Defining a surreal hyperexponential

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Conway’s class $\mathbf{No}$ of surreal numbers admits a rich structure: it forms a totally ordered real closed field with an exponential functions and a derivation. The aim of this note is to construct a surreal solution $E_\omega$ to the functional equation $E_\omega(a+1) = \exp E_\omega(a)$ with good properties.

1. INTRODUCTION

Let $\mathbf{No}$ be Conway’s class of surreal numbers. It is well known that $\mathbf{No}$ admits a rich structure: Conway showed that $\mathbf{No}$ forms a real closed field and Gonshor also defined an exponential function $\exp$ on $\mathbf{No}$ that satisfies the same first order theory as the usual exponential function on the reals. Following Conway’s tradition, all numbers will understood to be surreal in what follows.

The aim of this note is to define a bijective function $E_\omega: \mathbf{No}_{>,>} \rightarrow \mathbf{No}_{>,>}$ on the class $\mathbf{No}_{>,>}$ of positive infinitely large numbers, which is strictly increasing and satisfies the functional equation

$$E_\omega(a+1) = \exp E_\omega(a)$$

(1)

for all $a \in \mathbf{No}_{>,>}$. Since this equation admits many solutions, the main difficulty is to single out a particular solution that will be most “natural” in a way that needs to be made precise.
The function $E_\omega$ is said hyperexponential and it is the first non-trivial hyperexponential in the transfinite sequence $(E_\alpha)_{\alpha \in \text{Ord}}$ of iterated exponentials

$$E_0 := \text{Id}, \ E_1 := \exp, \ E_2 := \exp \circ \exp, \ldots, \ E_{\omega^r} E_{\omega+1} := E_{\omega \circ \exp}, \ldots, \ E_{2\omega} := E_{\omega \circ E_\omega} \ldots$$

The corresponding reciprocals are called hyperlogarithms:

$$L_0 := \text{Id}, \ L_1 := \log, \ L_2 := \log \circ \log, \ldots, \ L_{\omega^r} L_{\omega+1} := \log \circ L_{\omega}, \ldots, \ L_{2\omega} := L_{\omega \circ L_{\omega}} \ldots$$

It is natural to require such more general hyperexponentials to satisfy $E_{\alpha+1}(a+1) = E_\alpha(E_{\alpha+1}(a))$ for all ordinals $\alpha$ and $a \in \text{No}^{>\omega}$. Similarly, $L_{\alpha+1}(L_\alpha(a)) = L_{\alpha+1}(a) - 1$.

There are known real analytic solutions of (1) with good properties [12, 6], even though there does not seem to exist any meaningful “most natural” solution. It is also well known that fractional iterates of $\exp$ and $\log$ can be defined in terms of $E_\omega$ and $L_\omega$; given $c \in \mathbb{R}$, we take $\exp_c(x) := E_\omega(L_\omega(x) + c)$ and $\log_c(x) := E_\omega(L_\omega(x) - c)$.

From a formal perspective, hyperexponentials and hyperseries were studied in detail by Schmeling and van der Hoeven [14]: they generalized transseries to include formal counterparts $e_a, \ell_a$ of $E_a, L_a$ for $a < \omega^\omega$ [14]. This yields in particular a natural hyperexponential on the set of positive infinitely large transseries. More recently, van den Dries, van der Hoeven and Kaplan [7] constructed the field of logarithmic hyperseries $\mathbb{L}$ with $\ell_a$ for all $a$, with corresponding natural hyperlogarithms $L_a^\ell : \mathbb{L}^{>\omega} \rightarrow \mathbb{L}^{>\omega}$. The ultimate goal [11, 2] is to produce a field of hyperseries $\mathbb{H} \supset \mathbb{L}$ with all hyperexponentials and to construct an isomorphism $\mathbb{H} \cong \text{No}$. This work can be considered as another step in this direction, by constructing a natural $E_\omega \circ \text{No}^{>\omega} \rightarrow \text{No}^{>\omega}$.

**Surreal numbers and transseries** It is well known that surreal numbers $a \in \text{No}$ can be written as infinite series

$$a = \sum_{m \in \text{Mo}} a_m m,$$

where $\text{Mo}$ denotes the class of surreal monomials and the coefficients $a_m$ are real. In particular, $\text{No}$ is isomorphic to the Hahn field $\mathbb{R}[[\text{Mo}]]$ of formal power series. Together with the exponential function, $\text{No}$ even admits the structure of a field of transseries in the sense of [14]; see [4].

We will freely use notations from [1, 11] when dealing with such transseries. In particular, the support of $a \in \text{No}$ is defined as $\supp(a) := \{m \in \text{Mo} : a_m \neq 0\}$ and its infinite part as $a_{\infty} := \sum_{m \in \text{Mo}, m > 1} a_m m$. Here we used Hardy’s notation $<$ for $a < b \in \mathbb{R}$ if $|a| < |b|$. Similarly, we set $a_{\prec} := \sum_{m \in \text{Mo}, m < 1} a_m m$. We also define $\text{No}^{>\omega} := (a \in \text{No} : a > 1)$, $\text{No}^{>\omega} := (a_\prec : a \in \text{No})$, and $\text{No}^{<\omega} := (a \in \text{No} : a < 1) = (a_\prec : a \in \text{No})$. Finally, we write $a \leq b$ if $sup \ a > a - b$, in which case we say that $a$ is a truncation of $b$.

**Defining exponentials** Consider a number $a \in \text{No}$ and decompose it as $a = a_\succ + a_\prec + a_\succ_\prec$ with $a_\succ \in \text{No}^{>\omega}$, $a_\succ \in \mathbb{R}$, and $a_\prec \in \text{No}^{<\omega}$. Then the functional equation of exp yields

$$\exp a = e^{a_\succ} \exp a_\prec \exp a_\succ_\prec$$

where $\exp a_\succ$ is the usual exponential in $\mathbb{R}$ and $\exp a_\prec = 1 + a_\prec + \frac{1}{2} a_\prec^2 + \frac{1}{6} a_\prec^3 + \cdots$. In order to define $\exp$ on $\text{No}^{<\omega}$, this relation shows that it would have sufficed to define it on $\text{No}^{>\omega}$. In addition, it can be shown that $\exp$ bijectively maps the class $\text{No}^{>\omega}$ to the class $\text{Mo}$. Our process to define $E_\omega$ is similar, with different subclasses $\text{Tr}$ and $\text{La}$ in the roles of $\text{No}^{>\omega}$ and $\text{Mo}$. The class $\text{Tr}$ is defined below and $\text{La}$ is the class of log-atomic numbers, i.e. numbers $a \in \text{No}^{<\omega}$ such that $log^n a \in \text{Mo}$ for all $n \in \mathbb{N}$.

