

GUESSING SINGULAR DEPENDENCIES*

Joris van der Hoeven

CNRS, Département de Mathématiques
Bâtiment 425
Université Paris-Sud
91405 Orsay Cedex
France

Email: joris@texmacs.org

Web: <http://www.math.u-psud.fr/~vdhoeven>

December 9, 2008

Given d complex numbers z_1, \dots, z_d , it is classical that linear dependencies $\lambda_1 z_1 + \dots + \lambda_d z_d = 0$ with $\lambda_1, \dots, \lambda_d \in \mathbb{Z}$ can be guessed using the LLL-algorithm. Similarly, given d formal power series $f_1, \dots, f_d \in \mathbb{C}[[z]]$, algorithms for computing Padé-Hermite forms provide a way to guess relations $P_1 f_1 + \dots + P_d f_d = 0$ with $P_1, \dots, P_d \in \mathbb{C}[z]$. Assuming that f_1, \dots, f_d have a radius of convergence $r > 0$ and given a real number $R > r$, we will describe a new algorithm for guessing linear dependencies of the form $g_1 f_1 + \dots + g_d f_d = h$, where $g_1, \dots, g_d, h \in \mathbb{C}[[z]]$ have a radius of convergence $\geq R$. We will also present two alternative algorithms for the special cases of algebraic and Fuchsian dependencies.

KEYWORDS: guessing, asymptotic dependency, orthogonalization, analytic continuation, Fuchsian singularity, Padé-Hermite forms, algorithm

A.M.S. SUBJECT CLASSIFICATION: 30E10, 30B40, 41A21, 65B05

1. INTRODUCTION

Consider an infinite sequence f_0, f_1, \dots of complex numbers. If f_0, f_1, \dots are the coefficients of a formal power series $f \in \mathbb{C}[[z]]$, then it is well-known [Pól37, Wil04, FS96] that a lot of information about the behaviour of f near its dominant singularity can be obtained from the asymptotic behaviour of the sequence f_0, f_1, \dots . However, if f is the solution to some complicated equation, then it can be hard to compute the asymptotic behaviour using formal methods. On the other hand, the coefficients f_0, f_1, \dots of such a solution f can often be computed numerically up to a high order. This raises the question of how to *guess* the asymptotic behaviour of f_0, f_1, \dots , based on this numerical evidence.

In fact, the systematic integration of “guessing tools” into symbolic computation packages would be a useful thing. Indeed, current systems can be quite good at all kinds of formal manipulations. However, in the daily practice of scientific discovery, it would be helpful if these systems could also detect hidden properties, which may not be directly apparent or expected, and whose validity generally depends on heuristics. One well-known tool in this direction is the LLL-algorithm. Given d numbers $z_1, \dots, z_d \in \mathbb{C}$, it can be used in order to guess relations of the form

$$\lambda_1 z_1 + \dots + \lambda_d z_d = 0 \quad (\lambda_1, \dots, \lambda_d \in \mathbb{Z}). \quad (1)$$

*. This work was partially supported by the ANR Gecko project.

Given d formal power series $f_1, \dots, f_d \in \mathbb{C}[[z]]$, algorithms for the computation of Padé-Hermite forms [BL94, Der94] can be used in order to guess linear relations

$$P_1 f_1 + \dots + P_d f_d = 0 \quad (P_1, \dots, P_d \in \mathbb{C}[z]). \quad (2)$$

A well-known implementation is provided by the GFUN package [SZ94]. Given a finite number of coefficients of a formal power series $f \in \mathbb{C}[[z]]$, the GFUN package is able to guess a closed form formula for f or a linear differential equation with coefficients in $\mathbb{C}[z]$ satisfied by f .

Unfortunately, many interesting formal power series f do not admit closed form formulas and are not holonomic. In that case, we can still use asymptotic extrapolation [vdH06] in order to guess the asymptotic behaviour of the coefficients. However, this only provides us some rough idea about the behaviour of f at its dominant singularity. In practice, it often happens that f locally satisfies an algebraic or differential equation with analytic coefficients, even though these coefficients fail to be polynomials. In this paper, we will shall describe two approaches to deal with this situation.

In section 2, we first present a numerical algorithm for approximating the radius of convergence of f , assuming that only a finite number of its coefficients are known. In the case when f admits a unique dominant isolated singularity σ , we will also describe several approaches to find σ .

In section 3, we consider the favourable situation when there exists an algorithm for the analytic continuation of f . This algorithm should allow us to compute the Taylor series expansion of f not only at the origin, but at any point on its Riemann surface. This is typically the case when f is the solution of an initial value problem [vdH07]. We will show how to exploit the mechanism of analytic continuation in order to find so called algebraic or Fuchsian dependencies at singularities.

In the remainder of the paper, we let $f_1, \dots, f_d \in \mathbb{C}[[z]]$ be power series with radii of convergence at least $r > 0$. We assume that the coefficients of the f_i can be computed up to a high order and with great accuracy. This is typically the case if the f_i are given as explicit generating functions or solutions to functional equations. Given a radius $R > r$, we are interested in the determination of linear dependencies

$$g_1 f_1 + \dots + g_d f_d = h \quad (g_1, \dots, g_d, h \in \mathbb{C}[[z]]), \quad (3)$$

where g_1, \dots, g_d, h have radii of convergence strictly larger than R . In section 4, we will describe an algorithm for doing this, based on Gram-Schmidt orthogonalization. Modulo the change of variables $z \mapsto R z$, it actually suffices to consider the case when $R = 1$. In section 5, we will present some relations which were recognized by the algorithm. In section 6, we will also examine the behaviour of the algorithm in the case when f_1, \dots, f_d are analytically independent. Section 7 contains a discussion of the algorithm and perspectives.

2. LOCALIZATION OF SINGULARITIES

Let f be an analytic function which is given by its power series $f_0 + f_1 z + f_2 z^2 + \dots$ at the origin. A first natural question is how to compute the radius of convergence $\rho = \rho_f$ of f at the origin, assuming that we have an algorithm for computing the coefficients f_i . In what follows, we will only be interested in heuristic algorithms. In general, good lower bounds for ρ can often be obtained efficiently, but the computation of sharp upper bounds can be undecidable. For some results on the exact computation of ρ , we refer to [DL89, vdH07].

Theoretically speaking, the radius of convergence is given by

$$\rho^{-1} = \limsup_{i \rightarrow \infty} \sqrt[i]{|f_i|}.$$

After the computation of $n \geq 2$ coefficients f_0, \dots, f_{n-1} of f , this yields the approximation

$$\rho^{-1} \approx \max_{n/2 \leq i < n} \sqrt[i]{|f_i|}. \quad (4)$$

For most convergent power series f , this formula is ultimately exact in the sense that

$$\rho^{-1} = \lim_{n \rightarrow \infty} \max_{n/2 \leq i < n} \sqrt[i]{|f_i|}. \quad (5)$$

This is for instance the case when $\log_2 |f_i|$ is ultimately convex or ultimately concave. The set of f for which (5) holds is also stable under the transformation $f(z) \mapsto f(z^k)$ for any $k \in \mathbb{N}^> = \{n \in \mathbb{N} : n > 0\}$. Of course, we may replace $n/2$ by αn in (5) for any $\alpha \in (0, 1)$.

