

UNIFORMIZATION OF MULTIVARIATE POWER SERIES*

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In this paper, we describe an algorithm for the “uniformization” of a multivariate power series. Let $\mathbb{K}[[\mathfrak{T}]]$ be the field of “grid-based power series” over a sufficiently large non archimedean “monomial group” (or value group) \mathfrak{T} , such as $\mathfrak{T} = \{t_1^{\alpha_1} \dots t_n^{\alpha_n} : \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$ with the lexicographical ordering on $\alpha_1, \dots, \alpha_n$. We interpret power series $f \in \mathbb{K}[[x_1, \dots, x_n]]$ as functions $\mathbb{K}[[\mathfrak{T}^{\prec}]]^n \rightarrow \mathbb{K}[[\mathfrak{T}^{\prec}]]$. On certain “regions” \mathcal{R} of the space $\mathbb{K}[[\mathfrak{T}^{\prec}]]^n$, it may happen that the valuation of f can be read off from the valuations of the x_i . In that case, f is said to be “uniform” on \mathcal{R} . We will describe an algorithm for cutting $\mathbb{K}[[\mathfrak{T}^{\prec}]]^n$ into a finite number of regions, each on which f is uniform for a suitable change of coordinates, which preserves the elimination ordering on x_1, \dots, x_n . The algorithm can probably be seen as an effective counterpart of local uniformization in the sense of Zariski, even though this connection remains to be established in detail.

1. INTRODUCTION

Let \mathbb{K} be a field and $\hat{\mathbb{K}}$ an extension field of \mathbb{K} . Given a polynomial $f = \mathbb{K}[x_1, \dots, x_n]$, a natural question is to study the behaviour of f as a function $f: \hat{\mathbb{K}}^n \rightarrow \hat{\mathbb{K}}$. In order to capture all relevant properties of f , it is convenient to assume that $\hat{\mathbb{K}}$ is algebraically closed, or at least contains the algebraic closure \mathbb{K}^{alg} of \mathbb{K} . Using quantifier elimination, we may always cut $\hat{\mathbb{K}}^n$ into a finite number of “regions” (which are constructible subsets of $\hat{\mathbb{K}}^n$ in this case), each on which f has a “uniform behaviour”. For instance, outside a certain Zariski closed singular locus, the solutions to $f(x_1, \dots, x_n) = 0$ are given by a finite number of ramified non singular algebraic functions $x_n(x_1, \dots, x_{n-1})$.

A similar challenge can be stated for many other theories. For instance, if \mathbb{K} is a real field, then we may also want to study the real geometric properties of the function $f: \hat{\mathbb{K}}^n \rightarrow \hat{\mathbb{K}}$, such as those regions where f is positive. In this case, $\hat{\mathbb{K}}$ is rather taken to be the real closure of \mathbb{K} and cylindrical decomposition is a typical technique for studying the behaviour of the function f .

In this paper, we will consider a power series $f \in \mathbb{K}[[x_1, \dots, x_n]]$ over a field of characteristic zero, instead of a polynomial. We will see that f can again be regarded as a function $f: \hat{\mathbb{K}}[[\hat{\mathfrak{T}}^{\prec}]]^n \rightarrow \hat{\mathbb{K}}[[\hat{\mathfrak{T}}^{\prec}]]$ on a suitable space of grid-based series over a sufficiently large, non archimedean monomial group $\hat{\mathfrak{T}}$. At a first approximation, elements in the field $\hat{\mathbb{K}}[[\hat{\mathfrak{T}}]]$ might be taken to be series in t_1, \dots, t_n with real exponents, and with a lexicographical ordering on t_1, \dots, t_n . Series in $\hat{\mathbb{K}}[[\hat{\mathfrak{T}}^{\prec}]]$ and $\hat{\mathbb{K}}[[\hat{\mathfrak{T}}^{\preceq}]]$ correspond to infinitesimal resp. bounded series in $\hat{\mathbb{K}}[[\hat{\mathfrak{T}}]]$. We refer to section 2 for more precise definitions.

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A suitable language for the study of power series functions $f: \hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\succ}]]^n \rightarrow \hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\succ}]]$ is the usual language of fields, extended with an asymptotic neglectation relation \prec . In this language, $\varphi \prec \psi$ should be interpreted as $\varphi = o(\psi)$, $\varphi \preceq \psi$ should be interpreted as $\varphi = O(\psi)$ and $\varphi \asymp \psi$ corresponds to the case when $\varphi \preceq \psi \preceq \varphi$. If φ and ψ lie in a valued field, such as $\hat{\mathbb{K}}[[\hat{\mathcal{X}}]]$, then we also have $\varphi \prec \psi \Leftrightarrow v(\varphi) > v(\psi)$, $\varphi \preceq \psi \Leftrightarrow v(\varphi) \geq v(\psi)$ and $\varphi \asymp \psi \Leftrightarrow v(\varphi) = v(\psi)$. Even though $\mathbb{K}[[x_1, \dots, x_n]]$ is not a valued field, it does come with a partial neglectation relation \prec [vdH06, Chapter 1]. Furthermore, a relation such as $x_1 \prec x_2$ naturally corresponds to a subset $\{(\xi_1, \xi_2) \in \hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\prec}]]^2: \xi_1 \prec \xi_2\}$ of $\hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\prec}]]^2$.

From the asymptotic point of view, the simplest behaviour of a series $f \in \mathbb{K}[[x_1, \dots, x_n]]$ on a subset \mathcal{R} of $\hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\prec}]]^n$ is when $v(f(\hat{x}_1, \dots, \hat{x}_n))$ only depends on $v(\hat{x}_1), \dots, v(\hat{x}_n)$ for points $(\hat{x}_1, \dots, \hat{x}_n) \in \mathcal{R}$. If $\mathcal{R} = \hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\prec}]]^n$, this is the case if and only if f is a *uniform series* in the sense that its support $\text{supp } f \subseteq x^{\mathbb{N}^n} = x_1^{\mathbb{N}} \dots x_n^{\mathbb{N}}$ admits a unique \preceq -maximal element $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, called the *dominant monomial* of f . In other words, we may write $f = x^\alpha g$, where g is a unit in $\mathbb{K}[[x_1, \dots, x_n]]$. The series $f = x_1 + x_2 \in \mathbb{K}[[x_1, x_2]]$ also satisfies $v(f(\hat{x}_1, \hat{x}_2)) = v(\hat{x}_2)$ on the region where $x_1 \prec x_2$. Again, it is possible to regard f as a uniform series, but in a suitable ring $\mathbb{K}[[x_1/x_2, x_2]]$ of *conical series*, which corresponds precisely to the coordinate ring for the region on which $x_1 \prec x_2$.

Given an arbitrary series $f \in \mathbb{K}[[x_1, \dots, x_n]]$, our main objective is to present an algorithm which decomposes the space $\hat{\mathbb{K}}[[\hat{\mathcal{X}}^{\prec}]]^n$ into a finite number of well-described regions \mathcal{R}_i , endowed with suitable local conical coordinates, such that f becomes a uniform conical series on each of these regions, with respect to the local coordinates. Moreover, the necessary changes of coordinates will respect the elimination order on x_1, \dots, x_n . More precisely, if $\tilde{x}_1, \dots, \tilde{x}_n$ are the local coordinates on any of the regions \mathcal{R}_i , then \tilde{x}_k only depends on x_1, \dots, x_k for each k . In particular, at the end of the algorithm, it will be possible to read off the solutions $x_n(x_1, \dots, x_{n-1})$ to the equation $f(x_1, \dots, x_n) = 0$ from the answer.

Resolution of singularities [Hir64], which has recently been made effective [Vil89, Bod01, FKP06], is one means to solve our problem. However, the resolution process has to be carefully adapted in order to preserve the elimination ordering, which is non trivial. Moreover, resolution of singularities is really an overkill for our problem: for many practical applications, it is not necessary to glue the final regions together using birational maps. What is worse, insisting on global desingularization can be expected to artificially blow up the complexity, since the size of a description of the final non singular variety is usually huge.

In fact, our objective is closer to local uniformization in the sense of Zariski [Zar40]. In the future, we hope to provide a dictionary which will prove both approaches to be equivalent up to preservation of the elimination order. As we will see, one major advantage of our approach is that the general case is essentially reduced to the Newton-Puiseux method in dimension two.

An earlier version of our approach was first described in [vdH97, Chapter 10]. Apart from some minor changes (notational, terminological and the fact that we will not assume \mathbb{K} to be ordered), several improvements were made. First of all, section 4.2 contains a more geometric description of the uniform Newton degree, thereby simplifying our previous treatment of parallel descent of the Newton degree. We also corrected some errors concerning the use of so called pseudo-coefficients (see section 3.5). Finally, we adapted the method to series with coefficients in a \mathbb{K} -vector space \mathbb{V} instead of \mathbb{K} , where \mathbb{V} is typically of the form $\mathbb{V} = \mathbb{K}[[x_{i_1}, \dots, x_{i_k}]]$ for $i_1 < \dots < i_k$. Although this last change is not mandatory, it simplifies the overall treatment and also prepares for the uniformization of local vector fields.

Let us briefly outline the contents of the paper. In section 2, we introduce conical varieties and their function rings, whose elements are conical series. In order to deal with regions where $x_k \asymp x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}}$ for certain $\alpha_1, \dots, \alpha_{k-1}$, we will rewrite

$$x_k = x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}} (\lambda + \tilde{x}_k)$$

for a new parameter λ over \mathbb{K} and $\tilde{x}_k \prec 1$. For this reason, the coefficients of our conical series will always live in an algebraic extension \mathbb{K}_Λ of \mathbb{K} with a finite number of parameters. It is also possible, but less convenient, to directly work with serial coordinates x_i which are either infinitesimal $x_i \prec 1$ or bounded $x_i \preccurlyeq 1$. In section 2.4, we introduce refinements, which correspond to injective conical morphisms between conical varieties of the same dimension.

In section 3, we recall some general techniques for machine computations with conical series. First of all, we present a technique for computations with series with respect to changing coordinate systems. We next recall the formalism of non deterministic algorithms, which is suitable for modeling the situation in which a region has to be cut into several pieces. Finally, we briefly recall the notion of a local community. This concept aims at modeling certain subclasses of conical series, which are suitable for computations on a concrete computer. Of course, general power series are not even representable, since they may involve infinitely many coefficients. All effective computations assume that \mathbb{K} is an *effective field*. This means that there are algorithms for performing the field operations and the equality test in \mathbb{K} .

In section 4, we turn to algorithms for the uniformization of a conical series. We first recall the classical Newton polygon method in two dimensions, but for series whose coefficients lie in a more general field $\hat{\mathbb{K}}[[\hat{\mathfrak{X}}]]$ of generalized power series. We next consider uniform versions of this method, using evaluations of all but one variable. In particular, we will define an important invariant, the uniform Newton degree, which will strictly decrease every two refinements. We finally present our uniformization algorithm. Since a uniform series remains uniform under refinements, the algorithm can also be used for the simultaneous uniformization of several series.

Uniformization is a key ingredient for many computations with multivariate power series. Clearly, we need it in order to describe the asymptotic behaviour of such series. We typically also need it for the inversion of a power series, for expressing the zeros of $f(x_1, \dots, x_n)$ as a function $x_n(x_1, \dots, x_{n-1})$ in the remaining variables, and for many other problems which can be stated in the first order theory of fields with a neglection relation \prec . For instance, given two series $f, g \in \mathbb{K}[[x_1, x_2, x_3]]$, it allows us to determine the region of all x_3 for which $f \prec g$.

