Efficient accelero-summation of holonomic functions

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Let $L \in \mathbb{K}(z)[\partial]$ be a linear differential operator, where $\mathbb{K}$ is the field of algebraic numbers. A holonomic function over $\mathbb{K}$ is a solution f to the equation $L f = 0$. We will also assume that f admits initial conditions in $\mathbb{K}$ at a non-singular point $z \in \mathbb{K}$.

Given a broken-line path $\gamma = z \rightsquigarrow z'$ between z and z', which avoids the singularities of L and with vertices in $\mathbb{K}$, we have shown in a previous paper [Hoeven, 1999] how to compute n digits of the analytic continuation of f along γ in time $O(n \log^3 n \log \log n)$. In a second paper [Hoeven, 2001b], this result was generalized to the case when z' is allowed to be a regular singularity, in which case we compute the limit of f when we approach the singularity along γ.

In the present paper, we treat the remaining case when the end-point of γ is an irregular singularity. In fact, we will solve the more general problem to compute "singular transition matrices" between non standard points above a singularity and regular points in $\mathbb{K}$ near the singularity. These non standard points correspond to the choice of "non-singular directions" in Écalle's accelero-summation process.

We will show that the entries of the singular transition matrices may be approximated up to n decimal digits in time $O(n \log^4 n \log \log n)$. As a consequence, the entries of the Stokes matrices for L at each singularity may be approximated with the same time complexity.

1. INTRODUCTION

Definitions

Let $\mathbb{K}$ be a subfield of $\mathbb{C}$. A *holonomic function* over $\mathbb{K}$ is a solution f to a linear differential equation $L f = 0$, where $L = \partial^r + L_{r-1} \partial^{r-1} + \cdots + L_0 \in \mathbb{K}(z)[\partial]$ is a monic linear differential operator of order r. Many classical special functions, such as exp, log, sin, cos, erf, hypergeometric functions, Bessel functions, the Airy function, etc. are holonomic. Moreover, the class of holonomic functions is stable under many operations, such as addition, multiplication, differentiation, integration and postcomposition with algebraic functions. In the sequel, and unless stated otherwise, we will assume that $\mathbb{K}$ is the field of algebraic numbers. We will say that f has *initial conditions in* $\mathbb{K}$ if $F(z) = (f(z), \ldots, f^{(r-1)}(z)) \in \mathbb{K}^r$ for a certain non-singular point $z \in \mathbb{K}$.

In this paper, we will be concerned with the efficient multidigit evaluation of limits of holonomic functions at irregular singularities. For this, it will be convenient to introduce some terminology. We say that $z \in \mathbb{C}$ is *effective*, if there exists an *approximation algorithm*, which takes $\varepsilon \in \mathbb{Q}^{>}$ on input and which returns a dyadic approximation $\tilde{z} \in (\mathbb{Z} + \mathrm{i}\,\mathbb{Z})\, 2^{\mathbb{Z}}$ with $|\tilde{z} - z| < \varepsilon$. Inside a computer, an effective complex number z is represented as an object with a method which corresponds to its approximation algorithm [Hoeven, 2005]. We denote by $\mathbb{C}^{\text{eff}}$ the set of effective complex numbers.

The *time complexity* of $z \in \mathbb{C}^{\text{eff}}$ is the time complexity of its approximation algorithm, expressed in terms of $n = -\log \varepsilon$. If an approximation algorithm has time complexity $T(n)$, then we call it a $T(n)$-approximation algorithm. An effective number is said to be *fast*, if it admits an approximation algorithm with a time complexity of the form $O(n \log^{O(1)} n)$. We denote by $\mathbb{C}^{\text{fast}}$ the set of such numbers. A partial function $f: (\mathbb{C}^{\text{eff}})^n \rightarrowtail \mathbb{C}^{\text{eff}}$ is said to be *fast* if it maps $(\mathbb{C}^{\text{fast}})^n$ into $\mathbb{C}^{\text{fast}}$. For instance, multiplication is fast [Schönhage and Strassen, 1971], since two n-bit numbers can be multiplied in time $M(n) = O(n \log n \log \log n)$. Implicitly defined functions in terms of fast functions, like division, are also fast, as a result of Newton's method.

Whenever the coefficients of L admit singularities, then solutions f to $L f = 0$ are typically multivalued functions on a Riemann surface. From an effective point of view, points on such a Riemann surface may be addressed via *broken-line paths* $\gamma = z \rightsquigarrow z' = z_0 \rightarrow z_1 \rightarrow \cdots \rightarrow z_l$ starting at the point $z = z_0$ where we specified the initial conditions for f. Each *straight-line segment* $z_i \rightarrow z_{i+1}$ should be sufficiently short, so that the disk with center z_i and radius $|z_{i+1} - z_i|$ contains no singularities. Given such a path, we will denote by $f(\gamma)$ the evaluation of f at the endpoint z' of γ, as obtained via analytic continuation.

Previous work

It was first noticed by Brent [Brent, 1976a, Section 6] that the constant e admits an efficient $O(M(n) \log n)$-approximation algorithm based on binary splitting. This result was obtained by analogy with Schönhage's fast algorithm for radix conversion. The paper also mentions efficient algorithms for the computation of more general exponentials, although this direction was not investigated in more detail, probably because even more efficient $O(M(n) \log n)$-algorithms were discovered shortly afterwards [Brent, 1976b].

The binary splitting algorithm was generalized to arbitrary holonomic over $\mathbb{Q}$ in [Chudnovsky and Chudnovsky, 1990]. It was shown there that, given a holonomic function f over $\mathbb{Q}$ with initial conditions in $\mathbb{Q}$, and a broken-line path $\gamma = z \rightsquigarrow z'$ as above with $z, z' \in \mathbb{Q}$, the number $f(\gamma)$ admits an $O(M(n) \log^2 n)$-approximation algorithm. In the case when z' is a more general effective number with a $T(n)$-approximation algorithm, it was also shown that $f(\gamma)$ admits an $O(T(n + O(1)) + M(n) \log^3 n)$-approximation algorithm. In particular, the restriction of a holonomic function to an open domain of $\mathbb{C}^{\text{eff}}$ is fast. By what precedes, this result is extremely interesting for the efficient multidigit evaluation of many special functions. Special cases and a few extensions were rediscovered independently by several authors [Karatsuba, 1991; Karatsuba, 1993; Karatsuba, 1995; Karatsuba, 2000; Hoeven, 1997; Hoeven, 1999; Haible and Papanikolaou, 1997].

Remark 1.1. An early hint to the existence of fast algorithms for the evaluation of holonomic functions occurred in [Gosper and Schroeppel, 1972]. It is plausible that the authors had something like the binary splitting algorithm in mind (the announced complexity is the right one up to a factor $O(\log \log n)$), but no details are provided.

Our first paper [Hoeven, 1999] on the subject contained three improvements with respect to [Chudnovsky and Chudnovsky, 1990]. First, we noticed the possibility to work over the algebraic numbers $\mathbb{K}$ instead of $\mathbb{Q}$, which allows for the fast evaluation of constants like $\Gamma(\sqrt{2})$. Secondly, we improved the above factor of $\log^3 n$ (for the evaluation in arbitrary points) to $\log^2 n \log\log n$. Finally, the evaluation of $f(\gamma)$ depends on a certain number of bounds, which were assumed to exist empirically in [Chudnovsky and Chudnovsky, 1990]. In [Hoeven, 1999], it was shown that all necessary bounds can be computed effectively, as a function of the operator L and the path γ. Stated otherwise, we showed that there exists an algorithm which takes L, γ and the initial conditions for f at z on input, and which *computes* $f(\gamma)$ (as an object with a $O(M(n)\log^2 n)$-approximation algorithm).

In a second paper [Hoeven, 2001b], we continued our studies by showing how to efficiently evaluate the limit of f along a broken-line path γ which ends in a regular singular point z'. This extension allows for the efficient evaluation of multiple zeta values, Bessel functions (whose initial conditions are specified in a regular singular point) and many other interesting transcendental constants. Some special cases of this more general result were obtained before in [Karatsuba, 1993; Karatsuba, 1995; Haible and Papanikolaou, 1997].

A related problem to the evaluation of f at the end-point of a broken line path γ is the computation of "transition matrices" along γ. Given a path $\gamma = z \rightsquigarrow z'$ from z to z', the "initial conditions" $F(z') = (f(z'), \ldots, f^{(r-1)}(z'))$ of f at z' depend linearly on the "initial conditions" $F(z) = (f(z), \ldots, f^{(r-1)}(z))$ at z. Hence, when considering $F(z)$ and $F(z')$ as column vectors, there exists a unique scalar matrix $\Delta_{z\rightsquigarrow z'} = \Delta^L_{z\rightsquigarrow z'}$ with

$$F(z') = \Delta_{z\rightsquigarrow z'} F(z),$$

which is called the *transition matrix* along γ for L. The relation $\Delta_{z\rightsquigarrow z''} = \Delta_{z'\rightsquigarrow z''}\Delta_{z\rightsquigarrow z'}$ make transition matrices well-suited for the process of analytic continuation. Therefore, most algorithms from [Chudnovsky and Chudnovsky, 1990; Hoeven, 1999] rely on the computation of transition matrices. In [Hoeven, 2001b], this concept was further generalized to the case when γ is allowed to pass through regular singularities.

Main results

In this paper, we will be concerned with the computation of the limits of holonomic functions in irregular singularities and, more generally, with the computation of generalized transition matrices along paths which are allowed to pass through irregular singularities. The algorithms are based on an effective counterpart of the accelero-summation process, as introduced by Écalle [Écalle, 1987; Écalle, 1992; Écalle, 1993; Braaksma, 1991; Borel, 1928; Ramis, 1978]. Since this process is not completely straightforward, let us first motivate its use for our application.

Consider a holonomic function f with an irregular singularity at the origin. Assume that f admits a (usually divergent) asymptotic expansion $f = f_0 + f_1 z + \cdots \in \mathbb{K}[[z]]$ in a sector $\mathcal{S}$ near the origin. Assume also that we have a bound B for $|f(z)|$ on $\mathcal{S}$. Given $z_0 \in \mathcal{S} \cap \mathbb{K}$, we are interested in computing $I = \int_{z_0}^0 f(t)\,\mathrm{d}t$. Notice that $\varphi(z) = \int_{z_0}^z f(t)\,\mathrm{d}t$ is a holonomic function, so the computation of I is a particular instance of the problem of computing the limit of a holonomic function in an irregular singularity.

In order to find $\tilde{I} \in (\mathbb{Z} + \mathrm{i}\,\mathbb{Z})\,2^{\mathbb{Z}}$ with $|\tilde{I} - I| < \varepsilon$, for a given $\varepsilon \in \mathbb{Q}^>$, it clearly suffices to compute $\varphi(z_1)$ with precision $\varepsilon/2$ at a point z_1 with $|z_1| < \varepsilon/(2B)$. This can be done using the analytic continuation algorithm from [Chudnovsky and Chudnovsky, 1990; Hoeven, 1999]. However, since the equation $Lf = 0$ may have other solutions g with growth rates of the form $\log|g| = O(|1/z|^\kappa)$ at $z = 0$, the transition matrix between z_0 and z_1 may contain entries of size $\mathrm{e}^{O((1/\varepsilon)^\kappa)}$. The computation of $n = O(-\log\varepsilon)$ digits of $\tilde{I}$ may therefore require a time $\mathrm{e}^{O(n)}$.

The situation gets a bit better, if we want to compute $J = \int_{z_0}^{0} f(t)\, \mathrm{e}^{-1/t}\, \mathrm{d}t$ instead of I, where we assume that $z_0 \in \mathbb{R}^{>}$. In that case, using a similar method as above, we may choose $z_1 \in \mathbb{Q}^{>}$ with $z_1 = O(-\log \varepsilon)$. Consequently, the computation of $n = O(-\log \varepsilon)$ digits of J requires a time $O(n^{\kappa} \log^{O(1)} n)$, where $\kappa \geqslant 1$. Although this already yields a polynomial time algorithm, we are really interested in fast approximation algorithms.

Roughly speaking, the main result of this paper is that the computation of an arbitrary limit of a holonomic function at an irregular singularity may be reduced to the computation of a finite number of other, more special limits. These special limits, which are similar to J above, with $\kappa = 1$, will be shown to admit fast $O(M(n) \log^3 n)$-approximation algorithms. More generally, we will generalize the concept of transition matrices, so as to allow for broken-line paths through irregular singularities. In particular, Stokes matrices may be seen as such "singular transition matrices". We will both show that singular transition matrices may be computed as a function of L and a singular broken-line path γ, and that their entries admit $O(M(n) \log^3 n)$-approximation algorithms.

This result admits several interesting applications besides the computation of limits of holonomic functions in singularities. For instance, we may consider solutions f to $L f = 0$ with a prescribed asymptotic behaviour in one or several singularities and recover the function from these "singular initial conditions" and one or more singular transition matrices. In [Hoeven, 2007a], it has also been shown that the possibility to compute the entries of Stokes matrices can be used for the numeric computation of the differential Galois group of L. In particular, we obtained an efficient algorithm for factoring L.

Our results can be compared to the only previous work on effective resummation that we are aware of [Thomann, 1995]. First of all, the current paper has the advantage that all necessary error bounds for guaranteeing a certain precision are computed automatically. Secondly, the almost linear time complexity is far better than those achieved by other numerical algorithms, like Taylor series expansions (of complexity $\approx O(n^2)$, at best) or the Runge-Kutta method (of complexity $\mathrm{e}^{O(\sqrt[4]{n})}$).

Quick overview

Let us briefly outline the structure of this paper. In section 2, we begin with a survey of the accelero-summation process. The idea is to give a meaning to the evaluation of a divergent formal solution to $L f = 0$ *via* a succession of transformations. We first make the formal solution convergent at the origin by applying a formal Borel transform. We next apply a finite number of integral transforms called "accelerations" followed by an a Laplace transform. At the end, we obtain an analytic solution to $L f = 0$ in a sector near the origin, which admits the divergent formal solution as its asymptotic expansion.

The material in section 3 is more or less classical. We first recall the definition of the Newton polygon of L in a singularity, as well as the relationship between its slopes and the shape of formal solutions to $L f = 0$. In particular, the steepest slope gives us information about the maximal growth rate κ of solutions. We next study the Newton polygons of other operators related to L, like the operators which annihilate the Borel transforms of solutions to L.

In section 4, we recall several stability properties [Stanley, 1980] for holonomic functions and constants, as well as their effective counterparts. In particular, we will show that the integrands involved in the accelero-summation procedure are holonomic and how to compute vanishing operators for them. Using the results from section 3, these operators will be seen to have the required growth rates at infinity.

In sections 5, we show how to compute uniform bounds for the transition matrices in suitable sectors near infinity. In section 6, these bounds will be used for the efficient evaluation of integrals with exponential decrease. In section 7, the different techniques are assembled into an effective and efficient accelero-summation procedure.

None of the algorithms in this paper have been implemented yet. Nevertheless, at least some of the algorithms should be implemented inside the standard library of the upcoming Mathemagix system [Hoeven et al., 2002] and any help would be appreciated.

Notations

The following notations will frequently be used in this paper:

$\dot{\mathbb{C}}$	Riemann surface of log
$K^{\square}$	Subset $\{x \in K: x \,\square\, 0\}$ of K, with $\square \in \{\neq, >, \geqslant\}$
$\mathcal{D}_{c,r}, \bar{\mathcal{D}}_{c,r}$	Open and closed disks with center c and radius r
$\bar{\mathcal{S}}^0_{\theta,\alpha,R}$	Closed sector $\{z \in \dot{\mathbb{C}}: \lvert\arg z - \theta\rvert \leqslant \alpha, \lvert z\rvert \leqslant R\}$ at the origin
$\bar{\mathcal{S}}^\infty_{\theta,\alpha,R}$	Closed sector $\{z \in \dot{\mathbb{C}}: \lvert\arg z - \theta\rvert \leqslant \alpha, \lvert z\rvert \geqslant R\}$ at infinity
$\tilde{\mathcal{B}}_z$	Formal Borel transform w.r.t. z
$\hat{\mathcal{L}}^\theta_z, \check{\mathcal{L}}^\theta_z$	Analytic Laplace transform w.r.t. z (for minors and majors)
$\hat{\mathcal{A}}^\theta_{k,k'}, \check{\mathcal{A}}^\theta_{k,k'}$	Acceleration operators (for minors and majors)
$\mathcal{M}_\varphi L$	Multiplicative conjugation of L with $\mathrm{e}^{\int \varphi}$
$\mathcal{P}_p L$	Compositional conjugation of L with z^p
$\mathcal{Q}_\omega L$	Compositional conjugation of L with ωz
Δ^L_γ	Transition matrix for L along γ

The operators $\tilde{\mathcal{B}}_z$, $\hat{\mathcal{L}}^\theta_z$, $\check{\mathcal{L}}^\theta_z$, $\hat{\mathcal{A}}^\theta_{k,k'}$, $\check{\mathcal{A}}^\theta_{k,k'}$ are defined in sections 2.1 and 2.2. The transformations $\mathcal{M}_\varphi$, $\mathcal{P}_p$ and $\mathcal{Q}_\omega$ are introduced in sections 3.2 and 4.2.4. Transition matrices are defined in section 4.3.

2. Reminders on the accelero-summation process

In this section we survey the process of accelero-summation, give some explicit bounds for the acceleration kernels, as well as the interpretation of the accelero-summation process in terms of "majors". We have aimed to keep our survey as brief as possible. It is more elegant to develop this theory using resurgent functions and resurgence monomials [Écalle, 1985; Candelberger et al., 1993]. For a more complete treatment, we refer to [Écalle, 1987; Écalle, 1992; Écalle, 1993; Braaksma, 1991; Martinet and Ramis, 1991].

2.1. The accelero-summation process

Let $\mathbb{C}[[z^{\mathbb{Q}^>}]]$ be the differential $\mathbb{C}$-algebra of infinitesimal Puiseux series in z for $\delta = z\,\partial$ and consider a formal power series solution $\tilde{f} \in \mathbb{O} = \mathbb{C}[[z^{\mathbb{Q}^>}]][\log z]$ to a linear differential equation over $\mathbb{K}(z)$. *When applicable*, the process of accelero-summation enables to associate an analytic meaning f to $\tilde{f}$ in a sector near the origin of the Riemann surface $\dot{\mathbb{C}}$ of log, even in the case when $\tilde{f}$ is divergent. Schematically speaking, we obtain f through a succession of transformations:

$$\begin{array}{ccccccccc} \tilde{f} & & & & & & & & f \\ \tilde{\mathcal{B}}_{z_1}\big\downarrow & & & & & & & & \big\uparrow \hat{\mathcal{L}}^{\theta_p}_{z_p} \\ \hat{f}_1 & \xrightarrow[\hat{\mathcal{A}}^{\theta_1}_{z_1 \to z_2}]{} & \hat{f}_2 & \longrightarrow & \cdots & \longrightarrow & \hat{f}_{p-1} & \xrightarrow[\hat{\mathcal{A}}^{\theta_{p-1}}_{z_{p-1} \to z_p}]{} & \hat{f}_p \end{array} \qquad (2.1)$$

Each $\hat{f}_i$ is a "resurgent function" which realizes $\tilde{f}_i(z_i) = \tilde{f}(z)$ in the "convolution model" with respect to the i-th "critical time" $z_i = \sqrt[k_i]{z}$ (with $k_i \in \mathbb{Q}^>$ and $k_1 > \cdots > k_p$). In our case, $\hat{f}_i$ is an analytic function which admits only a finite number of singularities above $\mathbb{C}$. In general, the singularities of a resurgent function are usually located on a finitely generated grid. Let us describe the transformations $\tilde{\mathcal{B}}$, $\hat{\mathcal{A}}^{\theta_i}_{z_i \to z_{i+1}}$ and $\hat{\mathcal{L}}^{\theta_p}_{z_p}$ in more detail.

The Borel transform We start by applying the *formal Borel transform* to the series $\tilde{f}_1(z_1) = \tilde{f}(z) = \sum_{\sigma,r} \tilde{f}_{1,\sigma,r}\, z_1^{\sigma} \log^r z_1 \in \mathbb{C}[[z_1^{\mathbb{Q}^>}]][\log z_1]$. This transformation sends each $z_1^{\sigma} \log^r z_1$ to

$$(\tilde{\mathcal{B}}_{z_1} z_1^{\sigma} \log^r z_1)(\zeta_1) = \zeta_1^{\sigma-1} \sum_{i=0}^{r} \binom{r}{i} \gamma^{(r-i)}(\sigma) \log^i \zeta_1, \tag{2.2}$$

where $\gamma(\sigma) = 1/\Gamma(\sigma)$, and extends by strong linearity:

$$\hat{f}_1(\zeta_1) = (\tilde{\mathcal{B}}_{z_1} \tilde{f}_1)(\zeta_1) = \sum_{\substack{\sigma \in \mathbb{Q}^> \\ r \in \mathbb{N}}} \tilde{f}_{1,r,\sigma}\, (\tilde{\mathcal{B}}_{z_1} z_1^{\sigma} \log^r z_1)(\zeta_1),$$

The result is a formal series $\hat{f}_1 \in \zeta_1^{-1} \mathbb{C}[[\zeta_1^{\mathbb{Q}^>}]][\log \zeta_1]$ in ζ_1 which converges near the origin of the Riemann surface $\dot{\mathbb{C}}$ of the logarithm. The formal Borel transform is a morphism of differential algebras which sends multiplication to the convolution product, i.e. $\tilde{\mathcal{B}}_{z_1}(f g) = (\tilde{\mathcal{B}}_{z_1} f) * (\tilde{\mathcal{B}}_{z_1} g)$, and differentiation ∂_z to multiplication by $-\zeta$. Intuitively speaking, the Borel transform is inverse to the Laplace transform defined below.

Accelerations Given $i < p$, the function $\hat{f}_i$ is defined near the origin of $\dot{\mathbb{C}}$, can be analytically continued on the axis $\mathrm{e}^{\theta_i \mathrm{i}} \mathbb{R}^> \subseteq \dot{\mathbb{C}}$, and admits a growth of the form $\hat{f}_i(\zeta_i) = \exp O(|\zeta_i|^{k_i/(k_i - k_{i+1})})$ at infinity. The next function $\hat{f}_{i+1}$ is obtained from $\hat{f}_i$ by an *acceleration* of the form

$$\hat{f}_{i+1}(\zeta_{i+1}) = (\hat{\mathcal{A}}^{\theta_i}_{z_i \to z_{i+1}} \hat{f}_i)(\zeta_{i+1}) = \int_{\zeta_i \in \mathrm{e}^{\theta_i \mathrm{i}} \mathbb{R}^>} \hat{f}_i(\zeta_i)\, \hat{K}_{k_i,k_{i+1}}(\zeta_i, \zeta_{i+1})\, \mathrm{d}\zeta_i, \tag{2.3}$$

where the acceleration kernel $\hat{K}_{k_i,k_{i+1}}$ is given by

$$\begin{aligned} \hat{K}_{k_i,k_{i+1}}(\zeta_i, \zeta_{i+1}) &= \frac{1}{2\pi\mathrm{i}} \int_{c-\infty\mathrm{i}}^{c+\infty\mathrm{i}} \mathrm{e}^{\zeta_{i+1} z_{i+1} - \zeta_i z_i}\, \mathrm{d}z_{i+1} \\ &= \frac{1}{\zeta_{i+1}} \hat{K}_{k_{i+1}/k_i}\left(\frac{\zeta_i}{\zeta_{i+1}^{k_{i+1}/k_i}}\right) \qquad (2.4) \\ \hat{K}_{\lambda}(\zeta) &= \frac{1}{2\pi\mathrm{i}} \int_{c-\infty\mathrm{i}}^{c+\infty\mathrm{i}} \mathrm{e}^{z - \zeta z^{\lambda}}\, \mathrm{d}z. \qquad (2.5) \end{aligned}$$

For large $\zeta \in \mathbb{R}^{\geqslant}$, we will show in section 2.4 below that

$$\hat{K}_{\lambda}(\zeta) \leqslant B \exp(-C\, \zeta^{1/(1-\lambda)})$$

for some constants $B, C > 0$. It follows that the acceleration $\hat{f}_{i+1}$ of $\hat{f}_i$ is well-defined for small ζ_{i+1} on $\mathrm{e}^{\phi \mathrm{i}} \mathbb{R}^>$, where $\phi = \theta_i\, k_i / k_{i+1}$. The set $\mathcal{D}_i \subseteq \mathbb{R}$ of directions θ such $\hat{f}_i$ admits a singularity on $\mathrm{e}^{\theta \mathrm{i}} \mathbb{R}^>$ is called the set of *Stokes directions* at the i-th critical time. Accelerations are morphisms of differential $\mathbb{C}$-algebras which preserve the convolution product. Intuitively speaking, one has $\hat{\mathcal{A}}^{\theta_i}_{z_i \to z_{i+1}} = \mathcal{B}_{z_{i+1}} \circ \hat{\mathcal{L}}^{\theta_i}_{z_i}$, where the Laplace transform $\hat{\mathcal{L}}^{\theta_i}_{z_i}$ is defined below.

THE LAPLACE TRANSFORM The last function $\hat{f}_p$ is defined near the origin of $\dot{\mathbb{C}}$, can be analytically continued on the axis $\mathrm{e}^{\theta_i \mathrm{i}}\,\mathbb{R}^{>} \subseteq \dot{\mathbb{C}}$ and admits at most exponential growth at infinity. The function f is now obtained using the analytic *Laplace transform*

$$f(z) = f_p(z_p) = (\hat{\mathcal{L}}_{z_p}^{\theta_p}\,\hat{f}_p)(z_p) = \int_{\zeta_p \in \mathrm{e}^{\theta_p \mathrm{i}}\mathbb{R}^{>}} \hat{f}_p(\zeta_p)\,\mathrm{e}^{-\zeta_p/z_p}\,\mathrm{d}\,\zeta_p. \tag{2.6}$$

For any sufficiently small z_p with $|\arg z_p - \theta_p| < \pi/2$, the value $f(z) = f_p(z_p)$ is well defined. The set $\mathcal{D}_p$ of Stokes directions is defined in a similar way as in the case of accelerations. The Laplace transform is a morphism of differential $\mathbb{C}$-algebras which is inverse to the Borel transform and sends the convolution product to multiplication.