**Surreal substructures** It turns out that each of the classes $\text{No}^{>\omega}$, $\text{Mo}$, $\text{Tr}$, and $\text{La}$ are examples of so-called surreal substructures, which were extensively studied in [3]. Let us recall a few basic facts; see also section 2.
Besides the usual ordering, the class \( \mathbf{No} \) of surreal numbers admits a well-founded partial order \( \subseteq \) called the simplicity relation. A surreal substructure is a subclass \( \mathbf{S} \) of \( \mathbf{No} \) that is isomorphic to \( (\mathbf{No}, \leq, \subseteq) \) for the induced relations by \( \leq \) and \( \subseteq \) on \( \mathbf{S} \). Equivalently, this means that for any subsets \( L, R \) of \( \mathbf{S} \) with \( L < R \), the cut

\[
(L \upharpoonright R)_\mathbf{S} := \{ a \in \mathbf{S} : L < a < R \}
\]

in \( \mathbf{S} \) admits a \( \subseteq \)-minimum which is then denoted \( (L \upharpoonright R)_\mathbf{S} \). We call \( (L, R) \) a cut representation in \( \mathbf{S} \). We extend this notation to the case when \( L, R \) are classes, provided \( (L \upharpoonright R)_\mathbf{S} \) indeed exists. If \( a \in \mathbf{S} \), then we let

\[
a^\mathbf{S}_L := \{ a' \in \mathbf{S} : a' \subseteq a \land a' < a \}, \quad a^\mathbf{S}_R := \{ a'' \in \mathbf{S} : a'' \subseteq a \land a'' > a \}, \quad a^\mathbf{S} := a^\mathbf{S}_L \cup a^\mathbf{S}_R.
\]

Then \( a = (a^\mathbf{S}_L \mid a^\mathbf{S}_R)_\mathbf{S} \). Moreover, for any cut \( (L \upharpoonright R)_\mathbf{S} \) in \( \mathbf{S} \) containing \( a \) (such as \( a = (L \upharpoonright R)_\mathbf{S} \)), the set \( L \) (resp. \( R \)) is cofinal (resp. coinitial) with respect to \( a^\mathbf{S}_L \) (resp. \( a^\mathbf{S}_R \)).

Important examples of surreal substructures include \( \mathbf{No} \), the class \( \mathbf{No}^\succ \) of strictly positive numbers, the class \( \mathbf{No}^{\succ, \succ} \) of positive infinitely large numbers, the classes \( \mathbf{Mo}^{\succ} \) of monomials and infinite monomials, the class \( \mathbf{No}^\prec := \mathbb{R}[[\mathbf{Mo}^\succ]] \) of purely infinite numbers, and the class \( \mathbf{No}^\prec := \mathbb{R}[[\mathbf{Mo}^\succ]] \) of infinitesimal numbers.

**Truncated numbers** Assume that \( E_\omega.a \) has been defined for some positive purely infinite number \( a \in \mathbf{No}^\succ := \mathbf{No}^\prec \cap \mathbf{No}^{\succ, \succ} \). For sufficiently small \( \varepsilon \), we wish to define \( E_\omega.(a + \varepsilon) \) using Taylor series expansion:

\[
E_\omega(a + \varepsilon) := E_\omega.a + (E_\omega.a) \varepsilon + \frac{1}{2} (E_\omega'(a) \varepsilon^2 + \cdots)
\]

(3)

The successive derivatives \( E_\omega.a, E_\omega'(a), \ldots \) can be defined in \( \mathbf{No} \) as ordinary (and so-called logarithmic) transseries applied to \( b := E_\omega.a \):

\[
E_\omega'.a := b(L_1.b) (L_2.b) \cdots
\]

\[
E_\omega''a = (b(L_1.b)^2(L_2.b)^2 \cdots + b(L_1.b) (L_2.b)^2(L_3.b)^2 + \cdots)
\]

It can be shown that (3) converges formally, provided that \( \varepsilon < 1/E_\omega.a \). More generally, consider \( \delta \in \mathbf{No} \) with \( \delta < 1/L_k b \) for a certain \( k \in \mathbb{N} \). Assuming (1), this means that \( \delta < 1/E_\omega(a-k) \), which allows us to define

\[
E_\omega(a-k + \delta) := E_\omega.a - (E_\omega'(a-k) \delta + \frac{1}{2} (E_\omega''(a-k) \delta^2 + \cdots)
\]

(4)

and set

\[
E_\omega(a + \delta) := E_k(E_\omega(a-k+\delta)).
\]

(5)

It follows that it is sufficient to define \( E_\omega \) at numbers \( \varphi \in \mathbf{No}^{\succ, \succ} \) with

\[
\text{supp } \varphi > \frac{1}{L_N E_\omega \varphi^\succ},
\]

where \( L_N = (L_0, L_1, \ldots) \). Those numbers are said to be truncated and we will write \( \mathbf{Tr} \) for the class of all truncated numbers. It turns out that \( \mathbf{Tr} \) is a surreal substructure.

**Defining hyperexponentials** Our definition of \( E_\omega \) proceeds in three stages:

1. We first define \( E_\omega \) on \( \mathbf{No}^\succ \). For any two positive purely infinite numbers \( \theta_1, \theta_2 \) with \( \theta_1 < \theta_2 \), we have \( \theta_1 + \mathbb{N} < \theta_2 - \mathbb{N} \). By the functional equation, we should have

\[
E_\omega.(\theta_1 + \mathbb{N}) = E_{\mathbb{N}} E_\omega.\theta_1 < L_N E_\omega.\theta_2 = E_\omega.(\theta_2 + \mathbb{N}).
\]

We deduce that for \( \theta \in \mathbf{No}^\succ \), the number \( E_\omega.\theta \) should lie in the cut \( (L \mid R) \) in \( \mathbf{No} \) where

\[
L := E_{\mathbb{N}} \theta \cup E_{\mathbb{N}} E_\omega.\theta^\mathbf{No}^\succ, \quad R := L_N E_\omega.\theta^\mathbf{No}^\succ
\]
The simplest way to ensure this is to define
\[ E_\omega(\theta) := \{ E_N \theta, E_N E_\omega \theta_L^{N_\omega} | L_N E_\omega \theta_R^{N_\omega} \} \]
for all \( \theta \in N_\omega^\omega \).