2. CONICAL VARIETIES

2.1. Grid-based series

In this section, we start by recalling some terminology and basic facts from [vdH06, Chapter 2]. Let \mathbb{K} be a field of characteristic zero and \mathfrak{M} a *monomial monoid*, that is, a commutative multiplicative monoid with a compatible partial ordering \preccurlyeq . In this paper, we will always assume that \mathfrak{M} is torsion free and a lattice for the ordering \preccurlyeq .

A subset $\mathfrak{G} \subseteq \mathfrak{M}$ is said to be *grid-based* if

$$\mathfrak{G} \subseteq \mathfrak{m}_1^{\mathbb{N}} \cdots \mathfrak{m}_k^{\mathbb{N}} \mathfrak{n},$$

for certain monomials $\mathfrak{m}_1, \dots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m}_1, \dots, \mathfrak{m}_k \prec 1$. Given a formal sum $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with $f_{\mathfrak{m}} \in \mathbb{K}$, we call

$$\text{supp } f = \{\mathfrak{m} \in \mathfrak{M}: f_{\mathfrak{m}} \neq 0\}$$

the *support* of f . We say that f is a *grid-based* series if $\text{supp } f$ is grid-based. We denote by $\mathbb{K}[[\mathfrak{M}]]$ the set of all grid-based series.

Let $f \in \mathbb{K}[[\mathfrak{M}]]$. The finite set \mathfrak{D}_f of \prec -maximal elements in $\text{supp } f$ is called the set of *dominant monomials* of f . If \mathfrak{D}_f is a singleton, then f is said to be *uniform*. In that case, the unique element \mathfrak{d}_f of \mathfrak{D}_f is called the *dominant monomial* of f and the corresponding coefficient $c_f = f_{\mathfrak{d}_f}$ the *dominant coefficient* of f . We say that f is *infinitesimal* (and write $f \prec 1$) if $\mathfrak{m} \prec 1$ for all $\mathfrak{m} \in \text{supp } f$. Similarly, f is said to be *bounded* (and we write $f \preccurlyeq 1$) if $\mathfrak{m} \preccurlyeq 1$ for all $\mathfrak{m} \in \text{supp } f$. We write $f \succ 1$ if f is uniform and $1 \preccurlyeq \mathfrak{d}_f \preccurlyeq 1$; this is the case if and only if $f = c + \varepsilon$ with $c \in \mathbb{K}^\neq$ and $\varepsilon \prec 1$.

PROPOSITION 1. *The set $\mathbb{K}[[\mathfrak{M}]]$ forms a \mathbb{K} -algebra, whose units are those uniform elements whose dominant monomials are invertible.*

The above definitions and results generalize to the case when the coefficient field \mathbb{K} is replaced by a \mathbb{K} -vector space \mathbb{V} . In fact, this may even be seen as a special case, when regarding \mathbb{V} as a subset of the quotient field of the tensor algebra of \mathbb{V} . When working with coefficients in \mathbb{V} , proposition 1 becomes:

PROPOSITION 2. *The set $\mathbb{V}[[\mathfrak{M}]]$ forms a $\mathbb{K}[[\mathfrak{M}]]$ -module.*

A possibly infinite family $(f_i)_{i \in I} \in \mathbb{V}[[\mathfrak{M}]]^I$ is said to be *grid-based* if $\bigcup_{i \in I} \text{supp } f_i$ is grid-based and $\{i \in I: \mathfrak{m} \in \text{supp } f_i\}$ is finite for every $\mathfrak{m} \in \mathfrak{M}$. In that case, the sum $S = \sum_{i \in I} f_i$ with $S_{\mathfrak{m}} = \sum_{i \in I} f_{i,\mathfrak{m}}$ is a well-defined grid-based series in $\mathbb{V}[[\mathfrak{M}]]$. It is possible to redevelop basic algebraic notions for this so called *strong summation* [vdH06, Chapters 2 and 6]. For instance, a strongly linear map $\mathbb{V}[[\mathfrak{M}]] \rightarrow \mathbb{V}[[\mathfrak{N}]]$ is a linear map which preserves infinite summation in a suitable manner. Similarly, a morphism $\Phi: \mathbb{K}[[x_1, \dots, x_n]] \rightarrow \mathbb{K}[[\tilde{x}_1, \dots, \tilde{x}_n]]$ of strong algebras corresponds to the operator which substitutes $\Phi(x_i)$ for each x_i .

2.2. Grid-based series with parameters

Let $\hat{\mathbb{K}}$ be an algebraic extension of \mathbb{K} which contains the algebraic closure \mathbb{K}^{alg} of \mathbb{K} . A *system of parametric coordinates* Λ is determined by a finite number of parameters $\lambda_1, \dots, \lambda_l$, subject to a finite number of polynomial constraints and one polynomial inequality over \mathbb{K} :

$$\begin{aligned} P_i(\lambda_1, \dots, \lambda_l) &= 0 & (i = 1, \dots, l) \\ Q(\lambda_1, \dots, \lambda_l) &\neq 0 \end{aligned}$$

We denote by $\mathcal{V}_\Lambda \subseteq \hat{\mathbb{K}}^l$ the corresponding *parametric variety* of all points $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_l)$ which satisfy these constraints. We also denote by

$$\mathbb{K}_\Lambda = \mathbb{K}[\lambda_1, \dots, \lambda_l, \lambda^*] / (P_1, \dots, P_l, \lambda^* Q - 1).$$

the corresponding *parametric coordinate ring*. Given a \mathbb{K} -vector space \mathbb{V} , we let $\mathbb{V}_\Lambda = \mathbb{V} \otimes \mathbb{K}_\Lambda$. If \mathbb{K} is an effective field, then it is classical that the consistency of a finite system of polynomial constraints can be checked by algorithm, using Groebner basis techniques for instance.

It is classical that any point $\hat{\lambda} \in \mathcal{V}_\Lambda$ induces a unique morphism

$$\Pi: \mathbb{K}_\Lambda \rightarrow \hat{\mathbb{K}}$$

with $\Pi(\lambda_i) = \hat{\lambda}_i$ for all i , and *vice versa*. Given $f \in \mathbb{K}_\Lambda$, we will then write $f(\hat{\lambda}) = \Pi(f)$. Given a second system of parametric coordinates $\tilde{\Lambda}$, a morphism

$$\Phi: \mathbb{K}_\Lambda \rightarrow \mathbb{K}_{\tilde{\Lambda}}$$

induces a morphism $\Phi^*: \mathcal{V}_{\hat{\Lambda}} \rightarrow \mathcal{V}_{\Lambda}$ of parametric varieties by asking that $f(\Phi^*(\hat{\lambda})) = \Phi(f)(\hat{\lambda})$ for all $f \in \mathbb{K}_{\Lambda}$. If Φ^* is injective, then $\mathcal{V}_{\hat{\Lambda}|\Lambda} = \Phi^*(\mathcal{V}_{\Lambda})$ will be called a *subregion* of \mathcal{V}_{Λ} .

A grid-based series with parameters in Λ is simply a grid-based series f in $\mathbb{K}_{\Lambda} \llbracket \mathfrak{M} \rrbracket$ (or in $\mathbb{V}_{\Lambda} \llbracket \mathfrak{M} \rrbracket$). Given a point $\hat{\lambda} \in \mathcal{V}_{\Lambda}$, we let $f(\hat{\lambda}) \in \hat{\mathbb{K}} \llbracket \mathfrak{M} \rrbracket$ (or $f(\hat{\lambda}) \in \hat{\mathbb{V}} \llbracket \mathfrak{M} \rrbracket$ with $\hat{\mathbb{V}} = \mathbb{V} \otimes \hat{\mathbb{K}}$) be its evaluation at this point. We say that $f \in \mathbb{K}_{\Lambda} \llbracket \mathfrak{M} \rrbracket$ is *uniform* if f admits a unique dominant monomial and an invertible dominant coefficient. This is the case if and only if $f(\hat{\lambda})$ is uniform for all $\hat{\lambda} \in \mathcal{V}_{\Lambda}$, since $\hat{\mathbb{K}} \supseteq \mathbb{K}^{\text{alg}}$. More generally, we therefore say that $f \in \mathbb{V} \llbracket \mathfrak{M} \rrbracket$ is *uniform* if and only if $f(\hat{\lambda})$ is uniform for all $\hat{\lambda} \in \mathcal{V}_{\Lambda}$. We say that $f \in \mathbb{V} \llbracket \mathfrak{M} \rrbracket$ is Λ -uniform if there exists a finite set $\mathfrak{D} \subseteq \mathfrak{M}$ such that $\mathfrak{D}_{f(\hat{\lambda})} = \mathfrak{D}$ for all $\hat{\lambda} \in \mathcal{V}_{\Lambda}$.

PROPOSITION 3. *Let $f \in \mathbb{V}_{\Lambda} \llbracket \mathfrak{M} \rrbracket$. Then \mathcal{V}_{Λ} can be decomposed*

$$\mathcal{V}_{\Lambda} = \mathcal{V}_{\Lambda_1|\Lambda} \amalg \dots \amalg \mathcal{V}_{\Lambda_r|\Lambda}$$

into a finite number of subregions, such that for each $i \in \{1, \dots, r\}$, there exists a set \mathfrak{D}_i , such that for each $\hat{\lambda} \in \mathcal{V}_{\Lambda_i}$, we have $\mathfrak{D}_{f(\hat{\lambda})} = \mathfrak{D}_i$.

PROOF. Let \mathfrak{G} be the set of monomials $\mathfrak{m} \in \text{supp } f$, such that \mathfrak{m} is a dominant monomial of $f(\hat{\lambda})$ for some $\hat{\lambda} \in \mathcal{V}_{\Lambda}$. Assume for contradiction that \mathfrak{G} is infinite. Since \mathfrak{G} is grid-based, there exists an infinite decreasing sequence $\mathfrak{m}_1 \succ \mathfrak{m}_2 \succ \dots$ of elements in \mathfrak{G} . Since \mathbb{K}_{Λ} is Noetherian, the ideal generated by $f_{\mathfrak{m}_1}, f_{\mathfrak{m}_2}, \dots$ is generated by $f_{\mathfrak{m}_1}, \dots, f_{\mathfrak{m}_k}$ for some k . Now choose $\hat{\lambda} \in \mathcal{V}_{\Lambda}$ such that \mathfrak{m}_{k+1} is a dominant monomial of $f(\hat{\lambda})$. Then $f_{\mathfrak{m}_1}(\hat{\lambda}) = \dots = f_{\mathfrak{m}_k}(\hat{\lambda}) = 0$, whence $f_{\mathfrak{m}_{k+1}}(\hat{\lambda}) = 0$, a contradiction. This shows that \mathfrak{G} and $\{\mathfrak{D}_{f(\hat{\lambda})} : \hat{\lambda} \in \mathcal{V}_{\Lambda}\} \subseteq \mathcal{P}(\mathfrak{G})$ are finite. Given $\mathfrak{D} \in \mathcal{P}(\mathfrak{G})$, the set $\{\hat{\lambda} \in \mathcal{V}_{\Lambda} : \mathfrak{D}_{f(\hat{\lambda})} = \mathfrak{D}\}$ is determined by the polynomial equations $f_{\mathfrak{n}}(\hat{\lambda}) = 0$ for all \mathfrak{n} such that $\mathfrak{m} \not\preceq \mathfrak{n}$ for all $\mathfrak{m} \in \mathfrak{D}$, and one polynomial inequation $\prod_{\mathfrak{m} \in \mathfrak{D}} f_{\mathfrak{m}}(\hat{\lambda}) \neq 0$. \square

