# Errata

The following mistakes slipped into the final version of our book "*Transseries and Real Differential Algebra*". Please let us know if you find any other errors. (We also found some minor typos that are not included here.)

## General

- "stable" (under various operations) should be "closed" at various locations throughout the manuscript.

## Chapter 1

**P.17, Proposition 1.5.** In the proof that $(a)\Rightarrow(b)$, it is understood that the set of minimal elements $G$ is defined by $G=\{x\in F: \forall y\in F, y\leqslant x\Rightarrow y\equiv x\}$. Then $G/\equiv$ (and not $G$) is a finite anti-chain. Nevertheless, for every $x\in G$, the set $\{y\in F: y\equiv x\}$ is finite, so $G$ is indeed finite.

**P.18, Proposition 1.6(*c*).** We meant "Any quasi-ordering on $E$ [...]".

**P.18, definition "bad sequence".** "ordering" should be "quasi-ordering".

**P.22.** In **OM** (and **OA**), we should also assume the opposite directions, namely $x\,y\leqslant x\,y'\Rightarrow y\leqslant y'$ (and $x+y\leqslant x+y'\Rightarrow y\leqslant y'$). In particular, ordered monoids are "cancellative": $x\,y=x\,y'\Rightarrow y=y'$ (resp. $x+y=x+y'\Rightarrow y=y'$). (Recall the assumption that all monoids are commutative.)

**P.23, before Exercise 1.15.** Suppress unfinished sentence.

**P.27, l.11.** There exist totally ordered rings with nilpotent elements, such as $\mathbb{Q}[\varepsilon]/(\varepsilon^2)$ with $a+b\,\varepsilon>0\Leftrightarrow(a>0)\vee(a=0\wedge b>0)$; see Exercise 1.16 for more details. We should therefore assume $R$ to be a totally ordered domain. This also holds for Corollary 1.19 and Proposition 1.20.

**P.27.** $\mathscr{Q}(R)\otimes M$ should be $\mathscr{Q}(R)\otimes_R M$ (twice) and the positive elements of this set are sums of elements of the form $x\otimes y$ with $x\geqslant 0$ and $y\geqslant 0$, as in example 1.16. Moreover, we should have required $\lambda>0\wedge\lambda\,x>0\Rightarrow x>0$ for all $\lambda\in R$ and $x\in M$, also in Proposition 1.19.

**P.27.** In Proposition 1.20, the algebra should be torsion-free.

**P.29.** In the proof of Proposition 1.22, "saturated" should be "a Hahn space".

**P.31.** In Proposition 1.26, one should add the assumption that $\preccurlyeq$ and $\prec$ are associated for the second assertion. In the second paragraph of the proof, we obtain $y \preccurlyeq \varphi x$ and $x \prec \varphi^{-1} y$ for some $\varphi$ ($x$ and $y$ got inverted).

## Chapter 2

**P.34, Equation (2.1).** The notation $\mathfrak{S}^* = \{\mathfrak{m}_1 \cdots \mathfrak{m}_k \colon \mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \mathfrak{M}\}$ for subsets $\mathfrak{S} \subseteq \mathfrak{M}$ was not properly introduced here. It plays a similar role as the set of words, but the notation is a bit confusing since it is not completely the same thing, technically speaking. It would have been better to use a different notation like $E^{\mathrm{w}}$ for the set of words with letters in $E$.

**P.34, Definition of grid-based sets.** We can take $n = 1$ if $\mathfrak{M}$ is a totally ordered group ($n = 1$ does not necessarily suffice if $\mathfrak{M}$ is only a monoid).

**P.38.** For the examples ring $C[[x^{\mathbb{Z}}]]$ and $C[[x^{\mathbb{Q}}]]$ to be fields, we need to assume that $C$ is a field.

**P.40.** In section 2.3.1, we need to assume that $\mathfrak{M}$ is ordered (not merely quasi-ordered).

**P.40.** At the start of the last paragraph, we only need to assume that $\mathfrak{M}$ is totally ordered for what follows.

**P.41, Warning 2.6.** We should have defined $C[[\mathfrak{M}]]^{\succ} = \{f \in C[[\mathfrak{M}]] : f \succ 1\}$, $C[[\mathfrak{M}]]^{\prec} = \{f \in C[[\mathfrak{M}]] : f \prec 1\}$, etc. Note also that $0 \notin C[[\mathfrak{M}]]^{\succ}$, so we rather have $C[[\mathfrak{M}]]_{\succ} \setminus \{0\} \supsetneq C[[\mathfrak{M}]]^{\succ}$.