2. We next extend \( E_\omega \) to \( \text{Tr} \). Similar arguments and the simplicity heuristic impose
\[ E_\omega \varphi := (E_N \varphi, E_\omega \varphi_L^{\text{Tr}} | E_\omega \varphi_R^{\text{Tr}}) \in \text{La}, \]
where \( \mathcal{E} \) is a function group to be defined in Section 2. Here \( \text{Tr}, \text{La} \), and (3) play a similar role as \( N_\omega^\omega, \text{Mo} \), and (2) when extending the definition of exp.

3. We finally extend the definition of \( E_\omega \) to \( N_\omega^\omega \) by relying on (4) and (5).

2. EQUATIONS AND CONVEX PARTITIONS

Before we define \( E_\omega \), let us briefly recall a general method to define surreal substructures using convex partitions. For more details, see [3, Section 6].

**Uniform equations** Let \( T \) be a surreal substructure and \( F: S \rightarrow T \) be a function. Let \( \lambda, \rho \) be functions defined for cut representations in \( S \) and such that \((\lambda(L,R), \rho(L,R))\) is a cut representation in \( T \) whenever \((L,R)\) is a cut representation in \( S \). We say that \((\lambda, \rho)\) is an equation of \( F \) if, for all \( a \in S \), we have
\[ F(a) = (\lambda(a_L^S, a_R^S) | \rho(a_L^S, a_R^S))_T. \]
We say that the equation is uniform if we have \( F((L | R)_S) = (\lambda(L,R) | \rho(L,R))_T \) whenever \((L,R)\) is a cut representation in \( S \). For instance, by [9, Theorem 3.2], for \( r \in \mathbb{R} \), the following equation for the translation \( a + r \) by \( r \) is uniform:
\[ a + r = (a_L + r, a_R + r | a_L + r, a_R + r). \quad (6) \]

**Remark 1.** Assume that \( F \) has an equation \((\lambda, \rho)\) with
\[ \forall a \in S, \forall a' \in a_L^S, \forall a'' \in a_R^S, \exists b' \in \lambda(a_L, a_R), \exists b'' \in \rho(a_L, a_R), \quad F(a') \leq b' \wedge F(a'') \geq b''. \]
This is in particular the case if \( F(a_L^S) \subseteq \lambda(a_L^S, a_R^S) \) and \( F(a_R^S) \subseteq \rho(a_L^S, a_R^S) \) for all \( a \in S \). Then we claim that \( F \) is strictly increasing. To see this, consider \( a_0, a_1 \in S \) with \( a_0 < a_1 \). By [3, Proposition 4.6], there is a \( \leq \)-maximal element \( c \) of \( S \) with \( c \leq a \), and we have \( a < c \leq b \) or \( a \leq c < b \). We treat the first case, the other one being symmetric. Since \( a < c \) and \( c \leq a \), we have \( c \in a_R^S \) so there is \( b'' \in \rho(a_L^S, a_R^S) \) with \( b'' \leq F(c) \). We have \( F(a) < b'' \leq F(c) \). A similar argument yields \( F(c) \leq F(b) \), so \( F(a) < F(b) \).

**Convex partitions** Let \( S \) be a surreal substructure and let \( \Pi \) be a partition of \( S \) into convex subclasses, each of which admits a cofinal and colinear subset. We refer to \( \Pi \) as a thin convex partition of \( S \). For \( a \in S \), we let \( \Pi(a) \) denote the unique member of \( \Pi \) containing \( a \). We also write \( \Pi[X] := \bigcup_{x \in X} \Pi[x] \) for any subclass \( X \) of \( S \). We say that an element \( a \in S \) is \( \Pi \)-simple if it is the \( \leq \)-minimum of its class \( \Pi(a) \). This is equivalent to the existence of a cut \((L | R)_S \) in \( S \) with \( a = (L | R)_S \) and \( \Pi[L] < a < \Pi[R] \).

Then the class \( S_{\Pi} \) of \( \Pi \)-simple elements forms a surreal substructure which is contained in \( S \). For \( a \in S_{\Pi} \), we have \( a = (\Pi[a]_{\Pi} | \Pi[a]_{\Pi}^{\text{Smp}})_{\text{Smp}} \).

**Function groups** A function group \( \mathcal{G} \) on a surreal substructure \( S \) is a group of strictly increasing bijections \( S \rightarrow S \) under functional composition. We regard elements \( f, g \) of \( \mathcal{G} \) as actions on \( S \) and sometimes write \( f \circ g \) and \( f(a) \) rather than \( f \circ g \). For such a function group \( \mathcal{G} \), the collection \( \Pi[\mathcal{G}] := (\mathcal{G}[a])_{a \in S} \) of classes
\[ \mathcal{G}[a] := \{ b \in S : \exists f, g \in \mathcal{G}, fa \leq b \leq ga \}. \]
is a thin convex partition of $S$ and we define $\text{Smp}_{g} := \text{Smp}_{g|}$. For $f, g \in G$, the relation $f < g \iff \forall a \in S, (fa < ga)$ is a partial order on $G$. We will frequently rely on the elementary fact that $(G, <)$ is partially bi-ordered, i.e. that we have

$$\forall f, g, h \in G, \quad g > \text{id}_S \iff fh > fh.$$ 

**Common function groups** As an example, we can obtain the previous structures as the classes $\text{Smp}_{g}$ for actions of the following function groups $G$ acting on $N_{\omega}$, $N_{\omega} >$ or $N_{\omega} > >$. For $r \in \mathbb{R}$ and $s \in \mathbb{R}^\omega$, we define

$$T_r := a \rightarrow a + r \quad \text{acting on } N_{\omega} \text{ or } N_{\omega} >$$

$$H_s := a \rightarrow sa \quad \text{acting on } N_{\omega} > \text{ or } N_{\omega} > >$$

$$P_s := a \rightarrow a^s = \exp(s \log a) \quad \text{acting on } N_{\omega} > \text{ or } N_{\omega} > >$$

