Towards a Model Theory for Transseries

Matthias Aschenbrenner, Lou van den Dries,
and Joris van der Hoeven

For Anand Pillay, on his 60th birthday.

Abstract The differential field of transseries extends the field of real Laurent series, and occurs in various context: asymptotic expansions, analytic vector fields, o-minimal structures, to name a few. We give an overview of the algebraic and model-theoretic aspects of this differential field, and report on our efforts to understand its first-order theory.

Introduction

We shall describe a fascinating mathematical object, the differential field $\mathbb{T}$ of transseries. It is an ordered field extension of $\mathbb{R}$ and is a kind of universal domain for asymptotic real differential algebra. In the context of this paper, a *transseries* is what is called a *logarithmic-exponential series* or LE-series in [16]. Here is the main problem that we have been pursuing, intermittently, for more than 15 years.

Conjecture. *The theory of the ordered differential field $\mathbb{T}$ is model complete, and is the model companion of the theory of H-fields with small derivation.*

With slow progress during many years, our understanding of the situation has recently increased at a faster rate, and this is what we want to report on. In Section 1 we give an informal description of $\mathbb{T}$, in Section 2 we give some evidence for the conjecture and indicate some plausible consequences. In Section 3 we define H-fields, and explain their expected role in the story. Section 4 describes our recent partial results towards the conjecture, obtained since the publication of the survey [4]. (A full account is in preparation, and of course we hope to finish it with a proof of the conjecture.) Section 5 proves quantifier-free versions of the conjectural induced structure on the constant

Keywords: Transseries; Hardy fields; differential fields; model completeness; NIP.

field $\mathbb{R}$ of $\mathbb{T}$, of the asymptotic o-minimality of $\mathbb{T}$, and of $\mathbb{T}$ having NIP. In the last Section 6 we discuss what might be the right primitives to eliminate quantifiers for $\mathbb{T}$; this amounts to a strong form of the above conjecture.

This paper is mainly expository and programmatic in nature, and occasionally speculative. It is meant to be readable with only a rudimentary knowledge of model theory, valuations, and differential fields, and elaborates on talks by us on various recent occasions, in particular by the second-named author at the meeting in Oléron. For more background on the material in Sections 1–3 (for example, on Hardy fields) see [4], which can serve as a companion to the present paper.

Conventions. Throughout, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. For a field K we let $K^\times = K \setminus \{0\}$ be its multiplicative group. By a *Hahn field* we mean a field $\boldsymbol{k}((t^\Gamma))$ of generalized power series, with coefficients in the field $\boldsymbol{k}$ and exponents in a non-trivial ordered abelian group Γ, and we view it as a *valued* field in the usual way[1]. By *differential field* we mean a field extension K of $\mathbb{Q}$ equipped with a derivation $\partial\colon K \to K$. In our work the operation of taking the *logarithmic* derivative is just as basic as the derivation itself, and so we introduce a special notation: $y^\dagger := y'/y$ denotes the logarithmic derivative of a non-zero y in a differential field. Thus $(yz)^\dagger = y^\dagger + z^\dagger$ for non-zero y, z in a differential field. Given a differential field K and an element a in a differential field extension of K we let $K\langle a\rangle$ be the differential field generated by a over K. An *ordered differential field* is a differential field equipped with an ordering in the usual sense of "ordered field." A *valued differential field* is a differential field equipped with a (Krull) valuation that is trivial on its subfield $\mathbb{Q}$. The term *pc-sequence* abbreviates *pseudo-cauchy sequence*.

1 Transseries

The ordered differential field $\mathbb{T}$ of transseries arises as a natural remedy for certain shortcomings of the ordered differential field of formal Laurent series.

1.1 Laurent series. Recall that the field $\mathbb{R}(\!(x^{-1})\!)$ of formal Laurent series in powers of x^{-1} over $\mathbb{R}$ consists of all series of the form

$$f(x) \;=\; \underbrace{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x}_{\text{infinite part of } f} + a_0 + \underbrace{a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots}_{\text{infinitesimal part of } f}$$

with real coefficients $a_n, a_{n-1}, \dots$ We order $\mathbb{R}(\!(x^{-1})\!)$ by requiring $x > \mathbb{R}$, and make it a differential field by requiring $x' = 1$ and differentiating termwise.

The ordered differential field $\mathbb{R}(\!(x^{-1})\!)$ is too small for many purposes:

- x^{-1} has no antiderivative $\log x$ in $\mathbb{R}(\!(x^{-1})\!)$;
- there is no reasonable exponentiation $f \mapsto \exp(f)$.

Here "reasonable" means that it extends real exponentiation and preserves its key properties: the map $f \mapsto \exp(f)$ should be an isomorphism from the ordered additive group of $\mathbb{R}(\!(x^{-1})\!)$ onto its ordered multiplicative group of positive elements, and $\exp(x) > x^n$ for all n in view of $x > \mathbb{R}$. Note that exponentiation does make sense for the *finite* elements of $\mathbb{R}((x^{-1}))$:

$$\begin{aligned}
&\exp(a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots) \\
&= e^{a_0} \sum_{n=0}^{\infty} \frac{1}{n!}(a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots)^n \\
&= e^{a_0}(1 + b_1x^{-1} + b_2x^{-2} + \cdots) \quad \text{for suitable } b_1, b_2, \ldots \in \mathbb{R}.
\end{aligned}$$

The main *model-theoretic* defect of $\mathbb{R}((x^{-1}))$ as a differential field is that it defines the subset $\mathbb{Z}$; see [18, proof of Proposition 3.3, (i)]. Thus it has no "tame" model-theoretic features. (In contrast, $\mathbb{R}((x^{-1}))$ viewed as just a field is decidable by the work of Ax and Kochen [6].)

1.2 Transseries. To remove these defects we extend $\mathbb{R}((x^{-1}))$ to an ordered differential field $\mathbb{T}$ of *transseries*: series of *transmonomials* (or logarithmic-exponential monomials) arranged from left to right in decreasing order, each multiplied by a real coefficient, for example

$$e^{e^x} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x)^\pi + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + e^{-x} + x^{-1}e^{-x} .$$

The reversed order type of the set of transmonomials that occur in a given transseries can be any countable ordinal. (For the series displayed it is $\omega + 2$.) As with $\mathbb{R}((x^{-1}))$, the natural derivation of $\mathbb{T}$ is given by termwise differentiation of such series, and in the natural ordering on $\mathbb{T}$, a non-zero transseries is positive iff its leading ("leftmost") coefficient is positive.

Transseries occur in solving implicit equations of the form $P(x, y, e^x, e^y) = 0$ for y as $x \to +\infty$, where P is a polynomial in four variables over $\mathbb{R}$. More generally, transseries occur as asymptotic expansions of functions definable in o-minimal expansions of the real field; see [4] for more on this. Transseries also arise as formal solutions to algebraic differential equations and in many other ways. For example, the Stirling expansion for the Gamma function is a (very simple) transseries.

The terminology "transseries" is due to Écalle who introduced $\mathbb{T}$ in his solution of Dulac's Problem: a polynomial vector field in the plane can only have finitely many limit cycles; see [19]. (This is related to Hilbert's 16th Problem.) Independently, $\mathbb{T}$ was also defined by Dahn and Göring in [17], in connection with Tarski's problem on the real exponential field, and studied as such in [16], in the aftermath of Wilkie's famous theorem [40]. (Discussions of the history of transseries are in [28; 32].)

Transseries are added and multiplied in the usual way and form a ring $\mathbb{T}$, and this ring comes equipped with several other natural operations. Here are a few, each accompanied by simple examples and relevant facts about $\mathbb{T}$:

Taking the multiplicative inverse. Each non-zero $f \in \mathbb{T}$ has a multiplicative inverse in $\mathbb{T}$: for example,

$$\begin{aligned}
\frac{1}{x - x^2e^{-x}} = \frac{1}{x(1 - xe^{-x})} &= x^{-1}(1 + xe^{-x} + x^2e^{-2x} + \cdots) \\
&= x^{-1} + e^{-x} + xe^{-2x} + \cdots
\end{aligned}$$

As an ordered field, $\mathbb{T}$ is a real closed extension of $\mathbb{R}$. In particular, an algebraic closure of $\mathbb{T}$ is given by $\mathbb{T}[i]$ where $i^2 = -1$.

Formal differentiation. Each $f \in \mathbb{T}$ can be differentiated term by term, giving a derivation $f \mapsto f'$ on the field $\mathbb{T}$. For example:

$$(e^{-x} + e^{-x^2} + e^{-x^3} + \cdots)' \;=\; -(e^{-x} + 2xe^{-x^2} + 3x^2 e^{-x^3} + \cdots).$$

The field of constants of this derivation is $\{f \in \mathbb{T} : \ f' = 0\} = \mathbb{R}$.

Formal integration. For each $f \in \mathbb{T}$ there is some $F \in \mathbb{T}$ (unique up to addition of a constant from $\mathbb{R}$) with $F' = f$: for example,

$$\int \frac{e^x}{x}\,dx \;=\; \text{constant} + \sum_{n=0}^{\infty} n!\,x^{-1-n}e^x \quad \text{(diverges)}.$$

Formal composition. Given $f, g \in \mathbb{T}$ with $g > \mathbb{R}$, we can "substitute g for x in f" to obtain a transseries $f \circ g \in \mathbb{T}$. For example, let $f(x) = x + \log x$ and $g(x) = x \log x$; writing $f(g(x))$ for $f \circ g$, we have

$$\begin{aligned}
f(g(x)) &= x\log x + \log(x \log x) \;=\; x\log x + \log x + \log(\log x),\\
g(f(x)) &= (x + \log x)\log(x + \log x)\\
&= x\log x + (\log x)^2 + (x+\log x)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{\log x}{x}\right)^n\\
&= x\log x + (\log x)^2 + \log x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\frac{(\log x)^{n+1}}{x^n}.
\end{aligned}$$

The Chain Rule holds:

$$(f \circ g)' \;=\; (f' \circ g)\cdot g' \qquad \text{for all } f, g \in \mathbb{T},\ g > \mathbb{R}.$$

Compositional inversion. The set $\mathbb{T}^{>\mathbb{R}} := \{f \in \mathbb{T} : f > \mathbb{R}\}$ of positive infinite transseries is closed under the composition operation $(f, g) \mapsto f \circ g$, and forms a group with identity element x. For example, the transseries $g(x) = x \log x$ has a compositional inverse of the form

$$\frac{x}{\log x}\cdot F\left(\frac{\log\log x}{\log x}, \frac{1}{\log x}\right)$$

where $F(X, Y)$ is an ordinary convergent power series in the two variables X and Y over $\mathbb{R}$ with constant term 1. (This fact plays a certain role in the solution, using transseries, of a problem of Hardy dating from 1911, obtained independently in [15] and [26]; see [32].)

Exponentiation. We have a canonical isomorphism $f \mapsto \exp(f)$, with inverse $g \mapsto \log(g)$, between the ordered additive group of $\mathbb{T}$ and the ordered multiplicative group $\mathbb{T}^{>0}$; it extends the exponentiation of finite Laurent series described above. With $\sinh := \frac{1}{2}e^x - \frac{1}{2}e^{-x} \in \mathbb{T}^{>0}$ (sinus hyperbolicus),

$$\begin{aligned}
\exp(\sinh) &= \exp\left(\tfrac{1}{2}e^x\right)\cdot\exp\left(-\tfrac{1}{2}e^{-x}\right)\\
&= e^{\frac{1}{2}e^x}\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\tfrac{1}{2}e^{-x}\right)^n \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{n!2^n}e^{\frac{1}{2}e^x-nx},\\
\log(\sinh) &= \log\left(\tfrac{e^x}{2}\left(1-e^{-2x}\right)\right) \;=\; x-\log 2-\sum_{n=1}^{\infty}\frac{1}{n}e^{-2nx}.
\end{aligned}$$

As an exponential ordered field, $\mathbb{T}$ is an elementary extension of the real exponential field [15], and thus model complete and o-minimal [40]. The iterated exponentials

$$e_0 := x,\ e_1 := \exp x,\ e_2 := \exp(\exp(x)),\ \dots$$

form an increasing cofinal sequence in the ordering of $\mathbb{T}$. Likewise, their formal compositional inverses

$$\ell_0 := x,\ \ell_1 := \log x,\ \ell_2 := \log(\log(x)),\ \dots$$

form a decreasing coinitial sequence in $\mathbb{T}^{>\mathbb{R}}$.

A precise construction of $\mathbb{T}$ is in [16], where it is denoted by $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$. See also [19], [20] and [28] for other accounts. The *purely logarithmic transseries* are those which, informally speaking, do not involve exponentiation, and they make up an intriguing differential subfield $\mathbb{T}_{\log}$ of $\mathbb{T}$ that has a very explicit definition: First, setting $\ell_0 := x$ and $\ell_{m+1} = \log \ell_m$ yields the sequence (ℓ_m) of iterated logarithms of x. Next, let $\mathfrak{L}_m$ be the formal multiplicative group

$$\ell_0^{\mathbb{R}}\cdots\ell_m^{\mathbb{R}} = \left\{\ell_0^{r_0}\cdots\ell_m^{r_m} : r_0,\dots,r_m \in \mathbb{R}\right\},$$

made into an ordered group such that $\ell_0^{r_0}\cdots\ell_m^{r_m} > 1$ iff the exponents $r_0,\dots,r_m$ are not all zero, and $r_i > 0$ for the least i with $r_i \neq 0$. Of course, if $m \leqslant n$, then $\mathfrak{L}_m$ is naturally an ordered subgroup of $\mathfrak{L}_n$, and so we have a natural inclusion of Hahn fields $\mathbb{R}((\mathfrak{L}_m)) \subseteq \mathbb{R}((\mathfrak{L}_n))$. We now have

$$\mathbb{T}_{\log} = \bigcup_{n=0}^{\infty}\mathbb{R}((\mathfrak{L}_n)) \qquad \text{(increasing union of differential subfields).}$$

It is straightforward to define $\log f \in \mathbb{T}_{\log}$ for $f \in \mathbb{T}_{\log}^{>0}$.

