Errata

The following mistakes slipped into the final version of our book “Transseries and Real Differential Algebra”. Please let us know if you find any other errors. (We also found some minor typos that are not included here.)

General

• “stable” (under various operations) should be “closed” at various locations throughout the manuscript.

Chapter 1

P.17, Proposition 1.5. In the proof that \((a) \Rightarrow (b)\), it is understood that the set of minimal elements \(G\) is defined by \(G = \{ x \in F : \forall y \in F, y \leq x \Rightarrow y \equiv x \}\).

Then \(G/\equiv\) (and not \(G\)) is a finite anti-chain. Nevertheless, for every \(x \in G\), the set \(\{ y \in F : y \equiv x \}\) is finite, so \(G\) is indeed finite.

P.18, Proposition 1.6(e). We meant “Any quasi-ordering on \(E\) […].”

P.18, definition “bad sequence”. “ordering” should be “quasi-ordering”.

P.22. In \(OM\) (and \(OA\)), we should also assume the opposite directions, namely \(x y \leq x y' \Rightarrow y \leq y'\) (and \(x + y \leq x + y' \Rightarrow y \leq y'\)). In particular, ordered monoids are “cancellative”: \(x y = x y' \Rightarrow y = y'\) (resp. \(x + y = x + y' \Rightarrow y = y'\)). (Recall the assumption that all monoids are commutative.)

P.23, before Exercise 1.15. Suppress unfinished sentence.

P.27, 1.11. There exist totally ordered rings with nilpotent elements, such as \(Q[\varepsilon]/(\varepsilon^2)\) with \(a + b \varepsilon > 0 \Leftrightarrow (a > 0) \lor (a = 0 \land b > 0)\); see Exercise 1.16 for more details. We should therefore assume \(R\) to be a totally ordered domain. This also holds for Corollary 1.19 and Proposition 1.20.

P.27. \(\mathcal{D}(R) \otimes M\) should be \(\mathcal{D}(R) \otimes_R M\) (twice) and the positive elements of this set are sums of elements of the form \(x \otimes y\) with \(x \geq 0\) and \(y \geq 0\), as in example 1.16. Moreover, we should have required \(\lambda > 0 \land \lambda x > 0 \Rightarrow x > 0\) for all \(\lambda \in R\) and \(x \in M\), also in Proposition 1.19.

P.27. In Proposition 1.20, the algebra should be torsion-free.

P.29. In the proof of Proposition 1.22, “saturated” should be “a Hahn space”.
2. **Errata**

P.31. In Proposition 1.26, one should add the assumption that $\preceq$ and $\prec$ are associated for the second assertion. In the second paragraph of the proof, we obtain $y \preceq \varphi x$ and $x \prec \varphi^{-1} y$ for some $\varphi$ ($x$ and $y$ got inverted).

Chapter 2

P.34. **Equation (2.1).** The notation $S^* = \{m_1 \cdots m_k, m_1, \ldots, m_k \in \mathcal{M}\}$ for subsets $S \subseteq \mathcal{M}$ was not properly introduced here. It plays a similar role as the set of words, but the notation is a bit confusing since it is not completely the same thing, technically speaking. It would have been better to use a different notation like $E^w$ for the set of words with letters in $E$.

P.34. **Definition of grid-based sets.** We can take $n = 1$ if $\mathcal{M}$ is a totally ordered group ($n = 1$ does not necessarily suffice if $\mathcal{M}$ is only a monoid).

P.38. For the examples ring $C[[x^2]]$ and $C[[x^Q]]$ to be fields, we need to assume that $C$ is a field.

P.40. In section 2.3.1, we need to assume that $\mathcal{M}$ is ordered (not merely quasi-ordered).

P.40. At the start of the last paragraph, we only need to assume that $\mathcal{M}$ is totally ordered for what follows.

