>,\succ},\quad L_\beta L_{\beta/\omega}s=L_\beta s-1, \tag{4.2}$$

by [5, Proposition 4.13]. For $\gamma<\beta$, we have a strictly increasing function $L_\gamma\colon\mathbb{T}^{>,\succ}\longrightarrow\mathbb{T}^{>,\succ}$ obtained as a composition of functions L_{ω^η} with $\eta<\mu$, as in (4.1). By [5, Proposition 4.7], we have

$$\mathcal{E}_\beta[s] \;=\; \{t\in\mathbb{T}^{>,\succ}:\exists\gamma<\beta,L_\gamma t\asymp L_\gamma s\}.$$

4.6 Hyperexponentiation

In a traditional transseries field $\mathbb{T}$, the transmonomials are characterized by the fact that, for any $f\in\mathbb{T}^{>}$, we have

$$f\in\mathfrak{M}\iff\operatorname{supp}\log f\succ 1. \tag{4.3}$$

In particular, the logarithm $\log: \mathbb{T}^{>} \longrightarrow \mathbb{T}$ is surjective as soon as $\exp \varphi$ is defined for all $\varphi \in \mathbb{T}$ with $\operatorname{supp} \varphi \succ 1$. In hyperserial fields, similar properties hold for $L_{<\omega^\eta}$-atomic elements with respect to the hyperexponential E_{ω^η}, as we will recall now.

Given $\boldsymbol{\nu} \in \mathbf{On}^{>} \cup \{\mathbf{On}\}$, let $\mathbb{T}$ be a confluent hyperserial skeleton $\mathbb{T}$ of force $\boldsymbol{\nu}$. By [5, Theorem 4.1], we have a composition $\circ: \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$. Given $\eta < \boldsymbol{\nu}$, the extended function $L_{\omega^\eta}: \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ is strictly increasing and hence injective. Consequently, L_{ω^η} has a partially defined functional inverse that we denote by E_{ω^η}.

The characterization (4.3) generalizes as follows:

Definition 4.4. [5, Definition 7.10] *We say that $\varphi \in \mathbb{T}^{>,\succ}$ is* **1-truncated** *if*

$$\operatorname{supp} \varphi \succ 1.$$

Given $0 < \eta < \nu$, we say that a series $\varphi \in \mathbb{T}^{>,\succ}$ is **$\boldsymbol{\omega^\eta}$-truncated** *if*

$$\forall \mathfrak{m} \in \operatorname{supp} \varphi, \quad \mathfrak{m} \prec 1 \Longrightarrow \left(\forall \gamma < \omega^\eta, \varphi < \ell_{\omega^\eta}^{\uparrow \gamma} \circ \mathfrak{m}^{-1}\right).$$

For any $\beta = \omega^\eta < \omega^{\boldsymbol{\nu}}$, we write $\mathbb{T}_{\succ,\beta}$ for the class of β-truncated series in $\mathbb{T}$.

Proposition 4.5. [5, Corollary 7.21] *For $f \in \mathbb{T}^{>,\succ}$ and $\beta = \omega^\eta < \omega^{\boldsymbol{\nu}}$, we have*

$$f \in \mathfrak{M}_\beta \iff L_\beta f \in \mathbb{T}_{\succ,\beta}.$$

In general, we have $\mathbb{T}_{\succ,\beta} + \mathbb{R}^{\geqslant} \subseteq \mathbb{T}_{\succ,\beta}$. Whenever η is a successor ordinal, we even have

$$\mathbb{T}_{\succ,\beta} + \mathbb{R} = \mathbb{T}_{\succ,\beta} \tag{4.4}$$

Let φ be a series such that $E_\beta \varphi$ is defined. By [5, Lemma 7.14], the series φ is β-truncated if and only if

$$\forall \gamma < \beta, \quad \operatorname{supp} \varphi \succ (L_\gamma E_\beta \varphi)^{-1}.$$

For $\mu < \boldsymbol{\nu}$, the axiom $\mathbf{R}_\mu$ is therefore equivalent to the inclusion $L_{\omega^\mu} \mathfrak{M}_{\omega^\mu} \subseteq \mathbb{T}_{\succ,\omega^\mu}$. For $s \in \mathbb{T}^{>,\succ}$, there is a unique $\trianglelefteq$-maximal truncation $\sharp_\beta(s)$ of s which is β-truncated. By [5, Propositions 6.16 and 6.17], the classes

$$\mathscr{L}_\beta[s] := \left\{t \in s + \mathbb{T}^{\prec} : t = s, \text{ or } \exists \gamma < \beta, \left(t < \ell_\beta^{\uparrow \gamma} \circ |s - t|^{-1}\right)\right\} \tag{4.5}$$

with $s \in \mathbb{T}^{>,\succ}$ form a partition of $\mathbb{T}^{>,\succ}$ into convex subclasses. Moreover, the series $\sharp_\beta(s)$ is both the unique β-truncated element and the $\trianglelefteq$-minimum of $\mathscr{L}_\beta[s]$. If $E_\beta s$ is defined, then we have the following simplified definition [5, Proposition 7.19] of the class $\mathscr{L}_\beta[s]$:

$$\mathscr{L}_\beta[s] := \left\{t \in \mathbb{T}^{>,\succ} : \exists \gamma < \beta, t - s \prec \frac{1}{L_\gamma E_\beta s}\right\}. \tag{4.6}$$

The following shows that the existence of E_β on $\mathbb{T}^{>,\succ}$ is essentially equivalent to its existence on $\mathbb{T}_{\succ,\beta}$.

Proposition 4.6. [5, Corollary 7.24] *Let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ and assume that for $\eta < \boldsymbol{\mu}$, the function E_{ω^η} is defined on $\mathbb{T}_{\succ,\omega^\eta}$. Then each hyperlogarithm L_{ω^η} for $\eta < \boldsymbol{\mu}$ is bijective.*

If Proposition 4.6 holds, then we say that $\mathbb{T}$ is a (confluent) *hyperserial field of force* $(\boldsymbol{\nu}, \boldsymbol{\mu})$. Since every function $L_\gamma, \gamma < \omega^{\boldsymbol{\mu}}$ is then a strictly increasing bijection $\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$, we obtain

$$\mathscr{E}_\lambda[s] = \{t \in \mathbb{T}^{>,\succ} : \exists \gamma < \lambda, \exists r_0, r_1 \in \mathbb{R}^{>}, E_\gamma(r_0 L_\gamma s) < t < E_\gamma(r_1 L_\gamma s)\}, \tag{4.7}$$

for each ordinal $\lambda=\omega^{\iota}$ with $\iota\leqslant\boldsymbol{\mu}$. By [5, Corollary 7.23], for all $s\in\mathbb{T}^{>,\succ}$, we have

$$E_\beta(\sharp_\beta(s)) \;=\; \eth_\beta(E_\beta s). \tag{4.8}$$

5 The transseries field No

Recall that **No** is identified with the ordered field of well-based series $\mathbb{R}[[\mathbf{Mo}]]$. In this section, we describe, in the first level $\nu=1$ of our hierarchy, the properties of **No** equipped with the Kruskal-Gonshor logarithm.

5.1 Surreal exponentiation

In [19, Chapter 10], Gonshor defines the exponential function $\exp:\mathbf{No}\longrightarrow\mathbf{No}^{>}$, relying on partial Taylor sums of the real exponential function. For $a\in\mathbf{No}$ and $n\in\mathbb{N}$, write

$$[a]_n \;\coloneqq\; \sum_{k\leqslant n}\frac{a^k}{k!}.$$

We then have the recursive definition

$$\forall a\in\mathbf{No},\quad \exp a\coloneqq\left\{\exp(a_L)\,[a-a_L]_{\mathbb{N}},\exp(a_R)\,[a-a_R]_{2\mathbb{N}+1}\;\middle|\;\frac{\exp a_R}{[a_R-a]_{\mathbb{N}}},\frac{\exp a_L}{[a_L-a]_{2\mathbb{N}+1}}\right\}.$$

We will sometimes write e^a instead of $\exp a$. The function $\exp:(\mathbf{No},+,<)\longrightarrow(\mathbf{No}^{>},\times,<)$ is a bijective morphism [19, Corollary 10.1, Corollary 10.3], which satisfies:

- exp coincides with the natural exponential on $\mathbb{R}\subseteq\mathbf{No}$ [19, Theorem 10.2].
- $\mathrm{e}^{\mathbf{No}_{\succ}}=\mathbf{Mo}$ [19, Theorems 10.7, 10.8 and 10.9].

We define $\log:\mathbf{No}^{>}\longrightarrow\mathbf{No}$ to be the functional inverse of exp, and we set $L_1\coloneqq\log\upharpoonright\mathbf{Mo}^{\succ}$. Given an ordinal α, we understand that ω^α still stands for the α-th ordinal power of ω from section 2.2.2 and warn the reader that ω^α does not necessarily coincide with $\mathrm{e}^{\alpha\log\omega}$.

Together, the above facts imply that L_1 satisfies the axioms $\mathbf{FE}_0$, $\mathbf{A}_0$, $\mathbf{M}_0$, $\mathbf{R}_0$ and **SL**. Therefore, $(\mathbf{No},L_1)$ is a hyperserial skeleton of force 1. The extension of L_1 to $\mathbf{No}^{>}$ from section 4.5 coincides with log. It was shown in [13] that $(\mathbf{No},+,\times,<,\exp)$ is an elementary extension of $(\mathbb{R},+,\times,<,\exp)$. See [28, 7, 8] for more details on exp and log.

5.2 No as a transseries field

Berarducci and Mantova identified the class $\mathbf{Mo}_\omega$ of log-atomic numbers as $\mathbf{Mo}_\omega=\mathbf{Smp}_{\mathscr{C}}$ [9, Corollary 5.17] and showed that $(\mathbf{No},L_1)$ is 1-confluent [9, Corollary 5.11]. Thus $(\mathbf{No},L_1)$ is a confluent hyperserial skeleton of force $(1,1)$. Thanks to [5, Theorem 1.1], it is therefore equipped with a composition law $\mathbb{L}_{<\omega}\times\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}$. See [29, 10] for further details on extensions of this composition law to exponential extensions of $\mathbb{L}_{<\omega}$.

Berarducci and Mantova also proved [9, Theorem 8.10] that **No** is a field of transseries in the sense of [24, 29], i.e. that $(\mathbf{No},L_1)$ satisfies the axiom **T4** of [29, Definition 2.2.1]. We plan to prove in subsequent work that $(\mathbf{No},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ satisfies a generalized version of **T4**.

6 Hyperserial structure on No

We have seen in section 5 that $(\mathbf{No},L_1)$ is a confluent hyperserial skeleton of force (ν,ν) for $\nu=1$. The aim of this section is to extend this result to any ordinal ν. More precisely, we will define a sequence $(L_{\omega^\mu})_{\mu\in\mathbf{On}}$ of partial functions on **No** such that for each ordinal ν, the structure $(\mathbf{No},(L_{\omega^\mu})_{\mu<\nu})$ is a confluent hyperserial skeleton of force (ν,ν), and L_1 coincides with Gonshor's logarithm.

6.1 Remarkable group actions on No

Assume for the moment that we can define L_γ and E_γ as bijective strictly increasing functions on $\mathbf{No}^{>,\succ}$ for all ordinals γ. This is the case already for $\gamma<\omega$. Let us introduce several useful groups that act on $\mathbf{No}$, as well as several remarkable subclasses of $\mathbf{No}$.

Given an ordinal ν, we write $\alpha=\omega^\nu$ and we consider the function groups

$$\begin{aligned}\mathscr{E}_\alpha' &= \langle E_\gamma H_r L_\gamma : \gamma<\alpha \wedge r\in\mathbb{R}^>\rangle\\ \mathscr{E}_\alpha^* &= \langle E_\gamma, P_r : \gamma<\alpha\wedge r\in\mathbb{R}^>\rangle.\end{aligned}$$

where E_γ, H_s, P_s and L_γ act on $\mathbf{No}^{>,\succ}$. We also define

$$\begin{aligned}\mathscr{L}_\alpha' &= L_\alpha \mathscr{E}_\alpha' E_\alpha\\ \mathscr{L}_\alpha^* &= L_\alpha \mathscr{E}_\alpha^* E_\alpha.\end{aligned}$$

We write $L_{<\lambda}:=\{L_\gamma:\gamma<\lambda\}$ and $E_{<\lambda}:=\{E_\gamma:\gamma<\lambda\}$ for each $\lambda\leqslant\alpha$. In the case when $\alpha=1$, note that

$$\begin{aligned}\mathscr{E}_1' &= \mathscr{H}\\ \mathscr{E}_1^* &= \mathscr{P}\\ \mathscr{L}_1' &= \mathscr{T}\\ \mathscr{L}_1^* &= \mathscr{H}.\end{aligned}$$

By Proposition 3.3 and the fact the set-sized function groups $\mathscr{E}_\alpha'$, $\mathscr{E}_\alpha^*$, $\mathscr{L}_\alpha'$, and $\mathscr{L}_\alpha^*$ induce thin partitions of $\mathbf{No}^{>,\succ}$, we may define the following surreal substructures

$$\begin{aligned}\mathbf{Mo}_\alpha' &:= \mathbf{Smp}_{\mathscr{E}_\alpha'}\\ \mathbf{Mo}_\alpha^* &:= \mathbf{Smp}_{\mathscr{E}_\alpha^*}\\ \mathbf{Tr}_\alpha &:= \mathbf{Smp}_{\mathscr{L}_\alpha'}\\ \mathbf{Tr}_\alpha^* &:= \mathbf{Smp}_{\mathscr{L}_\alpha^*}.\end{aligned}$$

Here we note that $\mathbf{Mo}_1'$ corresponds to the class $\mathbf{Mo}^{\succ}=\mathbf{Mo}_1$ of infinite monomials in $\mathbf{No}$ and $\pi_{\mathscr{E}_1}'$ maps positive infinite numbers to their dominant monomial. Similarly, $\mathbf{Tr}_1$ coincides with $\mathbf{No}_{\succ}^{>}$ and $\pi_{\mathscr{L}_1'}$ maps $a\in\mathbf{No}^{>,\succ}$ to $a_\succ$. In sections 6 and 7, we will prove the following identities.

$$\begin{aligned}
\mathbf{Mo}_\alpha' &= \mathbf{Mo}_\alpha, && \text{[Proposition 6.18]}\\
\pi_{\mathscr{E}_\alpha'} &= \mathfrak{d}_\alpha, && \text{[Proposition 6.18]}\\
\mathbf{Tr}_\alpha &= \mathbf{No}_{\succ,\alpha}=L_\alpha\mathbf{Mo}_\alpha, && \text{[Proposition 7.6]}\\
\pi_{\mathscr{L}_\alpha'} &= \sharp_\alpha, && \text{[Proposition 7.6]}\\
\mathbf{Tr}_\alpha^* &= \mathbf{Tr}_\alpha \quad \text{if } \nu \text{ is a limit ordinal,} && \text{[Lemma 6.11]}\\
\mathbf{Tr}_\alpha^* &= \mathbf{No}_{\succ}^{>} \quad \text{if } \nu \text{ is a successor ordinal,} && \text{[Lemma 7.8]}\\
\forall r\in\mathbb{R}, \Xi_{\mathbf{No}_{\succ,\alpha}} T_r &= T_r\,\Xi_{\mathbf{No}_{\succ,\alpha}} \quad \text{if } \nu \text{ is a successor ordinal,} && \text{[Lemma 7.7]}\\
\forall r\in\mathbb{R}, \Xi_{\mathbf{Mo}_\alpha} T_r &= E_\alpha T_r L_\alpha\,\Xi_{\mathbf{Mo}_\alpha} \quad \text{if } \nu \text{ is a successor ordinal,} && \text{[Proposition 7.10]}\\
\mathbf{Mo}_\alpha^* &= \mathbf{Mo}_\alpha \prec \mathbf{No}_\succ \quad \text{if } \nu \text{ is a successor ordinal,} && \text{[Proposition 7.12]}\\
\mathbf{Mo}_\alpha^* &= E_\alpha \mathbf{Tr}_\alpha^*. && \text{[Proposition 7.13]}
\end{aligned}$$

The first and third identities imply in particular that the classes $\mathbf{Mo}_\alpha$ and $\mathbf{No}_{\succ,\alpha}$ from section 4 are in fact surreal substructures, when regarding $\mathbf{No}$ as a hyperserial field.