The inductive construction of $\mathbb{T}$ is more complicated, but also yields $\mathbb{T}$ as a directed union of Hahn subfields, each of which is also closed under the derivation. Hahn fields themselves (as opposed to suitable directed unions of Hahn fields) cannot be equipped with a reasonable exponential map; see [31].

Note that $\mathbb{T}_{\log}$ is a proper subfield of $\mathbb{R}((\mathfrak{L}))$, where $\mathfrak{L} := \bigcup_{n=0}^{\infty}\mathfrak{L}_n$ (directed union of ordered multiplicative subgroups): for example, the series

$$\frac{1}{\ell_0^2}+\frac{1}{(\ell_0\ell_1)^2}+\cdots+\frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}+\cdots$$

lies in $\mathbb{R}((\mathfrak{L}))$, but not in $\mathbb{T}_{\log}$, and in fact, not even in $\mathbb{T}$. (This series will be important in Section 4 below; see also Theorem 2.3.)

1.3 Analytic counterparts of $\mathbb{T}$. Convergent series in $\mathbb{R}((x^{-1}))$ define germs of real analytic functions at infinity. This yields an isomorphism of ordered differential fields between the subfield of convergent series in $\mathbb{R}((x^{-1}))$ and a Hardy field. It would be desirable to extend this to isomorphisms between larger differential subfields T of $\mathbb{T}$ and Hardy fields H which preserves as much structure as possible: the ordering, differentiation, and even integration and composition, whenever defined.

However, if T is sufficiently closed under integration (or solutions of other simple differential equations), then it will contain divergent power series in x^{-1}, as well as more general divergent transseries. A major difficulty is to give an analytic meaning to such transseries. In simple cases, Borel summation provides a systematic device for doing this. Borel's theory has been greatly extended by Écalle, who introduced a big subfield $\mathbb{T}^{\mathrm{as}}$ of $\mathbb{T}$. The elements of $\mathbb{T}^{\mathrm{as}}$ are called *accelero-summable transseries*, and $\mathbb{T}^{\mathrm{as}}$ is real closed, stable under differentiation, integration, composition, etc. The analytic counterparts of accelero-summable transseries are called *analysable functions*, and they appear naturally in Écalle's proof of the Dulac Conjecture. As a prelude to the $\mathbb{T}$-Conjecture in the next section, here are some sweeping statements[2] from Écalle's book [19] on these notions, indicating that $\mathbb{T}$ and its cousin $\mathbb{T}^{\mathrm{as}}$ might be viewed as *universal domains* for asymptotic analysis.

It seems [...] (*but I have not yet verified this in all generality*) *that* $\mathbb{T}^{\mathrm{as}}$ *is closed under resolution of differential equations, or, more exactly, that if a differential equation has formal solutions in* $\mathbb{T}$*, then these solutions are automatically in* $\mathbb{T}^{\mathrm{as}}$*.*

It seems [...] *that the algebra* $\mathbb{T}^{\mathrm{as}}$ *of accelero-summable transseries is truly the algebra-from-which-one-can-never-exit and that it marks an almost impassable horizon for "ordered analysis."* (*This sector of analysis is in some sense "orthogonal" to harmonic analysis.*)

This notion of analysable function represents probably the ultimate extension of the notion of (*real*) *analytic function, and it seems inclusive and stable to a degree unheard of.*

Accelero-summation requires a big machinery. If we just try to construct isomorphisms $T \to H$ which do not necessarily preserve composition (but do preserve the ordering and differentiation), then simpler arguments with a more model-theoretic flavor can be used to prove the following, from [29]:

Theorem 1.1. *Let* $\mathbb{T}^{\mathrm{da}} \subseteq \mathbb{T}$ *be the field of transseries that are differentially algebraic over* $\mathbb{R}$*. Then there is an isomorphism of ordered differential fields between* $\mathbb{T}^{\mathrm{da}}$ *and some Hardy field.*

In [29], this follows from general theorems about extending isomorphisms between suitable differential subfields of $\mathbb{T}$ and Hardy fields.

2 The $\mathbb{T}$-Conjecture

As explained above, the elementary theory of $\mathbb{T}$ as an exponential field is understood, but $\mathbb{T}$ is far more interesting when viewed as a *differential* field.

From now on we consider $\mathbb{T}$ as an ordered valued differential field.

$\mathbb{T}$-Conjecture. *$\mathbb{T}$ is model complete.*

Model completeness is fairly robust as to which first-order language is used, but to be precise, we consider $\mathbb{T}$ here as an $\mathcal{L}$-structure, where $\mathcal{L}$ is the language of ordered valued differential rings given by

$$\mathcal{L} \ := \ \{0,\ 1,\ +,\ -,\ \cdot\ ,\ \partial,\ \leqslant,\ \preccurlyeq\}$$

where the unary operation symbol ∂ names the derivation, and the binary relation symbol $\preccurlyeq$ names the valuation divisibility on the field $\mathbb{T}$ given by

$$f \preccurlyeq g \iff |f| \leqslant c|g| \text{ for some } c \in \mathbb{R}^{>0}.$$

For the $\mathbb{T}$-Conjecture, it doesn't really matter whether or not we include $\leqslant$ and $\preccurlyeq$, since the ordering and the valuation divisibility are existentially definable in terms of the other primitives: for $\preccurlyeq$, use that $\mathbb{R}$ is the field of constants for the derivation. (See also [3, Section 14].) A purely differential-algebraic formulation of the $\mathbb{T}$-Conjecture reads as follows:

For any differential polynomial P over $\mathbb{Q}$ in $m+n$ variables there exists a differential polynomial Q over $\mathbb{Q}$ in $m+p$ variables, for some p depending on P, such that for all $a \in \mathbb{T}^m$ the following equivalence holds:

$$P(a,b) = 0 \text{ for } \textit{some } b \in \mathbb{T}^n \quad \iff \quad Q(a,c) \neq 0 \text{ for } \textit{all } c \in \mathbb{T}^p.$$

In logical terms: every existential formula in the language of differential rings is equivalent in $\mathbb{T}$ to a universal formula in that language. In Section 5 we shall try to make it plausible that a strong form of the $\mathbb{T}$-Conjecture (elimination of quantifiers in a reasonable language) implies the following attractive and intrinsic model-theoretic properties of $\mathbb{T}$:

- If $X \subseteq \mathbb{T}^n$ is definable, then $X \cap \mathbb{R}^n$ is semialgebraic.
- $\mathbb{T}$ is *asymptotically o-minimal*: for each definable $X \subseteq \mathbb{T}$ there is a $b \in \mathbb{T}$ such that either $(b,+\infty) \subseteq X$ or $(b,+\infty) \subseteq \mathbb{T} \setminus X$.
- $\mathbb{T}$ has NIP. (What this means is explained in Section 5.)

2.1 Positive evidence. In Section 5 we establish quantifier-free versions of the last three statements. Over the years, evidence for the $\mathbb{T}$-Conjecture has accumulated. For example, the value group of $\mathbb{T}$ equipped with a certain function induced by the derivation of $\mathbb{T}$ (the "asymptotic couple" of $\mathbb{T}$ as defined in Section 3.3 below) is model complete; see [1]. The best evidence for the $\mathbb{T}$-Conjecture to date is the analysis by van der Hoeven in [28] of the set of zeros in $\mathbb{T}$ of any given differential polynomial in one variable over $\mathbb{T}$. Among other things, he proved the following Intermediate Value Theorem:

Theorem 2.1. *Given any differential polynomial $P(Y) \in \mathbb{T}\{Y\}$ and $f, h \in \mathbb{T}$ with $P(f) < 0 < P(h)$, there is $g \in \mathbb{T}$ with $f < g < h$ and $P(g) = 0$.*

Here and later $K\{Y\} = K[Y, Y', Y'', \dots]$ is the ring of differential polynomials in the indeterminate Y over a differential field K. The proofs in [28] make full use of the formal structure of $\mathbb{T}$ as an increasing union of Hahn fields. This makes it possible to apply analytic techniques (fixed point theorems, compact-like operators, etc.) for solving algebraic differential equations; see also [27]. Much of our work consists of recovering significant parts of [28] under weak first-order assumptions on valued differential fields.

2.2 The different flavors of $\mathbb{T}$. In any precise inductive construction of $\mathbb{T}$ we can impose various conditions on the so-called *support* of a transseries, which is the ordered set of transmonomials occurring in it with a non-zero coefficient. This leads to variants of the differential field $\mathbb{T}$; see for example the discussion in [19] and [28]. For the sake of definiteness, we take here $\mathbb{T}$ to be the field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ of logarithmic-exponential power series from [16], where supports are only required to be *anti-wellordered*; this is basically the weakest condition that can be imposed.

In [28], however, every transseries in $\mathbb{T}$ has *gridbased* support (which implies that it is contained in a finitely generated subgroup of the multiplicative group of transmonomials). This leads to a rather small differential subfield of our $\mathbb{T}$, but results such as the Intermediate Value Theorem in [28] proved there for the gridbased version of $\mathbb{T}$ are known to hold also for the $\mathbb{T}$ we consider here. Of course, we expect these variants of $\mathbb{T}$ all to be elementarily equivalent, and this is part of the motivation for our $\mathbb{T}$-Conjecture. For this expectation to hold we would need also an explicit first-order axiomatization of the theory of $\mathbb{T}$, and show that the various flavors of $\mathbb{T}$ all satisfy these axioms. At the end of Section 4 we conjecture such an axiomatization as part of a more explicit version of the $\mathbb{T}$-Conjecture.

Likewise, we expect Écalle's differential field $\mathbb{T}^{\mathrm{as}}$ of accelero-summable transseries to be an elementary submodel of $\mathbb{T}$. (By the way, $\mathbb{T}^{\mathrm{as}}$ comes in similar variants as $\mathbb{T}$ itself.)

2.3 Linear differential operators over $\mathbb{T}$. The Intermediate Value Property for differential polynomials over $\mathbb{T}$ resembles the behavior of ordinary one-variable polynomials over $\mathbb{R}$. There is another analogy in [28] between $\mathbb{T}$ and $\mathbb{R}$ which is much easier to establish: factoring linear differential operators over $\mathbb{T}$ is similar to factoring one-variable polynomials over $\mathbb{R}$. By a linear differential operator over $\mathbb{T}$ we mean an operator $A = a_0 + a_1\partial + \cdots + a_n\partial^n$ on $\mathbb{T}$ ($\partial =$ the derivation, all $a_i \in \mathbb{T}$); it defines the same function on $\mathbb{T}$ as the differential polynomial $a_0Y + a_1Y' + \cdots + a_nY^{(n)}$. The linear differential operators over $\mathbb{T}$ form a *non-commutative* ring $\mathbb{T}[\partial]$ under composition.

Theorem 2.2. *Every linear differential operator over $\mathbb{T}$ of order $n \geqslant 1$ is surjective as a map $\mathbb{T} \to \mathbb{T}$, and is a product (composition) of operators $a + b\partial$ of order 1 in $\mathbb{T}[i][\partial]$. Every such operator is a product of order 1 and order 2 operators in $\mathbb{T}[\partial]$.*

Thus coming to grips with linear differential operators over $\mathbb{T}$ reduces to some extent to understanding those of order 1 and order 2. Studying operators of order 1 is largely a matter of solving equations $y' = a$ and $z^\dagger = b$. Modulo solving such equations, order 2 operators can be reduced to those of the form $4\partial^2 + f$, where the next theorem is relevant.

Theorem 2.3. *Let $f \in \mathbb{T}$. Then the following are equivalent:*

(1) *the equation $4y'' + fy = 0$ has a non-zero solution in $\mathbb{T}$;*
(2) *$f < \frac{1}{(\ell_0)^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}$ for some n;*

(3) $f \neq 2(u^{\dagger\dagger})' - (u^{\dagger\dagger})^2 + (u^\dagger)^2$ *for all* $u > \mathbb{R}$ *in* $\mathbb{T}$.

The equivalence of (1) and (2) is analogous to a theorem of Boshernitzan [12] and Rosenlicht [37] in the realm of Hardy fields. (See the remarks following Theorem 1.12 in [4] for a correction of [37].) The equivalence of (1) and (3) has been known to us since 2002. Its model-theoretic significance is that the existential condition (1) on f is equivalent to a universal condition on f, namely (3), in accordance with the $\mathbb{T}$-Conjecture.

We note here that for a non-constant element u of a differential field,

$$2(u^{\dagger\dagger})' - (u^{\dagger\dagger})^2 + (u^\dagger)^2 \;=\; 2S(u), \quad \text{where}$$
$$S(u) \;:=\; (u'^{\dagger})' - \frac{1}{2}(u'^{\dagger})^2 \;=\; \frac{u^{(3)}}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2$$

is known as the Schwarzian derivative of u, which plays a role in the analytic theory of linear differential equations; see [25, Chapter 10].

3 *H*-Fields

Abraham Robinson taught us to think about model completeness and quantifier elimination in an abstract algebraic way. This approach as refined by Shoenfield and Blum suggests that the $\mathbb{T}$-Conjecture follows from an adequate extension theory for those ordered differential fields that share certain basic (universal) properties with $\mathbb{T}$. This involves a critical choice of the "right" class of ordered differential fields. Our choice: H-fields[3] as defined below. Then the challenge becomes to show that the "existentially closed" H-fields are exactly the H-fields that share certain deeper first-order properties with $\mathbb{T}$. If we can achieve this, then $\mathbb{T}$ will be model complete.

In practice this often amounts to the following: come up with the "right" extra primitives (these should be existentially as well as universally definable in $\mathbb{T}$); guess the "right" axioms characterizing existentially closed H-fields; and prove suitable embedding theorems for H-fields enriched with these primitives. If this works, one has a proof of a strong form of the $\mathbb{T}$-Conjecture, namely an elimination of quantifiers in the language $\mathcal{L}$ augmented by symbols for the extra primitives. Such an approach to understanding definability in a given mathematical structure often yields further payoffs, for example, a useful dimension theory for definable sets.