The formula (4) has the disadvantage that it has not been scaled appropriately: when replacing f by cf , where $c \in \mathbb{C}^\times$ is such that $|c| \neq 1$, we obtain different approximations for ρ_{cf} and ρ_f . Therefore, it is better to replace f_i by f_i/f_k for some appropriate coefficient $f_k \neq 0$ with $k < i$. One way to choose appropriate indices k is to consider the numerical Newton polygon of $F = f_a z^a + \dots + f_{n-1} z^{n-1}$, where $a = \lfloor \alpha n \rfloor$, $\alpha = 1/2$.

Let $P_i = (i, \log_2 |f_i|)$ for $a \leq i < n$, where we understand that $\log_2 0 = -\infty$. Then the Newton diagram of F is the convex hull of the half lines $P_i + \{0\} \times \mathbb{R}^{\leq}$ for $a \leq i < n$. There exists a minimal subset $\{P_{i_1}, \dots, P_{i_k}\} \subseteq \{P_a, \dots, P_{n-1}\}$ with $a = i_1 < \dots < i_k = n-1$, such that the Newton diagram is also the convex hull of the half lines $P_{i_j} + \{0\} \times \mathbb{R}^{\leq}$ for $1 \leq j \leq k$. Graphically speaking, the P_{i_j} are the vertices of the Newton diagram.

For a fixed $\beta \in (\alpha, 1)$, say $\beta = 3/4$, we may now determine the unique edge $P_{i_j} P_{i_{j+1}}$ of the Newton diagram such that $i_j \leq \beta n < i_{j+1}$, and replace the formula (4) by

$$\rho^{-1} \approx \left| \frac{f_{i_{j+1}}}{f_{i_j}} \right|^{\frac{1}{i_{j+1} - i_j}}. \quad (6)$$

For most convergent power series f , the formula (6) is again ultimately exact. The indices $i_1 < \dots < i_k$ can be computed from the coefficients f_a, \dots, f_{n-1} using a linear traversal in time $O(n)$. Modulo a tiny extra cost, this enables us to compute a more accurate approximation of ρ than (4). The formula (6) has been implemented in the MATHEMAGIX system [vdH02b]. For $n \approx 64$, the computed radius is usually correct up to one decimal digit.

The formula (6) usually yields a reasonable estimate for ρ , even in very degenerate cases when there are several singularities at distance ρ or close to ρ . If f admits an isolated singularity σ at distance ρ , with no other singularities at distance close to ρ , then the quotient f_{n-1}/f_{n-2} usually tends to σ^{-1} for large n . When the singularity at σ has a known type, then the approximation $\sigma^{-1} = f_{n-1}/f_{n-2}$ can be further improved.

For instance, in the case of algebraic singularities, we have

$$f_n \approx \sigma^{-n} n^{k/p} (a_0 + a_1 n^{-1/p} + a_2 n^{-2/p} + \dots),$$

with $k \in \mathbb{Z}$ and ramification index $p \in \mathbb{N}^>$, whence

$$\log f_n \approx (-\log \sigma) n + \frac{k}{p} \log n + \log a_0 + \frac{a_1}{a_0} n^{-1/p} + \dots \quad (7)$$

Using the E-algorithm [Wen01, BZ91], we may now compute simultaneous approximations for the first coefficients $-\log \sigma, k/p, \log a_0, a_1/a_0$, etc. of the expansion (7). It turns out that this strategy greatly improves the accuracy of the approximation of σ (see also [vdH06]).

Similarly, we say that f is Fuchsian at σ , if f satisfies a linear differential equation

$$L_r \delta^r f + \dots + L_0 f = 0,$$

where $\delta = z \partial / \partial z$ and L_0, \dots, L_r are analytic functions at σ with $L_r(\sigma) \neq 0$. In that case, the Taylor coefficients f_n satisfy the asymptotic expansion

$$f_n \approx \sigma^{-n} n^\lambda (a_0(\log n) + a_1(\log n) n^{-1/p} + a_2(\log n) n^{-2/p} + \dots),$$

where $\lambda \in \mathbb{C}$, $p \in \mathbb{N}^>$ and the a_i are polynomials in $\log n$ of degrees $< r$. Again, the E-algorithm or more general algorithms for asymptotic extrapolation [vdH06] can be used to compute σ with a high accuracy. Notice that these algorithms also provide estimates for the accuracies of the computed approximations.

In the case when the closest singularity σ is a simple pole, we may directly try to find the polynomial $P = z - \sigma$. This polynomial has the property that the radius of convergence of Pf is strictly larger than $|\sigma|$. More generally, if f is meromorphic on the compact disc \bar{D}_r of radius r , then we may search for a polynomial P such that the radius of convergence of Pf is strictly larger than r . This can be done using simple linear algebra, as follows:

Algorithm `denom`(f, r, N, D)

INPUT: the first $N > 2D$ coefficients of f , a radius r and a degree bound D

OUTPUT: an approximation of a monic polynomial P with $\rho_{Pf} > r$,
chosen of minimal degree $\deg P \leq D$, or failed

Step 1. [Initialize]

$d := 0$

Step 2. [Determine P]

Solve the linear system

$$\begin{pmatrix} f_{N-2d+1} & \cdots & f_{N-d} \\ \vdots & & \vdots \\ f_{N-d} & \cdots & f_{N-1} \end{pmatrix} \begin{pmatrix} P_{d-1} \\ \vdots \\ P_0 \end{pmatrix} + \begin{pmatrix} f_{N-2d} \\ \vdots \\ f_{N-d-1} \end{pmatrix} = 0 \quad (8)$$

Set $P := z^d + P_{d-1}z^{d-1} + \cdots + P_0$

Step 3. [Terminate or loop]

Heuristically determine $\rho := \rho_{Pf}$, based on the first N coefficients of Pf

If $\rho > r$ then return P

If $d = N$ then return failed

Set $d := d + 1$ and go to step 2

THEOREM 1. *The algorithm `denom` is correct.*

PROOF. Modulo a scaling $z = rz'$, we may assume without loss of generality that $r = 1$. Furthermore, the proof is clear if $d = 0$, so we will assume that $\rho_f \leq 1$. Assume that there exists a polynomial Q with $\rho_{Qf} > 1$ and choose

$$Q = z^d + Q_{d-1}z^{d-1} + \cdots + Q_0$$

monic and of minimal degree d . Notice that such a polynomial Q exists in particular, if (8) admits no solution for some d . We have

$$\begin{pmatrix} f_{N-2d+1} & \cdots & f_{N-d} \\ \vdots & & \vdots \\ f_{N-d} & \cdots & f_{N-1} \end{pmatrix} \begin{pmatrix} Q_{d-1} \\ \vdots \\ Q_0 \end{pmatrix} + \begin{pmatrix} f_{N-2d} \\ \vdots \\ f_{N-d-1} \end{pmatrix} = \begin{pmatrix} (Qf)_{N-d} \\ \vdots \\ (Qf)_{N-1} \end{pmatrix}. \quad (9)$$

Since we assumed that $\rho_{Qf} > 1$, the coefficients $(Qf)_N = O(\lambda^N)$ have exponential decay for some $\lambda < 1$. Now consider a polynomial R of degree $< d$ and let $g = Rf$, so that

$$H_N \begin{pmatrix} R_{d-1} \\ \vdots \\ R_0 \end{pmatrix} := \begin{pmatrix} f_{N-2d+1} & \cdots & f_{N-d} \\ \vdots & & \vdots \\ f_{N-d} & \cdots & f_{N-1} \end{pmatrix} \begin{pmatrix} R_{d-1} \\ \vdots \\ R_0 \end{pmatrix} = \begin{pmatrix} g_{N-d} \\ \vdots \\ g_{N-1} \end{pmatrix}.$$

By the minimality hypothesis of d , we have $\rho_{Rf} \leq 1$, whence the coefficients g_N have polynomial or exponential growth in N . Since this property holds for all R , it follows that the matrix norm $\|H_N^{-1}\|$ remains bounded for large N . In particular, the solution to (8) is only an exponentially small perturbation of the solution to (9) for large values of N . \square

REMARK 2. The algorithm uses a very simple d -step search for the optimal degree d . Using a binary search (doubling d at each step at a first stage, and using a dichotomic search at a second stage), the number of steps can be reduced to $O(\log d)$.