2.3. Conical power series

A *system of serial coordinates* X consists of a finite number of serial coordinates x_1, \dots, x_n , subject to a finite number of asymptotic constraints

$$x^{\alpha_i} = x_1^{\alpha_{i,1}} \dots x_n^{\alpha_{i,n}} \prec 1 \quad (\alpha_{i,j} \in \mathbb{Z}, i = 1, \dots, c).$$

These constraints can be encoded by the finite set $\mathfrak{C} = \{x^{\alpha_1}, \dots, x^{\alpha_c}\}$, which is said to be *consistent* if

$$\forall k_1, \dots, k_c \in \mathbb{N}, k_1 \alpha_1 + \dots + k_c \alpha_c = 0 \Rightarrow k_1 = \dots = k_c = 0.$$

In that case, $x_1^{\mathbb{Q}} \dots x_n^{\mathbb{Q}}$ is a monomial group for the partial ordering given by

$$x^{\beta} \preceq x^{\gamma} \iff (\exists k_1, \dots, k_c \in \mathbb{Q}^{\geq}, \beta - \gamma = k_1 \alpha_1 + \dots + k_c \alpha_c).$$

The set \mathfrak{C} generates a monomial submonoid

$$\mathfrak{X} = \mathfrak{C}^* = x^{\alpha_1 \mathbb{N} + \dots + \alpha_c \mathbb{N}},$$

which is said to be *conical*. Series in $\mathbb{K} \llbracket \mathfrak{X} \rrbracket$, $\mathbb{V} \llbracket \mathfrak{X} \rrbracket$, $\mathbb{K}_{\Lambda} \llbracket \mathfrak{X} \rrbracket$ and $\mathbb{V}_{\Lambda} \llbracket \mathfrak{X} \rrbracket$ are also said to be *conical*. For what follows, it will be convenient to always assume that

$$\mathfrak{C} \supseteq \mathfrak{I}_n := \{x_1, \dots, x_n\}.$$

The theory of linear programming provides us with efficient algorithms for testing consistency and whether a given constraint $x^{\beta} \preceq x^{\gamma}$ is implied by the constraints in \mathfrak{C} .

A system \mathcal{X} of *conical coordinates* consists of the combination of a system $\Lambda_{\mathcal{X}}$ of parametric coordinates and a system $X_{\mathcal{X}}$ of serial coordinates. In what follows, we will shortly call such an \mathcal{X} a *coordinate system*. We will denote the corresponding parameters by $\lambda_{\mathcal{X},1}, \dots, \lambda_{\mathcal{X},l_{\mathcal{X}}}$, the serial coordinates by $x_{\mathcal{X},1}, \dots, x_{\mathcal{X},n_{\mathcal{X}}}$, the asymptotic constraints by $\mathfrak{C}_{\mathcal{X}}$, and $\mathfrak{X}_{\mathcal{X}} = \mathfrak{C}_{\mathcal{X}}^*$. If \mathcal{X} is clear from the context, then we will drop all subscripts \mathcal{X} . When working with respect to a coordinate system $\tilde{\mathcal{X}}$ or \mathcal{X}' , we will again drop subscripts and rather use tildas or primes for distinguishing from \mathcal{X} . For instance, the serial coordinates with respect to $\tilde{\mathcal{X}}$ will be denoted by $\tilde{x}_1, \dots, \tilde{x}_{\tilde{n}}$.

The *coordinate ring* of \mathcal{X} is given by

$$\mathbb{K}_{\mathcal{X}} = \mathbb{K}_{\Lambda} \llbracket \mathfrak{X} \rrbracket.$$

Let $\hat{\mathfrak{X}}$ be a totally ordered monomial group such that any conical monomial group \mathfrak{X} can be embedded in $\hat{\mathfrak{X}}$. Given any infinite dimensional totally ordered vector space E over \mathbb{R} , one may take $\hat{\mathfrak{X}}$ to be the multiplicative monomial group t^E isomorphic to E . A morphism of strong algebras

$$\Pi: \mathbb{K}_{\mathcal{X}} \rightarrow \hat{\mathbb{K}} \llbracket \hat{\mathfrak{X}} \rrbracket$$

with $\Pi(\mathbb{K}_{\Lambda}) \subseteq \hat{\mathbb{K}}$ is entirely determined by the l -tuple $\hat{\lambda} = \Pi(\lambda) = (\Pi(\lambda_1), \dots, \Pi(\lambda_l)) \in \hat{\mathbb{K}}^l$ and the n -tuple $\hat{x} = \Pi(x) = (\Pi(x_1), \dots, \Pi(x_n)) \in \hat{\mathbb{K}} \llbracket \hat{\mathfrak{X}} \rrbracket^n$. The pair $\xi = (\hat{\lambda}, \hat{x})$ is called a *point* and the set $\mathcal{V}_{\mathcal{X}}$ of all such points the *conical variety* associated to \mathcal{X} . As before, we will write $f(\xi) = \Pi(f)$ for all $f \in \mathbb{K}_{\mathcal{X}}$.

Let $k \in \{1, \dots, n\}$ and let \mathcal{X}' the coordinate system obtained from \mathcal{X} by removing x_k and keeping the same parameters (see remark 4 below), so that $\mathfrak{X}' x_k^{\mathbb{N}} \subseteq \mathfrak{X}$. A series $f \in \mathbb{K}_{\mathcal{X}}$ is said to be *ordinary* in x_k if $\text{supp } f \subseteq \mathfrak{X}' x_k^{\mathbb{N}}$. Then f may also be regarded as a series in $\mathbb{K}_{\mathcal{X}'} \llbracket [x_k] \rrbracket$ or as a series in $\mathbb{K} \llbracket [x_k] \rrbracket_{\mathcal{X}'}$. If x_k is the only element in \mathfrak{C} which depends on x_k (so that $\mathfrak{X} = \mathfrak{X}' x_k^{\mathbb{N}}$), then x_k is called an *ordinary coordinate* of \mathcal{X} , and all series in $\mathbb{K}_{\mathcal{X}}$ are ordinary in x_k .

REMARK 4. A more precise construction of the coordinate system \mathcal{X}' goes as follows. We take $\Lambda' = \Lambda$. The serial coordinates of \mathcal{X}' are $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$. We finally enforce

$$\mathfrak{X}' = \mathfrak{X} \cap x_1^{\mathbb{Z}} \cdots x_{k-1}^{\mathbb{Z}} x_{k+1}^{\mathbb{Z}} \cdots x_n^{\mathbb{Z}},$$

by taking \mathfrak{C}' to be the subset of monomials $\mathfrak{m} \in \mathfrak{X}'$ which admit only trivial factorizations in \mathfrak{X}' . This minimal set \mathfrak{C}' of generators is finite and can be computed as follows. Using linear programming, we first compute a minimal set of generators $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ for the cone $(\mathfrak{X}')^{\mathbb{Q}_{\geq}}$. For each i , the minimal $\alpha_i \in \mathbb{Q}^>$ with $\mathfrak{m}_i^{\alpha} \in x^{\mathbb{Z}^n}$ yields a minimal generator $\mathfrak{n}_i = \mathfrak{m}_i^{\alpha_i}$ for \mathfrak{X}' . Then $\mathfrak{C}' \subseteq \mathfrak{B} := \{\mathfrak{n}_1^{\beta_1} \cdots \mathfrak{n}_r^{\beta_r} : \beta_i \in \mathbb{Q}, 0 \leq \beta_i \leq 1\} \cap x^{\mathbb{Z}^n}$, so we conclude by removing all elements which can be factored from \mathfrak{B} .

2.4. Refinements

Consider two coordinate systems \mathcal{X} and $\tilde{\mathcal{X}}$. A *conical morphism* is a morphism of strong algebras

$$\Phi: \mathbb{K}_{\mathcal{X}} \rightarrow \mathbb{K}_{\tilde{\mathcal{X}}},$$

with $\Phi(\mathbb{K}_{\Lambda}) \subseteq \mathbb{K}_{\tilde{\Lambda}}$. Notice that the image of a uniform series f under Φ is again uniform as soon as $\Phi(\mathfrak{d}_f) \neq 0$ and zero otherwise. The morphism Φ induces a mapping

$$\Phi^*: \mathcal{V}_{\tilde{\mathcal{X}}} \rightarrow \mathcal{V}_{\mathcal{X}}$$

on the associated varieties by

$$f(\Phi^*(\tilde{\xi})) = \Phi(f)(\tilde{\xi})$$

for all $f \in \mathbb{K}_{\mathcal{X}}$ and $\tilde{\xi} \in \mathcal{V}_{\tilde{\mathcal{X}}}$. If Φ^* is injective, then Φ will be called an *immersion*. If Φ^* is injective and the dimensions n and \tilde{n} of \mathcal{X} and $\tilde{\mathcal{X}}$ coincide, then Φ is called a *refinement* and

$$\mathcal{V}_{\tilde{\mathcal{X}}|\mathcal{X}} = \Phi^*(\mathcal{V}_{\tilde{\mathcal{X}}})$$

a *subregion* of $\mathcal{V}_{\mathcal{X}}$. A refinement Φ is said to be *triangular*, if $\Phi(x_k)$ only depends on $\tilde{x}_1, \dots, \tilde{x}_k$ and $\Phi(x_k)$ is ordinary in \tilde{x}_k , for $k = 1, \dots, n$. A *k-refinement* is a triangular refinement Φ , such that $\Phi(x_i) = \tilde{x}_i$ for $i = k + 1, \dots, n$. A refinement Φ is said to be *strict* in x_k if Φ is triangular and $\Phi(f)$ is ordinary in \tilde{x}_k for all $f \in \mathbb{K}_{\mathcal{X}}$. A *k-refinement* is said to be strict if it is strict in x_k .

The composition of two triangular refinements is again triangular, the composition of two *k-refinements* is again a *k-refinement*, and the composition of a refinement which is strict in x_k and a triangular refinement (or a triangular refinement and a refinement which is strict in x_k) is again strict in x_k . We will sometimes say that the coordinate x_k is *refined*, when applying a refinement which is strict in x_k .

EXAMPLE 5. Consider a conical morphism $\Phi: \mathbb{K}_{\mathcal{X}} \rightarrow \mathbb{K}_{\tilde{\mathcal{X}}}$ with $\tilde{\mathfrak{S}} = \mathfrak{S}$ and $\Phi(x_i) = x_i$ for all i . Then Φ is entirely determined by its restriction Φ_{Λ} to \mathbb{K}_{Λ} . In particular, Φ is a refinement if and only if $\mathbb{K}_{\tilde{\Lambda}}$ consists of fractions of elements in $\Phi_{\Lambda}(\mathbb{K}_{\Lambda})$. Such refinements correspond to the imposition of constraints on the parameters.

EXAMPLE 6. Consider a conical morphism $\Phi: \mathbb{K}_{\mathcal{X}} \rightarrow \mathbb{K}_{\tilde{\mathcal{X}}}$ such that $\Phi_{\Lambda} = \text{Id}$, $\tilde{n} = n$ and $\Phi(x_i) = \tilde{x}_i$ for all i . Then $\tilde{\mathfrak{X}} \supseteq \mathfrak{X}$ and Φ is an *n-refinement*, which corresponds to the imposition of new asymptotic constraints on serial coordinates.