Let K be an ordered differential field, and put

$C = \{a \in K : a' = 0\}$ (constant field of K)

$\mathcal{O} = \{a \in K : |a| \leqslant c \text{ for some } c \in C^{>0}\}$ (convex hull of C in K)

$\mathcal{o} = \{a \in K : |a| < c \text{ for all } c \in C^{>0}\}$ (maximal ideal of $\mathcal{O}$).

We call K an ***H*-field** if the following conditions are satisfied:

(H1) $\mathcal{O} \;=\; C + \mathcal{o}$,

(H2) $a > C \implies a' > 0$.

Examples of H-fields include any Hardy field containing $\mathbb{R}$, such as $\mathbb{R}(x, e^x)$; the ordered differential field $\mathbb{R}(\!(x^{-1})\!)$ of Laurent series; and $\mathbb{T}$. All these satisfy an extra axiom:

(H3) $a \in \mathcal{o} \implies a' \in \mathcal{o}$, which is also expressed by saying that the derivation is **small.**

An H-field K comes with a definable (Krull) valuation v whose valuation ring is the convex hull $\mathcal{O}$ of C. It will be useful to fix similar notation for any valued differential field K, not necessarily an H-field: C is the constant field, $\mathcal{O}$ is the valuation ring, $\scriptstyle\mathcal{O}$ is the maximal ideal of $\mathcal{O}$, and $v\colon K^\times \to \Gamma$ with $\Gamma = v(K^\times)$ is the valuation. If we need to indicate the dependence on K we use subscripts, so $C = C_K$, $\mathcal{O} = \mathcal{O}_K$, and so on. The valuation divisibility on K corresponding to its valuation is the binary relation $\preccurlyeq$ on K given by

$$f \preccurlyeq g \iff vf \geqslant vg.$$

Note that if K is an H-field, then for all $f, g \in K$,

$$f \preccurlyeq g \iff |f| \leqslant c|g| \text{ for some } c \in C^{>0}.$$

We also write $g \succcurlyeq f$ instead of $f \preccurlyeq g$, and we define

$$f \asymp g \iff f \preccurlyeq g \text{ and } g \preccurlyeq f.$$

Further, we introduce the binary relations $\prec$ and $\succ$ on K:

$$f \prec g \iff vf > vg, \qquad f \succ g \iff g \prec f.$$

If K is an H-field, then for $f, g \in K$ this means:

$$f \prec g \iff |f| < c|g| \text{ for all } c \in C^{>0}.$$

Rosenlicht gave a nice valuation-theoretic formulation of l'Hôpital's rule: if K is a Hardy field, then we have:

$$\text{for all } f, g \in K \text{ with } f, g \prec 1\colon\ f \prec g \iff f' \prec g'. \tag{$*$}$$

This rule $(*)$ goes through for H-fields. The ordering of an H-field determines its valuation, but plays otherwise a secondary role. Moreover, it is often useful to pass to algebraic closures like $\mathbb{T}[i]$, with the valuation extending uniquely, still obeying (H1) and $(*)$, but without ordering. Thus much of our work is in the setting of **asymptotic differential fields**: these are the valued differential fields satisfying $(*)$. Section 4 will show the benefits of coarsening the valuation of an H-field; the resulting object might not be an H-field anymore, but remains an asymptotic differential field. It is a non-trivial fact that any algebraic extension of an asymptotic differential field is also an asymptotic differential field.

An H-field K is **existentially closed** if every finite system of algebraic differential equations over K in several unknowns with a solution in an H-field extension of K has a solution in K. Including in these systems also differential inequalities (using $\leqslant$ and $<$) and asymptotic conditions (involving $\preccurlyeq$ and $\prec$) makes no difference. (See [3, Section 14].) A more detailed version of the $\mathbb{T}$-Conjecture now says:

Refined $\mathbb{T}$-Conjecture. *$\mathbb{T}$ is an existentially closed H-field, and there exists a set Σ of $\mathcal{L}$-sentences such that the existentially closed H-fields with small derivation are exactly the H-fields satisfying Σ.* (*In more model-theoretic jargon: the theory of H-fields with small derivation has a model companion, and $\mathbb{T}$ is a model of this model companion.*)

A comment on axiom (H1) for H-fields. It expresses that the *constant* field for the derivation is also in a natural way the *residue* field for the valuation. However, (H1) cannot be expressed by a universal sentence in the language $\mathcal{L}$ of ordered valued differential rings. We define a pre-H-field to be an ordered valued differential subfield of an H-field. There are pre-H-fields that are not H-fields, and the valuation of a pre-H-field is not always determined by its ordering, as is the case in H-fields. Fortunately, any pre-H-field K has an H-field extension $H(K)$, its **H-field closure**, that embeds uniquely over K into any H-field extension of K; see [2]. (Here and below, "extension" and "embedding" are meant in the sense of $\mathcal{L}$-structures.)

3.1 Liouville closed H-fields. The real closure of an H-field is again an H-field; see [2]. Going beyond algebraic adjunctions, we consider adjoining solutions to first-order linear differential equations $y' + ay = b$.

Call an H-field K **Liouville closed** if it is real closed and for all $a, b \in K$ there are $y, z \in K$ such that $y' = a$ and $z \neq 0$, $z^\dagger = b$; equivalently, K is real closed, and any equation $y' + ay = b$ with $a, b \in K$ has a *non-zero*[4] solution $y \in K$. For example, $\mathbb{T}$ is Liouville closed. Each existentially closed H-field is Liouville closed as a consequence of the next theorem. A **Liouville closure** of an H-field K is a minimal Liouville closed H-field extension of K. We can now state the main result from [2]:

Theorem 3.1. *Let K be an H-field. Then K has exactly one Liouville closure, or exactly two Liouville closures (up to isomorphism over K).*

Whether K has one or two Liouville closures is related to a trichotomy in the class of H-fields which pervades our work. In fact, it is a trichotomy that can be detected on the level of the value group; see below.

3.2 Trichotomy for H-fields. Let K be an H-field. Let v be the valuation defined above, and $\Gamma := v(K^\times)$ its value group. We set

$$\Gamma^{\neq} := \Gamma \setminus \{0\}, \quad \Gamma^{>} := \{\gamma \in \Gamma : \ \gamma > 0\}.$$

As a consequence of $(*)$, the derivation and logarithmic derivative of K induce functions on $\Gamma^{\neq}$:

$$\begin{aligned} v(a) = \gamma &\mapsto v(a') = \gamma' \ : \ \Gamma^{\neq} \to \Gamma, \\ v(a) = \gamma &\mapsto v(a^\dagger) = \gamma^\dagger := \gamma' - \gamma \ : \ \Gamma^{\neq} \to \Gamma, \end{aligned}$$

where $a \in K^\times$, $v(a) \neq 0$. The function $\gamma \mapsto \gamma' \colon \Gamma^{\neq} \to \Gamma$ is strictly increasing and $\gamma \mapsto \gamma^\dagger \colon \Gamma^{>} \to \Gamma$ is decreasing, with $(-\gamma)^\dagger = \gamma^\dagger$ for all $\gamma \in \Gamma^{\neq}$. Figure 1 shows the qualitative behavior of the functions $\gamma \mapsto \gamma'$ and $\gamma \mapsto \gamma^\dagger$. Some features are a little hard to indicate in such a picture, for example the fact that $\gamma^\dagger$ is constant on each archimedean class of $\Gamma^{\neq}$.

Following Rosenlicht [36], we put

$$\Psi \ = \ \Psi_K \ := \ \{\gamma^\dagger : \ \gamma \in \Gamma^{\neq}\}.$$

Then $\Psi < (\Gamma^{>})'$, and exactly one of the following holds:

Case 1: $\Psi < \gamma < (\Gamma^{>})'$ for some (necessarily unique) γ;
Case 2: Ψ has a largest element;

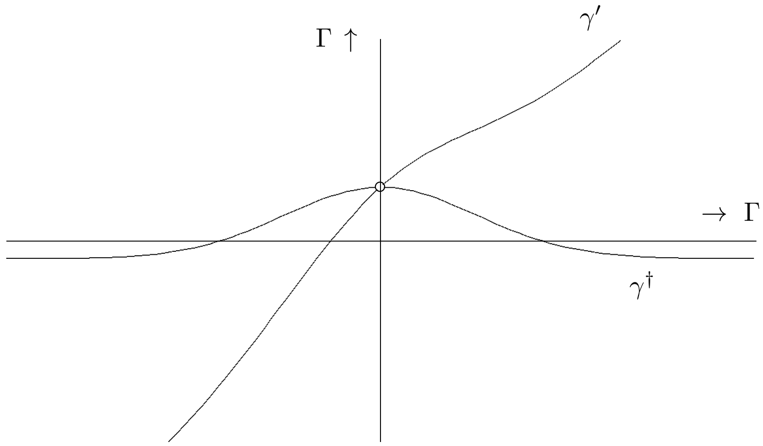

Figure 1

Case 3: $\sup\Psi$ does not exist; equivalently, $\Gamma = (\Gamma^{\neq})'$.

If $K = C$ we are in Case 1, with $\gamma = 0$; the Laurent series field $\mathbb{R}(\!(x^{-1})\!)$ falls under Case 2, and Liouville closed H-fields under Case 3. In Case 1 there are two Liouville closures of K; in Case 2 there is only one, but Case 3 requires finer distinctions for a definite answer. We now explain this in more detail.

Suppose K falls under Case 1. Then the element γ is called a **gap**, and there are two ways to remove the gap: with $v(a) = \gamma$, we have an H-field extension $K(y_1)$ with $y_1 \prec 1$ and $y_1' = a$, and we also have an H-field extension $K(y_2)$ with $0 \neq y_2 \prec 1$ and $y_2^\dagger = a$. Both of these extensions fall under Case 2, and they are incompatible in the sense that they cannot be embedded over K into a common H-field extension of K. (Any Liouville closed extension of K, however, contains either a copy of $K(y_1)$ or a copy of $K(y_2)$.) Instead of "K falls under Case 1" we say "K has a gap."

Suppose K falls under Case 2. Then so does every differential-algebraic H-field extension of K that is finitely generated over K as a differential field. Thus it takes an infinite generation process to construct the (unique) Liouville closure of K. An H-field falling under Case 3 is said to admit **asymptotic integration**. This is because Case 3 is equivalent to having for each non-zero a in the field an element y in the field such that $y' \sim a$.

3.3 Asymptotic couples. Let K be an H-field with valuation $v\colon K^\times \to \Gamma$ defined as before. The ordered group Γ equipped with the function

$$\gamma \mapsto \gamma^\dagger \colon \Gamma^{\neq} \to \Gamma$$

is an **asymptotic couple** in the terminology of Rosenlicht [33; 34; 35], who proved the first non-trivial facts about them as structures in their own right, independent of their origin in Hardy fields. Indeed, this function $\gamma \mapsto \gamma^\dagger$ has rather nice properties: it is a valuation on the abelian group Γ: for all $\alpha, \beta \in \Gamma$ (and $0^\dagger := \infty > \Gamma$),

(i) $(\alpha+\beta)^\dagger \geqslant \min(\alpha^\dagger, \beta^\dagger)$,
(ii) $(-\alpha)^\dagger = \alpha^\dagger$,

and this valuation is compatible with the group ordering: for all $\alpha, \beta \in \Gamma$,

$$0 < \alpha \leqslant \beta \implies \alpha^\dagger \geqslant \beta^\dagger. \tag{H}$$

The asymptotic couple of $\mathbb{T}$ has a good model theory: It allows elimination of quantifiers in its natural language augmented by a predicate for the subset Ψ of Γ, and its theory is axiomatized by adding to Rosenlicht's axioms[5] for asymptotic couples the following requirements:

(i) divisibility of the underlying abelian group;
(ii) compatibility with the ordering, as displayed in (H) above;
(iii) Ψ is downward closed and has no maximum;
(iv) there is no gap.

This result is in [1], which proves also the weak o-minimality of the asymptotic couple of $\mathbb{T}$. The trichotomy from the previous section is in [2], and holds for all asymptotic couples satisfying (H).

4 New Results

The above material raises some issues which turn out to be related. First, no H-subfield K of $\mathbb{T}$ with $\Gamma \neq \{0\}$ has a gap. Even to construct a *Hardy field* with a gap and $\Gamma \neq \{0\}$ takes effort. Nevertheless, the model theory of asymptotic couples strongly suggests that H-fields with a gap should play a key role, and so the question arises how a given H-field can be extended to one with a gap. The analogous issue for asymptotic couples is easy, but we only managed to show rather recently that every H-field can be extended to one with a gap. This is discussed in more detail in Section 4.7.

Recall: a valued field is *maximal* if it has no proper immediate extension; this is equivalent to the more geometric notion of *spherically complete*. For example, Hahn fields are maximal. Decisive results in the model theory of maximal valued fields are due to Ax & Kochen [7] and Eršov [21]. Among other things they showed that *henselian* is the exact first-order counterpart of *maximal*, at least in equicharacteristic zero. In later extensions (Scanlon's valued differential fields in [38] and the valued difference fields from [8; 9]), the natural models are still maximal. Here and below, "maximal" means "maximal as a valued field" even if the valued field in question has further structure like a derivation.

However, in our situation the expected natural models cannot be maximal: no maximal H-field can be Liouville closed, let alone existentially closed. Maximal H-fields do nevertheless exist in abundance, and turn out to be a natural source for creating H-fields with a gap. It also remains true that immediate extensions require close attention: $\mathbb{T}$ has proper immediate H-field extensions that embed over $\mathbb{T}$ into an elementary extension of $\mathbb{T}$; see the proof of Proposition 5.4. Thus we cannot bypass the immediate extensions of $\mathbb{T}$ in any model-theoretic account of $\mathbb{T}$ as we are aiming for.