6.2 Inductive setting

For the definition of the partial hyperlogarithm L_{ω^μ}, we will proceed by induction on μ. Let μ be an ordinal. Until the end of this section we make the following induction hypotheses:

Induction hypotheses

$\mathbf{I}_{1,\mu}$. For $\eta<\mu$, the partial hyperlogarithm L_{ω^η} is defined; we have $L_1=\log\upharpoonright\mathbf{Mo}^{\succ}$ and $(\mathbf{No},(L_{\omega^\eta})_{\eta<\mu})$ is a confluent hyperserial skeleton of force (μ,μ).

$\mathbf{I}_{2,\mu}$. For $r,s\in\mathbb{R}$ with $1<s$ and for $\gamma,\rho<\omega^\mu$ with $\gamma<\rho$, we have

$$\forall a\in\mathbf{No}^{>,\succ},\quad E_\gamma(rL_\gamma a)<E_\rho(sL_\rho a).$$

$\mathbf{I}_{3,\mu}$. For $\eta\leqslant\mu$, the class $\mathbf{Mo}'_{\omega^\eta}$ is that of $L_{<\omega^\eta}$-atomic surreal numbers, i.e. $\mathbf{Mo}'_{\omega^\eta}=\mathbf{Mo}_{\omega^\eta}$.

These induction hypotheses require a few additional explanations. Assuming that $\mathbf{I}_{1,\mu}$ holds, the partial functions L_{ω^η} with $\eta<\mu$ extend into strictly increasing bijections $L_{\omega^\eta}:\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}^{>,\succ}$, by the results from section 4. Using (1.3), this allows us to define a strictly increasing bijection $L_\gamma:\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}^{>,\succ}$ for any $\gamma<\mu$ and we denote by E_γ its functional inverse. In particular, this ensures that the hypotheses $\mathbf{I}_{2,\mu}$ and $\mathbf{I}_{3,\mu}$ make sense.

Remark 6.1. In addition to the above induction hypotheses, we will implicitly assume that our hyperlogarithms L_{ω^η} for $\eta<\mu$ are always defined by (6.1) below. In particular, our construction of L_{ω^μ} is *not* relative to any potential construction of the preceding hyperlogarithms L_{ω^η} with $\eta<\mu$ that would satify the induction hypotheses $\mathbf{I}_{1,\mu}$, $\mathbf{I}_{2,\mu}$, and $\mathbf{I}_{3,\mu}$. Instead, we define *one* specific family of functions $(L_{\omega^\eta})_{\eta\in\mathbf{On}}$ that satisfy our requirements, as well as the additional identities listed in subsection 6.1.

Proposition 6.2. *The axioms* $\mathbf{I}_{1,1}$, $\mathbf{I}_{2,1}$ *and* $\mathbf{I}_{3,1}$ *hold for* $(\mathbf{No},L_1)$.

Proof. Section 5 shows that $\mathbf{I}_{1,1}$ holds. Consider $r,s\in\mathbb{R}^>$ with $s>1$. On $\mathbf{No}^{>,\succ}$, we have $T_{\log r}<H_s$, hence $H_r=E_1\,T_{\log r}L_1<E_1\,H_sL_1$. It follows that we have $E_n\,H_rL_n<E_{n+1}\,H_sL_{n+1}$ on $\mathbf{No}^{>,\succ}$ for all $n\in\mathbb{N}$. This implies that $\mathbf{I}_{2,1}$ holds. Finally, $\mathbf{I}_{3,1}$ is valid because of the relation $\mathbf{Mo}_\omega=\mathbf{Smp}_{\mathscr{E}}$. □

Proposition 6.3. *Let ν be a limit ordinal and assume that* $\mathbf{I}_{1,\mu}$, $\mathbf{I}_{2,\mu}$, *and* $\mathbf{I}_{3,\mu}$ *hold for all* $\mu<\nu$. *Then* $\mathbf{I}_{1,\nu}$, $\mathbf{I}_{2,\nu}$, *and* $\mathbf{I}_{3,\nu}$ *hold.*

Proof. The statement $\mathbf{I}_{2,\nu}$ follows immediately by induction. Towards $\mathbf{I}_{3,\nu}$, note that we have $\mathbf{Mo}_\alpha=\bigcap_{\eta<\nu}\mathbf{Mo}_{\omega^\eta}=\bigcap_{\eta<\nu}\mathbf{Mo}'_{\omega^\eta}$ by $\mathbf{I}_{1,\eta}$ (and thus $\mathbf{DD}_\eta$) and $\mathbf{I}_{3,\eta}$ for all $\eta<\nu$. By [4, Proposition 6.28], we have $\mathbf{Mo}'_\alpha=\bigcap_{\eta<\nu}\mathbf{Mo}'_{\omega^\eta}=\mathbf{Mo}_\alpha$. So $\mathbf{I}_{3,\nu}$ holds.

By $\mathbf{I}_{1,\eta}$ for all $\eta<\nu$, we need only justify that $(\mathbf{No},(L_{\omega^\eta})_{\eta<\nu})$ is ν-confluent to deduce that $\mathbf{I}_{1,\nu}$ holds. For $a\in\mathbf{No}^{>,\succ}$, by $\mathbf{I}_{2,\nu}$, there are a $\mathfrak{a}\in\mathbf{Mo}'_\alpha=\mathbf{Mo}_\alpha$ and a $\beta:=\omega^\eta<\alpha$ with $E_\beta({}^1\!/_2\,L_\beta a)\leqslant\mathfrak{a}\leqslant E_\beta(2\,L_\beta a)$. We deduce that $L_\beta a\asymp L_\beta\mathfrak{a}$, thus $\mathfrak{a}\in\mathscr{E}_\beta[a]$. This concludes the proof. □

From now on, we assume that $\mathbf{I}_{1,\mu}$, $\mathbf{I}_{2,\mu}$, and $\mathbf{I}_{3,\mu}$ are satisfied for $\mu\geqslant1$ and we define

$$\begin{aligned}\nu&:=\mu+1\\ \alpha&:=\omega^\nu\\ \beta&:=\omega^\mu.\end{aligned}$$

The remainder of the section is dedicated to the definition of L_β and the proof of the inductive hypotheses $\mathbf{I}_{1,\nu}$, $\mathbf{I}_{2,\nu}$, and $\mathbf{I}_{3,\nu}$ for ν. In combination with Propositions 6.2 and 6.3, this will complete our induction and the proof of Theorem 1.1.

6.3 Defining the hyperlogarithm

Recall that we have $\mathbf{Mo}'_\beta=\mathbf{Mo}_\beta$ by $\mathbf{I}_{3,\mu}$. In particular $\mathbf{Mo}_\beta$ is a surreal substructure. Consider $\eta<\nu$. The skeleton $(\mathbf{No},(L_{\omega^\iota})_{\iota<\eta})$ is a confluent hyperserial skeleton of force (η,η) by $\mathbf{I}_{1,\mu}$. So for $a\in\mathbf{No}^{>,\succ}$, (4.7) and $\mathbf{I}_{2,\mu}$ yield $\mathscr{E}_{\omega^\eta}[a]=\mathscr{E}'_{\omega^\eta}[a]$.

In view of $\mathbf{A}_\mu$ and $\mathbf{M}_\mu$, the simplest way to define L_β is *via* the cut equation:

$$\forall \mathfrak{a} \in \mathbf{Mo}_\beta, \quad L_\beta \mathfrak{a} := \left\{ \mathbb{R}, L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta} \middle| L_\beta \mathfrak{a}_R^{\mathbf{Mo}_\beta} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a} \right\}. \tag{6.1}$$

Note the asymmetry between left and right options $L_\beta \mathfrak{a}' + (L_{<\beta} \mathfrak{a}')^{-1}$ and $L_\beta \mathfrak{a}'' - (L_{<\beta} \mathfrak{a})^{-1}$ (instead of $L_\beta \mathfrak{a}'' - (L_{<\beta} \mathfrak{a}'')^{-1}$) for generic $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$. In Corollary 7.4 below, we will derive a more symmetric but equivalent cut equation for L_β, as promised in the introduction. For now, we prove that (6.1) is warranted and that $\mathbf{A}_\mu$, $\mathbf{M}_\mu$, and $\mathbf{R}_\mu$ hold.

Proposition 6.4. *The function L_β is well-defined on $\mathbf{Mo}_\beta$ and, for $\mathfrak{a} \in \mathbf{Mo}_\beta$, we have*

$\mathbf{H}_\mathfrak{a}$: $\left(\forall \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}, L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}\right)$ *and* $\left(\forall \mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}, L_\beta \mathfrak{a} + \frac{1}{L_{<\beta} \mathfrak{a}} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}''}\right)$.

Proof. We prove this by induction on $(\mathbf{Mo}_\beta, \sqsubseteq)$. Let $\mathfrak{a} \in \mathbf{Mo}_\beta$ such that $\mathbf{H}_\mathfrak{b}$ holds for all $\mathfrak{b} \in \mathfrak{a}_\sqsubset^{\mathbf{Mo}_\beta}$. Let $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$. We have $\mathfrak{a}' \in (\mathfrak{a}'')_L^{\mathbf{Mo}_\beta}$ or $\mathfrak{a}'' \in (\mathfrak{a}')_R^{\mathbf{Mo}_\beta}$, so $\mathbf{H}_{\mathfrak{a}'}$ or $\mathbf{H}_{\mathfrak{a}''}$ yields

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}''}.$$

For $\gamma < \beta$, we have $\ell_{\gamma+1} \prec \frac{1}{2} \ell_\gamma$ and $\frac{1}{L_\gamma \mathfrak{a}'} \succ \frac{1}{L_\gamma \mathfrak{a}''}, \frac{1}{L_\gamma \mathfrak{a}}$, whence

$$\frac{1}{L_{\gamma+1} \mathfrak{a}'} > \frac{2}{L_\gamma \mathfrak{a}'} > \frac{1}{L_\gamma \mathfrak{a}'} + \frac{1}{L_\gamma \mathfrak{a}''} + \frac{1}{L_\gamma \mathfrak{a}},$$

for all $\gamma < \beta$. Hence,

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}}.$$

We clearly have $L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}} \asymp L_\beta \mathfrak{a}'' > \mathbb{R}$. Finally,

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} \asymp L_\beta \mathfrak{a}' \prec L_{<\beta} \mathfrak{a}',$$

so $L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_{<\beta} \mathfrak{a}$. This shows that $L_\beta \mathfrak{a}$ is defined and

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}}.$$

Since $\mathfrak{a}' < \mathfrak{a} < \mathfrak{a}''$, it follows that

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a} \pm \frac{1}{L_{<\beta} \mathfrak{a}} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}}.$$

By induction, this proves $\mathbf{H}_\mathfrak{a}$ for all $\mathfrak{a} \in \mathbf{Mo}_\beta$. □

Proposition 6.5. *The axiom $\mathbf{M}_\mu$ holds.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \mathbf{Mo}_\beta$ with $\mathfrak{a} < \mathfrak{b}$. Since $\mathbf{Mo}_\beta$ is a surreal substructure, there is a $\mathfrak{c} \in \mathbf{Mo}_\beta$ with $\mathfrak{c} \sqsubseteq \mathfrak{a}, \mathfrak{b}$ and $\mathfrak{a} \leqslant \mathfrak{c} \leqslant \mathfrak{b}$. If $\mathfrak{a} < \mathfrak{c}$, then we have $L_\beta \mathfrak{a} + (L_{<\beta} \mathfrak{a})^{-1} < L_\beta \mathfrak{c} - (L_{<\beta} \mathfrak{c})^{-1}$ by $\mathbf{H}_\mathfrak{a}$. If $\mathfrak{c} < \mathfrak{b}$, then we have $L_\beta \mathfrak{c} + (L_{<\beta} \mathfrak{c})^{-1} < L_\beta \mathfrak{b} - (L_{<\beta} \mathfrak{b})^{-1}$ by $\mathbf{H}_\mathfrak{b}$. We cannot have both $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{c} = \mathfrak{b}$, so this proves that $L_\beta \mathfrak{a} + (L_{<\beta} \mathfrak{a})^{-1} < L_\beta \mathfrak{b} - (L_{<\beta} \mathfrak{b})^{-1}$. Therefore $\mathbf{M}_\mu$ holds. □