EXAMPLE 7. The *ramification* $R_{p_1, \dots, p_n}: \mathbb{K}_{\mathcal{X}} \rightarrow \mathbb{K}_{\tilde{\mathcal{X}}}$ of orders p_1, \dots, p_n is the *n-refinement* defined by $\Phi_{\Lambda} = \text{Id}$ and

$$R_{p_1, \dots, p_n}(x_i) = \tilde{x}_i^{p_i} \quad (i = 1, \dots, n).$$

More precisely, $\tilde{\mathfrak{C}}$ consists of $\tilde{\mathfrak{J}}_n = \{\tilde{x}_1, \dots, \tilde{x}_n\}$, together with a constraint $\tilde{x}_1^{p_1 \alpha_1} \dots \tilde{x}_n^{p_n \alpha_n}$ for every constraint $x^{\alpha} \in \mathfrak{C} \setminus \mathfrak{J}_n$.

EXAMPLE 8. Given a coordinate system \mathcal{X} , we will denote by \mathcal{X}_k the *k-dimensional* coordinate system formed by the parameters and the first *k* serial coordinates x_1, \dots, x_k of \mathcal{X} . We denote by \mathfrak{X}_k the corresponding conical monomial group. Given a new invertible parameter $\tilde{\lambda}$ and an infinitesimal monomial $\mathfrak{m} \prec 1$ in \mathfrak{X}_{k-1} which is incomparable to x_k , let us show that the asymptotic change of variables

$$x_k := \mathfrak{m}(\tilde{\lambda} + \tilde{x}_k) \quad (\tilde{x}_k \prec 1) \tag{1}$$

gives rise to a strict *k-refinement*. Indeed, the parametric coordinate ring of the new coordinate system $\tilde{\mathcal{X}}$ is given by $\mathbb{K}_{\tilde{\Lambda}} = \mathbb{K}_{\Lambda}[\tilde{\lambda}, \tilde{\lambda}^{-1}]$. The new serial coordinates are $\tilde{x}_i = x_i$ for $i \neq k$ and a new coordinate \tilde{x}_k . The monomial group $\tilde{\mathfrak{X}}$ is generated by \tilde{x}_k and the monomials $x^{\alpha}(\mathfrak{m}/x_k)^{\alpha_k}$ with $x^{\alpha} \in \mathfrak{C}$. We take $\Phi(x_i) = x_i$ for $i \neq k$ and $\Phi(x_k) = \mathfrak{m}(\tilde{\lambda} + \tilde{x}_k)$. Given a point $\xi = \Phi^*(\tilde{\xi}) \in \mathcal{V}_{\tilde{\mathcal{X}}|\mathcal{X}}$, we have

$$\begin{aligned} \tilde{\lambda}(\tilde{\xi}) &= c_{x_k(\xi)/\mathfrak{m}(\xi)} \\ \tilde{x}_k(\tilde{\xi}) &= x_k(\xi)/\mathfrak{m}(\xi) - c_{x_k(\xi)/\mathfrak{m}(\xi)}, \end{aligned}$$

so Φ is indeed a *k-refinement*. Since \tilde{x}_k is an ordinary coordinate of $\tilde{\mathcal{X}}$, the refinement is strict in x_k .

EXAMPLE 9. Let $\varphi \in \mathbb{K}_{\mathcal{X}_{k-1}}$ be a uniform series with an invertible dominant coefficient c and infinitesimal dominant monomial $\mathfrak{m} \in \mathfrak{X}_{k-1}$. If \mathfrak{m} is incomparable to x_k , then it can be shown as above that the asymptotic change of coordinates

$$x_k := \varphi + \mathfrak{m} \tilde{x}_k \quad (\tilde{x}_k \prec 1) \tag{2}$$

gives rise to a strict k -refinement Φ .

3. MACHINE COMPUTATIONS WITH CONICAL SERIES

3.1. Computations with respect to changing coordinates

One technical difficulty concerning the upcoming algorithms is that we frequently have to change our coordinates. From a notational point of view, it would be cumbersome to explicitly rewrite all our objects with respect to the new coordinates after every change. Instead, we will rather assume that our coordinate system with the corresponding constraints is stored in a global variable and that our objects are automatically rewritten with respect to the most recent coordinate system when necessary.

More precisely, starting with the coordinate ring $\mathbb{K}_{\mathcal{X}_0}$, the execution of an algorithm up to a given “current execution point”, gives rise to a sequence of refinements:

$$\begin{aligned} \Phi_1: \mathbb{K}_{\mathcal{X}_0} &\rightarrow \mathbb{K}_{\mathcal{X}_1} \\ \Phi_2: \mathbb{K}_{\mathcal{X}_1} &\rightarrow \mathbb{K}_{\mathcal{X}_2} \\ &\vdots \\ \Phi_r: \mathbb{K}_{\mathcal{X}_{r-1}} &\rightarrow \mathbb{K}_{\mathcal{X}_r} \end{aligned}$$

The “current coordinates” $\mathcal{X} := \mathcal{X}_r$ at that point are encoded by the finite sets of parameters and serial coordinates, together with the constraints imposed upon them.

Now consider a series $f \in \mathbb{K}_{\mathcal{X}_i}$ introduced between the i -th and the $(i+1)$ -th refinement. We will encode such a series by the pair (\mathcal{X}_f, f) with $\mathcal{X}_f = \mathcal{X}_i$. Whenever an operation needs to be performed on f at the current execution point, we automatically replace this encoding by $(\mathcal{X}_{\tilde{f}}, \tilde{f})$, where $\mathcal{X}_{\tilde{f}} = \mathcal{X}_r$ and $\tilde{f} = (\Phi_r \circ \dots \circ \Phi_{i+1})(f)$. Similarly, a monomial $\mathfrak{m} \in \mathfrak{X}_{\mathcal{X}_i}$ will be encoded by the pair $(\mathcal{X}_{\mathfrak{m}}, \mathfrak{m})$, where $\mathcal{X}_{\mathfrak{m}} = \mathcal{X}_i$. When necessary, this encoding will be replaced by $(\mathcal{X}_{\tilde{\mathfrak{m}}}, \tilde{\mathfrak{m}})$, where $\mathcal{X}_{\tilde{\mathfrak{m}}} = \mathcal{X}_r$ and $\tilde{\mathfrak{m}}$ is the dominant monomial of $(\Phi_r \circ \dots \circ \Phi_{i+1})(\mathfrak{m})$; this will ensure that monomials remain monomials of the same asymptotic magnitude at any stage, even though their values as series may change.

Sometimes, we will also work with a system \mathcal{X}' of “subcoordinates” of the current coordinate system \mathcal{X} . This means that the serial coordinates of \mathcal{X}' are a subset $\{x_{i_1}, \dots, x_{i_{n'}}\}$ of $\{x_1, \dots, x_n\}$, with the constraints induced from \mathcal{X} . The parametric coordinates of \mathcal{X}' and \mathcal{X} are assumed to be identical. When working with respect to such a system \mathcal{X}' of subcoordinates, any change of coordinates for \mathcal{X}' lifts to a change of coordinates for \mathcal{X} . In particular, all changes of coordinates can always be performed on \mathcal{X} , even when we need to work with respect to subcoordinates.

REMARK 10. Various variants of the above strategy are possible. For instance, whenever an operation on two series takes place, then we may rewrite them with respect to the simplest common coordinate system. Instead of using a global variable for the current coordinate system, operations on the coordinate system (such as the imposition of constraints) may also be done on the series which are intended to use the new coordinates, thereby allowing for a more functional programming style.

3.2. Non deterministic algorithms

Another frequently occurring difficulty is that certain relations may not be satisfied uniformly on the current region $\mathcal{V}_{\mathcal{X}}$. For instance, a polynomial relation $P(\lambda) = 0$ for the parameters may be satisfied on a certain subregion, whereas the opposite relation $P(\lambda) \neq 0$ is satisfied on another subregion. If, at a certain point during the execution of an algorithm, we need to decide whether $P(\lambda) = 0$ or $P(\lambda) \neq 0$, we may thus have to decompose the current region into these two subregions and continue the execution on each of them.

A classical computer science approach to this situation is to allow for so called *non deterministic* algorithms. This non deterministic setting features an additional programming construct called *case separation*, which consists of selecting the way to continue the algorithm non deterministically, among a finite number of cases. For instance, when testing whether $P(\lambda)$ vanishes, one branch would correspond to the subregion of λ for which $P(\lambda) = 0$ and the other branch would correspond to the subregion of λ for which $P(\lambda) \neq 0$.

It is classical that a non deterministic computation can be represented by a tree: each inner node ν of the tree corresponds to a non deterministic choice in the algorithm and the children ν_1, \dots, ν_s of the node to the possible continuations of the algorithm. König's lemma implies that this computation tree is finite if and only if it contains no infinite chains. In other words, if the non deterministic algorithm terminates for all possible sequences of choices, then we obtain a deterministic algorithm by running the non deterministic algorithm several times, for each possible sequence of choices.

In our setting, each node of the computation tree also corresponds to a coordinate system \mathcal{X}_ν and the case separation induces a partition

$$\mathcal{V}_{\mathcal{X}_\nu} = \mathcal{V}_{\mathcal{X}_{\nu_1}|\mathcal{X}_\nu} \amalg \dots \amalg \mathcal{V}_{\mathcal{X}_{\nu_s}|\mathcal{X}_\nu}.$$

In particular, the root of the tree corresponds to the original coordinates \mathcal{X} and the leafs of the tree correspond to the final coordinates $\mathcal{X}_1, \dots, \mathcal{X}_r$ for which the algorithm provides uniform results. At the end, it will then be guaranteed that

$$\mathcal{V}_{\mathcal{X}} = \mathcal{V}_{\mathcal{X}_1|\mathcal{X}} \amalg \dots \amalg \mathcal{V}_{\mathcal{X}_r|\mathcal{X}}.$$

Throughout our algorithms, we will assume that branches which correspond to empty regions are automatically eliminated. In other words, a non deterministic process is killed as soon as contradictory constraints are imposed.

REMARK 11. The approach of non deterministic algorithms is known under various other names, depending on the area. In computer algebra, it is sometimes referred to as *dynamic evaluation*. In basic programming languages, non deterministic algorithms are best implemented simply by rerunning the algorithm several times from its start and exhausting all sequences of non deterministic choices. In high order programming which support so called *continuations*, this rerunning can be avoided.

3.3. Local communities

Assume now that \mathbb{K} is an effective field. Since power series in $\mathbb{K}[[x]]$ and more general grid-based power series in $\mathbb{K}[[\mathfrak{M}]]$ may contain infinitely many coefficients, we need to restrict our attention to suitable subclasses of series in order to compute with them. In this section, we recall the concept of a “local community”, which axiomatizes such computationally interesting classes of series. In fact, it also models other interesting classes of series, such as the class of convergent multivariate power series over \mathbb{C} .

We will only recall the main definitions and facts about local communities and similarly for Cartesian representations in the next subsection. More details can be found in [vdH06, Sections 3.4 and 3.5] and [vdH97].

A *local community* over an effective \mathbb{K} -algebra \mathbb{A} is a family $\mathbb{L} = (\mathbb{L}_n)_{n \in \mathbb{N}}$ of \mathbb{A} -subalgebras $\mathbb{L}_n \subseteq \mathbb{A}[[z_1, \dots, z_n]]$, which satisfies the following properties:

- LC1.** For all $1 \leq k \leq n$, we have $z_k \in \mathbb{L}_n$.
- LC2.** Given $n \geq 1$ and $f \in \mathbb{L}_n$, such that f is divisible by z_1 , we have $f/z_1 \in \mathbb{L}_n$.
- LC3.** Given a strong algebra morphism $\Phi: \mathbb{A}[[z_1, \dots, z_n]] \rightarrow \mathbb{A}[[\tilde{z}_1, \dots, \tilde{z}_{\tilde{n}}]]$ with $\Phi(z_1), \dots, \Phi(z_n) \in \mathbb{L}_{\tilde{n}}$, we have $\Phi(\mathbb{L}_n) \subseteq \mathbb{L}_{\tilde{n}}$.

