On a conjecture of Hardy

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The conjecture

L-functions

– An L-function is a function constructed from $\mathbb{R}$ and x by $+, -, \times, /$, exp, log and algebraic functions. Example:

$$\frac{\exp e^{e^x \log^{24} x + x}}{\log^8 \log x + e^x} + e^{x^x \log^{1998} x}$$

– Hardy: germs of L-functions at ∞ form a totally ordered field.

– Many functions can be expanded w.r.t. scale of L-functions.

– Hardy: solutions to $E(x+1) = e^{E(x)}$ grow faster than every iterated exponential.

Question

– Is there an L-function, asymptotic to $(\log x \log\log x)^{inv}$?

– Liouville: $(\log x \log\log x)^{inv}$ is not equal to an L-function.

Grid-based series

Asymptotic scales

- S: ordered group (by $\prec\!\!\prec$) of positive germs at infinity, stable under exponentiation by reals.

- S finitely generated by $B = \{b_1, \ldots, b_n\}$, if

$$S = \{b_1^{\alpha_1} \cdots b_n^{\alpha_n} | \alpha_1, \ldots, \alpha_n \in \mathbb{R}\},$$

- B base, if $1 \prec\!\!\prec b_1 \prec\!\!\prec \cdots \prec\!\!\prec b_n$ and $\log b_1 \prec\!\!\prec \cdots \prec\!\!\prec \log b_n$.

- Example: $S = \{x^\alpha e^{x\beta} | \alpha, \beta \in \mathbb{R}\}$.

Grid-based series over field C

- $C[\![S]\!] = C[\![b_1; \cdots; b_n]\!]$ field of series

$$f = \sigma_0 \varphi(\sigma_1, \ldots, \sigma_k),$$

where $\varphi \in C[[\sigma_1, \ldots, \sigma_k]]$, $\sigma_0, \ldots, \sigma_k \in S$ and $\sigma_i \prec\!\!\prec 1$ for $1 \leqslant i \leqslant k$.

- Example: $e^x (1 - x^{-1} - x^{-x})^{-1} \in \mathbb{R}[\![x; e^x; x^x]\!]$.

- $\mathbb{R}[\![S]\!]^{conv}$: subfield of convergent series (i.e. φ convergent).

Lexicographical expansions

Lexicographical expansion of $f \in \mathbb{R}[\![b_1; \cdots ; b_n]\!]$

$$\begin{aligned} f &= \sum_{\alpha_n \in \mathbb{R}} f_{\alpha_n} b_n^{\alpha_n} \\ &\vdots \\ f_{\alpha_n, \ldots, \alpha_2} &= \sum_{\alpha_1 \in \mathbb{R}} f_{\alpha_n, \ldots, \alpha_1} b_1^{\alpha_1}. \end{aligned}$$

$f_{\alpha_n, \ldots, \alpha_{i+1}}$ both in $\mathbb{R}[\![b_1; \cdots ; b_i]\!]$ and $\mathbb{R}[\![b_1; \cdots ; b_{i-1}]\!][\![b_i]\!]$.

$$\begin{aligned} \frac{1}{(1-x^{-1})(1-e^{-x})} &= 1 + x^{-1} + x^{-2} + x^{-3} + \cdots \\ &+ e^{-x} + x^{-1}e^{-x} + x^{-2}e^{-x} + x^{-3}e^{-x} + \cdots \\ &\vdots \end{aligned}$$

Canonical decomposition

$$f = f^{\uparrow} + f^{c} + f^{\downarrow} = \sum_{\sigma \not\succ 1} f_s + f_1 + \sum_{\sigma \not\prec 1} f_s.$$

Lexicographically,

$$\begin{aligned} f^{\uparrow} &= \sum_{\alpha_n > 0} f_{\alpha_n} b_n^{\alpha_n} + \cdots + \sum_{\alpha_1 > 0} f_{0,\ldots,0,\alpha_1} b_0^{\alpha_1}; \\ f^{c} &= f_{0,\ldots,0} \\ f^{\downarrow} &= \sum_{\alpha_n < 0} f_{\alpha_n} b_n^{\alpha_n} + \cdots + \sum_{\alpha_1 < 0} f_{0,\ldots,0,\alpha_1} b_0^{\alpha_1}. \end{aligned}$$

Example:

$$\left[\frac{x^{100} e^{x}}{(1-x^{-1})(1-e^{-x})} \right]^{\uparrow} = \frac{x^{100} e^{x}}{1-x^{-1}} + x^{100} + x^{99} + \cdots + x.$$

Canonical bases

L-series

– $\mathbb{R}[\![S]\!]^L$: series constructed from $\mathbb{R}$, monomials $b_i^{\alpha_i}$, the field operations and left composition of infinitesimal L-series by $\exp z$, $\log(1+z)$ or algebraic series.

– L-series are both expressions and convergent series in $\mathbb{R}[\![S]\!]^{conv}$.

