Defining a surreal hyperexponential

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Conway's class **No** of surreal numbers admits a rich structure: it forms a totally ordered real closed field with an exponential functions and a derivation. The aim of this note is to construct a surreal solution E_ω to the functional equation $E_\omega(a+1) = \exp E_\omega(a)$ with good properties.

1. INTRODUCTION

Let **No** be Conway's class of surreal numbers. It is well known that **No** admits a rich structure: Conway showed that **No** forms a real closed field and Gonshor also defined an exponential function exp on **No** that satisfies the same first order theory as the usual exponential function on the reals. Following Conway's tradition, all *numbers* will understood to be surreal in what follows.

The aim of this note is to define a bijective function $E_\omega: \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ on the class $\mathbf{No}^{>,\succ}$ of positive infinitely large numbers, which is strictly increasing and satisfies the functional equation

$$E_\omega(a+1) = \exp E_\omega a \tag{1}$$

for all $a \in \mathbf{No}^{>,\succ}$. Since this equation admits many solutions, the main difficulty is to single out a particular solution that will be most "natural" in a way that needs to be made precise.

The function E_ω is said *hyperexponential* and it is the first non-trivial hyperexponential in the transfinite sequence $(E_\alpha)_{\alpha \in \mathbf{Ord}}$ of iterated exponentials

$E_0 := \mathrm{Id},\ E_1 := \exp,\ E_2 := \exp \circ \exp,\ \ldots,\ E_\omega,\ E_{\omega+1} := E_\omega \circ \exp,\ \ldots,\ E_{2\omega} := E_\omega \circ E_\omega,\ \ldots$

The corresponding reciprocals are called *hyperlogarithms*:

$L_0 := \mathrm{Id},\ L_1 := \log,\ L_2 := \log \circ \log,\ \ldots,\ L_\omega,\ L_{\omega+1} := \log \circ L_\omega,\ \ldots,\ L_{2\omega} := L_\omega \circ L_\omega,\ \ldots$

It is natural to require such more general hyperexponentials to satisfy $E_{\omega^{\alpha+1}}(a+1) = E_{\omega^\alpha}(E_{\omega^{\alpha+1}}(a))$ for all ordinals α and $a \in \mathbf{No}^{>,\succ}$. Similarly, $L_{\omega^{\alpha+1}}(L_{\omega^\alpha}(a)) = L_{\omega^{\alpha+1}}(a) - 1$.

There are known real analytic solutions of (1) with good properties [12, 6], even though there does not seem to exist any meaningful "most natural" solution. It is also well known that fractional iterates of exp and log can be defined in terms of E_ω and L_ω: given $c \in \mathbb{R}$, we take $\exp_c(x) := E_\omega(L_\omega(x) + c)$ and $\log_c(x) := E_\omega(L_\omega(x) - c)$.

From a formal perspective, hyperexponentials and hyperseries were studied in detail by Schmeling and van der Hoeven [14]: they generalized transseries to include formal counterparts e_α, ℓ_α of E_α, L_α for $\alpha < \omega^\omega$ [14]. This yields in particular a natural hyperexponential on the set of positive infinitely large transseries. More recently, van den Dries, van der Hoeven and Kaplan [7] constructed the field of *logarithmic hyperseries* $\mathbb{L}$ with ℓ_α for all α, with corresponding natural hyperlogarithms $L_\alpha\colon \mathbb{L}^{>,\succ} \to \mathbb{L}^{>,\succ}$. The ultimate goal [11, 2] is to produce a field of hyperseries $\mathbb{H} \supseteq \mathbb{L}$ with all hyperexponentials and to construct an isomorphism $\mathbb{H} \cong \mathbf{No}$. This work can be considered as another step in this direction, by constructing a natural $E_\omega\colon \mathbf{No}^{>,\succ} \to \mathbf{No}^{>,\succ}$.

Surreal numbers and transseries It is well known that surreal numbers $a \in \mathbf{No}$ can be written as infinite series

$$a = \sum_{\mathfrak{m} \in \mathbf{Mo}} a_\mathfrak{m}\, \mathfrak{m},$$

where $\mathbf{Mo}$ denotes the class of surreal *monomials* and the coefficients $a_\mathfrak{m}$ are real. In particular, $\mathbf{No}$ is isomorphic to the Hahn field $\mathbb{R}[[\mathbf{Mo}]]$ of formal power series. Together with the exponential function, $\mathbf{No}$ even admits the structure of a field of *transseries* in the sense of [14]; see [4].

We will freely use notations from [1, 11] when dealing with such transseries. In particular, the *support* of $a \in \mathbf{No}$ is defined as $\operatorname{supp} a := \{\mathfrak{m} \in \mathbf{Mo} : a_\mathfrak{m} \neq 0\}$ and its *infinite part* as $a_\succ := \sum_{\mathfrak{m} \in \mathbf{Mo}, \mathfrak{m} \succ 1} a_\mathfrak{m}\, \mathfrak{m}$. Here we used Hardy's notation $\prec$ for $a \prec b \Leftrightarrow \mathbb{R}^> |a| < |b|$. Similarly, we set $a_\prec := \sum_{\mathfrak{m} \in \mathbf{Mo}, \mathfrak{m} \prec 1} a_\mathfrak{m}\, \mathfrak{m}$. We also define $\mathbf{No}^\succ := \{a \in \mathbf{No} : a \succ 1\}$, $\mathbf{No}_\succ := \{a_\succ : a \in \mathbf{No}\}$, and $\mathbf{No}^\prec := \{a \in \mathbf{No} : a \prec 1\} = \{a_\prec : a \in \mathbf{No}\}$. Finally, we write $a \trianglelefteq b$ if $\operatorname{supp} a \succ a - b$, in which case we say that a is a *truncation* of b.

Defining exponentials Consider a number $a \in \mathbf{No}$ and decompose it as $a = a_\succ + a_\asymp + a_\prec$ with $a_\succ \in \mathbf{No}_\succ$, $a_\asymp \in \mathbb{R}$, and $a_\prec \in \mathbf{No}^\prec$. Then the functional equation of exp yields

$$\exp a = \mathrm{e}^{a_\succ} \exp a_\asymp \exp a_\prec, \tag{2}$$

where $\exp a_=$ is the usual exponential in $\mathbb{R}$ and $\exp a_\prec = 1 + a_\prec + {}^1\!/_2\, a_\prec^2 + {}^1\!/_6\, a_\prec^3 + \cdots$. In order to define exp on $\mathbf{No}^>$, this relation shows that it would have sufficed to define it on $\mathbf{No}_\succ$. In addition, it can be shown that exp bijectively maps the class $\mathbf{No}_\succ$ to the class $\mathbf{Mo}$. Our process to define E_ω is similar, with different subclasses $\mathbf{Tr}$ and $\mathbf{La}$ in the roles of $\mathbf{No}_\succ$ and $\mathbf{Mo}$. The class $\mathbf{Tr}$ is defined below and $\mathbf{La}$ is the class of *log-atomic numbers*, i.e. numbers $\mathfrak{a} \in \mathbf{No}^{>,\succ}$ such that $\log^{\circ n} \mathfrak{a} \in \mathbf{Mo}$ for all $n \in \mathbb{N}$.