P.41. **Warning 2.6.** We should have defined $C[[\mathcal{M}]]^{\succ} = \{f \in C[[\mathcal{M}]] : f \succ 1\}$, $C[[\mathcal{M}]]^{\prec} = \{f \in C[[\mathcal{M}]] : f \prec 1\}$, etc. Note also that $0 \notin C[[\mathcal{M}]]^{\succ}$, so we rather have $C[[\mathcal{M}]]^{\succ} \setminus \{0\} \supseteq C[[\mathcal{M}]]^{\succ}$.

P.46. To the axioms of strong abelian groups, one should add: for all $F \in \mathcal{F}(A)$, we have $\sum (-F) = -\sum F$. On the other hand, it would be better to remove the axiom $\text{SA6}$, since it excludes the possibility of strong groups with torsion elements.

P.49. **Line 6.** It should have been made precise that the couples $(v, w)$ form an anti-chain for the ordering $\preceq'$.

P.50. In the proof of Proposition 2.14, note that well-based families were only defined in Exercise 2.7.

P.52. **Proposition 2.17.** It would have increased readability to invert the roles of $\varphi$ and $\psi$, to make notations compatible with the role of $\varphi$ below Proposition 2.17.

P.52. In the displayed equation above Proposition 2.18, one should assume that $g \prec 1$.

P.53. At the very start of section 2.6, we need to assume that $C$ is a field of characteristic zero in order to define $(1 + z)^{\lambda}$ for $\lambda \in R$.

P.53. In the proof of Proposition 2.19, it should have been mentioned upfront that $\varphi \circ \varphi$ is strictly increasing and that $\varphi$ preserves infinitesimals.

Chapter 4

P.81. **Proposition 4.1.** The ring $R$ should be an ordered domain and the partial exponential function should satisfy $\text{E1}$, $\text{E2}$, and $\text{E3}$.
P.81. In the proof of Proposition 4.1, a few extra precisions are welcome: the first displayed formula in particular shows that \( x \neq -(2n + 1) \). If we also have \( x \neq 0 \), then the second displayed formula yields \( 0 \geq x^{4h}(2n + 1 + x)^2 > 0 \), which is impossible.

P.81. **Proposition 4.3.** Statement (c) should read: If \( R \) contains the ordered field \( \mathbb{Q} \) and \( \text{dom exp} \) is a \( \mathbb{Q} \)-module, then
\[
\forall n \in \mathbb{N}, \forall x \in \text{dom exp}, \quad x > (2n)^2 \Rightarrow \exp x > x^n.
\]

P.84. Just before T1, T2, and T3: we assume that the logarithm extends the one on \( C^{	ext{op}} \) and that it is compatible with the \( C \)-power structure on \( \mathbb{T} \).

P.84. **Example 4.5.** \( x \in \mathbb{T}_x \) should have been \( x \in \mathbb{T}^{	ext{op}} \). “Stable under exponentiation” means \( \text{im log} = \mathbb{T} \), and \( x^2 / (1 - x^{-1}) \notin \mathbb{T}_x \).

P.85. The fact that \( f \in \mathbb{T}^{*,r} \Rightarrow \log f \in \mathbb{T}^{*,r} \) is implicitly used in the proof (c). The fact that \( \log f \prec f \) for \( f \in \mathbb{T}^{*,r} \) follows more directly from (1).

P.86. In the proof of L3, the displayed equation \( f - E_{2n}(\log f) \sim cf - \log cf - 1 > 0 \) should be \( f - E_{2n}(\log f) \sim cf - E_{2n}(\log cf) > 0 \).

P.88. **Section 4.3.2.** Replace \( \mathbb{T}_x \) by \( \mathbb{T}_x \) in the displayed formula \( \Sigma_{\exp} = \exp \Sigma_x \) and on two other occasions just below.

P.88. In the statement of Proposition 4.9, \( R_{\exp} \) should be \( \mathbb{T}_{\exp} \).