3. DETECTING DEPENDENCIES VIA ANALYTIC CONTINUATION

Let f be an analytic function which is given by its power series $f_0 + f_1 z + f_2 z^2 + \dots$ at the origin. Assume that f can be continued analytically on a Riemann surface \mathcal{R} above the closed disk $\bar{\mathcal{D}}_r$ of radius r minus a finite set of points $\sigma_1, \dots, \sigma_s$. Let \mathbb{A} be the set of analytic functions on $\bar{\mathcal{D}}_r$. We say that f is algebraic on $\bar{\mathcal{D}}_r$ if f only admits algebraic singularities inside $\bar{\mathcal{D}}_r$. This is the case if and only if there exists a polynomial dependency

$$P_d f^d + \dots + P_0 = 0, \tag{10}$$

where $P \in \mathbb{A}[F]$. In that case, we may normalize the relation such that P has minimal degree and such that P_d is a monic polynomial of minimal degree. In particular, all roots of P_d are inside the disc $\bar{\mathcal{D}}_r$. We say that f is Fuchsian on $\bar{\mathcal{D}}_r$, if f only admits Fuchsian singularities inside $\bar{\mathcal{D}}_r$. This implies the existence of a dependency

$$L_r f^{(r)} + \dots + L_0 f = 0, \tag{11}$$

where $L \in \mathbb{A}[\partial]$. Again, we may normalize (11) such that L has minimal order r and such that L_r is a monic polynomial of minimal degree. Assuming that we have a high accuracy algorithm for the analytic continuation of f , we will give heuristic algorithms for the computation of dependencies of the form (10) or (11), if they exist.

REMARK 3. Algorithms for analytic continuation typically exist when f is the solution of an initial value problem. For a precise definition of a computable analytic function, we refer to [vdH07]. For the heuristic purposes of this section, the existence of a numerical algorithm for the continuation of f will be sufficient.

Assume that f is algebraic on $\bar{\mathcal{D}}_r$. For each singularity σ_i , choose a path φ_i from the origin to a point near σ_i which avoids the other singularities, and let C_i be the operator which performs an analytic continuation along φ_i , one turn around σ_i , followed by an analytic continuation along φ_i^{-1} . In the following algorithm for the detection of algebraic dependencies, we assume a heuristic equality test for analytic functions at a point, for instance by checking p -bit equality of the first n terms of the Taylor series expansions.

Algorithm `alg_dep`($f, r, (\sigma_1, \dots, \sigma_s), N, D, B$)

INPUT: an analytic function f above $\bar{\mathcal{D}}_r \setminus \{\sigma_1, \dots, \sigma_s\}$ and bounds N, D and B

OUTPUT: a normalized algebraic dependency (10) with $d \leq B$ and $\deg P_d \leq D$, or failed

Step 1. [Initialize]

Set $\Phi := \{f\}$

Step 2. [Saturate]

If $C_i \Phi \subseteq \Phi$ for all i , then go to step 3

If $\text{card}(\Phi) > N$ then return failed

$\Phi := C_1 \Phi \cup \dots \cup C_s \Phi$
 Repeat step 2

Step 3. [Terminate]

Denote $\Phi := \{\varphi_1, \dots, \varphi_k\}$
 Compute $Q := (F - \varphi_1) \dots (F - \varphi_k) = Q_k F^k + \dots + Q_0$
 For each $i \in \{0, \dots, k\}$, compute $D_i := \text{denom}(Q_i, r, D)$
 If $D_i = \text{failed}$ for some i , then return failed
 Return $\text{lcm}(D_0, \dots, D_k) Q$

THEOREM 4. *The algorithm `alg_dep` is correct.*

PROOF. Assume that f satisfies a normalized relation (10), with $d = \deg P \leq B$ and $\deg P_d \leq D$. Since Φ only contains distinct roots of P , we have $(\prod_{\varphi \in \Phi} F - \varphi) | P$ in $\mathbb{C}((z))[F]$ throughout the algorithm. In particular $\text{card}(\Phi) \leq d$, and we ultimately obtain stabilization $C_1 \Phi \cup \dots \cup C_s \Phi \subseteq \Phi$.

At this point, analytic continuation around any of the points σ_i leaves the polynomial Q invariant, so the coefficients of Q are analytic and single-valued on $\bar{D}_r \setminus \{\sigma_1, \dots, \sigma_s\}$. On the other hand, given a singularity σ_i , each solution $\varphi \in \Phi$ is also given by a convergent Puiseux series near σ_i , whence so are the coefficients of Q . Since the only Puiseux series without monodromy around σ_i are Laurent series, it follows that the coefficients of Q are meromorphic on \bar{D}_r .

By the minimality assumption on $\deg P$, it follows that $\deg Q = \deg P$ and $P = P_d Q$. Since each coefficient $Q_i = P_i/P_d$ is meromorphic on \bar{D}_r , we may use the algorithm `denom` from the previous section in order to compute a polynomial D_i of minimal degree $\deg D_i \leq \deg P_d \leq B$ such that $Q_i D_i \in \mathbb{A}$. The monic least common multiple $\text{lcm}(D_0, \dots, D_k)$ is nothing but the monic polynomial P_d of minimal degree such that $P_d Q \in \mathbb{A}[F]$. \square

REMARK 5. As a safeguard, we may heuristically compute the radii of convergence of P_0, \dots, P_d and check whether they are indeed superior to r .