4.1 Immediate extensions of H-fields.

Theorem 4.1. *Every real closed H-field has an immediate maximal H-field extension.*

This was not even known when the value group is $\mathbb{Q}$. A difference with the situation for valued fields of equicharacteristic 0 (without derivation) is the lack of uniqueness of the maximal immediate extension. (The proof of Proposition 5.4 shows such non-uniqueness in the case of $\mathbb{T}$.)

Here are some comments on our proof of Theorem 4.1. First, this involves a change of derivation as follows. Let K be a differential field with derivation ∂, and let $\varphi \in K^{\times}$. Then we define K^{φ} to be the differential field obtained from K by taking $\varphi^{-1}\partial$ as its derivation instead of ∂. Then the constant field C of K is also the constant field of K^{φ}, and so $C\{Y\}$ is a common differential subring of $K\{Y\}$ and $K^{\varphi}\{Y\}$. Given a differential polynomial $P \in K\{Y\}$ we let $P^{\varphi} \in K^{\varphi}\{Y\}$ be the result of rewriting P in terms of the derivation $\varphi^{-1}\partial$, so $P^{\varphi}(y) = P(y)$ for all $y \in K$. (For example, $Y'^{\varphi} = \varphi Y'$ in $K^{\varphi}\{Y\}$.) This change of derivation is called **compositional conjugation**. A suitable choice of φ can often drastically simplify things. Also, if K is an H-field and $\varphi > 0$, then K^{φ} is still an H-field, with $\Psi^{\varphi} = \Psi - v\varphi$.

Next, given any valued differential field K, we extend its valuation v to a valuation on the domain $K\{Y\}$ of differential polynomials by

$$vP := \min\{va : a \in K \text{ is a coefficient of } P\}.$$

Let now K be a real closed H-field with value group $\Gamma \neq \{0\}$, and suppose first that K does not admit asymptotic integration. Then $\sup \Psi$ exists in Γ, and by compositional conjugation we can arrange that $\sup \Psi = 0$. One can show that then K is **flexible**, by which we mean that it has the following property: for any $P \in K\{Y\}$ with $vP(0) > vP$ and any $\gamma \in \Gamma^{>}$, the set $\{vP(y) : \ y \in K,\ |vy| < \gamma\}$ is infinite. This property then plays a key role in constructing an immediate maximal H-field extension of K. (It is worth mentioning that the notion of flexibility makes sense for any valued differential field. There are indeed other kinds of flexible valued differential fields such as those considered in [38] where this property can be used for similar ends.)

The case that the real closed H-field K does admit asymptotic integration is harder and uses compositional conjugation in a more delicate way. We say more on this in the next subsection.

4.2 The Newton polynomial.

In this subsection K is a real closed H-field with asymptotic integration. To simulate the favorable case $\sup \Psi = 0$ from the previous subsection, we use compositional conjugation by φ with $v\varphi < (\Gamma^{>})'$ as large as possible. Call $\varphi \in K$ **admissible** if $v\varphi < (\Gamma^{>})'$.

Theorem 4.2. *Let $P \in K\{Y\}$, $P \neq 0$. Then there is a differential polynomial $N_P \in C\{Y\}$, $N_P \neq 0$, such that for all admissible $\varphi \in K$ with sufficiently large $v\varphi$ we have $a \in K^{\times}$ and $R \in K^{\varphi}\{Y\}$ with*

$$P^{\varphi} = aN_P + R \quad \text{in } K^{\varphi}\{Y\}, \qquad vR > va.$$

We call N_P the **Newton polynomial** of P. As described here, N_P is only determined up to multiplication by an element of $C^{\times}$, but the key fact is that N_P is independent of the admissible φ for high enough $v\varphi$. We now have a modified version of the flexibility property of the previous subsection: given any non-zero $P \in K\{Y\}$ with $N_P(0) = 0$ and any $\gamma \in \Gamma^>$, the set $\{vP(y) : \ y \in K, \ |vy| < \gamma\}$ is infinite. This can then be used to prove Theorem 4.1 for real closed H-fields with asymptotic integration.

4.3 Differential-newtonian $\boldsymbol{H}$-fields. An important fact about $\mathbb{T}$ from [28] is that if the Newton polynomial of $P \in \mathbb{T}\{Y\}$ has degree 1, then P has a zero in the valuation ring. Let us define an H-field K to be **differential-newtonian** if it is real closed, admits asymptotic integration, and every non-zero $P \in K\{Y\}$ whose Newton polynomial has degree 1 has a zero in the valuation ring. Thus $\mathbb{T}$ is differential-newtonian. A more basic example of a differential-newtonian H-field is $\mathbb{T}_{\log}$. If K is a differential-newtonian H-field, then so is each compositional conjugate K^φ with $\varphi > 0$.

4.4 Differential-henselian valued differential fields. If an H-field K is differential-newtonian, then certain coarsenings of suitable compositional conjugates of K are differential-henselian. To explain this, let K be any valued differential field with small derivation, that is, $\partial \smallo \subseteq \smallo$. It is not hard to see that then $\partial\mathcal{O} \subseteq \mathcal{O}$, and so the residue field $\boldsymbol{k} = \mathcal{O}/\smallo$ is a differential field. In the spirit of [38] we define K to be **differential-henselian** if the following conditions are satisfied:

(DH1) every linear differential equation $a_0y + a_1y' + \cdots + a_ny^{(n)} = b$ with $a_0, \dots, a_n, b \in \boldsymbol{k}$, $a_n \neq 0$, has a solution in $\boldsymbol{k}$;

(DH2) for every $P \in \mathcal{O}\{Y\}$ with $vP_0 > 0$ and $vP_1 = 0$, there is $y \in \smallo$ such that $P(y) = 0$.

Here P_d is the homogeneous part of degree d of P, so

$$P_0 = P(0), \qquad P_1 = \sum_i \frac{\partial P}{\partial Y^{(i)}}(0)Y^{(i)}.$$

Without further assumptions on K this doesn't give much, but it is enough to get an analogue of the familiar lifting of residue fields in henselian valued fields: if K is differential-henselian, then every maximal differential subfield of $\mathcal{O}$ maps isomorphically (as a differential field) onto $\boldsymbol{k}$ under the residue map.

If K is an H-field with $\partial\smallo \subseteq \smallo$, then the derivation on its residue field $\boldsymbol{k}$ is trivial, so (DH1) fails. To make the notion of differential-henselian relevant for H-fields we need to consider coarsenings: Suppose K is a differential-newtonian H-field and $\partial\smallo \subseteq \smallo$. Then the value group $\Gamma = v(K^\times)$ has a distinguished non-trivial convex subgroup

$$\Delta \ := \ \{\gamma \in \Gamma : \ \gamma^\dagger > 0\},$$

and K with the coarsened valuation $v_\Delta \colon K^\times \to \Gamma/\Delta$ is differential-henselian. Moreover, by passing to suitable compositional conjugates of K, we can make this distinguished non-trivial convex subgroup Δ as small as we like, and in this way we can make the coarsened valuation approximate the original valuation as close as needed. We call K with v_Δ the **flattening of** K.

These coarsenings are asymptotic differential fields, as defined in Section 3. Let us consider more generally any asymptotic differential field K with $\partial\smallo \subseteq \smallo$. Then "differential-henselian" does have some further general consequences:

Lemma 4.3. *If K is differential-henselian and $a_0, \dots, a_n, b \in K$, $a_n \neq 0$, then $a_0 y + a_1 y' + \cdots + a_n y^{(n)} = b$ for some $y \in K$.*

This has a useful sharper version where we assume that $a_0, \dots, a_n, b \in \mathcal{O}$ and $a_i \notin \smallo$ for some i, with the solution y also required to be in $\mathcal{O}$.

Proposition 4.4. *If K is maximal as a valued field, and its differential residue field $\boldsymbol{k}$ satisfies* (DH1)*, then K is differential-henselian.*

It follows in particular from Lemma 4.3 that if K is a differential-newtonian H-field, then every linear differential equation $a_0 y + a_1 y' + \cdots + a_n y^{(n)} = b$ over K, with $a_n \neq 0$, has a solution in K.

While the AKE paradigm[6] does not apply directly to H-fields, it may well be relevant for certain coarsenings of compositional conjugates of H-fields. Here we have of course in mind that "differential-henselian" should take over the role of "henselian" in the AKE-theory.

It is worth mentioning that in dealing with a pc-sequence (a_λ) in an H-field with asymptotic integration we can reduce to two very different types of behavior: one kind of behavior is when (a_λ) is **fluent**, that is, it remains a pc-sequence upon coarsening the valuation by some non-trivial convex subgroup of the value group Γ, and the other type of behavior is when (a_λ) is **jammed**, that is, for every $\gamma \in \Gamma^{>}$ there is an index λ_0 such that $|v(a_\mu - a_\lambda)| < \gamma$ for all $\mu > \lambda > \lambda_0$. In differential-henselian matters it is enough to deal with fluent pc-sequences. Jammed pc-sequences are considered in Section 4.7.

4.5 Consequences for existentially closed $\boldsymbol{H}$-fields. Using the results above, some important facts about $\mathbb{T}$ can be shown to go through for existentially closed H-fields. Thus existentially closed H-fields are not only Liouville closed, but also differential-newtonian. As to linear differential equations, let us go into a little more detail.

Let K be a differential field, and consider a *linear differential operator*

$$A = a_0 + a_1\partial + \cdots + a_n\partial^n \qquad (a_0, \dots, a_n \in K)$$

over K; here ∂ stands for the derivation operator on K. These operators form a ring $K[\partial]$ under composition, with $\partial a = a\partial + a'$ for $a \in K$.

Theorem 4.5. *If K is an existentially closed H-field, $n \geqslant 1$, $a_n \neq 0$, then $A\colon K \to K$ is surjective, and A is a product of operators $a + b\partial$ of order* 1 *in $K[i][\partial]$* (*and thus a product of order* 1 *and order* 2 *operators in $K[\partial]$*).

4.6 The Equalizer Theorem. This is an important technical tool, needed, for example, in proving Proposition 4.4.

Let K be a valued differential field with small derivation and value group Γ. Let $P = P(Y) \in K\{Y\}$, $P \neq 0$. Then we have for $g \in K^\times$ the non-zero differential polynomial $P(gY) \in K\{Y\}$, and it turns out that its valuation $vP(gY)$ depends only on vg (not on g). Thus P induces a function

$$v_P:\ \Gamma \to \Gamma, \quad v_P(\gamma)\ :=\ vP(gY) \text{ for } g \in K^\times \text{ with } vg = \gamma.$$

Moreover, if $P(0) = 0$, this function is strictly increasing. These facts are easy to prove, but the following "equalizer" theorem lies deeper:

Theorem 4.6. *Let K be an asymptotic differential field with small derivation and divisible value group Γ. Let $P \in K\{Y\}$, $P \neq 0$, be homogeneous of degree $d > 0$. Then $v_P\colon \Gamma \to \Gamma$ is a bijection. If also $Q \in K\{Y\}$, $Q \neq 0$, is homogeneous of degree $e \neq d$, then there is a unique $\gamma \in \Gamma$ with $v_P(\gamma) = v_Q(\gamma)$.*

In combination with compositional conjugation and Newton polynomials, the equalizer theorem plays a role in detecting the $\gamma \in \Gamma$ for which there can exist $y \in K^\times$ with $vy = \gamma$ and $P(y) = 0$.

4.7 Two important pseudo-cauchy sequences. We consider here jammed pc-sequences. Recall that in $\mathbb{T}$ we have the iterated logarithms ℓ_n with

$$\ell_0 = x, \quad \ell_{n+1} = \log \ell_n,$$

and that this sequence is coinitial in $\mathbb{T}^{>\mathbb{R}}$. By a straightforward computation,

$$-\ell_n^{\dagger\dagger}\ =\ \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}.$$

Then $(-\ell_n^{\dagger\dagger})$ is a (jammed) pc-sequence in $\mathbb{T}$, but has no pseudolimit in $\mathbb{T}$. It does have a pseudolimit in a suitable immediate H-field extension, and such a limit can be thought of as $\sum_{n=0}^{\infty} \frac{1}{\ell_0\ell_1\cdots\ell_n}$.

The fact that the pc-sequence $(\ell_n^{\dagger\dagger})$ has no pseudolimit in $\mathbb{T}$ is related to a key elementary property of $\mathbb{T}$. To explain this we assume in the rest of this subsection: *K is a real closed H-field with asymptotic integration.*

To mimick the above iterated logarithms, first take for any $f \succ 1$ in K an $\mathrm{L}\, f \succ 1$ in K such that $(\mathrm{L}\, f)' \asymp f^\dagger$. (Think of $\mathrm{L}\, f$ as a substitute for $\log f$.) Next, pick a sequence of elements $\ell_\lambda \succ 1$ in K, indexed by the ordinals λ less than some infinite limit ordinal κ: take any $\ell_0 \succ 1$ in K, and set $\ell_{\lambda+1} := \mathrm{L}(\ell_\lambda)$; if μ is an infinite limit ordinal such that all ℓ_λ with $\lambda < \mu$ have been chosen, then take any element $\ell_\mu \succ 1$ in K such that $\ell_\mu \prec \ell_\lambda$ for all $\lambda < \mu$, if there is such an ℓ_μ, while if there is no such ℓ_μ, set $\kappa := \lambda$. Thus

(i) $\ell_\mu \prec \ell_\lambda$ whenever $\mu > \lambda$;
(ii) (ℓ_λ) is coinitial in $K^{\succ 1}$, that is, for each $f \succ 1$ in K there is an index λ with $\ell_\lambda \prec f$.