Surreal substructures It turns out that each of the classes $\mathbf{No}_\succ$, $\mathbf{Mo}$, $\mathbf{Tr}$, and $\mathbf{La}$ are examples of so-called surreal substructures, which were extensively studied in [3]. Let us quickly recall a few basic facts; see also section 2.

Besides the usual ordering, the class **No** of surreal numbers admits a well-founded partial order $\sqsubseteq$ called the *simplicity relation*. A *surreal substructure* is a subclass $\mathbf{S}$ of **No** that is isomorphic to $(\mathbf{No}, \leqslant, \sqsubseteq)$ for the induced relations by $\leqslant$ and $\sqsubseteq$ on $\mathbf{S}$. Equivalently, this means that for any subsets L, R of $\mathbf{S}$ with $L < R$, the *cut*

$$(L \mid R)_{\mathbf{S}} \; := \; \{a \in \mathbf{S} : L < a < R\}$$

in $\mathbf{S}$ admits a $\sqsubseteq$-minimum which is then denoted $\{L \mid R\}_{\mathbf{S}}$. We call (L, R) a *cut representation* in $\mathbf{S}$. We extend this notation to the case when $\mathbf{L}, \mathbf{R}$ are classes, provided $\{\mathbf{L} \mid \mathbf{R}\}_{\mathbf{S}}$ indeed exists. If $a \in \mathbf{S}$, then we let

$$a_L^{\mathbf{S}} \; := \; \{a' \in \mathbf{S} : a' \sqsubseteq a \wedge a' < a\}, \qquad a_R^{\mathbf{S}} \; := \; \{a'' \in \mathbf{S} : a'' \sqsubseteq a \wedge a'' > a\}, \qquad a_{\sqsubset}^{\mathbf{S}} \; := \; a_L^{\mathbf{S}} \sqcup a_R^{\mathbf{S}}.$$

Then $a = \{a_L^{\mathbf{S}} \mid a_R^{\mathbf{S}}\}_{\mathbf{S}}$. Moreover, for any cut $(L \mid R)_{\mathbf{S}}$ in $\mathbf{S}$ containing a (such as $a = \{L \mid R\}_{\mathbf{S}}$), the set L (resp. R) is cofinal (resp. coinitial) with respect to $a_L^{\mathbf{S}}$ (resp. $a_R^{\mathbf{S}}$).

Important examples of surreal substructures include **No**, the class $\mathbf{No}^{>}$ of strictly positive numbers, the class $\mathbf{No}^{>,\succ}$ of positive infinitely large numbers, the classes **Mo** and $\mathbf{Mo}^{\succ}$ of monomials and infinite monomials, the class $\mathbf{No}_{\succ} := \mathbb{R}[[\mathbf{Mo}^{\succ}]]$ of purely infinite numbers, and the class $\mathbf{No}^{\prec} := \mathbb{R}[[\mathbf{Mo}^{\succ}]]$ of infinitesimal numbers.

Truncated numbers Assume that $E_\omega a$ has been defined for some positive purely infinite number $a \in \mathbf{No}_{\succ}^{>} := \mathbf{No}_{\succ} \cap \mathbf{No}^{>}$. For sufficiently small ε, we wish to define $E_\omega(a + \varepsilon)$ using Taylor series expansion:

$$E_\omega(a + \varepsilon) \; := \; E_\omega a + (E'_\omega a)\, \varepsilon + \tfrac{1}{2}(E''_\omega a)\, \varepsilon^2 + \cdots. \tag{3}$$

The successive derivatives $E'_\omega a, E''_\omega a, \ldots$ can be defined in **No** as ordinary (and so-called logarithmic) transseries applied to $b := E_\omega a$:

$$\begin{aligned} E'_\omega a \; &:= \; b\,(L_1 b)\,(L_2 b) \cdots \\ E''_\omega a \; &= \; b\,(L_1 b)^2 (L_2 b)^2 \cdots + b\,(L_1 b)\,(L_2 b)^2 (L_3 b)^2 + \cdots \\ &\vdots \end{aligned}$$

It can be shown that (3) converges formally, provided that $\varepsilon \prec 1 / E_\omega a$. More generally, consider $\delta \in \mathbf{No}$ with $\delta \prec 1 / L_k b$ for a certain $k \in \mathbb{N}$. Assuming (1), this means that $\delta \prec 1 / E_\omega(a - k)$, which allows us to define

$$E_\omega(a - k + \delta) \; := \; E_\omega(a - k) + (E'_\omega(a - k))\, \delta + \tfrac{1}{2}(E''_\omega(a - k))\, \delta^2 + \cdots \tag{4}$$

and set

$$E_\omega(a + \delta) \; := \; E_k(E_\omega(a - k + \delta)). \tag{5}$$

It follows that it is sufficient to define E_ω at numbers $\varphi \in \mathbf{No}^{>,\succ}$ with

$$\operatorname{supp} \varphi \; \succ \; \frac{1}{L_{\mathbb{N}}\, E_\omega\, \varphi_{\succ}},$$

where $L_{\mathbb{N}} = \{L_0, L_1, \ldots\}$. Those numbers are said to be *truncated* and we will write **Tr** for the class of all truncated numbers. It turns out that **Tr** is a surreal substructure.

Defining hyperexponentials Our definition of E_ω proceeds in three stages:

1. We first define E_ω on $\mathbf{No}_{\succ}^{>}$. For any two positive purely infinite numbers θ_1, θ_2 with $\theta_1 < \theta_2$, we have $\theta_1 + \mathbb{N} < \theta_2 - \mathbb{N}$. By the functional equation, we should have $E_\omega(\theta_1 + \mathbb{N}) = E_{\mathbb{N}}\, E_\omega\, \theta_1 < L_{\mathbb{N}}\, E_\omega\, \theta_2 = E_\omega(\theta_2 + \mathbb{N})$. We deduce that for $\theta \in \mathbf{No}_{\succ}^{>}$, the number $E_\omega\, \theta$ should lie in the cut $(L \mid R)$ in **No** where

$$L \; := \; E_{\mathbb{N}}\, \theta \cup E_{\mathbb{N}}\, E_\omega\, \theta_L^{\mathbf{No}_{\succ}^{>}}, \qquad R \; := \; L_{\mathbb{N}}\, E_\omega\, \theta_R^{\mathbf{No}_{\succ}^{>}}$$

The simplest way to ensure this is to define

$$E_\omega(\theta) \ := \ \left\{E_{\mathbb{N}}\,\theta, E_{\mathbb{N}}\,E_\omega\,\theta_L^{\mathbf{No}^{\gtrsim}_{\succ}} \mid L_{\mathbb{N}}\,E_\omega\,\theta_R^{\mathbf{No}^{\gtrsim}_{\succ}}\right\}$$

for all $\theta \in \mathbf{No}^{\gtrsim}_{\succ}$.

2. We next extend E_ω to $\mathbf{Tr}$. Similar arguments and the simplicity heuristic impose

$$E_\omega\,\varphi \ := \ \{E_{\mathbb{N}}\,\varphi, \mathcal{E}E_\omega\,\varphi_L^{\mathbf{Tr}} \mid \mathcal{E}E_\omega\,\varphi_R^{\mathbf{Tr}}\} \ \in \ \mathbf{La},$$

where $\mathcal{E}$ is a function group to be defined in Section 2. Here $\mathbf{Tr}$, $\mathbf{La}$, and (3) play a similar role as $\mathbf{No}_{\succ}$, $\mathbf{Mo}$, and (2) when extending the definition of exp.