Given tuples $\boldsymbol{k} = (k_1, \ldots, k_p)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$ of critical times $k_1 > \cdots > k_p$ in $\mathbb{Q}^{>}$ and directions $\theta_1 \in \mathcal{R}_1 := \mathbb{R} \setminus \mathcal{D}_1, \ldots, \theta_p \in \mathcal{R}_p := \mathbb{R} \setminus \mathcal{D}_p$, we say that a formal power series $\tilde{f} \in \mathbb{O}$ is *accelero-summable* in the multi-direction $\boldsymbol{\theta}$ if the above scheme yields an analytic function $f := \operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}} \tilde{f}$. For any $\alpha < k_p\,\pi/2$, this function is defined in a sufficiently small sector near $\dot{0}$ of the form $\bar{\mathcal{S}}_{k_p\theta_p,\alpha,R}$. We denote the set of accelero-summable power series of this kind by $\mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}}$.

The set $\mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}}$ forms a differential subring of $\mathbb{O}$ and the map $\tilde{f} \mapsto f$ for $\tilde{f} \in \mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}}$ is injective. If $\boldsymbol{k}'$ and $\boldsymbol{\theta}'$ are obtained from $\boldsymbol{k}$ and $\boldsymbol{\theta}$ by inserting a new critical time and an arbitrary direction, then we have $\mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}} \subsetneq \mathbb{O}_{\boldsymbol{k}',\boldsymbol{\theta}'}$. In particular, $\mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}}$ contains $\mathbb{O}_{\mathrm{cv}} = \mathbb{C}\{\{z^{\mathbb{Q}^{>}}\}\}[\log z]$, where $\mathbb{C}\{\{z^{\mathbb{Q}^{>}}\}\}$ denotes the ring of convergent infinitesimal Puiseux series. Assuming that each $\mathcal{D}_i$ is finite modulo $2\,\pi$, and setting $\boldsymbol{\mathcal{R}} := \mathcal{R}_1 \times \cdots \times \mathcal{R}_p$, we also denote $\mathbb{O}_{\boldsymbol{k},\boldsymbol{\mathcal{R}}} = \bigcap_{\boldsymbol{\theta} \in \boldsymbol{\mathcal{R}}} \mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}}$, $\mathbb{O}_{\boldsymbol{k}} = \bigcup_{\boldsymbol{\mathcal{R}}} \mathbb{O}_{\boldsymbol{k},\boldsymbol{\mathcal{R}}}$ and $\mathbb{O}_{\mathrm{as}} = \bigcup_{\boldsymbol{k}} \mathbb{O}_{\boldsymbol{k}}$.

Let $\mathfrak{E}$ be the group of elements e^P with $P \in \mathbb{K}[z^{\mathbb{Q}^{<}}]$ and denote by $\mathbb{S} = \mathbb{O}\,z^{\mathbb{K}}[\mathfrak{E}]$ the ring of all polynomials of the form $\tilde{f} = \sum_{\mathfrak{e} \in \mathfrak{E}} \tilde{f}_{\mathfrak{e}}\,\mathfrak{e}$ with $f_{\mathfrak{e}} \in \mathbb{O}\,z^{\mathbb{K}}$. The notion of accelero-summation extends to elements in $\mathbb{S}$ instead of $\mathbb{O}$. Indeed, given $\tilde{g} \in \mathbb{O}_{\boldsymbol{k},\boldsymbol{\theta}}$, $\sigma \in \mathbb{C}$, $\mathfrak{e} = \mathrm{e}^{P(1/\sqrt[p]{z})} \in \mathfrak{E}$, we may simply take $(\operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}}\,\tilde{g}\,z^{\sigma}\,\mathfrak{e})(z) = (\operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}}\,\tilde{g})(z)\,z^{\sigma}\,\mathfrak{e}$. It can be checked that this definition is coherent when replacing $\tilde{g}\,z^{\sigma}$ by $(z^k\,\tilde{g})\,z^{\sigma-k}$ for some $k \in \mathbb{Q}$. By linearity, we thus obtain a natural differential subalgebra $\mathbb{S}_{\boldsymbol{k},\boldsymbol{\theta}} \subseteq \mathbb{S}$ of accelero-summable transseries with critical times $\boldsymbol{k}$ and in the multi-direction $\boldsymbol{\theta}$. We also have natural analogues $\mathbb{S}_{\boldsymbol{k}}$ and $\mathbb{S}_{\mathrm{as}}$ of $\mathbb{O}_{\boldsymbol{k}}$ and $\mathbb{O}_{\mathrm{as}}$.

2.2. Majors and minors

In general, the acceleration and Laplace integrands are both singular at zero and at infinity. Much of the remainder of this paper is directly or indirectly concerned with the efficient integration near infinity. This leaves us with the integration at zero. A classical trick is to replace the integrand by a so called *major*. This allows us to replace the integral from zero to a point u close to zero by a contour integral around zero from $\mathrm{e}^{-2\pi\mathrm{i}}\,u$ to u. We will rapidly review this technique and refer to [Écalle, 1985; Candelberger et al., 1993; Écalle, 1992; Écalle, 1993] for details.

Consider an analytic germ $\hat{f}$ near the origin $\dot{0}$ of the Riemann surface $\dot{\mathbb{C}}$ of log. A *major* for $\hat{f}$ is an analytic germ $\check{f}$ with

$$\hat{f}(\zeta) = \check{f}(\zeta) - \check{f}(\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}).$$

The *minor* $\hat{f}$ only depends on the class $\bar{f}$ of $\check{f}$ modulo the set of regular germs at 0. We call $\bar{f}$ a *microfunction*. Given a regular germ φ, $\sigma \in \mathbb{Q}^{\geqslant}$ and $k \in \mathbb{N}$, the minor

$$\hat{f}(\zeta) = \varphi(\zeta)\,\zeta^{\sigma}\log^k \zeta$$

admits the major

$$\check{f} = \begin{cases} \varphi(\zeta)\,\zeta^{\sigma}\,P_k(\log\zeta) & \text{if } \sigma\in\mathbb{N}\\ \frac{1}{1-\mathrm{e}^{-2\pi\mathrm{i}\sigma}}\,\varphi(\zeta)\,\zeta^{\sigma}\,P_{\sigma,k}(\log\zeta) & \text{if } \sigma\notin\mathbb{N}\end{cases} \tag{2.7}$$

for certain polynomials $P_k(\log\zeta) = \frac{1}{2\pi\mathrm{i}\,(k+1)}\log^{k+1}\zeta + \cdots$ and $P_{\sigma,k}(\log\zeta) = \log^k\zeta + \cdots$. More generally, if $\hat{f}$ is locally integrable in a sector containing a point u near $\dot{0}$, then

$$\check{f}(\zeta) = \frac{-1}{2\pi\mathrm{i}}\int_0^u \frac{\hat{f}(\xi)}{\xi-\zeta}\,\mathrm{d}\xi \tag{2.8}$$

is a major for $\hat{f}$. The class of $\check{f}$ does not depend on the choice of u.

Given majors $\check{f}_i$ for the $\hat{f}_i$ from section 2.1, we may now replace (2.3) and (2.6) by

$$\hat{f}_{i+1}(\zeta_{i+1}) \;=\; \int_{\mathcal{H}_{\theta_i}} \check{f}_i(\zeta_i)\,\hat{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\zeta_i \tag{2.9}$$

$$f_p(z_p) = (\check{\mathcal{L}}_{z_p}^{\theta_p} f_p)(z_p) \;=\; \int_{\mathcal{H}_{\theta_p}} \check{f}_p(\zeta_p)\,\mathrm{e}^{-\zeta_p/z_p}\,\mathrm{d}\zeta_p, \tag{2.10}$$

where $\mathcal{H}_\theta$ stands for the contour (see figure 2.1 below) which consists of $\mathcal{H}_\theta^-$ from $\mathrm{e}^{(\theta-2\pi)\mathrm{i}}\infty$ to $\mathrm{e}^{(\theta-2\pi)\mathrm{i}}\varepsilon$ (for some small $\varepsilon>0$), followed by $\mathcal{C}_\theta$ from $\mathrm{e}^{(\theta-2\pi)\mathrm{i}}\varepsilon$ around 0 to $\mathrm{e}^{\theta\mathrm{i}}\varepsilon$, and $\mathcal{H}_\theta^+$ from $\mathrm{e}^{\theta\mathrm{i}}\varepsilon$ to $\mathrm{e}^{\theta\mathrm{i}}\infty$.

Using the formula (2.8) in combination with (2.9) leads to the direct expression

$$\check{f}_{i+1}(\zeta_{i+1}) = (\check{\mathcal{A}}_{k_i,k_{i+1}}^{\theta_i}\,\check{f}_i)(\zeta_{i+1}) = \int_{\mathcal{H}_{\theta_i}} \check{f}_i(\zeta_i)\,\check{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\zeta_i \tag{2.11}$$

of $\check{f}_{i+1}$ in terms of $\check{f}_i$, where

$$\check{K}(\zeta_i,\zeta_{i+1}) = {}^u\check{K}(\zeta_i,\zeta_{i+1}) = \frac{-1}{2\pi\mathrm{i}}\int_0^u \frac{\hat{K}_{k_i,k_{i+1}}(\zeta_i,\xi)}{\xi-\zeta_{i+1}}\,\mathrm{d}\xi.$$

The integrals (2.11) and (2.10) further decompose into

$$\begin{aligned}\check{f}_{i+1}(\zeta_{i+1}) \;&=\; \int_{\mathcal{C}_{\theta_i}} \check{f}_i(\zeta_i)\,\check{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\zeta_i + \\ &\quad\; \int_{\mathcal{H}_{\theta_i}^+} \hat{f}_i(\zeta_i)\,\check{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\zeta_i \end{aligned}\tag{2.12}$$

$$f_p(z_p) \;=\; \int_{\mathcal{C}_{\theta_p}} \check{f}_p(\zeta_p)\,\mathrm{e}^{-\zeta_p/z_p}\,\mathrm{d}\zeta_p + \int_{\mathcal{H}_{\theta_p}^+} \hat{f}_p(\zeta_p)\,\mathrm{e}^{-\zeta_p/z_p}\,\mathrm{d}\zeta_p. \tag{2.13}$$

More generally, differentiating $m\in\mathbb{N}$ times w.r.t. ζ_{i+1}, we obtain the following formulas, on which we will base our effective accelero-summation algorithms:

$$\begin{aligned}\check{f}_{i+1}^{(m)}(\zeta_{i+1}) \;&=\; \int_{\mathcal{C}_{\theta_i}} \check{f}_i(\zeta_i)\,\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\zeta_i + \\ &\quad\; \int_{\mathcal{H}_{\theta_i}^+} \hat{f}_i(\zeta_i)\,\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\zeta_i \end{aligned}\tag{2.14}$$

$$f_p^{(m)}(z_p) \;=\; \int_{\mathcal{C}_{\theta_p}} \check{f}_p(\zeta_p)\,\frac{\partial^m\,\mathrm{e}^{-\zeta_p/z_p}}{\partial z_p^m}\,\mathrm{d}\zeta_p + \int_{\mathcal{H}_{\theta_p}^+} \hat{f}_p(\zeta_p)\,\frac{\partial^m\,\mathrm{e}^{-\zeta_p/z_p}}{\partial z_p^m}\,\mathrm{d}\zeta_p. \tag{2.15}$$

In section 2.4 below, we will show that for u small enough, the kernel $\check{K}(\zeta_i, \zeta_{i+1})$ and its derivatives in ζ_{i+1} have the same order of decrease at infinity as $\hat{K}(\zeta_i, \zeta_{i+1})$.

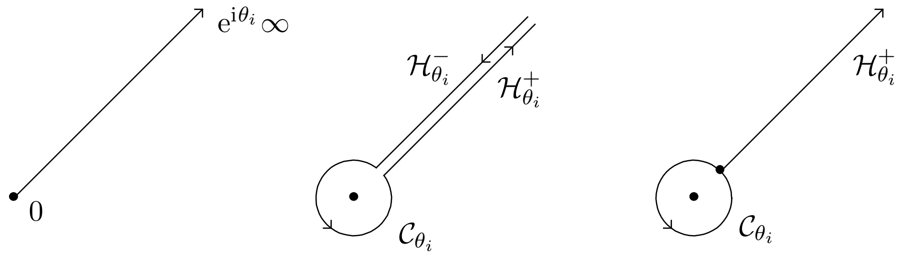

Figure 2.1. Illustrations of the contours for the acceleration and Laplace integrals. At the left, the contour for the direct integrals (2.3) and (2.6) using minors. In the middle, the contour in the case of majors (2.9) and (2.10). At the right hand side, we use majors for integration at 0 and minors for the integration at infinity, as in (2.12) and (2.13).

2.3. Some elementary bounds

LEMMA 2.1. *Given $\alpha \in \mathbb{R}$ and $X > 0$ with $X \geqslant 2\alpha$, we have*

$$\int_X^\infty x^\alpha \mathrm{e}^{-x} \,\mathrm{d}x \leqslant 2 X^\alpha \mathrm{e}^{-X}.$$

Proof. In the case when $\alpha \leqslant 0$, we have

$$\int_X^\infty x^\alpha \mathrm{e}^{-x} \,\mathrm{d}x \leqslant X^\alpha \int_X^\infty \mathrm{e}^{-x} \,\mathrm{d}x = X^\alpha \mathrm{e}^{-X} \leqslant 2 X^\alpha \mathrm{e}^{-X}.$$

If $\alpha > 0$, then consider the function $t = \varphi(x) = x - \alpha \log x$ and its inverse $x = \psi(t)$. Given $X \geqslant 2\alpha$, we obtain

$$\begin{aligned} \int_X^\infty x^\alpha \mathrm{e}^{-x} \,\mathrm{d}x &= \int_{\varphi(X)}^\infty \psi'(t) \mathrm{e}^{-t} \,\mathrm{d}t = \int_{\varphi(X)}^\infty \frac{\mathrm{e}^{-t} \,\mathrm{d}t}{1 - \frac{\alpha}{\psi(t)}} \,\mathrm{d}t \\ &\leqslant 2 \int_{\varphi(X)}^\infty \mathrm{e}^{-t} \,\mathrm{d}t = 2 X^\alpha \mathrm{e}^{-X}. \end{aligned}$$

□

LEMMA 2.2. *Given $\alpha \geqslant 0$, $\lambda > 0$, $\beta \geqslant \lambda - 1$ and $X > 0$ with $X \geqslant \left(\frac{2\beta + 2}{\alpha\lambda}\right)^{1/\lambda}$, we have*

$$\begin{aligned} \int_X^\infty x^\beta \mathrm{e}^{-\alpha x^\lambda} \,\mathrm{d}x &\leqslant \frac{2}{\alpha\lambda} X^{\beta+1-\lambda} \mathrm{e}^{-\alpha X^\lambda}; \\ \int_X^\infty \mathrm{e}^{-\alpha x^\lambda} \,\mathrm{d}x &\leqslant X \mathrm{e}^{-\alpha X^\lambda}. \end{aligned}$$

Proof. For $X \geqslant \left(\frac{2\beta + 2}{\alpha\lambda}\right)^{1/\lambda} \geqslant \left(2 \frac{\beta + 1 - \lambda}{\alpha\lambda}\right)^{1/\lambda}$, the above lemma implies

$$\int_X^\infty x^\beta \mathrm{e}^{-\alpha x^\lambda} \,\mathrm{d}x = \frac{1}{\lambda \alpha^{\frac{\beta+1}{\lambda}}} \int_{\alpha X^\lambda}^\infty x^{\frac{\beta+1-\lambda}{\lambda}} \mathrm{e}^{-x} \,\mathrm{d}x \leqslant \frac{2}{\alpha\lambda} X^{\beta+1-\lambda} \mathrm{e}^{-\alpha X^\lambda}.$$

The second relation easily follows from the first one by setting $\beta = 0$. □

LEMMA 2.3. *Let $\alpha, \beta, \varepsilon, X > 0$. Then*

$$X^{\beta}\,\mathrm{e}^{\alpha X} \leqslant \left(\frac{\beta}{\varepsilon\,\mathrm{e}}\right)^{\beta} \mathrm{e}^{(\alpha+\varepsilon)X}.$$

Proof. This follows from the fact that the function $x^{\beta}\,\mathrm{e}^{-\varepsilon x}$ admits its minimum at $x = \beta / \varepsilon$. □

LEMMA 2.4. *Given $\alpha \geqslant 1$ and $X \leqslant 2\,\alpha$, we have*

$$\int_X^{\infty} x^{\alpha}\,\mathrm{e}^{-x}\,\mathrm{d}x \leqslant 4\,\mathrm{e}^{3\alpha}\,\Gamma(\alpha)\,\mathrm{e}^{-X}$$

Proof. By lemma 2.1, we have

$$\int_X^{\infty} x^{\alpha}\,\mathrm{e}^{-x}\,\mathrm{d}x \leqslant 2\,\alpha^{\alpha+1}\,\mathrm{e}^{-\alpha} + 2\,(2\,\alpha)^{\alpha}\,\mathrm{e}^{-2\alpha},$$

since $x^{\alpha}\,\mathrm{e}^{-x}$ admits its maximum in $x = \alpha$. Furthermore,

$$2\,\alpha^{\alpha+1}\,\mathrm{e}^{-\alpha} + 2\,(2\,\alpha)^{\alpha}\,\mathrm{e}^{-2\alpha} \leqslant 2\,(\mathrm{e}^{\alpha}\,\alpha^{\alpha+1} + 2^{\alpha}\,\alpha^{\alpha})\,\mathrm{e}^{-X} \leqslant 4\,\mathrm{e}^{3\alpha}\,\Gamma(\alpha)\,\mathrm{e}^{-X}.$$

The second inequality can be checked for small α by drawing the graph and it holds for large α because of Stirling's formula. □

LEMMA 2.5. *Given $\lambda \in (0,1)$, $\alpha \geqslant 1$, $\beta > 0$ and $X \leqslant [2\,(\alpha + 1 - \lambda) / \lambda]^{1/\lambda}$, we have*

$$\int_X^{\infty} x^{\alpha}\,\mathrm{e}^{-\beta x^{\lambda}}\,\mathrm{d}x \leqslant 4\,\mathrm{e}^{\frac{4\alpha+4}{\lambda}}\,\Gamma\!\left(\tfrac{\alpha}{\lambda}\right)\mathrm{e}^{-\beta X^{\lambda}}.$$

Proof. Application of the previous lemma, after a change of variables. □

2.4. Explicit bounds for the acceleration kernels at infinity

LEMMA 2.6. *Let $\lambda \in (0,1)$ and $\zeta > 0$. Denote*

$$\begin{aligned} s &= (\lambda\,\zeta)^{\frac{1}{1-\lambda}} \\ \alpha &= \tfrac{\pi}{2}\,\lambda \end{aligned}$$

and assume $s \geqslant 14$. Then

$$|\hat{K}_{\lambda}(\zeta)| \leqslant \frac{8\,s}{\sqrt{1-\lambda}\,(\cos\alpha)^{1/\lambda}}\,\mathrm{e}^{-\frac{1-\lambda}{\lambda}s}.$$

Proof. Let $\varphi(z) = z - \zeta\,z^{\lambda}$. We will evaluate the integral (2.5) using the saddle point method. In the neighbourhood of the saddlepoint s, we have

$$\begin{aligned} \varphi(s + \mathrm{i}\,\varepsilon) &= \frac{\lambda - 1}{\lambda}\,s - \frac{1-\lambda}{2\,s}\,\varepsilon^2 - \frac{\mathrm{i}}{2}\int_0^{\varepsilon} \varphi'''(s + \mathrm{i}\,t)\,(\varepsilon - t)^2\,\mathrm{d}t, \\ \varphi'''(z) &= -\zeta\,\lambda\,(1-\lambda)\,(2-\lambda)\,z^{\lambda-3}. \end{aligned}$$

For z on $[s, s+\mathrm{i}\sqrt{s}]$, we also have

$$\begin{aligned}|\varphi'''(z)| &\leqslant \left(\frac{1}{1-s^{-1/2}}\right)^3 \zeta\,\lambda\,(1-\lambda)\,(2-\lambda)\,s^{\lambda-3}\\ &= \left(\frac{1}{1-s^{-1/2}}\right)^3 \frac{(1-\lambda)\,(2-\lambda)}{s^2},\end{aligned}$$

For $|\varepsilon| \leqslant \sqrt{s}$, it follows that

$$\begin{aligned}\left|\frac{\mathrm{i}}{2}\int_0^{\varepsilon} \varphi'''(s+\mathrm{i}\,t)\,(t-\varepsilon)^2\,\mathrm{d}t\right| &\leqslant \left(\frac{1}{1-s^{-1/2}}\right)^3 \frac{(1-\lambda)\,(2-\lambda)}{6\,s^2}\,\varepsilon^3\\ &\leqslant \frac{1-\lambda}{3\,s^{3/2}}\left(\frac{1}{1-s^{-1/2}}\right)^3 \varepsilon^2\\ &\leqslant \frac{1-\lambda}{4\,s}\,\varepsilon^2,\end{aligned}$$

where the last bound follows from our assumption $s \geqslant 14$. We infer that

$$\Re(\varphi(s+\mathrm{i}\,\varepsilon)-\varphi(s)) \leqslant -\frac{1-\lambda}{4\,s}\,\varepsilon^2,$$

whence

$$\left|\int_{-\sqrt{s}}^{\sqrt{s}} \mathrm{e}^{\varphi(s+\mathrm{i}x)-\varphi(s)}\,\mathrm{d}\,x\right| \leqslant \left|\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1-\lambda}{4s}x^2}\,\mathrm{d}x\right| = \sqrt{\frac{4\,\pi\,s}{1-\lambda}}. \tag{2.16}$$

Now let $\omega = s\,(\cos\alpha)^{-1/\lambda} > \sqrt{s}$. We have

$$\left|\left[\int_{-\omega}^{-\sqrt{s}} + \int_{\sqrt{s}}^{\omega}\right] \mathrm{e}^{\varphi(s+\mathrm{i}x)-\varphi(s)}\,\mathrm{d}\,x\right| \leqslant 2\,\omega, \tag{2.17}$$

since $\Re\,\varphi(s+\mathrm{i}\,x)$ admits a unique maximum at $x=0$. Furthermore,

$$\Re\,(\zeta\,(s+\mathrm{i}\,x)^{\lambda}) \geqslant \zeta\,|x|^{\lambda}\cos\alpha,$$

for all $x \in \mathbb{R}$. Lemma 2.2 therefore implies

$$\left|\left[\int_{-\infty}^{-\omega} + \int_{\omega}^{\infty}\right] \mathrm{e}^{\varphi(s+\mathrm{i}x)-\varphi(s)}\,\mathrm{d}\,x\right| \leqslant 2\,\omega\,\mathrm{e}^{\zeta(s^{\lambda}-(\cos\alpha)\omega^{\lambda})} \leqslant 2\,\omega, \tag{2.18}$$

since $\omega^{\lambda} \geqslant s^{\lambda}/\cos\alpha \geqslant 2\,s^{\lambda-1}/\cos\alpha = 2/(\lambda\,\zeta\,\cos\alpha)$. Putting the relations (2.16), (2.17) and (2.18) together, we obtain

$$|\hat{K}_{\lambda}(\zeta)|\,\mathrm{e}^{\frac{1-\lambda}{\lambda}s} \leqslant \sqrt{\frac{4\,\pi\,s}{1-\lambda}} + \frac{4\,s}{(\cos\alpha)^{1/\lambda}} \leqslant \frac{8\,s}{\sqrt{1-\lambda}\,(\cos\alpha)^{1/\lambda}}.$$

This completes the proof of our lemma. □

LEMMA 2.7. *Let* $\lambda = k_{i+1}/k_i$ *and assume that* $\arg\zeta_i = \lambda\arg\zeta_{i+1}$, $\arg u = \arg\zeta_{i+1}$, $0 < |u| < |\zeta_{i+1}|/2$ *and*

$$|\zeta_i/u^{\lambda}| \geqslant \max\left(\frac{14^{1-\lambda}}{\lambda}, \left(\frac{2}{1-\lambda}\right)^{1-\lambda}\lambda^{-\lambda}\right). \tag{2.19}$$

Then

$$|\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,\zeta_{i+1})| \leqslant \left|\frac{2^{2-m}\,m!}{\pi\,(\cos\frac{\lambda\pi}{2})^{1/\lambda}\sqrt{1-\lambda}\,\zeta_{i+1}^{m+1}}\right| \mathrm{e}^{-\frac{1-\lambda}{\lambda}(\lambda\zeta_i/u^{\lambda})^{1/(1-\lambda)}}. \tag{2.20}$$

Proof. We first observe that

$$\begin{aligned}\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,\zeta_{i+1}) &= \frac{-m!}{2\,\pi\,\mathrm{i}}\int_0^u \frac{\hat{K}_\lambda(\zeta_i/\xi^\lambda)}{(\xi-\zeta_{i+1})^{m+1}}\frac{\mathrm{d}\xi}{\xi} \\ &= \frac{-m!}{2\,\pi\,\mathrm{i}\,\lambda}\int_{\zeta_i/u^\lambda}^\infty \frac{\hat{K}_\lambda(\xi)}{((\zeta_i/\xi)^{1/\lambda}-\zeta_{i+1})^{m+1}}\frac{\mathrm{d}\xi}{\xi}.\end{aligned}$$

For $\xi \geqslant \zeta_i/u^\lambda$, we also have $|\zeta_i/\xi|^{1/\lambda}\leqslant|u|\leqslant|\zeta_{i+1}|/2$, so that

$$|\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,\zeta_{i+1})|\leqslant\left|\frac{m!}{2^{m+2}\,\pi\,\lambda\,\zeta_{i+1}^{m+1}}\right|\int_{\zeta_i/u^\lambda}^\infty \hat{K}_\lambda(\xi)\frac{\mathrm{d}\xi}{\xi}. \tag{2.21}$$

Setting $\alpha=(\lambda\,\pi)/2$, the lemmas 2.6 and 2.2 now imply

$$\begin{aligned}\int_{\zeta_i/u^\lambda}^\infty \hat{K}_\lambda(\xi)\frac{\mathrm{d}\xi}{\xi} &\leqslant \frac{8}{\sqrt{1-\lambda}\,(\cos\alpha)^{1/\lambda}}\int_{\zeta_i/u^\lambda}^\infty (\lambda\,\xi)^{\frac{1}{1-\lambda}}\,\mathrm{e}^{-\frac{1-\lambda}{\lambda}(\lambda\xi)^{1/(1-\lambda)}}\frac{\mathrm{d}\xi}{\xi} \\ &\leqslant \frac{16\,\lambda}{(\cos\alpha)^{1/\lambda}\sqrt{1-\lambda}}\,\mathrm{e}^{-\frac{1-\lambda}{\lambda}(\lambda\zeta_i/u^\lambda)^{1/(1-\lambda)}},\end{aligned}$$

because of the assumption (2.19). Combining this bound with (2.21), we obtain (2.20). □

3. DIFFERENTIAL OPERATORS AND NEWTON POLYGONS

3.1. Definition of the Newton polygon

Let $\mathbb{K}[z^{\mathbb{Q}}]$ be the set of polynomials of the form $P=P_{\alpha_1}z^{\alpha_1}+\cdots+P_{\alpha_l}z^{\alpha_l}$ with $P_{\alpha_1},\ldots,P_{\alpha_l}\in\mathbb{K}^{\neq}$ and $\alpha_1>\cdots>\alpha_l\in\mathbb{Q}$. If $l\neq 0$, then we call $v^\infty(P)=-\alpha_1$ the *valuation* of P at infinity and $\alpha_l=v^0(P)$ the *valuation* of P at zero. If $l=0$, then $v^\infty(P)=v^0(P)=+\infty$. We write $v=v^\infty$ or $v=v^0$ when it is clear from the context whether we are working near $z=\infty$ or $z=0$.