LC4. Given $n \geq 1$ and $f \in \mathbb{L}_n$, such that $f_{z_1^0 \dots z_n^0} = 0$ and $f_{z_1^0 \dots z_{n-1}^0 z_n^1} = 1$, let g be the unique power series $g \in \mathbb{A}[[z_1, \dots, z_{n-1}]]$ such that the substitution of g for z_n in f vanishes. Then $g \in \mathbb{L}_{n-1}$.

Given $f \in \mathbb{L}_n$ and $k \leq n$, we notice that

$$\frac{\partial f}{\partial z_k}(z_1, \dots, z_n) = \lim_{z_{n+1} \rightarrow 0} \frac{f(z_1, \dots, z_k + z_{n+1}, \dots, z_n) - f(z_1, \dots, z_n)}{z_{n+1}} \in \mathbb{L}_n,$$

since computing the limit for $z_{n+1} \rightarrow 0$ reduces to substituting z_{n+1} by zero. In particular, when expanding f as a series in z_k , then each of its coefficients is again in \mathbb{L}_n .

EXAMPLE 12. Let \mathbb{L}_n be the set of all algebraic power series in z_1, \dots, z_n over \mathbb{K} for each n . Then $\mathbb{L} = (\mathbb{L}_n)_{n \in \mathbb{N}}$ is a local community over \mathbb{K} , which is actually the smallest local community over \mathbb{K} .

EXAMPLE 13. Assuming $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let \mathbb{L}_n be the set of all convergent power series in z_1, \dots, z_n over \mathbb{K} . Then $\mathbb{L} = (\mathbb{L}_n)_{n \in \mathbb{N}}$ is a local community.

The local community \mathbb{L} is said to be *effective* if there are algorithms for carrying out the \mathbb{A} -algebra operations in \mathbb{L}_n , for performing the division by z_1 in **LC2**, for the substitution Φ in **LC3**, and for computing g as a function of f in **LC4**. For instance, the local community of all algebraic power series over \mathbb{K} is effective.

Given a local community \mathbb{L} over \mathbb{K} and a system Λ of parametric coordinates over \mathbb{K} , we denote by \mathbb{L}_Λ the smallest local community over \mathbb{K}_Λ which contains \mathbb{L} . Given another system $\tilde{\Lambda}$ of parametric coordinates, any morphism $\Phi: \mathbb{K}_\Lambda \rightarrow \mathbb{K}_{\tilde{\Lambda}}$ induces a natural componentwise morphism $\Phi_\mathbb{L}: \mathbb{L}_\Lambda \rightarrow \mathbb{L}_{\tilde{\Lambda}}$. We say that \mathbb{L} is *parametrically effective* if \mathbb{L}_Λ is effective for each system of parametric coordinates over \mathbb{K} and if $\Phi_\mathbb{L}$ is computable whenever $\Phi: \mathbb{K}_\Lambda \rightarrow \mathbb{K}_{\tilde{\Lambda}}$ is computable. The local community of all algebraic power series over \mathbb{K} is parametrically effective.

REMARK 14. It can be shown that \mathbb{L} is parametrically effective as soon as \mathbb{L} is effective. The idea is first to represent elements in \mathbb{K}_Λ by algebraic series in $\mathbb{K}[[z_1, \dots, z_d]]$ after substitutions $\lambda_i \rightarrow c_i + z_i$ for a suitable point $c \in \mathcal{V}_\Lambda$ and transcendence basis $\lambda_1, \dots, \lambda_d$ of \mathbb{K}_Λ . Then elements in $\mathbb{K}_\Lambda \otimes \mathbb{L}_n$ can be regarded as elements in \mathbb{L}_{d+n} .

3.4. Cartesian representations

A *multivariate Laurent series* in z_1, \dots, z_k over \mathbb{A} is an element $f \in \mathbb{A}((z_1, \dots, z_k)) := \mathbb{A}[[z^{\mathbb{Z}^k}]]$ with the componentwise partial ordering on $z^{\mathbb{Z}^k}$. We may always rewrite $f = z^\alpha g$ with $\alpha \in \mathbb{Z}^k$ and $g \in \mathbb{A}[[z_1, \dots, z_k]]$.

Let \mathfrak{M} be an arbitrary monomial monoid and consider a grid-based series $f \in \mathbb{A}[[\mathfrak{M}]]$. A *Cartesian representation* for f is a series $\check{f} \in \mathbb{A}((z_1, \dots, z_k))$ with $f = \hat{\check{f}}$ for some strong algebra morphism $\hat{\cdot}: \mathbb{A}((z_1, \dots, z_k)) \rightarrow \mathbb{A}[[\mathfrak{M}]]$ with $z_i \in \mathfrak{M}^\prec$ for all i . A family $(\check{f}_i)_{i \in I}$ of Cartesian representations is said to be *compatible* if the strong morphism $\hat{\cdot}$ is the same for all its components \check{f}_i . It can be shown that any finite family of grid-based series admits a family of compatible Cartesian representations.

Let \mathbb{L} be a local community over \mathbb{A} . A Cartesian representation $\check{f} \in z^{\mathbb{Z}^k} \mathbb{L}_k$ of f is called an \mathbb{L} -*representation* of f . The set $\mathbb{A}[[\mathfrak{M}]]_\mathbb{L}$ of all series with an \mathbb{L} -representation is clearly an \mathbb{A} -subalgebra of $\mathbb{A}[[\mathfrak{M}]]$. A Cartesian representation \check{f} of f is said to be *faithful* if for every dominant monomial \check{m} of \check{f} , there exists a dominant monomial m of f with $\check{m} \preccurlyeq m$. Any $f \in \mathbb{A}[[\mathfrak{M}]]_\mathbb{L}$ can be shown to admit a faithful \mathbb{L} -representation. Every uniform $f \in \mathbb{A}[[\mathfrak{M}]]_\mathbb{L}$ also admits a uniform \mathbb{L} -representation. Moreover, if \mathbb{L} is effective, then there are algorithms for computing faithful and uniform \mathbb{L} -representations.

The above definitions and properties extend to the case when we take our coefficients in a strong \mathbb{A} -module \mathbb{V} of the form $\mathbb{V} = \mathbb{A}[[t_1, \dots, t_r]]^s$. In that case, a Cartesian representation $\check{f} \in \mathbb{V}((z_1, \dots, z_k))$ can be rewritten as an s -tuple of series

$$\check{f} = (\check{f}_1, \dots, \check{f}_s) \in z^{\mathbb{Z}^k} \mathbb{A}[[z_1, \dots, z_k]][[t_1, \dots, t_r]]^s.$$

Under the identifications $t_i = z_{k+i}$, we then say that \check{f} is an \mathbb{L} -representation if $\check{f}_i \in z^{\mathbb{Z}^k} \mathbb{L}_{k+r}$ for all i . The corresponding subspace $\mathbb{V}[[\mathfrak{M}]]_{\mathbb{L}}$ of $\mathbb{V}[[\mathfrak{M}]]$ is an $\mathbb{A}[[\mathfrak{M}]]_{\mathbb{L}}$ -module and the results about faithful and uniform representations extend.

Given a faithful Cartesian representation \check{f} of a conical series $f \in \mathbb{V}_{\mathcal{X}, \mathbb{L}}$, the dominant monomials of f can be read off from the dominant monomials of \check{f} . In particular, if \mathbb{L} is effective, then we have an algorithm for the computation of \mathfrak{D}_f . In general, f is not Λ -uniform, so case separations may be necessary in order to enforce this property. As long as f is not uniform, it suffices to pick a non uniform dominant coefficient c of f , distinguish the two cases when $c = 0$ and $c \neq 0$, and keep computing the dominant monomials of f in both cases. In a similar way as in proposition 3, it can be shown that this algorithm terminates.

Given a finite subset $\mathfrak{D} \subseteq \mathfrak{X}$ of monomials, the associated *Newton polytope* is the subset \mathfrak{N} of all monomials $\mathfrak{n} \in \mathfrak{D}$ for which there exists a strong morphism $\Phi: \mathbb{K}_{\mathcal{X}} \rightarrow \check{\mathbb{K}}[[\check{\mathfrak{X}}]]$ such that $\Phi(\mathfrak{n}) \succ \Phi(\mathfrak{m})$ for all $\mathfrak{m} \in \mathfrak{D}$. The computation of \mathfrak{N} as a function of \mathfrak{D} is classical and corresponds to the computation of a convex hull. If $\mathfrak{D} = \mathfrak{D}_f$, then we call $\mathfrak{N} = \mathfrak{N}_f$ the *Newton polytope* of f . If f is Λ -uniform, then we call \mathfrak{N}_f the Λ -uniform Newton polytope of f . By what precedes, and modulo case separations, we have an algorithm for the computation of the Λ -uniform Newton polytope of f . In the sequel, this algorithm will be called *polytope $_{\mathbb{V}}$* .

3.5. Diagonal communities and how to avoid them

Let us consider a conical series f in x_1, \dots, x_n which admits an \mathbb{L} -representation \check{f} with respect to a fixed local community \mathbb{L} . Given $\alpha_k, \dots, \alpha_n \in \mathbb{Q}$, it is natural to ask for an \mathbb{L} -representation of the coefficient $g = [x_k^{\alpha_k} \dots x_n^{\alpha_n}] f$ of $x_k^{\alpha_k} \dots x_n^{\alpha_n}$ in f . Unfortunately, such an \mathbb{L} -representation does not always exist. For instance, assume that f admits an \mathbb{L} -representation of the form

$$\check{f} = \sum_{\alpha_1, \alpha_2 \in \mathbb{N}} \check{f}_{\alpha_1, \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2},$$

where $x_1 \prec x_2$, $z_1 = x_1/x_2$ and $z_2 = x_2$. Then the coefficient $g = [x_2^0] f$ admits a Cartesian representation

$$\check{g} = \sum_{\alpha \in \mathbb{N}} \check{f}_{\alpha, \alpha} z_1^{\alpha} z_2^{\alpha}. \tag{3}$$

However, there is no reason for this diagonal to be in \mathbb{L} . A local community \mathbb{L} is said to be a *diagonal community* if it satisfies

DC. For any $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$, the set \mathbb{L}_d is closed under taking diagonals

$$\check{f} \longmapsto \Delta_{\lambda} \check{f} = \sum_{\substack{\alpha \in \mathbb{N}^d \\ \lambda \cdot \alpha = 0}} f_{\alpha} z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

The ring of conical series with \mathbb{L} -representations in a diagonal community is closed under the extraction of coefficients with respect to x_1, \dots, x_n .

Many classical local communities, such as the communities of algebraic or convergent power series, are actually diagonal. However, local communities are not always diagonal, and even if they are, then proving this fact may be non trivial. Fortunately, for our purpose of uniformization, we can do with a suitable approximate version of the extraction of coefficients with respect to x_1, \dots, x_n : given $\alpha_k, \dots, \alpha_n \in \mathbb{Q}$, we say that g^* is a *pseudo-coefficient* of $x_k^{\alpha_k} \dots x_n^{\alpha_n}$ in f , if the set of dominant monomials of $f - g^* x_k^{\alpha_k} \dots x_n^{\alpha_n}$ contains no monomial of the form x^β with $\beta_k = \alpha_k, \dots, \beta_n = \alpha_n$. In [vdH97, Section 9.4.4], we have given an algorithm for the computation of such a pseudo-coefficient g^* .