3. We finally extend the definition of E_ω to $\mathbf{No}^{>,\succ}$ by relying on (4) and (5).

2. EQUATIONS AND CONVEX PARTITIONS

Before we define E_ω, let us briefly recall a general method to define surreal substructures using convex partitions. For more details, see [3, Section 6].

Uniform equations Let $\mathbf{T}$ be a surreal substructure and $F\colon \mathbf{S} \longrightarrow \mathbf{T}$ be a function. Let λ, ρ be functions defined for cut representations in $\mathbf{S}$ and such that $(\lambda(L,R), \rho(L,R))$ is a cut representation in $\mathbf{T}$ whenever (L,R) is a cut representation in $\mathbf{S}$. We say that (λ,ρ) is an *equation* of F if, for all $a \in \mathbf{S}$, we have

$$F(a) \ = \ \{\lambda(a_L^{\mathbf{S}}, a_R^{\mathbf{S}}) \mid \rho(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})\}_{\mathbf{T}}.$$

We say that the equation is *uniform* if we have $F(\{L \mid R\}_{\mathbf{S}}) = \{\lambda(L,R) \mid \rho(L,R)\}_{\mathbf{T}}$ whenever (L,R) is a cut representation in $\mathbf{S}$. For instance, by [9, Theorem 3.2], for $r \in \mathbb{R}$, the following equation for the translation $a \in \mathbf{No} \longmapsto a + r$ by r is uniform:

$$a + r \ = \ \{a_L + r, a + r_L \mid a + r_R, a_R + r\}. \tag{6}$$

Remark 1. Assume that F has an equation (λ,ρ) with

$$\forall a \in \mathbf{S}, \forall a' \in a_L^{\mathbf{S}}, \forall a'' \in a_R^{\mathbf{S}}, \exists b' \in \lambda(a_L, a_R), \exists b'' \in \rho(a_L, a_R), \qquad F(a') \ \leqslant \ b' \ \wedge \ F(a'') \ \geqslant \ b''.$$

This is in particular the case if $F(a_L^{\mathbf{S}}) \subseteq \lambda(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$ and $F(a_R^{\mathbf{S}}) \subseteq \rho(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$ for all $a \in \mathbf{S}$. Then we claim that F is strictly increasing. To see this, consider $a_0, a_1 \in \mathbf{S}$ with $a_0 < a_1$. By [3, Proposition 4.6], there is a $\sqsubseteq$-maximal element c of $\mathbf{S}$ with $c \sqsubseteq a, b$, and we have $a < c \leqslant b$ or $a \leqslant c < b$. We treat the first case, the other one being symmetric. Since $a < c$ and $c \sqsubseteq a$, we have $c \in a_R^{\mathbf{S}}$ so there is $b'' \in \rho(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$ with $b'' \leqslant F(c)$. We have $F(a) < b'' \leqslant F(c)$. A similar argument yields $F(c) \leqslant F(b)$, so $F(a) < F(b)$.

Convex partitions Let $\mathbf{S}$ be a surreal substructure and let $\mathbf{\Pi}$ be a partition of $\mathbf{S}$ into convex subclasses, each of which admits a cofinal and coinitial subset. We refer to $\mathbf{\Pi}$ as a *thin convex partition* of $\mathbf{S}$. For $a \in \mathbf{S}$, we let $\mathbf{\Pi}[a]$ denote the unique member of $\mathbf{\Pi}$ containing a. We also write $\mathbf{\Pi}[\mathbf{X}] := \bigcup_{x \in \mathbf{X}} \mathbf{\Pi}[x]$ for any subclass $\mathbf{X}$ of $\mathbf{S}$. We say that an element $a \in \mathbf{S}$ is $\mathbf{\Pi}$-*simple* if it is the $\sqsubseteq$-minimum of its class $\mathbf{\Pi}[a]$. This is equivalent to the existence of a cut $(L \mid R)_{\mathbf{S}}$ in $\mathbf{S}$ with $a = \{L \mid R\}_{\mathbf{S}}$ and $\mathbf{\Pi}[L] < a < \mathbf{\Pi}[R]$.

Then the class $\mathbf{Smp}_{\mathbf{\Pi}}$ of $\mathbf{\Pi}$-simple elements forms a surreal substructure which is contained in $\mathbf{S}$. For $a \in \mathbf{Smp}_{\mathbf{\Pi}}$, we have $a = \{\mathbf{\Pi}[a_L^{\mathbf{Smp}_{\Pi}}] \mid \mathbf{\Pi}[a_R^{\mathbf{Smp}_{\Pi}}]\}_{\mathbf{Smp}_{\mathbf{\Pi}}}$.

Function groups A *function group* $\mathcal{G}$ on a surreal substructure $\mathbf{S}$ is a group of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$ under functional composition. We regard elements f, g of $\mathcal{G}$ as actions on $\mathbf{S}$ and sometimes write fg and fa for $a \in \mathbf{S}$ rather than $f \circ g$ and $f(a)$. For such a function group $\mathcal{G}$, the collection $\mathbf{\Pi}_{\mathcal{G}} := (\mathcal{G}[a])_{a \in \mathbf{S}}$ of classes

$$\mathcal{G}[a] \ := \ \{b \in \mathbf{S} : \exists f, g \in \mathcal{G}, fa \leqslant b \leqslant ga\}$$

is a thin convex partition of $\mathbf{S}$ and we define $\mathbf{Smp}_{\mathcal{G}} := \mathbf{Smp}_{\Pi_{\mathcal{G}}}$.

For $f, g \in \mathcal{G}$, the relation $f < g \Longleftrightarrow \forall a \in \mathbf{S}, (fa < ga)$ is a partial order on $\mathcal{G}$. We will frequently rely on the elementary fact that $(\mathcal{G}, <)$ is *partially bi-ordered*, i.e. that we have

$$\forall f, g, h \in \mathcal{G}, \qquad g > \mathrm{id}_{\mathbf{S}} \Longleftrightarrow fgh > fh.$$

Common function groups As an example, we can obtain the previous structures as the classes $\mathbf{Smp}_{\mathcal{G}}$ for actions of the following function groups $\mathcal{G}$ acting on $\mathbf{No}$, $\mathbf{No}^{>}$ or $\mathbf{No}^{>,\succ}$. For $r \in \mathbb{R}$ and $s \in \mathbb{R}^{>}$, we define

$$\begin{aligned} T_r &:= a \longmapsto a + r && \text{acting on } \mathbf{No} \text{ or } \mathbf{No}^{>,\succ} \\ H_s &:= a \longmapsto s a && \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ} \\ P_s &:= a \longmapsto a^s = \exp(s \log a) && \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ} \end{aligned}$$

We define

$$\begin{aligned} \mathcal{T} &:= \{T_r : r \in \mathbb{R}\} \\ \mathcal{H} &:= \{H_s : s \in \mathbb{R}^{>}\} \\ \mathcal{P} &:= \{P_s : s \in \mathbb{R}^{>}\} \\ \mathcal{E} &:= \langle E_n H_s L_n : n \in \mathbb{N}, s \in \mathbb{R}^{>} \rangle \\ \mathcal{E}^* &:= \{E_n, L_n : n \in \mathbb{N}\}. \end{aligned}$$

We then have the following list of identities [3, Section 7.1]:

- The action of $\mathcal{T}$ on $\mathbf{No}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{No}_{\succ}$ (resp. $\mathbf{No}_{\succ}^{>}$).
- The action of $\mathcal{H}$ on $\mathbf{No}^{>}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{Mo}$ (resp. $\mathbf{Mo}^{\succ}$).
- The action of $\mathcal{P}$ on $\mathbf{No}^{>,\succ}$ yields $\exp \mathbf{Mo}^{\succ}$.
- The action of $\mathcal{E}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{La}$ [4, Corollary 5.17].
- The action of $\mathcal{E}^*$ on $\mathbf{No}^{>,\succ}$ yields the class $\mathbf{K}$ of [13].