Now consider a differential operator

$$L=L_r\,\delta^r+\cdots+L_0\in\mathbb{K}[z^{\mathbb{Q}}][\delta] \qquad (L_r\neq 0),$$

where $\delta=z\,\partial=z\,\frac{\partial}{\partial z}$. The *support* $\operatorname{supp}L$ of L is defined to be the set of all pairs $(i,\alpha)\in\mathbb{N}\times\mathbb{Q}$ with $L_{i,\alpha}=(L_i)_\alpha\neq 0$. The Newton polygon (see figure 3.1) of L at infinity (resp. zero) is the convex hull of

$$\{(x,\alpha+\epsilon\,y)\colon(i,\alpha)\in\operatorname{supp}L, 0\leqslant x\leqslant i, y\geqslant 0\},$$

where $\epsilon=-1$ (resp. $\epsilon=1$).

The boundary of the Newton polygon consists of two vertical halflines and a finite number of edges. The *outline* of (the Newton polygon of) L is the sequence $(i_0,\alpha_0),\ldots,(i_l,\alpha_l)$ of points with $0=i_0<\cdots<i_l=r$, such that the j-th edge of the Newton polygon is precisely the segment which joins (i_{j-1},α_{j-1}) to (i_j,α_j). We call

$$\sigma_j=\frac{\alpha_j-\alpha_{j-1}}{i_j-i_{j-1}}$$

the *slope* of the j-th edge. From the definition of the Newton polygon as a convex hull, it follows that

$$v(L_k)-v(L_{i_j})\geqslant\epsilon\,\sigma_j\,(k-i_j)$$

for all k. We call $\kappa=\kappa_L=\epsilon\,\sigma_{l-1}$ the *growth rate* of L.

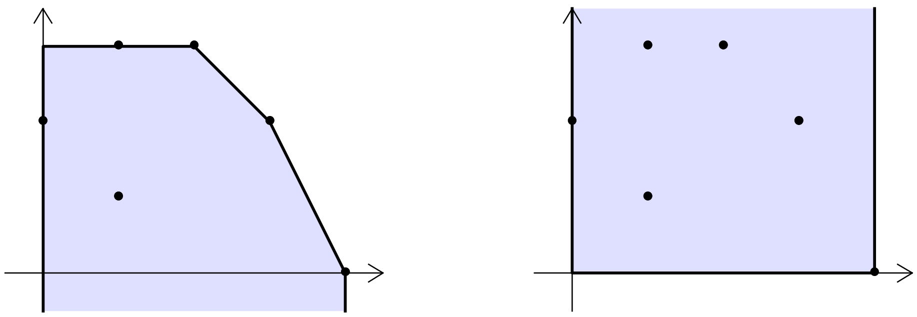

Figure 3.1. Illustration of the Newton polygons at infinity and zero of the operator $L = \delta^4 + 2 z^2 \delta^3 - z^3 \delta^2 + (7 z - 3 z^3) \delta + 11 z^2$.

3.2. Operations on differential operators

3.2.1. Multiplicative conjugation

Given $L \in \mathbb{K}[z^{\mathbb{Q}}][\delta]$ and $\varphi \in \mathbb{K}[z^{\mathbb{Q}}]$, we define $\mathcal{M}_{\varphi} L$ to be the operator which is obtained by substituting $\delta + \varphi$ for δ in L. For all f, we have

$$(\mathcal{M}_{\varphi} L)(f) = \mathrm{e}^{-\int \varphi / z} L(\mathrm{e}^{\int \varphi / z} f).$$

In the case when $\varphi \in \mathbb{K}$, we have

$$\operatorname{supp} \mathcal{M}_{\varphi} L \subseteq \operatorname{supp} L + (-\mathbb{N}, 0).$$

In particular, the Newton polygon of $\mathcal{M}_{\varphi} L$ and L coincide, both at zero and infinity (see figure 3.2). In general, only the slopes which are steeper than the exponent of the dominant monomial of φ coincide.

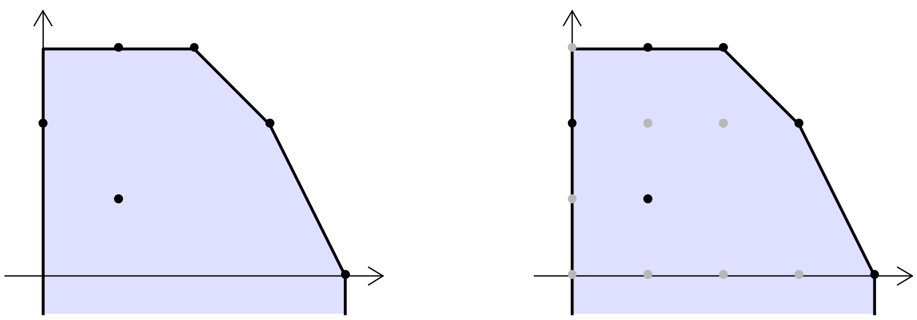

Figure 3.2. Illustration of the Newton polygons at infinity of L from figure 3.1 and $\mathcal{M}_2 M = \delta^4 + (2 z^2 + 8) \delta^3 + (12 z^2 - z^3 + 24) \delta^2 + (24 z^2 - 7 z^3 + 7 z + 32) \delta + 27 z^2 - 10 z^3 + 14 z + 16$.

3.2.2. Compositional conjugation

Let $\tau \in \mathbb{Q}^{\neq}$ and consider the transformation $\mathcal{P}_{\tau}: z \mapsto z^{\tau}$. If $z = u^{\tau}$, then

$$z \frac{\partial}{\partial z} = u^{\tau} \frac{\partial}{\partial u^{\tau}} = \frac{1}{\tau} u \frac{\partial}{\partial u},$$

so the transformation $\mathcal{P}_\tau$ naturally extends to $\mathbb{K}[z^{\mathbb{Q}}][\delta]$ by sending δ to $\tau^{-1}\delta$. We have

$$\operatorname{supp} \mathcal{P}_\tau L = \{(i, \tau\alpha): (i, \alpha) \in \operatorname{supp} L\}.$$

Consequently, if

$$(i_0, \alpha_0), \ldots, (i_l, \alpha_l)$$

is the outline of L, then

$$(i_0, \tau\alpha_0), \ldots, (i_l, \tau\alpha_l)$$

is the outline of $\mathcal{P}_\tau L$. In particular, $\kappa_{\mathcal{P}_\tau L} = |\tau| \kappa_L$. Of course, if $\tau < 0$, then we understand that the roles of infinity and zero are interchanged. In figure 3.3, we have illustrated the effect of the transformation $\mathcal{P}_\tau$ on the Newton polygon.

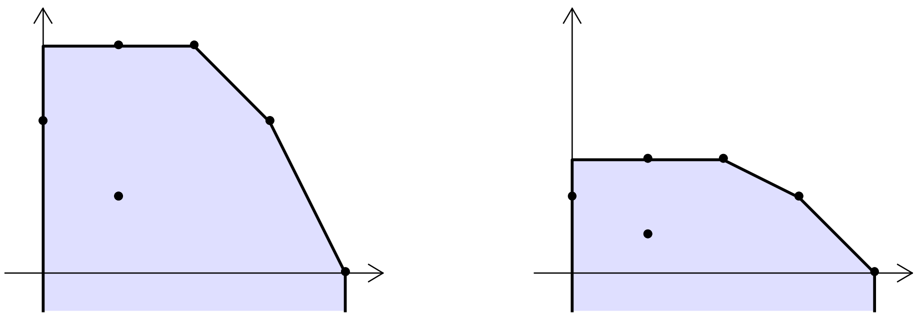

Figure 3.3. Illustration of the Newton polygons at infinity of L from figure 3.1 and $\mathcal{P}_{1/2} L = 16\,\delta^4 + 16\,z\,\delta^3 - 4\,z^{3/2}\,\delta^2 + (14\,z^{1/2} - 6\,z^{3/2})\,\delta + 11\,z$.

3.3. The Borel transform

Let us now consider the analogue of the formal Borel transform $\tilde{\mathcal{B}}$ from section 2.1 for differential operators. It is classical that the formal Borel transform satisfies

$$\begin{aligned} \tilde{\mathcal{B}}(z^2 \partial_z f) &= \zeta \tilde{\mathcal{B}} f; \\ \tilde{\mathcal{B}}(z^{-1} f) &= \partial_\zeta \tilde{\mathcal{B}} f. \end{aligned}$$

for $f \in z\,\mathbb{K}[[z]]$. Rewritten in terms of the operators $\delta_z = z\,\partial_z$ and $\delta_\zeta = \zeta\,\partial_\zeta$, this yields

$$\begin{aligned} \tilde{\mathcal{B}}(\delta_z f) &= (\delta_\zeta + 1)\,\tilde{\mathcal{B}} f; \\ \tilde{\mathcal{B}}(z^{-1} f) &= (\zeta^{-1}\delta_\zeta)\,\tilde{\mathcal{B}} f. \end{aligned}$$

This induces a natural $\mathbb{K}$-algebra morphism $\mathcal{B}: \mathbb{K}[z^{-1}][\delta_z] \to \mathbb{K}[\zeta^{-1}][\delta_\zeta]$, by setting

$$\begin{aligned} \mathcal{B} z^{-1} &= \zeta^{-1}\,\delta_\zeta; \\ \mathcal{B}\delta_z &= \delta_\zeta + 1. \end{aligned}$$

Each term $L_{i,j}\,z^j\,\delta^i$ of an operator $L \in \mathbb{K}[z^{-1}][\delta_z]$ gives rise to a contribution

$$\mathcal{B}(L_{i,j}\,z^j\,\delta_z^i) = \zeta^j\,(L_{j,i}\,\delta_\zeta^{i-j} + c_{i-j-1}\,\delta_\zeta^{i-j-1} + \cdots + c_0)$$

to $\mathcal{B}L$, for suitable constants $c_{i-j-1}, \ldots, c_0 \in \mathbb{K}$. In particular,

$$\operatorname{supp} \mathcal{B}(L_{i,j}\,z^j\,\delta_z^i) \subseteq (i - j, j) + (-1, 0)\,\mathbb{N}.$$

Let $(i_0, \alpha_0), \ldots, (i_l, \alpha_l)$ be the outline of L at infinity and for all j, let

$$\hat{\sigma}_j = \frac{\sigma_j}{1 - \sigma_j}.$$

If $0 < \sigma_j < 1$, then the j-th edge gives rise to an edge with slope $\hat{\sigma}_j$ in the Newton polygon of $\mathcal{B}L$ at zero. If $\sigma_j > 1$, then it gives rise to an edge with slope $\hat{\sigma}_j$ in the Newton polygon of $\mathcal{B}L$ at infinity (see figure 3.4). In addition, if L_r contains several terms, then the Newton polygon of $\mathcal{B}L$ at infinity also contains an edge with slope -1.

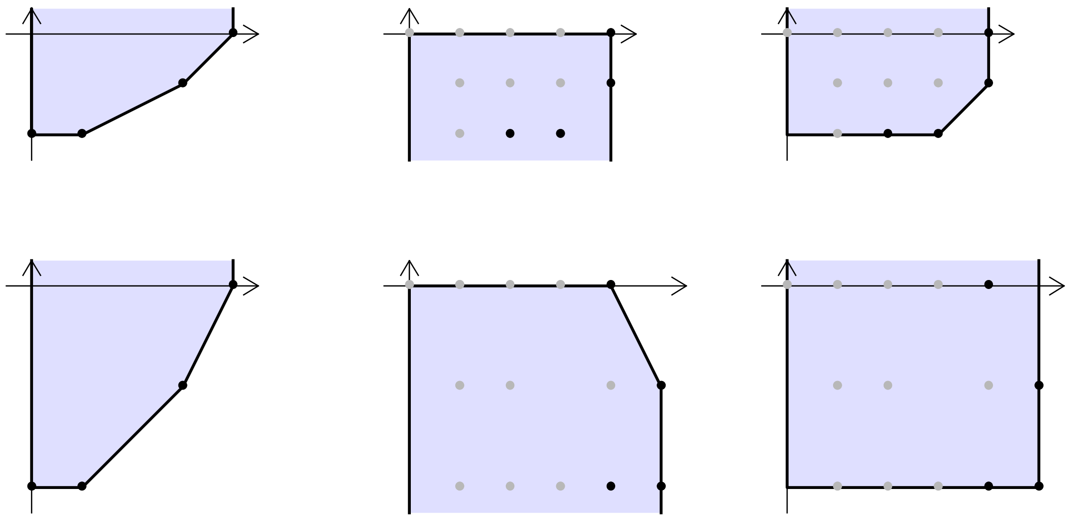

Figure 3.4. The left hand column shows the Newton polygons at infinity of the operators $L = \delta^4 - \frac{6}{z}\delta^3 - \frac{1}{z^2}\delta - \frac{7}{z^2}$ and $\mathcal{P}_2 L = \frac{1}{16}\delta^4 - \frac{3}{4z^2}\delta^3 - \frac{1}{2z^4}\delta - \frac{7}{z^4}$. At the right hand side, we have drawn the Newton polygons of their Borel transforms $\mathcal{B}L = \left(1 - \frac{6}{\zeta}\right)\delta^4 + \left(4 - \frac{18}{\zeta} - \frac{1}{\zeta^2}\right)\delta^3 + \left(6 - \frac{18}{\zeta} - \frac{7}{\zeta^2}\right)\delta^2 + \left(4 - \frac{6}{\zeta} + \frac{8}{\zeta^2}\right)\delta + 1$ and $\mathcal{B}\mathcal{P}_2 L = \left(-\frac{3}{4\zeta^2} - \frac{1}{2\zeta^4}\right)\delta^5 + \left(\frac{1}{16} - \frac{3}{2\zeta^2} - \frac{9}{2\zeta^4}\right)\delta^4 + \left(\frac{1}{4} + \frac{79}{2\zeta^4}\right)\delta^3 + \left(\frac{3}{8} + \frac{3}{2\zeta^2} - \frac{159}{2\zeta^4}\right)\delta^2 + \left(\frac{1}{4} + \frac{3}{\zeta^2} + \frac{45}{\zeta^4}\right)\delta + \frac{1}{16}$ at infinity (the middle column) and at zero (the right hand column).

3.4. Formal solutions

Having chosen whether we work near infinity or near the origin, let

$$\begin{aligned} \mathbb{O} &= \mathbb{K}[[z^{\epsilon \mathbb{Q}^>}]][\log z]; \\ \mathfrak{E} &= \exp \mathbb{K}[z^{\epsilon \mathbb{Q}^<}]; \\ \mathbb{S} &= (\mathbb{O}\, z^{\mathbb{Q}})[\mathfrak{E}]. \end{aligned}$$

Given $f \in \mathbb{S}$, the set $\mathcal{E}_f = \{\mathfrak{e} \in \mathfrak{E}: f_{\mathfrak{e}} \neq 0\}$ is called the set of *exponential parts* of f, and the number $\kappa_f = \max\{-\epsilon\,\alpha: \mathrm{e}^{P(z)} \in \mathcal{E}_f, P_\alpha \neq 0\} \cup \{0\}$ the *growth rate* of f. More generally given a subvector space $\mathcal{V}$ of $\mathbb{S}$, we denote $\mathcal{E}_{\mathcal{V}} = \{\mathcal{E}_f: f \in \mathcal{V}\}$ and $\kappa_{\mathcal{V}} = \max\{\kappa_f: f \in \mathcal{V}\}$.

The Newton polygon provides a lot of information about the structure of the subvector space $\mathcal{V}_L \subseteq \mathbb{S}$ of formal solutions to $L f = 0$. In the sequel, we will use the following classical consequences of the Newton polygon method:

THEOREM 3.1. *Let $L \in \mathbb{K}(z)[\delta]^{\neq}$ be monic, of order r and assume that $\mathbb{K}$ is algebraically closed. Then the equation $L f$ admits a full basis of solutions in $\mathbb{S}$, i.e. $\dim \mathcal{V}_L = r$. Moreover, each basis element may be chosen so as to have a unique exponential part.* □

THEOREM 3.2. *Let $\sigma_1 < \cdots < \sigma_l$ be the slopes of the Newton polygon of L. Then*

a) $\{\kappa_f : f \in \mathcal{V}_L^{\neq}\} = \{\epsilon\,\sigma_1, \ldots, \epsilon\,\sigma_l\}$.

b) $\kappa_{\mathcal{V}_L} = \kappa_L$. □

4. HOLONOMY

4.1. Holonomic functions in several variables

Let $\mathbb{K}$ be an algebraically closed subfield of $\mathbb{C}$. Consider the coordinates $\boldsymbol{z} = (z_1, \ldots, z_n)$ and corresponding derivatives $\boldsymbol{\partial} = (\partial_1, \ldots, \partial_n)$ w.r.t. $z_1, \ldots, z_n$. An analytic function f in $\boldsymbol{z}$ is said to be *holonomic* over $\mathbb{K}$, if it satisfies a non-trivial linear differential equation $L_i\, f = 0$ with $L_i \in \mathbb{K}(\boldsymbol{z})[\partial_i]$ for each $i \in \{1, \ldots, n\}$. Equivalently, we may require that $\mathbb{K}[\boldsymbol{\partial}]\, f$ is a finitely generated module over $\mathbb{K}(\boldsymbol{z})$. The second criterion implies the following classical proposition [Stanley, 1980]:

PROPOSITION 4.1. *Let f and g be holonomic functions in $\boldsymbol{z}$. Then*

a) Any rational function in $\mathbb{K}(\boldsymbol{z})$ is holonomic.

b) $f + g$ is a holonomic function.

c) $f\,g$ is a holonomic function.

d) $\partial_i\, f$ is a holonomic function for all $i \in \{1, \ldots, n\}$.

e) Given a point $\boldsymbol{u}$ on the Riemann-surface $\mathcal{R}$ of f, the specialization $f(z_n = u_n)$ is holonomic.

f) Given algebraic functions $g_1, \ldots, g_n$ over $\mathbb{K}(\boldsymbol{z})$ the composition

$$f \circ (g_1, \ldots, g_n) : \boldsymbol{z} \mapsto f(g_1(\boldsymbol{z}), \ldots, g_n(\boldsymbol{z}))$$

is holonomic.

Proof. The property (c) follows from the inclusion

$$\mathbb{K}[\boldsymbol{\partial}](f\,g) \subseteq (\mathbb{K}[\boldsymbol{\partial}]\, f)\, (\mathbb{K}[\boldsymbol{\partial}]\, g)$$

and the fact that the dimension of the right-hand side is finite over $\mathbb{K}(\boldsymbol{z})$. All other properties are proved in a similar way. □

A more interesting closure property is the stability under definite integration. Consider a holonomic function f in $\boldsymbol{z}$ and a point $\boldsymbol{u}$ on its Riemann surface $\mathcal{R}$. Let $\mathcal{R}_n$ be the Riemann surface of the specialization $f(\boldsymbol{z}' = \boldsymbol{u}')$, where $\boldsymbol{z}' = (z_1, \ldots, z_{n-1})$ and $\boldsymbol{u}' = (u_1, \ldots, u_{n-1})$. Consider a path $\gamma : (0, 1) \to \mathcal{R}_n$ on $\mathcal{R}_n$ with possibly singular end-points. If γ is singular at $\epsilon \in \{0, 1\}$, then we assume that there exists a neighbourhood $\boldsymbol{\mathcal{U}}'$ of $\boldsymbol{u}'$, such that $(\boldsymbol{z}', \gamma) : (0, 1) \to \mathcal{R}$ is a path on $\mathcal{R}$ for all $\boldsymbol{z}' \in \boldsymbol{\mathcal{U}}'$ and $\lim_{t \to \epsilon} f(\boldsymbol{z}', \gamma(t)) = 0$. We now have:

PROPOSITION 4.2. *The integral $g(\boldsymbol{z}') = \int_\gamma f(\boldsymbol{z}', z_n)\, \mathrm{d} z_n$ is a holonomic function.*

Proof. It suffices to show that g is holonomic in a neighbourhood of $\boldsymbol{u}'$. Let $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ be such that

$$\mathbb{K}[\boldsymbol{\partial}]\, f \subseteq \mathbb{K}[\boldsymbol{\partial}]_{<\boldsymbol{p}}\, f = \mathsf{Vect}(\boldsymbol{\partial}^{\boldsymbol{k}} f : 0 \leqslant k_1 < p_1, \ldots, 0 \leqslant k_n < p_n).$$

Let f^+ and f^- be the specializations of f in z_n at the end-point resp. starting point of γ. Notice that f^+ and f^- are defined in a neighbourhood of $\boldsymbol{u}'$. Setting $\boldsymbol{\partial}' = (\partial_1, \ldots, \partial_{n-1})$, the space

$$\mathcal{R} := (\mathbb{K}[\boldsymbol{\partial}']_{<\boldsymbol{p}'} f^+) + (\mathbb{K}[\boldsymbol{\partial}']_{<\boldsymbol{p}'} f^-)$$

is finite dimensional over $\mathbb{K}(\boldsymbol{z}')$. For each $\boldsymbol{k} \in \mathbb{N}^n$ and $l \in \mathbb{N}$, let

$$I_{\boldsymbol{k};l} = \int_\gamma z_n^l \, (\boldsymbol{\partial}^{\boldsymbol{k}} f)(\boldsymbol{z}) \, \mathrm{d} z_n.$$

The differential equation for f in z_n yields a finite relation

$$\sum_{k_n=0}^{p_n} \sum_i C_{k_n,i} \, I_{(\boldsymbol{k}',k_n);l+i} = 0,$$

with $C_{k_n,i} \in \mathbb{K}(\boldsymbol{z}')$ for all k_n. Partial integration also yields a relation

$$I_{(\boldsymbol{k}',i);l} - \frac{1}{l+1} I_{(\boldsymbol{k}',i+1);l+1} \in \mathcal{R}$$

for every i. Combining these relations, we obtain a non-trivial relation

$$A_{\boldsymbol{k}',0} \, I_{(\boldsymbol{k}',0);l} + \cdots + A_{\boldsymbol{k}',q} \, I_{(\boldsymbol{k}',0);l-q} \in \mathcal{R},$$

where $A_{\boldsymbol{k}',0}, \ldots, A_{\boldsymbol{k}',q} \in \mathbb{K}(\boldsymbol{z}')[l]$. For l which are not a root of $A_{\boldsymbol{k}',0}$, we thus obtain a recurrence relation for $I_{(\boldsymbol{k}',0);l}$. Therefore, the space

$$\mathcal{I} = \mathsf{Vect}(I_{\boldsymbol{k};l} \colon 0 \leqslant k_1 < p_1, \ldots, 0 \leqslant k_{n-1} < p_{n-1}, k_n = 0, l \in \mathbb{N}) + \mathcal{R}$$

is again finite dimensional over $\mathbb{K}(\boldsymbol{z}')$. We conclude our proof with the observation that $\mathcal{I}$ is stable under $\boldsymbol{\partial}'$. □

4.2. Computation of vanishing operators

Let us now turn our attention to the one-dimensional case. Given a monic differential operator $L \in \mathbb{K}(z)[\partial]$, we denote by $\mathcal{H}_L$ the space of solutions to the equation $L f = 0$ at a given point. In the case of formal solutions at zero or infinity, we will also write $\mathcal{E}_L = \mathcal{E}_{\mathcal{H}_L}$. Inversely, given a vector space V of formal series, analytic germs or analytic functions on some domain, we say that $L \in \mathbb{K}(z)[\partial]$ *vanishes* on V if $L V = 0$. We say that L is a *vanishing operator* for V if $\mathcal{H}_L = V$, in which case V is said to be *closed*.