One can show that this yields a jammed pc-sequence $\left(\ell_\lambda^{\dagger\dagger}\right)$ in K and that the pseudolimits of this pc-sequence in H-field extensions of K do not depend on the choice of the sequence (ℓ_λ): different choices yield "equivalent" pc-sequences in the sense of [9]. Here is a useful fact about this pc-sequence:

Theorem 4.7. *If $s \in K$ is a pseudolimit of $\left(\ell_\lambda^{\dagger\dagger}\right)$, then there is an H-field extension $K(y)$ such that $y^\dagger = s$ and $K(y)$ has a gap.*

Every H-field has an extension to a real closed H-field with asymptotic integration (for example, a Liouville closure). By Theorem 4.1 we can further arrange that this extension is maximal, so that all pc-sequences in it have a pseudolimit. Thus the last theorem has the following consequence:

Corollary 4.8. *Every H-field has an H-field extension with a gap.*

If K is Liouville closed, then $\left(\ell_\lambda^{\dagger\dagger}\right)$ has no pseudolimit in K.

Theorem 4.9. *The following conditions on K are equivalent:*

(1) $K \models \forall a \exists b \left[v(a - b^\dagger) \ \leqslant vb \ < (\Gamma^>)' \right]$;
(2) $\left(\ell_\lambda^{\dagger\dagger}\right)$ *has no pseudolimit in K.*

Since $\mathbb{T}$ satisfies (2), it also satisfies (1). Our discussion preceding Corollary 4.8 made it clear that not all real closed H-fields with asymptotic integration satisfy (2). We call attention to the first-order nature of condition (1).

There is a related and even more important pc-sequence. To define it, set

$$\varrho(z) \ := \ 2z' + z^2 \qquad \text{for } z \in K.$$

Then in $\mathbb{T}$ we have

$$\varrho(-\ell_n^{\dagger\dagger}) \ = \ -\left(\frac{1}{(\ell_0)^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2} \right),$$

so $\left(\varrho(-\ell_n^{\dagger\dagger})\right)$ is also a jammed pc-sequence in $\mathbb{T}$ without any pseudolimit in $\mathbb{T}$. Likewise, for our real closed H-field K with asymptotic integration, $\left(\varrho(-\ell_\lambda^{\dagger\dagger})\right)$ is a jammed pc-sequence. (If $\left(\ell_\lambda^{\dagger\dagger}\right)$ pseudoconverges in K, then so does $\left(\varrho(-\ell_\lambda^{\dagger\dagger})\right)$, but [5] has a Liouville closed example where the converse fails.) Here is an analogue of Theorem 4.9:

Theorem 4.10. *The following conditions on K are equivalent:*

(1) $K \models \forall a \exists b \left[vb < (\Gamma^>)' \text{ and } v(a - \varrho(-b^\dagger)) \ \leqslant \ 2vb \right]$;
(2) $\left(\varrho(-\ell_\lambda^{\dagger\dagger})\right)$ *has no pseudolimit in K;*
(3) $\left(\varrho(-\ell_\lambda^{\dagger\dagger})\right)$ *has no pseudolimit in any differentially algebraic H-field extension of K.*

The equivalence of (1) and (2) is relatively easy, but to show that (2) implies (3) is much harder. Since $\mathbb{T}$ satisfies (2), it also satisfies (1) and (3). The first-order nature of condition (1) will surely play a role in our quest to characterize the existentially closed H-fields by first-order axioms. The equivalence of (2) and (3) is related to the following important fact:

Theorem 4.11. *Suppose $\left(\varrho(-\ell_\lambda^{\dagger\dagger})\right)$ has no pseudolimit in K. Then $\left(\ell_\lambda^{\dagger\dagger}\right)$ has a pseudolimit a in an immediate H-field extension $K\langle a\rangle$ of K with the following universal property: if b is any pseudolimit of $\left(\ell_\lambda^{\dagger\dagger}\right)$ in any immediate pre-H-field extension of K, then there is a unique isomorphism $K\langle a\rangle \to K\langle b\rangle$ over K of ordered valued differential fields sending a to b.*

We define an H-field to be **amenable** if it is real closed, admits asymptotic integration, and satisfies the first-order condition in (1) of Theorem 4.10. Any real closed H-field that admits asymptotic integration and is a directed union of H-subfields F for which Ψ_F has a largest element is amenable. Amenability is invariant under compositional conjugation.

4.8 Simple Newton polynomials. As shown in [28], the Newton polynomials of differential polynomials over $\mathbb{T}$ have the very special form

$$(c_0 + c_1 Y + \cdots + c_m Y^m) \cdot (Y')^n \qquad (c_0, \dots, c_m \in \mathbb{R} = C).$$

This fails for some other real closed H-fields with asymptotic integration:

Example. Consider the immediate H-field extension $K = \mathbb{R}(\!(\mathfrak{L})\!)$ of $\mathbb{T}_{\log}$, where $\mathfrak{L} = \bigcup_{n=0}^{\infty} \mathfrak{L}_n$ (see the end of Section 1.2). Then K is closed under asymptotic integration, and is not amenable, since it contains a pseudolimit $\varrho := -\sum_{n=0}^{\infty} \frac{1}{(\ell_0 \ell_1 \cdots \ell_n)^2}$ of the pc-sequence $\big(\varrho(-\ell_n^{\dagger\dagger})\big)$. We set

$$P := N + \varrho \cdot (Y')^2 \in K\{Y\} \quad \text{where } N(Y) := 2Y'Y^{(3)} - 3(Y'')^2 \in \mathbb{R}\{Y\}.$$

A somewhat lengthy computation yields $N_P = N \notin \mathbb{R}[Y](Y')^{\mathbb{N}}$.

Amenability turns out to be exactly what makes Newton polynomials to have the above simple form:

Theorem 4.12. *Let K be a real closed H-field with asymptotic integration. Then K is amenable if and only if the Newton polynomial of any non-zero differential polynomial $P \in K\{Y\}$ has the form*

$$(c_0 + c_1 Y + \cdots + c_m Y^m) \cdot (Y')^n \qquad (c_0, \dots, c_m \in C).$$

This has a nice consequence for the behavior of a differential polynomial near the constant field:

Corollary 4.13. *Let K be an amenable H-field and $P \in K\{Y\}$, $P \neq 0$. Then there are $\alpha \in \Gamma$, $a \in K^{>C}$ and m, n such that*

$$C_L < y < a \implies v(P(y)) \; = \; \alpha + mvy + nvy'$$

for all y in all H-field extensions L of K, where C_L = constant field of L.

We also have the following converse to a result from Section 4.4:

Corollary 4.14. *Suppose the H-field K is amenable, and there are K-admissible $\varphi > 0$ with arbitrarily high $v\varphi < (\Gamma^{>})'$ such that the flattening of K^{φ} is differential-henselian. Then K is differential-newtonian.*

4.9 Conjectural characterization of existentially closed H-fields. We can show that every existentially closed H-field with small derivation is amenable. We already mentioned earlier that they are Liouville closed, and differential-newtonian, and that their linear differential operators factor completely after adjoining $\mathrm{i} = \sqrt{-1}$ to the field. Maybe this is the full story:

Optimistic Conjecture. *An H-field K with small derivation is existentially closed if and only if it satisfies the following first-order conditions:*

(i) *K is Liouville closed,*
(ii) *every $A \in K[\partial]$ is a product of operators of order 1 in $K[\mathrm{i}][\partial]$,*
(iii) *K is amenable,*
(iv) *K is differential-newtonian.*

This conjecture makes the $\mathbb{T}$-Conjecture more precise. It is also conceivable that the H-field $\mathbb{T}_{\log}$ has a good model theory. It satisfies (ii), (iii), (iv), and has some other attractive properties. On the other hand, the H-field $\mathbb{T}_{\exp}$ of purely exponential transseries defines $\mathbb{Z}$; see [3, Section 13].

5 Quantifier-free Definability

In Section 2 we considered three intrinsic model-theoretic statements about $\mathbb{T}$:

(1) If $X \subseteq \mathbb{T}^n$ is definable, then $X \cap \mathbb{R}^n$ is semialgebraic.
(2) $\mathbb{T}$ is *asymptotically o-minimal*: for each definable $X \subseteq \mathbb{T}$ there is a $b \in \mathbb{T}$ such that either $(b, +\infty) \subseteq X$ or $(b, +\infty) \subseteq \mathbb{T} \setminus X$.
(3) $\mathbb{T}$ has NIP.

In this section we prove quantifier-free versions of these statements. First we do this in the easy case when the language is the natural language $\mathcal{L}$ of ordered valued differential rings. ("Easy" means here that it follows with very little work from results in the literature.) Next we exhibit a basic obstruction[7] showing that $\mathbb{T}$ does not eliminate quantifiers in $\mathcal{L}$. This obstruction can be lifted by extending $\mathcal{L}$ to a language $\mathcal{L}^*$ which has a unary function symbol naming a certain integration operator on $\mathbb{T}$. (This operator is existentially definable in $\mathbb{T}$ using $\mathcal{L}$.) We then show that (1), (2), (3) also hold for quantifier-free definable relations on $\mathbb{T}$ when the latter is construed as an $\mathcal{L}^*$-structure.

Thus (1), (2), (3) would follow from the strong form of the $\mathbb{T}$-Conjecture which says that $\mathbb{T}$ admits quantifier elimination in the language $\mathcal{L}^*$. This form of the $\mathbb{T}$-Conjecture is unfortunately too strong: In Section 6 we discuss further obstacles, and speculate on how these might be dealt with.

5.1 Quantifier-free definable sets in $\mathbb{T}$ using $\mathcal{L}$. Recall:

$$\mathcal{L} \;=\; \{0,\ 1,\ +,\ -,\ \cdot\ ,\ \partial,\ \leqslant,\ \preccurlyeq\}.$$

In this subsection we view any H-field K as an $\mathcal{L}$-structure in the natural way, and so "quantifier-free definable" means "definable in K by a quantifier-free formula of the language $\mathcal{L}$ augmented by names for the elements of K." The next three propositions state the quantifier-free versions of (1)–(3) above, each in a somewhat more general form.

Proposition 5.1. *Let K be a real closed H-field. If $X \subseteq K^n$ is quantifier-free definable, then its trace $X \cap C^n$ in the field C of constants is semialgebraic.*

Proof Let $P = P(Y_1, \dots, Y_n) \in K\{Y_1, \dots, Y_n\}$ be a differential polynomial. Removing from P the terms involving any $Y_i^{(r)}$ with $r \geqslant 1$ we obtain an ordinary polynomial $p \in K[Y_1, \dots, Y_n]$ such that for all $y_1, \dots, y_n \in C \subseteq K$,

$$P(y_1, \dots, y_n) \;=\; p(y_1, \dots, y_n).$$

Recall also that for all $f, g \in K$ we have

$$f \preccurlyeq g \iff |f| \leqslant c|g| \text{ for some } c \in C^{>0}.$$

It follows that if $X \subseteq K^n$ is quantifier-free definable, then $X \cap C^n$ is definable (with parameters) in the pair (K, C) construed here as the real closed field K (forgetting its derivation and valuation), with C as a distinguished subset. This pair (K, C) is a model of $\mathrm{RCF}_{\mathrm{tame}}$, as defined in [13]. By Proposition 8.1 of [13] applied to $T = \mathrm{RCF}$, a subset of C^n which is definable (with parameters) in the pair (K, C) is semialgebraic in the sense of C. $\square$

We now turn to quantifier-free asymptotic o-minimality. This follows easily from the *logarithmic decomposition* of a differential polynomial in [28], as we explain now. Let K be a differential field. For $y \in K$, we set $y^{\langle 0 \rangle} := y$, and inductively, if $y^{\langle n \rangle} \in K$ is defined and non-zero, $y^{\langle n+1 \rangle} := (y^{\langle n \rangle})^\dagger$ (and otherwise $y^{\langle n+1 \rangle}$ is not defined). Thus in the differential fraction field $K\langle Y \rangle$ of the differential polynomial ring $K\{Y\}$ each $Y^{\langle n \rangle}$ is defined, the elements $Y^{\langle 0 \rangle}, Y^{\langle 1 \rangle}, Y^{\langle 2 \rangle}, \dots$ are algebraically independent over K, and

$$K\langle Y \rangle = K\big(Y^{\langle n \rangle} : \ n = 0, 1, 2, \dots\big).$$

If $y^{\langle n \rangle}$ is defined and $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{1+n}$, we set

$$y^{\langle \boldsymbol{i} \rangle} \ := \ (y^{\langle 0 \rangle})^{i_0} (y^{\langle 1 \rangle})^{i_1} \cdots (y^{\langle n \rangle})^{i_n}.$$

One can show that any $P \in K\{Y\}$ of order $\leqslant r$ has a unique decomposition

$$P \ = \ \textstyle\sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i} \rangle} Y^{\langle \boldsymbol{i} \rangle} \quad \textbf{(logarithmic decomposition)},$$

with $\boldsymbol{i}$ ranging over $\mathbb{N}^{1+r}$, all $P_{\langle \boldsymbol{i} \rangle} \in K$, and $P_{\langle \boldsymbol{i} \rangle} \neq 0$ for only finitely many $\boldsymbol{i}$.

Consider the case $y \in K := \mathbb{T}$. Then $y^{\langle 1 \rangle} = y^\dagger$ is defined for $y \neq 0$, and if $y > \exp(x^2)$, then $y^{\langle 1 \rangle} > 2x$ and $y > (y^{\langle 1 \rangle})^m$ for all m. By induction on n, if $y > \exp^{n+1}(x^2)$, with the exponent $n+1$ referring to compositional iteration, then $y^{\langle n+1 \rangle}$ is defined, $y^{\langle n \rangle} > \exp^n(x^2)$, and $y^{\langle n+1 \rangle} > (y^{\langle n \rangle})^m$ for all m.