P.89. The last formula in the proof of Proposition 4.9 should read \( \log m \asymp \exp((\log \log m)_x) \prec \exp((\log m)_x) \asymp m \).

P.89. **L.-7.** In the definition of \( \mathbb{E}_n \), replace \( \log_n x \) by \( C[(\log_n x)C] \).

P.90. In the definition of level, replace “smallest number \( n \in \mathbb{Z} \) by “largest number \( n \in \mathbb{Z} \).” Recall that \( \log_n = \exp_{-n} \) for \( n < 0 \) in the definition of \( \mathbb{E}_n \).

P.90-91. In subsection 4.3.5, the subscripts and superscripts got inverted in the expressions of the form \( C^p \) and \( \mathbb{E}_q^p \).

P.91. **Exercise 4.9.** It should be \( \log_C^p x \) instead of \( \log_C^p x \).

P.92. **Exercise 4.12(a).** It should be \( f_\alpha = \sqrt{x} - \sum_{0 < \beta < \alpha} e^{f_\beta \log} \). By transfinite induction, one shows that \( (f_\alpha) \) is a strictly decreasing sequence of transseries with purely large support (whence each \( e^{f_\beta \log} \) is a transmonomial) and such that the order type of \( \text{supp } f_\alpha \) is precisely \( \alpha \).

P.92. In the definition of transbasis, we understand that \( \mathcal{B} = (b_1, \ldots, b_n) \) is a finite basis of an asymptotic scale in \( \mathbb{T} \) with \( n \geq 1 \). Note that TB2 and TB3 implicitly imply that \( b_1, \ldots, b_n \succ 1 \).

P.92. **Example 4.14.** The second transbasis should read \( (x, e^{(x + \gamma_2)^2}) \).

P.93. We should have said “Then \( \text{supp } f \) is contained in a set of the form \( (\exp_\gamma x)^C e^{g_0 + g_1 N + \cdots + g_k N} \), where \( g_0, \ldots, g_k \in \mathbb{C}^p \}_{-1} \exp x \mathbb{T}_x \),” (while dropping the assumption that \( e^{g_1}, \ldots, e^{g_k} \prec 1 \), for the later reduction “without loss of generality”). Note also Exercise 4.9(d).

**Chapter 5**

P.102. In the displayed formulas for the proof of Proposition 5.5, replace \( (f \circ \exp)^' \) and \( (g \circ \exp)^' \) by \( (f \circ \exp)' \log_l \) and \( (g \circ \exp)' \circ \log_l \). Also replace \( \log_l x \) by \( \log_{l-1} x \) in the last formula.
Errata

P.105, Top. It would be better to say: “By what precedes, for fixed $i$, the family $T_m$ is grid-based and the same for any $m \in S_i$. Hence $\bigcup_{1 \leq i \leq n} \bigcup_{m \in S_i} T_m m$ is again grid-based and $f$ is a grid-based mapping...”

P.105. The last displayed formula of the proof of Proposition 5.7 should read

$$(f \circ \pi f \mid_x \xi) = (f \circ \pi f \mid_x \xi) = 0.$$ 

P.112. At the end of the proof of Theorem 5.13, the last displayed equation should be $g_{[M,N]} + f_{[N,M]} = \cdots$.

Chapter 6

P.125, Section 6.4.2, Second displayed equation. Suppress the $c$.

P.133, Exercise 6.17. “extensive” should be “strictly extensive”.

Chapter 7

P.164, Exercise 7.28. The coefficients of $L$ should be exponential.

Chapter 8

P.172. Equation (8.13) should read

$$D_P(F) = \sum_{\omega} \left( \sum_{\|\tau\| = \omega} s_{\tau,\omega} D_P,|\tau| \right) F[\omega].$$

P.195. In the output of Algorithm unravel, suppress “with dominant term $\tau$”.

Chapter 9

P.207–208. The paragraph numbers in Section 9.2.2 should be removed.