Given two operators $K, L \in \mathbb{K}(z)[\partial]$, we know by proposition 4.1 that there exists an operator $M \in \mathbb{K}(z)[\partial]$ which vanishes on $\mathcal{H}_K + \mathcal{H}_L$. It turns out that the operator $K \boxplus L$ of minimal order with this property is actually a vanishing operator for $\mathcal{H}_K + \mathcal{H}_L$. A similar property holds for the operators $K \boxtimes L$, $L^{\boxed{\prime}}$ and $L^{\boxed{\circ} p}$ of minimal orders which vanish on $\mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$, $\mathcal{H}_L'$, resp. $\mathsf{Vect}(\mathcal{H}_L \circ \varphi \colon \varphi^p = z)$, where $p \in \mathbb{N}^>$. What is more, there exist algorithms for computing these vanishing operators.

In this section, we will briefly recall these algorithms, and thereby give an effective proof of lemma 4.3 below. The algorithms are all more or less classical, but we could not find a reference where they are all described together. We will also prove a slightly weaker result for the operation (2.8) which associates a major to a minor.

LEMMA 4.3. *Let K, L be monic differential operators in $\mathbb{K}(z)[\partial]$ and $p \in \mathbb{N}^{\neq}$.*

- *a) There exists a unique monic $K \boxplus L \in \mathbb{K}(z)[\partial]$ with $\mathcal{H}_{K \boxplus L} = \mathcal{H}_K + \mathcal{H}_L$.*
- *b) There exists a unique monic $K \boxtimes L \in \mathbb{K}(z)[\partial]$ with $\mathcal{H}_{K \boxtimes L} = \mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$.*
- *c) There exists a unique monic $L^{\boxed{\prime}} \in \mathbb{K}(z)[\partial]$ with $\mathcal{H}_{L^{\boxed{\prime}}} = \mathcal{H}_L'$.*
- *d) There exists a unique monic $L^{\boxed{\circ} p} \in \mathbb{K}(z)[\partial]$ with $\mathcal{H}_{L^{\boxed{\circ} p}} = \mathsf{Vect}(\mathcal{H}_L \circ \varphi \colon \varphi^p = z)$.*

4.2.1. Addition

We notice that $K \boxplus L$ coincides with the least common left multiple of K and L in the Ore ring $\mathbb{K}(z)[\partial]$. Indeed, any common left multiple vanishes on $\mathcal{H}_K + \mathcal{H}_L$ and any operator which vanishes on $\mathcal{H}_K$ resp. $\mathcal{H}_L$ right divides K resp. L. One may thus compute $K \boxplus L$ using any classical algorithm for the computation of least common left multiples, such as the Euclidean algorithm.

4.2.2. Multiplication

Given formal solutions F and G to $KF = 0$ and $LG = 0$, the product FG and its successive derivatives $F' G + FG'$, $F'' G + 2 F' G' + FG''$, etc. may all be reduced using the relations $KF = LG = 0$. In other words, $(FG)^{(k)} \in \mathcal{V} = \bigoplus_{i<r,j<s} \mathbb{K}(z) F^{(i)} G^{(j)}$, for all k, where r and s denote the orders of K resp. L. Consequently, there exists a $\mathbb{K}(z)$-linear relation among $FG, \ldots, (FG)^{(rs)}$ in $\mathcal{V}$. By linear algebra, we may compute the monic operator M of smallest order with $M(FG) = 0$ in $\mathcal{V}$. Using an adaptation of the proof of [Hendriks and Singer, 1999, Lemma 6.8], we will show that $M = K \boxtimes L$.

Let $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ be fundamental systems of solutions to $Kf = 0$ resp. $Lg = 0$ at a non-singular point, considered as elements of the field $\mathcal{K}$ of convergent Laurent series at this point. Let $C_1, \ldots, C_r$ and $D_1, \ldots, D_s$ be formal indeterminates. Then the substitutions

$$\begin{aligned} F^{(i)} &\mapsto C_1 f_1^{(i)} + \cdots + C_r f_r^{(i)} \qquad (i < r)\\ G^{(j)} &\mapsto D_1 g_1^{(j)} + \cdots + D_s g_s^{(j)} \qquad (j < s) \end{aligned}$$

yield an isomorphism

$$\varphi: \mathcal{A} = \mathcal{K}[F, \ldots, F^{(r-1)}, G, \ldots, G^{(s-1)}] \rightarrow \mathcal{B} = \mathcal{K}[C_1, \ldots, C_r, D_1, \ldots, D_s].$$

Now consider a monic operator $N \in \mathbb{K}(z)[\partial]$ of smaller order than M. Using the relations $KF = LG = 0$, we may rewrite $N(FG)$ as a non-zero element of $\mathcal{V} \subseteq \mathcal{A}$. It follows that $\varphi(N(FG)) \neq 0$. Consequently, there exist constants $c_1, \ldots, c_r, d_1, \ldots, d_s \in \mathbb{K}$ with $\varphi(N(FG))(c_1, \ldots, c_r, d_1, \ldots, d_s) \neq 0$. Setting $f = c_1 f_1 + \cdots + c_r f_r$ and $g = d_1 g_1 + \cdots + d_s g_s$, we infer that $N(fg) \neq 0$, so N is not a vanishing operator of $\mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$. This shows that M is indeed the differential operator of lowest order which vanishes on $\mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$.

The proof that $\mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$ is closed is based on differential Galois theory [van der Put and Singer, 2003]: when computing the solutions to operators in $\mathbb{K}(z)[\partial]$ in a suitable Picard-Vessiot or D-algebraic closure $\mathcal{K}$, any differential automorphism of $\mathcal{K}$ over $\mathbb{K}(z)$ leaves both $\mathcal{H}_K$ and $\mathcal{H}_L$, whence $\mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$, invariant. But, given a finite dimensional-subvector space $\mathcal{V}$ of $\mathcal{K}$ which is invariant under any differential automorphism, we may explicitly construct an operator $\Omega \in \mathbb{K}(z)[\partial]$ with $\mathcal{H}_\Omega = \mathcal{V}$, e.g. [Hoeven, 2007a, proposition 21(b)]. This shows that $\mathsf{Vect}(\mathcal{H}_K \mathcal{H}_L)$ is closed.

4.2.3. Differentiation

If $L(1) = 0$, then L is right divisible by ∂, so we must have $L = L^{◫} \partial$. Otherwise, the least common multiple of L and ∂ in $\mathbb{K}(z)[\partial]$ has order $r + 1$, so there exist operators A of order 1 and B of order r and with $AL = B\partial$. These operators may again be computed using a modified version of the Euclidean algorithm. Since $\dim \mathcal{H}_{L^{◫}} = \dim \mathcal{H}_L$ and $B\mathcal{H}_L' = 0$, we have $L^{◫} = B$.

4.2.4. Ramification

In order to compute the operator $L^{\boxdot p}$, it is more convenient to work with the derivation δ instead of ∂. It is easy to see that this changes the definitions operators $K \boxplus L$, $K \boxtimes L$, $L^{◫}$ and $L^{\boxdot p}$ only up to a multiple by a power of z.

Given a primitive p-th root of unity $\omega \in \mathbb{K}$, let $\mathcal{Q}_\omega L$ be the operator with $(\mathcal{Q}_\omega L)_i(z) = L_i(\omega z)$ for all i. Then we have $(\mathcal{Q}_\omega L)(f \circ (\omega z)) = L(f) \circ (\omega z)$ for all f, whence $f \circ (\omega z)$ is a root of $\mathcal{Q}_\omega L$ if and only if f is a root of L. By what precedes, it follows that $\Omega = L \boxplus \mathcal{Q}_\omega L \boxplus \cdots \boxplus \mathcal{Q}_{\omega^{p-1}} L$ satisfies $\mathcal{H}_\Omega = \mathcal{H}_L + \mathcal{H}_L \circ (\omega z) + \cdots + \mathcal{H}_L \circ (\omega^{p-1} z)$. Furthermore, $\mathcal{Q}_\omega \Omega = \Omega$ implies that $\Omega_i \in \mathbb{K}(z^p)$ for all i. Consequently, $\mathcal{P}_{1/p} \Omega \in \mathbb{K}(z)[\delta]$ and we conclude that $L^{\boxdot_p} = \mathcal{P}_{1/p} \Omega$.

4.2.5. Majors

Consider the operation $\mathcal{M}$ which associates

$$\check{f} = \mathcal{M} \hat{f} = \frac{-1}{2 \pi \mathrm{i}} \int_0^u \frac{\hat{f}(\xi)}{\xi - \zeta} \mathrm{d} \xi$$

to $\hat{f}$. We have

$$\begin{aligned} \mathcal{M} \hat{f}' &= (\mathcal{M} \hat{f})' + \frac{1}{2 \pi \mathrm{i}} \left(\frac{\hat{f}(u)}{\zeta - u} - \frac{\hat{f}(0)}{\zeta} \right) \\ \mathcal{M}(\zeta \hat{f}) &= \zeta \mathcal{M} \hat{f} - \frac{1}{2 \pi \mathrm{i}} \int_0^u \hat{f}(\xi) \mathrm{d} \xi. \end{aligned}$$

Given a relation $\hat{L} \hat{f} = 0$ for $\hat{f}$, where $L \in \mathbb{K}[\zeta][\partial]$ has order r, we thus obtain a relation

$$\hat{L} \check{f} = \frac{P(\zeta)}{\zeta^r (\zeta - u)^r}$$

for some polynomial P with transcendental coefficients. Setting

$$\check{L} := \partial^{\deg P + 1} [\zeta^r (\zeta - u)^r L], \tag{4.1}$$

it follows that $\check{L} \check{f} = 0$. By theorem 3.2, we notice that the growth rate of $\check{L}$ at zero or infinity is the same as the growth rate of $\hat{L}$ at zero resp. infinity, since $\mathbb{O}\, \mathfrak{e}$ is stable under differentiation and integration without constant term, for each $\mathfrak{e} \in \mathfrak{E}$.

4.2.6. Applications

Lemma 4.3 admits several useful consequences for what follows.

COROLLARY 4.4. *If the coefficients of K and L are analytic on an open or closed subset $\mathcal{U}$ of $\mathbb{C}$, then the same thing holds for the coefficients of $K \boxplus L$, $K \boxtimes L$ and $L^{\boxplus}$.*

Proof. Given functions $h_1, \ldots, h_r$, let $W_{h_1, \ldots, h_r}$ denote their Wronskian. If $h_1, \ldots, h_r$ is a basis of the solution space $\mathcal{H}_L$ of a monic operator $L \in \mathbb{K}(z)[\partial]$, then we recall that the operator L is determined in terms of $h_1, \ldots, h_r$ by the formula

$$L f = \frac{W_{f, h_1, \ldots, h_r}}{W_{h_1, \ldots, h_r}} \tag{4.2}$$

In particular, if $h_1, \ldots, h_r$ are analytic on $\mathcal{U}$, then so are the coefficients of L, as is seen by expanding the right-hand side of (4.2). It now suffices to apply this observation to $K \boxplus L$, $K \boxtimes L$ and $L^{\boxplus}$. □

COROLLARY 4.5. *Let $K, L \in \mathbb{K}(\zeta)[\delta]$ be monic and $p \in \mathbb{N}^{\neq}$. Then*

a) $\mathcal{E}_{K \boxplus L} = \mathcal{E}_K \cup \mathcal{E}_L$.

b) $\mathcal{E}_{K \boxtimes L} = \mathcal{E}_K \mathcal{E}_L$.

c) $\mathcal{E}_{L^{\boxminus}} = \mathcal{E}_L$.

d) $\mathcal{E}_{L^{\boxdot p}} = \mathcal{E}_L^{1/p}$.

Proof. This follows directly from the lemma together with theorem 3.2. □

4.3. Transition matrices

4.3.1. Classical transition matrices

Consider a monic differential operator $L = \partial^r + L_{r-1} \partial^{r-1} + \cdots + L_0$ whose coefficients are analytic function on a Riemann surface $\mathcal{R}$. Given a point $z \in \mathcal{R}$ it is well known that there exists a unique *canonical fundamental system*

$$\boldsymbol{f}^z = (\; f_0^z \quad \cdots \quad f_{r-1}^z \;)$$

of analytic solutions to $L f = 0$ at z with the property that $f_j^{(i)} = \delta_{i,j}$ for all i, j. Since L is linear, an arbitrary solution f to $L f = 0$ is uniquely determined by the vector

$$F(z) = \begin{pmatrix} f(z) \\ \vdots \\ f^{(r-1)}(z) \end{pmatrix}$$

of its *initial conditions* at z by

$$f = \boldsymbol{f}^z F(z). \tag{4.3}$$

More generally, given a path $z \rightsquigarrow z'$ on $\mathcal{R}$ from z to another point z', the values of the analytic continuations of $f, \ldots, f^{(r-1)}$ along the path also linearly depend on $F(z)$. Consequently, there exists a unique scalar matrix $\Delta_{z \rightsquigarrow z'} = \Delta_{z \rightsquigarrow z'}^L$ with

$$F(z') = \Delta_{z \rightsquigarrow z'} F(z). \tag{4.4}$$

We call $\Delta_{z \rightsquigarrow z'}^L$ the *transition matrix* for L along the path $z \rightsquigarrow z'$. Dually, we have

$$\boldsymbol{f}^z = \boldsymbol{f}^{z'} \Delta_{z \rightsquigarrow z'}, \tag{4.5}$$

because of (4.3). Also, if $z' \rightsquigarrow z''$ is a second path, then

$$\Delta_{z \rightsquigarrow z' \rightsquigarrow z''} = \Delta_{z' \rightsquigarrow z''} \Delta_{z \rightsquigarrow z'} \tag{4.6}$$

and in particular

$$\Delta_{z' \rightsquigarrow z} = \Delta_{z \rightsquigarrow z'}^{-1}. \tag{4.7}$$

4.3.2. Singular transition matrices

The notion of transition matrices can be generalized to allow for paths which pass through regular or irregular singularities of the operator L. In order to do this, we start by generalizing the idea of a canonical fundamental system of formal solutions $\boldsymbol{f}^z$ in the singularity z.

In the case when the coefficients of L are in $\mathbb{K}(z)$, then theorem 3.1 tells us that there exists a fundamental system of solutions at $z = 0$. This result is refined in [Hoeven, 2001a], where we show how to compute a canonical basis $f_0, \ldots, f_{r-1}$ of so called "complex transseries" solutions, which is uniquely characterized by suitable asymptotic properties. In particular,

- Each f_i is of the form $f_i = \varphi_i z^{\sigma_i} \mathfrak{e}_i$ with $\varphi_i \in \mathbb{O}$, $\sigma_i \in \mathbb{K}$ and $\mathfrak{e}_i \in \mathfrak{E}$.
- Whenever $\mathfrak{e}_i = \mathfrak{e}_j$ and $\sigma_i \in \sigma_j + \mathbb{Z}$ for $i \neq j$, then $\varphi_{i, v(\varphi_j) + \sigma_j - \sigma_i} = 0$.

Notice that there are other definitions of "canonical" systems of solutions [van Hoeij, 1997], which share the property that they can be computed effectively in terms of the operator L.

Given a notion of a "canonical system of formal solutions at a singularity z", we obtain a dual notion of "initial conditions at z" for arbitrary formal solutions, via the relation (4.3). Now assume in addition that, for a suitable sectorial neighbourhood $\mathcal{U}\subseteq\mathcal{R}$ of z, we are able to associate a genuine analytic function $\rho(f)$ to any formal solution f at z. Then either (4.4) or (4.5) yields a definition for the transition matrix along a straight-line from z to z'. In general, the association $\rho: f\mapsto\rho(f)$ depends on one or several parameters, like the non-singular directions in the accelero-summation procedure. We will now show how to encode these parameters in a suitable generalization of a broken-line path.

Assume from now on that $L\in\mathbb{K}(z)[\partial]$. We define a *singular broken-line path* as being a path $z_0\to z_1\to\cdots\to z_l$, where each z_i is either

- A non singular point σ_i in $\mathbb{K}$.
- A regular singular point $\sigma_i\in\mathbb{K}$ with a direction θ (and we denote $z_i=(\sigma_i)_\theta$).
- An irregular singular point $\sigma_i\in\mathbb{K}$ with critical times $\boldsymbol{k}$ and directions $\boldsymbol{\theta}$ (and we denote $z_i=(\sigma_i)_{\boldsymbol{k},\boldsymbol{\theta}}$). Furthermore, we assume that $\boldsymbol{f}^{\sigma_i}_{+\sigma_i}\in\mathbb{S}^r_{\boldsymbol{k},\boldsymbol{\theta}}$ (where $\boldsymbol{f}^{\sigma_i}_{+\sigma_i}(\varepsilon)=\boldsymbol{f}^{\sigma_i}(\sigma_i+\varepsilon)$ for ε with $|\arg\varepsilon-k_p\,\theta_p|<\pi/2$), $|\arg(\sigma_{i\pm1}-\sigma_i)-k_p\,\theta_p|<\pi/2$.

Moreover, for each $i<l$, the open ball with center σ_i and radius $|\sigma_{i+1}-\sigma_i|$ is assumed to contain no other singularities than σ_i. If the σ_i are all non singular or regular singular, then we call $z_0\to z_1\to\cdots\to z_l$ a *regular singular broken-line path*.

Now given an irregular singular point $\sigma\in\mathbb{K}$, such that $\boldsymbol{f}^\sigma_{+\sigma}\in\mathbb{S}^r_{\boldsymbol{k},\boldsymbol{\theta}}$ for critical times $\boldsymbol{k}$ and directions $\boldsymbol{\theta}$, we define the transition matrix

$$\Delta_{\sigma_{\boldsymbol{k},\boldsymbol{\theta}}\to z}=\begin{pmatrix}\operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}}f^\sigma_{+\sigma,0}(z) & \cdots & \operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}}f^\sigma_{\sigma,r-1}(z)\\ \vdots & & \vdots\\ \operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}}(f^\sigma_{+\sigma,0})^{(r-1)}(z) & \cdots & \operatorname{sum}_{\boldsymbol{k},\boldsymbol{\theta}}(f^\sigma_{+\sigma,r-1})^{(r-1)}(z)\end{pmatrix},$$

for any z with $|\arg(z-\sigma)-k_p\,\theta_p|<\pi/2$ and such that z is sufficiently close to σ. For regular singular points $\sigma\in\mathbb{K}$, a similar definition was given in [Hoeven, 2001b].

In view of (4.6) and (4.7), we may extend this definition to arbitrary singular broken-line paths. In particular, it can be checked that the Stokes matrices for L are all of the form

$$\Sigma_{\sigma,\boldsymbol{k},\boldsymbol{\theta},\boldsymbol{\theta}'}=\Delta_{\sigma_{\boldsymbol{k},\boldsymbol{\theta}}\to\sigma+\varepsilon\to\sigma_{\boldsymbol{k},\boldsymbol{\theta}'}}=\Delta^{-1}_{\sigma_{\boldsymbol{k},\boldsymbol{\theta}'}\to\sigma+\varepsilon}\,\Delta_{\sigma_{\boldsymbol{k},\boldsymbol{\theta}}\to\sigma+\varepsilon}.$$

Notice that this definition does not depend on the choice of ε. In a similar way as in [Hoeven, 2001b], it is also possible to construct a suitable extension $\hat{\mathcal{R}}$ of $\mathcal{R}$ with "irregular singular points", in such a way that singular broken-line paths may be lifted to $\hat{\mathcal{R}}$. However, such an extension will not be needed in what follows.

4.3.3. Transition matrices for the multivariate case

It is well known that the theory of Gröbner bases generalizes to partial differential operators in the ring $\mathbb{K}(z_1,\ldots,z_n)[\partial_1,\ldots,\partial_n]$. Consider a zero-dimensional system of such operators given by a Gröbner basis $\boldsymbol{L}=(L_1,\ldots,L_s)$. Let $\mathcal{K}$ be the set of tuples $(k_1,\ldots,k_n)$, such that $l_1\leqslant k_1,\ldots,l_n\leqslant k_n$ holds for no leading monomial $\partial_1^{l_1}\cdots\partial_n^{l_n}$ of one of the L_i. We may enumerate $\mathcal{K}=\{\boldsymbol{k}_0,\ldots,\boldsymbol{k}_{r-1}\}$, with $\boldsymbol{k}_0<\cdots<\boldsymbol{k}_{r-1}$ for a fixed total ordering $<$ on the monoid $\mathbb{N}^n$.

Given a non-singular point $\boldsymbol{z}\in\mathbb{C}^n$ for $\boldsymbol{L}$, there again exists a unique *canonical fundamental system*

$$\boldsymbol{f}^{\boldsymbol{z}}=(\;f_0^{\boldsymbol{z}}\;\;\cdots\;\;f_{r-1}^{\boldsymbol{z}}\;)$$

of analytic solutions to $\boldsymbol{L} f = 0$ at z with the property that $\boldsymbol{\partial}^{\boldsymbol{k}_i} f_j = \delta_{i,j}$ for all i, j. Also, an arbitrary solution f to $L f = 0$ is uniquely determined by the vector

$$F(\boldsymbol{z}) = \left(\begin{array}{c} \boldsymbol{\partial}^{\boldsymbol{k}_0} f(z) \\ \vdots \\ \boldsymbol{\partial}^{\boldsymbol{k}_{r-1}} f(z) \end{array} \right)$$

of its initial conditions at $\boldsymbol{z}$ by $f = \boldsymbol{f}^z F(z)$. Consequently, the definitions and properties (4.4–4.7) naturally generalize to the multidimensional paths $\boldsymbol{z} \rightsquigarrow \boldsymbol{z}'$ which avoid the singularities of $\boldsymbol{L}$.

4.4. Holonomic constants

Recall that $\mathcal{D}_{c,r}$ and $\bar{\mathcal{D}}_{c,r}$ stand for the open and closed disks of center c and radius r. A constant α in $\mathbb{C}$ is said to be *holonomic* over $\mathbb{K}$ if there exists a linear differential operator $L = \partial^n + L_{n-1} \partial^{n-1} + \cdots + L_0 \in \mathbb{K}(z)[\partial]$ and a vector of initial conditions $\boldsymbol{v} \in \mathbb{K}^n$, such that the L_i are defined on $\bar{\mathcal{D}}_{0,1}$ and $\alpha = f(1)$, where f is the unique solution to $L f = 0$ with $f^{(i)}(0) = v_{i+1}$ for $i < n$. We denote by $\mathbb{K}^{\mathrm{hol}}$ the set of holonomic constants over $\mathbb{K}$.

PROPOSITION 4.6.

a) $\mathbb{K}^{\mathrm{hol}}$ *is a subring of* $\mathbb{C}$.

b) *Let* L *be a linear differential operator of order* n *in* $\mathbb{K}(z)[\partial]$. *Then* $\Delta_\gamma^L \in \mathsf{Mat}_r(\mathbb{K}^{\mathrm{hol}})$ *for any non singular broken-line path* γ *with end-points in* $\mathbb{K}$.

c) *Let* $\boldsymbol{L} = (L_1, \ldots, L_s)$ *be a Gröbner basis for a zero-dimensional system of differential operators in* $\mathbb{K}(\boldsymbol{z})[\boldsymbol{\partial}]$. *Then for any non singular broken-line path* γ *with end-points in* $\mathbb{K}^n$, *we have* $\Delta_\gamma^{\boldsymbol{L}} \in \mathsf{Mat}_r(\mathbb{K}^{\mathrm{hol}})$.

Proof. Consider holonomic constants $\alpha = f(1)$ and $\beta = g(1)$, where f and g are solutions to $K f = 0$ and $L g = 0$ with initial conditions in $\mathbb{K}^m$ resp. $\mathbb{K}^n$ and where the coefficients of K and L are defined on $\bar{\mathcal{D}}_{0,1}$. By the corollary 4.4, the coefficients of $K \boxtimes L$ are again defined on $\bar{\mathcal{D}}_{0,1}$ and $\alpha \beta = h(1)$, where h is the unique solution with initial conditions $h^{(i)}(0) = \sum_{j=0}^{i} \binom{j}{i} f^{(j)}(0)\, g^{(i-j)}(0) \in \mathbb{K}$ for $i < m n$. A similar argument shows the stability of $\mathbb{K}^{\mathrm{hol}}$ under addition.

As to (b), we first observe that the transition matrix $\Delta_{0 \rightarrow 1}$ along the straight-line path from 0 to 1 has holonomic entries, provided that the coefficients of L are defined on $\bar{\mathcal{D}}_{0,1}$. Indeed, by corollary 4.4, the coefficients of the monic operators $L^{\boxplus^i}$ with solution spaces $\mathcal{H}_L^{(i)}$ are defined on $\bar{\mathcal{D}}_{0,1}$. Using a transformation $z \mapsto (\mu - \lambda) z + \lambda$ with $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}$, it follows that $\Delta_{\lambda \rightarrow \mu}$ has holonomic entries whenever the L_i are defined on the closed disk $\bar{\mathcal{D}}_{\lambda, |\mu - \lambda|}$. Now any broken-line path γ is homotopic to a broken-line path $\lambda_1 \rightarrow \cdots \rightarrow \lambda_l$ such that the L_i are defined on the closed disks $\bar{\mathcal{D}}_{\lambda_j, |\lambda_{j+1} - \lambda_j|}$. From (a), we therefore conclude that $\Delta_\gamma = \Delta_{\lambda_{l-1} \rightarrow \lambda_l} \cdots \Delta_{\lambda_1 \rightarrow \lambda_2}$ has holonomic entries.