Let a non-zero $P \in \mathbb{T}\{Y\}$ of order $\leqslant r$ be given with the logarithmic decomposition displayed before. Take $\boldsymbol{j} \in \mathbb{N}^{1+r}$ lexicographically maximal with $P_{\langle \boldsymbol{j} \rangle} \neq 0$. It follows from the above that we can take $b \in \mathbb{T}$ so large that if $y > b$, then $y^{\langle r \rangle}$ is defined and $P(y) \sim \ P_{\langle \boldsymbol{j} \rangle} y^{\langle \boldsymbol{j} \rangle}$. In particular, if $P_{\langle \boldsymbol{j} \rangle} > 0$, then $P(y) > 0$ for all $y > b$, and if $P_{\langle \boldsymbol{j} \rangle} < 0$, then $P(y) < 0$ for all $y > b$. By similar reasoning, given any non-zero $P, Q \in \mathbb{T}\{Y\}$, there is $b \in \mathbb{T}$ such that either $P(y) \preccurlyeq Q(y)$ for all $y > b$ in $\mathbb{T}$, or $P(y) \succ Q(y)$ for all $y > b$ in $\mathbb{T}$. Thus:

Proposition 5.2. *If $X \subseteq \mathbb{T}$ is quantifier-free definable, then there is $b \in \mathbb{T}$ such that either $(b, +\infty) \subseteq X$, or $(b, +\infty) \subseteq \mathbb{T} \setminus X$.*

This proposition holds for any Liouville closed H-field K instead of $\mathbb{T}$: we can define on such K a substitute for the exponential function exp as used in the proof above, see [3, Section 1.1].

A relation $R \subseteq A \times B$ is said to be **independent** if for every $N \geqslant 1$ there are elements $a_1, \dots, a_N \in A$ and $b_I \in B$, for each $I \subseteq \{1, \dots, N\}$, such that

$$R(a_i, b_I) \iff i \in I \qquad (\text{for } i = 1, \dots, N, \ \text{ and all } I \subseteq \{1, \dots, N\}).$$

A (one-sorted) structure $\boldsymbol{M} = (M; \dots)$ is said to have **NIP** (the **Non-Independence Property**) if there is no independent definable relation $R \subseteq M^m \times M^n$. This is a robust model-theoretic tameness condition on a structure, and follows from o-minimality as well as from stability. It was introduced early on by Shelah [39]; there is also a substantial body of recent work around this notion, see for example [30].

Proposition 5.3. *Let K be an H-field. No quantifier-free definable relation $R \subseteq K^m \times K^n$ is independent.*

Proof Let OVDF be the $\mathcal{L}$-theory of ordered, valued, differential fields where the only axiom relating the ordering, valuation, and derivation is

$$\forall x \forall y \ (0 \leqslant x \leqslant y \ \rightarrow \ x \preccurlyeq y).$$

Guzy and Point [23, Corollary 6.4] show that OVDF has a model completion OVDF^c, and that OVDF^c has NIP. Now use an embedding of K into some model of OVDF^c. $\square$

5.2 $\mathbb{T}$ does not admit quantifier elimination in $\mathcal{L}$. Let K be an H-field with small derivation. Then we have the ideal

$$\mathrm{I}(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \smallo\}$$

of its valuation ring $\mathcal{O}$. If K is also Liouville closed, then

$$\mathrm{I}(K) = \partial\smallo = \{y \in K : y \prec f^\dagger \text{ for all non-zero } f \in \smallo\},$$

so $\mathrm{I}(K)$ is existentially as well as universally definable in the $\mathcal{L}$-structure K. Still considering $\mathbb{T}$ as an $\mathcal{L}$-structure, we have:

Proposition 5.4. *The subset* $\mathrm{I}(\mathbb{T})$ *of* $\mathbb{T}$ *is not quantifier-free definable in* $\mathbb{T}$.

Proof Recall from Section 4.7 the pc-sequence $\big(\ell_n^{\dagger\dagger}\big)$ in $\mathbb{T}$:

$$\ell_n^{\dagger\dagger} = \left(\frac{1}{\ell_n}\right)^{\dagger\dagger} = -\left(\frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}\right).$$

It has no pseudolimit in $\mathbb{T}$. Fix some $\aleph_0$-saturated elementary extension K of $\mathbb{T}$ and take $\ell \in K$ such that $\ell > C$ but $\ell < \ell_n$ for all n. Then $a := \ell^{\dagger\dagger} = (1/\ell)^{\dagger\dagger}$ is a pseudolimit of $\big(\ell_n^{\dagger\dagger}\big)$. Also, one computes easily that

$$(1/\ell_n)'^{\dagger} = (1/\ell_n)^{\dagger\dagger} - (1/\ell_0\cdots\ell_n),$$

so the sequence $\big((1/\ell_n)'^{\dagger}\big)$ is also a pc-sequence, and has the same pseudolimits in K as $\big(\ell_n^{\dagger\dagger}\big)$. Moreover, $b := (1/\ell)'^{\dagger}$ is a pseudolimit of $\big((1/\ell_n)'^{\dagger}\big)$. Thus by Theorem 4.11, the H-subfields $\mathbb{T}\langle a\rangle$ and $\mathbb{T}\langle b\rangle$ of K are immediate extensions of $\mathbb{T}$, and we have an isomorphism $\mathbb{T}\langle a\rangle \to \mathbb{T}\langle b\rangle$ over $\mathbb{T}$ that sends a to b. The element $f = (1/\ell)^\dagger$ of K satisfies $f^\dagger = a$ and $h' \prec f \prec h^\dagger$ for all $h \in \mathbb{T}^\times$ with $h \not\asymp 1$, and the real closure $\mathbb{T}\langle a\rangle^{\mathrm{rc}}$ of $\mathbb{T}\langle a\rangle$ in K is an immediate extension of $\mathbb{T}$. Hence in the terminology of [3, Section 12], a creates a gap over $\mathbb{T}\langle a\rangle^{\mathrm{rc}}$, by [3, Proposition 12.4]. Since $g = (1/\ell)'$ satisfies $g^\dagger = b$, by [3, Lemma 12.3] and the uniqueness statement in [2, Lemma 5.3] the above isomorphism extends to an isomorphism

$$\mathbb{T}\langle a, f\rangle \to \mathbb{T}\langle b, g\rangle$$

of $\mathcal{L}$-structures which sends f to g. Now, if $\mathrm{I}(\mathbb{T})$ were defined in $\mathbb{T}$ by a quantifier-free formula $\varphi(y)$ in the language $\mathcal{L}$ augmented by names for the elements of $\mathbb{T}$, then we would have $K \models \neg\varphi(f)$ and $K \models \varphi(g)$, and so $\mathbb{T}\langle a, f\rangle \models \neg\varphi(f)$ and $\mathbb{T}\langle b, g\rangle \models \varphi(g)$, which violates the above isomorphism between $\mathbb{T}\langle a, f\rangle$ and $\mathbb{T}\langle b, g\rangle$. $\square$

For later use it is convenient to extend the language $\mathcal{L}$ to the language $\mathcal{L}_{\mathbb{T},\mathrm{a}}$, by adding for each semialgebraic function $\mathbb{T}^n \to \mathbb{T}$ an n-ary function symbol naming that function. In particular, by taking $n = 0$ we see that $\mathcal{L}_{\mathbb{T},\mathrm{a}}$ has a name (constant symbol) for each $f \in \mathbb{T}$. Also, $\mathbb{T}$, and any real closed valued differential field extension of $\mathbb{T}$, is naturally construed as a structure for the language $\mathcal{L}_{\mathbb{T},\mathrm{a}}$. For example, the function $y \mapsto y^{-1} \colon \mathbb{T} \to \mathbb{T}$, with $0^{-1} := 0$ by convention, is named by a function symbol of $\mathcal{L}_{\mathbb{T},\mathrm{a}}$. So is, for each integer

$d \geqslant 1$, the function $y \mapsto y^{1/d} \colon \mathbb{T} \to \mathbb{T}$, taking the value 0 for $y \leqslant 0$ by convention.

Proposition 5.5. *If $X \subseteq \mathbb{T}^n$ is quantifier-free definable in $\mathbb{T}$ as $\mathcal{L}_{\mathbb{T},\mathrm{a}}$-structure, then X is quantifier-free definable in $\mathbb{T}$ as $\mathcal{L}$-structure.*

One can view this as a partial quantifier elimination: it is obvious how to eliminate occurrences of function symbols of $\mathcal{L}_{\mathbb{T},\mathrm{a}} \setminus \mathcal{L}$ from a quantifier-free $\mathcal{L}_{\mathbb{T},\mathrm{a}}$-formula at the cost of introducing existentially quantified new variables, and Proposition 5.5 says that we can eliminate those quantifiers again *without reintroducing these function symbols.* This fact can be proved by explicit means, but we prefer a model-theoretic argument that we can use also in later situations where explicit elimination would be very tedious.

To formulate this in sufficient generality, let L be a sublanguage of the (one-sorted) first-order language L^*, and assume that L has a constant symbol. Let $\boldsymbol{A}^* = (A; \dots)$ and $\boldsymbol{B}^*$ range over L^*-structures, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be their L-reducts. Let T^* be an L^*-theory. Then we have the following criterion:

Lemma 5.6. *Let $\varphi^*(x)$ with $x = (x_1, \dots, x_n)$ be an L^*-formula. Then $\varphi^*(x)$ is T^*-equivalent to some quantifier-free L-formula $\varphi(x)$ iff for all $\boldsymbol{A}^*, \boldsymbol{B}^* \models T^*$, common L-substructures $\boldsymbol{C} = (C; \dots)$ of $\boldsymbol{A}$ and $\boldsymbol{B}$, and $c \in C^n$:*

$$\boldsymbol{A}^* \models \varphi^*(c) \iff \boldsymbol{B}^* \models \varphi^*(c).$$

This criterion is well-known (at least for $L = L^*$), and follows by a standard model-theoretic compactness argument. Typically, the criterion gets used via its corollary below. To state that corollary, we define T^* to have **closures of L-substructures** if for all $\boldsymbol{A}^*, \boldsymbol{B}^* \models T^*$ with a common L-substructure $\boldsymbol{C} = (C; \dots)$ of $\boldsymbol{A}$ and $\boldsymbol{B}$, there is a (necessarily unique) isomorphism from the L^*-substructure of $\boldsymbol{A}^*$ generated by C onto the L^*-substructure of $\boldsymbol{B}^*$ generated by C which is the identity on C.

Corollary 5.7. *If T^* has closures of L-substructures, then every quantifier-free L^*-formula is T^*-equivalent to a quantifier-free L-formula.*

Proof of Lemma 5.5 We are going to apply Corollary 5.7 with

$$L := \mathcal{L} \text{ augmented by a name (constant symbol) for each element of } \mathbb{T},$$
$$L^* := \mathcal{L}_{\mathbb{T},\mathrm{a}}, \qquad T^* := \text{ the elementary theory of } \mathbb{T} \text{ as } L^*\text{-structure.}$$

Indeed, we show that T^* has closures of L-substructures. Let $E, F \models T^*$ have a common L-substructure D. Thus D is an ordered differential subring of both E and F such that for all $f, g \in D$ we have $f \preccurlyeq_E g \iff f \preccurlyeq_D g \iff f \preccurlyeq_F g$, where $\preccurlyeq_D$, $\preccurlyeq_E$, $\preccurlyeq_F$ are the interpretations of the symbol $\preccurlyeq$ of L in D, E, F, respectively. Let K_E and K_F be the fraction fields of the integral domain D in E and F respectively. Then K_E is the underlying ring of an L-substructure of E, to be denoted also by K_E. Likewise, K_F denotes the corresponding L-substructure of F, and we have a unique L-isomorphism $K_E \to K_F$ that is the identity on D. Let K_E^{rc} and K_F^{rc} be the real closures of the ordered fields K_E and K_F in E and F, respectively. Then K_E^{rc} is the underlying ring of an L^*-substructure of E, to be denoted also by K_E^{rc}. Likewise, K_F^{rc} denotes the corresponding L^*-substructure of F, and the above L-isomorphism $K_E \to K_F$

extends uniquely to an L^*-isomorphism $K_E^{\mathrm{rc}} \to K_F^{\mathrm{rc}}$. It remains to note that K_E^{rc} is the L^*-substructure of E generated by D. □

Given a real closed field extension K of $\mathbb{T}$, a set $X \subseteq K^n$ is **$\mathbb{T}$-semialgebraic** if it is defined in K by some (quantifier-free) formula in the language of ordered rings augmented by names for the elements of $\mathbb{T}$, and a function $K^n \to K$ is **$\mathbb{T}$-semialgebraic** if its graph is.