Algorithm `fuch_dep`($f, r, (\sigma_1, \dots, \sigma_s), N, D, B$)

INPUT: an analytic function f above $\bar{D}_r \setminus \{\sigma_1, \dots, \sigma_s\}$ and bounds N, D and B

OUTPUT: a normalized Fuchsian dependency (11) with $r \leq B$ and $\deg L_r \leq D$, or failed

Step 1. [Initialize]

Set $\Phi := \{f\}$

Step 2. [Saturate]

If $\text{Vect}(C_i \Phi) \subseteq \text{Vect}(\Phi)$ for all i , then go to step 3
 If $\text{card}(\Phi) > N$ then return failed
 $\Phi := \Phi \cup \{C_i \varphi\}$ for i and $\varphi \in \Phi$ with $C_i \varphi \notin \text{Vect}(\Phi)$
 Repeat step 2

Step 3. [Terminate]

Denote $\Phi := \{\varphi_1, \dots, \varphi_k\}$
 Compute $K := \text{lcm}(\partial - \varphi_1^\dagger, \dots, \partial - \varphi_k^\dagger) = K_k \partial^k + \dots + K_0$
 in the skew polynomial ring $\mathbb{C}((z))[\partial]$, where φ^\dagger denotes φ'/φ
 For each $i \in \{0, \dots, k\}$, compute $D_i := \text{denom}(K_i, r, D)$
 If $D_i = \text{failed}$ for some i , then return failed
 Return $\text{lcm}(D_0, \dots, D_k) K$

THEOREM 6. *The algorithm `fuch_dep` is correct.*

PROOF. Assume that f satisfies a normalized Fuchsian relation (11), with $r \leq B$ and $\deg L_r \leq D$. Throughout the algorithm, the set Φ only contains linearly independent solutions to $L\varphi = 0$. Therefore, the smallest operator $\text{lcm}_{\varphi \in \Phi}(\partial - \varphi^\dagger)$ which vanishes on Φ divides L in $\mathbb{C}((z))[\partial]$. In particular $\text{card}(\Phi) \leq r$, and we ultimately obtain stabilization $C_1\Phi + \dots + C_s\Phi \subseteq \Phi$.

Consider one of the singularities σ_i . Since f is Fuchsian at σ_i , the equation $Lf = 0$ admits a fundamental system of solutions of the form

$$\varphi(\sigma_i + u) = \left(\varphi_{r-1}(u^{1/p})(\log u)^{r-1} + \dots + \varphi_0(u^{1/p}) \right) u^\lambda,$$

where $\lambda \in \mathbb{C}$ and $\varphi_0, \dots, \varphi_{r-1}$ are convergent power series. The coefficients of K lay in the field \mathbb{K} generated by all generalized power series of this form. Again, the only elements of \mathbb{K} with a trivial monodromy around σ_i are convergent Laurent series at σ_i . Since analytic continuation around σ_i leaves the operator K unchanged, it follows that the coefficients of K are meromorphic on \bar{D}_r . We conclude in a similar way as in the proof of theorem 4. \square

REMARK 7. The hypothesis that f admits at worse a Fuchsian singularity at σ_i is essential for the algorithm to work. For instance, in the case of the function

$$f = e^{\frac{1}{z-\sigma}} + e^{-\frac{1}{z-\sigma}},$$

the monodromy around σ is trivial, whence applying the algorithm for $r > |\sigma|$ would simply result in having $\Phi = \{f\}$ at step 3. Although f^\dagger has no monodromy around σ , this function is no longer a Laurent series. In fact, the desired vanishing operator has second order in this case, but it cannot be read off directly from Φ .

So far, our algorithms assume that we know the locations of the singularities $\sigma_1, \dots, \sigma_s$ inside the disc \bar{D}_r . Using the techniques from the previous section, we still need an algorithm to determine these locations.

Assume that we have localized some of the singularities $\sigma_1, \dots, \sigma_s$ and that Φ has been stabilized under the corresponding operators C_i . Given one of the singularities σ_i , one subtask is to determine a small radius ε_i such that Φ does not admit other singularities above the disc with center σ_i and radius ε_i . For a given candidate ε_i , this condition can be checked as follows. We choose a sufficient number $A = 17$ of equally spaced points

$$\zeta_k = \sigma_i + e^{\frac{2\pi i k}{A}} \varepsilon_i$$

on the Riemann surface of Φ above the circle with center σ_i and radius ε_i . For each $\varphi \in \Phi$ and each point ζ_k , we may use the techniques from the previous section in order to determine the closest singularity $\zeta_k + \delta_{\varphi,k}$ of φ to ζ_k . Now our check succeeds if $\zeta_k + \delta_{\varphi,k} \approx \sigma_i$ or $|\zeta_k + \delta_{\varphi,k} - \sigma_i| > \varepsilon_i$ for all φ and ζ_k . If it fails, then we retry with a smaller ε_i . As soon as ε_i becomes smaller than a fixed threshold, then we abort.

A second subtask is to guarantee the absence of singularities on the remaining part of the Riemann surface of Φ above \bar{D}_r . We construct a sequence of open balls $\mathcal{B}_1, \mathcal{B}_2, \dots$ as follows. For $i \leq s$, we let \mathcal{B}_i be the ball with center σ_i and radius ε_i . The remaining \mathcal{B}_i are constructed by induction over i . We arbitrarily chose the center ζ_i of \mathcal{B}_i in $\bar{D}_r \setminus (\mathcal{B}_1 \cup \dots \cup \mathcal{B}_{i-1})$ and heuristically determine the minimum ρ of the convergence radii of the elements in Φ above ζ_i . If ρ is significantly smaller than $\delta = \min(|\zeta_i - \sigma_1|, \dots, |\zeta_i - \sigma_s|, |r - |\zeta_i||)$, then this indicates a missing singularity: we heuristically determine the closest singularity σ to ζ_i and add it to the set $\{\sigma_1, \dots, \sigma_s\}$. Since ζ_i was chosen arbitrarily, there usually is at most one closest singularity. As an additional security check, we may also determine the closest singularity to $(\sigma + \zeta_i)/2$ and check that it coincides with σ . If $\rho > \delta/2$, then we take $\rho/2$ to be the radius of \mathcal{B}_i and continue our construction. Since \bar{D}_r is compact, the construction will eventually terminate, unless we find a new singularity.

The unknown series g_1, \dots, g_d give rise to a row vector

$$G = (g_{1,n-1} \ \cdots \ g_{d,n-1} \ \cdots \ \cdots \ g_{1,0} \ \cdots \ g_{d,0}).$$

Setting $h = g_1 f_1 + \cdots + g_d f_d$, we then have

$$GM = (h_{n-1} \ \cdots \ h_0 \ G),$$

whence $\Phi = GM$ encodes the relation $(\varphi_1, \dots, \varphi_{d+1}) = (g_1, \dots, g_d, h)$. This reduces the problem to finding those vectors G for which $\|GM\|$ is small, provided that at least one of the coefficients $g_{1,0}, \dots, g_{d,0}$ is reasonably large.

We start with the computation of a thin LQ decomposition of M . This can for instance be done using the Gram-Schmidt process: starting with the first row, we orthogonally project each row on the vector space spanned by the previous rows. This results in a decomposition

$$M = LQ,$$

where L is a lower triangular $nd \times nd$ matrix with ones on the diagonal and Q is an $nd \times (n + nd)$ matrix, whose rows are mutually orthogonal (i.e. $QQ^* = \text{Id}$). Since L^{-1} coincides with the righthand side of Q , the decomposition can be done in-place. Now consider the matrix Λ formed by the d last rows of L^{-1} . Then each row $G = \Lambda_i$ gives rise to a relation (13), encoded by $\Phi = \Lambda_i M$. Moreover, this relation is *normal* or *i-normal*, in the sense that $g_{i,0} = 1$ and $g_{j,0} = 0$ for all $j > i$. Since $\Lambda_i M$ is the shortest vector in $M_i + \text{Vect}(M_0, \dots, M_{i-1})$, the relation is also minimal in norm, among all *i-normal* relations. Choosing the row Λ_i for which $\|\Lambda_i M\|$ is minimal, our algorithm simply returns the corresponding relation. Then our algorithm has the following fundamental property:

PROPOSITION 8. *The algorithm returns a normal relation g for which $\|GM\|$ is minimal.*

Let us now return to the case when $f_1, \dots, f_d \in \mathbb{C}[[z]]$ are no longer truncated, but all have a radius of convergence $\geq r$. A relation (12) is again said to be *normal* or *i-normal* if $g_{i,0} = 1$ and $g_{j,0} = 0$ for all $j > i$. Under the limit $n \rightarrow \infty$, we claim that our algorithm finds a minimal normal relation, if there exists a relation of the form (12):

THEOREM 9.