As to the last property, we first notice that the function $f(\boldsymbol{u} + t \boldsymbol{v})$ is holonomic in t for any fixed $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{K}^n$. In a similar way as above, it follows that the multivariate transition matrix from section 4.3.3 along a straight-line path $\boldsymbol{u} \rightarrow \boldsymbol{v}$ has entries in $\mathbb{K}$ for sufficiently close $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{K}^n$. Since any non singular broken-line path is homotopic to the finite composition of straight-line paths of this kind, we conclude by the multivariate analogue of (4.6) and (a). □

A number α in $\mathbb{C}$ is said to be a *singular holonomic constant* over $\mathbb{K}$ if there exists a linear differential operator $L = \partial^n + L_{n-1}\partial^{n-1} + \cdots + L_0 \in \mathbb{K}(z)[\partial]$ and a vector of initial conditions $\boldsymbol{v} \in \mathbb{K}^n$, such that the L_i are defined on $\mathcal{D}_{0,1}$ and $\alpha = \lim_{z \to 1} f(z)$, where f is the unique solution to $L f = 0$ with $f^{(i)}(0) = v_{i+1}$ for $i < n$. We understand that the limit $z \to 1$ is taken on the straight-line path from 0 to 1. If L is regular singular at 1, then we call α a *regular singular holonomic constant* over $\mathbb{K}$. We denote by $\mathbb{K}^{\mathrm{shol}}$ the class of singular holonomic constants over $\mathbb{K}$ and by $\mathbb{K}^{\mathrm{rhol}}$ the class of regular singular holonomic constants over $\mathbb{K}$.

PROPOSITION 4.7.

a) $\mathbb{K}^{\mathrm{rhol}}$ *is a subring of* $\mathbb{C}$.

b) $\mathbb{K}^{\mathrm{shol}}$ *is a subring of* $\mathbb{C}$.

c) Let L be a linear differential operator of order n in $\mathbb{K}(z)[\partial]$. Then $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{rhol}})$ for any regular singular broken-line path γ as in section 4.3.2.

d) Let L be a linear differential operator of order n in $\mathbb{K}(z)[\partial]$. Then $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{shol}})$ for any singular broken-line path γ as in section 4.3.2.

Proof. Several errors slipped into the original proof. We provided an updated proof in Appendix ?. Note that we only prove $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{hola}})$ in (c). □

5. BOUNDS FOR THE TRANSITION MATRICES

5.1. Integral formula for transition matrices

Consider a linear differential operator

$$L = \partial^r + L_{r-1}\partial^{r-1} + \cdots + L_0$$

whose coefficients are analytic functions on an open or closed subset $\mathcal{R}$ of $\mathbb{C}$. We will give an explicit formula for the transition matrix $\Delta_\gamma = \Delta_\gamma^L$ along a path γ in $\mathcal{R}$.

Let us first rewrite the equation $L f = 0$ as a first order system and give an alternative characterization for the transition matrix. Let

$$M = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ \vdots & & \ddots & \\ 0 & \mathbf{0} & & 1 \\ -L_0 & -L_1 & \cdots & -L_{r-1} \end{pmatrix}$$

Then the differential equation

$$\Phi' = M\,\Phi \tag{5.1}$$

admits a unique solution Φ with $\Phi(\zeta) = I$. Given a path $\zeta \rightsquigarrow \zeta'$ in $\mathcal{R}$, it is not hard to see that $\Delta_{\zeta \rightsquigarrow \zeta'}$ coincides with the analytic continuation of Φ along $\zeta \rightsquigarrow \zeta'$.

Given an analytic function f on $\mathcal{R}$, we will denote by $\int_\zeta f$ the unique analytic function on $\mathcal{R}$ given by

$$\left(\int_\zeta f\right)(\zeta') = \int_\zeta^{\zeta'} f(\xi)\,\mathrm{d}\xi.$$

Then the system (5.1) with our initial condition admits a natural solution

$$\Delta_{\zeta \rightsquigarrow \zeta'} = \left(I + \int_\zeta M + \int_\zeta M \int_\zeta M + \cdots\right)(\zeta'). \tag{5.2}$$

We will show below that this "integral series" indeed converges when $\zeta \rightsquigarrow \zeta'$ is a straight-line path. In fact, using a similar technique, one can show that the formula is valid in general, but we will not need that in what follows.

5.2. Majorants

Let $\mathcal{C}(\mathcal{R}, \mathbb{C})$ and $\mathcal{C}(\mathcal{R}, \mathbb{R}^{\geqslant})$ denote the spaces of continuous $\mathbb{C}$-valued resp. $\mathbb{R}^{\geqslant}$-valued functions on $\mathcal{R}$. Given matrices A and B of the same sizes and with coefficients in $\mathcal{C}(\mathcal{R}, \mathbb{C})$ resp. $\mathcal{C}(\mathcal{R}, \mathbb{R}^{\geqslant})$, we say that A is *majored* by B, and we write $A \trianglelefteq B$, if

$$|A_{i,j}(\zeta)| \leqslant B_{i,j}(\zeta)$$

for all i, j. Given majorations $A \trianglelefteq B$ and $\tilde{A} \trianglelefteq \tilde{B}$, we clearly have majorations

$$A + \tilde{A} \ \trianglelefteq \ B + \tilde{B} \tag{5.3}$$

$$A \tilde{A} \ \trianglelefteq \ B \tilde{B} \tag{5.4}$$

Assuming that every point in $\mathcal{R}$ can be reached by a straight-line path starting at ζ, we also have

$$\int_\zeta A \ \trianglelefteq \ \int_\zeta^{\text{real}} B, \tag{5.5}$$

where

$$\left(\int_\zeta^{\text{real}} B\right)(\zeta') = \int_0^{|\zeta'-\zeta|} B\left(\zeta + \frac{\xi - \zeta}{|\xi - \zeta|}\,\xi\right) \mathrm{d}\,\xi.$$

Assume now that M is bounded on $\mathcal{R}$. Then there exist constants $\beta_0, \ldots, \beta_{r-1} \geqslant 0$ with

$$M \trianglelefteq B = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ \vdots & & \ddots & \\ 0 & \mathbf{0} & & 1 \\ \beta_0 & \beta_1 & \cdots & \beta_{r-1} \end{pmatrix}$$

and we may assume without loss of generality that B admits pairwise distinct eigenvalues. From the rules (5.3), (5.4) and (5.5), it follows that

$$\Delta_{\zeta \rightsquigarrow \zeta'} \trianglelefteq \left(I + \int_\zeta^{\text{real}} B + \int_\zeta^{\text{real}} B \int_\zeta^{\text{real}} B + \cdots\right)(\zeta').$$

The right-hand side of this majoration can be rewritten as $\Psi(|\zeta' - \zeta|)$, where Ψ is the unique solution on $\mathbb{R}^{\geqslant}$ to the equation

$$\Psi' = B\,\Psi,$$

such that $\Psi(0) = I$. Now let U and D be matrices with

$$B = U^{-1} D\, U,$$

where

$$D = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_r \end{pmatrix}.$$

Then we have

$$\Psi(x) = U^{-1} \begin{pmatrix} \mathrm{e}^{\lambda_1 x} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathrm{e}^{\lambda_r x} \end{pmatrix} U.$$

This shows in particular that (5.2) converges when $\zeta \rightsquigarrow \zeta'$ is a straight-line path, since it suffices to replace $\mathcal{R}$ by a compact convex subset which contains a neighbourhood of $\zeta \to \zeta'$.

5.3. Bounds for the transition matrices

Given an operator L with coefficients in $\mathbb{K}(\zeta^{\mathbb{Q}})$ which are bounded at infinity, it is not hard to explicitly compute a sector $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$ with $\alpha < \pi/2$ on which the L_i have no poles and a majorating matrix B with coefficients in $\mathbb{K}$. The aperture α may chosen as close to $\pi/2$ as desired. Then the results from the previous section yield:

THEOREM 5.1. *There exists an algorithm which, given an operator*

$$L = \partial^r + L_{r-1}\,\partial^{r-1} + \cdots + L_0 \in \mathbb{K}(\zeta^{\mathbb{Q}})[\partial]$$

with $L_i = O(1)$ for all i at infinity, computes a sector $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$ and constants $K, \lambda \in \mathbb{R}^{>}$ with

$$\|\Delta_{\zeta \to \zeta'}\| \leqslant K\,\mathrm{e}^{\lambda|\zeta' - \zeta|}$$

for all straight-line path inside $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$.

More generally, given an operator $L \in \mathbb{K}(\zeta^{\mathbb{Q}})[\delta]$ of growth rate $\kappa > 0$, the operator $\tilde{L} = \mathcal{P}_{1/\kappa}\,L$ has growth rate one and we have

$$\Delta^{\tilde{L}}_{\zeta \to \zeta'} = \Delta^{L}_{\zeta^{\kappa} \to (\zeta')^{\kappa}}$$

for all straight-line paths $\zeta \to \zeta'$ whose image under $\zeta \mapsto \zeta^{\kappa}$ is homotopic to the straight-line path $\zeta^{\kappa} \to (\zeta')^{\kappa}$. Moreover, after replacing δ by $\zeta\,\partial$ in $\tilde{L}$ and dividing by a suitable power of ζ, we observe that $\tilde{L}$ fulfills the conditions of theorem 5.1. We thus obtain:

THEOREM 5.2. *There exists an algorithm which, given an operator*

$$L = \partial^r + L_{r-1}\,\partial^{r-1} + \cdots + L_0 \in \mathbb{K}(\zeta^{\mathbb{Q}})[\partial]$$

with growth rate $\kappa > 0$ at infinity, computes a sector $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$ and constants $K, \lambda \in \mathbb{R}^{>}$ with

$$\|\Delta_{\zeta \to \zeta'}\| \leqslant K\,\mathrm{e}^{\lambda|\zeta'^{\kappa} - \zeta^{\kappa}|}$$

for all straight-line path inside $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$.

Remark 5.3. In fact, the hypothesis that $\zeta \rightsquigarrow \zeta'$ is a straight-line path is not really necessary in theorem 5.1. With some more work, one may actually consider sectors of $\dot{\mathbb{C}}$ at infinity with aperture larger than $\pi/2$. In theorem 5.2, this allows you to impose the aperture of α to be as large as desired.

6. EFFECTIVE INTEGRAL TRANSFORMS

Consider an operator

$$L = L_r\,\delta^r + \cdots + L_0 \in \mathbb{K}[\zeta][\delta]$$

with growth rate $\kappa > 0$ at infinity. Let $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$ be a sector of aperture $\alpha < \pi/2$ such that L_r does not vanish on $\bar{\mathcal{S}}^{\infty}_{\theta,\alpha,R}$ and such that we have a bound

$$\|\Delta_{\zeta \to \zeta'}\| \leqslant K\,\mathrm{e}^{\lambda|\zeta'^{\kappa} - \zeta^{\kappa}|} \tag{6.1}$$

for all $\zeta, \zeta' \in \bar{\mathcal{S}}_{\theta,\alpha,R}^{\infty}$. Let

$$\rho = \frac{\sin\alpha}{1-\sin\alpha} R,$$

so that the ball centered at $(R+\rho)\,\mathrm{e}^{\mathrm{i}\theta}$ with radius ρ is just contained in $\bar{\mathcal{S}}_{\theta,\alpha,R}^{\infty}$ (see figure 6.1), and let $\nu \in \mathbb{N}\,2^{\mathbb{Z}}$ be a fixed constant of small bit-size, with $1<\nu<\mu=1+\rho/(R+\rho)$.

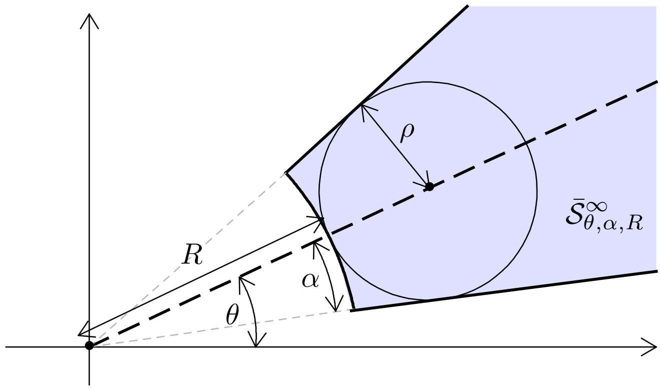

Figure 6.1. The sector $\bar{\mathcal{S}}_{\theta,\alpha,R}^{\infty}$ and the associated constants R, θ, α and ρ.

6.1. Uniformly fast approximation of transition matrices

Let $\zeta, \zeta' \in \mathbb{R}^{\geqslant}\mathrm{e}^{\mathrm{i}\theta}$ with $|\zeta'|>|\zeta|\geqslant R+\rho$ and $\varepsilon>0$. Assuming that $\mathrm{e}^{\mathrm{i}\theta}, \zeta, \zeta', \varepsilon \in \mathbb{K}$, we may now use the algorithm `approx` below in order to approximate $\Delta_{\zeta\to\zeta'}$ at precision ε. The computation of $\tilde{\Delta} := \Delta_0 + \cdots + \Delta_{k-1}(\zeta'-\zeta)^{k-1}$ is done using the binary splitting algorithm from [Chudnovsky and Chudnovsky, 1990; Hoeven, 1999].

Algorithm `approx(ı ; ı′; ″)`
Input: $\zeta, \zeta', \varepsilon \in \mathbb{K}$ as above
Output: a matrix $\tilde{\Delta}$ with $\|\tilde{\Delta} - \Delta_{\zeta\to\zeta'}\| < \varepsilon$

if $|\zeta'| \leqslant \nu\,|\zeta|$ **then**
  Let $k \in \mathbb{N}$ be minimal with $K\,\mathrm{e}^{\lambda(\mu^{\kappa}-1)|\zeta|^{\kappa}}\frac{\beta^k}{1-\beta} < \frac{\varepsilon}{2}$, where $\beta = \frac{|\zeta'-\zeta|}{\mu\,|\zeta|}$
  Consider the expansion $\Delta_{\zeta\to\zeta+t} = \Delta_0 + \Delta_1 t + \Delta_2 t^2 + \cdots$
  Compute $\tilde{\Delta} := \Delta_0 + \Delta_1(\zeta'-\zeta) + \cdots + \Delta_{k-1}(\zeta'-\zeta)^{k-1}$ at precision $\varepsilon/2$
  Return $\tilde{\Delta}$
else
  Let $M_2 := K\,\mathrm{e}^{\lambda|\zeta'^{\kappa} - (\nu\zeta)^{\kappa}|}$
  Compute $\tilde{\Delta}_1 := \mathtt{approx}(\zeta, \nu\zeta, \varepsilon/(2M_2))$
  Compute $\tilde{\Delta}_2 := \mathtt{approx}(\nu\zeta, \zeta', \varepsilon/(2\|\tilde{\Delta}_1\|))$
  Return $\tilde{\Delta}_2\tilde{\Delta}_1$

THEOREM 6.1.

a) *The algorithm* `approx` *is correct.*

b) *Let* $n = \max(|\zeta'|^{\kappa}, -\log\varepsilon)$ *and let* s *be the sum of the bit-sizes of* ζ *and* ζ'. *Then the running time of the algorithm is uniformly bounded by* $O(M(n)\log^2 n\,(\log n + s))$.

Proof. The correctness of the algorithm in the "single-step case" when $|\zeta'| \leqslant \nu\, |\zeta|$ follows from (6.1) and Cauchy's formula, since

$$\begin{aligned}\|\tilde{\Delta} - \Delta_{\zeta \to \zeta'}\| &\leqslant \sum_{i \geqslant k} \|\Delta_i\|\, |\zeta' - \zeta|^k \\ &\leqslant \sum_{i \geqslant k} K\, \mathrm{e}^{\lambda |(\mu\zeta)^\kappa - \zeta^\kappa|} \frac{|\zeta' - \zeta|^i}{|\mu\, \zeta|^i} \\ &= K\, \mathrm{e}^{\lambda(\mu^\kappa - 1)|\zeta|^\kappa} \frac{\beta^k}{1 - \beta}.\end{aligned}$$

In the "multi-step case" when $|\zeta'| > \nu\, |\zeta|$, we have

$$\begin{aligned}\|\Delta_{\nu\zeta \to \zeta'}\, \Delta_{\zeta \to \nu\zeta} - \tilde{\Delta}_2\, \tilde{\Delta}_1\| &\leqslant \|\Delta_{\nu\zeta \to \zeta'}\, (\Delta_{\zeta \to \nu\zeta} - \tilde{\Delta}_1)\| + \|(\Delta_{\nu\zeta \to \zeta'} - \tilde{\Delta}_2)\, \tilde{\Delta}_1\| \\ &\leqslant M_2\, \|\Delta_{\zeta \to \nu\zeta} - \tilde{\Delta}_1\| + \|\Delta_{\nu\zeta \to \zeta'} - \tilde{\Delta}_2\|\, \|\tilde{\Delta}_1\|,\end{aligned}$$

and the result follows by induction.

As to the complexity analysis, let l be minimal such that $|\zeta|\, \nu^l \geqslant |\zeta'|$ and denote

$$\begin{aligned}\zeta_i &= \zeta \nu^i \quad (i < l) \\ \zeta_l &= \zeta'.\end{aligned}$$

Then the recursive application of the algorithm gives rise to l single-step cases for each $\Delta_{\zeta_i \to \zeta_{i+1}}$ with $i < l$. We have $l = O(\log |\zeta'|) = O(\log n)$ and claim that the precision ε_i at which we approximate each $\Delta_{\zeta_i \to \zeta_{i+1}}$ satisfies $\varepsilon_i \geqslant \varepsilon / (2^l\, M)$, where $M = K\, \mathrm{e}^{\lambda |\zeta'^\kappa - \zeta^\kappa|}$.

Indeed, using induction over l, this is clear in the case when $l = 1$. In the multi-step case, we have $M_2 \leqslant M$ and $\|\tilde{\Delta}_1\| \leqslant M_1 = K\, \mathrm{e}^{\lambda |(\nu\zeta)^\kappa - \zeta^\kappa|}$. Hence, $\Delta_{\zeta_0 \to \zeta_1}$ is approximated at precision $\varepsilon / (2\, M_2) \geqslant \varepsilon / (2^l\, M)$. The induction hypothesis also implies that each $\Delta_{\zeta_i \to \zeta_{i+1}}$ is approximated at precision $\varepsilon_i \geqslant \varepsilon' / (2^{l-1}\, M')$, where $\varepsilon' = \varepsilon / (2\, M_1)$ and $M' = K\, \mathrm{e}^{\lambda |\zeta'^\kappa - (\nu\zeta)^\kappa|}$. We conclude that $\varepsilon_i \geqslant \varepsilon' / (2^{l-1}\, M') = \varepsilon / (2^l\, M_1\, M') = \varepsilon / (2^l\, M)$.

Having proved our claim, let us now estimate the cost of each single-step approximation of $\Delta_{\zeta_i \to \zeta_{i+1}}$ at precision $\varepsilon_i \geqslant \varepsilon / (2^l\, M)$. Since $0 < \beta < (\nu - 1) / \mu < 1$, the minimal k satisfies

$$\begin{aligned}k &= O\!\left(-\log\left(\frac{\varepsilon}{2^l\, M\, \mathrm{e}^{\lambda(\mu^\kappa - 1)|\zeta_i|^\kappa}}\right)\right) \\ &= O(-\log \varepsilon) + O(l) + O(\log M) + O(|\zeta_i|^\kappa) \\ &= O(n).\end{aligned}$$

Furthermore, the entries of $\tilde{\Delta}$ are $O(n)$-digit numbers, since

$$\Delta_{\zeta_i \to \zeta_{i+1}} \leqslant K\, \mathrm{e}^{\lambda |\zeta_{i+1}^\kappa - \zeta_i^\kappa|}$$

and the size of ζ_i is bounded by $O(s) + O(i) = O(s + \log n)$. By a similar argument as in the start of section 4.1 of [Hoeven, 1999], it follows that the ε_i-approximation of $\tilde{\Delta}$ is computed in time $O(M(n) \log n\, (\log n + s))$ using binary splitting. Since $l = O(\log n)$, the overall running time is therefore bounded by $O(M(n) \log^2 n\, (\log n + s))$. □

6.2. Fast approximation of integral transforms

Consider a second differential operator $\Omega \in \mathbb{K}[\zeta][\delta]$ with growth rate κ at infinity. Let f be a solution to $\Omega\, f = 0$ with initial conditions in $\mathbb{K}$ at a point $\zeta \in \mathbb{K}$ with $\arg \zeta = \theta$ and $|\zeta| \geqslant R + \rho$. Assume that f satisfies a bound

$$|f(\xi)| \leqslant K'\, \mathrm{e}^{-\lambda' |\xi|^\kappa} \tag{6.2}$$

on $[\zeta, \mathrm{e}^{\mathrm{i}\theta}\infty]$, where $K', \lambda' > 0$. Our aim is to compute

$$\Phi = \int_{\zeta}^{\mathrm{e}^{\mathrm{i}\theta}\infty} f(\xi)\,\mathrm{d}\xi.$$

Now the primitive

$$\grave{}f(\zeta') = \int_{\zeta}^{\zeta'} f(\xi)\,\mathrm{d}\xi$$

satisfies the equation $(\grave{}\Omega)(\grave{}f) = 0$, where the operator $\grave{}\Omega := (\mathcal{M}_{-1}\Omega)\,\delta \in \mathbb{K}[\zeta][\delta]$ has growth rate κ at infinity. Moreover, $\grave{}f$ admits initial conditions in $\mathbb{K}$ at ζ.

Assuming that we chose $L = \grave{}\Omega$ and that the bound (6.1) holds for the transition matrices associated to L, we may now use the following simple algorithm for the approximation of Φ.

Algorithm `integral_approx(")`
Input: $\varepsilon \in \mathbb{K}^{>}$
Output: an approximation $\tilde{\Phi}$ for Φ with $|\tilde{\Phi} - \Phi| < \varepsilon$

Let I be the vector of initial conditions for $\grave{}f$ at ζ, so that $\grave{}f(\zeta') = \Delta^{L}_{\zeta \to \zeta'} I$
Take $\zeta' \in \mathbb{K}$ with $\arg \zeta' = \theta$ such that $|\int_{|\zeta'|}^{\infty} K'\,\mathrm{e}^{-\lambda' t^{\kappa}}\,\mathrm{d}t| < \varepsilon/2$
Return `approx`$(\zeta, \zeta', \varepsilon/(2\,\|I\|))\,I$

In the case when $\kappa \geqslant 1$, we notice that

$$\int_{T}^{\infty} K'\,\mathrm{e}^{-\lambda' t^{\kappa}}\,\mathrm{d}t = \int_{T^{\kappa}}^{\infty} \frac{K'}{\kappa\, t^{1-1/\kappa}}\,\mathrm{e}^{-\lambda' t}\,\mathrm{d}t \leqslant \int_{T^{\kappa}}^{\infty} \frac{K'}{\kappa}\,\mathrm{e}^{-\lambda' t}\,\mathrm{d}t = \frac{K'}{\kappa\,\lambda'}\,\mathrm{e}^{-\lambda' T^{\kappa}}$$

for all $T \geqslant 1$, so we may take

$$|\zeta'| = \max\left(\operatorname{lround} \sqrt[\kappa]{\frac{\max\left(\log(2\,K'/(\kappa\,\lambda'\,\varepsilon)), 0\right)}{\lambda'}}, 1\right), \tag{6.3}$$

where $\operatorname{lround}(x)$ is the largest number in $2^{\mathbb{Z}}\,\{0, \ldots, 2^{32} - 1\}$ below x. In the case when $\kappa < 1$, we may use lemma 2.3 to replace the bound (6.2) by a bound of the form

$$|f(\xi)| \leqslant K'\,\mathrm{e}^{-\lambda' |\xi|^{\kappa}} \leqslant K''\,|\xi|^{1-1/\kappa}\,\mathrm{e}^{-\lambda'' |\xi|^{\kappa}},$$

with $0 < \lambda'' < \lambda$. Then

$$\int_{T}^{\infty} K'\,\mathrm{e}^{-\lambda' t^{\kappa}}\,\mathrm{d}t \leqslant \int_{T}^{\infty} K''\,t^{1-1/\kappa}\,\mathrm{e}^{-\lambda'' t^{\kappa}}\,\mathrm{d}t \leqslant \int_{T^{\kappa}}^{\infty} \frac{K''}{\kappa}\,\mathrm{e}^{-\lambda'' t}\,\mathrm{d}t = \frac{K''}{\kappa\,\lambda''}\,\mathrm{e}^{-\lambda'' T^{\kappa}}$$

and we may take

$$|\zeta'| = \operatorname{lround} \sqrt[\kappa]{\frac{\max\left(\log(2\,K''/(\kappa\,\lambda''\,\varepsilon)), 0\right)}{\lambda''}}. \tag{6.4}$$

For both formulas (6.3) and (6.4), we have $|\zeta'| = O(\sqrt[\kappa]{-\log \varepsilon})$. Applying theorem 6.1, it follows that

THEOREM 6.2. *The algorithm* `integral_approx` *is correct and its running time is bounded by* $O(M(n) \log^3 n)$*, where* $n = -\log \varepsilon$*.* □

Example 6.3. Consider the formulas

$$\begin{aligned}\Gamma(z) &= \int_0^\infty x^{z-1}\,\mathrm{e}^{-x}\,\mathrm{d}x\\ \frac{1}{\Gamma(z)} &= \frac{\mathrm{i}}{2\,\pi}\int_{\mathcal{H}}(-t)^{-z}\,\mathrm{e}^{-t}\,\mathrm{d}t\\ \gamma^{(m)}(z) &= \frac{\mathrm{i}}{2\,\pi}\int_{\mathcal{H}}(-\log(-t))^m\,(-t)^{-z}\,\mathrm{e}^{-t}\,\mathrm{d}t,\end{aligned}$$

where $\gamma(z)=1/\Gamma(z)$, $m\in\mathbb{N}$, and $\mathcal{H}$ is a Hankel contour from ∞ around 0 and then back to ∞. For any $z\in\mathbb{K}$, these integrals can be evaluated using our fast algorithm, whenever they are defined. So the values of these integrals are in $\mathbb{K}^{\mathrm{shol}}$ and they have $O(M(n)\log^2 n)$-approximation algorithms. Using Euler's reflection formula

$$\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin(\pi\,z)},$$

and Proposition 4.7(b), we also see that $\sin(\pi z)^{-1}\in\mathbb{K}^{\mathrm{shol}}$ and $(1-\mathrm{e}^{-2\pi\mathrm{i}z})^{-1}\in\mathbb{K}^{\mathrm{shol}}$ for all $z\in\mathbb{K}\setminus\mathbb{Z}$. Note that $(1-\mathrm{e}^{-2\pi\mathrm{i}z})^{-1}$ has an $O(M(n)\log^2 n)$-approximation algorithm.