Proposition 6.6. *The axiom $\mathbf{A}_\mu$ holds.*

Proof. The rightmost options in (6.1) directly yield $\mathbf{A}_\mu$. □

Proposition 6.7. *The axiom* $\mathbf{R}_\mu$ *holds.*

Proof. Let $\mathfrak{a} \in \mathbf{Mo}_\beta$ and write $\varphi := L_\beta \mathfrak{a}$. Let $\mathfrak{m} \in \operatorname{supp} \varphi$ with $\mathfrak{m} \prec 1$. We have $\varphi < L_{<\beta} \mathfrak{a}$ and $\varphi_{\succ \mathfrak{m}} \asymp \varphi$ so $\varphi_{\succ \mathfrak{m}} < L_{<\beta} \mathfrak{a}$. Moreover $\varphi_{\succ \mathfrak{m}}$ is positive infinite. The number $\varphi_{\succ \mathfrak{m}}$ is strictly simpler than φ, so $\varphi_{\succ \mathfrak{m}}$ does not lie in the cut which defines $L_\beta \mathfrak{a}$ in (6.1). Therefore, there is an $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ or an $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$ and an ordinal $\gamma < \beta$ with $\varphi_{\succ \mathfrak{m}} \leqslant L_\beta \mathfrak{a}' + (L_\gamma \mathfrak{a}')^{-1}$ or $\varphi_{\succ \mathfrak{m}} \geqslant L_\beta \mathfrak{a}'' - (L_\gamma \mathfrak{a})^{-1}$. Consider the first case. We have $L_\beta \mathfrak{a}' + (L_{<\beta} \mathfrak{a}')^{-1} < \varphi \leqslant \varphi_{\succ \mathfrak{m}} + \varphi_{\mathfrak{m}} \mathfrak{m} + \delta$ for a certain $\delta \prec \mathfrak{m}$. So $\varphi_{\mathfrak{m}} > 0$ and

$$\frac{1}{L_{<\beta} \mathfrak{a}'} < \frac{1}{L_\gamma \mathfrak{a}'} + \varphi_{\mathfrak{m}} \mathfrak{m}.$$

For $\rho < \beta$ with $\gamma < \rho$, we have $(L_\rho \mathfrak{a}')^{-1} \succ (L_\gamma \mathfrak{a}')^{-1}$ so $(L_\rho \mathfrak{a}')^{-1} - (L_\gamma \mathfrak{a}')^{-1} \asymp (L_\rho \mathfrak{a}')^{-1}$. We deduce that $(L_\rho \mathfrak{a}')^{-1} \preccurlyeq \mathfrak{m}$ for all such ρ. It follows that $(L_\rho \mathfrak{a})^{-1} \preccurlyeq \mathfrak{m}$ for all $\rho < \beta$. In the second case, we directly get $\mathfrak{m} \succ (L_\gamma \mathfrak{a})^{-1}$. This proves that we always have $\mathfrak{m} \succ (L_{<\beta} \mathfrak{a})^{-1}$. In other words $\operatorname{supp} \varphi \succ (L_{<\beta} \mathfrak{a})^{-1}$, whence $\mathbf{R}_\mu$ holds. □

Proposition 6.8. *If* μ *is a successor ordinal, then the cut equation* (6.1) *is uniform.*

Proof. Let $(\mathfrak{L}_\mathfrak{a}, \mathfrak{R}_\mathfrak{a})$ be a cut representation in $\mathbf{Mo}_\beta$ and write $\mathfrak{a} := \{\mathfrak{L}_\mathfrak{a} \mid \mathfrak{R}_\mathfrak{a}\}_{\mathbf{Mo}_\beta}$. For $\mathfrak{l} \in \mathfrak{L}_\mathfrak{a}$, we have $L_\beta \mathfrak{l} < L_\beta \mathfrak{a} < L_{<\beta} \mathfrak{a}$ so $L_\beta \mathfrak{l} < L_{<\beta} \mathfrak{a}$. For $\mathfrak{r} \in \mathfrak{R}_\mathfrak{a}$, we have $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{r}$ by $\mathbf{M}_\mu$. Since $\mathfrak{l} \prec \mathfrak{a}$, it follows that $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{r} - (L_{<\beta} \mathfrak{a})^{-1}$. We may thus define the number

$$\varphi := \left\{ \mathbb{R}, L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a} \,\middle|\, L_\beta \mathfrak{R}_\mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a} \right\}.$$

In order to show that (6.1) is uniform, we need to prove that $L_\beta \mathfrak{a} = \varphi$, for any choice of the cut representation $(\mathfrak{L}_\mathfrak{a}, \mathfrak{R}_\mathfrak{a})$. We will do so by proving that $L_\beta \mathfrak{a} \sqsubseteq \varphi$ and $\varphi \sqsubseteq L_\beta \mathfrak{a}$.

Recall that $(\mathfrak{L}_\mathfrak{a}, \mathfrak{R}_\mathfrak{a})$ is cofinal with respect to $(\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid \mathfrak{a}_R^{\mathbf{Mo}_\beta})$ and that L_β is strictly increasing. Consequently, we have

$$\varphi < L_\beta \mathfrak{a}_R^{\mathbf{Mo}_\beta} - (L_{<\beta} \mathfrak{a})^{-1}.$$

Given $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$, there is an $\mathfrak{l} \in \mathfrak{L}_\mathfrak{a}$ with $\mathfrak{a}' \leqslant \mathfrak{l}$. By $\mathbf{M}_\mu$, we have $L_\beta \mathfrak{a}' + (L_\gamma \mathfrak{a}')^{-1} \leqslant L_\beta \mathfrak{l} + (L_\gamma \mathfrak{l})^{-1}$ for all $\gamma < \beta$, so $\varphi > \{L_\beta \mathfrak{a}' + (L_{<\beta} \mathfrak{a}')^{-1} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}\}$. This proves that φ lies in the cut defining $L_\beta \mathfrak{a}$ as per (6.1), whence $L_\beta \mathfrak{a} \sqsubseteq \varphi$.

Conversely, in order to prove that $\varphi \sqsubseteq L_\beta \mathfrak{a}$, it suffices to show that $L_\beta \mathfrak{a}$ lies in the cut

$$\left(L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a} \,\middle|\, L_\beta \mathfrak{R}_\mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}} \right).$$

Let $\mathfrak{l} \in \mathfrak{L}_\mathfrak{a}$ and let $\mathfrak{b} \in \mathbf{Mo}_\beta$ be $\sqsubseteq$-maximal with $\mathfrak{b} \sqsubseteq \mathfrak{l}, \mathfrak{a}$. We have $\mathfrak{l} \leqslant \mathfrak{b} \leqslant \mathfrak{a}$, whence $L_\beta \mathfrak{b} \leqslant L_\beta \mathfrak{a}$, by $\mathbf{M}_\mu$. If $\mathfrak{b} \sqsubset \mathfrak{l}$, then $\mathfrak{b} \in \mathfrak{l}_R^{\mathbf{Mo}_\beta}$, so $\mathbf{H}_\mathfrak{l}$ yields $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{b}$ and $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{a}$. Otherwise $\mathfrak{l} = \mathfrak{b} \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$, so $\mathbf{H}_\mathfrak{a}$ yields $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{a}$. This proves that $\{L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a}\} < L_\beta \mathfrak{a}$.

Let $\mathfrak{r} \in \mathfrak{R}_\mathfrak{a}$ and let $\mathfrak{c} \in \mathbf{Mo}_\beta$ be $\sqsubseteq$-maximal with $\mathfrak{c} \sqsubseteq \mathfrak{r}, \mathfrak{a}$. As above, if $\mathfrak{c} \sqsubset \mathfrak{a}$, then $\mathfrak{c} \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$ so $\mathbf{H}_\mathfrak{a}$ yields $L_\beta \mathfrak{a} < L_\beta \mathfrak{c} - (L_{<\beta} \mathfrak{a})^{-1}$, whence $L_\beta \mathfrak{a} < L_\beta \mathfrak{r} - (L_{<\beta} \mathfrak{a})^{-1}$. Otherwise $\mathfrak{a} = \mathfrak{c} \in \mathfrak{r}_L^{\mathbf{Mo}_\beta}$ so $\mathbf{H}_\mathfrak{r}$ yields $L_\beta \mathfrak{r} > L_\beta \mathfrak{a} + (L_{<\beta} \mathfrak{a})^{-1}$. Hence $L_\beta \mathfrak{a} < L_\beta \mathfrak{R}_\mathfrak{a} - (L_{<\beta} \mathfrak{a})^{-1}$ and we conclude by induction. □

6.4 Functional equation

In this subsection we derive $\mathbf{FE}_\mu$, under the assumption that μ is a successor ordinal. We start with the following inequality.

Lemma 6.9. *If $\mu>1$, then we have $E_{<\beta_{/\omega}}<E_{\beta_{/\omega}}H_2 L_{\beta_{/\omega}}$ on $\mathbf{No}^{>,\succ}$.*

Proof. For $\gamma<\beta_{/\omega}$, there are $\eta<\mu_-$ and $n<\omega$ with $\gamma<\omega^\eta n$. We have

$$E_\gamma \;<\; E_{\omega^\eta n} \;=\; E_{\omega^{\eta+1}} T_n L_{\omega^{\eta+1}} \;<\; E_{\omega^{\eta+1}} H_2 L_{\omega^{\eta+1}}$$

on $\mathbf{No}^{>,\succ}$ by (4.2). Note that $\eta+1\leqslant\mu_-<\mu$, so $\mathbf{I}_{2,\mu}$ yields

$$E_{\omega^{\eta+1}} H_2 L_{\omega^{\eta+1}} \;\leqslant\; E_{\beta_{/\omega}} H_2 L_{\beta_{/\omega}},$$

whence $E_\gamma<E_{\beta_{/\omega}}H_2 L_{\beta_{/\omega}}$. □

Let $\mathfrak{a}\in\mathbf{Mo}_\beta$. Since $\mathbf{Mo}_\beta$ is a surreal substructure, we may consider the $L_{<\beta}$-atomic number

$$\mathfrak{b} \;:=\; \{L_{\beta_{/\omega}}\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta_{/\omega}}\mathfrak{a}_R^{\mathbf{Mo}_\beta},\mathfrak{a}\}_{\mathbf{Mo}_\beta}.$$

We claim that $\mathfrak{b}=L_{\beta_{/\omega}}\mathfrak{a}$. Assume that $\mu=1$ and write $\mathfrak{a}=\Xi_{\mathbf{Mo}_\omega}a$. We have

$$\begin{aligned}
\log\mathfrak{a} &= \Xi_{\mathbf{Mo}_\omega}(a-1) && \text{(by [2, Proposition 2.5])}\\
&= \Xi_{\mathbf{Mo}_\omega}\{a_L-1 \mid a_R-1,a\} && \text{(by (3.1))}\\
&= \{\Xi_{\mathbf{Mo}_\omega}(a_L-1) \mid \Xi_{\mathbf{Mo}_\omega}(a_R-1),\Xi_{\mathbf{Mo}_\omega}a\}_{\mathbf{Mo}_\omega}\\
&= \{\log\Xi_{\mathbf{Mo}_\omega}a_L \mid \log\Xi_{\mathbf{Mo}_\omega}a_R,\Xi_{\mathbf{Mo}_\omega}a\}_{\mathbf{Mo}_\omega} && \text{(by [2, Proposition 2.5])}\\
&= \{\log\mathfrak{a}_L^{\mathbf{Mo}_\omega} \mid \log\mathfrak{a}_R^{\mathbf{Mo}_\omega},\mathfrak{a}\}_{\mathbf{Mo}_\omega}\\
&= \mathfrak{b}.
\end{aligned}$$

Assume now that $\mu>1$. The function $L_{\beta_{/\omega}}$ is strictly increasing with $L_{\beta_{/\omega}}<\mathrm{id}_{\mathbf{No}^{>,\succ}}$. Therefore

$$L_{\beta_{/\omega}}\mathfrak{a} \;\in\; (L_{\beta_{/\omega}}\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta_{/\omega}}\mathfrak{a}_R^{\mathbf{Mo}_\beta},\mathfrak{a})_{\mathbf{Mo}_\beta},$$

so $\mathfrak{b}\sqsubseteq L_{\beta_{/\omega}}\mathfrak{a}$. Since $\mathfrak{a}\in\mathbf{Mo}_\beta$, the cut equation (6.1) for μ_- yields

$$L_{\beta_{/\omega}}\mathfrak{a} \;=\; \{\mathbb{R},L_{\beta_{/\omega}}\mathfrak{a}'+(L_{<\beta}\mathfrak{a}')^{-1}:\mathfrak{a}'\in\mathfrak{a}_L^{\mathbf{Mo}_{\beta_{/\omega}}} \mid L_{\beta_{/\omega}}\mathfrak{a}_R^{\mathbf{Mo}_{\beta_{/\omega}}}-(L_{<\beta}\mathfrak{a})^{-1},L_{<\beta_{/\omega}}\mathfrak{a}\}. \tag{6.2}$$

Given $\mathfrak{a}'\in\mathfrak{a}_L^{\mathbf{Mo}_{\beta_{/\omega}}}$, we have $\mathfrak{d}_\beta(\mathfrak{a}')\in\mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}'\in\mathcal{E}_\beta[\mathfrak{d}_\beta(\mathfrak{a}')]$. We deduce that

$$L_{\beta_{/\omega}}\mathfrak{a}' \;\in\; L_{\beta_{/\omega}}\mathcal{E}_\beta[\mathfrak{d}_\beta(\mathfrak{a}')] \;=\; \mathcal{E}_\beta[L_{\beta_{/\omega}}\mathfrak{d}_\beta(\mathfrak{a}')].$$

Moreover, by definition, we have

$$\mathfrak{b} \;>\; \mathcal{E}_\beta'[L_{\beta_{/\omega}}\mathfrak{d}_\beta(\mathfrak{a}')] \;=\; \mathcal{E}_\beta[L_{\beta_{/\omega}}\mathfrak{d}_\beta(\mathfrak{a}')],$$

so $\mathfrak{b}>L_{\beta_{/\omega}}\mathfrak{a}'$. Symmetric arguments yield $\mathfrak{b}<L_{\beta_{/\omega}}\mathfrak{a}_R^{\mathbf{Mo}_{\beta_{/\omega}}}$. Lemma 6.9 implies that $L_{<\beta_{/\omega}}\mathfrak{a}\subseteq\mathcal{E}_\beta[\mathfrak{a}]$, whence $\mathfrak{d}_\beta(L_{<\beta_{/\omega}}\mathfrak{a})=\{\mathfrak{a}\}$. We get $\mathfrak{b}<\mathcal{E}_\beta\mathfrak{d}_\beta(L_{<\beta_{/\omega}}\mathfrak{a})$, whence $\mathfrak{b}<L_{<\beta_{/\omega}}\mathfrak{a}$. Thus $\mathfrak{b}$ lies in the cut defining $L_{\beta_{/\omega}}\mathfrak{a}$ in (6.2), so $L_{\beta_{/\omega}}\mathfrak{a}\sqsubseteq\mathfrak{b}$. This proves our claim that

$$\forall\mathfrak{a}\in\mathbf{Mo}_\beta,\quad L_{\beta_{/\omega}}\mathfrak{a}=\{L_{\beta_{/\omega}}\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta_{/\omega}}\mathfrak{a}_R^{\mathbf{Mo}_\beta},\mathfrak{a}\}_{\mathbf{Mo}_\beta}. \tag{6.3}$$

We now derive $\mathbf{FE}_\mu$.