- a) Assume that $\Phi_{;\infty} \neq 0$. Then $\Phi_{;\infty}$ contains a minimal normal relation.
- b) Assume that $\varphi \in \Phi_{;\infty}$ is a minimal *i-normal* relation. For each $n \in \mathbb{N}$, let $f_{;n}$ be the truncation of f at order n and consider the corresponding minimal *i-normal* relation $\varphi_{;n} \in \Phi_{;n}$. Then the relations $\varphi_{;n}$ converge to φ in $\mathbb{C}[[z]]_{;\infty}^{d+1}$.
- c) If $\Phi_{;\infty} = 0$, then the norms $\|\varphi_{;n}\|$, with $\varphi_{;n}$ as in (b), are unbounded for $n \rightarrow \infty$.

PROOF. A non trivial relation $\varphi = (g, h)$ is easily normalized: we first divide g by z^v , where $v = \min \{\text{val } g_1, \dots, \text{val } g_d\}$. We next divide by $g_{i,0}$, where i is largest with $g_{i,0} \neq 0$. Now the set of all *i-normal* relations is a closed affine subspace of $\mathbb{C}[[z]]_{;\infty}^d$. The orthogonal projection of 0 on this subspace yields an *i-normal* relation of minimal norm. This proves (a).

Assume that there exists an *i-normal* relation (12). Given an order $n \in \mathbb{N}$, consider the minimal *i-normal* relation $\varphi_{;n}$ at this order. Truncation of this relation at a smaller order $m < n$ yields an *i-normal* relation $\varphi_{;n;m}$ at order m with $\varphi_{;n;m} \perp \varphi_{;n} - \varphi_{;n;m}$, whence

$$\|\varphi_{;n}\|^2 = \|\varphi_{;n;m}\|^2 + \|\varphi_{;n} - \varphi_{;n;m}\|^2. \tag{14}$$

Moreover, since $\varphi_{;m}$ is the projection of 0 on the affine space of i -normal relations at order n , we have $\varphi_{;m} \perp \varphi_{;n;m} - \varphi_{;m}$ and

$$\|\varphi_{;n;m}\|^2 = \|\varphi_{;n;m} - \varphi_{;m}\|^2 + \|\varphi_{;m}\|^2. \quad (15)$$

In particular, $\|\varphi_{;m}\| \leq \|\varphi_{;n;m}\| \leq \|\varphi_{;n}\|$ and

$$\|\varphi_{;0}\| \leq \|\varphi_{;1}\| \leq \dots \leq \|\varphi\|.$$

For large m , it follows that $\|\varphi_{;n}\| - \|\varphi_{;m}\| \rightarrow 0$, whence

$$\|\varphi_{;n} - \varphi_{;m}\| \leq \|\varphi_{;n} - \varphi_{;n;m}\| + \|\varphi_{;n;m} - \varphi_{;m}\| \rightarrow 0.$$

In other words, $\varphi_{;n}$ is a Cauchy sequence, which converges to a limit $\tilde{\varphi} \in \mathbb{C}[[z]]_{\infty}^d$. By the minimality hypothesis of φ , we have $\|\tilde{\varphi}\| = \|\varphi\|$. Passing (14) and (15) to the limit, we get $\|\tilde{\varphi} - \varphi\| = 0$, whence $\tilde{\varphi} = \varphi$ and (b).

In general, the existence of a bound B with

$$\|\varphi_{;0}\| \leq \|\varphi_{;1}\| \leq \dots \leq B$$

still ensures $\varphi_{;n}$ to be a Cauchy sequence, and its limit yields an i -normal relation (12). This proves the last assertion (c). \square

5. EXAMPLES OF RECOGNIZED RELATIONS

A first implementation of our guessing algorithm has been made in the MATHEMAGIX system [vdH02b]. It is instructive to test the algorithm on examples for which the existence of linear dependencies is known. We have done this for three examples of increasing difficulty. In order to avoid problems due to numerical instability, we have used a large working precision, of 1024 or 2048 bits.

Single poles. The easiest example on which we can run the algorithm is

$$f_1 = \frac{1}{1 - \lambda z} \quad (\lambda > 1).$$

Here follow the values for $g_{;n}$ at different orders and $\lambda = 2$:

$$\begin{aligned} g_{;4,1} &= 1.0000 - 1.6000 z - 0.60000 z^2 - 0.20000 z^3 \\ g_{;16,1} &= 1.0000 - 1.6180 z - 0.61803 z^2 - \dots - 5.8340 10^{-6} z^{14} - 1.9447 10^{-6} z^{15} \\ g_{;64,1} &= 1.0000 - 1.6180 z - 0.61803 z^2 - \dots - 5.0484 10^{-26} z^{62} - 1.6828 10^{-26} z^{63} \\ g_{;256,1} &= 1.0000 - 1.6180 z - 0.61803 z^2 - \dots - 2.8307 10^{-106} z^{254} - 9.4357 10^{-107} z^{255} \end{aligned}$$

The $g_{;n,1}$ clearly convergence to a limit g_1 with radius of convergence > 1 . It should be noticed that we do not have $g_1 = 1 - \lambda z$. In other words, we did not find the “best” relation, as in the case of Padé-Hermite approximation. A closer examination of the result shows that

$$\begin{aligned} g_1 &= \frac{1 - \lambda z}{1 - \alpha z} \\ \alpha + \frac{1}{\alpha} &= \lambda + \frac{2}{\lambda} \quad (\alpha < 1) \end{aligned}$$

In particular, α decreases if λ increases.

Logarithmic singularity. Our second example concerns a logarithmic singularity:

$$\begin{aligned} f_1 &= \cos(z) \log(1 - 2z) \\ f_2 &= f_1' \end{aligned}$$

We obtain:

$$\begin{aligned}
 g_{;32,1} &= 0.070136 + 0.66832 z - 1.1491 z^2 - \dots - 0.010473 z^{29} + 0.0033638 z^{30} \\
 g_{;32,2} &= 1.0000 - 1.6886 z - 0.35145 z^2 - \dots - 0.0071095 z^{30} + 0.0033638 z^{31} \\
 g_{;64,1} &= 0.013789 + 0.88839 z - 1.3375 z^2 - \dots - 0.0083494 z^{61} + 0.0012131 z^{62} \\
 g_{;64,2} &= 1.0000 - 1.6782 z - 0.42492 z^2 - \dots - 0.0071362 z^{62} + 0.0012131 z^{63} \\
 g_{;128,1} &= 1.3819 \cdot 10^{-4} + 0.99781 z - 1.6176 z^2 - \dots - 2.3908 \cdot 10^{-4} z^{125} + 1.6114 \cdot 10^{-5} z^{126} \\
 g_{;128,2} &= 1.0000 - 1.6326 z - 0.54349 z^2 - \dots - 2.2297 \cdot 10^{-4} z^{126} + 1.6114 \cdot 10^{-5} z^{127} \\
 g_{;256,1} &= 4.7212 \cdot 10^{-9} + 1.00000 z - 1.6204 z^2 - 0.24832 z^3 - 0.75949 z^4 - 0.31293 z^5 - \dots \\
 &\quad - 9.8386 \cdot 10^{-5} z^{128} - 9.5406 \cdot 10^{-5} z^{129} - 9.3630 \cdot 10^{-5} z^{130} - 9.3108 \cdot 10^{-5} z^{131} - \dots \\
 &\quad - 2.3728 \cdot 10^{-6} z^{251} + 2.7475 \cdot 10^{-7} z^{252} - 1.9342 \cdot 10^{-8} z^{253} + 6.2727 \cdot 10^{-10} z^{254} \\
 g_{;256,2} &= 1.0000 - 1.6204 z - 0.58164 z^2 - 0.21949 z^3 - 0.25182 z^4 - 0.016358 z^5 - \dots \\
 &\quad - 1.0928 \cdot 10^{-4} z^{128} - 9.8597 \cdot 10^{-5} z^{129} - 8.6828 \cdot 10^{-5} z^{130} - 7.3899 \cdot 10^{-5} z^{131} - \dots \\
 &\quad 2.1091 \cdot 10^{-6} z^{252} + 2.5583 \cdot 10^{-7} z^{253} - 1.8714 \cdot 10^{-8} z^{254} + 6.2727 \cdot 10^{-10} z^{255}
 \end{aligned}$$

The convergence only becomes apparent at higher orders. Contrary to the previous example, it seems that the series in the computed relation all have a radius of convergence 1. The norms of the computed results are given by $\|\varphi_{;32}\| = 3.5577$, $\|\varphi_{;64}\| = 3.5955$, $\|\varphi_{;128}\| = 3.6739$ and $\|\varphi_{;256}\| = 3.6862$.

Number of alcohols. A more interesting example from combinatorics is the enumeration of the number of alcohols of the form $C_n H_{2n+1} O H$ [Pól37]. Its generating series satisfies the functional equation

$$s(z) = 1 + z \frac{s(z)^3 + 2s(z^3)}{3}.$$

Using asymptotic extrapolation [vdH06], this series is found to have a radius of convergence $r \approx 0.304218409$. In order to investigate s close to its dominant singularity, we apply our algorithm to

$$\begin{aligned}
 f_1 &= s(0.15(z + 0.25)) \\
 f_2 &= f_1'
 \end{aligned}$$

The translation $z \mapsto z + 0.25$ is done using power series evaluation until stabilization at the working precision. At different orders, we obtain:

$$\begin{aligned}
 g_{;32,1} &= 1.6836 - 0.29080 z + 0.0013079 z^2 + \dots + 0.022645 z^{30} - 0.0029284 z^{31} \\
 g_{;32,2} &= 1.0000 - 2.4006 z - 0.88670 z^2 - \dots + 0.0061880 z^{30} - 9.9191 \cdot 10^{-4} z^{31} \\
 g_{;64,1} &= 1.7104 - 0.36445 z + 0.23778 z^2 - \dots + 1.6968 \cdot 10^{-4} z^{62} - 1.0104 \cdot 10^{-5} z^{63} \\
 g_{;64,2} &= 1.0000 - 2.3257 z - 1.2027 z^2 - \dots + 5.2359 \cdot 10^{-5} z^{62} - 3.4225 \cdot 10^{-6} z^{63} \\
 g_{;96,1} &= 1.7105 - 0.36226 z + 0.23722 z^2 - \dots + 9.4113 \cdot 10^{-7} z^{94} - 3.6428 \cdot 10^{-8} z^{95} \\
 g_{;96,2} &= 1.0000 - 2.3236 z - 1.2135 z^2 - \dots + 3.0034 \cdot 10^{-7} z^{94} - 1.2339 \cdot 10^{-8} z^{95} \\
 g_{;128,1} &= 1.7105 - 0.36215 z + 0.23696 z^2 - 0.052535 z^3 + 0.033518 z^4 - \dots \\
 &\quad + 6.3727 \cdot 10^{-5} z^{96} - 3.4222 \cdot 10^{-5} z^{97} - 1.2120 \cdot 10^{-4} z^{98} - 1.5462 \cdot 10^{-4} z^{99} - \dots \\
 &\quad + 7.2542 \cdot 10^{-7} z^{124} - 7.4945 \cdot 10^{-8} z^{125} + 4.6860 \cdot 10^{-9} z^{126} - 1.3437 \cdot 10^{-10} z^{127} \\
 g_{;128,2} &= 1.0000 - 2.3235 z - 1.2138 z^2 - 0.0067226 z^3 - 0.080434 z^4 + \dots \\
 &\quad - 2.6028 \cdot 10^{-5} z^{96} - 3.4490 \cdot 10^{-5} z^{97} - 2.9260 \cdot 10^{-5} z^{98} - 1.1090 \cdot 10^{-5} z^{99} + \dots \\
 &\quad + 2.1258 \cdot 10^{-7} z^{124} - 2.3161 \cdot 10^{-8} z^{125} + 1.5192 \cdot 10^{-9} z^{126} - 4.5516 \cdot 10^{-11} z^{127}
 \end{aligned}$$

The corresponding norms are given by $\|\varphi_{;32}\| = 4.3276$, $\|\varphi_{;64}\| = 4.3845$, $\|\varphi_{;96}\| = 4.3863$ and $\|\varphi_{;128}\| = 4.3863$. Again, the convergence is rather slow, and the computed series all seem to have radius of convergence 1.

6. RATE OF DIVERGENCE IN CASE OF INDEPENDENCE

It is also instructive to run the algorithm on examples where the f_i are known to be analytically independent. In that case, it is important to examine the behaviour of the norms $\|\varphi_{;n}\|$ for $n \rightarrow \infty$ and find more precise criteria which will enable us to discard the existence of a nice dependency with a reasonable degree of certainty.

Logarithmic singularity. Let us return to the case of a simple logarithmic singularity

$$f_1 = \log(1 - \lambda z),$$

where $\lambda > 1$. Running our algorithm directly on this function yields the following results for $\lambda = 2$:

$$\begin{aligned} g_{;32,1} &= 1.0000 - 4.2642 z + 4.3396 z^2 + \dots + 1.3621 z^{29} - 0.68197 z^{30} \\ g_{;64,1} &= 1.0000 - 8.1566 z + 23.882 z^2 - \dots + 4.1515 z^{61} - 0.67813 z^{62} \\ g_{;128,1} &= 1.0000 - 16.183 z + 112.02 z^2 - \dots + 9.2483 z^{125} - 0.65589 z^{126} \\ g_{;256,1} &= 1.0000 - 32.213 z + 480.89 z^2 - \dots + 19.957 z^{253} - 0.66367 z^{254} \end{aligned}$$

The results indeed do not converge and the corresponding sequence of norms $\|\varphi_{;32}\| = 12.324$, $\|\varphi_{;64}\| = 77.306$, $\|\varphi_{;128}\| = 3383.9$ and $\|\varphi_{;256}\| = 6.4461 \cdot 10^6$ diverges at moderate speed.