We define

$$\mathcal{T} := \{T_r : r \in \mathbb{R}\}$$

$$\mathcal{H} := \{H_s : s \in \mathbb{R}^\omega\}$$

$$\mathcal{D} := \{P_s : s \in \mathbb{R}^\omega\}$$

$$\mathcal{E} := \langle E_n H_s L_n : n \in \mathbb{N}, s \in \mathbb{R}^\omega \rangle$$

$$\mathcal{E}^* := \langle E_n H_s L_n : n \in \mathbb{N} \rangle.$$ 

We then have the following list of identities [3, Section 7.1]:

- The action of $\mathcal{T}$ on $N_{\omega}$ (resp. $N_{\omega} > >$) yields $N_{\omega}$ (resp. $N_{\omega} > >$).
- The action of $\mathcal{H}$ on $N_{\omega} >$ (resp. $N_{\omega} > >$) yields $M_{\omega}$ (resp. $M_{\omega} >$).
- The action of $\mathcal{D}$ on $N_{\omega} > >$ yields $\exp M_{\omega} >$.
- The action of $\mathcal{E}$ on $N_{\omega} > >$ yields $L_{\omega}$ [4, Corollary 5.17].
- The action of $\mathcal{E}^*$ on $N_{\omega} > >$ yields the class $K$ of $[13]$.

For $a \in N_{\omega} > >$, we will denote $\delta_{\omega}(a)$ the unique log-atomic element of $\mathcal{E}[a]$ and we will denote $\delta_{\omega}^*(a)$ the unique element of $\mathcal{E}^*[a]$ lying in $K$. We have $\delta_{\omega}^*(a) \subseteq \delta_{\omega}(a) \subseteq a$.

### 3. LOGARITHMIC TRANSSERIES AND HYPERSERIES

Let $\mathbb{L}_{<\omega} := \mathbb{R}[[\mathcal{L}_{<\omega}]]$ denote the field of logarithmic hyperseries from [7]. For each $n \in \mathbb{N}$, we define $\ell_n := L_{n} x \in \mathbb{L}_{<\omega}$. Recall that $\mathbb{R}[[\mathcal{L}_{<\omega} \times x_0]]$ stands for the set of series with Noetherian support in the partially ordered set $\mathcal{L}_{<\omega} \times x_0$. We may consider elements of $\mathbb{R}[[\mathcal{L}_{<\omega} \times x_0]]$ as bivariate series that are logarithmic transseries with respect to $x$ and ordinary series with respect to $\varepsilon$.

**Lemma 2.** Consider a series $f \in \mathbb{R}[[\mathcal{L}_{<\omega} \times x_0]]$ such that the substitution $f(\varepsilon)$ of $\varepsilon$ by $\varepsilon$ in $f$ vanishes for all $\varepsilon < 1$ in $\mathbb{L}_{<\omega}$. Then $f = 0$.

**Proof.** Assume for contradiction that $f \neq 0$. Write $f = f_0 + f_1 \varepsilon + f_2 \varepsilon^2 + \cdots$ and let $k \in \mathbb{N}$ be minimal with $f_k \neq 0$. Since $f$ is Noetherian as a series in $\mathbb{R}[[\mathcal{L}_{<\omega} \times x_0]]$, the set $\bigcup_{i \in \mathbb{N}} \text{supp } f_i$ is well based and admits a largest element $m$. Taking $\varepsilon \in x_0^\infty$ sufficiently small such that $m \varepsilon < f_k$, it follows that $f_{k+1} \varepsilon^{k+1} + f_{k+2} \varepsilon^{k+2} + \cdots < f_k \varepsilon^k$, whence $f(\varepsilon) = f_k \varepsilon^k + f_{k+1} \varepsilon^{k+1} + f_{k+2} \varepsilon^{k+2} + \cdots \sim f_k \varepsilon^k \neq 0$. \hfill $\square$
LEMMA 3. Given integers \( p \geq k \geq 0 \) and \( \delta(\epsilon) = L_{p-k}(L_k x + \epsilon) - L_p x \), we have

\[
(L'_\omega L_k x) \epsilon + \frac{1}{2} (L''_\omega L_k x) \epsilon^2 + \cdots = (L'_\omega L_p x) \delta(\epsilon) + \frac{1}{2} (L''_\omega L_p x) \delta(\epsilon)^2 + \cdots \tag{7}
\]
as an identity in \( \mathbb{R}[[L_{\leq \omega} \times \epsilon^N]] \).

**Proof.** The left and right hand sides of (7) are clearly Noetherian series in \( \mathbb{R}[[L_{\leq \omega} \times \epsilon^N]] \). For any \( \tilde{\epsilon} < L_k x \in L_{\leq \omega} \), the following Taylor series expansions hold in \( L_{\leq \omega} \):

\[
L'_\omega (L_k x + \tilde{\epsilon}) = L'_\omega L_k x + (L''_\omega L_k x) \tilde{\epsilon}^2 + \cdots \tag{9}
\]

\[
L'_\omega (L_p x + \delta(\tilde{\epsilon})) = L'_\omega L_p x + (L''_\omega L_p x) \delta(\tilde{\epsilon}) + \frac{1}{2} (L''_\omega L_p x) \delta(\tilde{\epsilon})^2 + \cdots \tag{10}
\]

Subtracting both expansions, the identity (7) holds for \( \epsilon \) substituted by \( \tilde{\epsilon} < 1 \) in \( L_{\leq \omega} \). We conclude by Lemma 2. \( \square \)

In [7], we defined the field \( L_{\leq \omega} = \mathbb{R}[[L_{\leq \omega}]] \), as well as a hyperlogarithmic function \( L_\omega \) on \( L_{\leq \omega}^\omega \) for which \( L_{\leq \omega} \cong \mathbb{R}[[t_0^\omega]][[L_{\leq \omega}]] \) with \( t_0 = L_\omega x \). Let \( u := L_\omega x \in L_{\leq \omega} \). Then logarithmic transseries in \( u \) can be considered as elements in \( L_{\leq \omega} \circ L_\omega \subseteq L_{\leq \omega} \) and the successive derivatives of \( E_\omega(u) \) with respect to \( u \) are given by

\[
E^{(k)}(u) = \theta^k x \in L_{\leq \omega} \quad \theta := \gamma^{-1} \frac{\partial}{\partial x} \quad \gamma := \frac{1}{\prod_{k \leq \omega} t_k}.
\]