Proposition 6.10. *For $\mathfrak{a}\in\mathbf{Mo}_\beta$, we have $L_\beta L_{\beta_{/\omega}}\mathfrak{a}=L_\beta\mathfrak{a}-1$.*

Proof. We prove this by induction on $(\mathbf{Mo}_\beta,\sqsubseteq)$. Let $\mathfrak{a}\in\mathbf{Mo}_\beta$ be such that the result holds on $\mathfrak{a}_\sqsubset^{\mathbf{Mo}_\beta}$. By (6.3), we have

$$L_{\beta_{/\omega}}\mathfrak{a} \;=\; \{L_{\beta_{/\omega}}\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta_{/\omega}}\mathfrak{a}_R^{\mathbf{Mo}_\beta},\mathfrak{a}\}_{\mathbf{Mo}_\beta}.$$

Let $\mathfrak{a}'$ and $\mathfrak{a}''$ range in $\mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}_R^{\mathbf{Mo}_\beta}$ respectively. Proposition 6.8 and our induction hypothesis yield:

$$\begin{aligned} L_\beta L_{\beta_{/\omega}} \mathfrak{a} &= \left\{ \mathbb{R}, L_\beta L_{\beta_{/\omega}} \mathfrak{a}' + \frac{1}{L_{<\beta} L_{\beta_{/\omega}} \mathfrak{a}'} \,\middle|\, L_\beta L_{\beta_{/\omega}} \mathfrak{a}'' - \frac{1}{L_{<\beta} L_{\beta_{/\omega}} \mathfrak{a}}, L_\beta \mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} L_{\beta_{/\omega}} \mathfrak{a} \right\} \\ &= \left\{ \mathbb{R}, L_\beta \mathfrak{a}' - 1 + \frac{1}{L_{<\beta} \mathfrak{a}'} \,\middle|\, L_\beta \mathfrak{a}'' - 1 - \frac{1}{L_{<\beta} \mathfrak{a}}, L_\beta \mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a} \right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L_\beta \mathfrak{a} - 1 &= \left\{ \mathbb{R} - 1, L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} - 1 \,\middle|\, L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}} - 1, L_{<\beta} \mathfrak{a} - 1, L_\beta \mathfrak{a} \right\} \\ &= \left\{ \mathbb{R}, L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} - 1 \,\middle|\, L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}} - 1, L_\beta \mathfrak{a}, L_{<\beta} \mathfrak{a} \right\}. \end{aligned}$$

In order to conclude that $L_\beta L_{\beta_{/\omega}} \mathfrak{a} = L_\beta \mathfrak{a} - 1$, it remains to show that $L_\beta \mathfrak{a} - 1 < L_\beta \mathfrak{a} - (L_{<\beta} \mathfrak{a})^{-1}$ and that $L_\beta L_{\beta_{/\omega}} \mathfrak{a} < L_\beta \mathfrak{a}$. The first inequality holds because $(L_{<\beta} \mathfrak{a})^{-1}$ is a set of infinitesimal numbers. An easy induction shows that $L_{\beta_{/\omega}} a < a$ for all $a \in \mathbf{No}^{>,\succ}$. The second inequality follows, because L_β is strictly increasing on $\mathbf{Mo}_\beta$. This completes our inductive proof. □

Combining our results so far, we have proved that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is a hyperserial skeleton of force ν.

6.5 Confluence

We next prove that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is ν-confluent.

Lemma 6.11. *If μ is a non-zero limit ordinal, then the function groups $\mathscr{E}_\beta'$ and $\mathscr{E}_\beta^*$ are mutually pointwise cofinal. In particular, we have $\mathbf{Mo}_\beta = \mathbf{Mo}_\beta^*$ and $\mathbf{Tr}_\beta = \mathbf{Tr}_\beta^*$.*

Proof. For $\gamma \in (0, \beta)$ and $r \in \mathbb{R}^>$, we have $E_\gamma H_r L_\gamma < E_\gamma$ since $H_r < E_\gamma$. We have

$$\{L_\rho, E_\rho : \rho \in (0, \beta)\} \;\lneqq\; \mathscr{E}_\beta^*,$$

whereas $\mathbf{I}_{2,\mu}$ yields

$$\{E_\rho H_r L_\rho : \rho \in (0, \beta)\} \;\lneqq\; \mathscr{E}_\beta'.$$

Therefore $\mathscr{E}_\beta' \leqq \mathscr{E}_\beta^*$. For $\rho < \beta$, there is $\eta < \mu$ with $\rho < \omega^\eta$. By (4.2), we have

$$E_\rho \;<\; E_{\omega^\eta} \;=\; E_{\omega^{\eta+1}} T_1 L_{\omega^{\eta+1}} \;<\; E_{\omega^{\eta+1}} H_2 L_{\omega^{\eta+1}},$$

which proves the inequality $\mathscr{E}_\beta^* \leqq \mathscr{E}_\beta'$. □

Lemma 6.12. *For each $a \in \mathbf{No}^{>,\succ}$, any $\sqsubseteq$-minimal element of $\mathscr{E}_\alpha[a]$ is $L_{<\alpha}$-atomic.*

Proof. Let $\mathfrak{A}$ denote the class of numbers $\mathfrak{a} \in \mathbf{No}^{>,\succ}$ that are $\sqsubseteq$-minimal in $\mathscr{E}_\alpha[\mathfrak{a}]$. Any such $\sqsubseteq$-minimal number $\mathfrak{a}$ is also $\sqsubseteq$-minimal in $\mathscr{E}_\beta'[\mathfrak{a}] = \mathscr{E}_\beta[\mathfrak{a}] \subseteq \mathscr{E}_\alpha[\mathfrak{a}]$, hence $L_{<\beta}$-atomic. Thus L_β is defined on $\mathfrak{A}$. It is enough to prove that $\mathfrak{A}$ is closed under L_β in order to obtain that $\mathfrak{A} \subseteq \mathbf{Mo}_\alpha$.

Consider $\mathfrak{a} \in \mathfrak{A}$, and recall that we have

$$L_\beta \mathfrak{a} \;=\; \left\{ \mathbb{R}, L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta} \,\middle|\, L_\beta \mathfrak{a}_R^{\mathbf{Mo}_\beta} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a} \right\}. \tag{6.4}$$

Assume for contradiction that $L_\beta \mathfrak{a}$ is not $\sqsubseteq$-minimal in $\mathscr{E}_\alpha[L_\beta \mathfrak{a}]$. So there is a $\mathfrak{b} \in \mathscr{E}_\alpha[L_\beta \mathfrak{a}]$ with $\mathfrak{b} \sqsubset L_\beta \mathfrak{a}$. This implies that $\mathfrak{b}$ lies outside the cut defining $L_\beta \mathfrak{a}$, so $\mathfrak{b}$ is larger than a right option of (6.4) or smaller than a left option of (6.4).

Assume first that $\mathfrak{b} < L_\beta \mathfrak{a}$. So there is an $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ with $\mathfrak{b} \preccurlyeq L_\beta \mathfrak{a}'$. We have $\mathfrak{d}_\alpha(\mathfrak{a}) = \mathfrak{d}_\alpha(\mathfrak{b})$ so there is an $n \in \mathbb{N}$ with

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{b}) \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(L_\beta \mathfrak{a}).$$

Thus

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ (n+1)}(\mathfrak{a}') \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ (n+1)}(\mathfrak{a}).$$

This contradicts the $\sqsubseteq$-minimality of $\mathfrak{a}$.

Now consider the other case when $\mathfrak{b} > L_\beta \mathfrak{a}$. In particular, $\mathfrak{b}$ must be larger than a right option of (6.4). Symmetric arguments imply that we cannot have $\mathfrak{b} \succcurlyeq L_\beta \mathfrak{a}''$ for some $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$. So there must exist a $\gamma < \beta$ with $\mathfrak{b} \geqslant L_\gamma \mathfrak{a}$. If μ is a limit ordinal, then $\gamma < \mu_-$ so Lemma 6.11 yields $\mathfrak{d}_\beta(L_\gamma \mathfrak{a}) = \mathfrak{a}$, whence $\mathfrak{d}_\beta(\mathfrak{b}) \succcurlyeq \mathfrak{a}$. If μ is a successor ordinal, then there is a $k \in \mathbb{N}$ with $\gamma \leqslant \beta_{/\omega} k$, so

$$\mathfrak{d}_\beta(\mathfrak{b}) \geqslant \mathfrak{d}_\beta(L_{(\beta_{/\omega})k} \mathfrak{a}) = L_{(\beta_{/\omega})k} \mathfrak{a}$$

and Proposition 6.10 yields $L_\beta \mathfrak{d}_\beta(\mathfrak{b}) \succcurlyeq L_\beta \mathfrak{a} - k \succcurlyeq L_\beta \mathfrak{a}$. In both cases, we thus have $L_\beta \mathfrak{d}_\beta(\mathfrak{b}) \succcurlyeq L_\beta \mathfrak{a}$. For any integer $n > 1$, we deduce that

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{b}) \succcurlyeq (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{a}) > (L_\beta \circ \mathfrak{d}_\beta)^{\circ (n+1)}(\mathfrak{a}) = (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(L_\beta \mathfrak{a}).$$

This contradicts the fact that $\mathfrak{b}$ lies in $\mathscr{E}_\alpha[L_\beta \mathfrak{a}]$.

We have shown that the cases $\mathfrak{b} < L_\beta \mathfrak{a}$ and $\mathfrak{b} > L_\beta \mathfrak{a}$ both lead to a contradiction. Consequently, $L_\beta \mathfrak{a}$ is $\sqsubseteq$-minimal in $\mathscr{E}_\alpha[L_\beta \mathfrak{a}]$ and we conclude that $L_\beta \mathfrak{A} \subseteq \mathfrak{A}$, as claimed. □

Corollary 6.13. *$(\mathbf{No}, (L_{\omega^\eta})_{\eta < \nu})$ is ν-confluent.*

Proof. We already know that $(\mathbf{No}, (L_{\omega^\eta})_{\eta < \mu})$ is μ-confluent by $\mathbf{I}_{1,\mu}$. Recall that $(\mathbf{No}, \sqsubseteq)$ is well-founded, so each class $\mathscr{E}_\alpha[a]$ for $a \in \mathbf{No}^{>,\succ}$ contains a $\sqsubseteq$-minimal element. Lemma 6.12 therefore implies that $\mathbf{No}$ is ν-confluent. □

The corollary implies that $(\mathbf{No}, (L_{\omega^\eta})_{\eta < \nu})$ is a confluent hyperserial skeleton of force ν. Moreover, the class $\mathbf{No}_{\succ,\beta}$ is that of $\trianglelefteq$-minima and thus $\sqsubseteq$-minima in the convex classes

$$\mathscr{L}_\beta[a] = \{b \in a + \mathbf{No}^{\prec} : b = a \vee (\exists \gamma < \beta, a < \ell_\beta^{\uparrow \gamma} \circ |a - b|^{-1})\},$$

for $a \in \mathbf{No}^{>,\succ}$. In other words, we have $\mathbf{No}_{\succ,\beta} = \mathbf{Smp}_{\mathscr{L}_\beta}$. In order to conclude that $\mathbf{No}_{\succ,\beta}$ is a surreal substructure, we still need to prove that the convex partition $\mathscr{L}_\beta$ is thin. This will be done at the end of section 6.6 below.

Proposition 6.14. *The cut equation (6.1) is uniform.*

Proof. Let $(\mathfrak{L}_\mathfrak{a}, \mathfrak{R}_\mathfrak{a})$ be a cut representation in $\mathbf{Mo}_\beta$ and write $\mathfrak{a} := \{\mathfrak{L}_\mathfrak{a} \mid \mathfrak{R}_\mathfrak{a}\}_{\mathbf{Mo}_\beta}$. We have

$$\mathscr{L}_\beta[L_\beta \mathfrak{L}_\mathfrak{a}] < \mathscr{L}_\beta[L_\beta \mathfrak{a}] < \mathscr{L}_\beta[L_\beta \mathfrak{R}_\mathfrak{a}].$$

By (4.6), this shows that

$$L_\beta \mathfrak{a} \in \left(\mathbb{R}, L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a} \,\middle|\, L_\beta \mathfrak{R}_\mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a}\right).$$

In particular, the number

$$\varphi := \left\{\mathbb{R}, L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a} \,\middle|\, L_\beta \mathfrak{R}_\mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a}\right\}$$

is well-defined, with $\varphi \sqsubseteq L_\beta \mathfrak{a}$. As in the proof of Proposition 6.8, we have $L_\beta \mathfrak{a} \sqsubseteq \varphi$, whence $\varphi = L_\beta \mathfrak{a}$. We conclude that the cut equation (6.1) is uniform. □

6.6 Hyperexponentials

We have shown that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is a hyperserial skeleton of force (ν, μ). In order to prove that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ has force (ν, ν), it remains to prove that every β-truncated number φ has a hyperexponential $E_\beta \varphi$. This is the purpose of this subsection.

Proposition 6.15. *We have $L_\beta \mathbf{Mo}_\beta = \mathbf{No}_{\succ,\beta}$, and E_β has the following cut equation on $\mathbf{No}_{\succ,\beta}$:*

$$\forall \varphi \in \mathbf{No}_{\succ,\beta}, \quad E_\beta \varphi = \left\{ E_{<\beta} \varphi, E_{<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right), \mathscr{E}'_\beta E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \,\middle|\, \mathscr{E}'_\beta E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}} \right\}. \tag{6.5}$$

Proof. We prove the result by induction on $(\mathbf{No}_{\succ,\beta}, \sqsubseteq)$. Let $\varphi \in \mathbf{No}_{\succ,\beta}$ such that E_β is defined on $\varphi_{\sqsubset}^{\mathbf{No}_{\succ,\beta}}$ with the given equation. We will first show that the number

$$\mathfrak{a} \ := \ \left\{ E_{<\beta} \varphi, E_{<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right), \mathscr{E}'_\beta E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \,\middle|\, \mathscr{E}'_\beta E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}} \right\} \tag{6.6}$$

is well-defined. We will then prove that $L_\beta \mathfrak{a} = \varphi$.