For $a \in \mathbf{No}^{>,\succ}$, we will denote $\mathfrak{d}_\omega(a)$ the unique log-atomic element of $\mathcal{E}[a]$ and we will denote $\mathfrak{d}_\omega^*(a)$ the unique element of $\mathcal{E}^*[a]$ lying in $\mathbf{K}$. We have $\mathfrak{d}_\omega^*(a) \sqsubseteq \mathfrak{d}_\omega(a) \sqsubseteq a$.

3. LOGARITHMIC TRANSSERIES AND HYPERSERIES

Let $\mathbb{L}_{<\omega} = \mathbb{R}[[\mathfrak{L}_{<\omega}]]$ denote the field of logarithmic hyperseries from [7]. For each $n \in \mathbb{N}$, we define $\ell_n := L_n x \in \mathbb{L}_{<\omega}$. Recall that $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ stands for the set of series with Noetherian support in the partially ordered set $\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}$. We may consider elements of $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ as bivariate series that are logarithmic transseries with respect to x and ordinary series with respect to ε.

LEMMA 2. *Consider a series $f \in \mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ such that the substitution $f(\bar{\varepsilon})$ of ε by $\bar{\varepsilon}$ in f vanishes for all $\bar{\varepsilon} \prec 1$ in $\mathbb{L}_{<\omega}$. Then $f = 0$.*

Proof. Assume for contradiction that $f \neq 0$. Write $f = f_0 + f_1 \varepsilon + f_2 \varepsilon^2 + \cdots$ and let $k \in \mathbb{N}$ be minimal with $f_k \neq 0$. Since f is Noetherian as a series in $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$, the set $\bigcup_{i \in \mathbb{N}} \operatorname{supp} f_i$ is well based and admits a largest element $\mathfrak{m}$. Taking $\bar{\varepsilon} \in x^{\mathbb{R}}$ sufficiently small such that $\mathfrak{m}\, \bar{\varepsilon} \prec f_k$, it follows that $f_{k+1} \bar{\varepsilon}^{k+1} + f_{k+2} \bar{\varepsilon}^{k+2} + \cdots \prec f_k \bar{\varepsilon}^k$, whence $f(\bar{\varepsilon}) = f_k \bar{\varepsilon}^k + f_{k+1} \bar{\varepsilon}^{k+1} + f_{k+2} \bar{\varepsilon}^{k+2} + \cdots \sim f_k \bar{\varepsilon}^k \neq 0$. ◻

LEMMA 3. *Given integers* $p \geqslant k \geqslant 0$ *and* $\delta(\varepsilon) = L_{p-k}(L_k x + \varepsilon) - L_p x$, *we have*

$$(L'_\omega L_k x)\,\varepsilon + \tfrac{1}{2}(L''_\omega L_k x)\,\varepsilon^2 + \cdots \;=\; (L'_\omega L_p x)\,\delta(\varepsilon) + \tfrac{1}{2}(L''_\omega L_p x)\,\delta(\varepsilon)^2 + \cdots \tag{7}$$

as an identity in $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$.

Proof. The left and right hand sides of (7) are clearly Noetherian series in $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$. For any $\bar{\varepsilon} \prec L_k x$ in $\mathbb{L}_{<\omega}$, the following Taylor series expansions hold in $\mathbb{L}_{\leqslant\omega}$:

$$\begin{aligned} L_\omega(L_k x + \bar{\varepsilon}) &= L_\omega L_k x + (L'_\omega L_k x)\,\bar{\varepsilon} + \tfrac{1}{2}(L''_\omega L_k x)\,\bar{\varepsilon}^2 + \cdots \\ L_\omega(L_p x + \delta(\bar{\varepsilon})) &= L_\omega L_p x + (L'_\omega L_p x)\,\delta(\bar{\varepsilon}) + \tfrac{1}{2}(L''_\omega L_p x)\,\delta(\bar{\varepsilon})^2 + \cdots \end{aligned}$$

Subtracting both expansions, the identity (7) holds for ε substituted by $\bar{\varepsilon} \prec 1$ in $\mathbb{L}_{<\omega}$. We conclude by Lemma 2. □

In [7], we defined the field $\mathbb{L}_{\leqslant\omega} = \mathbb{R}[[\mathfrak{L}_{\leqslant\omega}]]$, as well as a hyperlogarithmic function L_ω on $\mathbb{L}_{\leqslant\omega}^{>,\succ}$ for which $\mathbb{L}_{\leqslant\omega} \cong \mathbb{R}[[\ell_\omega^{\mathbb{R}}]][[\mathfrak{L}_{<\omega}]]$ with $\ell_\omega = L_\omega x$. Let $u := L_\omega x \in \mathbb{L}_{\leqslant\omega}$. Then logarithmic transseries in u can be considered as elements in $L_{<\omega} \circ L_\omega \subseteq \mathbb{L}_{\leqslant\omega}$ and the successive derivatives of $E_\omega(u)$ with respect to u are given by

$$E_\omega^{(k)}(u) \;=\; \vartheta^k x \;\in\; \mathbb{L}_{<\omega}, \qquad \vartheta \;:=\; \gamma^{-1}\frac{\partial}{\partial x}, \qquad \gamma \;:=\; \ell'_\omega \;=\; \prod_{k<\omega} \frac{1}{\ell_k}.$$

For any $f \in \mathbb{L}_{<\omega}$, we have

$$\operatorname{supp} \vartheta f \;\subseteq\; \operatorname{supp} \vartheta + \operatorname{supp} f, \qquad \operatorname{supp} \vartheta \;=\; \gamma^{-1} \operatorname{supp} \frac{\partial}{\partial x} \;=\; \gamma^{-1}\left\{\frac{1}{\ell_0}, \frac{1}{\ell_0 \ell_1}, \ldots\right\},$$

whence

$$\operatorname{supp} E_\omega^{(k)}(u) \;=\; \operatorname{supp} \vartheta^k x \;\subseteq\; \left(\frac{\gamma}{\ell_0}\right)^{-k} \left\{1, \frac{1}{\ell_1}, \frac{1}{\ell_1 \ell_2}, \ldots\right\}^\infty \ell_0.$$

For $\delta \prec 1/x$ in $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$, this allows us to define $E_\omega(u+\delta)$ using the Taylor series expansion

$$E_\omega(u+\delta) \;:=\; E_\omega(u) + E'_\omega(u)\,\delta + \tfrac{1}{2} E''_\omega(u)\,\delta^2 + \cdots. \tag{8}$$