7. EFFECTIVE ACCELERO-SUMMATION

Let us now show how to put the results from the previous sections together into an accelero-summation algorithm for holonomic functions. Let $\tilde{f}\in\mathbb{O}$ be a formal solution with initial conditions in $\mathbb{K}$ at the origin to the equation $L f=0$ with $L\in\mathbb{K}[z][\delta]$. We will first show how to determine the critical times $k_1>\cdots>k_p$ in $\mathbb{Q}^{>}$ and the Stokes directions at each critical time. Having fixed $\theta_1\in\mathcal{R}_1:=\mathbb{R}\setminus\mathcal{D}_1,\ldots,\theta_p\in\mathcal{R}_p:=\mathbb{R}\setminus\mathcal{D}_p$, we next detail the effective acceleration-procedure and show how to efficiently evaluate $f=\mathrm{sum}_{\boldsymbol{k},\boldsymbol{\theta}}\tilde{f}$ in a sector close to the origin.

7.1. Setting up the framework

Normalization Without loss of generality, we may assume that the valuation of f at zero is larger than the degree d of L in z. Indeed, it suffices to replace f by $f z^n$ and L by $\mathcal{M}_n L$ for a sufficiently large n.

Critical times Let $\sigma_1<\cdots<\sigma_p$ be the non-horizontal slopes of the Newton polygon of L at the origin. We take $k_1=1/\sigma_1,\ldots,k_p=1/\sigma_p$, so the critical times are $z_1=z^{\sigma_1},\ldots,z_p=z^{\sigma_p}$. For example, in figure 3.4, the critical times are $z_1=\sqrt{z}$ and $z_2=z$.

Equations for $\hat{f}_i$ and $\check{f}_i$ For each critical time z_i, let us show how to compute vanishing operators for $\hat{f}_i$ and $\check{f}_i$. Let $a,b\in\mathbb{N}^{\neq}$ be relatively prime with $k_i=a/b$. Since $b\leqslant d$, we notice that the valuation of f_i in $z_i=0$ is larger than one.

1. We first compute $L^{\boxdot b}\in\mathbb{K}[z][\delta]$ and $\mathcal{P}_a L^{\boxdot b}\in\mathbb{K}[z][\delta]$. We may reinterpret $\mathcal{P}_a L^{\boxdot b}$ as an operator in $\mathbb{K}[z_i][\delta]$ and notice that $(\mathcal{P}_a L^{\boxdot b})(f_i)=0$.

2. Let n be minimal, such that $z_i^{-n}\mathcal{P}_a L^{\boxdot b}\in\mathbb{K}[z_i^{-1}][\delta]$. We compute the Borel transform $\hat{L}_i=\mathcal{B}(z_i^{-n}\mathcal{P}_a L^{\boxdot b})\in\mathbb{K}[\zeta_i^{-1}]\,[\delta]$. Since $v^{z_i=0}(f_i)>1$, we formally have $\hat{L}_i\,\hat{f}_i=0$. In fact, since the accelero-summation process preserves differentially algebraic relations, we will also have $\hat{L}_i\,\hat{f}_i=0$.

3. Compute $\check{L}_i$ with $\check{L}_i\,\check{f}_i=0$ using the procedure from section 4.2.5.

Singular directions For our accelero-summation process to work, it will suffice to avoid the non-zero singularities of the operator $\check{L}_i$ at each critical time z_i. In other words, denoting by $\check{r}_i$ the order of $\check{L}_i$, we take $\mathcal{D}_i = \{\arg u : u \in \mathbb{K}^{\neq}, \check{L}_{i,\check{r}_i}(u) = 0\}$.

Growth rates of $\hat{L}_i$ and $\check{L}_i$ Given a critical time z_i, let us now study the growth rates of $\hat{L}_i$ and $\check{L}_i$ at zero and infinity. By corollary 4.5, and with a, b as above, the slopes of the Newton polygon of $\mathcal{P}_a L^{\boxdot b}$ are $\sigma_1 k_i = k_i / k_1, \ldots, \sigma_p k_i = k_i / k_p$. By section 3.3 and formula (4.1), it follows that the non-horizontal slopes of the Newton polygons of $\hat{L}_i$ and $\check{L}_i$ at the origin are

$$\frac{k_i}{k_1 - k_i} < \cdots < \frac{k_i}{k_{i-1} - k_i}.$$

In particular, if $i = 1$, then $\check{L}_i$ is regular singular at 0 and [Hoeven, 2001b] shows how to compute the values of $\check{f}_1$ in the neighbourhood of $\dot{0}$. We also infer that the non-horizontal slopes of the Newton polygon of $\hat{L}_i$ and $\check{L}_i$ at infinity are

$$\frac{k_i}{k_{i+1} - k_i} < \cdots < \frac{k_i}{k_p - k_i}$$

and possibly -1. In particular, if $i < p$, then the growth rate of $\check{L}_i$ at infinity is $\frac{k_i}{k_i - k_{i+1}}$. In view of theorem 5.2, we may thus apply $\check{\mathcal{A}}_{k_i, k_{i+1}}^{\theta_i}$ to $\check{f}_i$ (see below for further details). Also, if $i = p$, then the growth rate of $\check{L}_i$ at infinity is zero or one and theorem 5.2 shows that we are allowed to apply $\check{\mathcal{L}}_{z_p}^{\theta_p}$ to $\check{f}_p$.

The acceleration kernels Given a critical time z_i with $i < p$ and $\lambda = k_{i+1} / k_i$, consider the acceleration kernel

$$\begin{aligned}\check{K}_{k_i, k_{i+1}}(\zeta_i, \zeta_{i+1}) &= \frac{-1}{2\pi\mathrm{i}} \int_0^u \frac{\hat{K}_{k_i, k_{i+1}}(\zeta_i, \xi)}{\xi - \zeta_{i+1}} \mathrm{d}\xi \\ &= \frac{1}{4\pi^2} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \int_0^u \frac{\mathrm{e}^{\xi t - \zeta_i t^\lambda}}{\xi - \zeta_{i+1}} \mathrm{d}\xi\, \mathrm{d}t\end{aligned}$$

The choices of ζ_{i+1} and u will be detailed in the next section. In order to compute (2.15), we need an equation for $\check{K}$ in ζ_i, of growth rate $1/(1-\lambda) = k_i/(k_i - k_{i+1})$ at infinity. Setting

$$\varphi(t) = \frac{-1}{2\pi i} \int_0^u \frac{\mathrm{e}^{\xi t}}{\xi - \zeta_{i+1}} \mathrm{d}\xi,$$

we have

$$\varphi'(t) = \zeta_{i+1}\, \varphi(t) - \frac{\mathrm{e}^{tu} - 1}{2\pi\mathrm{i}\, t},$$

whence $(t\,\varphi' - \zeta_{i+1}\, t\, \varphi)'' = u\, (t\,\varphi' - \zeta_{i+1}\, t\,\varphi)'$ and

$$\Omega\, \varphi = t\, \varphi''' - ((\zeta_{i+1} + u)\, t - 2)\, \varphi'' + (u\, \zeta_{i+1}\, t - (2\, \zeta_{i+1} + u))\, \varphi' + u\, \zeta_{i+1}\, f = 0$$

By looking at the Newton polygon, we observe that Ω has growth rate 1 at $t = \infty$. Now

$$\begin{aligned}\check{K}_{k_i, k_{i+1}}(\zeta_i, \zeta_{i+1}) &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \varphi(t)\, \mathrm{e}^{-\zeta_i t^\lambda} \mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}\,\lambda} \int_{\mathcal{C}} (-t)^{1/\lambda - 1}\, \varphi((-t)^{1/\lambda})\, \mathrm{e}^{\zeta_i t} \mathrm{d}t, \end{aligned} \qquad (7.1)$$

for a suitable contour $\mathcal{C}$. Applying a suitable ramification, followed by $\mathcal{M}_{1-1/\lambda}$ and $\mathcal{Q}_{-1}$ to Ω, we obtain a vanishing operator A_i for $(-t)^{1/\lambda - 1}\, \varphi((-t)^{1/\lambda})$, with growth rate $1/\lambda$ at infinity. Although (7.1) is not really a Borel transform (at $t = \infty$), it does satisfy the formal properties of a Borel transform. In other words, $\check{A}_i = \mathcal{B}\mathcal{P}_{-1} A_i$ is a vanishing operator for $\check{K}$ with respect to ζ_i, of growth rate $1/(1-\lambda)$ at $\zeta_i = \infty$.

Equations for the integrands We finally need equations for the integrands of (2.14) and (2.15). If $i<p$, then we have shown above how to construct a vanishing operator $\check{A}_i$ for $\check{K}_{k_i,k_{i+1}}$ at infinity. In section 4.2.3, we have also shown how to construct a vanishing operator $(\check{A}_i)^{\boxplus m}$ for each $\check{K}_{k_i,k_{i+1}}^{(m)}$. It follows that $\check{\Xi}_{i,m}=(\check{A}_i)^{\boxplus m}\boxtimes\check{L}_i$ and $\hat{\Xi}_{i,m}=(\check{A}_i)^{\boxplus m}\boxtimes\hat{L}_i$ are vanishing operators for the first and second integrands in (2.14). Moreover, the operator $(\check{A}_i)^{\boxplus m}\boxtimes\hat{L}_i$ has growth rate $k_i/(k_i-k_{i+1})$ at infinity, by lemma 4.3. Similarly, $\check{\Xi}_{p,m}=(\delta+z_p^{-1})^{\boxplus m}\boxtimes\check{L}_p$ and $\hat{\Xi}_{p,m}=(\delta+z_p^{-1})^{\boxplus m}\boxtimes\hat{L}_p$ are vanishing operators for the first and second integrands in (2.15), and $(\delta+z_p^{-1})^{\boxplus m}\boxtimes\hat{L}_p$ has growth rate 1 at infinity.

7.2. Calibration

Assume now that $\theta_1\in\mathcal{R}_1,\ldots,\theta_p\in\mathcal{R}_p$ are fixed non singular directions with $\mathrm{e}^{\mathrm{i}\theta_1},\ldots,\mathrm{e}^{\mathrm{i}\theta_p}\in\mathbb{K}$. In order to approximate $f(z)$ for z close to $\dot{0}$, we first have to precompute a certain number of parameters for the acceleration process, which do not depend on the precision of the desired approximation for $f(z)$. In particular, we will compute a suitable sector $\mathcal{S}_{\mathrm{geom}}$ near the origin, such that the effective accelero-summation procedure will work for every $z\in\mathcal{S}_{\mathrm{geom}}$. Besides $\mathcal{S}_{\mathrm{geom}}$, for each critical time z_i, we precompute

- The operators $\hat{L}_i$, $\check{L}_i$, $\hat{\Xi}_{i,m}$ and $\check{\Xi}_{i,m}$ from the previous section, for $m<\check{r}_i:=\operatorname{order}(\check{L}_i)$ if $i<p$ and $m<r:=\operatorname{order}(L)$ if $i=p$.
- The starting point $a_i\in\dot{\mathbb{K}}$ for $\mathcal{C}_{\theta_i}$ and $\mathcal{H}_{\theta_i}^{+}$ in (2.14) resp. (2.15). If $i>1$, then we will require that $\arg a_i=k_{i-1}\theta_{i-1}/k_i$.
- A sector $\mathcal{S}_i=\bar{\mathcal{S}}_{\theta_i,\alpha_i,R_i}^{\infty}$ near infinity as in section 6.
- The point $b_i=R_i\,\mathrm{e}^{\mathrm{i}\theta_i}/(1-\sin\alpha_i)\in\dot{\mathbb{K}}$, which corresponds to the center of the ball in figure 6.1.
- A point u_{i+1} above $\mathbb{K}$ such that $\check{K}_i(\zeta_i,\zeta_{i+1})={}^{u_{i+1}}\check{K}_i(\zeta_i,\zeta_{i+1})$, for $i<p$.

Let us show more precisely how to do this.

Computing a_1 If ω is the smallest non-zero singularity of $\check{L}_1$, then we may take a_1 arbitrarily with $|a_1|<\omega$. By construction, $\hat{L}_1$ is (at worst) regular singular at 0, whence so is $\check{L}_1$, as we see from (4.1). Using the algorithms from [Hoeven, 2001b], it follows that the entries of the transition matrices for $\hat{L}_1$ and $\check{L}_1$ between 0 and a_1 can be approximated in time $O(M(n)\log^2 n)$; these entries belong to $\mathbb{K}^{\mathrm{hola}}$ using the terminology from Appendix B. From (2.2), we also see that $\hat{f}_1$ is a $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})]$-linear combination of the canonical solutions of $\hat{L}_1$ at the origin, where $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K})]$ is the smallest $\mathbb{K}$-algebra that contains all constants of the form $\gamma^{(m)}(\sigma)$ with $\gamma(z)=1/\Gamma(z)$, $m\in\mathbb{N}$, and $\sigma\in\mathbb{K}$. Similarly, using (2.7), we deduce that $\check{f}_1$ is a $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K}),(1-\mathrm{e}^{-2\pi\mathrm{i}\mathbb{K}^{\neq}})^{-1}]$-linear combination of the canonical solutions of $\check{L}_1$ at the origin. In view of Example 6.3, it follows that $\check{f}_1^{(m)}(a_1)$ has an $O(M(n)\log^3 n)$-approximation algorithm for each $m\in\mathbb{N}$. Moreover, $\check{f}_1^{(m)}(a_1)\in\mathbb{K}^{\mathrm{shola}}$ using the terminology from Appendix B.

Computing $\mathcal{S}_i$, a_{i+1} and u_{i+1} Given $i<p$, and setting $\kappa=k_i/(k_i-k_{i+1})$, we use theorem 5.2 to compute a sector $\mathcal{S}_i^{\mathrm{pre}}=\bar{\mathcal{S}}_{\theta_i,\alpha_i^{\mathrm{pre}},R_i^{\mathrm{pre}}}^{\infty}$ and constants K, λ with

$$\|\Delta_{\xi\to\zeta_i}^{\hat{L}_i}\|\leqslant K\,\mathrm{e}^{\lambda|\zeta_i^{\kappa}-\xi^{\kappa}|}\leqslant K\,\mathrm{e}^{\lambda|\zeta_i^{\kappa}|}$$

for all straight-line paths $\xi\to\zeta_i$ in $\mathcal{S}_i^{\mathrm{pre}}$. By lemmas 2.7 and 2.3, we may compute a subsector $\mathcal{S}_i=\bar{\mathcal{S}}_{\theta_i,\alpha_i,R_i}^{\infty}\subseteq\mathcal{S}_i^{\mathrm{pre}}$ and small a_{i+1} and u_{i+1} with $\arg a_{i+1}=\arg u_{i+1}=k_i\,\theta_i/k_{i+1}$, such that we have a bound

$$|{}^{u_{i+1}}\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,a_{i+1})|\leqslant K'\,\mathrm{e}^{\lambda'|\zeta_i^{\kappa}|}\qquad(\lambda'<-\lambda)$$

for all $m < \check{r}_{i+1}$ and all $\zeta_i \in \mathcal{S}_i$. We notice that $\check{K}_{k_i,k_{i+1}}(\cdot, a_{i+1})$ is regular singular at the origin (for the same reason as $\check{L}_1$ above) with initial conditions in $\mathbb{K}[\gamma^{(\mathbb{N})}(\mathbb{K}),(1-\mathrm{e}^{-2\pi\mathrm{i}\mathbb{K}^{\neq}})^{-1}]$. We thus have $O(M(n)\log^3 n)$-approximation algorithms for $\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})$ for any $\zeta_i \in \dot{\mathbb{K}}^{\neq}$ and $m \in \mathbb{N}$.

Computing $\boldsymbol{\mathcal{S}_p}$ and $\boldsymbol{\mathcal{S}_{\mathrm{geom}}}$ By theorem 5.2, we may also compute a sector $\mathcal{S}_p^{\mathrm{pre}} = \bar{\mathcal{S}}^{\infty}_{\theta_p,\alpha_p^{\mathrm{pre}},R_p^{\mathrm{pre}}}$ and constants K, λ with

$$\|\Delta_{\xi\to\zeta_p}^{\hat{L}_p}\| \leqslant K\,\mathrm{e}^{\lambda|\zeta_p-\xi|} \leqslant K\,\mathrm{e}^{\lambda|\zeta_p|}$$

for all straight-line paths $\xi \to \zeta_p$ in $\mathcal{S}_p^{\mathrm{pre}}$. Choosing R_{geom} sufficiently small and R_p sufficiently large, we obtain a subsector $\mathcal{S}_p = \bar{\mathcal{S}}^{\infty}_{\theta_p,\alpha_p,R_p} \subseteq \mathcal{S}_p^{\mathrm{pre}}$ with

$$|(\mathrm{e}^{-\zeta_p/z_p})^{(m)}| \leqslant K'\,\mathrm{e}^{-\lambda'|\zeta_p|}$$

for all $m < r$, $\zeta_p \in \mathcal{S}_p$ and $z_p \in \mathcal{S}_{\mathrm{geom}} = \bar{\mathcal{S}}^{0}_{\theta_p,\alpha_{\mathrm{geom}},R_{\mathrm{geom}}}$, with α_{geom} as close to $\frac{\pi}{2}$ as desired.

7.3. Approximation of $f(z)$

For each $i \in \{1,\ldots,p\}$ and $j < \check{r}_i$, let $\check{\varphi}_{i,j}$ be the unique solution to $\check{L}_i(\check{\varphi}_{i,j}) = 0$ with $\check{\varphi}_{i,j}^{(m)}(a_i) = \delta_{j,m}$ for all $m < \check{r}_i$. Using the analytic continuation algorithm from [Hoeven, 1999], we may efficiently evaluate all derivatives of $\check{\varphi}_{i,j}$ and its minor $\hat{\varphi}_{i,j}$ at any non-singular point above $\mathbb{K}$. For each $j < r$, we also denote by φ_j the unique solution to $L\varphi_j = 0$ with $\varphi_j^{(m)}(z_r) = \delta_{j,m}$ for all $m < r$.

Given $i < p$ and $m < \check{r}_{i+1}$, there now exist $O(M(n)\log^3 n)$-approximation algorithms for the integrals.

$$\begin{aligned}
\mathrm{A}^1_{i,m,j} &= \int_{\mathcal{C}_{\theta_i}} \check{\varphi}_{i,j}(\zeta_i)\,\check{K}^{(m)}_{k_i,k_{i+1}}(\zeta_i,a_{i+1})\,\mathrm{d}\,\zeta_i;\\
\mathrm{A}^2_{i,m,j} &= \int_{a_i}^{b_i} \hat{\varphi}_{i,j}(\zeta_i)\,\check{K}^{(m)}_{k_i,k_{i+1}}(\zeta_i,a_{i+1})\,\mathrm{d}\,\zeta_i;\\
\mathrm{A}^3_{i,m,j} &= \int_{b_i}^{\mathrm{e}^{\mathrm{i}\theta_i}\infty} \hat{\varphi}_{i,j}(\zeta_i)\,\check{K}^{(m)}_{k_i,k_{i+1}}(\zeta_i,a_{i+1})\,\mathrm{d}\,\zeta_i.
\end{aligned}$$

Indeed, the first two integrals can be approximated using the algorithm from [Hoeven, 1999], applied to the operators $\partial\,\check{\Xi}_{i,m}$ and $\partial\,\hat{\Xi}_{i,m}$. The last one is computed using the algorithm `integral_approx`. Notice that the path in the second integral consists of a circular arc composed with a straight-line segment of constant argument. We regard the numbers

$$\begin{aligned}
\mathrm{A}_{i,m,j} &= \mathrm{A}^1_{i,m,j} + \mathrm{A}^2_{i,m,j} + \mathrm{A}^3_{i,m,j}\\
&= \int_{\mathcal{C}_{\theta_i}} \check{\varphi}_{i,j}(\zeta_i)\,\check{K}^{(m)}_{k_i,k_{i+1}}(\zeta_i,a_{i+1})\,\mathrm{d}\,\zeta_i + \int_{\mathcal{H}^+_{\theta_i}} \hat{\varphi}_{i,j}(\zeta_i)\,\check{K}^{(m)}_{k_i,k_{i+1}}(\zeta_i,a_{i+1})\,\mathrm{d}\,\zeta_i
\end{aligned}$$

as the entries of a matrix

$$\mathrm{A}_i = \begin{pmatrix} \mathrm{A}_{i,0,0} & \cdots & \mathrm{A}_{i,0,\check{r}_i-1}\\ \vdots & & \vdots\\ A_{i,\check{r}_{i+1}-1,0} & \cdots & \mathrm{A}_{i,\check{r}_{i+1}-1,\check{r}_i-1}\end{pmatrix}.$$

By construction, we thus have

$$\check{\mathcal{A}}^{\theta_i}_{k_i,k_{i+1}}\begin{pmatrix}\check{\varphi}_{i,0} & \cdots & \check{\varphi}_{i,\check{r}_i-1}\end{pmatrix} = \begin{pmatrix}\check{\varphi}_{i+1,0} & \cdots & \check{\varphi}_{i+1,\check{r}_{i+1}-1}\end{pmatrix}\mathrm{A}_i\,. \tag{7.2}$$

Similarly, if $i=p$, then there exist $O(M(n)\log^3 n)$-approximation algorithms for

$$\Lambda_{m,j}=\int_{\mathcal{C}_{\theta_p}}\check{\varphi}_{i,j}(\zeta_i)\,(\mathrm{e}^{-\zeta_p/z_p})^{(m)}\,\mathrm{d}\,\zeta_p+\int_{\mathcal{H}^+_{\theta_p}}\hat{\varphi}_{i,j}(\zeta_i)\,(\mathrm{e}^{-\zeta_p/z_p})^{(m)}\,\mathrm{d}\,\zeta_i,$$

and these numbers again form the entries of a matrix Λ. By construction, we have

$$\check{\mathcal{L}}^{\theta_p}\begin{pmatrix}\check{\varphi}_{p,0} & \cdots & \check{\varphi}_{p,\check{r}_p-1}\end{pmatrix}=\begin{pmatrix}\varphi_0 & \cdots & \varphi_{r-1}\end{pmatrix}\Lambda\,. \tag{7.3}$$

Now we already observed in section 7.2 that we have $O(M(n)\log^3 n)$-approximation algorithms for the entries of the vector

$$\check{\Phi}_1=\begin{pmatrix}\check{f}_1(a_1)\\ \vdots\\ \check{f}_1^{(\check{r}_1-1)}(a_1)\end{pmatrix}.$$

From (7.2) and (7.3), it follows that

$$\Lambda\,\mathrm{A}_{p-1}\cdots\mathrm{A}_1\,\check{\Phi}_1=\begin{pmatrix}f_p(z_p)\\ \vdots\\ f_p^{(r-1)}(z_p)\end{pmatrix}$$

and the entries of this vector admit $O(M(n)\log^3 n)$-approximation algorithms.

7.4. Main results

Summarizing the results from the previous sections, we have proved:

THEOREM 7.1.

a) *There exists an algorithm which takes $L\in\mathbb{K}[z][\delta]$ with an irregular singularity at $z=0$ on input and which computes the critical times $z_1=\sqrt[k_1]{z},\ldots,z_p=\sqrt[k_p]{z}$ for L, together with the sets of singular directions $\mathcal{D}_1,\ldots,\mathcal{D}_p$ modulo $2\,\pi$. In addition, given $\alpha<k_p\pi/2$, $\theta_1\in\mathbb{R}\setminus\mathcal{D}_1,\ldots,\theta_p\in\mathbb{R}\setminus\mathcal{D}_p$ with $\mathrm{e}^{\mathrm{i}\alpha},\mathrm{e}^{\mathrm{i}\theta_1},\ldots,\mathrm{e}^{\mathrm{i}\theta_p}\in\mathbb{K}$, the algorithm computes a sector $\bar{\mathcal{S}}^0_{k_p\theta_p,\alpha,\eta}$ with $\eta\in\mathbb{Q}^>$ to be used below.*

b) *There exists an algorithm which takes the following data on input:*

- *L, α, $\theta_1,\ldots,\theta_p$ and η as above;*
- *A formal solution $\tilde{f}\in\mathbb{O}$ to $L\,\tilde{f}=0$ (determined by initial conditions in $\mathbb{K}$);*
- *$z\in\bar{\mathcal{S}}^0_{k_p\theta_p,\alpha,\eta}$ above $\mathbb{K}$, $m\in\mathbb{N}$ and $\varepsilon\in\mathbb{Q}^>$.*

Setting $f=\mathrm{sum}_{\boldsymbol{k},\boldsymbol{\theta}}\tilde{f}$, the algorithm computes $\tilde{v}\in\mathbb{K}$ with $|f^{(m)}(z)-\tilde{v}|<\varepsilon$. Moreover, setting $n=-\log\varepsilon$, this computation takes a time $O(M(n)\log^3 n)$.