Let $\varphi' \in \varphi_L^{\mathbf{No}_{\succ,\beta}}$ and $\varphi'' \in \varphi_R^{\mathbf{No}_{\succ,\beta}}$. If $\varphi' \in (\varphi'')_L^{\mathbf{No}_{\succ,\beta}}$, then $E_\beta \varphi'' > \mathscr{E}'_\beta E_\beta \varphi'$ by the definition of $E_\beta \varphi''$. So $\mathscr{E}'_\beta E_\beta \varphi' < \mathscr{E}'_\beta E_\beta \varphi''$. Otherwise, we have $\varphi'' \in (\varphi')_R^{\mathbf{No}_{\succ,\beta}}$, whence $\mathscr{E}'_\beta E_\beta \varphi'' > E_\beta \varphi'$ by definition of $E_\beta \varphi'$, so $\mathscr{E}'_\beta E_\beta \varphi' < \mathscr{E}'_\beta E_\beta \varphi''$. So we always have

$$\mathscr{E}'_\beta E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \ < \ \mathscr{E}'_\beta E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}.$$

We also have $E_{<\beta} \varphi'' < E_\beta \varphi''$, so $E_{<\beta} \varphi < \mathscr{E}'_\beta E_\beta \varphi''$. This proves that $E_{<\beta} \varphi < \mathscr{E}'_\beta E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}$. It remains to show that

$$E_{<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right) \ < \ \mathscr{E}'_\beta E_\beta (\varphi_R^{\mathbf{No}_{\succ,\beta}}).$$

Note that $\varphi_R^{\mathbf{No}_{\succ,\beta}} > \mathscr{L}_\beta[\varphi]$, so by the definition of $\mathscr{L}_\beta[\varphi]$, we have

$$L_\beta^{\uparrow<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right) \ < \ \varphi \ < \ \varphi_R^{\mathbf{No}_{\succ,\beta}}. \tag{6.7}$$

Hence $E_{<\beta}((\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi)^{-1}) < E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}$, which completes the proof that $\mathfrak{a}$ is well-defined.

Let us now prove that $L_\beta \mathfrak{a} = \varphi$. Note that $\mathfrak{a} \in \mathbf{Mo}_\beta$ by Proposition 3.2. First assume that μ is a limit ordinal. Lemma 6.11 yields $\langle E_{<\beta} \rangle \precneqq \mathscr{E}_\beta$, so we may write

$$\mathfrak{a} \ = \ \left\{ \mathfrak{d}_\beta(\varphi), \mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right), E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \,\middle|\, E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}} \right\}_{\mathbf{Mo}_\beta}.$$

By (4.6), for $b \in \mathbf{No}^{>,\succ}$ the classes that $\mathscr{L}_\beta[L_\beta b]$ and $L_\beta b \pm (L_{<\beta} b)^{-1}$ are mutually cofinal and coinitial. Moreover, we have $L_\beta E_\beta \psi = \psi$ for all $\psi \in \varphi_{\sqsubset}^{\mathbf{No}_{\succ,\beta}}$, by our hypothesis on φ. Hence, Proposition 6.14 and (4.6) imply

$$L_\beta \mathfrak{a} \ = \ \left\{ \mathbb{R}, \mathscr{L}_\beta[L_\beta \mathfrak{d}_\beta(\varphi)], \mathscr{L}_\beta\left[L_\beta \mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right) \right], \mathscr{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}] \,\middle|\, \varphi_R^{\mathbf{No}_{\succ,\beta}} - \frac{1}{L_{<\beta} \mathfrak{a}} \right\}.$$

Note that $L_\beta \mathfrak{a} \in (\varphi_L^{\mathbf{No}_{\succ,\beta}} \mid \varphi_R^{\mathbf{No}_{\succ,\beta}})_{\mathbf{No}_{\succ,\beta}}$, so $\varphi \sqsubseteq L_\beta \mathfrak{a}$. Now $L_\beta \mathfrak{d}_\beta(\varphi) \in \mathscr{L}_\beta[L_\beta \varphi] < \varphi$. We also have

$$L_\beta \mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right) \ \in \ L_\beta \mathscr{E}'_\beta\left[\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi} \right],$$

where

$$
\begin{aligned}
L_\beta \mathscr{E}'_\beta\left[\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right] &= L_\beta \mathscr{E}^*_\beta\left[\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right] && \text{(by Lemma 6.11)}\\
&\lneqq L_\beta^{\uparrow<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right)\\
&< \varphi. && \text{(by (6.7))}
\end{aligned}
$$

So $L_\beta \mathfrak{d}_\beta(\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi)^{-1} < \varphi$. Since $\varphi \in \mathbf{No}_{\succ,\alpha}$, the inequality $\mathscr{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}] < \varphi$ follows from Proposition 3.2. Finally, we have by definition that $\mathfrak{a} > E_{<\beta}((\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi)^{-1})$, so $\varphi_R^{\mathbf{No}_{\succ,\beta}} - (L_{<\beta}\,\mathfrak{a})^{-1} > \varphi$. This proves that $L_\beta\,\mathfrak{a} \sqsubseteq \varphi$, so $L_\beta\,\mathfrak{a} = \varphi$.

Assume now that μ is a successor ordinal. For all $b \in \mathbf{No}^{>,\succ}$, the sets $E_{<\beta}\varphi$, $E_{<\beta}\mathfrak{d}_\beta(\varphi)$, and $E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta(\varphi)$ are mutually cofinal. So we can rewrite (6.6) as

$$
\begin{aligned}
\mathfrak{a} &= \left\{E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta(\varphi), E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right), \mathscr{E}'_\beta E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \,\middle|\, \mathscr{E}'_\beta E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}\right\}\\
&= \left\{E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta(\varphi), E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right), E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \,\middle|\, E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}\right\}_{\mathbf{Mo}_\beta}.
\end{aligned}
$$

As in the limit case, Proposition 6.14 yields

$$
L_\beta\,\mathfrak{a} = \left\{\mathbb{R}, \mathscr{L}_\beta\left[L_\beta^{\uparrow<\beta}\mathfrak{d}_\beta(\varphi)\right], \mathscr{L}_\beta\left[L_\beta^{\uparrow<\beta}\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right)\right], \mathscr{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}] \,\middle|\, \varphi_R^{\mathbf{No}_{\succ,\beta}} - \frac{1}{L_{<\beta}\,\mathfrak{a}}\right\}.
$$

Let $\gamma < \beta$. There is an $n \in \mathbb{N}$ with $\gamma < \beta_{/\omega} n$. Since $L_\beta \varphi < \varphi - (n+1)$, we have

$$
\varphi > L_\beta^{\uparrow \beta_{/\omega}(n+1)}\mathfrak{d}_\beta(\varphi) \geqslant L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta(\varphi) + 1.
$$

In particular $\varphi > \mathscr{L}_\beta\left[L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta(\varphi)\right]$. We saw in (6.7) that $L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta((\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi)^{-1}) < \varphi$, whence $\mathscr{L}_\beta\left[L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta((\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi)^{-1})\right] < \varphi$. We also obtain the inequalities

$$
\mathscr{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}] < \varphi < \varphi_R^{\mathbf{No}_{\succ,\beta}} - (L_{<\beta}\,\mathfrak{a})^{-1}
$$

in a similar way as in the limit case.

We conclude that $\varphi = L_\beta\,\mathfrak{a}$ holds in general. It follows by induction that the formula for E_β is valid. In particular $L_\beta\colon \mathbf{Mo}_\beta \longrightarrow \mathbf{No}_{\succ,\beta}$ is surjective. □

With Proposition 6.15, we have completed the proof of $\mathbf{I}_{1,\nu}$. By (4.7), we have $\mathscr{E}_{\beta\omega}[a] = \mathscr{E}'_{\beta\omega}[a]$ for all $a \in \mathbf{No}^{>,\succ}$. Given $a \in \mathbf{No}_{\succ,\beta}$, we also deduce from (4.6) that the set $a \pm (L_{<\beta} E_\beta\, a)^{-1}$ is cofinal and coinitial in $\mathscr{L}_\beta[a]$. The convex partition defined by $\mathscr{L}_\beta$ is thus thin. By Proposition 3.3, the class $\mathbf{No}_{\succ,\beta}$ is a surreal substructure with uniform cut equation

$$
\forall a \in \mathbf{No}, \quad \Xi_{\mathbf{No}_{\succ,\beta}} a = \{\mathbb{R}, \mathscr{L}_\beta[\Xi_{\mathbf{No}_{\succ,\beta}} a_L] \mid \mathscr{L}_\beta[\Xi_{\mathbf{No}_{\succ,\beta}} a_R]\} \tag{6.8}
$$

For $a \in \mathbf{No}$, we have $\mathscr{L}_\beta[\Xi_{\mathbf{No}_{\succ,\beta}} a] < \Xi_{\mathbf{No}_{\succ,\beta}} a_R$, so $\Xi_{\mathbf{No}_{\succ,\beta}} a < \Xi_{\mathbf{No}_{\succ,\beta}} a_R - (L_{<\beta} E_\beta \Xi_{\mathbf{No}_{\succ,\beta}} a)^{-1}$. We deduce that the following equivalent is equivalent to (6.8):

$$
\Xi_{\mathbf{No}_{\succ,\beta}} a = \left\{\mathbb{R}, \Xi_{\mathbf{No}_{\succ,\beta}} a' + \frac{1}{L_{<\beta} E_\beta \Xi_{\mathbf{No}_{\succ,\beta}} a'} : a' \in a_L \,\middle|\, \Xi_{\mathbf{No}_{\succ,\beta}} a_R - \frac{1}{L_{<\beta} E_\beta \Xi_{\mathbf{No}_{\succ,\beta}} a}\right\}. \tag{6.9}
$$

6.7 End of the inductive proof

We now prove $\mathbf{I}_{2,\nu}$, $\mathbf{I}_{3,\nu}$ and Theorem 1.1.

Lemma 6.16. *If μ is a limit ordinal, then we have $E_\beta T_1 L_\beta > E_{<\beta}$ on $\mathbf{No}^{>,\succ}$.*

Proof. Let $a \in \mathbf{No}^{>,\succ}$. We have $\sharp_\beta(L_\beta a+1) > \sharp_\beta(L_\beta a)$, so (4.8) yields

$$\mathfrak{d}_\beta(E_\beta(L_\beta a+1)) \;=\; E_\beta(\sharp_\beta(L_\beta a+1)) \;>\; E_\beta(\sharp_\beta(L_\beta a)) \;=\; \mathfrak{d}_\beta(a).$$

We deduce that $E_\beta(L_\beta a+1) > \mathscr{E}_\beta a$ so $E_\beta(L_\beta a+1) > E_{<\beta} a$ by Lemma 6.11. □

Proposition 6.17. *For $r,s \in \mathbb{R}$ with $s>1$ and $\gamma<\rho<\alpha$, we have $E_\gamma H_r L_\gamma < E_\rho H_s L_\rho$ on $\mathbf{No}^{>,\succ}$, i.e. $\mathbf{I}_{2,\nu}$ holds.*

Proof. Throughout this proof, we consider inequalities and equalities of functions on $\mathbf{No}^{>,\succ}$. Write $\gamma = \beta m + \iota$ and $\rho = \beta n + \theta$ where $m,n<\omega$ and $\iota,\theta<\beta$. We have

$$\begin{aligned} E_\gamma H_r L_\gamma &= E_{\beta m} E_\iota H_r L_\iota L_{\beta m} \quad \text{and}\\ E_\rho H_r L_\rho &= E_{\beta n} E_\theta H_s L_\theta L_{\beta n}. \end{aligned}$$

If $m=n$, then $\iota<\theta$, so $\mathbf{I}_{2,\mu}$ yields $E_\iota H_r L_\iota < E_\theta H_s L_\theta$, whence $E_\gamma H_r L_\gamma < E_\rho H_s L_\rho$. Assume that $m<n$. If μ_- is a successor ordinal, then there is $p<\omega$ with $\iota < \beta_{/\omega} p$. By $\mathbf{I}_{2,\mu}$, we have $E_\theta H_s L_\theta \geqslant H_s > T_p$. So $E_\beta (E_\theta H_s L_\theta) L_\beta > E_\beta T_p L_\beta = E_{\beta_{/\omega} p}$. We conclude by noting that $E_{\beta_{/\omega} p} > E_\iota > E_\iota H_r L_\iota$. If μ_- is a limit ordinal, then $E_\theta H_s L_\theta > T_1$ so $E_\beta (E_\theta H_s L_\theta) L_\beta > E_\iota > E_\iota H_r L_\iota$ by Lemma 6.16. It follows that for $k \in \mathbb{N}^>$, we have $E_{\beta(k+1)} E_\theta H_s L_\theta L_{\beta(k+1)} > E_{\beta k} E_\iota H_r L_\iota L_{\beta k}$. An easy induction on k yields the result. □

Proposition 6.18. *$\mathbf{Mo}'_\alpha$ is the class of $L_{<\alpha}$-atomic numbers, i.e. $\mathbf{I}_{3,\nu}$ holds.*

Proof. Let $a \in \mathbf{No}^{>,\succ}$. By Corollary 6.13, the simplest element of $\mathscr{E}_\alpha[a]$ is $L_{<\alpha}$-atomic. Since $\mathscr{E}_\alpha[a] = \mathscr{E}'_\alpha[a]$, we deduce that $\mathbf{Mo}'_\alpha \subseteq \mathbf{Mo}_\alpha$.

Conversely, given $\mathfrak{a} \in \mathbf{Mo}_\alpha$, we have $\mathfrak{b} := \pi_{\mathscr{E}'_\alpha}(\mathfrak{a}) \in \mathbf{Mo}'_\alpha \subseteq \mathbf{Mo}_\alpha$. Now $\mathfrak{b} \in \mathscr{E}'_\alpha[\mathfrak{a}]$, so by $\mathbf{I}_{2,\nu}$, there are $r,s \in \mathbb{R}^>$ and $\gamma<\alpha$ with $E_\gamma(r L_\gamma \mathfrak{a}) < \mathfrak{b} < E_\gamma(s L_\gamma \mathfrak{a})$. Hence, $L_\gamma \mathfrak{b} \asymp L_\gamma \mathfrak{a}$, $L_\gamma \mathfrak{b} = L_\gamma \mathfrak{a}$ and $\mathfrak{b} = \mathfrak{a}$. We conclude that $\mathfrak{a} \in \mathbf{Mo}'_\alpha$. □

In particular, the class $\mathbf{Mo}_\alpha$ is a surreal substructure. We have proved $\mathbf{I}_{1,\nu}, \mathbf{I}_{2,\nu}$, and $\mathbf{I}_{3,\nu}$, so we obtain the following by induction:

Theorem 6.19. *The field $(\mathbf{No}, (L_{\omega^\eta})_{\eta \in \mathbf{On}})$ is a confluent hyperserial skeleton of force $(\mathbf{On}, \mathbf{On})$.*

Combining this with Propositions 4.3 and 4.6, we obtain Theorem 1.1. Let us finally show that $(\mathbf{No}, \circ)$ contains only one $L_{<\mathbf{On}}$-atomic element.