**P.46.** To the axioms of strong abelian groups, one should add: for all $\mathcal{F} \in \mathscr{S}(A)$, we have $\sum (-\mathcal{F}) = -\sum \mathcal{F}$. On the other hand, it would be better to remove the axiom **SA6**, since it excludes the possibility of strong groups with torsion elements.

**P.49, Line 6.** It should have been made precise that the couples $(\mathfrak{v}, \mathfrak{w})$ form an anti-chain for the ordering $\preccurlyeq^!$.

**P.50.** In the proof of Proposition 2.14, note that well-based families were only defined in Exercise 2.7.

**P.52, Proposition 2.17.** It would have increased readability to invert the roles of $\varphi$ and $\psi$, to make notations compatible with the role of $\varphi$ below Proposition 2.17.

**P.52.** In the displayed equation above Proposition 2.18, one should assume that $g \prec 1$.

**P.53.** At the very start of section 2.6, we need to assume that $C$ is a field of characteristic zero in order to define $(1 + z)^{\lambda}$ for $\lambda \in R$.

**P.53.** In the proof of Proposition 2.19, it should have been mentioned upfront that $\mathfrak{d} \circ \varphi$ is strictly increasing and that $\hat{\varphi}$ preserves infinitesimals.

## Chapter 4

**P.81, Proposition 4.1.** The ring $R$ should be an ordered domain and the partial exponential function should satisfy **E1**, **E2**, and **E3**.

**P.81.** In he proof of Proposition 4.1, a few extra precisions are welcome: the first displayed formula in particular shows that $x \neq -(2n+1)$. If we also have $x \neq 0$, then the second displayed formula yields $0 \geqslant x^{4n}(2n+1+x)^2 > 0$, which is impossible.

**P.81, Proposition 4.3.** Statement $(c)$ should read: If $R$ contains the ordered field $\mathbb{Q}$ and $\operatorname{dom}\exp$ is a $\mathbb{Q}$-module, then

$$\forall n \in \mathbb{N}, \forall x \in \operatorname{dom}\exp, \quad x > (2n)^2 \Rightarrow \exp x > x^n. \tag{1}$$

**P.84.** Just before **T1**, **T2**, and **T3**: we assume that the logarithm extends the one on $C^>$ and that it is compatible with the $C$-power structure on $\mathbb{T}$.

**P.84, Example 4.5.** $x \in \mathfrak{T}_{\succ}$ should have been $x \in \mathfrak{T}^{\succ}$, "stable under exponentiation" means $\operatorname{im}\log = \mathbb{T}$, and $x^2/(1-x^{-1}) \notin \mathbb{T}_{\succ}$.

**P.85.** The fact that $f \in \mathbb{T}^{>,\succ} \Rightarrow \log f \in \mathbb{T}^{>,\succ}$ is implicitly used in the proof $(c)$. The fact that $\log f \prec f$ for $f \in \mathbb{T}^{>,\succ}$ follows more directly from (1).

**P.86.** In the proof of **L3**, the displayed equation $f - E_{2n}(\log f) \sim c_f - \log c_f - 1 > 0$ should be $f - E_{2n}(\log f) \sim c_f - E_{2n}(\log c_f) > 0$.

**P.88, Section 4.3.2.** Replace $\mathfrak{T}_{\succ}$ by $\mathbb{T}_{\succ}$ in the displayed formula $\mathfrak{T}_{\exp} = \exp \mathfrak{T}_{\succ}$ and on two other occasions just below.

**P.88.** In the statement of Proposition 4.9, $\mathbb{R}_{\exp}$ should be $\mathbb{T}_{\exp}$.

**P.89.** The last formula in the proof of Proposition 4.9 should read $\log \mathfrak{m} \asymp \exp((\log\log \mathfrak{m})_{\succ}) \prec \exp((\log \mathfrak{m})_{\succ}) \asymp \mathfrak{m}$.