LEMMA 4. *The following identity holds in* $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$:

$$E_\omega L_\omega(x+\varepsilon) \;=\; x + \varepsilon. \tag{9}$$

Proof. The left hand side is well defined by (8) for $\delta = L_\omega(x+\varepsilon) - u = L_\omega(x+\varepsilon) - L_\omega x = \gamma\,\varepsilon + {}^1\!/_2\,\gamma'\,\varepsilon^2 + \cdots \prec 1/x$ in $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$. The fact that (9) holds for ε substituted by $\bar{\varepsilon}$ in $\mathbb{L}_{<\omega}^{\prec}$ follows from the usual rules of iterated derivatives of inverse functions. For a detailed proof, we refer to [14, section 6.4]. We conclude by Lemma 2. □

4. HYPEREXPONENTIALS OF TRUNCATED NUMBERS

We recursively define E_ω for positive purely infinite numbers $\theta \in \mathbf{No}_{\succ}^{>}$ by

$$E_\omega\,\theta \;:=\; \left\{\mathfrak{d}^*_\omega(\theta), E_\omega\,\theta_L^{\mathbf{No}_{\succ}^{>}} \,\middle|\, E_\omega\,\theta_R^{\mathbf{No}_{\succ}^{>}}\right\}_{\mathbf{K}}. \tag{10}$$

PROPOSITION 5. *The function* E_ω *defines a strictly increasing bijection* $\mathbf{No}_{\succ}^{>} \longrightarrow \mathbf{K}$. *Moreover, the previous equation is uniform.*

Proof. The function E_ω is well-defined and strictly increasing by Remark 1. The uniformity of the equation follows immediately.

Let L_ω denote the partial inverse function of E_ω and prove that L_ω is defined on $\mathbf{K}$ by induction on $\sqsubseteq$. Let $\kappa \in \mathbf{K}$ such that $\kappa_{\sqsubset}^{\mathbf{K}}$ is contained $E_\omega \mathbf{No}^{\succ}_{>}$. Since E_ω is injective, its inverse is defined on $\kappa_{\sqsubset}^{\mathbf{K}}$. Let

$$\theta \;:=\; \{L_\omega \kappa_L^{\mathbf{K}} \mid L_\omega \kappa_R^{\mathbf{K}}, \kappa\}_{\mathbf{No}^{\succ}_{>}}.$$

This number is well defined since $L_\omega \colon \kappa_{\sqsubset}^{\mathbf{K}} \longrightarrow \mathbf{No}^{\succ}_{>}$ is strictly increasing and for $\kappa' \in \kappa_L^{\mathbf{K}}$, we have $L_\omega \kappa' < \kappa' < \kappa$. By uniformity, we have $E_\omega \theta = \{\mathfrak{d}_\omega^*(\theta), \kappa_L^{\mathbf{K}} \mid \kappa_R^{\mathbf{K}}, E_\omega \kappa\}_{\mathbf{K}}$ where $\kappa = \{\kappa_L^{\mathbf{K}} \mid \kappa_R^{\mathbf{K}}\}_{\mathbf{K}}$. In order to conclude that $E_\omega \theta = \kappa$, it therefore suffices to show that κ lies in the cut $(\mathfrak{d}_\omega^*(\theta) \mid E_\omega \kappa)$. We have $E_\omega \kappa > \kappa$ by (10) and $\theta < \kappa$ by definition of θ, whence $\mathfrak{d}_\omega^*(\theta) < \kappa$ since $\kappa \in \mathbf{K}$. We conclude by induction that $E_\omega \colon \mathbf{No}^{\succ}_{>} \longrightarrow \mathbf{K}$ is surjective. □

We next identify the class of truncated numbers. For $a \in \mathbf{No}^{>,\succ}$, we consider the following convex class

$$\mathbf{\Pi}[a] \;:=\; \left\{ b \in \mathbf{No}^{>,\succ} : \exists n \in \mathbb{N}, a - b \prec \frac{1}{L_n E_\omega a_{\succ}} \right\}.$$

PROPOSITION 6. *The classes $\mathbf{\Pi}[a]$ for $a \in \mathbf{No}^{>,\succ}$ form a thin convex partition of $\mathbf{No}^{>,\succ}$.*

Proof. Given $a \in \mathbf{No}^{>,\succ}$, it is clear that the class $\mathbf{\Pi}[a]$ is convex and that it contains a. Note that for $a \in \mathbf{No}^{>,\succ}$, we have $\mathbf{\Pi}[a] \subseteq a + \mathbf{No}^{\prec}$. Let $a, b \in \mathbf{No}^{>,\succ}$ with $\mathbf{\Pi}[a] \neq \mathbf{\Pi}[b]$. We claim that $\mathbf{\Pi}[a] \cap \mathbf{\Pi}[b] = \emptyset$. If $a_{\succ} \neq b_{\succ}$, then we have $a + \mathbf{No}^{\prec} \cap b + \mathbf{No}^{\prec} = \emptyset$, which yields the result. Assume that $a_{\succ} = b_{\succ}$. Assume for contradiction that there are $c \in \mathbf{No}^{>,\succ}$ and $m, n \in \mathbb{N}$ with $a - c \prec 1/_{L_m E_\omega a_{\succ}}$ and $b - c \prec 1/_{L_n E_\omega a_{\succ}}$. Given $d \in \mathbf{\Pi}[a]$, there is a number $p \geqslant m$, n with $a - d \prec 1/_{L_p E_\omega a_{\succ}}$. Therefore $a - c, b - c, a - d$ are dominated by $1/_{L_p E_\omega a_{\succ}}$, whence $b - d = b - c - (a - c) + (a - d) \prec 1/_{L_p E_\omega a_{\succ}}$. This proves that $\mathbf{\Pi}[a] \subseteq \mathbf{\Pi}[b]$ and symmetric arguments yield $\mathbf{\Pi}[a] \supseteq \mathbf{\Pi}[b]$: a contradiction. This proves our claim. It only remains to see that the class $\mathbf{\Pi}[a]$ admits a cofinal and coinitial subset for any $a \in \mathbf{No}^{>,\succ}$. Indeed, we can take $a \pm 1/_{L_{\mathbb{N}} E_\omega a_{\succ}}$ as examples of such sets. □

COROLLARY 7. *The class $\mathbf{Tr} := \mathbf{Smp}_{\mathbf{\Pi}}$ is a surreal substructure.*

Let $a \in \mathbf{No}^{>,\succ}$ and let φ denote the $\trianglelefteq$-supremum of truncations ψ of a (i.e. series with $\psi \trianglelefteq a$) with $\operatorname{supp} \psi \succ 1/_{L_n E_\omega a_{\succ}}$. In particular, we have $a_{\succ} \trianglelefteq \varphi$ since $\operatorname{supp} a_{\succ} \succ 1$. We see that φ satisfies $\varphi \trianglelefteq a$ and $\operatorname{supp} \varphi \succ 1/_{L_n E_\omega a_{\succ}}$. Write $\varphi = a_{\succ} + \delta$ and $a = \varphi + \varepsilon = a_{\succ} + \delta + \varepsilon$. By $\trianglelefteq$-maximality of φ, we have $\varepsilon \prec 1/_{L_n E_\omega a_{\succ}} = 1/_{L_n E_\omega \varphi_{\succ}}$ so $a \in \mathbf{\Pi}[\varphi]$, or equivalently $\varphi \in \mathbf{\Pi}[a]$. We deduce that φ is the $\trianglelefteq$-minimum, hence $\sqsubseteq$-minimum of $\mathbf{\Pi}[a]$, so $\varphi \in \mathbf{Tr}$. We also see that for $\theta \in \mathbf{No}^{\succ}_{>}$ and $r \in \mathbb{R}$, we have $\theta + r \subseteq \mathbf{Tr}$. Since $\mathbf{Tr}$ is a surreal substructure, we may use recursion on $\varphi \in \mathbf{Tr}$ to define

$$\hat{E}_\omega(\varphi) \;:=\; \{E_{\mathbb{N}} \varphi, \mathcal{E} \hat{E}_\omega(\varphi_L^{\mathbf{Tr}}) \mid \mathcal{E} \hat{E}_\omega(\varphi_R^{\mathbf{Tr}})\}. \tag{11}$$