For any \( f \in L_{\leq \omega} \) we have

\[
\text{supp } \theta^k f \subseteq \text{supp } \theta + \text{supp } f, \quad \text{supp } \theta = \gamma^{-1} \text{supp } \frac{\partial}{\partial x} = \gamma^{-1} \left\{ \frac{1}{t_0}, \frac{1}{t_0 t_1}, \ldots \right\},
\]

whence

\[
\text{supp } E^{(k)}(u) = \text{supp } \theta^k x \subseteq \left( \frac{\gamma}{t_0} \right)^{-k} \left\{ \frac{1}{t_0}, \frac{1}{t_0 t_1}, \ldots \right\} \infty t_0.
\]

For \( \delta < 1/x \) in \( \mathbb{R}[[L_{\leq \omega} \times \epsilon^N]] \), this allows us to define \( E_\omega(u + \delta) \) using the Taylor series expansion

\[
E_\omega(u + \delta) := E_\omega(u) + E'_\omega(u) \delta + \frac{1}{2} E''_\omega(u) \delta^2 + \cdots \tag{8}
\]

**LEMMA 4.** The following identity holds in \( \mathbb{R}[[L_{\leq \omega} \times \epsilon^N]] \):

\[
E_\omega L_\omega(x + \epsilon) = x + \epsilon. \tag{9}
\]

**Proof.** The left hand side is well defined by (8) for \( \delta = L_\omega(x + \epsilon) - u = L_\omega(x + \epsilon) - L_\omega x = \gamma \epsilon + \frac{1}{2} \gamma' \epsilon^2 + \cdots < 1/x \) in \( \mathbb{R}[[L_{\leq \omega} \times \epsilon^N]] \). The fact that (9) holds for \( \epsilon \) substituted by \( \tilde{\epsilon} \) in \( L_{\leq \omega} \) follows from the usual rules of iterated derivatives of inverse functions. For a detailed proof, we refer to [14, section 6.4]. We conclude by Lemma 2. \( \square \)

### 4. HYPEREXPONENTIALS OF TRUNCATED NUMBERS

We recursively define \( E_\omega \) for positive purely infinite numbers \( \theta \in \text{No}^\omega \) by

\[
E_\omega \theta := \left\{ \delta^\omega(\theta), E_\omega \theta^\text{No}^\omega, E_\omega \theta^\text{No}^\omega \right\}_{\bar{K}} \tag{10}
\]

**PROPOSITION 5.** The function \( E_\omega \) defines a strictly increasing bijection \( \text{No}^\omega \rightarrow \bar{K} \). Moreover, the previous equation is uniform.
Proof. The function $E_\omega$ is well-defined and strictly increasing by Remark 1. The uniformity of the equation follows immediately.

Let $L_\omega$ denote the partial inverse function of $E_\omega$ and prove that $L_\omega$ is defined on $K$ by induction on $\mathbb{C}$. Let $a \in K$ such that $\kappa^K_a$ is contained $E_\omega \mathbb{N}_\omega^\prec$. Since $E_\omega$ is injective, its inverse is defined on $\kappa^K_a$. Let

$$\theta := (L_\omega, \kappa^K | L_\omega, \kappa^K, \kappa)_{\mathbb{N}_\omega^\prec}.$$ 

This number is well defined since $L_\omega, \kappa^K : \mathbb{N} \rightarrow \mathbb{N}_\omega^\prec$ is strictly increasing and for $\kappa' \in \kappa^K$, we have $L_\omega, \kappa' < \kappa' < \kappa$. By uniformity, we have $E_\omega, \theta = (\delta_\omega^\kappa(\theta), \kappa^K | \kappa^K, E_\omega, \kappa)_K$ where $\kappa = (\kappa^K | \kappa^K)_K$. In order to conclude that $E_\omega, \theta = \kappa$, it therefore suffices to show that $\kappa$ lies in the cut $(\delta_\omega^\kappa(\theta) | E_\omega, \kappa)$. We have $E_\omega, \kappa > \kappa$ by (10) and $\theta < \kappa$ by definition of $\theta$, whence $\delta_\omega^\kappa(\theta) < \kappa$ since $\kappa \in K$. We conclude by induction that $E_\omega, \mathbb{N}_\omega^\prec \rightarrow K$ is surjective.

We next identify the class of truncated numbers. For $a \in \mathbb{N}_\omega^\prec$, we consider the following convex class

$$\Pi[a] := \left\{ b \in \mathbb{N}_\omega^\prec \mid \exists n \in \mathbb{N}, a - b < \frac{1}{L_n E_\omega, a} \right\}.$$ 

PROPOSITION 6. The classes $\Pi[a]$ for $a \in \mathbb{N}_\omega^\prec$ form a thin convex partition of $\mathbb{N}_\omega^\prec$.

Proof. Given $a \in \mathbb{N}_\omega^\prec$, it is clear that the class $\Pi[a]$ is convex and that it contains $a$. Note that for $a \in \mathbb{N}_\omega^\prec$, we have $\Pi[a] \subseteq a + \mathbb{N}_\omega^\prec$. Let $a, b \in \mathbb{N}_\omega^\prec$ with $\Pi[a] \neq \Pi[b]$. We claim that $\Pi[a] \cap \Pi[b] = \emptyset$. If $a > b$, then we have $a + \mathbb{N}_\omega^\prec \cap b + \mathbb{N}_\omega^\prec = \emptyset$, which yields the result. Assume that $a > b$. Assume for contradiction that there are $c \in \mathbb{N}_\omega^\prec$ and $m, n \in \mathbb{N}$ with $a - c < \frac{1}{L_m E_\omega, a}$ and $b - c < \frac{1}{L_n E_\omega, b}$. Given $d \in \Pi[a]$, there is a number $p \geq m$, $n$ with $a - d < \frac{1}{L_p E_\omega, a}$. Therefore $a - c, b - c, a - d$ are dominated by $\frac{1}{L_p E_\omega, a}$, whence $b - d = b - c - (a - c) + (a - d) < \frac{1}{L_p E_\omega, a}$. This proves that $\Pi[a] \subseteq \Pi[b]$ and symmetric arguments yield $\Pi[b] \subseteq \Pi[a]$: a contradiction. This proves our claim. It only remains to see that the class $\Pi[a]$ admits a cofinal and coinitial subset for any $a \in \mathbb{N}_\omega^\prec$. Indeed, we can take $a \pm \frac{1}{L_n E_\omega, a}$ as examples of such sets.