**P.89, L.-7.** In the definition of $\mathbb{E}_n$, replace $\log_n x$ by $C[(\log_n x)^C]$.

**P.90.** In the definition of level, replace "smallest number $n \in \mathbb{Z}$" by "largest number $n \in \mathbb{Z}$". Recall that $\log_n = \exp_{-n}$ for $n < 0$ in the definition of $\mathbb{E}_n$.

**P.90-91.** In subsection 4.3.5, the subscripts and superscripts got inverted in the expressions of the form $C^p$ and $C_q^p$.

**P.91, Exercise 4.9.** It should be $\log_q^C x$ instead of $\log_p^C x$.

**P.92, Ecercise 4.12$(a)$.** It should be $f_\alpha = \sqrt{x} - \sum_{0<\beta<\alpha} \mathrm{e}^{f_\beta \circ \log}$. By transfinite induction, one shows that $(f_\alpha)$ is a strictly decreasing sequence of transseries with purely large support (whence each $\mathrm{e}^{f_\beta \circ \log}$ is a transmonomial) and such that the order type of $\operatorname{supp} f_\alpha$ is precisely $\alpha$.

**P.92.** In the definition of transbasis, we understand that $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ is a finite basis of an asymptotic scale in $\mathbb{T}$ with $n \geqslant 1$. Note that **TB2** and **TB3** implicitly imply that $\mathfrak{b}_1, \ldots, \mathfrak{b}_n \succ 1$.

**P.92, Example 4.14.** The second transbasis should read $(x, \mathrm{e}^{(x+3/2)\sqrt{x}})$.

**P.93.** We should have said "Then $\operatorname{supp} f$ is contained in a set of the form $(\exp_l x)^C \mathrm{e}^{g_0 + g_1 \mathbb{N} + \cdots + g_k \mathbb{N}}$, where $g_0, \ldots, g_k \in C_{p-1}^0 [\![\exp_l x]\!]_{\succ}$" (while dropping the assumption that $\mathrm{e}^{g_1}, \ldots, \mathrm{e}^{g_k} \prec 1$, for the later reduction "without loss of generality"). Note also Exercise 4.9$(d)$.

## Chapter 5

**P.102.** In the displayed formulas for the proof of Proposition 5.5, replace $(f \circ \exp_l)'$ and $(g \circ \exp_l)'$ by $(f \circ \exp_l)' \log_l$ and $(g \circ \exp_l)' \circ \log_l$. Also replace $\log_l x$ by $\log_{l-1} x$ in the last formula.

**P.105, Top.** It would be better to say: "By what precedes, for fixed $i$, the family $\mathcal{T}_{\mathfrak{m}}$ is grid-based and the same for any $\mathfrak{m} \in \mathfrak{G}_i$. Hence $\bigcup_{1 \leqslant i \leqslant n} \bigcup_{\mathfrak{m} \in \mathfrak{G}_i} \mathcal{T}_{\mathfrak{m}} \mathfrak{m}$ is again grid-based and $\int$ is a grid-based mapping…"
**P.105.** The last displayed formula of the proof of Proposition 5.7 should read $(\int \mathrm{e}^x f\uparrow)\downarrow_{\asymp} = (\int \mathrm{e}^x f\uparrow)_{\asymp} = 0$.
**P.112.** At the end of the proof of Theorem 5.13, the last displayed equation should be $g_{[M,N']} + f_{[N,M']} = \cdots$.

## Chapter 6

**P.125, Section 6.4.2, Second displayed equation.** Suppress the $c$.
**P.133, Exercise 6.17.** "extensive" should be "strictly extensive".

## Chapter 7

**P.164, Exercise 7.28.** The coefficients of $L$ should be exponential.

## Chapter 8

**P.172.** Equation (8.13) should read

$$D_{P\uparrow}(F) = \sum_{\boldsymbol{\omega}} \left( \sum_{\substack{\boldsymbol{\tau} \geqslant \boldsymbol{\omega} \\ \|\boldsymbol{\tau}\| = \mathrm{wv}\, D_P}} s_{\boldsymbol{\tau},\boldsymbol{\omega}}\, D_{P,[\boldsymbol{\tau}]} \right) F^{[\boldsymbol{\omega}]}.$$

**P.195.** In the output of A;gorithm `unravel`, suppress "with dominant term $\tau$".

## Chapter 9

**P.207–208.** The paragraph numbers in Section 9.2.2 should be removed.