Various singularities. In the above example, it is interesting to study the rate of divergence of the sequence of norms for other values of λ . More generally, we can consider various types of singularities, such as

$$\begin{aligned} \psi_{1,\lambda} &= \log(1 - \lambda z) \\ \psi_{2,\lambda} &= \sqrt{1 - \lambda z} \\ \psi_{3,\lambda} &= e^{\frac{\lambda z}{1 - \lambda z}} \\ \psi_{4,\lambda} &= \text{random}(\lambda z) \end{aligned}$$

The series $\text{random}(z) = \rho_0 + \rho_1 z + \rho_2 z^2 + \dots$ is a series whose coefficients are chosen according to a random uniform distribution on $[0, 1]$. The results are shown in table 1 below. For $\psi_{1,\lambda}$, $\psi_{2,\lambda}$ and $\psi_{3,\lambda}$, it seems that the norm does not much depend on the precise type of singularity, but only on λ and the truncation order n . For the last series $\psi_{4,\lambda}$, the dependencies on λ and n approximately seem to follow the law

$$\log \|\varphi_{;n}\| \approx \frac{n}{2} \log \lambda. \quad (16)$$

This idealized law needs to be adjusted for functions $\psi_{1,\lambda}$, $\psi_{2,\lambda}$, $\psi_{3,\lambda}$ with more common types of singularities. Although (16) remains asymptotically valid for large values of $\log \lambda$, the factor $\log \lambda$ needs to be replaced by a smaller number for moderate values. For $\sqrt{2} \leq \lambda \leq 4$, this factor rather seems to be of the form $\alpha (\log \lambda)^2$, where the constant α depends on the nature of the singularity. Also, the linear dependency of $\log \|\varphi_{;n}\|$ on n is only reached for large values of n .

	32	64	128	256
$\psi_{1,\sqrt{2}}$	2.5897	4.2958	$1.1107 \cdot 10^1$	$7.5308 \cdot 10^1$
$\psi_{1,2}$	$1.2324 \cdot 10^1$	$7.7306 \cdot 10^1$	$3.3839 \cdot 10^3$	$6.4461 \cdot 10^6$
$\psi_{1,4}$	$2.7074 \cdot 10^3$	$3.1503 \cdot 10^6$	$5.0806 \cdot 10^{12}$	$1.3215 \cdot 10^{25}$
$\psi_{1,8}$	$5.7101 \cdot 10^6$	$1.6078 \cdot 10^{13}$	$1.2829 \cdot 10^{26}$	$8.6064 \cdot 10^{51}$
$\psi_{1,16}$	$6.0964 \cdot 10^{10}$	$1.7814 \cdot 10^{21}$	$1.5612 \cdot 10^{42}$	$1.2279 \cdot 10^{84}$
$\psi_{1,32}$	$1.1152 \cdot 10^{15}$	$9.1674 \cdot 10^{29}$	$4.9020 \cdot 10^{59}$	$1.6662 \cdot 10^{119}$
$\psi_{3,\sqrt{2}}$	2.2960	3.7208	9.7394	$6.5890 \cdot 10^1$
$\psi_{2,2}$	9.3539	$6.1551 \cdot 10^1$	$2.6678 \cdot 10^3$	$5.0111 \cdot 10^6$
$\psi_{2,4}$	$1.7232 \cdot 10^3$	$2.1500 \cdot 10^6$	$3.4149 \cdot 10^{12}$	$8.7128 \cdot 10^{24}$
$\psi_{2,8}$	$3.5980 \cdot 10^6$	$1.0031 \cdot 10^{13}$	$7.9396 \cdot 10^{25}$	$5.2284 \cdot 10^{51}$
$\psi_{3,\sqrt{2}}$	6.2155	$1.3295 \cdot 10^1$	$4.1722 \cdot 10^1$	$4.2679 \cdot 10^2$
$\psi_{3,2}$	$4.0164 \cdot 10^1$	$3.9310 \cdot 10^2$	$1.4047 \cdot 10^4$	$1.1091 \cdot 10^7$
$\psi_{3,4}$	$6.9272 \cdot 10^3$	$6.8304 \cdot 10^6$	$4.7564 \cdot 10^{11}$	$5.1308 \cdot 10^{20}$
$\psi_{3,8}$	$3.6660 \cdot 10^6$	$9.4896 \cdot 10^{11}$	$4.2745 \cdot 10^{21}$	$1.1809 \cdot 10^{39}$
$\psi_{4,2}$	$8.0487 \cdot 10^4$	$3.5565 \cdot 10^9$	$2.1354 \cdot 10^{19}$	$4.8792 \cdot 10^{38}$
$\psi_{4,4}$	$5.0774 \cdot 10^9$	$1.1548 \cdot 10^{19}$	$2.9916 \cdot 10^{38}$	$1.2335 \cdot 10^{77}$
$\psi_{4,8}$	$3.2564 \cdot 10^{14}$	$4.3151 \cdot 10^{28}$	$5.1348 \cdot 10^{57}$	$2.9654 \cdot 10^{115}$

Table 1. Computed values of $\|\varphi_n\|$ for various types of singularities $\psi_{i,\lambda}$ and orders n .

Various singularities and $d > 1$. We have also applied our algorithm jointly to several $\psi_{i,\lambda}$ as above and studied the dependency of $\|\varphi_n\|$ on d . The results are shown in table 2 below. In the bottom rows, $\psi_{4,\lambda}^*$, $\psi_{4,\lambda}^{**}$, ... stand for distinct uncorrelated random series $\text{random}(\lambda z)$. In that case, the relation (16) generalizes to

$$\log \|\varphi_n\| \approx \frac{n}{d+1} \log \lambda. \tag{17}$$

It also seems that the law can be adapted to functions with more common types of singularities, along similar lines as before.

	32	64	128
$\psi_{1,8}$	$5.7101 \cdot 10^6$	$1.6078 \cdot 10^{13}$	$1.2829 \cdot 10^{26}$
$\psi_{1,8}, \psi_{2,8}$	$4.8124 \cdot 10^3$	$4.1720 \cdot 10^7$	$2.0911 \cdot 10^{15}$
$\psi_{1,8}, \psi_{2,8}, \psi_{1,8}(-z)$	$1.4867 \cdot 10^3$	$3.8890 \cdot 10^6$	$1.5664 \cdot 10^{13}$
$\psi_{1,8}, \psi_{2,8}, \psi_{1,8}(-z), \psi_{2,8}(-z)$	$7.9704 \cdot 10^2$	$1.4235 \cdot 10^6$	$4.9236 \cdot 10^{12}$
$\psi_{4,8}$	$3.2564 \cdot 10^{14}$	$4.3151 \cdot 10^{28}$	$5.1348 \cdot 10^{57}$
$\psi_{4,8}, \psi_{4,8}^*$	$2.3631 \cdot 10^9$	$2.8202 \cdot 10^{18}$	$1.9011 \cdot 10^{38}$
$\psi_{4,8}, \psi_{4,8}^*, \psi_{4,8}^{**}$	$5.4571 \cdot 10^6$	$5.2439 \cdot 10^{13}$	$4.0431 \cdot 10^{28}$
$\psi_{4,8}, \psi_{4,8}^*, \psi_{4,8}^{**}, \psi_{4,8}^{***}$	$2.4329 \cdot 10^5$	$2.4748 \cdot 10^{11}$	$2.4206 \cdot 10^{22}$

Table 2. Computed values of $\|\varphi_n\|$ for different input vectors of increasing dimensions d .