Proposition 6.20. *The number ω is the only $L_{<\mathbf{On}}$-atomic element in $\mathbf{No}$. For all $a \in \mathbf{No}^{>,\succ}$, there is $\gamma \in \mathbf{On}$ with $L_\gamma a \asymp L_\gamma \omega$.*

Proof. The number ω lies in $\mathbf{Mo}_{\omega^\mu}$ for all $\mu \in \mathbf{On}$, so it is $L_{<\mathbf{On}}$-atomic. For $\nu \in \mathbf{On}$, the number $E_{\omega^\nu} \omega = \{E_{<\omega^\nu} \omega \mid \varnothing\}$ is an ordinal. As a sign sequence, the number $L_{\omega^\nu} \omega = \{\varnothing \mid L_{<\omega^\nu} \omega\}_{\mathbf{No}^{>,>}}$ is ω followed by a string containing only minuses [2, Lemma 2.6]. Since the sequences $(E_{\omega^\nu} \omega)_{\nu \in \mathbf{On}}$ and $(L_{\omega^\nu} \omega)_{\nu \in \mathbf{On}}$ are strictly increasing and strictly decreasing respectively, the classes $\{E_{\omega^\nu} \omega : \nu \in \mathbf{On}\}$ and $\{L_{\omega^\nu} \omega : \nu \in \mathbf{On}\}$ are respectively cofinal and coinitial in $\mathbf{No}^{>,\succ} = \{a \in \mathbf{No} : \omega \sqsubseteq a\}$. Thus for $a \in \mathbf{No}^{>,\succ}$, there is $\nu \in \mathbf{On}$ with $E_{\omega^\nu} \omega > a > L_{\omega^\nu} \omega$, whence $L_{\omega^{\nu+1}} \omega \asymp L_{\omega^{\nu+1}} a$. □

7 Remarkable identities

In this section, we give various identities regarding the function groups introduced in Section 6.1. In what follows, ν is a non-zero ordinal and $\alpha := \omega^\nu$.

7.1 Simplified cut equations for L_α and E_α

Given $\varphi \in \mathbf{No}^{>,>}$, let $E_{\lhd\alpha} := \{E_{(\alpha_{/\omega})n}\,\varphi : n \in \mathbb{N}\}$ if ν is a successor ordinal and $E_{\lhd\alpha}\,\varphi := \{\varphi\}$ if ν is a limit ordinal. In this subsection, we will derive the following simplified cut equations for L_α on $\mathbf{Mo}_\alpha$ and E_α on $\mathbf{No}_{>,\alpha}$:

$$
\begin{aligned}
\forall \mathfrak{a} \in \mathbf{Mo}_\alpha,\ L_\alpha\,\mathfrak{a} &= \{L_\alpha\,\mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha}\,\mathfrak{a}\}_{\mathbf{No}_{>,\alpha}} && (7.1)\\
&= \left\{\mathbb{R}, L_\alpha\,\mathfrak{a}' + \frac{1}{L_{<\alpha}\,\mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \,\middle|\, L_\alpha\,\mathfrak{a}'' - \frac{1}{L_{<\alpha}\,\mathfrak{a}''}, L_{<\alpha}\,\mathfrak{a} : \mathfrak{a}'' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha}\right\}, && (7.2)\\
\forall \varphi \in \mathbf{No}_{>,\alpha},\ E_\alpha\,\varphi &= \{E_{\lhd\alpha}\,\mathfrak{d}_\alpha(\varphi), E_\alpha\,\varphi_L^{\mathbf{No}_{>,\alpha}} \mid E_\alpha\,\varphi_R^{\mathbf{No}_{>,\alpha}}\}_{\mathbf{Mo}_\alpha} && (7.3)\\
&= \{E_{<\alpha}\,\varphi, \mathscr{E}_\alpha E_\alpha\,\varphi_L^{\mathbf{No}_{>,\alpha}} \mid \mathscr{E}_\alpha E_\alpha\,\varphi_R^{\mathbf{No}_{>,\alpha}}\}. && (7.4)
\end{aligned}
$$

For all $a \in \mathbf{No}^{>,>}$, the set $E_{\lhd\alpha}\,\mathfrak{d}_\alpha(a)$ contains only $L_{<\alpha}$-atomic numbers, so (7.3) is indeed a cut equation of the form $\{\rho \mid \lambda\}_{\mathbf{Mo}_\alpha}$.

Remark 7.1. The changes with respect to (6.1) and (6.5) lie in the occurrence of $\mathfrak{a}''$ instead of $\mathfrak{a}$ in (7.2) and the (related) absence of the left option $E_{<\alpha}((\varphi_R^{\mathbf{No}_{>,\alpha}} - \varphi)^{-1})$ in (7.4). So (7.2) and (7.4) give lighter sets of conditions than those in (6.1) and (6.5) to define L_α and E_α. This seemingly meager simplification will be crucial in further work. Indeed, combined with Proposition 3.1, this allows one to determine large classes of numbers a, b with $a \sqsubseteq b \Longrightarrow E_\alpha\,a \sqsubseteq E_\alpha\,b$.

First note that the cut equations (7.1) and (7.3) if they hold are uniform (see [6, Remark 1]). Moreover, we claim that (7.1,7.2) are equivalent and that (7.3,7.4) are equivalent. Indeed, recall that for a thin convex partition $\mathbf{\Pi}$ of a surreal substructure $\mathbf{S}$ and any cut representation (L, R) in $\mathbf{Smp}_{\mathbf{\Pi}}$, one has

$$\{L \mid R\}_{\mathbf{Smp}_{\mathbf{\Pi}}} = \{\mathbf{\Pi}[L] \mid \mathbf{\Pi}[R]\}_{\mathbf{S}}.$$

For $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\alpha}$ the classes $L_\alpha\,\mathfrak{a}' + (L_{<\alpha}\,\mathfrak{a}')^{-1}$ and $\mathscr{L}_\alpha[L_\alpha\,\mathfrak{a}']$ are mutually cofinal by (4.6). Similarly, $L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a}'')^{-1}$ and $\mathscr{L}_\alpha[L_\alpha\,\mathfrak{a}'']$ are mutually coinitial. By Lemma 6.11, the classes $E_{<\alpha}\varphi$ and $\mathscr{E}_\alpha[E_{\lhd\alpha}\,\mathfrak{d}_\alpha(\varphi)]$ are mutually cofinal. So it is enough to prove that (7.1) and (7.3) are valid cut equations for L_α and E_α respectively.

Lemma 7.2. *If ν is a successor ordinal, then the identities* (7.1) *and* (7.3) *hold.*

Proof. Let $\mathfrak{a} \in \mathbf{Mo}_\alpha$ and set

$$
\begin{aligned}
\varphi &:= \{L_\alpha\,\mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha}\,\mathfrak{a}\}_{\mathbf{No}_{>,\alpha}}\\
&= \left\{\mathbb{R}, L_\alpha\,\mathfrak{a}' + \frac{1}{L_{<\alpha}\,\mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha\,\mathfrak{a}'' - \frac{1}{L_{<\alpha}\,\mathfrak{a}''}, L_{<\alpha}\,\mathfrak{a} : \mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\alpha}\right\}.
\end{aligned}
$$

We have $\mathscr{L}_\alpha[L_\alpha\,\mathfrak{a}_L^{\mathbf{Mo}_\alpha}] < \varphi < L_{<\alpha}\,\mathfrak{a}$ so in view of (6.1), it is enough to prove that $\varphi < L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha} - (L_{<\alpha}\,\mathfrak{a})^{-1}$ to conclude that $\varphi = L_\alpha\,\mathfrak{a}$. Let $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\alpha}$. If $\mathfrak{a}'' \in \mathscr{E}_\alpha^*[\mathfrak{a}]$, then the inequality $\varphi < L_\alpha\,\mathfrak{a}''$ entails $\varphi < \mathscr{L}_\alpha[L_\alpha\,\mathfrak{a}'']$ whence $\varphi < L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a}'')^{-1}$ and $\varphi < L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a})^{-1}$. Otherwise, we have $\mathfrak{a} < L_{<\alpha}\,\mathfrak{a}''$, so $L_\alpha\,\mathfrak{a} < L_\alpha\,\mathfrak{a}'' - 2$, and $L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a})^{-1} > L_\alpha\,\mathfrak{a} + 1$. It is enough to prove that $L_\alpha\,\mathfrak{a} + 1 \geqslant \varphi$. Recall that

$$L_\alpha\,\mathfrak{a} + 1 = \left\{L_\alpha\,\mathfrak{a}, L_\alpha\,\mathfrak{a}' + \frac{1}{L_{<\alpha}\,\mathfrak{a}'} + 1 : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha} - \frac{1}{L_{<\alpha}\,\mathfrak{a}} + 1, L_{<\alpha}\,\mathfrak{a}\right\}$$

by (3.1). We see that $L_\alpha\,\mathfrak{a}' + \frac{1}{L_{<\alpha}\,\mathfrak{a}'} < L_\alpha\,\mathfrak{a} + 1$ for all $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha}$. We have $1 - \frac{1}{L_{<\alpha}\,\mathfrak{a}} > \frac{1}{L_{<\alpha}\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha}}$ so $L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha} - \frac{1}{L_{<\alpha}\,\mathfrak{a}} + 1 > \varphi$. Thus $\varphi \leqslant L_\alpha\,\mathfrak{a} + 1$. So (7.1) holds.

Now let $\psi \in \mathbf{No}_{>,\alpha}$ and set

$$\mathfrak{b} := \{E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi), E_\alpha\,\psi_L^{\mathbf{No}_{>,\alpha}} \mid E_\alpha\,\psi_R^{\mathbf{No}_{>,\alpha}}\}_{\mathbf{Mo}_\alpha}.$$

By uniformity of (7.1), we have

$$L_\alpha\,\mathfrak{b} = \{L_\alpha E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi), \psi_L^{\mathbf{No}_{>,\alpha}} \mid \psi_R^{\mathbf{No}_{>,\alpha}}, L_{<\alpha}\,\mathfrak{b}\}_{\mathbf{No}_{>,\alpha}},$$

whence $L_\alpha\,\mathfrak{b} \sqsupseteq \{\psi_L^{\mathbf{No}_{>,\alpha}} \mid \psi_R^{\mathbf{No}_{>,\alpha}}\}_{\mathbf{No}_{>,\alpha}} = \psi$. Conversely, $\mathfrak{b} > E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi)$ and $\mathfrak{b} > E_{<\alpha}\,\psi$, so $\psi < L_{<\alpha}\,\mathfrak{b}$. We have $L_\alpha E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi) = L_\alpha\,\mathfrak{d}_\alpha(\psi) + \mathbb{N}$. Since $L_\alpha\,\mathfrak{d}_\alpha(\psi) < L_{\alpha_{/\omega}}\,\mathfrak{d}_\alpha(\psi) \prec \psi$, this yields $L_\alpha E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi) < \psi$. This proves that ψ lies in the cut defining $L_\alpha\,\mathfrak{b}$. We conclude that $\psi = L_\alpha\,\mathfrak{b}$, hence (7.3) holds. □

We now assume that ν is a limit ordinal. For $z \in \mathbf{No}$, define

$$\begin{aligned} F(z) &:= \{\mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{>,\alpha}} z), F(z_L) \mid F(z_R)\}_{\mathbf{Mo}_\alpha}, \quad \text{and} \\ \Xi z &:= \{\mathbb{R}, \Xi z' + (L_{<\alpha} F(z'))^{-1} : z' \in z_L \mid \Xi z_R - (L_{<\alpha} F(z))^{-1}\}. \end{aligned}$$

Lemma 7.3. *For all $z \in \mathbf{No}$, we have*

$$F(z) \text{ is defined} \tag{7.5}$$

$$\Xi z \text{ is defined} \tag{7.6}$$

$$\Xi z = \Xi_{\mathbf{No}_{>,\alpha}} z \tag{7.7}$$

$$F(z) = E_\alpha\,\Xi z \tag{7.8}$$

Proof. We prove the result by induction on $(\mathbf{No}, \sqsubseteq)$. Let $z \in \mathbf{No}$ be such that (7.5), (7.6), (7.7) and (7.8) hold for all $y \in \mathbf{No}$ with $y \sqsubset z$.

For $z'' \in z_R$ and $z' \in z_L$, we have $\mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{>,\alpha}} z) \leqslant \mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{>,\alpha}} z'') < F(z'')$. We have $F(z') < F(z'')$ by definition of $F(z'')$ if $z' \in (z'')_L$ and by definition of $F(z')$ if $z'' \in (z')_R$. This proves that $F(z)$ is defined.

Let $z' \in z_L$ and $z'' \in z_R$. If $z' \in (z'')_L$, then we have $\Xi z'' > \Xi z' + (L_{<\alpha} F(z'))^{-1}$ by definition of $\Xi z''$. Since $F(z') < F(z)$ and $F(z), F(z') \in \mathbf{Mo}_\alpha$, we have $L_\gamma F(z') \prec L_\gamma F(z)$ for all $\gamma < \alpha$. We deduce that $\Xi z'' - (L_{<\alpha} F(z))^{-1} > \Xi z' + (L_{<\alpha} F(z'))^{-1}$. If $z'' \in (z')_L$, then $\Xi z' < \Xi z'' - (L_{<\alpha} F(z'))^{-1}$ by definition of $\Xi z'$. Since $F(z') < F(z)$, we obtain $\Xi z'' - (L_{<\alpha} F(z))^{-1} > \Xi z' + (L_{<\alpha} F(z'))^{-1}$. This proves that Ξz is defined.

Since (7.7) and (7.8) hold on $z_\sqsubset$, we have

$$\Xi z = \{\mathbb{R}, \Xi_{\mathbf{No}_{>,\alpha}} z' + (L_{<\alpha} E_\alpha\,\Xi_{\mathbf{No}_{>,\alpha}} z')^{-1} : z' \in z_L \mid \Xi_{\mathbf{No}_{>,\alpha}} z_R - (L_{<\alpha} E_\alpha\,\Xi_{\mathbf{No}_{>,\alpha}} z)^{-1}\}$$

By (6.9), this yields $\Xi z = \Xi_{\mathbf{No}_{>,\alpha}} z$, so (7.7) holds for z.