5.3 Adding a new primitive. Let $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$ be the language $\mathcal{L}_{\mathbb{T},\mathrm{a}}$ augmented by a unary predicate symbol I. We construe $\mathbb{T}$ as an $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure by interpreting I as $\mathrm{I}(\mathbb{T})$. In view of Proposition 5.4 and Lemma 5.5 this genuinely changes what can be defined quantifier-free in $\mathbb{T}$. Nevertheless, Propositions 5.1, 5.2, 5.3 (in the case $K = \mathbb{T}$) go through when "quantifier-free" is with respect to $\mathbb{T}$ as $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure. For "quantifier-free NIP" we can almost repeat the previous argument:

Proposition 5.8. *No quantifier-free definable relation $R \subseteq \mathbb{T}^m \times \mathbb{T}^n$ on $\mathbb{T}$ as an $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure is independent.*

Proof The set $\mathrm{I}(\mathbb{T})$ is convex in $\mathbb{T}$. Embedding the $\mathcal{L}$-structure $\mathbb{T}$ in a sufficiently saturated model $\boldsymbol{M}$ of OVDF^c, we can take $a > 0$ in $\boldsymbol{M}$ such that $(-a, a) \cap \mathbb{T} = \mathrm{I}(\mathbb{T})$, where the interval $(-a, a)$ is with respect to $\boldsymbol{M}$. Now use that $\boldsymbol{M}$ has NIP. □

Let K be an H-field extension of $\mathbb{T}$, and $r \in \mathbb{N}$. For $y = (y_1, \dots, y_m) \in K^m$ we set $y' := (y_1', \dots, y_m')$, and define the r**th prolongation** of y to be

$$\big(y, y', \dots, y^{(r)}\big) := \big(y_1, \dots, y_m, y_1', \dots, y_m', \dots, y_1^{(r)}, \dots, y_m^{(r)}\big) \in K^{m(1+r)}.$$

A **∂-covering** (of order r) of a function $g \colon K^m \to K$ consists of a finite covering $\mathcal{C}$ of $K^{m(1+r)}$ by $\mathbb{T}$-semialgebraic sets and for each $S \in \mathcal{C}$ a $\mathbb{T}$-semialgebraic function $g_S \colon K^{m(1+r)} \to K$ such that

$$g(y) = g_S\big(y, y', \dots, y^{(r)}\big) \quad \text{for all } y \in K^m \text{ with } \big(y, y', \dots, y^{(r)}\big) \in S.$$

For example, if $P \in \mathbb{T}\{Y_1, \dots, Y_m\}$ is a differential polynomial of order $\leqslant r$, then the function $y \mapsto P(y) \colon K^m \to K$ has a ∂-covering of order r consisting just of a single set, namely $K^{m(1+r)}$. It is easy to see that if $f \colon K^n \to K$ is $\mathbb{T}$-semialgebraic and $g_1, \dots, g_n \colon K^m \to K$ have ∂-coverings (of various orders), then $f(g_1, \dots, g_n) \colon K^m \to K$ has a ∂-covering. In particular, the sum $g_1 + g_2$ of functions $g_1, g_2 \colon K^m \to K$ with ∂-coverings has a ∂-covering, and so does their product $g_1 g_2$. Less obviously:

Lemma 5.9. *If $g \colon K^m \to K$ has a ∂-covering, then so does the function*

$$y \mapsto g(y)' \colon K^m \to K.$$

Proof Let $\mathcal{C}$ be a ∂-covering of g of order r with, for each set $S \in \mathcal{C}$, the witnessing function $g_S \colon K^{m(1+r)} \to K$. By further partitioning we can arrange that each set $S \in \mathcal{C}$ is a $\mathbb{T}$-semialgebraic cell which is of class C^1 in the sense that the standard projection map $p_S \colon S \to p(S)$ onto an open cell $p(S) \subseteq K^d$, with $d = \dim S$, is not just a homeomorphism, but even a diffeomorphism of class C^1 (in the sense of the real closed field K). In addition we can arrange that for each witnessing map $g_S \colon K^{m(1+r)} \to K$ the restriction of g_S to S is of class C^1. Let us now focus on one particular $S \in \mathcal{C}$, and first consider the case that S is open in $K^{m(1+r)}$. Then by the $\mathbb{T}$-semialgebraic version of Lemma 4.4 in [3], and Remark (2) following its proof, we have $\mathbb{T}$-semialgebraic functions $h, h_1, \ldots, h_{mr} \colon K^{m(1+r)} \to K$, such that for all $\vec{y} = (y_{10}, \ldots, y_{m0}, \ldots, y_{1r}, \ldots, y_{mr}) \in S$,

$$g_S(\vec{y})' \;=\; h(\vec{y}) + \sum_{i=1}^{m} \sum_{j=0}^{r} h_{ij}(\vec{y}) y'_{ij}.$$

If S is not open, of dimension d, this statement remains true, as one can see by reducing to the case of the open cell $p(S) \subseteq K^d$ via the C^1-diffeomorphism $p_S \colon S \to p(S)$. It follows easily that the function $y \mapsto g(y)'$ has a ∂-covering of order $r+1$, whose sets are the products $S \times K^m$ with $S \in \mathcal{C}$. $\square$

It follows that if $t(x_1, \ldots, x_n)$ is an $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-term, then the function

$$y \mapsto t(y) \colon K^n \to K$$

has a ∂-covering. This is now used to prove:

Proposition 5.10. *If $X \subseteq \mathbb{T}^n$ is quantifier-free definable in $\mathbb{T}$ as an $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure, then $X \cap \mathbb{R}^n$ is semialgebraic.*

Proof By the remark preceding the proposition, and the arguments in the proof of Proposition 5.1, it suffices to show the following. Let $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ be $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-terms in which the function symbol ∂ does not occur. Then the sets

$$\{y \in \mathbb{R}^n : t(y) = 0\}, \qquad \{y \in \mathbb{R}^n : t(y) > 0\},$$
$$\{y \in \mathbb{R}^n : s(y) \preccurlyeq t(y)\}, \qquad \{y \in \mathbb{R}^n : t(y) \in \mathrm{I}(\mathbb{T})\}$$

are semialgebraic subsets of $\mathbb{R}^n$. Since the function $y \mapsto t(y) \colon \mathbb{T}^n \to \mathbb{T}$ is $\mathbb{T}$-semialgebraic, this holds for the first three sets by the argument at the end of the proof of Proposition 5.1. For the last set, take some real closed field extension K of $\mathbb{T}$ with a positive element a such that $\mathrm{I}(\mathbb{T}) = \mathbb{T} \cap (-a, a)$, where the interval $(-a, a)$ is in the sense of K. Then

$$\{y \in \mathbb{R}^n : t(y) \in \mathrm{I}(\mathbb{T})\} = \{y \in \mathbb{R}^n : |t(y)| < a\},$$

which is the trace in $\mathbb{R}$ of a semialgebraic subset of K^n. Such traces are known to be semialgebraic in $\mathbb{R}$. $\square$

In proving next that $\mathbb{T}$ qua $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure is quantifier-free asymptotically o-minimal, we shall use the easily verified fact that if K is an H-field that admits asymptotic integration, and L is an H-field extension of K, then $\mathrm{I}(L) \cap K = \mathrm{I}(K)$.

Proposition 5.11. *If $X \subseteq \mathbb{T}$ is quantifier-free definable in $\mathbb{T}$ as $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure, then for some $f \in \mathbb{T}$, either $(f, +\infty) \subseteq X$ or $(f, +\infty) \subseteq \mathbb{T} \setminus X$.*

Proof Let K be an elementary extension of $\mathbb{T}$ as $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure and $a, b \in K$, $a > \mathbb{T}$, $b > \mathbb{T}$. By familiar model-theoretic arguments it suffices to show that then there is an isomorphism of $\mathcal{L}$-structures $\mathbb{T}\langle a\rangle^{\mathrm{rc}} \to \mathbb{T}\langle b\rangle^{\mathrm{rc}}$ over $\mathbb{T}$ that sends a to b, and maps $\mathrm{I}(K) \cap \mathbb{T}\langle a\rangle^{\mathrm{rc}}$ onto $\mathrm{I}(K) \cap \mathbb{T}\langle b\rangle^{\mathrm{rc}}$. (Here rc refers to the real closure in K.) Proposition 5.2 gives an isomorphism of $\mathcal{L}$-structures

$\mathbb{T}\langle a\rangle \to \mathbb{T}\langle b\rangle$ over $\mathbb{T}$ sending a to b, and this isomorphism extends uniquely to an $\mathcal{L}$-isomorphism $\mathbb{T}\langle a\rangle^{\mathrm{rc}} \to \mathbb{T}\langle b\rangle^{\mathrm{rc}}$. The arguments preceding Proposition 5.2 show that for all $m, n \geqslant 1$,

$$v(a^{\langle n-1\rangle}) \;<\; mv(a^{\langle n\rangle}) \;<\; v(\mathbb{T}^{\times}),$$

so a is differentially transcendental over $\mathbb{T}$, and the value group of $\mathbb{T}\langle a\rangle^{\mathrm{rc}}$ is

$$v(\mathbb{T}^{\times}) \oplus \bigoplus_n \mathbb{Q}v(a^{\langle n\rangle}) \qquad \text{(internal direct sum of } \mathbb{Q}\text{-subspaces)},$$

which contains $v(\mathbb{T}^{\times})$ as a convex subgroup. It also follows that $\mathbb{T}\langle a\rangle^{\mathrm{rc}}$ is an H-field, with the same constant field $\mathbb{R}$ as $\mathbb{T}$. Therefore, $\mathbb{T}\langle a\rangle^{\mathrm{rc}}$ admits asymptotic integration, so $\mathrm{I}(K) \cap \mathbb{T}\langle a\rangle^{\mathrm{rc}} = \mathrm{I}(\mathbb{T}\langle a\rangle^{\mathrm{rc}})$. Likewise, $\mathrm{I}(K) \cap \mathbb{T}\langle b\rangle^{\mathrm{rc}} = \mathrm{I}(\mathbb{T}\langle b\rangle^{\mathrm{rc}})$, hence our isomorphism $\mathbb{T}\langle a\rangle^{\mathrm{rc}} \to \mathbb{T}\langle b\rangle^{\mathrm{rc}}$ maps $\mathrm{I}(K)\cap\mathbb{T}\langle a\rangle^{\mathrm{rc}}$ onto $\mathrm{I}(K)\cap\mathbb{T}\langle b\rangle^{\mathrm{rc}}$ as required. ☐

This proposition tells us how a quantifier-free definable $X \subseteq \mathbb{T}$ behaves near $+\infty$. Using fractional linear transformations we get analogous behavior to the left as well as to the right of any point in $\mathbb{T}$. In other words, the $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure $\mathbb{T}$ is quantifier-free locally o-minimal. (In this connection we note that local o-minimality by itself does not imply NIP [22, Example 6.19].)

5.4 Expanding by small integration. Next we show that "small integration" can be eliminated from quantifier-free formulas. This is a further partial quantifier elimination in the style of Proposition 5.5.

Let K be an H-field. We have $\partial\mathcal{O} \subseteq \mathrm{I}(K)$, and we say that K **admits small integration** if $\partial\mathcal{O} = \mathrm{I}(K)$. So $\mathbb{T}$ admits small integration. It follows from Section 3 and Proposition 4.3 in [2] that K has an immediate H-field extension $\mathrm{si}(K)$ that is henselian as a valued field and admits small integration, with the following universal property: for any H-field extension L of K that is henselian as a valued field and admits small integration there is a unique K-embedding of $\mathrm{si}(K)$ into L. We call $\mathrm{si}(K)$ the **closure of K under small integration**.

Let K be an H-field admitting small integration. The derivation ∂ is injective on $\mathcal{O}$, so we can define $\int\colon K \to K$ by

$$\int a' = a \;\text{ for } a \in \mathcal{O}, \qquad \int b = 0 \;\text{ for } b \notin \partial\mathcal{O}.$$

Note that the standard part map $\mathrm{st}\colon K \to K$ defined by

$$\mathrm{st}(c+\varepsilon) = c \text{ for } c \in C,\ \varepsilon \prec 1, \qquad \mathrm{st}(a) = a \text{ for } a \succ 1,$$

can be expressed in terms of $\int$ by $\mathrm{st}(a) = a - \int a'$. The reason for mentioning this fact is that such a standard part map is used to eliminate quantifiers in certain expansions of o-minimal fields; see [14, (5.9)].

Real closed H-field extensions of $\mathbb{T}$ admitting small integration are construed below as $\mathcal{L}^*$-structures where $\mathcal{L}^*$ is a language extending $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$ by a new unary function symbol $\int$, to be interpreted as indicated above.

Let T^* be the $\mathcal{L}^*$-theory of real closed H-field extensions of $\mathbb{T}$ admitting small integration. Then we have the following elimination result:

Proposition 5.12. *T^* has closures of $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-substructures. Therefore, by Corollary 5.7, every quantifier-free $\mathcal{L}^*$-formula is T^*-equivalent to a quantifier-free $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-formula.*

Proof Let K be a model of T^* and let E be an $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-substructure of K. Then E is a real closed pre-H-field, and we may consider the H-field closure $H(E)$ of E as an H-subfield of K, with real closure $H(E)^{\mathrm{rc}}$ in K. We let $E^* := \mathrm{si}\big(H(E)^{\mathrm{rc}}\big)$ be the closure under small integrals of $H(E)^{\mathrm{rc}}$, viewed as an H-subfield of K. In fact, E^* is real closed and closed under small integrals, hence an $\mathcal{L}^*$-substructure of K. Let L be another model of T^* containing E as $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-substructure; we need to show that the natural inclusion $E \to L$ extends to an embedding of $\mathcal{L}^*$-structures $E^* \to L$. By the universal properties of H-field closure, real closure, and closure under small integrals, there is an embedding of $\mathcal{L}_{\mathbb{T},\mathrm{a}}$-structures $E^* \to L$ which extends the inclusion $E \to L$. This embedding also preserves the interpretations of the symbol $\int$ in K respectively L; so after identifying E^* with its image under this embedding, it remains to show that $\mathrm{I}(K) \cap E^* = \mathrm{I}(L) \cap E^*$. For this we distinguish two cases:

Case 1: there is $r \in \mathcal{O}_E \setminus C_E$ with $v(r') \notin (\Gamma_E^{>})'$. Take such r, and take $y \in H(E)$ with $y' = r'$ and $\alpha := v(y) > 0$. Then by Corollary 4.5, (1) in [2], $\Gamma_{H(E)} = \Gamma_E \oplus \mathbb{Z}\alpha$ with $0 < n\alpha < \Gamma_E^{>}$ for all $n \geqslant 1$. Also, $\max \Psi_{H(E)} = \alpha^\dagger$. It follows easily that

$$\mathrm{I}(K) \cap H(E) = \big\{f \in H(E) : vf > \alpha^\dagger\big\} = \mathrm{I}(L) \cap H(E).$$

This remains true when we replace $H(E)$ by E^*, since

$$\Gamma_{E^*} = \text{divisible hull of } \Gamma_{H(E)} = \Gamma_E \oplus \mathbb{Q}\alpha,$$

thus $\max \Psi_{E^*} = \alpha^\dagger$.