PROPOSITION 8. *The equation (11) is uniform and $\hat{E}_\omega$ is a strictly increasing function $\mathbf{Tr} \longrightarrow \mathbf{La}$.*

Proof. Since $\mathbf{Tr}$ is a surreal substructure, the definition, strict monotonicity and uniformity follow by Remark 1. For $\varphi \in \mathbf{Tr}$, we have $\hat{E}_\omega(\varphi) > \mathcal{E} E_{\mathbb{N}} \varphi$ since $\mathcal{E} < \exp$ on $\mathbf{No}^{>,\succ}$. We deduce that $\hat{E}_\omega(\varphi)$ is $\mathcal{E}$-simple, hence log-atomic. □

By [4, Lemma 2.4], for every infinite monomial $\mathfrak{m} \in \mathbf{Mo}^{\succ}$, we have

$$\exp \mathfrak{m} \;=\; \{\mathcal{D} \mathfrak{m}, \mathcal{D} \exp \mathfrak{m}_L^{\mathbf{Mo}^{\succ}} \mid \mathcal{D} \exp \mathfrak{m}_R^{\mathbf{Mo}^{\succ}}\}. \tag{12}$$

PROPOSITION 9. *We have* $\forall \varphi \in \mathbf{Tr}, \hat{E}_\omega(\varphi+1) = \mathrm{e}^{\hat{E}_\omega(\varphi)}$.

Proof. We prove this by induction on $(\mathbf{Tr}, \sqsubseteq)$. Let $\varphi \in \mathbf{Tr}$ such that this holds on $\varphi_{\sqsubset}^{\mathbf{Tr}}$. Note that $\mathbf{\Pi}[a+r] = \mathbf{\Pi}[a] + r$ for all $a \in \mathbf{No}^{>,\succ}$ and $r \in \mathbb{R}$. We have $\varphi = \{\mathbb{R}, \mathbf{\Pi}[\varphi_L^{\mathbf{Tr}}] \mid \mathbf{\Pi}[\varphi_R^{\mathbf{Tr}}]\}$, so

$$\begin{aligned}\varphi+1 &= \{\varphi, \mathbf{\Pi}[\varphi_L^{\mathbf{Tr}}]+1 \mid \mathbf{\Pi}[\varphi_R^{\mathbf{Tr}}]+1\}\\ &= \{\mathbf{\Pi}[\varphi], \mathbf{\Pi}[\varphi_L^{\mathbf{Tr}}+1] \mid \mathbf{\Pi}[\varphi_R^{\mathbf{Tr}}+1]\}\\ &= \{\varphi, \varphi_L^{\mathbf{Tr}}+1 \mid \varphi_R^{\mathbf{Tr}}+1\}_{\mathbf{Tr}}.\end{aligned}$$

We deduce that

$$\begin{aligned}\hat{E}_\omega(\varphi+1) &= \{E_{\mathbb{N}}(\varphi+1), \mathcal{E}\,\hat{E}_\omega(\varphi_L^{\mathbf{Tr}}+1) \mid \mathcal{E}\,\hat{E}_\omega(\varphi_R^{\mathbf{Tr}}+1)\}\\ &= \{E_{\mathbb{N}}\,\varphi, \mathcal{E}\exp\hat{E}_\omega(\varphi_L^{\mathbf{Tr}}) \mid \mathcal{E}\exp\hat{E}_\omega(\varphi_R^{\mathbf{Tr}})\}.\end{aligned}$$

Since $\hat{E}_\omega(\varphi) \in \mathbf{La} \subseteq \mathbf{Mo}^{\succ}$, we may apply (12). We also note that $\exp \mathcal{E}a$ and $\mathcal{E}\exp a$ are mutually cofinal and coinitial for all $a \in \mathbf{No}^{>,\succ}$ to obtain

$$\begin{aligned}\exp\hat{E}_\omega(\varphi) &= \{\mathcal{P}\,\hat{E}_\omega(\varphi), \mathcal{P}\exp E_{\mathbb{N}}\,\varphi, \mathcal{P}\exp\mathcal{E}\,\hat{E}_\omega(\varphi_L^{\mathbf{Tr}}) \mid \mathcal{P}\exp\mathcal{E}\,\hat{E}_\omega(\varphi_R^{\mathbf{Tr}})\}\\ &= \{\mathcal{P}\,\hat{E}_\omega(\varphi), E_{\mathbb{N}}\,\varphi, \mathcal{P}\mathcal{E}\exp\hat{E}_\omega(\varphi_L^{\mathbf{Tr}}) \mid \mathcal{P}\mathcal{E}\exp\hat{E}_\omega(\varphi_R^{\mathbf{Tr}})\}\\ &= \{\mathcal{P}\,\hat{E}_\omega(\varphi), E_{\mathbb{N}}\,\varphi, \mathcal{E}\exp\hat{E}_\omega(\varphi_L^{\mathbf{Tr}}) \mid \mathcal{E}\exp\hat{E}_\omega(\varphi_R^{\mathbf{Tr}})\}.\end{aligned}$$

We have $\hat{E}_\omega(\varphi+1) > \hat{E}_\omega(\varphi)$, so $\hat{E}_\omega(\varphi+1) > \mathcal{E}\,\hat{E}_\omega(\varphi)$ and $\hat{E}_\omega(\varphi+1) > \mathcal{P}\,\hat{E}_\omega(\varphi)$. We clearly have $\exp\hat{E}_\omega(\varphi) > E_{\mathbb{N}}\,\varphi$. We deduce that $\exp\hat{E}_\omega(\varphi) = \hat{E}_\omega(\varphi+1)$. By induction, the relation is valid on $\mathbf{Tr}$. □

PROPOSITION 10. *For* $\theta \in \mathbf{No}_{\succ}^{\gtrsim}$, *we have* $\hat{E}_\omega(\theta) = E_\omega\,\theta$.