REMARK 15. The main idea behind the computation of g^* is to hack the algorithm for the computation of a faithful \mathbb{L} -representation of f by replacing all zero tests by so called diagonal tests. This idea is based on the observation that, even though we cannot compute \check{g} in (3), we *can* check whether $\check{f} - \check{g} = 0$:

$$\check{f} = \check{g} \iff \check{f}(z_1 z_2, z_1 z_2) = \check{f}(z_1^2, z_2^2).$$

In general, denoting by g the genuine coefficient of $x_k^{\alpha_k} \dots x_n^{\alpha_n}$ in f , a similar trick may be used to test whether $f = g x_k^{\alpha_k} \dots x_n^{\alpha_n}$ [vdH06, Section 9.4.2]. Using such diagonal tests for f and truncations of f , it then becomes possible to compute the set of dominant monomials \mathfrak{D} of $f - g x_k^{\alpha_k} \dots x_n^{\alpha_n}$. The truncation $g^* = f_{\neq \mathfrak{D}}$ yields the desired pseudo-coefficient.

4. UNIFORMIZATION OF CONICAL SERIES

In this section, we will assume that we have fixed a local community \mathbb{L} and that all conical series to be considered admit \mathbb{L} -representations. In particular, the set $\mathbb{V}_{\mathcal{X}}$ will correspond to $\mathbb{V}_{\Delta} \llbracket \mathcal{X} \rrbracket_{\mathbb{L}}$ instead of $\mathbb{V}_{\Delta} \llbracket \mathcal{X} \rrbracket$. All our algorithms on conical series will only rely on operations that can be performed from within the local community \mathbb{L} . Therefore, if \mathbb{L} is parametrically effective, then our algorithms also become fully effective.

The only \mathbb{K} -vector spaces \mathbb{V} over which we will work are strong vector spaces of the form $\mathbb{V} = \mathbb{K}[[t_1, \dots, t_r]]^s$. If $\mathbf{e}_1, \dots, \mathbf{e}_s$ is the canonical basis of such a vector space \mathbb{V} over $\mathbb{K}[[t_1, \dots, t_r]]$, then the vectors $t_1^{\alpha_1} \dots t_r^{\alpha_r} \mathbf{e}_i$ with $\alpha_1, \dots, \alpha_r \in \mathbb{N}$ form a strong basis for \mathbb{V} . Given such a basis element \mathbf{b} and a vector $c \in \mathbb{V}$, we denote by $c_{\mathbf{b}}$ the coefficient of \mathbf{b} in c . Similarly, if $f \in \mathbb{V} \llbracket \mathfrak{M} \rrbracket$ is a grid-based series, then we denote $f_{\mathbf{b}} = \sum_{\mathbf{m} \in \mathfrak{M}} f_{\mathbf{m}, \mathbf{b}} \mathbf{m} \in \mathbb{K} \llbracket \mathfrak{M} \rrbracket$.

4.1. The Newton polygon method in dimension two

Let \mathfrak{M} be a totally ordered monomial group with \mathbb{Q} -powers (i.e. for any $\mathbf{m} \in \mathfrak{M}$ and $k \in \mathbb{N}^>$, there exists an $\mathbf{n} \in \mathfrak{M}$ with $\mathbf{n}^k = \mathbf{m}$). Given a series $f \in \mathbb{K}[[x]] \llbracket \mathfrak{M} \rrbracket^{\neq}$, the solutions in $\mathbb{K} \llbracket \mathfrak{M} \rrbracket$ to the equation

$$f(x) = 0 \quad (x \prec 1) \tag{4}$$

can be computed using the Newton polygon method. This method is explained in detail in [vdH06, Chapter 3] in the case when f is a polynomial in x and readily adapts to the case when f is a power series (see [vdH06, Exercises 3.1 and 3.7]). Let us briefly recall the main definitions and results.

The series f may also be regarded as a series in $\mathbb{K}[[x]] \llbracket \mathfrak{M} \rrbracket$ and its dominant coefficient $N_f = c_f \in \mathbb{K}[[x]]$ in this representation is called the *Newton series* of f . If N_f is a polynomial, then we rather call it the *Newton polynomial* of f . The valuation of N_f in x is called the *Newton degree* and we denote it by $\deg_{\prec 1} f$. If $\varphi \in \mathbb{K} \llbracket \mathfrak{M} \rrbracket^{\prec}$ is infinitesimal, then the additive and multiplicative conjugates $f_{+\varphi}, f_{\times \mathbf{m}} \in \mathbb{K} \llbracket \mathfrak{M} \rrbracket[[x]]^{\neq}$ are defined by

$$\begin{aligned} f_{+\varphi}(x) &= f(\varphi + x) \\ f_{\times \varphi}(x) &= f(\varphi x). \end{aligned}$$

For $\mathfrak{m} \in \mathfrak{M}^{\prec}$, this allows us to define the Newton degree associated to the ‘‘slope’’ \mathfrak{m} by $\deg_{\prec \mathfrak{m}} f = \deg_{\prec 1} f_{\times \mathfrak{m}}$. The following properties are easy or proved in [vdH06, Chapter 3]:

PROPOSITION 16. *Given $f, g \in \mathbb{K}[[\mathfrak{M}]][[x]]^{\neq}$, $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}^{\prec}$ with $\mathfrak{n} \prec \mathfrak{m}$, $\varphi \in \mathbb{K}[[\mathfrak{M}]]^{\prec}$ with $\varphi \prec \mathfrak{m}$ and $l < \deg_{\prec \mathfrak{m}} f$, we have*

$$\begin{aligned} \deg_{\prec \mathfrak{m}} fg &= \deg_{\prec \mathfrak{m}} f + \deg_{\prec \mathfrak{m}} g \\ \deg_{\prec \mathfrak{n}} f &\leq \deg_{\prec \mathfrak{m}} f \\ \deg_{\prec \mathfrak{m}} f_{+\varphi} &= \deg_{\prec \mathfrak{m}} f \\ \deg_{\prec \mathfrak{m}} \frac{\partial^l f}{\partial x^l} &= \deg_{\prec \mathfrak{m}} f - l \end{aligned}$$

PROPOSITION 17. *If $\deg_{\prec 1} f = 1$, then (4) admits a unique solution in $\mathbb{K}[[\mathfrak{M}]]^{\prec}$.*

PROPOSITION 18. *Let $\varphi \in \mathbb{K}[[\mathfrak{M}]]^{\prec, \neq}$, $c = c_{\varphi}$, $\mathfrak{m} = \mathfrak{d}_{\varphi}$, $N = N_{f_{\times \mathfrak{m}}} \in \mathbb{K}[x]$ and $\mu = \text{val } N_{+c}$. Then*

$$\deg_{\prec \mathfrak{m}} f_{+\varphi} = \mu \leq \deg_{\prec 1} f.$$

Moreover, if $\mu = d = \deg_{\prec 1} f$ and φ is the unique solution to the equation

$$\frac{\partial^{d-1} f}{\partial x^{d-1}}(\varphi) = 0 \quad (\varphi \prec 1), \quad (5)$$

then for any non zero $\tilde{\varphi} \in \mathbb{K}[[\mathfrak{M}]]^{\prec}$ with dominant monomial $\tilde{\mathfrak{m}}$, we have

$$\deg_{\prec \tilde{\mathfrak{m}}} f_{+\varphi+\tilde{\varphi}} < \deg_{\prec 1} f.$$

The theory adapts with minor modifications to the case when $f \in \mathbb{V}[[\mathfrak{M}]][[x]]$ admits its coefficients in a \mathbb{K} -vector space \mathbb{V} . In that case, we are still looking for solutions of (4) in $\mathbb{K}[[\mathfrak{M}]]$. However, in proposition 17 it is only guaranteed that (4) admits at most solution. In particular, the equation (5) cannot necessarily be solved in proposition 18. Nevertheless, there exists a strong basis element \mathfrak{b} of \mathbb{V} such that the coefficient $N_{\mathfrak{b}} \in \mathbb{K}[x]$ of \mathfrak{b} in $N \in \mathbb{V}[x]$ has Newton degree d . Then proposition 18 still holds if we take φ to be the solution of $(\partial^{d-1} f_{\mathfrak{b}} / \partial x^{d-1})(\varphi) = 0$. Indeed, if $\tilde{\mathfrak{m}} = \mathfrak{d}_{\tilde{\varphi}}$ and $\tilde{N} = N_{f_{+\varphi, \times \tilde{\mathfrak{m}}}}$, then \tilde{N} has the property that $\tilde{N}_{\mathfrak{b}, d-1} = 0$. Consequently, $\tilde{N}_{\mathfrak{b}}$ does not admit a root of multiplicity d and neither does \tilde{N} .

4.2. Uniform aspects of the Newton polygon method

Let us now return to the case of a multivariate series $f \in \mathbb{V}_{\mathcal{X}}$. We will assume that x_k is an ordinary coordinate in \mathcal{X} and let \mathcal{X}' be the coordinates in \mathcal{X} except x_k . In particular, $\mathfrak{X} = \mathfrak{X}' x_k^{\mathbb{N}}$ and $\mathbb{V}_{\mathcal{X}} = \mathbb{V}_{\mathcal{X}'}[[x_k]]$. Considering $f = f_0 + f_1 x_k + \dots$ as a power series in x_k , we may then evaluate f at a point $\xi' \in \mathcal{X}'$ using

$$f(\xi') = f_0(\xi') + f_1(\xi') x_k + \dots \in \hat{\mathbb{V}}[[x_k]] [[\hat{\mathfrak{X}}]].$$

Now the uniform Newton degree of f in x_k is defined by

$$\deg_{x_k \prec 1} f = \sup_{\substack{\xi' \in \mathcal{V}_{\mathcal{X}'} \\ f(\xi') \neq 0}} \deg_{\prec 1} f(\xi') \in \mathbb{N} \cup \{+\infty\}.$$

This definition extends to the case when the coordinate x_k is just ordinary in f : it suffices to replace \mathcal{X} by \mathcal{X}' and an infinitesimal coordinate x_k which satisfies no constraints. The non trivial fact that $\deg_{x_k \prec 1} f$ is actually finite will follow later from the possibility to uniformize f . Nevertheless, the finiteness is trivially guaranteed in one particular case:

PROPOSITION 19. Let $\mathbb{V}' = \mathbb{V}[[x_k]]$ and assume that f is uniform as a series in $\mathbb{V}'_{\mathcal{X}'}$. Denoting by $N \in \mathbb{V}'_{\Lambda} \cong \mathbb{V}_{\Lambda}[[x_k]]$ the dominant coefficient of f , we have

$$\deg_{x_k \prec 1} f = \text{val}_{x_k} N.$$

In fact, $N_{f(\xi')} = N(\xi')$ and $\deg_{\prec 1} f(\xi') = \text{val}_{x_k} N(\xi')$ for all $\xi' \in \mathcal{V}_{\mathcal{X}'}$ with $f(\xi') \neq 0$.

The properties stated in proposition 16 admit natural uniform analogues:

PROPOSITION 20. Given $f, g \in \mathbb{V}_{\mathcal{X}}^{\neq}$, $\mathfrak{m}, \mathfrak{n} \in \mathfrak{X}'$ with $\mathfrak{n} \prec \mathfrak{m}$, $\varphi \in \mathbb{V}_{\mathcal{X}'}$ with $\varphi \prec \mathfrak{m}$ and $l < \deg_{x_k \prec \mathfrak{m}} f$, we have

$$\begin{aligned} \deg_{x_k \prec \mathfrak{m}} f g &= \deg_{x_k \prec \mathfrak{m}} f + \deg_{x_k \prec \mathfrak{m}} g \\ \deg_{x_k \prec \mathfrak{n}} f &\leq \deg_{x_k \prec \mathfrak{m}} f \\ \deg_{x_k \prec \mathfrak{m}} f_{x_k + \varphi} &= \deg_{x_k \prec \mathfrak{m}} f \\ \deg_{x_k \prec \mathfrak{m}} \frac{\partial^l f}{\partial x_k^l} &= \deg_{x_k \prec \mathfrak{m}} f - l, \end{aligned}$$

where $f_{x_k + \varphi}(x_1, \dots, x_n) = f(x_1, \dots, x_{k-1}, x_k + \varphi, x_{k+1}, \dots, x_n)$.