From (7.7), we get $\mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{>,\alpha}} z) = \mathfrak{d}_\alpha(\Xi z)$. By Proposition 6.14 and our assumption that (7.8) holds on $z_\sqsubset$, we have

$$\begin{aligned} L_\alpha F(z) &= \{\mathbb{R}, \mathscr{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)], \mathscr{L}_\alpha[L_\alpha F(z_L)] \mid L_\alpha F(z_R) - (L_{<\alpha} F(z))^{-1}, L_{<\alpha} F(z)\} \\ &= \{\mathbb{R}, \mathscr{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)], \mathscr{L}_\alpha[\Xi z_L] \mid \Xi z_R - (L_{<\alpha} F(z))^{-1}, L_{<\alpha} F(z)\}. \end{aligned}$$

Recall that $\Xi z = \{\mathbb{R}, \mathscr{L}_\alpha[\Xi z_L] \mid \Xi z_R - (L_{<\alpha} F(z))^{-1}\}$. Therefore it suffices to show that Ξz lies in the cut $(\mathscr{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)] \mid L_{<\alpha} F(z))$ to conclude that $L_\alpha F(z) = \Xi z$ and thus that $F(z) = E_\alpha\,\Xi z$. Now $L_\alpha\,\mathfrak{d}_\alpha(\Xi z) < \mathscr{E}_\alpha^*[\Xi z]$ so $L_\alpha\,\mathfrak{d}_\alpha(\Xi z) \prec \Xi z$ and $\mathscr{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)] < \Xi z$. We have $F(z) > \mathfrak{d}_\alpha(\Xi z)$, where $F(z) \in \mathbf{Mo}_\alpha$. Since ν is a limit ordinal, Lemma 6.11 implies that $F(z) > E_{<\alpha}\,\Xi z$, so $\Xi z < L_{<\alpha} F(z)$. This completes the proof that $F(z) = E_\alpha\,\Xi z$. □

Corollary 7.4. *The identities (7.1), (7.2), (7.3), and (7.4) all hold.*

Proof. It is enough to prove (7.1) and (7.3). The identity (7.3) follows from (7.7) and (7.8). In order to obtain (7.1), we consider $\mathfrak{a} \in \mathbf{Mo}_\alpha$, set $\psi := \{L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha} \mathfrak{a}\}_{\mathbf{No}_{>,\alpha}}$, and we show that $\mathfrak{a} = E_\alpha \psi$. Since (7.3) is uniform, we have

$$\begin{aligned} E_\alpha \psi &= \{\mathfrak{d}_\alpha(\psi), E_\alpha L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid E_\alpha L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, E_\alpha L_{<\alpha} \mathfrak{a}\}_{\mathbf{Mo}_\alpha} \\ &= \{\mathfrak{d}_\alpha(\psi), \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, E_\alpha L_{<\alpha} \mathfrak{a}\}_{\mathbf{Mo}_\alpha}. \end{aligned}$$

We have $\mathfrak{d}_\alpha(\psi) < \mathfrak{a}$ because $\psi < L_{<\alpha} \mathfrak{a}$, and $E_\alpha L_{<\alpha} \mathfrak{a} > \mathfrak{a}$ because $E_\alpha > E_{<\alpha}$ on $\mathbf{No}^{>,\succ}$. Since $\mathfrak{a} = \{\mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid \mathfrak{a}_R^{\mathbf{Mo}_\alpha}\}_{\mathbf{Mo}_\alpha}$, we deduce that $E_\alpha \psi = \mathfrak{a}$. □

Remark 7.5. The simplified cut equations for E_α, L_α can be viewed as alternative definitions for those functions, since they hold inductively on their domain of definition. It is unclear how to develop our theory directly upon these alternative definitions. In particular, does there exists a direct way to see that the cut equation (7.2) is warranted, and that the corresponding function satisfies $\mathbf{R}_\mu$ and $\mathbf{M}_\mu$?

7.2 Identities involving $\mathbf{Tr}_\alpha$ and $\mathbf{Tr}_\alpha^\star$.

Proposition 7.6. *Defining* $\mathbf{Tr}_\alpha := \mathbf{Smp}_{\mathscr{L}_\alpha'}$ *as in Section 6.1, we have* $\mathbf{Tr}_\alpha = \mathbf{No}_{>,\alpha}$.

Proof. Let $\varphi \in \mathbf{No}_{>,\alpha}$. We have $E_\alpha \mathscr{L}_\alpha[\varphi] = \mathscr{E}_\alpha[E_\alpha \varphi]$ by [5, Proposition 7.22]. Recall that $\mathscr{E}_\alpha[a] = \mathscr{E}_\alpha'[a]$ for all $a \in \mathbf{No}^{>,\succ}$. Now $\mathscr{E}_\alpha' \circ E_\alpha = E_\alpha \circ \mathscr{L}_\alpha'$ by definition of $\mathscr{L}_\alpha'$, so $E_\alpha \mathscr{L}_\alpha[\varphi] = E_\alpha \mathscr{L}_\alpha'[\varphi]$ and $\mathscr{L}_\alpha[\varphi] = \mathscr{L}_\alpha'[\varphi]$. By definition of $\mathbf{Tr}_\alpha$, we conclude that $\mathbf{Tr}_\alpha = \mathbf{Smp}_{\mathscr{L}_\alpha} = \mathbf{No}_{>,\alpha}$. □

Assume that ν is a successor ordinal. Then we have $\mathbf{No}_{>,\alpha} = \mathbf{No}_{>,\alpha} + \mathbb{R}$ by (4.4), so the functions $T_r \, \Xi_{\mathbf{No}_{>,\alpha}}$ and $\Xi_{\mathbf{No}_{>,\alpha}} T_r$ are both strictly increasing bijections from $\mathbf{No}$ onto $\mathbf{No}_{>,\alpha}$.

Lemma 7.7. *Assume that* ν *is a successor ordinal. Then for* $r \in \mathbb{R}$*, we have* $T_r \, \Xi_{\mathbf{No}_{>,\alpha}} = \Xi_{\mathbf{No}_{>,\alpha}} T_r$ *on* $\mathbf{No}$.

Proof. Let us abbreviate $\Xi := \Xi_{\mathbf{No}_{>,\alpha}}$. We prove the lemma by induction on $(\mathbf{No}, \sqsubseteq) \times (\mathbb{R}, \sqsubseteq)$. Let $(z, r) \in \mathbf{No} \times \mathbb{R}$ with

$$\Xi y + s = \Xi(y + s)$$

whenever $(y, s) \in \mathbf{No} \times \mathbb{R}$ is strictly simpler than (z, r). We let z', z'', r', r'' denote generic elements of z_L, z_R, r_L, r_R and we note that $r', r'' \in \mathbb{R}$. By (6.8), we have

$$\begin{aligned} \Xi(z + r) &= \left\{ \Xi(z' + r) + \frac{1}{L_{<\alpha} E_\alpha \Xi(z' + r)}, \; \Xi(z + r') + \frac{1}{L_{<\alpha} E_\alpha \Xi(z + r')} \right| \\ &\qquad \left. \Xi(z + r'') - \frac{1}{L_{<\alpha} E_\alpha \Xi(z + r'')}, \; \Xi(z'' + r) - \frac{1}{L_{<\alpha} E_\alpha \Xi(z'' + r)} \right\}_{\mathbf{No}^{>,\succ}} \\ &= \left\{ T_r \Xi z' + \frac{1}{L_{<\alpha} E_\alpha T_r \Xi z'}, \; T_{r'} \Xi z + \frac{1}{L_{<\alpha} E_\alpha T_{r'} \Xi z} \right| \\ &\qquad \left. T_{r''} \Xi z - \frac{1}{L_{<\alpha} E_\alpha T_{r''} \Xi z}, \; T_r \Xi z'' - \frac{1}{L_{<\alpha} E_\alpha T_r \Xi z''} \right\}_{\mathbf{No}^{>,\succ}}. \end{aligned}$$

Recall that ν is a successor ordinal. Since (4.2) holds for all $a \in \mathbf{No}^{>,\succ}$, the sets $L_{<\alpha} E_\alpha \mathscr{T} a$ and $L_{<\alpha} E_\alpha a$ are mutually cofinal and coinitial. Moreover $T_s(z + b) = T_s z + b$ for all $s \in \mathbb{R}$ and $b \in \mathbf{No}$, so

$$\begin{aligned} \Xi(z + r) &= \left\{ T_r\left(\Xi z' + \frac{1}{L_{<\alpha} E_\alpha \Xi z'}\right), \; T_{r'}\left(\Xi z + \frac{1}{L_{<\alpha} E_\alpha \Xi z}\right) \right| \\ &\qquad \left. T_{r''}\left(\Xi z - \frac{1}{L_{<\alpha} E_\alpha \Xi z}\right), \; T_r\left(\Xi z'' - \frac{1}{L_{<\alpha} E_\alpha \Xi z''}\right) \right\}_{\mathbf{No}^{>,\succ}}. \end{aligned}$$

By (3.1), we have

$$T_r \Xi z \;=\; \left\{T_r\!\left(\Xi z' + \frac{1}{L_{<\alpha} E_\alpha \Xi z'}\right),\ T_{r'} \Xi z \;\middle|\; T_{r''} \Xi z,\ T_r\!\left(\Xi z'' - \frac{1}{L_{<\alpha} E_\alpha \Xi z''}\right)\right\}_{\mathbf{No}^{>,\succ}}.$$

The numbers $T_r \Xi z, T_{r'} \Xi z$ and $T_{r''} \Xi z$ are α-truncated so $T_r \Xi z$ lies in the cut

$$\left(\bigcup_{r'} T_{r'}\!\left(\Xi z + \frac{1}{L_{<\alpha} E_\alpha \Xi z}\right) \;\middle|\; \bigcup_{r''} T_{r''}\!\left(\Xi z - \frac{1}{L_{<\alpha} E_\alpha \Xi z}\right)\right)_{\mathbf{No}^{>,\succ}}.$$

We deduce that $T_r \Xi z = \Xi T_r z$. The result follows by induction. □

Lemma 7.8. *If ν is a successor ordinal, then we have $\mathscr{T} \lneqq \mathscr{L}_\alpha^*$ on $\mathbf{No}^{>,\succ}$. Consequently, $\mathbf{Tr}_\alpha^* = \mathbf{No}^{>}_{\succ}$.*

Proof. The set $E_{<\alpha}$ is pointwise cofinal in $\mathscr{E}_\alpha^*$. So $L_\alpha E_{<\alpha} E_\alpha$ is pointwise cofinal in $\mathscr{L}_\alpha^*$. For $\gamma < \alpha$, there is $n \in \mathbb{N}$ such that $\gamma \leqslant \alpha_{/\omega} n$. We have

$$L_\alpha E_\gamma E_\alpha \;\leqslant\; L_\alpha E_{\alpha_{/\omega} n} E_\alpha \;=\; (L_\alpha E_{\alpha_{/\omega}} E_\alpha)^{\circ n} \;=\; (L_\alpha E_\alpha T_1)^{\circ n} \;=\; T_1^{\circ n} \;=\; T_n \;\in\; \mathscr{T}.$$

We deduce that $\mathscr{T} \lneqq \mathscr{L}_\alpha^*$ on $\mathbf{No}^{>,\succ}$, whence $\mathbf{Tr}_\alpha^* = \mathbf{Smp}_{\mathscr{T}} = \mathbf{No}^{>}_{\succ}$. □

7.3 Identities involving $\mathbf{Mo}_\alpha$ and $\mathbf{Mo}_\alpha^\star$.

Lemma 7.9. *If ν is a successor ordinal, then for $z \in \mathbf{No}$ we have*

$$\Xi_{\mathbf{Mo}_\alpha}(z-1) \;=\; L_{\alpha_{/\omega}} \Xi_{\mathbf{Mo}_\alpha} z.$$

Proof. This can be seen as a converse to the proof of the identity (6.3). We proceed by induction on $(\mathbf{No}, \sqsubseteq)$. Let z be such that the relation holds on $z_{\sqsubset}$. By (6.3), we have

$$\begin{aligned}
L_{\alpha_{/\omega}} \Xi_{\mathbf{Mo}_\alpha} z \;&=\; \{L_{\alpha_{/\omega}} (\Xi_{\mathbf{Mo}_\alpha} z)_L^{\mathbf{Mo}_\alpha} \mid L_{\alpha_{/\omega}} (\Xi_{\mathbf{Mo}_\alpha} z)_R^{\mathbf{Mo}_\alpha}, \Xi_{\mathbf{Mo}_\alpha} z\}_{\mathbf{Mo}_\alpha}\\
&=\; \{L_{\alpha_{/\omega}} \Xi_{\mathbf{Mo}_\alpha} z_L \mid L_{\alpha_{/\omega}} \Xi_{\mathbf{Mo}_\alpha} z_R, \Xi_{\mathbf{Mo}_\alpha} z\}_{\mathbf{Mo}_\alpha}\\
&=\; \{\Xi_{\mathbf{Mo}_\alpha}(z_L - 1) \mid \Xi_{\mathbf{Mo}_\alpha}(z_R - 1), \Xi_{\mathbf{Mo}_\alpha} z\}_{\mathbf{Mo}_\alpha} \qquad \text{(by the inductive hypothesis)}\\
&=\; \Xi_{\mathbf{Mo}_\alpha}\{z_L - 1 \mid z_R - 1, z\}\\
&=\; \Xi_{\mathbf{Mo}_\alpha}(z-1) \quad \text{by (3.1)}.
\end{aligned}$$

We conclude by induction. □

Noting that $E_{\alpha_{/\omega}} = E_\alpha T_1 L_\alpha$ on $\mathbf{No}^{>,\succ}$, the previous relation further generalizes as follows.