Case 2: there is no such r. Then by [2, Corollary 4.5, (2)] we have $\Gamma_{H(E)} = \Gamma_E$, and hence $\Gamma_{E^*} = \Gamma_E$, so

$$\begin{aligned} \mathrm{I}(K) \cap E = \mathrm{I}(L) \cap E \quad &\Longleftrightarrow \quad (\Gamma_K^{>})' \cap \Gamma_E = (\Gamma_L^{>})' \cap \Gamma_E \\ &\Longleftrightarrow \quad (\Gamma_K^{>})' \cap \Gamma_{E^*} = (\Gamma_L^{>})' \cap \Gamma_{E^*} \\ &\Longleftrightarrow \quad \mathrm{I}(K) \cap E^* = \mathrm{I}(L) \cap E^*, \end{aligned}$$

as required. $\square$

We can do the same for *small exponentiation*: given $a \prec 1$ in $\mathbb{T}$, its exponential e^a is the unique element $1+y$ with $y \prec 1$ in $\mathbb{T}$ such that $y' = (1+y)a'$. Thus the bijection $a \mapsto e^a \colon \scriptstyle\mathcal{O} \to 1 + \scriptstyle\mathcal{O}$ is (existentially and universally) definable in the $\mathcal{L}$-structure $\mathbb{T}$. Arguments as in the proof of Proposition 5.12 show that expanding the $\mathcal{L}^*$-structure $\mathbb{T}$ by this operation (taking the value 0 on $\mathbb{T} \setminus \scriptstyle\mathcal{O}$, by convention) does not change what is quantifier-free definable.

6 Further Obstructions to Quantifier Elimination

The language $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$ is rather strong as to what it can express quantifier-free about $\mathbb{T}$, as we have seen. However, $\mathbb{T}$ does not admit QE in this language. To discuss this, let K be an H-field, and consider the subset

$$\mathrm{Z}(K) \;:=\; -(K^{>C})^{\dagger\dagger} \;=\; \{-a^{\dagger\dagger} :\ a \in K^{>C}\}$$

of K. In $\mathbb{T}$ the sequence (z_n) given by

$$z_n \ := \ -\ell_n^{\dagger\dagger} \ = \ \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}$$

is cofinal in $\mathrm{Z}(\mathbb{T})$. As with most of this paper we omit proofs for what we claim below: these proofs are either straightforward, or very similar to proofs of analogous results in Section 5, or would require many extra pages.

Lemma 6.1. *Suppose K is Liouville closed. Then $\mathrm{Z}(K)$ is closed downward: whenever $f \in K$ and $f < g \in \mathrm{Z}(K)$, then $f \in \mathrm{Z}(K)$. For $f \in K^\times$ we have*

$$f \in \mathrm{I}(K) \iff -f^\dagger \notin \mathrm{Z}(K).$$

This follows easily from results in [2] and [3]. In particular, $\mathrm{I}(\mathbb{T})$ is quantifier-free definable from $\mathrm{Z}(\mathbb{T})$ in the $\mathcal{L}_{\mathbb{T},\mathrm{a}}$-structure $\mathbb{T}$. A refinement of the proof of Proposition 5.4 shows that we cannot reverse here the roles of $\mathrm{I}(\mathbb{T})$ and $\mathrm{Z}(\mathbb{T})$:

Lemma 6.2. $\mathrm{Z}(\mathbb{T})$ *is not quantifier-free definable in the $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure* $\mathbb{T}$.

Let $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{Z}}$ be the language $\mathcal{L}_{\mathbb{T},\mathrm{a}}$ augmented by a unary predicate symbol Z, to be interpreted in $\mathbb{T}$ as $Z(\mathbb{T})$. We have not checked the details, but we expect that what we proved in Section 5 for $\mathbb{T}$ as $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$-structure goes through for $\mathbb{T}$ as $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{Z}}$-structure. However, we run into a new obstruction involving the function ϱ. To explain this, we first summarize some basic facts about this function on $\mathbb{T}$:

Lemma 6.3. *The restriction of $\varrho\colon \mathbb{T} \to \mathbb{T}$ to $\mathrm{Z}(\mathbb{T})$ is strictly decreasing and has the intermediate value property. Also, $\varrho(\mathbb{T}) = \varrho(\mathrm{Z}(\mathbb{T}))$, and so the sequence $\big(\varrho(z_n)\big)$ is strictly decreasing and coinitial in $\varrho(\mathbb{T})$.*

We need the following strengthening of Theorem 4.11, where the "iterated logarithms" ℓ_λ are as in that theorem:

Theorem 6.4. *Suppose the H-field K is amenable. Then $\big(\varrho(-\ell_\lambda^{\dagger\dagger})\big)$ has a pseudolimit a in an immediate H-field extension $K\langle a\rangle$ of K with the following universal property: if b is any pseudolimit of $\big(\varrho(-\ell_\lambda^{\dagger\dagger})\big)$ in any immediate pre-H-field extension of K, then there is a unique isomorphism $K\langle a\rangle \to K\langle b\rangle$ over K of ordered valued differential fields sending a to b.*

Using also a result from [5], this theorem has the following consequence:

Corollary 6.5. $\varrho(\mathbb{T})$ *is not quantifier-free definable in the $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{Z}}$-structure* $\mathbb{T}$.

The next candidate of a language in which $\mathbb{T}$ might eliminate quantifiers is the extension $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{Z},\mathrm{R}}$ of $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{Z}}$ by a unary predicate symbol R, interpreted in $\mathbb{T}$ by $\varrho(\mathbb{T})$. At this stage we do not know any obstruction to this possibility.

Propositions 5.8 and 5.10 go through with $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{Z},\mathrm{R}}$ replacing $\mathcal{L}_{\mathbb{T},\mathrm{a},\mathrm{I}}$: the same proofs work since $\mathrm{Z}(\mathbb{T})$ and $\varrho(\mathbb{T})$ are convex subsets of $\mathbb{T}$.

Notes

1. Strictly speaking, any valued field isomorphic to such a generalized power series field is also considered as a Hahn field in this paper.

2. The English translation given here is ours; the original sentences are on p. 148. We also used our notations $\mathbb{T}$ and $\mathbb{T}^{\mathrm{as}}$ instead of Écalle's $\mathbb{R}[[[x]]]$ and $\mathbb{R}\{\{\{x\}\}\}$.

3. The prefix H honors the pioneers Hahn, Hardy, and Hausdorff. Arguably, Borel's work [11] in this vein is even more significant, but his name doesn't start with H. One could go still further back, to du Bois-Reymond's paper [10], a source of inspiration for Hardy [24].

4. This non-zero requirement was inadvertently dropped on p. 580 of [2].

5. Formally, an asymptotic couple is an ordered abelian group Γ equipped with a valuation $\psi\colon \Gamma^{\neq} \to \Gamma$ such that $\psi(\alpha) < \beta + \psi(\beta)$ for all $\alpha, \beta \in \Gamma^{>}$.

6. "AKE" stands for "Ax-Kochen-Eršov".

7. The term "obstruction" is often used to refer to a non-trivial (co)homology class. In fact, the vanishing of a homology group leads to the elimination of a quantifier: this vanishing means that the existential condition on a cycle c to be a boundary is equivalent to the quantifier-free condition on c that its boundary vanishes.

References

[1] M. Aschenbrenner and L. van den Dries, *Closed asymptotic couples,* J. Algebra **225** (2000), 309–358. 7, 13

[2] M. Aschenbrenner and L. van den Dries, *H-fields and their Liouville extensions,* Math. Z. **242** (2002), 543–588. 11, 13, 22, 26, 27, 28, 29

[3] M. Aschenbrenner and L. van den Dries, *Liouville closed H-fields,* J. Pure Appl. Algebra **197** (2005), 83-139. 7, 10, 19, 21, 22, 25, 28

[4] M. Aschenbrenner and L. van den Dries, *Asymptotic differential algebra,* in: O. Costin, M. D. Kruskal, A. Macintyre (eds.), *Analyzable Functions and Applications,* 49–85, Contemp. Math., vol. 373, Amer. Math. Soc., Providence, RI (2005). 1, 2, 3, 9

[5] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven, *Differentially algebraic gaps,* Selecta Math. **11** (2005), 247–280. 18, 28

[6] J. Ax and S. Kochen, *Diophantine problems over local fields,* III. *Decidable fields,* Ann. of Math. (2) **83** (1966), 437–456. 3

[7] J. Ax and S. Kochen, *Diophantine problems over local fields,* I*,* Amer. J. Math. **87** (1965), 605–630. 13

[8] S. Azgin and L. van den Dries, *Elementary theory of valued fields with a valuation-preserving automorphism,* J. Inst. Math. Jussieu **10** (2011), no. 1, 1–35. 13

[9] L. Bélair, A. Macintyre, and T. Scanlon, *Model theory of the Frobenius on the Witt vectors,* Amer. J. Math. **129** (2007), no. 3, 665–721. 13, 17

[10] P. du Bois-Reymond, *Ueber asymptotische Werthe, infinitäre Approximationen und infinitäre Auflösung von Gleichungen,* Math. Ann. **8** (1875), 362–414. 29

[11] É. Borel, *Mémoire sur les séries divergentes,* Ann. Sci. École Norm. Sup. **16** (1899), 9–131. 29

[12] M. Boshernitzan, *New "orders of infinity",* J. Analyse Math. **41** (1982), 130–167. 9

[13] L. van den Dries, *Limit sets in o-minimal structures*, in: M. Edmundo et al. (eds.), *Proceedings of the RAAG Summer School Lisbon 2003: O-minimal Structures,* 172–215, Lecture Notes in Real Algebraic and Analytic Geometry, Cuvillier Verlag, Göttingen, 2005. 20

[14] L. van den Dries and A. Lewenberg, *T-convexity and tame extensions*, J. Symbolic Logic **60** (1995), no. 1, 74–102. 26

[15] L. van den Dries, A. Macintyre, and D. Marker, *Logarithmic-exponential power series,* J. London Math. Soc. (2) **56** (1997), no. 3, 417–434. 4, 5

[16] L. van den Dries, A. Macintyre, and D. Marker, *Logarithmic-exponential series,* Ann. Pure Appl. Logic **111** (2001), 61–113. 1, 3, 5, 8

[17] B. Dahn and P. Göring, *Notes on exponential-logarithmic terms,* Fund. Math. **127** (1987), no. 1, 45–50. 3

[18] J. Denef and L. Lipshitz, *Power series solutions of algebraic differential equations,* Math. Ann. **267** (1984), no. 2, 213–238. 3

[19] J. Écalle, *Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac,* Actualités Mathématiques, Hermann, Paris, 1992. 3, 5, 6, 8

[20] G. A. Edgar, *Transseries for beginners,* Real Anal. Exchange **35** (2010), no. 2, 253–309. 5

[21] Ju. L. Eršov, *On the elementary theory of maximal normed fields*, Soviet Math. Dokl. **6** (1965), 1390–1393. 13

[22] A. Fornasiero, *Tame structures and open cores,* preprint (2010). 26

[23] N. Guzy and F. Point, *Topological differential fields,* Ann. Pure Appl. Logic **161** (2010), no. 4, 570–598. 22

[24] G. H. Hardy, *Orders of Infinity*, 2nd ed., Cambridge Univ. Press, Cambridge (1924). 29

[25] E. Hille, *Ordinary Differential Equations in the Complex Domain,* Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney, 1976. 9

[26] J. van der Hoeven, *Asymptotique Automatique,* Thèse, École Polytechnique, Paris (1997). 4

[27] J. van der Hoeven, *Operators on generalized power series*, Illinois J. Math. **45** (2001), no. 4, 1161–1190. 8

[28] J. van der Hoeven, *Transseries and Real Differential Algebra,* Lecture Notes in Math., vol. 1888, Springer-Verlag, Berlin, 2002. 3, 5, 7, 8, 15, 19, 20

[29] J. van der Hoeven, *Transserial Hardy fields*, in: F. Cano et al. (eds.), *Differential Equations and Singularities. 60 years of J.M. Aroca,* 453–487, Astérisque **323**, S.M.F., 2009. 6

[30] E. Hrushovski, K. Peterzil, A. Pillay, *Groups, measures, and the NIP,* J. Amer. Math. Soc. **21** (2008), no. 2, 563–596. 21

[31] F.-V. Kuhlmann, S. Kuhlmann, and S. Shelah, *Exponentiation in power series fields,* Proc. Amer. Math. Soc. **125** (1997), 3177–3183. 5

[32] J.-P. Ressayre, *La théorie des modèles, et un petit problème de Hardy,* in: J.-P. Pier (ed.), *Development of Mathematics 1950–2000,* 925–938, Birkhäuser Verlag, Basel, 2000. 3, 4

[33] M. Rosenlicht, *On the value group of a differential valuation,* Amer. J. Math. **101** (1979), 258–266. 12

[34] M. Rosenlicht, *Differential valuations,* Pacific J. Math. **86** (1980), 301–319. 12

[35] M. Rosenlicht, *On the value group of a differential valuation,* II, Amer. J. Math. **103** (1981), 977–996. 12

[36] M. Rosenlicht, *The rank of a Hardy field,* Trans. Amer. Math. Soc. **280** (1983), no. 2, 659–671. 11

[37] M. Rosenlicht, *Asymptotic solutions of* $Y'' = F(x)Y$, J. Math. Anal. Appl. **189** (1995), no. 3, 640–650. 9

[38] T. Scanlon, *A model complete theory of valued D-fields,* J. Symbolic Logic **65** (2000), no. 4, 1758–1784. 13, 14, 15

[39] S. Shelah, *Stability, the f.c.p., and superstability; model theoretic properties of formulas in first-order theories,* Ann. Math. Logic **3** (1971), no. 3, 271–362. 21

[40] A. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function,* J. Amer. Math. Soc. **9** (1996), 1051–1094. 3, 5

Acknowledgments

The first-named author was partially supported by NSF Grants DMS-0556197 and DMS-0969642. All three authors thank the Fields Institute, Toronto, for its hospitality during the Thematic Program on o-minimal Structures and Real Analytic Geometry (2009). Some of the work reported on here was done on that occasion.

University of California, Los Angeles
Los Angeles, CA 90955
U.S.A.
matthias@math.ucla.edu

University of Illinois at Urbana-Champaign
Urbana, IL 61801
U.S.A.
vddries@math.uiuc.edu

École Polytechnique
91128 Palaiseau Cedex
France
vdhoeven@lix.polytechnique.fr