7. DISCUSSION AND PERSPECTIVES

In the case when a linear relation (12) exists, we have observed in section 5 that our algorithm usually returns a relation whose series all have radius of convergence 1. Larger radii of convergence R can be obtained simply by running the algorithm on $f(Rz)$ and scaling back the result.

In the example of the enumeration of alcohols, we have also mentioned the fact that the detection of dependencies is improved by zooming in on the singularity. Although we used a shift $z \mapsto z + c$ for doing so, this is quite expensive in general: if $r = |\sigma|$ is the module of the singularity σ , then approximately $(1 + \log_2^{-1}(r/|c|))n$ Taylor coefficients of $f(z)$ are needed in order to accurately compute n coefficients of $f(z + c)$. When possible, it is better to postcompose f with a suitable analytic function $\phi(z) = \phi_1 z + \phi_2 z^2 + \dots$. Here ϕ should be chosen in such a way that $f \circ \phi$ has $\phi^{-1}(\sigma)$ as its only singularity in the unit disk, and such that $|\phi^{-1}(\sigma)|$ is as small as possible. Also, if f satisfies a differential equation, then f can be expanded efficiently at c by integrating the equation from its initial conditions [BK78, vdH02a].

Our empirical observations in the section 6 suggest a few criteria for detecting the non-existence of dependencies (12). First of all, when running the algorithm for different orders n , we should observe a more or less geometric increase of the norm $\|\varphi_{;n}\|$. If we know the norm $r = \lambda^{-1}$ of the smallest singularity, then this idea may be refined by comparing the computed values of $\|\varphi_{;n}\|$ with the expected values, as given by the law (17), or a suitable adaptation of this law when $\log \lambda$ becomes small. This method is most effective when λ is large. When possible, it is therefore recommended to zoom in on the singularity, as discussed above.

For any numerical checks based on the law (17) or a refinement of it, we also recommend to precondition the input series f_1, \dots, f_d . In particular, we recommend to multiply each f_i by a suitable constant, ensuring that

$$\max_{j < n} |f_{i,j}| r^j = 1,$$

whenever we apply our algorithm at order n . Here r is computed using one of the algorithms from section 2.

Let us now analyze the cost of the algorithm from section 4. The current working precision p should in general be taken larger than $n \log_2 \lambda$ in order to keep the method numerically stable. Denoting by $M(p)$ the cost for multiplying two p bit numbers, a naive implementation of the Gram-Schmidt orthogonalization procedure yields a total cost of $O(d^3 n^3 M(p))$. Denoting by $MM(k, p)$ the cost of multiplying two $k \times k$ matrices with p bit entries, and using a blockwise Gram-Schmidt procedure, we obtain the better bound $O(MM(dn, p))$. However, the matrix M from section 4 has a very special form. With more work, it might therefore be possible to save an additional factor $O(n)$, but we have not actively tried to do so yet.

Since it is often possible to zoom in on the singularity, we may evaluate the computational cost in terms of the desired “output quality”. As a definition of the output quality, we may take the expected value $q \approx \frac{n}{d+1} \log \lambda$ of $\log \|\varphi_{;n}\|$ in the case when f_1, \dots, f_d are independent. In terms of q , the time complexity then becomes

$$O\left(\text{MM}\left(\frac{q d^2}{\log \lambda}, q d\right)\right) = \tilde{O}\left(\frac{q^{\omega+1} d^{2\omega+1}}{(\log \lambda)^\omega}\right) = \tilde{O}\left(\frac{q^4 d^7}{(\log \lambda)^3}\right),$$

where $\omega < 2.376 \leq 3$ is the exponent of matrix multiplication. The complexity bound makes it clear that we should put a lot of effort into keeping $\log \lambda$ large. For instance, zooming in on the singularity using a shift makes it possible to replace λ by $\lambda' = \kappa \lambda$ modulo the evaluation of f at a larger order $n' \approx (1 + \log_2 \kappa)n$ instead of n (and a bit precision $p' \approx (1 + \log_2 \kappa)p$ instead of p). In many practical cases, this can be done in time $\tilde{O}(n' p')$, which allows for a drastic reduction of the complexity. Of course, a possible analytic relation will only be obtained on a small disk around the singularity of interest.

To go short, our algorithm from section 4 mainly works well if the following conditions are satisfied:

1. The number d should not be too large.
2. There exists a large constant $\lambda > 1$ for which all singularities of f are concentrated inside the disk of radius λ^{-1} or outside the disk of radius λ .

If we are interested in dependencies near an isolated singularity, then the second condition can often be achieved by zooming in on the singularity.

As a final note, we mention the fact that linear analytic dependencies can sometimes be obtained using the process of asymptotic extrapolation [vdH06]. For instance, given a function f with an isolated smallest singularity at 1 of the form

$$f(z) = a(z) + b(z) \log \frac{1}{1-z},$$

where a and b are analytic at 1, the asymptotic extrapolation of $f = f_0 + f_1 z + \dots$ yields an asymptotic expansion of the form

$$f_n \approx \frac{c_1}{n} + \frac{c_2}{n^2} + \dots.$$

Using singularity analysis [FS96], we may then recover the function b from the coefficients c_0, c_1, \dots . However, this technique only works in special cases, since the first ω terms of the asymptotic expansion may hide other terms, which need to be taken into account when searching for exact dependencies.

BIBLIOGRAPHY

- [BK78] R.P. Brent and H.T. Kung. Fast algorithms for manipulating formal power series. *Journal of the ACM*, 25:581–595, 1978.
- [BL94] B. Beckermann and G. Labahn. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Analysis and Applications*, pages 804–823, 1994.
- [BZ91] C. Brezinski and R. Zaglia. *Extrapolation Methods. Theory and Practice*. North-Holland, Amsterdam, 1991.
- [Der94] H. Derksen. An algorithm to compute generalized padé-hermite forms. Technical Report Rep. 9403, Catholic University Nijmegen, January 1994.
- [DL89] J. Denef and L. Lipshitz. Decision problems for differential equations. *The Journ. of Symb. Logic*, 54(3):941–950, 1989.
- [FS96] P. Flajolet and R. Sedgewick. *An introduction to the analysis of algorithms*. Addison Wesley, Reading, Massachusetts, 1996.
- [Pól37] G. Pólya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Mathematica*, 68:145–254, 1937.
- [SZ94] B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Trans. on Math. Software*, 20(2):163–177, 1994.
- [vdH02a] J. van der Hoeven. Relax, but don't be too lazy. *JSC*, 34:479–542, 2002.
- [vdH02b] J. van der Hoeven et al. Mathemagix, 2002. <http://www.mathemagix.org>.
- [vdH06] J. van der Hoeven. Algorithms for asymptotic extrapolation. *JSC*, 2006. Submitted.
- [vdH07] J. van der Hoeven. On effective analytic continuation. *MCS*, 1(1):111–175, 2007.
- [Wen01] E. J. Weniger. Nonlinear sequence transformations: Computational tools for the acceleration of convergence and the summation of divergent series. Technical Report math.CA/0107080, Arxiv, 2001.
- [Wil04] H.S. Wilf. *Generatingfunctionology*. Academic Press, 2nd edition, 2004. <http://www.math.upenn.edu/~wilf/DownldGF.html>.