Proposition 7.10. *Assume that ν is a successor ordinal and let $r \in \mathbb{R}$. Then*

$$\Xi_{\mathbf{Mo}_\alpha} T_r \;=\; E_\alpha T_r L_\alpha \Xi_{\mathbf{Mo}_\alpha} \tag{7.9}$$

Proof. We proceed by induction. Let $(z, r) \in \mathbf{No} \times \mathbb{R}$ be such that

$$\Xi_{\mathbf{Mo}_\alpha} T_s y \;=\; E_\alpha T_s L_\alpha \Xi_{\mathbf{Mo}_\alpha} y$$

for all strictly simpler $(y, s) \in \mathbf{No} \times \mathbb{R}$ with respect to the product order $\sqsubseteq \times \sqsubseteq$. For $s \in \mathbb{R}$, let ϕ_s be the function $b \longmapsto E_\alpha T_s L_\alpha b$ on $\mathbf{No}^{>,\succ}$ and let $\mathfrak{a} := \Xi_{\mathbf{Mo}_\alpha} z$. By (3.1) and (3.2), we have

$$\begin{aligned}
\Xi_{\mathbf{Mo}_\alpha}(z+r) \;&=\; \{\mathbb{R}, \mathscr{E}_\alpha \Xi_{\mathbf{Mo}_\alpha}(z_L + r), \mathscr{E}_\alpha \Xi_{\mathbf{Mo}_\alpha}(z + r_L) \mid \mathscr{E}_\alpha \Xi_{\mathbf{Mo}_\alpha}(z_R + r), \mathscr{E}_\alpha \Xi_{\mathbf{Mo}_\alpha}(z + r_R)\}\\
&=\; \{\mathbb{R}, \mathscr{E}_\alpha \phi_r(\mathfrak{a}_L^{\mathbf{Mo}_\alpha}), \mathscr{E}_\alpha \phi_{r_L}(\mathfrak{a}) \mid \mathscr{E}_\alpha \phi_r(\mathfrak{a}_R^{\mathbf{Mo}_\alpha}), \mathscr{E}_\alpha \phi_{r_R}(\mathfrak{a})\}.
\end{aligned}$$

By (7.1), Lemma 7.7 and (3.1), we have:

$$T_r L_\alpha \mathfrak{a} \;=\; \{T_r L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha}, T_{r_L} L_\alpha \mathfrak{a} \mid T_{r_R} L_\alpha \mathfrak{a}, T_r L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha} \mathfrak{a}\}_{\mathbf{Tr}_\alpha}.$$

We deduce that

$$\begin{aligned}\phi_r(\mathfrak{a}) &= \{E_{<\alpha} T_r L_\alpha \mathfrak{a}, \mathscr{E}_\alpha \phi_r(\mathfrak{a}_L^{\mathbf{Mo}_\alpha}), \mathscr{E}_\alpha \phi_{r_L}(\mathfrak{a}) \mid \mathscr{E}_\alpha \phi_{r_R}(\mathfrak{a}), \mathscr{E}_\alpha \phi_r(\mathfrak{a}_R^{\mathbf{Mo}_\alpha}), \mathscr{E}_\alpha E_\alpha L_{<\alpha} \mathfrak{a}\} \\ &= \{E_{<\alpha} L_\alpha \mathfrak{a}, \mathscr{E}_\alpha \phi_r(\mathfrak{a}_L^{\mathbf{Mo}_\alpha}), \mathscr{E}_\alpha \phi_{r_L}(\mathfrak{a}) \mid \mathscr{E}_\alpha \phi_{r_R}(\mathfrak{a}), \mathscr{E}_\alpha \phi_r(\mathfrak{a}_R^{\mathbf{Mo}_\alpha}), E_\alpha L_{<\alpha} \mathfrak{a}\}.\end{aligned}$$

It is enough to prove that $E_{<\alpha} L_\alpha \mathfrak{a} < \Xi_{\mathbf{Mo}_\alpha}(z+r) < E_\alpha L_{<\alpha} \mathfrak{a}$ to conclude that $\phi_r(\mathfrak{a}) = \Xi_{\mathbf{Mo}_\alpha}(z+r)$. Towards this, fix an $n \in \mathbb{N}$ with $-n \leqslant r \leqslant n$. Lemma 7.9 yields

$$\begin{aligned}\Xi_{\mathbf{Mo}_\alpha}(z+r) &\leqslant \Xi_{\mathbf{Mo}_\alpha}(z+n) = E_{\alpha/\omega^n} \mathfrak{a} < E_\alpha L_{<\alpha} \mathfrak{a} \\ \Xi_{\mathbf{Mo}_\alpha}(z+r) &\geqslant \Xi_{\mathbf{Mo}_\alpha}(z-n) = L_{\alpha/\omega^n} \mathfrak{a} > E_{<\alpha} L_\alpha \mathfrak{a}.\end{aligned}$$

We conclude by induction that (7.9) holds. □

Remark 7.11. For $r, s \in \mathbb{R}$, we have $\phi_{r+s} = \phi_r \circ \phi_s$, and $\phi_1 = E_{\alpha/\omega}$. Therefore we can see $(\phi_r)_{r \in \mathbb{R}}$ as a system of fractional and real iterates of the hyperexponential function $E_{\alpha/\omega}$ on $\mathbf{No}^{>,\succ}$. The previous proposition shows that the action of those iterates on $L_{<\alpha}$-atomic numbers reduces to translations, modulo the parametrization $\Xi_{\mathbf{Mo}_\alpha}$. In particular, one can compute the functional square root of exp on $\mathbf{Mo}_\omega$ in terms of sign sequences using the material from [3].

Proposition 7.12. *If ν is a successor ordinal, then $\mathbf{Mo}_\alpha^* = \mathbf{Mo}_\alpha {\prec} \mathbf{No}_\succ$.*

Proof. For $\theta \in \mathbf{No}_\succ$, we have $\theta_L + \mathbb{N} < \theta < \theta_R - \mathbb{N}$. By Lemma 7.9, it follows that $E_{\alpha/\omega \mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_L < \Xi_{\mathbf{Mo}_\alpha} \theta < L_{\alpha/\omega \mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_R$. This implies that $\mathscr{E}_\alpha^* \Xi_{\mathbf{Mo}_\alpha} \theta_L < \Xi_{\mathbf{Mo}_\alpha} \theta < \mathscr{E}_\alpha^* \Xi_{\mathbf{Mo}_\alpha} \theta_R$, so $\Xi_{\mathbf{Mo}_\alpha} \theta$ is $\mathscr{E}_\alpha^*$-simple.

Conversely, consider $\theta \in \mathbf{No}^{>,\succ}$ such that $\Xi_{\mathbf{Mo}_\alpha} \theta$ is $\mathscr{E}_\alpha^*$-simple. We have $\Xi_{\mathbf{Mo}_\alpha} \theta_L \subseteq (\Xi_{\mathbf{Mo}_\alpha} \theta)_L$ and $\Xi_{\mathbf{Mo}_\alpha} \theta_R \subseteq (\Xi_{\mathbf{Mo}_\alpha} \theta)_R$, whence $E_{\alpha/\omega \mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_L < \Xi_{\mathbf{Mo}_\alpha} \theta < L_{\alpha/\omega \mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_R$. We obtain $\theta_L + \mathbb{N} < \theta < \theta_R - \mathbb{N}$, which proves that $\theta \in \mathbf{No}_\succ$. □

Proposition 7.13. *We have $E_\alpha \mathbf{Tr}_\alpha^* = \mathbf{Mo}_\alpha^*$.*

Proof. Let $\varphi \in \mathbf{Tr}_\alpha^*$. So $\varphi \in \mathbf{Tr}_\alpha$. By Proposition 3.1, the number $E_\alpha \varphi$ is simplest in

$$E_\alpha(\mathscr{E}_\alpha^*[\varphi] \cap \mathbf{Tr}_\alpha) = \mathscr{E}_\alpha^*[E_\alpha \varphi] \cap \mathbf{Mo}_\alpha.$$

Since $\mathbf{Mo}_\alpha^* \subseteq \mathbf{Mo}_\alpha$, we have $E_\alpha \varphi \sqsubseteq \mathscr{E}_\alpha^*[E_\alpha \varphi] \cap \mathbf{Mo}_\alpha^*$ so $E_\alpha \varphi \sqsubseteq \mathfrak{d}_\alpha^*(E_\alpha \varphi)$. We deduce that $E_\alpha \varphi = \mathfrak{d}_\alpha^*(E_\alpha \varphi)$, so $E_\alpha \varphi$ is $\mathscr{E}_\alpha^*$-simple. Conversely, let $\mathfrak{a} \in \mathbf{Mo}_\alpha^*$. By Proposition 3.1 the number $L_\alpha \mathfrak{a}$ is simplest in $L_\alpha(\mathscr{E}_\alpha^*[\mathfrak{a}] \cap \mathbf{Mo}_\alpha) = \mathscr{L}_\alpha^*[L_\alpha \mathfrak{a}] \cap \mathbf{No}_{\succ,\alpha}$. Since $\mathbf{Tr}_\alpha^* \subseteq \mathbf{No}_{\succ,\alpha}$, we have $L_\alpha \mathfrak{a} \sqsubseteq \mathscr{L}_\alpha^*[L_\alpha \mathfrak{a}] \cap \mathbf{Tr}_\alpha^*$ so $L_\alpha \mathfrak{a} \sqsubseteq \sharp_\alpha^*(L_\alpha \mathfrak{a})$. We deduce that $L_\alpha \mathfrak{a} \sqsubseteq \sharp_\alpha^*(L_\alpha \mathfrak{a})$ is $\mathscr{L}_\alpha^*$-simple. □

Corollary 7.14. *If ν is a successor ordinal, then $\mathbf{Mo}_\alpha^* = E_\alpha \mathbf{No}_\succ^{>}$.*

BIBLIOGRAPHY

[1] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. On numbers, germs, and transseries. In *Proc. Int. Cong. of Math. 2018*, volume 1, pages 1–24. Rio de Janeiro, 2018.

[2] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. The surreal numbers as a universal H-field. *J. Eur. Math. Soc.*, 21(4):1179–1199, 2019.

[3] V. Bagayoko. Sign sequences of log-atomic numbers. Hal-02952455, 2020.

[4] V. Bagayoko and J. van der Hoeven. Surreal substructures. HAL-02151377 (pre-print), 2019.

[5] V. Bagayoko, J. van der Hoeven, and E. Kaplan. Hyperserial fields. (notes non publiées), 2020.

[6] V. Bagayoko, J. van der Hoeven, and V. Mantova. Defining a surreal hyperexponential. HAL-02861485 (pre-print), 2020.

[7] A. Berarducci. Surreal numbers, exponentiation and derivations. https://arxiv.org/abs/2008.06878, 08 2020.

[8] A. Berarducci, S. Kuhlmann, V. Mantova, and M. Matusinski. Exponential fields and Conway's omega-map. *Proceedings of the American Mathematical Society*, 2019.

[9] A. Berarducci and V. Mantova. Surreal numbers, derivations and transseries. *JEMS*, 20(2):339–390, 2018.

[10] A. Berarducci and V. Mantova. Transseries as germs of surreal functions. *Trans. of the AMS*, 371:3549–3592, 2019.

[11] J. H. Conway. *On numbers and games*. Academic Press, 1976.

[12] B. I. Dahn and P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae*, 127:45–50, 1986.

[13] L. van den Dries and Ph. Ehrlich. Fields of surreal numbers and exponentiation. *Fundamenta Mathematicae*, 167(2):173–188, 2001.

[14] L. van den Dries, J. van der Hoeven, and E. Kaplan. Logarithmic hyperseries. *Trans. of the AMS*, 372(7):5199–5241, 2019.

[15] P. du Bois-Reymond. Sur la grandeur relative des infinis des fonctions. *Annali di Matematica Pura ed Applicata (1867-1897)*, 4(1):338–353, 1870.

[16] P. du Bois-Reymond. Über asymptotische Werte, infinitäre Approximationen und infinitäre Auflösung von Gleichungen. *Math. Ann.*, 8:363–414, 1875.

[17] P. du Bois-Reymond. Über die Paradoxen des Infinitärscalcüls. *Math. Ann.*, 11:149–167, 1877.

[18] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Hermann, collection: Actualités mathématiques, 1992.

[19] H. Gonshor. *An Introduction to the Theory of Surreal Numbers*. Cambridge Univ. Press, 1986.

[20] H. Hahn. Über die nichtarchimedischen Größensysteme. *Sitz. Akad. Wiss. Wien*, 116:601–655, 1907.

[21] G. H. Hardy. *Orders of infinity*. Cambridge Univ. Press, 1910.

[22] G. H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London Mathematical Society*, 10(2):54–90, 1911.

[23] D. Harvey and J. van der Hoeven. Faster polynomial multiplication over finite fields using cyclotomic coefficient rings. Accepted for publication in J. of Complexity, 2019.

[24] J. van der Hoeven. *Automatic asymptotics*. PhD thesis, École polytechnique, Palaiseau, France, 1997.

[25] J. van der Hoeven. Transséries fortement monotones. Chapter 1 of unpublished CNRS activity report, `https://www.texmacs.org/joris/schmeling/rap1-2000.pdf`, 2000.

[26] J. van der Hoeven. *Transseries and real differential algebra*, volume 1888 of *Lecture Notes in Mathematics*. Springer-Verlag, 2006.

[27] J. van der Hoeven. *The Jolly Writer. Your Guide to GNU TeXmacs*. Scypress, 2020.

[28] V. Mantova and M. Matusinski. Surreal numbers with derivation, Hardy fields and transseries: a survey. *Contemporary Mathematics*, pages 265–290, 2017.

[29] M. C. Schmeling. *Corps de transséries*. PhD thesis, Université Paris-VII, 2001.

GLOSSARY

$\{L \mid R\}$	simplest number between L and R	4
$\mathbb{R}[[\mathfrak{M}]]$	field of well-based series with real coefficients over $\mathfrak{M}$	5
$\operatorname{supp} f$	support of a series	5
$\mathfrak{d}_f$	$\max \operatorname{supp} f$	5
$f_{\succ \mathfrak{m}}$	truncation $\sum_{\mathfrak{n} \succ \mathfrak{m}} f_{\mathfrak{n}} \mathfrak{n}$ of f	5
$f_{\succ}$	$f_{\succ 1}$	5
$f \trianglelefteq g$	$\operatorname{supp} f \succ g - f$	5
$f \prec g$	$\mathbb{R}^{>} \lvert f \rvert < \lvert g \rvert$	5
$f \preccurlyeq g$	$\exists r \in \mathbb{R}^{>}, \lvert f \rvert < r \lvert g \rvert$	5
$f \asymp g$	$f \preccurlyeq g$ and $g \preccurlyeq f$	5
$\mathbb{S}_{\succ}$	series $f \in \mathbb{S}$ with $\operatorname{supp} f \succ 1$	5
$\mathbb{S}^{\prec}$	series $f \in \mathbb{S}$ with $f \prec 1$	5
$\mathbb{S}^{>,\succ}$	series $f \in \mathbb{S}$ with $f \geqslant 0$ and $f \succ 1$	5
On	class of ordinals	6
$\sqsubseteq$	simplicity relation	6
$\dot{\omega}^{\gamma}$	ordinal exponentiation with base ω at γ	6
μ_-	$\mu = \mu_- + 1$ if μ is a successor ordinal and $\mu_- = \mu$ if μ is a limit ordinal	6
$\alpha_{/\omega}$	$\dot{\omega}^{\mu_-}$ for $\alpha = \dot{\omega}^{\mu}$	6
$\mathbf{U} \prec \mathbf{V}$	the surreal substructure $\Xi_{\mathbf{U}} \mathbf{V}$	7
$\mathbf{Smp}_{\mathbf{\Pi}}$	class of $\mathbf{\Pi}$-simple elements	8
$\pi_{\mathbf{\Pi}}$	projection $\mathbf{S} \longrightarrow \mathbf{Smp}_{\mathbf{\Pi}}$	8
$\mathscr{G}[a]$	class of numbers b with $\exists g, h \in \mathscr{G}, g a \preccurlyeq b \preccurlyeq h a$	8