COROLLARY 7.2. *Singular holonomic constants in $\mathbb{K}^{\mathrm{shol}}$ admit $O(M(n)\log^3 n)$-approximation algorithms.*

The theorem 7.1 in particular applies to the fast approximation of singular transition matrices from section 4.3.2. Indeed, let $f_i=\varphi_i\,z^{\sigma_i}\,\mathfrak{e}_i$ with $\varphi_i\in\mathbb{O}$, $\sigma_i\in\mathbb{K}$ and $\mathfrak{e}_i\in\mathfrak{E}$ be one of the canonical solutions to $L\,f=0$ at the origin. Then φ_i may be accelero-summed by theorem 7.1 and $z^{\sigma_i}\,\mathfrak{e}_i$ may be evaluated at points above $\mathbb{K}$ using fast exponentiation and logarithms. We thus obtain:

COROLLARY 7.3. *There exists an algorithm which takes the following data on input:*

- *An operator* $L \in \mathbb{K}[z][\delta]$.
- *A singular broken-line path* γ.
- *A precision* $\varepsilon \in \mathbb{Q}^>$.

The algorithm computes a matrix $\tilde{\Delta}$ *with entries in* $\mathbb{K}$ *and* $\|\tilde{\Delta} - \Delta_\gamma^L\| < \varepsilon$. *Moreover, setting* $n = -\log \varepsilon$, *the algorithm takes a time* $O(M(n) \log^3 n)$. □

We have summarized the complexities of efficient evaluation algorithms for holonomic functions in table 7.1 below. In the rightmost column, the complexity bound for divergent series follows from corollary 7.3, when composing the transition matrix between zero and a point $\tilde{z} \in \mathbb{K}$ close to z with the non singular transition matrix from $\tilde{z}$ to z.

series of type	evaluation in $z \in \mathbb{K}$	evaluation in general z
$\sum_{n=0}^{\infty} \frac{f_n}{(n!)^\kappa} z^n$	$O(M(n) \log n)$	$O(M(n) \log^2 n \log \log n)$
$\sum_{n=0}^{\infty} f_n z^n$	$O(M(n) \log^2 n)$	$O(M(n) \log^2 n \log \log n)$
$\sum_{n=0}^{\infty} f_n (n!)^\kappa z^n$	$O(M(n) \log^3 n)$	$O(M(n) \log^3 n)$

Table 7.1. Summary of the complexities of evaluation of different types of holonomic series. We assume that $\kappa \in \mathbb{Q}^>$ and that the f_n satisfy $|f_n| \leqslant K \alpha^n$ for certain $K, \alpha > 0$. For the series in the last row, we assume that "evaluation" is done using an appropriate accelero-summation scheme. For the rightmost column, we do not count the cost of the approximation of the constant z itself.

Remark 7.4. A mistake slipped into the present remark in the published version of this paper. In the meantime, Marc Mezzarobba has improved the upper two entries of the right-most column in Table 7.1, which can now be replaced by $O(M(n) \log^2 n)$; see [Mezzarobba, 2011].

Remark 7.5. In [Hoeven, 1999], we assumed that $\mathbb{K}$ is an algebraic number field (i.e. a finite dimensional field extension of $\mathbb{Q}$) rather than the field $\mathbb{Q}^{\text{alg}}$ of all algebraic numbers over $\mathbb{Q}$. Of course, both point of views are equivalent, since given a finite number of algebraic numbers $x_1, \ldots, x_k \in \mathbb{Q}^{\text{alg}}$, there exists an algebraic number field $\mathbb{K}$ with $x_1, \ldots, x_k \in \mathbb{K}$.

It is convenient to work w.r.t. a fixed algebraic number field $\mathbb{K}$ in order to have an algorithm for fast multiplication. For instance, given a basis $x_1, \ldots, x_k$ of $\mathbb{K}$, we may assume without loss of generality that

$$x_i x_j = a_1^{i,j} x_1 + \cdots + a_k^{i,j} x_k, \quad (a_l^{i,j} \in \mathbb{Z}) \tag{7.4}$$

after multiplication of the x_i by suitable integers. Then we represent elements of $\mathbb{K}$ as non-simplified fractions $(p_1 x_1 + \cdots + p_k x_k) / q$, where $p_1, \ldots, p_k \in \mathbb{Z}$ and $q \in \mathbb{N}^>$. In this representation, and using (7.4), we see that two fractions of size n can be multiplied in time $O(M(n))$.

Remark 7.6. In the case when $\mathbb{K}$ is a subfield of $\mathbb{C}^{\text{eff}}$ which is not contained in the field $\mathbb{Q}^{\text{alg}}$ of algebraic numbers, the algorithms from this paper and [Hoeven, 1999; Hoeven, 2001b] still apply, except that the complexity bounds have to be adjusted. Let us make this more precise, by using the idea from [Chudnovsky and Chudnovsky, 1990] for the computation of Taylor series coefficients of holonomic functions. We first observe that the efficient evaluation of holonomic functions essentially boils down to the efficient evaluation of matrix products

$$M_{m-1} \cdots M_0,$$

where M_k is a matrix with entries in $\mathbb{K}[k]$ (in the regular singular case, one also has a finite number of exceptional values of k for which M_k is explicitly given and with entries in $\mathbb{K}$). Even if $\mathbb{K} \subsetneq \mathbb{Q}^{\mathrm{alg}}$, then we may still compute the matrix products

$$M_{k;l} = M_{k+l-1} \cdots M_k$$

using dichotomy

$$M_{k;l_1+l_2} = M_{k+l_1;l_2} M_{k;l_1}$$

as polynomials in $\mathbb{K}[k]$ of degree $O(l)$. This requires a time $O(M(n\,l)\log l)$, when working with a precision of n digits. Assuming for simplicity that m is a perfect square, and taking $l = \sqrt{m}$, we next use an efficient evaluation algorithm [Moenck and Borodin, 1972; Borodin and Moenck, 1974] for the substitution of $k = \{0, l, \ldots, m-l\}$ in $M_{k;l}$. This requires a time $O(M(n\sqrt{m})\log(m))$. We finally compute

$$M_{0;m} = M_{m-l;l} \cdots M_{0;l}$$

in time $O(M(n)\sqrt{m})$. Assuming that $\log n \asymp \log m$, this yields an algorithm for the n-digit evaluation of $M_{0;m}$ of complexity $O(M(n\sqrt{m}\log m))$. In table 7.1, the complexities in the three different rows should therefore be replaced by $O(M(n^{3/2})\sqrt{\log n})$, $O(M(n^{3/2})\log n)$ resp. $O(M(n^{3/2})\log^2 n)$. Indeed, for the first two cases, we have $m = O(n/\log n)$ resp. $m = O(n)$. In the last case, we have the usual $O(\log n)$ overhead. Notice that there is no need to distinguish between the columns.

8. Conclusion

This last paper in a series [Hoeven, 1999; Hoeven, 2001b] on the efficient evaluation of holonomic functions deals with the most difficult case of limit computations in irregular singularities, where the formal solutions are generally divergent. We have not only shown how to compute such limits and so called singular transition matrices in terms of the equation and broken-line paths, but we have also shown that the resulting constants are comprised in the very special class $\mathbb{C}^{\mathrm{fast}}$ of complex numbers whose digits can be computed extremely fast.

Since it is quite remarkable for a number to belong to $\mathbb{C}^{\mathrm{fast}}$, an interesting question is whether there are any other "interesting constants" in $\mathbb{C}^{\mathrm{fast}}$ which cannot be obtained using the currently available systematic techniques: the resolution of implicit equations using Newton's method and the evaluation of holonomic functions, including their "evaluation" in singular points.

Because of the emphasis in this paper on fast approximation algorithms, we have not yet investigated in detail the most efficient algorithms for obtaining approximations with limited precision. Indeed, given an initial operator $L \in \mathbb{K}[z][\delta]$ of order r and degree d in z, ramification, the Borel transform and the multiplication with the acceleration kernel lead to vanishing operators of far larger (although polynomial) size $O((d\,r)^3)$. If only limited precision is required, one may prefer to use a naive $O(n^2)$-algorithm for computing the integral transforms, but which avoids the computation of large vanishing operators. In some cases, one may also use summation up to the least term, as sketched in the appendix below.

In this paper, we have restricted ourselves to the very special context of holonomic functions, even though Écalle's accelero-summation process has a far larger scope. Of course, the results in our paper are easily generalized to the case of more general algebraically closed subfields $\mathbb{K}$ of $\mathbb{C}$, except that we only get $O(n^2 \log^{O(1)} n)$-approximation algorithms; using improvements from [Chudnovsky and Chudnovsky, 1990], this can be reduced to $O(n^{3/2} \log^{O(1)} n)$. Following [Écalle, 1987; Braaksma, 1991; Braaksma, 1992], it should also be possible to give algorithms for the accelero-summation of solutions to nonlinear differential equations.

APPENDIX A. SUMMATION UP TO THE LEAST TERM

Let $L \in \mathbb{K}(z)[\delta]$ and let f be a solution to $L f = 0$ with a formal power series expansion $\tilde{f} = \tilde{f}_0 + \tilde{f}_1 z + \cdots$. It is well known [Poincaré, 1886] that the truncated sum

$$(\mathrm{sum}_N \tilde{f})(z) = \tilde{f}_0 + \cdots + \tilde{f}_N z^N$$

up to the term $\tilde{f}_N z^N$ for which $|\tilde{f}_N z^N|$ is minimal usually provides an exponentially good approximation for $f(z)$. Even though such truncations do not allow for the computation of an arbitrarily good approximation of the value $f(z)$ for fixed z, it is well adapted to the situation in which only a limited precision is required. Indeed, for any $N \in \mathbb{N}$, in order to compute $(\mathrm{sum}_N \tilde{f})(z)$, we may directly apply the binary splitting algorithm from [Chudnovsky and Chudnovsky, 1990; Hoeven, 1999].

In this appendix, we will sketch how summation up to the least term can be made more precise using the accelero-summation process. We start from a formal solution $\tilde{f} = \tilde{f}_0 + \cdots + \tilde{f}_l \log^l z \in \mathbb{O}$ to $L f = 0$. Given $N \in \mathbb{N}$, we define

$$\tilde{g}(z) = (\mathrm{sum}_N \tilde{f})(z) = \sum_{0 \leqslant i \leqslant l} \sum_{0 \leqslant n \leqslant N} (\tilde{f}_i)_n z^n \log^i z.$$

Our aim is to compute an explicit bound for $\tilde{g}(z) - (\mathrm{sum}_{\boldsymbol{k}, \boldsymbol{\theta}} \tilde{f})(z)$ for a suitable non singular multi-direction $\boldsymbol{\theta}$. Modulo a change of variables $z \to \omega z$, we may take $\boldsymbol{\theta} = \mathbf{0}$.

Consider a critical time z_i. If $i = 1$, then $\tilde{\mathcal{B}}_{z_1} \tilde{f}$ is convergent at the origin, so we may compute a bound of the form

$$|(\hat{g}_1 - \hat{f}_1)(\zeta_1)| \leqslant B_1 C_1^N \zeta_1^{k_1 N - 1} \tag{A.1}$$

on an interval $(0, c_1]$ at the origin, using [Hoeven, 2001b]. For $i > p$, we assume by induction that we have a bound

$$|(\hat{g}_i - \hat{f}_i)(\zeta_i)| \leqslant B_i C_i^N \frac{\Gamma(k_1 N)}{\Gamma(k_i N)} \left(\zeta_i^{k_i N - 1} + \exp\left(-D_i \zeta_i^{-\frac{k_i}{k_{i-1} - k_i}}\right) \right) \tag{A.2}$$

on a sector $(0, c_i]$ at the origin and for sufficiently large $N \geqslant N_i$. Using [Hoeven, 2001b] a second time, we may also compute bounds for the coefficients of $\hat{f}_1$ as a polynomial in $\log \zeta_1$. At each critical time z_i, this leads to further bounds

$$|\hat{g}_i(\zeta_i)| \leqslant B_i' (C_i')^N \zeta_i^{k_i N - 1} \frac{\Gamma(k_1 N)}{\Gamma(k_i N)}, \tag{A.3}$$

for $\zeta_i \in [c_i, \infty)$.

Assuming that $i < p$, we now have

$$\begin{aligned}
|(\hat{g}_{i+1} - \hat{f}_{i+1})(\zeta_{i+1})| &\leqslant I_1 + I_2 + I_3; \\
I_1 &= \left| \int_{c_i}^{\infty} \hat{g}_i(\zeta_i) \hat{K}_{k_i, k_{i+1}}(\zeta_i, \zeta_{i+1}) \, \mathrm{d}\zeta_i \right|; \\
I_2 &= \left| \int_0^{c_i} (\hat{g}(\zeta_i) - \hat{f}(\zeta_i)) \hat{K}_{k_i, k_{i+1}}(\zeta_i, \zeta_{i+1}) \, \mathrm{d}\zeta_i \right|; \\
I_3 &= \left| \int_{c_i}^{\infty} \hat{f}_i(\zeta_i) \hat{K}_{k_i, k_{i+1}}(\zeta_i, \zeta_{i+1}) \, \mathrm{d}\zeta_i \right|.
\end{aligned}$$

We may further decompose

$$\begin{aligned} I_2 &\leqslant I_4+I_5+I_6; \\ I_4 &= B_i\,C_i^N\,\frac{\Gamma(k_1\,N)}{\Gamma(k_i\,N)}\left|\int_0^\infty \zeta_i^{k_i N-1}\,\hat{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\,\zeta_i\right| \\ &= B_i\,C_i^N\,\zeta_{i+1}^{k_{i+1}N-1}\,\frac{\Gamma(k_1\,N)}{\Gamma(k_{i+1}\,N)}; \qquad (A.4) \\ I_5 &= B_i\,C_i^N\,\frac{\Gamma(k_1\,N)}{\Gamma(k_i\,N)}\left|\int_{c_i}^\infty \zeta_i^{k_i N-1}\,\hat{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\,\zeta_i\right|; \\ I_6 &= B_i\,C_i^N\,\frac{\Gamma(k_1\,N)}{\Gamma(k_i\,N)}\left|\int_0^{c_i} \exp\left(-D_i\,\zeta_i^{-\frac{k_i}{k_{i-1}-k_i}}\right)\hat{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\,\zeta_i\right|, \end{aligned}$$

if $i>1$ and similarly with $I_6=0$ if $i=1$.

By lemmas 2.6 and 2.5, we may compute c'_{i+1}, N_{i+1}, $A_{i,1}$, $A_{i,2}$ and $A_{i,3}$ with

$$\left|\int_{c_i}^\infty \zeta_i^{k_i N-1}\,\hat{K}_{k_i,k_{i+1}}(\zeta_i,\zeta_{i+1})\,\mathrm{d}\,\zeta_i\right|\leqslant A_{i,1}\,\frac{\Gamma(k_i\,N)}{\Gamma(k_{i+1}\,N)}\,A_{i,2}^N\exp\left(-A_{i,3}\,\zeta_{i+1}^{-\frac{k_{i+1}}{k_i-k_{i+1}}}\right),$$

for $\zeta_{i+1}\in(0,c'_{i+1}]$ and $N\geqslant N_{i+1}$. Using (A.3), we thus get

$$I_1+I_5\leqslant A_{i,1}\,A_{i,2}^N\,(B_i\,C_i^N+B'_i\,(C'_i)^N)\,\frac{\Gamma(k_1\,N)}{\Gamma(k_{i+1}\,N)}\exp\left(-A_{i,3}\,\zeta_{i+1}^{-\frac{k_{i+1}}{k_i-k_{i+1}}}\right). \qquad (A.5)$$

Using the techniques from section 7, we may also compute a bound

$$|\hat{f}_i(\zeta_i)|\leqslant A_{i,4}\,\mathrm{e}^{A_{i,5}\zeta_i^{k_i/k_{i+1}}},$$

for $\zeta_i\in[c_i,\infty)$. Using lemma 2.6 and (A.5), we may thus compute c_{i+1}, $A_{i,6}$, $A_{i,7}$ and $A_{i,8}$ with

$$I_1+I_3+I_5+I_6\leqslant A_{i,6}\,A_{i,7}^N\,\frac{\Gamma(k_1\,N)}{\Gamma(k_{i+1}\,N)}\exp\left(-A_{i,8}\,\zeta_{i+1}^{-\frac{k_{i+1}}{k_i-k_{i+1}}}\right), \qquad (A.6)$$

for $\zeta_{i+1}\in(0,c_{i+1}]$ and $N\geqslant N_{i+1}$. Combining (A.4) and (A.6), we may therefore compute B_{i+1} and C_{i+1} such that (A.2) recursively holds at stage $i+1$.

In the case when $i=p$, similar computations yield constants B, C, D, N_{geom} and a small sector $\mathcal{S}=\mathcal{S}_{0,\alpha,R}$ with aperture $\alpha<\pi/(2\,k_p)$, such that

$$|(g-f)(z)|\leqslant B\,C^N\,\Gamma(k_1\,N)\,(|z|^N+\exp(-D\,|z|^{-1/k_p})). \qquad (A.7)$$

for all $z\in\mathcal{S}$ and all $N\geqslant N_{\mathrm{geom}}$. The optimal $N=N_{\mathrm{opt}}$ for which this bound is minimal satisfies

$$N_{\mathrm{opt}}\sim k_1^{-1}\,(C\,|z|)^{-1/k_1}.$$

We thus have

$$|(g-f)(z)|\leqslant B'\,\mathrm{e}^{-(C|z|)^{-1/k_1}},$$

for some explicitly computable B'. This completes the proof of the following:

THEOREM A.1. *There exists an algorithm which takes on input*

- *A differential operator $L\in\mathbb{K}(z)[\delta]$ with an irregular singularity at $z=0$;*
- *The critical times $\boldsymbol{k}$ and non singular directions $\boldsymbol{\theta}$ with $k_i\,\theta_i=k_{i+1}\,\theta_{i+1}$ for all i,*

and which computes $B,C,R,\alpha>0$ and $N_{\mathrm{geom}}\in\mathbb{N}$, such that the bound

$$|(\mathrm{sum}_N\,\tilde{f}-\mathrm{sum}_{\boldsymbol{k},\boldsymbol{\theta}}\,\tilde{f})(z)|\leqslant B\,C^N\,\Gamma(k_1\,N)\,(|z|^N+\exp(-D\,|z|^{-1/k_p}))$$

holds for any $z \in \mathcal{S}_{k_p \theta_p, \alpha, R}$ *and* $N \geqslant N_{\text{geom}}$. *In particular, for some computable constant* n_0 *and precisions* $\varepsilon = \mathrm{e}^{-n}$ *with*

$$n \leqslant (C\,|z|)^{-1/k_1} - n_0 \tag{A.8}$$

we may compute an ε*-approximation of* $(\mathrm{sum}_{\boldsymbol{k},\boldsymbol{\theta}}\,\tilde{f})(z)$ *for* $z \in \mathbb{K} \cap \mathcal{S}_{k_p \theta_p, \alpha, R}$ *in time* $O(M(n)\log^2 n)$, *where the complexity bound is uniform in* z, *provided that the bit-size of* z *is bounded by* $O(\log n)$.

Remark A.2. In the published version of the paper, we forgot to mention the restriction that the bit-size of z should remain bounded by $O(\log n)$. In fact, this restriction can be removed using a bit more work: see [Hoeven, 2016, section 3]. Another afterthought: if $\mathrm{size}(z) = O(\log n)$, then the bound can actually be reduced to $O(\mathsf{M}(n)\log n)$ through a careful analysis of the binary splitting algorithm in this specific context.

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APPENDIX B. ERRATA: NOTES ON HOLONOMIC CONSTANTS

There were several problems with the proofs of proposition 4.7(c) and (d). In this appendix, we present corrected proofs (in the case of proposition 4.7(c), we slightly modified the statement), as well as a theorem that the class $\mathbb{K}^{\text{rhol}}$ regular singular holonomic constants is essentially the same as the class of $\mathbb{K}^{\text{hol}}$ of ordinary holonomic constants. Until subsection B.8, we will assume that $\mathbb{K} = \mathbb{Q}^{\text{alg}}$ is the field of algebraic numbers. In subsection B.8, we also discuss a few related questions and results from [Fischler and Rivoal, 2011]; we are grateful to Marc Mezzarobba for this reference.

B.1. Notations

Let $\mathcal{L}^{\text{hol}}$ and $\mathcal{L}^{\text{shol}}$ denote for the sets of monic $L \in \mathbb{K}(z)[\partial]$ whose coefficients are respectively defined on $\bar{\mathcal{D}}_{0,1}$ and $\mathcal{D}_{0,1}$. Let $\mathcal{L}^{\text{rhol}}$ be the set of $L \in \mathcal{L}^{\text{shol}}$ such that L is at worst regular singular at $z = 1$. We define $\mathcal{F}^{\text{hol}}$, $\mathcal{F}^{\text{rhol}}$, and $\mathcal{F}^{\text{shol}}$ to be the sets of solutions $f \in \mathbb{K}\{\{z\}\}$ to an equation $Lf = 0$, where $L \in \mathcal{L}^{\text{hol}}$, $L \in \mathcal{L}^{\text{rhol}}$, or $L \in \mathcal{L}^{\text{shol}}$, respectively, and such that $\lim_{z\to 1} f(z)$ exists. We recall that $\mathbb{K}^{\text{hol}} = \{f(1) : f \in \mathcal{F}^{\text{hol}}\}$, $\mathbb{K}^{\text{rhol}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{rhol}}\}$, and $\mathbb{K}^{\text{shol}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{shol}}\}$.

It will be convenient to also introduce the variants $\mathcal{L}^{\text{hola}}$, $\mathcal{L}^{\text{rhola}}$, and $\mathcal{L}^{\text{shola}}$ of $\mathcal{L}^{\text{hol}}$, $\mathcal{L}^{\text{rhol}}$, and $\mathcal{L}^{\text{shol}}$ for which we allow L to be at most regular singular at $z = 0$. For instance, $\mathcal{L}^{\text{hola}}$ consists of monic operators $L \in \mathbb{K}(z)[\partial]$ whose coefficients are defined on $\bar{\mathcal{D}}_{0,1} \setminus \{0\}$ and such that L is at worst regular singular at $z = 0$. The counterparts $\mathcal{F}^{\text{hola}}$, $\mathcal{F}^{\text{rhola}}$, and $\mathcal{F}^{\text{shola}}$ are defined in a similar way as before; we still require analytic solutions $f \in \mathbb{K}\{\{z\}\}$ of $Lf = 0$ at $z = 0$. We again set $\mathbb{K}^{\text{hola}} = \{f(1) : f \in \mathcal{F}^{\text{hola}}\}$, $\mathbb{K}^{\text{rhol}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{rhola}}\}$, and $\mathbb{K}^{\text{shola}} = \{\lim_{z\to 1} f(z) : f \in \mathcal{F}^{\text{shola}}\}$.

B.2. Ring structure

PROPOSITION B.1. $\mathbb{K}^{\text{hol}}$, $\mathbb{K}^{\text{rhol}}$, $\mathbb{K}^{\text{shol}}$, $\mathbb{K}^{\text{hola}}$, $\mathbb{K}^{\text{rhola}}$, *and* $\mathbb{K}^{\text{shola}}$ *are all subrings of* $\mathbb{C}$.

Proof. This is proved in a similar way as proposition 4.6(a). For instance, in order to see that $\mathbb{K}^{\text{rhol}}$ is closed under multiplication, consider solutions f and g of $Kf = 0$ and $Lg = 0$ with initial conditions in $\mathbb{K}^m$ resp. $\mathbb{K}^n$, where the coefficients of K and L are defined on $\mathcal{D}_{0,1}$, where K and L are regular singular at $z = 1$, and such that the limits of f and g at $z = 1$ exist. Then corollary 4.4 implies that $K \boxtimes L$ is defined on $\mathcal{D}_{0,1}$ and corollary 4.5 implies that $K \boxtimes L$ is regular singular at $z = 1$. Consequently, $\lim_{z\to 1}(fg)(z) = (\lim_{z\to 1} f(z))\,(\lim_{z\to 1} g(z))$ belongs to $\mathbb{K}^{\text{rhol}}$. □

This proposition also allows us to consider initial conditions in $\mathbb{K}^{\mathrm{hol}}$ instead of $\mathbb{K}$ in many circumstances. For instance, by definition, the value of a function $f \in \mathcal{F}^{\mathrm{hol}}$ at a point in $\bar{\mathcal{D}}_{0,1} \cap \mathbb{K}$ lies in $\mathbb{K}^{\mathrm{hol}}$. Thanks to the proposition, this even holds for solutions $f \in \mathbb{K}^{\mathrm{hol}}\{\{z\}\}$ to an equation $L f = 0$ with $f \in \mathcal{L}^{\mathrm{hol}}$. Indeed, given $z \in \mathbb{K}$, we have $F(z) = \Delta_{0 \to z} F(0)$; since $\Delta_{0 \to z}$ and $F(0)$ both have coefficients in $\mathbb{K}^{\mathrm{hol}}$, the same holds for $F(z)$.