COROLLARY 7. The class $\text{Tr} := \text{Smp}_\Pi$ is a surreal substructure.

Let $a \in \mathbb{N}_\omega^\prec$ and let $\varphi$ denote the $\ll$-supremum of truncations $\psi$ of $a$ (i.e. series with $\psi \ll a$) with $\supp \varphi > \frac{1}{L_n E_\omega, a}$. In particular, we have $a \ll \varphi$ since $\sup a > 1$. We see that $\varphi$ satisfies $\varphi \ll a$ and $\supp \varphi > \frac{1}{L_n E_\omega, a}$. Write $\varphi = a + \delta$ and $a = \varphi + \varepsilon = a + \delta + \varepsilon$. By $\ll$-maximality of $\varphi$, we have $\varepsilon < \frac{1}{L_n E_\omega, a}$, $\frac{1}{L_m E_\omega, \varphi}$, so $a \in \Pi[\varphi]$, or equivalently $\varphi \in \Pi[a]$. We deduce that $\varphi$ is the $\ll$-minimum, hence $\ll$-minimum of $\Pi[a]$, so $\varphi \in \text{Tr}$. We also see that for $\theta \in \mathbb{N}_\omega^\prec$ and $r \in \mathbb{R}$, we have $\theta + r \subseteq \text{Tr}$. Since $\text{Tr}$ is a surreal substructure, we may use recursion on $\varphi \in \text{Tr}$ to define

$$\hat{E}_\omega(\varphi) := \{ E_\omega \varphi, E \hat{E}_\omega(\varphi \text{Tr}) \mid E \in \hat{E}_\omega(\varphi \text{Tr}) \}. \quad (11)$$

PROPOSITION 8. The equation (11) is uniform and $\hat{E}_\omega$ is a strictly increasing function $\text{Tr} \rightarrow L_a$.

Proof. Since $\text{Tr}$ is a surreal substructure, the definition, strict monotonicity and uniformity follow by Remark 1. For $\varphi \in \text{Tr}$, we have $\hat{E}_\omega(\varphi) > E E_\omega \varphi$ since $E \ll \exp$ on $\mathbb{N}_\omega^\prec$. We deduce that $\hat{E}_\omega(\varphi)$ is $E$-simple, hence log-atomic.

By [4, Lemma 2.4], for every infinite monomial $m \in \text{Mo}^\omega$, we have

$$\exp m = \{ D m, D \exp m \text{Mo}^\omega \mid D \exp m \text{Mo}^\omega \}. \quad (12)$$
PROPOSITION 9. We have \( \forall \varphi \in \text{Tr}, \hat{E}_{\omega}(\varphi + 1) = e^{\hat{E}_{\omega}(\varphi)} \).

Proof. We prove this by induction on \((\text{Tr}, \subseteq)\). Let \( \varphi \in \text{Tr} \) such that this holds on \( \varphi_{\text{Tr}}^+ \). Note that \( \Pi [a + r] = \Pi[a] + r \) for all \( a \in \mathbb{N}^{>\omega} \) and \( r \in \mathbb{R} \). We have \( \varphi = (\mathbb{R}, \Pi[\varphi_{\text{L}}^+], \Pi[\varphi_{\text{R}}^+]) \), so
\[
\varphi + 1 = (\varphi, \Pi[\varphi_{\text{L}}^+] + 1 | \Pi[\varphi_{\text{R}}^+] + 1) = (\Pi[\varphi], \Pi[\varphi_{\text{L}}^+] + 1 | \Pi[\varphi_{\text{R}}^+] + 1) = (\varphi, \varphi_{\text{L}}^+ + 1 | \varphi_{\text{R}}^+ + 1)_{\text{Tr}}.
\]
We deduce that
\[
\hat{E}_{\omega}(\varphi + 1) = (E_{\mathbb{N}}(\varphi + 1), E_{\hat{E}_{\omega}(\varphi_{\text{L}}^+ + 1) | E_{\hat{E}_{\omega}(\varphi_{\text{R}}^+ + 1)}) = (E_{\mathbb{N}} \varphi, E_{\exp} \hat{E}_{\omega}(\varphi_{\text{L}}^+) | E_{\exp} \hat{E}_{\omega}(\varphi_{\text{R}}^+)).
\]
Since \( \hat{E}_{\omega}(\varphi) \in \mathbb{La} \subseteq \mathbb{Mo}^\omega \), we may apply (12). We also note that \( \exp \mathbb{A} \) and \( \exp \mathbb{A} \) are mutually cofinal and coinitial for all \( a \in \mathbb{N}^{>\omega} \) to obtain
\[
\exp \hat{E}_{\omega}(\varphi) = (D \hat{E}_{\omega}(\varphi), D \exp E_{\mathbb{N}} \varphi, D \exp E_{\hat{E}_{\omega}(\varphi_{\text{L}}^+) | E_{\exp} \hat{E}_{\omega}(\varphi_{\text{R}}^+)) = (D \hat{E}_{\omega}(\varphi), E_{\mathbb{N}} \varphi, D \exp \hat{E}_{\omega}(\varphi_{\text{L}}^+) | D \exp \hat{E}_{\omega}(\varphi_{\text{R}}^+)) = (D \hat{E}_{\omega}(\varphi), E_{\mathbb{N}} \varphi, \exp \hat{E}_{\omega}(\varphi_{\text{L}}^+) | \exp \hat{E}_{\omega}(\varphi_{\text{R}}^+)).
\]
We have \( \hat{E}_{\omega}(\varphi + 1) > \hat{E}_{\omega}(\varphi) \), so \( \hat{E}_{\omega}(\varphi + 1) > E_{\hat{E}_{\omega}(\varphi + 1)} \) and \( \hat{E}_{\omega}(\varphi + 1) > D \hat{E}_{\omega}(\varphi) \). We clearly have \( \exp \hat{E}_{\omega}(\varphi) > E_{\mathbb{N}} \varphi \). We deduce that \( \exp \hat{E}_{\omega}(\varphi) = \hat{E}_{\omega}(\varphi + 1) \). By induction, the relation is valid on \( \text{Tr} \).