For the remaining propositions 17 and 18, it is useful to define an additional concept: we say that $f \in \mathbb{V}[[\mathfrak{X}]]$ is *Newton prepared* if there exist two monomials $\mathfrak{m}, \mathfrak{n} \in \mathfrak{X}$, such that f admits a Λ -uniform Newton polytope \mathfrak{N}_f , which is contained in $(\mathfrak{m}/\mathfrak{n})^{\mathbb{Q}} \mathfrak{n}$. If \mathfrak{N}_f contains at least two elements, then there exist unique $k \in \mathbb{N}$, $\mathfrak{m} \in x_1^{\mathbb{Q}} \dots x_{k-1}^{\mathbb{Q}}$ and $\mathfrak{n} \in \mathfrak{X}$ with $\mathfrak{N}_f \subseteq (x_k/\mathfrak{m})^{\mathbb{Z}} \mathfrak{n}$. We will call x_k the *principal coordinate* for f and \mathfrak{m} its associated *slope*. If, moreover, we have $\mathfrak{N}_f \subseteq (x_k/\mathfrak{m})^{\mathbb{N}} \mathfrak{n}$, then we say that f is *k-Newton prepared*, with associated Newton polynomial $N = \sum_i f_{(x_k/\mathfrak{m})^i \mathfrak{n}} x_k^i \in \mathbb{V}_{\Lambda}[x_k]$. Notice that proposition 19 applies as soon as $\mathfrak{m} \prec 1$.

PROPOSITION 21. Assume that $f \in \mathbb{K}_{\mathcal{X}}$ is *k-Newton prepared*, with principal coordinate x_k and $\deg_{x_n \prec 1} f = 1$. Then $f(x_k) = 0$ admits a unique infinitesimal solution in $\mathbb{K}_{\mathcal{X}'}$.

PROOF. This is a direct application of [vdH06, Theorem 3.3 and Exercise 3.1]. \square

Proposition 21 is not good enough though, since the unique solution φ may depend on x_{k+1}, \dots, x_n . Consequently, we will have to replace φ by the coefficient $\varphi_{\prec x_k} = [x_{k+1}^0 \dots x_n^0] \varphi$ of $x_{k+1}^0 \dots x_n^0$ in φ . Unless \mathbb{L} is a diagonal community, we have no means to compute this coefficient. In practice, we therefore rather compute a pseudo-coefficient $x_{k+1}^0 \dots x_n^0$ in φ , which we will denote by $\varphi_{\prec x_k}^*$.

PROPOSITION 22. Assume that $f \in \mathbb{V}_{\mathcal{X}}$ is *k-Newton prepared*, with slope $\mathfrak{m} \prec 1$ and Newton polynomial N . Let $\varphi \in \mathbb{K}_{\mathcal{X}'_{k-1}}$ be uniform with dominant term $c \mathfrak{m}$. Then

$$\deg_{x_k \prec \mathfrak{m}} f_{x_k + \varphi} = \mu = \max_{\hat{\lambda} \in \mathcal{V}_{\Lambda}} \text{val} N_{+c}(\hat{\lambda}).$$

Moreover, in the case when $\mu = d = \deg_{x_k \prec 1} f$, let \mathfrak{b} be a strong basis element for \mathbb{V} such that $N_{\mathfrak{b}} \neq 0$, and assume that $\varphi = \psi_{\prec x_k}^*$, where ψ satisfies

$$\frac{\partial^{d-1} f_{\mathfrak{b}}}{\partial x_k^{d-1}}(\psi) = 0 \quad (\psi \prec 1).$$

Assume that $f_{x_k + \varphi}$ is *k-Newton prepared*, with slope $\tilde{\mathfrak{m}} \prec \mathfrak{m}$, and let $\tilde{\varphi} \in \mathbb{K}_{\mathcal{X}'}$ be uniform, with dominant monomial $\tilde{\mathfrak{m}}$. If $f_{x_k + \varphi + \tilde{\varphi}}$ is *k-Newton prepared*, then

$$\deg_{x_k \prec \tilde{\mathfrak{m}}} f_{x_k + \varphi + \tilde{\varphi}} < \deg_{x_k \prec 1} f.$$

PROOF. The first assertion follows directly from the first assertion of proposition 18. As to the second assertion, let \mathbf{n} be the monomial, such that the dominant monomials of $f_{\mathbf{b}}$ are $(x_k/\mathbf{m})^i \mathbf{n}$ with $i=0, \dots, d$. Then the dominant monomials of $f_{\mathbf{b}, x_k + \varphi}$ are $(x_k/\tilde{\mathbf{m}})^i \mathbf{n} (\tilde{\mathbf{m}}/\mathbf{m})^d$ with $i=0, \dots, d$. Writing $\varphi = \psi + \eta$, the definition of pseudo-coefficients states that each dominant monomial of η depends on at least one of the coordinates x_{k+1}, \dots, x_n . Now

$$\begin{aligned} [x_k^{d-1}] f_{\mathbf{b}, x_k + \varphi} &= \frac{\partial^{d-1} f_{\mathbf{b}}}{\partial x_k^{d-1}}(\psi + \eta) \\ &= \frac{\partial^d f_{\mathbf{b}}}{\partial x_k^d}(\psi) \eta + \dots \end{aligned}$$

Since

$$\frac{\partial^d f_{\mathbf{b}}}{\partial x_k^d}(\psi) \asymp \mathbf{n}/\mathbf{m}^d,$$

it follows that the unique dominant monomial $\mathbf{n} \tilde{\mathbf{m}}/\mathbf{m}^d$ of $[x_k^{d-1}] f_{\mathbf{b}, x_k + \varphi}$ is a dominant monomial of $(\mathbf{n}/\mathbf{m}^d) \eta$. In other words, $\tilde{\mathbf{m}}$ must be a dominant monomial of η , even though $\tilde{\mathbf{m}}$ is free from x_{k+1}, \dots, x_n . This contradiction completes the proof. \square

4.3. Polarization

A first useful subalgorithm which we will need is *polarization*. Given an arbitrary monomial $\mathbf{m} \in x_1^{\mathbb{Q}} \cdots x_n^{\mathbb{Q}}$, we will frequently have to decide whether $\mathbf{m} \succ 1$, $\mathbf{m} \asymp 1$ or $\mathbf{m} \prec 1$. In general, it may happen that none of these conditions are satisfied globally on $\mathcal{V}_{\mathcal{X}}$. In such cases, we will use polarization in order to decompose $\mathcal{V}_{\mathcal{X}}$ into at most three subregions, such that on each subregion we have either $\mathbf{m} \succ 1$, $\mathbf{m} \asymp 1$ or $\mathbf{m} \prec 1$.

Now the subregions where $\mathbf{m} \prec 1$ and $\mathbf{m} \succ 1$ simply correspond to the imposition of the corresponding asymptotic constraints. In the remaining case, we write $\mathbf{m} = (x_k/\mathbf{n})^\alpha$ with $\alpha \in \mathbb{Q}$ and $\mathbf{n} \in x_1^{\mathbb{Q}} \cdots x_{k-1}^{\mathbb{Q}}$. For a new invertible parameter λ , and modulo a suitable ramification, it then suffices to perform the refinement $x_k := \mathbf{n}(\lambda + \tilde{x}_k)$ with $\tilde{x}_k \prec 1$.

Algorithm polarize(\mathbf{m})

INPUT: a monomial $\mathbf{m} = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ with $\alpha_1, \dots, \alpha_k \in \mathbb{Q}$ and $\alpha_k \neq 0$

ACTION: refine the coordinates such that we either get $\mathbf{m} \succ 1$, $\mathbf{m} \asymp 1$ or $\mathbf{m} \prec 1$

Ramify the coordinates, such that $\alpha_1, \dots, \alpha_k, \alpha_1/\alpha_k, \dots, \alpha_{k-1}/\alpha_k \in \mathbb{Z}$

If neither $\mathbf{m} \succ 1$, $\mathbf{m} = 1$, nor $\mathbf{m} \prec 1$, then separate the following cases:

1. Impose the constraint $\mathbf{m} \succ 1$
2. Introduce an invertible parameter λ .
Refine $x_k = x_1^{-\alpha_1/\alpha_k} \cdots x_{k-1}^{-\alpha_{k-1}/\alpha_k} (\lambda + \tilde{x}_k)$ with $\tilde{x}_k \prec 1$.
3. Impose the constraint $\mathbf{m} \prec 1$

4.4. Newton preparation

In section 4.2, we have seen the usefulness of Newton prepared series. Assuming that we have an algorithm `uniformize` for the uniformization of series in at most $n - 1$ variables, we will now present an algorithm for the Newton preparation of a series. Our algorithm assumes given an ordinary coordinate x_k and will only use refinements in the remaining coordinates \mathcal{X}' . Hence, the refinements never involve x_k , although they may involve x_{k+1}, \dots, x_n . If f does not become uniform after the preparation, then it should be noticed that the principal coordinate x_l of f may satisfy $l > k$.

Algorithm $\text{prepare}_{\mathbb{V}}(f, x_k)$ INPUT: a non zero series $f \in \mathbb{V}_{\mathcal{X}}$ and an ordinary coordinate x_k for f ACTION: refine the coordinates other than x_k such that f becomes Newton preparedLet \mathcal{X}' the coordinates other than x_k and $\mathbb{V}' = \mathbb{V}[[x_k]]$, so that $f \in \mathbb{V}'_{\mathcal{X}'}$ $\text{uniformize}_{\mathbb{V}'}(f)$ Let c be the dominant monomial of $f \in \mathbb{V}'_{\mathcal{X}'}$ $\text{uniformize}_{\mathbb{V}}(c)$ and let d be the valuation in x_k of c Expand $f = f_0 + f_1 x_k + \dots$ as a series in x_k For $i \in \{0, \dots, d-1\}$ do $\text{uniformize}_{\mathbb{V}'}(f_k)$ Let $\{\mathfrak{d}_1 x_k^{\alpha_1}, \dots, \mathfrak{d}_r x_k^{\alpha_r}\} := \text{polytope}_{\mathbb{V}}(f)$, with $\alpha_1 < \dots < \alpha_r = d$ and $\mathfrak{d}_1, \dots, \mathfrak{d}_r \in \mathfrak{X}'$ For all $i < j < r$ do $\text{polarize}((\mathfrak{d}_r/\mathfrak{d}_i)^{1/(\alpha_r-\alpha_i)}/(\mathfrak{d}_r/\mathfrak{d}_j)^{1/(\alpha_r-\alpha_j)})$

The algorithm for performing the Newton preparation is quite straightforward. We first uniformize f as a series in $\mathbb{V}'_{\mathcal{X}'}$ and let $d := \deg_{x_k < 1} f$. Writing $f = f_0 + f_1 x_k + \dots$ as a series in x_k , the tail $f_d x_k^d + f_{d+1} x_k^{d+1} + \dots$ will then be uniform. After uniformizing the remaining coefficients f_0, \dots, f_{d-1} , it follows that the Newton polytope of f has the form $\mathfrak{N}_f = \{\mathfrak{d}_1 x_k^{\alpha_1}, \dots, \mathfrak{d}_r x_k^{\alpha_r}\}$, with $\alpha_1 < \dots < \alpha_r = d$ and $\mathfrak{d}_1, \dots, \mathfrak{d}_r \in \mathfrak{X}'$. Now a sufficient condition for f to be Newton prepared is that all the slopes $(\mathfrak{d}_r/\mathfrak{d}_i)^{1/(\alpha_r-\alpha_i)}$ are pairwise comparable for \preceq . This is forced using polarization, where we notice that only refinements which do not involve x_k are needed.