B.3. Regular singular transition matrices

Lemma B.2. *Let $L \in \mathcal{L}^{\mathrm{hola}}$, $\alpha \in \mathbb{K}$, and consider a solution $f \in z^{\alpha_i} \mathbb{K}\{\{z\}\}[\log z]$ of the equation $L f = 0$. Then $f \in z^{\alpha_i} \mathcal{F}^{\mathrm{hola}}[\log z]$.*

Proof. Without loss of generality, we may assume that $\alpha = 0$. Now write $f = f_d (\log z)^d + \cdots + f_0$ with $f_0, \ldots, f_d \in \mathbb{K}\{\{z\}\}$. Then $f(z \mathrm{e}^{2 \pi \mathrm{i}}) = f_d (\log z + 2 \pi \mathrm{i})^d + \cdots + f_0 \in \mathbb{K}\{\{z\}\}[\log z][2 \pi \mathrm{i}]$ is also annihilated by L. Since $2 \pi \mathrm{i}$ is transcendental, each of the coefficients of $f(z \mathrm{e}^{2 \pi \mathrm{i}})$ as a polynomial in $2 \pi \mathrm{i}$ is again annihilated by L; these coefficients are

$$f_d, \quad d f_d \log z + f_{d-1}, \quad \ldots, \quad f_d (\log z)^d + \cdots + f_0.$$

It follows that

$$f_d \in \mathcal{F}^{\mathrm{hola}}, \quad f_{d-1} \in \mathcal{F}^{\mathrm{hola}} + \mathcal{F}^{\mathrm{hola}} \log z, \quad \ldots, \quad f_0 \in \mathcal{F}^{\mathrm{hola}} + \cdots + \mathcal{F}^{\mathrm{hola}} (\log z)^d,$$

whence $f \in \mathcal{F}^{\mathrm{hola}}[\log z]$. □

Proposition B.3. *Let L be a linear differential operator of order n in $\mathbb{K}(z)[\partial]$. Then $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{hola}})$ for any regular singular broken-line path γ as in section 4.3.2.*

Proof. In view of (4.6), it suffices to prove the result for paths of the form $\sigma_\theta \to \sigma + z$ and for paths of the form $\sigma + z \to \sigma_\theta$. Without loss of generality we may assume that $\sigma = 0$. By what precedes, the entries of $\Delta_{0_\theta \to z}$ as functions in z are all in $\mathcal{F}^{\mathrm{hola}}[z^{\mathbb{K}}][\log z]$. Now values of functions $z^\alpha (\log z)^k$ with $\alpha \in \mathbb{K}$ and $k \in \mathbb{N}$ at points $z \in \mathbb{K}^{\neq}$ are in $\mathbb{K}^{\mathrm{hol}}$. Consequently, values of entries of $\Delta_{0_\theta \to z}$ at $z \in \mathbb{K}^{\neq}$ are in $\mathbb{K}^{\mathrm{hola}}$. Let $h_1, \ldots, h_r$ be the canonical basis of solutions of $L f = 0$ at the origin. Then we recall that

$$\Delta_{0_\theta \to z} = \begin{pmatrix} h_1(z) & \cdots & h_r(z) \\ \vdots & & \vdots \\ h_1^{(r-1)}(z) & \cdots & h_r^{(r-1)}(z) \end{pmatrix}.$$

The determinant $W = W_{h_1, \ldots, h_r}$ of this matrix satisfies the equation $W' + L_{r-1} W = 0$ and its inverse W^{-1} satisfies $(W^{-1})' - L_{r-1} W^{-1} = 0$. Since L is at worst regular singular at $z = 0$, we have $L_{r-1} = \frac{\alpha}{z} + Q$, where $\alpha \in \mathbb{K}$ and $Q \in \mathbb{K}(z)$ is analytic at $z = 0$. It follows that $W = c^{-1} z^{-\alpha} \mathrm{e}^{-\int Q}$ and $W^{-1} = c z^\alpha \mathrm{e}^{\int Q}$ for some $c \in \mathbb{K}$, where $(\int Q)(0) = 0$ (here $c \in \mathbb{K}$ follows from the fact that the coefficients of all $h_i^{(j)}$ are in $\mathbb{K}$ as oscillatory transseries). Given $z \in \mathbb{K}^{\neq}$ where W^{-1} is defined, it follows that $W^{-1}(z) \in \mathbb{K}^{\mathrm{hol}}$. Since $\mathbb{K}^{\mathrm{hola}}$ is a ring, it follows that the coefficients of

$$\Delta_{z \to 0_\theta} = \Delta_{0_\theta \to z}^{-1} = W^{-1}(z) \operatorname{adj}(\Delta_{0_\theta \to z})$$

are in $\mathbb{K}^{\mathrm{hola}}$. □

Corollary B.4. *We have $\mathbb{K}^{\mathrm{rhol}} \subseteq \mathbb{K}^{\mathrm{hola}}$.*

Proof. Given $c \in \mathbb{K}^{\mathrm{rhol}}$, let $L \in \mathcal{L}^{\mathrm{rhol}}$ and $f \in \mathcal{F}^{\mathrm{rhol}}$ such that $L f = 0$ and $c = \lim_{z \to 1} f(z)$. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{K}^{\mathrm{hola}}$ be the entries of $\Delta_{0 \to 1} F(0)$. Then $f = \lambda_1 h_1 + \cdots + \lambda_r h_r$, where $h_1, \ldots, h_r$ is the canonical fundamental system of solutions of $L f = 0$. Since $\lim_{z \to 1} f(z)$ exists, we must have $h_i = O(1)$ whenever $\lambda_i \neq 0$ and $f(1) = \sum_{i, \lambda_i \neq 0} \lambda_i h_i(0) \in \mathbb{K}^{\mathrm{hola}}$. □

B.4. Alien operators

Given an analytic function f defined on a neighbourhood of the origin on the Riemann surface of the logarithm, we define $(\nabla f)(z) = f(z) - f(z \mathrm{e}^{-2\pi\mathrm{i}})$. Setting $\Lambda = (2\pi\mathrm{i})^{-1} \log z$, the operator $f(z) \mapsto f(z \mathrm{e}^{-2\pi\mathrm{i}})$ acts on $\mathbb{C}\{\{z\}\}[\Lambda]$ by sending Λ to $\Lambda - 1$, whence $\nabla = 1 - \mathrm{e}^{-\partial_\Lambda} = \partial_\Lambda - 1/2\, \partial_\Lambda + \cdots$. Given $d \in \mathbb{N}$, let $\mathbb{C}\{\{z\}\}[\Lambda]_{<d}$ be the set of $f \in \mathbb{C}\{\{z\}\}[\Lambda]$ of degree $<d$ in Λ. For $0 \leqslant i \leqslant j$, we note that the operator

$$\mathrm{Z}^{i;j} := (i - \Lambda \nabla) \cdots (j - 1 - \Lambda \nabla)$$

sends $\mathbb{C}\{\{z\}\}[\Lambda]_{<j}$ into $\mathbb{C}\{\{z\}\}[\Lambda]_{<i}$. We also note that $\mathrm{Z}^{i;j}(\Lambda^{i-1}) \sim (i - j)!\, \Lambda^{i-1}$.

For each $\alpha \in \mathbb{K}$, let $\nabla_\alpha := z^\alpha \nabla z^{-\alpha}$ and $\mathbb{L}_\alpha := z^\alpha\, \mathbb{C}\{\{z\}\}[\log z]$. Then $\nabla_\alpha \mathbb{L}_\beta \subseteq \mathbb{L}_\beta$ for all β and ∇_α acts like multiplication by $1 - \mathrm{e}^{-2\pi\mathrm{i}(\beta - \alpha)}$ on $z^\beta\, \mathbb{C}\{\{z\}\}$. Moreover, given $\varphi \in \mathbb{L}_\alpha$ of degree $<d$ in $\log z$, we have $\nabla_\alpha^d \varphi = 0$. We define $\mathcal{X}$ to be the monoid of power products $(1 - \mathrm{e}^{-2\pi\mathrm{i}\alpha_1})^{k_1} \cdots (\mathrm{e}^{1 - 2\pi\mathrm{i}\alpha_\ell})^{k_\ell}$ with $\alpha_1, \ldots, \alpha_\ell \in (\mathbb{K} \cap \mathbb{R}) \setminus \mathbb{Q}$ and $k_1, \ldots, k_\ell \in \mathbb{N}$. For $0 \leqslant i \leqslant j$, we also define

$$\mathrm{Z}_\alpha^{i;j} = z^\alpha\, \mathrm{Z}^{i;j} z^{-\alpha} = (i - \Lambda \nabla_\alpha) \cdots (j - 1 - \Lambda \nabla_\alpha).$$

We note that $\mathrm{Z}_\alpha^{i;j}$ sends $z^\alpha\, \mathbb{C}\{\{z\}\}[\Lambda]_{<j}$ into $z^\alpha\, \mathbb{C}\{\{z\}\}[\Lambda]_{<i}$.

Let $\mathcal{H}$ be the space of holonomic functions f on $\bar{\mathcal{D}}_{0,1} \setminus \{0\}$ that are regular singular at the origin and such that $F(1) = (f(1), \ldots, f^{(r-1)}(1)) \in (\mathbb{K}^{\mathrm{hol}})^r$. Such a function f satisfies an equation $L f = 0$ with $L \in \mathcal{L}^{\mathrm{hola}}$. We regard $F(1)$ as a column vector, as usual, and recall that $f(z \mathrm{e}^{-2\pi\mathrm{i}})$ is another solution of $L f = 0$ with $F(\mathrm{e}^{-2\pi\mathrm{i}}) = \Delta_{1 \circlearrowleft 1} F(1) \in (\mathbb{K}^{\mathrm{hol}})^r$. Indeed, the monodromy matrix $\Delta_{1 \circlearrowleft 1}$ of L around $z = 0$ with end-points at $z = 1$ has coefficients in $\mathbb{K}^{\mathrm{hol}}$. It follows that $\nabla f \in \mathcal{H}$. Moreover, $\mathcal{H}$ is a ring with $z^{\mathbb{K}} \subseteq \mathcal{H}$ and $\log z \in \mathcal{H}$. It follows that $\nabla_\alpha f \in \mathcal{H}$ and $\mathrm{Z}_\alpha^{i;j} f \in \mathcal{H}$ for all $\alpha \in \mathbb{K}$ and $0 \leqslant i \leqslant j$. Note that $\nabla_\alpha f$ always satisfies the same equation as f, contrary to $\mathrm{Z}_\alpha^{i;j} f$.

B.5. Eschewing regular singularities

THEOREM B.5. *We have*

$$\mathbb{K}^{\mathrm{hol}} \subseteq \mathbb{K}^{\mathrm{rhol}} \subseteq \mathbb{K}^{\mathrm{hola}} \subseteq \mathcal{X}^{-1}\, \mathbb{K}^{\mathrm{hol}}.$$

Proof. The inclusion $\mathbb{K}^{\mathrm{hol}} \subseteq \mathbb{K}^{\mathrm{rhol}}$ is trivial and we already proved that $\mathbb{K}^{\mathrm{rhol}} \subseteq \mathbb{K}^{\mathrm{hola}}$, so we focus on the remaining inclusion $\mathbb{K}^{\mathrm{hola}} \subseteq \mathcal{X}^{-1}\, \mathbb{K}^{\mathrm{hol}}$.

Consider a monic $L \in \mathcal{L}^{\mathrm{hola}}$ of order r. Then $L h = 0$ has a canonical fundamental system of solutions

$$h_{i,j} \in z^{\alpha_i}\, \mathbb{K}\{\{z\}\}[\log z]_{\leqslant j}, \qquad h_{i,j} \sim z^{\alpha_i} (\log z)^j, \qquad i = 1, \ldots, \ell, \quad j = 0, \ldots, \nu_i - 1.$$

In particular, we have $r = \nu_1 + \cdots + \nu_\ell$. For each $i \in \{1, \ldots, \ell\}$, let

$$\Pi_i := \nabla_{\alpha_1}^{\nu_1} \cdots \nabla_{\alpha_{i-1}}^{\nu_{i-1}} \nabla_{\alpha_{i+1}}^{\nu_{i+1}} \cdots \nabla_{\alpha_\ell}^{\nu_\ell},$$

so that $\Pi_i f \in \mathbb{L}_{\alpha_i}$ for any solution f of $L f = 0$. We also define

$$u_i := \prod_{i' \neq i} (1 - \mathrm{e}^{-2\pi\mathrm{i}(\alpha_{i'} - \alpha_i)})^{\nu_{i'}},$$

so that $\Pi_i f = u_i f$ whenever $f \in z^{\alpha_i}\, \mathbb{C}\{\{z\}\}$.

Let f be a solution of $L f = 0$ with $F(1) = (f(1), \ldots, f^{(r-1)}(1)) \in \mathbb{K}^{\mathrm{hol}}$ and let $\lambda_{i,j} \in \mathbb{C}$ be such that $f = \sum_{i,j} \lambda_{i,j} h_{i,j}$. We need to show that $\lambda_{i,j} \in \mathcal{X}^{-1} \mathbb{K}^{\mathrm{hol}}$ for all i and j. Given $i \in \{1, \ldots, \ell\}$, let us show by induction on j that $\lambda_{i,j} \in u_i^{-1} \mathbb{K}^{\mathrm{hol}}$. To this effect, given $j \in \{0, \ldots, \nu_i - 1\}$, assume that $\lambda_{i,j+1}, \ldots, \lambda_{i,\nu_i-1} \in u_i^{-1} \mathbb{K}^{\mathrm{hol}}$, and let us show that $\lambda_{i,j} \in u_i^{-1} \mathbb{K}^{\mathrm{hol}}$.

Let $g := \mathrm{Z}_\alpha^{j+1;\nu_i} f$ and let $f_\alpha = \lambda_{i,0} h_{i,0} + \cdots + \lambda_{i,\nu_i-1} h_{i,\nu_i-1}$ and $g_\alpha = \mathrm{Z}_\alpha^{j+1;\nu_i} f_\alpha$ be the components of f and g in $z^\alpha \mathbb{C}\{\{z\}\}[\log z]$. By construction, g_α has degree at most j in $\log z$ and the coefficient $(g_\alpha)_j$ of degree j is of the form

$$(g_\alpha)_j = (\nu_i - j - 1)!\, \lambda_{i,j} + c_{j+1} \lambda_{i,j+1} + \cdots + c_{\nu_i-1} \lambda_{i,\nu_i-1} + o(1)$$

for constants $c_{j+1}, \ldots, c_{\nu_i-1} \in \mathbb{K}[(2\pi \mathrm{i})^{-1}]$ that can be computed explicitly.

We next consider the function $\varphi = z^{-\alpha_i} \Pi_i \nabla_\alpha^j g$. By construction, $\varphi \in \mathbb{C}\{\{z\}\}$ and

$$\varphi = j!\, u_i ((\nu_i - j - 1)!\, \lambda_{i,j} + c_{j+1} \lambda_{i,j+1} + \cdots + c_{\nu_i-1} \lambda_{i,\nu_i-1} + o(1)).$$

Moreover, both g and φ belong to $\mathcal{H}$, so the value of the contour integral

$$\varphi(0) \;=\; \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\varphi(z)}{z} \mathrm{d} z$$

actually lies in $\mathbb{K}^{\mathrm{hol}}$. By our assumption that $\lambda_{i,j+1}, \ldots, \lambda_{i,\nu_i-1} \in u_i^{-1} \mathbb{K}^{\mathrm{hol}}$, it follows that

$$\varphi(0) - j!\, u_i (c_{j+1} \lambda_{i,j+1} + \cdots + c_{\nu_i-1} \lambda_{i,\nu_i-1}) \in \mathbb{K}^{\mathrm{hol}},$$

whence $u_i \lambda_{i,j} \in \mathbb{K}^{\mathrm{hol}}$. By induction on j, this shows that $\lambda_{i,j} \in u_i^{-1} \mathbb{K}^{\mathrm{hol}}$, for all j. □

Remark B.6. In the special case when $\ell = 1$ or when $\alpha_i - \alpha_j \in \mathbb{Q}$ for all i, j, we note that the numbers u_i are all in $\mathbb{K}$. It follows that $\Delta_{0^\theta \to z}$ and $\Delta_{z \to 0^\theta}$ have coefficients in $\mathbb{K}^{\mathrm{hol}}$.

B.6. Irregular singularities

PROPOSITION B.7. *Let L be a linear differential operator of order n in $\mathbb{K}(z)[\partial]$. Then $\Delta_\gamma \in \mathsf{Mat}_n(\mathbb{K}^{\mathrm{shola}})$ for any singular broken-line path γ as in section 4.3.2.*

Proof. In view of (4.6), it suffices to prove the result for paths of the form $\sigma_{\boldsymbol{k},\boldsymbol{\theta}} \to \sigma + z$ or $\sigma + z \to \sigma_{\boldsymbol{k},\boldsymbol{\theta}}$. In fact, it suffices to consider paths of the form $\sigma_{\boldsymbol{k},\boldsymbol{\theta}} \to \sigma + z$, by using a similar argument as in the regular singular case, based on the Wronskian. Without loss of generality we may assume that $\sigma = 0$.

Now, as shown in detail in section 7.3, the matrix $\Delta_{0_{\boldsymbol{k},\boldsymbol{\theta}} \to z}$ can be expressed as a product of matrices whose entries are either in $\mathbb{K}^{\mathrm{shola}}$, or of the form

$$\int_{b_i}^{\mathrm{e}^{\mathrm{i}\theta_i}\infty} \hat{\varphi}_i(\zeta_i)\, \check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})\, \mathrm{d}\zeta_i, \tag{B.1}$$

or

$$\int_{b_p}^{\mathrm{e}^{\mathrm{i}\theta_p}\infty} \hat{\varphi}_p(\zeta_p)\, (\mathrm{e}^{-\zeta_p/z_p})^{(m)}\, \mathrm{d}\zeta_i, \tag{B.2}$$

where $a_{i+1}, b_i, z_p \in \mathbb{K}$, $m \in \mathbb{N}$ and $\hat{\varphi}_i$ is holonomic with initial conditions in $\mathbb{K}^{\mathrm{shola}}$. Moreover, b_i and b_p may be chosen as large as desired. By the results from section 4.2, the kernels $\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i, a_{i+1})$, $(\mathrm{e}^{-\zeta_p/z_p})^{(m)}$ and the integrands are all holonomic, with initial conditions in $\mathbb{K}^{\mathrm{shola}}$ at b_i. Note that the m-th derivatives are taken with respect to a_{i+1} and z_p, so they amount to multiplying the integrands with a polynomial in ζ_i or ζ_p of degree m.

Let us focus on the integrals of type (B.1); the integrals of type (B.2) are treated similarly. The function $\hat{\varphi}_i$ satisfies a holonomic equation (i.e. a monic linear differential equation with coefficients in $\mathbb{K}(\zeta_i)$) of which all solutions have a growth bounded by $B\,\mathrm{e}^{C\,|\zeta_i|^{k_i/(k_i-k_{i+1})}}$, for some fixed constant $C>0$ and a constant B that depends on the solution. Likewise, as shown in section 7.1, $\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,a_{i+1})$ satisfies a holonomic equation of which all solutions are bounded by $B\,\mathrm{e}^{-C\,|\zeta_i|^{k_i/(k_i-k_{i+1})}}$ for a fixed constant $C>0$ that can be made arbitrarily large (by taking b_i large). By Lemma 4.3(b), it follows that the same holds for the integrand $I(\zeta):=\hat{\varphi}_i(\zeta_i)\,\check{K}_{k_i,k_{i+1}}^{(m)}(\zeta_i,a_{i+1})$.

Given such a holonomic equation satisfied by I, consider the canonical fundamental basis $h_1,\ldots,h_s$ of solutions to this equation at $\zeta_i=b_i$. For each j, the function h_j has initial conditions in $\mathbb{K}$ at b_i and the integral

$$\int_{b_i}^{\mathrm{e}^{\mathrm{i}\theta_i}\infty} h_j(\zeta_i)\,\mathrm{d}\,\zeta_i,$$

converges. Taking b_i sufficiently large and applying a change of variables of the form $(\zeta_i/b_i)^c=(1-\xi)^{-1}$, we see that the value of the integral lies in $\mathbb{K}^{\mathrm{shol}}$. Since I can be rewritten as a $\mathbb{K}^{\mathrm{shola}}$-linear combination of $h_1,\ldots,h_s$, we conclude that (B.1) also takes a value in $\mathbb{K}^{\mathrm{shola}}$. □

B.7. Invertible elements

Given an integral domain R, let $R^\times$ be its subgroup of invertible elements. An interesting question is to determine the sets $(\mathbb{K}^{\mathrm{hol}})^\times$, $(\mathbb{K}^{\mathrm{rhol}})^\times$, etc. Obviously, $\mathbb{K}^{\neq}\subseteq(\mathbb{K}^{\mathrm{hol}})^\times$ and $\mathrm{e}^{\mathbb{K}}\subseteq(\mathbb{K}^{\mathrm{hol}})^\times$. We also know that $\pi^{\mathbb{Z}}\subseteq(\mathbb{K}^{\mathrm{hol}})^\times$, since

$$\begin{aligned}\pi &= 4\left(\arctan\frac{1}{2}+\arctan\frac{1}{3}\right)\in\mathbb{K}^{\mathrm{hol}}\\ \frac{1}{\pi} &= \frac{2\sqrt{2}}{9801}\sum_{k\in\mathbb{N}}\frac{(4\,k)!\,(1103+26390\,k)}{(k!)^4\,396^{4k}}\in\mathbb{K}^{\mathrm{hol}},\end{aligned}$$

and $\pi^{1/2\,\mathbb{Z}}\subseteq(\mathbb{K}^{\mathrm{shol}})^\times$, since

$$\sqrt{\pi}=\Gamma\left(\frac{1}{2}\right).$$

It would be interesting to know whether $\pi^\alpha\in(\mathbb{K}^{\mathrm{shol}})^\times$ for other rational numbers $\alpha\in\mathbb{Q}\setminus(1/2\,\mathbb{Z})$. From

$$\begin{aligned}\Gamma(z) &= \int_0^\infty x^{z-1}\,\mathrm{e}^{-x}\,\mathrm{d}x\\ \frac{1}{\Gamma(z)} &= \frac{\mathrm{i}}{2\,\pi}\int_{\mathcal{H}}(-t)^{-z}\,\mathrm{e}^{-t}\,\mathrm{d}t,\end{aligned}$$

we deduce that $\Gamma(\mathbb{K}\setminus\mathbb{Z})\subseteq(\mathbb{K}^{\mathrm{shol}})^\times$, where $\mathcal{H}$ is a Hankel contour from ∞ around 0 and then back to ∞. From the above facts and Euler's reflection formula

$$\Gamma(1-z)\,\Gamma(z)=\frac{\pi}{\sin(\pi\,z)},$$

we also deduce that $\sin(\pi\,(\mathbb{K}\setminus\mathbb{Z}))\subseteq(\mathbb{K}^{\mathrm{shol}})^\times$. This is noteworthy, since $\sin z$ is a well known example of a holonomic function whose inverse $\frac{1}{\sin z}$ is *not* holonomic.

Apart from the invertible elements that directly follow from the above list of examples, the author is not aware of any other invertible holonomic constants. In particular, the precise status of $\mathcal{X}$ is unclear. From $\sin(\pi(\mathbb{K}\setminus\mathbb{Z}))\subseteq\mathbb{K}^{\mathrm{shol}}$, it follows that $\mathcal{X}^{-1}\subseteq\mathbb{K}^{\mathrm{shol}}$, whence

$$\mathbb{K}^{\mathrm{shol}}=\mathbb{K}^{\mathrm{shola}}.$$

In combination with proposition B.7, this actually provides a correct proof of [Hoeven, 2007b, proposition 4.7 (d)]. If $\mathcal{X}^{-1}\subseteq\mathbb{K}^{\mathrm{hol}}$, then this would also imply $\mathbb{K}^{\mathrm{hol}}=\mathbb{K}^{\mathrm{rhol}}=\mathbb{K}^{\mathrm{hola}}=\mathbb{K}^{\mathrm{rhola}}$. It seems plausible though that $\mathcal{X}\cap\mathbb{K}^{\mathrm{hol}}=\{1\}$ and $\mathbb{K}^{\mathrm{hol}}=\mathbb{K}^{\mathrm{rhol}}=\mathbb{K}^{\mathrm{hola}}=\mathbb{K}^{\mathrm{rhola}}$ both hold.

B.8. Further comments

For simplicity, we have assumed that $\mathbb{K}=\mathbb{Q}^{\mathrm{alg}}$ is the field of algebraic numbers, throughout our exposition. Most results go through without much change for arbitrary algebraically closed fields $\mathbb{K}$. Only in the proof of Lemma B.2, we used the assumption that $2\pi\mathrm{i}\notin\mathbb{K}$; if $2\pi\mathrm{i}\in\mathbb{K}$, then the same conclusion can be obtained by induction on d, by applying the induction hypothesis on $f(z\,\mathrm{e}^{2\pi\mathrm{i}})-f(z)$, when $d>0$.

Another interesting direction of generalization would be to consider holonomic functions that are completely defined over $\mathbb{Q}^{\mathrm{alg}}$, but to consider values at points in larger fields $\mathbb{K}$. Such classes of constants contain numbers like $\mathrm{e}^{\mathrm{e}^{\pi}}$, $\sin\Gamma(\sqrt{2})$, etc.

In [Fischler and Rivoal, 2011], the authors consider values of so-called Siegel G-functions, which are a particular type of Fuchsian holonomic functions. They prove an analogue of theorem B.5 in this setting. Their proof is significantly simpler, thanks to special properties of G-functions [Fischler and Rivoal, 2011, theorem 3], and based on similar arguments as our proof of proposition B.3. The paper [Fischler and Rivoal, 2011] also contains several results about fraction fields of fields of values of G-functions. It would be interesting to investigate analogues of these results in our setting.

Still in [Fischler and Rivoal, 2011], the authors study the case when $\mathbb{K}\subsetneq\mathbb{Q}^{\mathrm{alg}}$ is an algebraic number field that is strictly contained in $\mathbb{Q}^{\mathrm{alg}}$. They showed that any real algebraic number can be obtained as the value at $z=1$ of a G-function over $\mathbb{Q}$ that is defined on $\bar{\mathcal{D}}_{0,1}$. In our setting, this immediately implies that $\mathbb{Q}^{\mathrm{alg}}\subseteq\mathbb{Q}^{\mathrm{hol}}[\mathrm{i}]$, whence $(\mathbb{Q}^{\mathrm{alg}})^{\mathrm{hol}}=\mathbb{Q}^{\mathrm{hol}}[\mathrm{i}]$. More generally, for any algebraic number field $\mathbb{K}$, we obtain $(\mathbb{Q}^{\mathrm{alg}})^{\mathrm{hol}}=\mathbb{K}^{\mathrm{hol}}[\mathrm{i}]$ if $\mathbb{K}\subseteq\mathbb{R}$ and $(\mathbb{Q}^{\mathrm{alg}})^{\mathrm{hol}}=\mathbb{K}^{\mathrm{hol}}$ if $\mathbb{K}\subsetneq\mathbb{R}$.

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