Proof. We prove this by induction on $(\mathbf{No}_{\succ}^{\gtrsim}, \sqsubseteq)$. Let $\theta \in \mathbf{No}_{\succ}^{\gtrsim}$ be such that this holds on $\theta_{\sqsubset}^{\mathbf{No}_{\succ}^{\gtrsim}}$. For $\varphi \in \theta_L^{\mathbf{Tr}}$, we have $\varphi_{\succ} \in \theta_L^{\mathbf{No}_{\succ}^{\gtrsim}}$, and there is $n \in \mathbb{N}$ with $\varphi \leqslant \varphi_{\succ} + n$. We deduce that $\hat{E}_\omega(\varphi) \leqslant E_n\,\hat{E}_\omega(\varphi_{\succ}) = E_n\,E_\omega\,\varphi_{\succ}$. In particular, we have $\hat{E}_\omega(\varphi) < E_{n+1}\,E_\omega\,\varphi_{\succ}$ so $\mathcal{E}\,\hat{E}_\omega(\varphi) < E_{n+2}\,E_\omega\,\varphi_{\succ}$. This proves that $E_{\mathbb{N}}\,\hat{E}_\omega\,\theta_L^{\mathbf{No}_{\succ}^{\gtrsim}}$ is cofinal with respect to $\mathcal{E}\,\hat{E}_\omega(\theta_L^{\mathbf{Tr}})$. For $\theta' \in \theta_L^{\mathbf{No}_{\succ}^{\gtrsim}}$ and $n \in \mathbb{N}$, we have $E_n\,E_\omega\,\theta' = E_n\,\hat{E}_\omega(\theta') = \hat{E}_\omega(\theta'+n)$ where $\theta'+n \in \theta_L^{\mathbf{Tr}}$, so $\mathcal{E}\,\hat{E}_\omega(\theta_L^{\mathbf{Tr}})$ is cofinal with respect to $E_{\mathbb{N}}\,E_\omega\,\theta_L^{\mathbf{No}_{\succ}^{\gtrsim}}$. Symmetric arguments yield that $L_{\mathbb{N}}\,E_\omega\,\theta_R^{\mathbf{No}_{\succ}^{\gtrsim}}$ and $\mathcal{E}\,\hat{E}_\omega(\theta_R^{\mathbf{Tr}})$ are mutually coinitial. We conclude that $\hat{E}_\omega(\theta) = \left\{E_{\mathbb{N}}\,\theta, E_{\mathbb{N}}\,E_\omega\,\theta_L^{\mathbf{No}_{\succ}^{\gtrsim}} \,\middle|\, L_{\mathbb{N}}\,E_\omega\,\theta_R^{\mathbf{No}_{\succ}^{\gtrsim}}\right\} = E_\omega\,\theta$. □

PROPOSITION 11. *The function* $E_\omega: \mathbf{Tr} \longrightarrow \mathbf{La}$ *is bijective. Its reciprocal* L_ω *admits the following uniform equation on* $\mathbf{La}$*:*

$$L_\omega\,\mathfrak{a} = \{L_\omega\,\mathfrak{a}_L^{\mathbf{La}} \mid L_\omega\,\mathfrak{a}_R^{\mathbf{La}}, L_{\mathbb{N}}\,\mathfrak{a}\}_{\mathbf{Tr}}.$$

Proof. Noticing that $E_\omega\,\varphi = \hat{E}_\omega(\varphi) = \{E_{\mathbb{N}}\,\mathfrak{d}_\omega(\varphi), E_\omega\,\varphi_L^{\mathbf{Tr}} \mid E_\omega\,\varphi_R^{\mathbf{Tr}}\}_{\mathbf{La}}$ for all $\varphi \in \mathbf{Tr}$, this follows from the same arguments as in Proposition 5. □

5. HYPEREXPONENTIALS OF ARBITRARY NUMBERS

The field $\mathbb{L}_{<\omega} = \mathbb{R}[[\mathfrak{L}_{<\omega}]]$ of logarithmic hyperseries of [7] is a subfield of the class of all well-based transseries in an infinitely large variable x. Both $\mathbb{L}_{<\omega}$ and the class of all transseries are closed under derivation and under composition [8, 10, 14]. For every positive infinite number $a \in \mathbf{No}^{>,\succ}$, there also exists an evaluation embedding $\mathbb{L}_{<\omega} \longrightarrow \mathbf{No}; f \longmapsto f(a)$ such that $f(g(a)) = (f \circ g)(a)$ for all $f, g \in \mathbb{L}_{<\omega}$: see [5].

Given $a \in \mathbf{No}^{>,\succ}$, let $\varphi = \varphi_a \in \mathbf{Tr}$ be the unique truncated series with $a \in \mathbf{\Pi}[\varphi]$. If $a \neq \varphi$, then there is a smallest number $n = n_a \in \mathbb{Z}$ with

$$a - \varphi \prec \frac{1}{L_n E_\omega \varphi} = \frac{1}{E_\omega(\varphi - n)}.$$

Write $\varepsilon := a - \varphi$. With ϑ as in section 3, we define for every $k \in \mathbb{N}$:

$$E_\omega^{(k)}(\varphi - n) := (\vartheta^k x)(E_\omega(\varphi - n)).$$

Substitution of $E_\omega(\varphi - n)$ for x in (8) allows us to extend the definition of E_ω by

$$E_\omega(a - n) := \sum_{k \in \mathbb{N}} \frac{1}{k!} E_\omega^{(k)}(\varphi - n)\, \varepsilon^k,$$

and

$$E_\omega(a) := E_n(E_\omega(a - n)).$$

PROPOSITION 12. *For all $a \in \mathbf{No}^{>,\succ}$, we have*

$$E_\omega(a + 1) = E_1 E_\omega a.$$

Proof. If $\varphi_a = a$, then this is Proposition 9. Otherwise, we have $\varphi_{a+1} = \varphi_a + 1 \neq a + 1$ and $n_{a+1} = n_a + 1$, whence $E_\omega(a + 1 - n_{a+1}) = E_\omega(a - n_a)$ and

$$E_\omega(a + 1) = E_{n_a+1}(E_\omega(a - n_a)) = E_1 E_{n_a}(E_\omega(a - n_a)) = E_1 E_\omega a. \qquad \square$$

Inversely, consider an arbitrary positive infinite number $b \in \mathbf{No}^{>,\succ}$. Then there exists a $k \in \mathbb{N}$ such that $L_k b = L_k \mathfrak{a} + \varepsilon$ for some log-atomic $\mathfrak{a} \in \mathbf{La}$ and $\varepsilon \prec L_k \mathfrak{a}$. We extend the definition of L_ω to any such number b by

$$L_\omega b := L_\omega \mathfrak{a} + (\ell_\omega' \circ \ell_k(\mathfrak{a}))\, \varepsilon + \tfrac{1}{2} (\ell_\omega'' \circ \ell_k(\mathfrak{a}))\, \varepsilon^2 + \cdots.$$

In view of Lemma 3, the value of $L_\omega b$ does not depend on the choice of k. Note also that this definition indeed extends our previous definition of L_ω on $\mathbf{La}$.