4.5. Blowing up edges

Assume now that f is Newton prepared, but not uniform, and let x_k be its principal coordinate with slope \mathfrak{m} . At this point, we might in principle proceed by polarizing x_k/\mathfrak{m} . However, in order to force a strict decrease of the Newton degree (using proposition 22), we have to slightly modify the procedure for polarization and introduce a new case for handling the situation when the Newton polynomial has non zero roots of maximal multiplicity d . In this new case, which will only be needed if x_k is an ordinary coordinate in f (see also remark 23 below), we apply a Tschirnhausen transformation.

Algorithm $\text{blowup}_{\mathbb{V}}(f)$ INPUT: a non uniform Newton prepared series $f \in \mathbb{V}_{\mathcal{X}}$ ACTION: refine the coordinates such that f becomes “more uniform”Let $\mathfrak{N} := \text{polytope}_{\mathbb{V}}(f)$ Let $x_k, \mathfrak{m} \in x_1^{\mathbb{Q}} \dots x_{k-1}^{\mathbb{Q}}$ and $\mathfrak{n} \in \mathfrak{X}$ be such that $\mathfrak{N} \subseteq (x_k/\mathfrak{m})^{\mathbb{Z}} \mathfrak{n}$ $\text{polarize}(\mathfrak{m})$ and return whenever $\mathfrak{m} \succ 1$ Ramify the coordinates, such that $\mathfrak{m} \in \mathfrak{X}_{k-1}$ Let $N := \sum_{\alpha \in \mathbb{Z}} f_{(x_k/\mathfrak{m})^{\alpha}} x_k^{\alpha} \in \mathbb{K}_{\Lambda}[x_k^{\mathbb{Z}}]$ and let $d \in \mathbb{Z}$ be largest with $N_d \neq 0$ If x_k is ordinary in f then separate the following cases

1. Impose the constraint $x_k \succ \mathfrak{m}$
2. Impose the constraint $x_k \prec \mathfrak{m}$
3. Introduce an invertible parameter λ with $N \neq N_d (x_k - \lambda)^d$
Refine $x_k := \mathfrak{m} (\lambda + \tilde{x}_k)$ with $\tilde{x}_k \prec 1$
4. Introduce an invertible parameter λ with $N = N_d (x_k - \lambda)^d$
Let \mathfrak{b} be a strong basis element for \mathbb{V} such that $N_{\mathfrak{b}} \neq 0$
Let \mathcal{X}' be the coordinates in \mathcal{X} except x_k
Let $\varphi \in \mathbb{K}_{\mathcal{X}'}$ be the unique solution to $(\partial^{d-1} f_{\mathfrak{b}}/\partial x_k^{d-1})(\varphi) = 0$
Refine $x_k := \text{pseudo}(\varphi, k) + \mathfrak{m} \tilde{x}_k$ with $\tilde{x}_k \prec 1$

Else $\text{polarize}(x_k/\mathfrak{m})$

In the last step of 4, the subalgorithm $\text{pseudo}(\varphi, k)$ computes a pseudo-coefficient $\varphi_{<x_k}^*$ of $x_{k+1}^0 \cdots x_n^0$ in φ . In addition, modulo additional refinements and recursive calls of uniformize , we enforce the difference $\varphi - \varphi_{<x_k}^*$ to be uniform. This will ensure that the $\varphi_{<x_k}^*$ keeps its status of being a pseudo-coefficient, even after subsequent refinements of x_1, \dots, x_{k-1} . In the exceptional case that one of the coordinates x_{k+1}, \dots, x_n is refined during the uniformization of $\varphi - \varphi_{<x_k}^*$, we do not require $\varphi - \varphi_{<x_k}^*$ to be uniform on exit. Let us finally notice that, in the case when \mathbb{L} is an effective diagonal community, we may simply take $\text{pseudo}(\varphi, k) := \varphi_{<x_k}$ to be the genuine coefficient of $x_{k+1}^0 \cdots x_n^0$ in φ .

Algorithm $\text{pseudo}(f, k)$

INPUT: a series $f \in \mathbb{K}_{\mathcal{X}'}$, where \mathcal{X}' are the coordinates in \mathcal{X}_k except x_k

OUTPUT: a pseudo-coefficient $f_{<x_k}^*$ of $x_{k+1}^0 \cdots x_n^0$ in f . Moreover, if no refinement on x_{k+1}, \dots, x_n occurs during the execution, then $f - f_{<x_k}^*$ is guaranteed to be uniform on exit.

Repeat

Let φ be a pseudo-coefficient of $x_{k+1}^0 \cdots x_n^0$ in f .

If $f - \varphi$ is regular or a refinement on x_{k+1}, \dots, x_n has occurred, then return φ

$\text{uniformize}_{\mathbb{K}}(f - \varphi)$

4.6. Uniformization

We are now in a position to state the main algorithm for the uniformization of f . As long as f is not uniform, we keep Newton preparing f and blowing up the resulting edge until f becomes uniform. If no ordinary coordinate exists, then we perform a sequence of suitable polarizations which will either make f uniform or induce a refinement which makes one of the coordinates ordinary.

Algorithm $\text{uniformize}_{\mathbb{V}}(f)$

INPUT: a series $f \in \mathbb{V}_{\mathcal{X}}$

ACTION: refine the coordinates such that f becomes uniform

While f is not uniform do

If there exists an ordinary coordinate x_k in f then take k maximal and

$\text{prepare}_{\mathbb{V}}(f, x_k)$

If f is not uniform, then $\text{blowup}_{\mathbb{V}}(f)$

Else

$\mathfrak{N} := \text{polytope}_{\mathbb{V}}(f)$

Let k be minimal such that $|\mathfrak{N} \cap \mathfrak{X}_k x_{k+1}^{\alpha_{k+1}} \cdots x_n^{\alpha_n}| \geq 2$ for some $\alpha_{k+1}, \dots, \alpha_n$

For all $\mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2, \mathfrak{n}_2 \in \mathfrak{N}$ with $\mathfrak{m}_i/\mathfrak{n}_i = x_1^{\alpha_{i,1}} \cdots x_k^{\alpha_{i,k}}$ and $\alpha_{i,k} \neq 0$ do

$\text{polarize}((x_1^{\alpha_{1,1}} \cdots x_{k-1}^{\alpha_{1,k-1}})^{1/\alpha_{1,k}} / (x_1^{\alpha_{2,1}} \cdots x_{k-1}^{\alpha_{2,k-1}})^{1/\alpha_{2,k}})$

Pick $\mathfrak{m}, \mathfrak{n} \in \mathfrak{N}$ with $\mathfrak{m}/\mathfrak{n} = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ and $\alpha_i \neq 0$ and

$\text{polarize}(x_1^{\alpha_1} \cdots x_k^{\alpha_k})$

REMARK 23. We do not know of any effective test whether a given coordinate x_k is ordinary in f . For this reason, we will use a slightly weaker test. For every series f , we maintain a set O_f of coordinates which are ensured to be ordinary for f , and simply check whether x_k is in this set. Whenever x_k is refined, we add x_k to O_f . The coordinate x_k is removed from O_f , whenever an l -refinement $x_l := \mathfrak{m}(\tilde{\lambda} + \tilde{x}_l)$ or $x_l := \varphi + \mathfrak{m}\tilde{x}_l$ occurs, where $l > k$ and \mathfrak{m} or φ is not ordinary in x_k . Our slightly weaker test is still sufficient for making the termination proof below work.

THEOREM 24. *The algorithm uniformize is correct and terminates.*

PROOF. The algorithm is clearly correct. Let us first prove the termination modulo the termination of the subalgorithm `pseudo`. Assume for contradiction that it does not terminate on a given input f , but that each of its recursive invocations for lower dimensional series terminates.

We observe that all coordinates cannot indefinitely remain non ordinary. Indeed, the polarizations in the second part either strictly reduce the number of elements in \mathfrak{N}_f or end up by refining one of the coordinates, thereby making it ordinary in f .

Let k be maximal such that the coordinate x_k is refined infinitely many times. Modulo picking up the computation at a later point, we may assume without loss of generality that the coordinates x_{k+1}, \dots, x_n are never refined (although new asymptotic constraints may be imposed upon them). After one refinement of x_k , the coordinate x_k will then remain ordinary in f .

Consider the series expansion $f = f_0 + f_1 x_k + \dots$ of f in x_k after a call of `prepare $_{\mathbb{V}}$ (f, x_k)`. Then each of the coefficients f_i for which $\mathfrak{N}_f \cap x_1^{\mathbb{Z}} \cdots z_{k-1}^{\mathbb{Z}} x_k^i x_{k+1}^{\mathbb{Z}} \cdots x_n^{\mathbb{Z}} \neq \emptyset$ is uniform. In particular, if f is not uniform, then \mathfrak{N}_f contains at least two elements whose exponents in x_k differ. Consequently, the principal coordinate x_l for f satisfies $l \geq k$. We claim that we cannot have $l > k$. Otherwise, `blowup $_{\mathbb{V}}$ (f)` necessarily falls in a case which results in the uniformization of f , since the coordinate x_l is never refined. After return, the main algorithm then terminates.

Consequently, we are infinitely often in the situation that `blowup $_{\mathbb{V}}$ (f)` is called with x_k as the principal coordinate of f . By proposition 22, the Newton degree $\deg_{x_k < 1} f$ at least decreases by one for every two such calls. Since $\deg_{x_k < 1} f \in \mathbb{N}$, this cannot happen infinitely many times. This contradiction proves the termination of `uniformize` modulo the termination of `pseudo`.

To complete the proof, let us consider one of the recursive calls `pseudo(f, k)` and prove its termination. Without loss of generality, we may assume that none of the coordinates x_{k+1}, \dots, x_n is refined during the recursive calls of `uniformize $_{\mathbb{K}}$ ($f - \varphi$)`. Intuitively speaking, the termination of `pseudo(f, k)` is equivalent to the termination of `uniformize $_{\mathbb{K}}$ ($f - f_{< x_k}$)`, interspersed with a sequence of additional refinements.

More precisely, assume that the loop does not terminate. In a similar way as above, let $k' < k$ be maximal such that $x_{k'}$ is refined infinitely many times. Without loss of generality, we may assume that $x_{k'}$ remains ordinary in $f - f_{< x_k}$. At each recursive call of `uniformize $_{\mathbb{K}}$ ($f - \varphi$)`, each of the dominant monomials of $f - \varphi$ involves one of the coordinates x_{k+1}, \dots, x_n . At the first subsequent call of `blowup $_{\mathbb{K}}$ ($f - \varphi$)`, it follows that the dominant edge of $f - \varphi$ is actually a dominant edge of $f - f_{< x_k}$. Now in a similar way as above, it occurs infinitely often that $x_{k'}$ is the principal coordinate of $f - \varphi$ (and thus of $f - f_{< x_k}$) for this call. But then $\deg_{x_{k'} < 1} (f - f_{< x_k})$ decreases every two such calls, which leads to the desired contradiction. \square

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