Reliable homotopy methods for solving polynomial and analytic systems of equations

Both in computer algebra and in numerical analysis, the resolution of systems of polynomial equations is a central problem. In computer algebra, this problem is often tackled using rewriting techniques, such as the computation of Groebner bases [4, 6] or regular chains [14]. In numerical analysis, one of the most successful methods is based on numerical homotopies [21, 20, 1]. The idea is quite simple and goes as follows.

Starting with an input system, one first constructs a "simpler" system with the "same characteristics". For instance, given the system

$$\begin{cases} P_1(x,y) = x^2 - y^2 + x + 3 = 0 \\ P_2(x,y) = x^2 + 2\,x\,y + 7\,y^2 - 8\,y + 2 = 0, \end{cases} \tag{1}$$

we may for instance take

$$\begin{cases} Q_1(x,y) = x^2 - \alpha_1 = 0 & (\alpha_1 = 1 + \mathrm{i}) \\ Q_2(x,y) = y^2 - \alpha_2 = 0 & (\alpha_2 = 2 - \mathrm{i}) \end{cases} \tag{2}$$

for our simpler system. Both systems indeed have the "same characteristics" in the sense that $\deg Q_1 = \deg P_1 = 2$ and $\deg Q_2 = \deg P_2 = 2$. In particular, we expect both systems to admit the same number of solutions. By construction, the solutions $(x,\, y) \in \{(\sqrt{\alpha_1},\, \sqrt{\alpha_2}),\, (\sqrt{\alpha_1},\, -\sqrt{\alpha_2}),\, (-\sqrt{\alpha_1},\, \sqrt{\alpha_2}),\, (-\sqrt{\alpha_1},\, -\sqrt{\alpha_2})\}$ of (2) are easy to compute. Now the main idea is to continuously deform the second system into the first system using a homotopy:

$$\begin{cases} H_1^t(x,y) = (1-t)\,P_1(x,y) + t\,Q_1(x,y) = 0 \\ H_2^t(x,y) = (1-t)\,P_2(x,y) + t\,Q_2(x,y) = 0. \end{cases} \tag{3}$$

Indeed, at $t=1$ and $t=0$, the homotopy system (3) reduces to (2) and (1), respectively. In order to obtain the solutions of (1), it thus suffices to follow the solutions of (3) from $t=1$ until $t=0$, using standard numerical algorithms. For instance, knowing the approximate solutions of (1) at a given time $t = t_0$, we may find approximate solutions at $t = t_0 + \delta$ using Newton's method, starting with the approximations at $t = t_0$ as our *ansatz*. It is also possible to design higher order (e.g. Euler-type) methods, which allow us to take larger steps δ.

For generic non singular systems of a given dimension, it has been shown that numerical homotopy methods converge after a small number of steps [18, 19, 3]. From the practical point of view, existing implementations often outperform other known methods for the resolution of systems of polynomial equations [21, 20, 1]. However, traditional implementations of homotopy continuation were purely numeric, with the major drawback that the computations were not certified. One way to overcome this problem is to apply homotopy methods formally. A suitable algebraic framework in which this is possible was developed and implemented in [7, 8, 5]. However, this approach suffers from the fact that all computations have to be done with a potentially large precision.

More recently, various methods have been proposed and implemented to certify the numerical methods or results [15, 17, 11, 2, 9]. However, a lot of work remains to be done in this area. One major problem is that current implementations either tend to be fast and non certified, or slow and certified. Other efficiency problems occur when the system admits multiple solutions with medium or large multiplicities. Being a specialist in fast arithmetic and reliable computations, van der Hoeven is well placed to supervise the design of faster methods for reliable homotopy continuation. The design and implementation of such methods and possible generalizations is the main goal of the present PhD proposal.

More specifically, building on [11], the following theoretical questions are worth investigating:

- How to perform reliable homotopy continuations in the most efficient way?
- What are the most efficient methods to numerically compute multiple zeros and certify them?
- How to develop mixed numeric-exact homotopy methods? The typical output of such a method would be an algebraic representation for the zero set and small disjoint ball enclosures for the corresponding numerical solutions.
- How to generalize the existing theory to analytic instead of polynomial equations?

From the implementation point of view, there are also various interesting challenges. Some basic tools for reliable arithmetic have already been implemented in the MATHEMAGIX system [10]. We plan to work on the following practical issues:

- Development of algorithms for the efficient evaluation of the input polynomials for various numerical and reliable data types, possibly using automatic code generation.
- Implementation of additional reliable data types for the efficient certification of numeric homotopy steps, such as special kinds of Taylor models [16].
- Development of multi-threaded homotopy solvers inside the MATHEMAGIX system.
- Make the homotopy solvers benefit from hardware SIMD (Single Instruction Multiple Data) vector instructions using existing tools in the MATHEMAGIX system [13, 12].

Being part of the MATHEMAGIX system, we intend to distribute all implementations as free software under the GNU General Public Licence.

The proposal involves methods from various areas, each on which the adviser and the MAX team have renowned expertise : computer algebra, reliable numerical methods, fast arithmetic, computable complex analysis, singularity theory, high performance computing, software engineering, etc. We do not expect our candidate to master and contribute to each of these aspects. However, we do request expertise in computer algebra or in reliable computing, as well as good programming skills.

Depending on the background and interests of the PhD candidate, the project can take various directions, which we would all consider satisfactory :

- The main challenge remains the design and implementation of faster reliable methods for homotopy continuation. Our current competitors are [2, 9]. If the PhD candidate manages to design and implement faster algorithms (at least for certain types of polynomial systems), then we would consider this work to be very successful.
- For a more theoretically minded candidate, reliable algorithms for the resolution of analytic systems of equations might be another interesting outcome of the project. This would constitute a major step towards the automation of multivariate complex analysis.
- A very skilled programmer might yet give another turn to the project, by implementing high performance algorithms for numerical homotopy continuation which outperform all existing implementations. Depending on the algorithmic and technical novelties, this could be another useful contribution to the area.

The development of reliable computational software is a major theme for both DIGITEO and DIGICOSME. We think that our project will both lead to better theoretical insight into the difficult question of solving non linear equations in a reliable way, as well as better implementations which can freely be used by others.

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