PROPOSITION 13. *For all $b \in \mathbf{No}^{>,\succ}$, we have*

$$L_\omega L_1 b = L_\omega b - 1.$$

Proof. With $L_k b = L_k \mathfrak{a} + \varepsilon$ as above (while taking $k > 0$), we have

$$\begin{aligned} L_\omega L_1 b &= L_\omega L_1 \mathfrak{a} + (L_\omega' L_{k-1} L_1 \mathfrak{a})\, \varepsilon + \tfrac{1}{2} (L_\omega'' L_{k-1} L_1 \mathfrak{a})\, \varepsilon^2 + \cdots \\ &= L_\omega \mathfrak{a} - 1 + (L_\omega' L_k \mathfrak{a})\, \varepsilon + \tfrac{1}{2} (L_\omega'' L_k \mathfrak{a})\, \varepsilon^2 + \cdots \\ &= L_\omega b - 1, \end{aligned}$$

where $L_\omega L_1 \mathfrak{a} = L_\omega \mathfrak{a} - 1$ because of Proposition 9. $\square$

PROPOSITION 14. *For any $b \in \mathbf{No}^{>,\succ}$, we have $E_\omega L_\omega b = b$.*

Proof. Let $k \in \mathbb{N}$ be such that $L_k b = L_k \mathfrak{a} + \bar{\varepsilon}$, where $\mathfrak{a} \in \mathbf{La}$ and $\bar{\varepsilon} \prec L_k \mathfrak{a}$. Let us first consider the special case when $k = 0$. Since $\mathfrak{a}$ is log-atomic, we have $\mathbb{L}_{<\omega} \cong \mathbb{L}_{<\omega}(\mathfrak{a})$. From Lemma 4, it therefore follows that $E_\omega L_\omega(\mathfrak{a} + \varepsilon) = \mathfrak{a} + \varepsilon$ inside $\mathbb{R}[[\mathfrak{L}_{<\omega}(\mathfrak{a}) \times \varepsilon^{\mathbb{N}}]]$. The result follows by specializing this relation at $\bar{\varepsilon}$. If $k > 0$, then $L_\omega b = L_\omega L_k b + k = L_\omega(L_k \mathfrak{a} + \bar{\varepsilon}) + k$ by Proposition 13. Applying the result for the special case when $k = 0$, we have $E_\omega (L_\omega b - k) = L_k \mathfrak{a} + \bar{\varepsilon} = L_k b$. We conclude by Proposition 12. $\square$

In particular, the function $E_\omega : \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ is surjective. We next prove that it is strictly increasing, concluding our proof that $E_\omega : \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ is a strictly increasing bijection with reciprocal L_ω.

LEMMA 15. *For $\varphi, \psi \in \mathbf{Tr}$ with $\varphi < \psi$, we have $\mathcal{E} E_\omega(\mathbf{\Pi}[\varphi]) < \mathcal{E} E_\omega(\mathbf{\Pi}[\psi])$.*

Proof. Note that $\mathbf{La} \ni E_\omega(\varphi) < E_\omega(\psi) \in \mathbf{La}$, so it is enough to prove that $E_\omega(a) \in \mathcal{E}[E_\omega(\varphi)]$ for all $a \in \mathbf{\Pi}[\varphi]$. For such a, there is $n \in \mathbb{N}$ with $\varepsilon := a - \varphi \prec E_\omega(\varphi - n)^{-1}$, and

$$\begin{aligned} L_n E_\omega(a) &= \sum_{k \in \mathbb{N}} \frac{E_\omega^{(k)}(\varphi - n)}{k!} \varepsilon^k \\ &= E_\omega(\varphi - n) + \delta, \end{aligned}$$

where $\delta := \sum_{k>0} \frac{E_\omega^{(k)}(\varphi - n)}{k!} \varepsilon^k$ is infinitesimal. So $L_n E_\omega(a) \sim L_n E_\omega(\varphi)$, whence $E_\omega(a) \in \mathcal{E}[E_\omega(\varphi)]$. □

LEMMA 16. *For $\varphi \in \mathbf{Tr}$ and $a, b \in \mathbf{No}^{>,\succ}$ with $a, b \in \mathbf{\Pi}[\varphi]$, there is $n \in \mathbb{N}$ with*

$$L_n E_\omega(b) - L_n E_\omega(a) \sim E'_\omega(\varphi - n)\,(b - a).$$

Proof. Write $a = \varphi + \varepsilon_a$ and $b = \varphi + \varepsilon_b$ where $\varepsilon_a, \varepsilon_b \prec 1$ and let $n \in \mathbb{N}$ with $\varepsilon_a, \varepsilon_b \prec E_\omega(\varphi - n)^{-1}$. Writing $\varepsilon_k := \varepsilon_b^k - \varepsilon_a^k$ for $k \in \mathbb{N}^>$, we have $\varepsilon_k \prec E_\omega(\varphi - n)^{-k}$. We deduce that

$$\begin{aligned} L_n E_\omega(b) - L_n E_\omega(a) &= \sum_{k>0} \frac{E_\omega^{(k)}(\varphi - n)}{k!} \varepsilon_k \\ &\sim E'_\omega(\varphi - n)\,(\varepsilon_b - \varepsilon_a) \\ &\sim E'_\omega(\varphi - n)\,(b - a). \end{aligned}$$

□

PROPOSITION 17. *The function E_ω is strictly increasing on $\mathbf{No}^{>,\succ}$.*

Proof. Let $a, b \in \mathbf{No}^{>,\succ}$ with $a < b$. If $a < \mathbf{\Pi}[b]$, then we get $E_\omega(a) < E_\omega(b)$ by Lemma 15. Otherwise, we have $a \in \mathbf{\Pi}[b]$ so by Lemma 16, there are $\varphi \in \mathbf{Tr}$ and $n \in \mathbb{N}$ with

$$L_n E_\omega(b) - L_n E_\omega(a) \sim E'_\omega(\varphi - n)\,(b - a).$$

Since $E'_\omega(\varphi - n) > 0$, we conclude that $L_n E_\omega(b) > L_n E_\omega(a)$, whence $E_\omega(b) > E_\omega(a)$. □

COROLLARY 18. *The function E_ω is bijective, with reciprocal L_ω.*

BIBLIOGRAPHY

[1] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. *Asymptotic Differential Algebra and Model Theory of Transseries*. Number 195 in Annals of Mathematics studies. Princeton University Press, 2017.

[2] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. On numbers, germs, and transseries. In *Proc. Int. Cong. of Math. 2018*, volume 1, pages 1–24. Rio de Janeiro, 2018.

[3] V. Bagayoko and J. van der Hoeven. Surreal substructures. Technical Report, HAL, 2019. `http://hal.archives-ouvertes.fr/hal-02151377`.

[4] A. Berarducci and V. Mantova. Surreal numbers, derivations and transseries. *JEMS*, 20(2):339–390, 2018.

[5] A. Berarducci and V. Mantova. Transseries as germs of surreal functions. *Trans. of the AMS*, 371:3549–3592, 2019.

[6] M. Boshernitzan. Hardy fields, existence of transexponential functions. *Æquationes Math.*, 30:258–280, 1986.

[7] L. van den Dries, J. van der Hoeven, and E. Kaplan. Logarithmic hyperseries. *Trans. of the AMS*, 372(7):5199–5241, 2019.

[8] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Hermann, collection: Actualités mathématiques, 1992.

[9] H. Gonshor. *An Introduction to the Theory of Surreal Numbers*. Cambridge Univ. Press, 1986.

[10] J. van der Hoeven. *Automatic asymptotics*. PhD thesis, École polytechnique, Palaiseau, France, 1997.

[11] J. van der Hoeven. *Transseries and real differential algebra*, volume 1888 of *Lecture Notes in Mathematics*. Springer-Verlag, 2006.

[12] H. Kneser. Reelle analytische Lösungen der Gleichung $\phi(\phi(x)) = e^x$ und verwandter Funktionalgleichungen. *Jour. f. d. reine und angewandte Math.*, 187(1/2):56–67, 1950.

[13] S. Kuhlmann and M. Matusinski. The exponential-logarithmic equivalence classes of surreal numbers. *Order 32*, pages 53–68, 2015.

[14] M. C. Schmeling. *Corps de transséries*. PhD thesis, Université Paris-VII, 2001.