PROPOSITION 10. For \( \theta \in \mathbb{N}^{>\omega} \), we have \( \hat{E}_{\omega}(\theta) = E_{\omega} \theta \).

Proof. We prove this by induction on \( (\mathbb{N}^{>\omega}, \subseteq) \). Let \( \theta \in \mathbb{N}^{>\omega} \) be such that this holds on \( \theta_{\mathbb{N}}^{>\omega} \). For \( \varphi \in \theta_{\mathbb{L}}^{>\omega} \), we have \( \varphi_{\mathbb{L}} > \varphi_{\mathbb{N}}^{>\omega} \), and there is \( n \in \mathbb{N} \) with \( \varphi < \varphi_{\mathbb{L}} + n \). We deduce that \( \hat{E}_{\omega}(\varphi) \leq E_{\omega} \hat{E}_{\omega}(\varphi_{\mathbb{L}}^+) = E_{\omega} \varphi_{\mathbb{L}} + n \). In particular, we have \( \hat{E}_{\omega}(\varphi) < E_{\omega + 1} \varphi_{\mathbb{L}}^+ \) so \( \hat{E}_{\omega}(\varphi) < E_{n+2} \omega_{\omega} \varphi_{\mathbb{L}}^+ \). This proves that \( E_{\omega} \hat{E}_{\omega}(\theta_{\mathbb{L}}^{>\omega}) \) is cofinal with respect to \( E_{\omega} \hat{E}_{\omega}(\theta_{\mathbb{L}}^{>\omega}) \). For \( \theta' \in \theta_{\mathbb{N}}^{>\omega} \) and \( n \in \mathbb{N} \), we have \( E_{\omega} \theta' = E_{\omega} (\theta' + n) \) where \( \theta' + n \in \theta_{\mathbb{L}}^{>\omega} \). Symmetric arguments yield that \( E_{\omega} \hat{E}_{\omega}(\theta_{\mathbb{N}}^{>\omega}) \) and \( E_{\omega} \hat{E}_{\omega}(\theta_{\mathbb{R}}^{>\omega}) \) are mutually coinitial. We conclude that \( \hat{E}_{\omega}(\theta) = \{E_{\omega} \theta, E_{\omega} \varphi_{\mathbb{L}} \} | L_{\omega} E_{\omega} \varphi_{\mathbb{R}}^{>\omega} \} = E_{\omega} \theta \).

PROPOSITION 11. The function \( E_{\omega} : \text{Tr} \to \mathbb{La} \) is bijective. Its reciprocal \( L_{\omega} \) admits the following uniform equation on \( \mathbb{La} \):
\[ L_{\omega} \alpha = (L_{\omega} \alpha_{\mathbb{L}} | L_{\omega} \alpha_{\mathbb{R}} | L_{\omega} \alpha)_{\text{Tr}}. \]

Proof. Noticing that \( E_{\omega} \varphi = \hat{E}_{\omega}(\varphi) = (E_{\mathbb{N}} \hat{E}_{\omega}(\varphi), E_{\omega} \varphi_{\mathbb{L}} | E_{\omega} \varphi_{\mathbb{R}})_{\mathbb{La}} \) for all \( \varphi \in \text{Tr} \), this follows from the same arguments as in Proposition 5.

5. HYPEREXPOENTIALS OF ARBITRARY NUMBERS

The field \( \mathbb{L}_{<\omega} = \mathbb{R} [[[\mathbb{L}_{<\omega}]]] \) of logarithmic hyperseries of [7] is a subfield of the class of all well-based transseries in an infinitely large variable \( x \). Both \( \mathbb{L}_{<\omega} \) and the class of all transseries are closed under derivation and under composition [8, 10, 14]. For every positive infinite number \( a \in \mathbb{N}^{>\omega} \), there also exists an evaluation embedding \( \mathbb{L}_{<\omega} \to \mathbb{N}^{>\omega} ; f \to f(a) \) such that \( f(g(a)) = (f \circ g)(a) \) for all \( f, g \in \mathbb{L}_{<\omega} \); see [5].
Given $a \in \mathbb{No}^{>\omega}$, let $\varphi = q_{a} \in \text{Tr}$ be the unique truncated series with $a \in \Pi[\varphi]$. If $a \neq \varphi$, then there is a smallest number $n = n_{a} \in \mathbb{Z}$ with

$$a - \varphi < \frac{1}{L_{n} E_{\omega} \varphi} = \frac{1}{E_{\omega}(\varphi - n)}.$$ 

Write $\varepsilon := a - \varphi$. With $\theta$ as in section 3, we define for every $k \in \mathbb{N}$:

$$E_{\omega}^{(k)}(\varphi - n) := (\theta^{k} x)(E_{\omega}(\varphi - n)).$$

Substitution of $E_{\omega}(\varphi - n)$ for $x$ in $(8)$ allows us to extend the definition of $E_{\omega}$ by

$$E_{\omega}(a-n) := \sum_{k \in \mathbb{N}} \frac{1}{\omega^{k}} E_{\omega}^{(k)}(\varphi - n) \varepsilon^{k},$$

and

$$E_{\omega}(a) := E_{\omega}(E_{\omega}(a-n)).$$

**Proposition 12.** For all $a \in \mathbb{No}^{>\omega}$, we have

$$E_{\omega}(a+1) = E_{1} E_{\omega} a.$$ 

**Proof.** If $q_{a} = a$, then this is Proposition 9. Otherwise, we have $q_{a+1} = a_1 + 1 \neq a + 1$ and $n_{a+1} = n_{a} + 1$, whence $E_{\omega}(a+1-n_{a+1}) = E_{\omega}(a-n_{a})$ and

$$E_{\omega}(a+1) = E_{n_{a}+1}(E_{\omega}(a-n_{a})) = E_{1} E_{n_{a}}(E_{\omega}(a-n_{a})) = E_{1} E_{\omega} a.$$  

Inversely, consider an arbitrary positive infinite number $b \in \mathbb{No}^{>\omega}$. Then there exists a $k \in \mathbb{N}$ such that $L_{\omega} b = L_{k} a + \varepsilon$ for some log-atomic $a \in \text{La}$ and $\varepsilon < L_{k} a$. We extend the definition of $L_{\omega}$ to any such number $b$ by

$$L_{\omega} b := L_{\omega} a + (L'_{\omega} \circ \ell_{k}(a)) \varepsilon + \frac{1}{2} (L''_{\omega} \circ \ell_{k}(a)) \varepsilon^{2} + \cdots.$$ 

In view of Lemma 3, the value of $L_{\omega} b$ does not depend on the choice of $k$. Note also that this definition indeed extends our previous definition of $L_{\omega}$ on $\text{La}$.

**Proposition 13.** For all $b \in \mathbb{No}^{>\omega}$, we have

$$L_{\omega} L_{1} b = L_{\omega} b - 1.$$ 

**Proof.** With $L_{k} b = L_{k} a + \varepsilon$ as above (while taking $k > 0$), we have

$$L_{\omega} L_{1} b = L_{\omega} L_{1} a + (L'_{\omega} L_{k-1} L_{1} a) \varepsilon + \frac{1}{2} (L''_{\omega} L_{k-1} L_{1} a) \varepsilon^{2} + \cdots$$

$$= L_{\omega} a - 1 + (L'_{\omega} L_{k} a) \varepsilon + \frac{1}{2} (L''_{\omega} L_{k} a) \varepsilon^{2} + \cdots$$

$$= L_{\omega} b - 1,$$

where $L_{\omega} L_{1} a = L_{\omega} a - 1$ because of Proposition 9.

**Proposition 14.** For any $b \in \mathbb{No}^{>\omega}$, we have $E_{\omega} L_{\omega} b = b$.

**Proof.** Let $k \in \mathbb{N}$ be such that $L_{k} b = L_{k} a + \bar{\varepsilon}$, where $a \in \text{La}$ and $\bar{\varepsilon} < L_{k} a$. Let us first consider the special case when $k = 0$. Since $a$ is log-atomic, we have $L_{<\omega} \equiv L_{<\omega}(a)$. From Lemma 4, it therefore follows that $E_{\omega} L_{\omega}(a + \varepsilon) = a + \varepsilon$ inside $\mathbb{R}[[\mathcal{E}_{<\omega}(a) \times \varepsilon^{\mathbb{N}}]]$. The result follows by specializing this relation at $\bar{\varepsilon}$. If $k > 0$, then $L_{\omega} b = L_{\omega} L_{k} b + k = L_{\omega}(L_{k} a + \bar{\varepsilon}) + k$ by Proposition 13. Applying the result for the special case when $k = 0$, we have $E_{\omega} (L_{\omega} b - k) = L_{k} a + \bar{\varepsilon} = L_{k} b$. We conclude by Proposition 12.
In particular, the function $E_{\omega}: \mathbb{N}^{>\omega} \rightarrow \mathbb{N}^{>\omega}$ is surjective. We next prove that it is strictly increasing, concluding our proof that $E_{\omega}: \mathbb{N}^{>\omega} \rightarrow \mathbb{N}^{>\omega}$ is a strictly increasing bijection with reciprocal $L_{\omega}$.

**Lemma 15.** For $\varphi, \psi \in \mathcal{T}$ with $\varphi < \psi$, we have $E_{\omega}(\Pi[\varphi]) < E_{\omega}(\Pi[\psi])$.

**Proof.** Note that $\mathcal{L} \ni E_{\omega}(\varphi) < E_{\omega}(\psi) \in \mathcal{L}$, so it is enough to prove that $E_{\omega}(a) \in E[ E_{\omega}(\varphi) ]$ for all $a \in \Pi[\varphi]$. For such $a$, there is $n \in \mathbb{N}$ with $\varepsilon := a - \varphi < E_{\omega}(\varphi-n)^{-1}$, and

$$L_n E_{\omega}(a) = \sum_{k \in \mathbb{N}} \frac{E_{\omega}(k) (\varphi-n)}{k!} \varepsilon^k = E_{\omega}(\varphi-n) + \delta,$$

where $\delta := \sum_{k>0} \frac{E_{\omega}(k) (\varphi-n)}{k!} \varepsilon^k$ is infinitesimal. So $L_n E_{\omega}(a) \sim L_n E_{\omega}(\varphi)$, whence $E_{\omega}(a) \in E[ E_{\omega}(\varphi) ]$. \qed

**Lemma 16.** For $\varphi \in \mathcal{T}$ and $a, b \in \mathbb{N}^{>\omega}$ with $a, b \in \Pi[\varphi]$, there is $n \in \mathbb{N}$ with

$$L_n E_{\omega}(b) - L_n E_{\omega}(a) \sim E_{\omega}(\varphi-n) (b-a).$$

**Proof.** Write $a = \varphi + \varepsilon_a$ and $b = \varphi + \varepsilon_b$ where $\varepsilon_a, \varepsilon_b < 1$ and let $n \in \mathbb{N}$ with $\varepsilon_a, \varepsilon_b < E_{\omega}(\varphi-n)^{-1}$. Writing $\varepsilon_k := \varepsilon_{b_k} - \varepsilon_{a_k}$ for $k \in \mathbb{N}$, we have $\varepsilon_k < E_{\omega}(\varphi-n)^{-k}$. We deduce that

$$L_n E_{\omega}(b) - L_n E_{\omega}(a) = \sum_{k>0} \frac{E_{\omega}(k) (\varphi-n)}{k!} \varepsilon_k 
\sim E_{\omega}(\varphi-n) (\varepsilon_b - \varepsilon_a) 
\sim E_{\omega}(\varphi-n) (b-a). \quad \Box$$

**Proposition 17.** The function $E_{\omega}$ is strictly increasing on $\mathbb{N}^{>\omega}$.

**Proof.** Let $a, b \in \mathbb{N}^{>\omega}$ with $a < b$. If $a \in \Pi[b]$, then we get $E_{\omega}(a) < E_{\omega}(b)$ by Lemma 15. Otherwise, we have $a \in \Pi[b]$ so by Lemma 16, there are $\varphi \in \mathcal{T}$ and $n \in \mathbb{N}$ with

$$L_n E_{\omega}(b) - L_n E_{\omega}(a) \sim E_{\omega}(\varphi-n) (b-a).$$

Since $E_{\omega}(\varphi-n) > 0$, we conclude that $L_n E_{\omega}(b) > L_n E_{\omega}(a)$, whence $E_{\omega}(b) > E_{\omega}(a)$. \qed

**Corollary 18.** The function $E_{\omega}$ is bijective, with reciprocal $L_{\omega}$.

**Bibliography**