– Straightforward expansion algorithm for L-series.

– Iterated coefficients of L-series again L-series.

– If f is an L-series, then so are $f^{\uparrow}, f^{c}, f^{\downarrow}$.

B canonical base if

B1. $b_1 = \log_l x$ is an l-th iterated logarithm.

B2. $\log b_i \in \mathbb{R}[\![b_1; \cdots ; b_{i-1}]\!]^L$ et $(\log b_i)^{\uparrow} = \log b_i$, for all $i > 1$.

Example:

$$B = \{\log x, x, \exp[\tfrac{x}{\log x - 1}]\},$$

but not

$$\{x, e^{e^x}\} \quad \text{nor} \quad \{x, e^{x+x^{-1}}, e^{x^2}\}.$$

$B:$ dynamic canonical base containing x.

ALGORITHM expand

INPUT: An L-function f.

OUTPUT: f rewritten as an L-series in $\mathbb{R}[\![S]\!]^L$.

case $f \in \mathbb{R}$ or $f = x$.

Return f.

case $f = g \,\square\, h, \quad \square \in \{+, -, \times, /\}$.

Return expand$(g) \,\square\,$ expand(h).

case $f = \log(g)$.

Set $g :=$ expand(g).

Rewrite $g = c b_1^{\alpha_1} \cdots b_n^{\alpha_n} (1 + \varepsilon)$, where $c \in \mathbb{R}^*$ and $\varepsilon \prec\!\!\prec 1$.

If $\alpha_1 \neq 0$, add $\log b_1$ to B.

Return $\log c + \alpha_1 \log b_1 + \cdots + \alpha_n b_n + \log(1 + \varepsilon)$.

case $f = \exp(g)$.

Set $g :=$ expand(g).

If $l = \lim g \in \mathbb{R}$, return $e^l e^{g-l}$.

Test whether $g \asymp \log b_i$ for some $2 \leqslant i \leqslant n$.

Yes $\longrightarrow$ return b_i^lexpand$(e^{g - l \log b_i})$, where $l = \lim g/(\log b_i)$.

No $\longrightarrow$ add $e^{|g^\uparrow|}$ to B and return $(e^{|g^\uparrow|})^{\mathrm{sign}(g)} e^{g_0} e^{g^\downarrow}$.

case $f = \varphi(g)$, with φ algebraic.

Set $g := \text{expand}(g)$ and $l := \lim g$.

If $|l| = \infty$, return $\text{expand}(\psi(g^{-1}))$, where $\psi(z) \stackrel{\text{def}}{=} \varphi(z^{-1})$.

If $l \neq 0$, return $\text{expand}(\psi(g - l))$, où $\psi(z) \stackrel{\text{def}}{=} \varphi(z + l)$.

Rewrite $\varphi(z) = z^\alpha \psi(z^\beta)$, with $\alpha \in \mathbb{Q}, \beta \in \mathbb{Q}_+^*$ et $\psi \in \mathbb{R}[[z]]$.

If $\psi \neq 1$, return $\text{expand}(g^\alpha)\psi(\text{expand}(g^\beta))$.

Rewrite $g = c\sigma(1 + \varepsilon)$, with $c \in \mathbb{R}^*, \sigma \in S$ et $\varepsilon \prec\!\!\prec 1$.

Return $c^\alpha \sigma^\alpha [(1 + z)^\alpha \circ \varepsilon]$.

Theorem. *Let f be an L-function and B_0 a canonical base containing x. Then there exists a canonical base $B = \{b_1, \ldots, b_n\} \supseteq B_0$, such that f can be rewritten as an L-series in $\mathbb{R}[\![S]\!]^L$.*

Proof of the conjecture

Assume $g = (\log x \log\log x)^{inv} \asymp f$ for an L-function f.
There exists $B \supseteq \{\log\log x, \log x, x\}$, such that $\log f \in \mathbb{R}[\![S]\!]^L$.
Moreover, $(\log f)^{\uparrow}$ is an L-function.

Classical convergent expansion for $\log\log g = (xe^x)^{inv}$:

$$\log\log g = \log x - \log\log x + \frac{\log\log x}{\log x} + \cdots = \log x + \sum_{n=0}^{\infty} \frac{g_n}{\log^n x},$$

with coefficients $g_n \in \mathbb{R}[\log\log x]$. Hence

$$\log\log g \in \mathbb{R}[\![\log\log x; \log x]\!]^{conv}$$

and

$$\log g = \tfrac{x}{\log x} \exp g^{\downarrow} \in \tfrac{x}{\log x}\mathbb{R}[\![\log\log x; \log x]\!]^{conv}$$

is such that

$$(\log g)^{\uparrow} = \log g.$$

Thus $\log f$ and $\log g$ are both in $\mathbb{R}[\![S]\!]^{conv}$ and

$$e^{\varphi} \asymp e^{\psi} \Leftrightarrow \varphi^{\uparrow} = \psi^{\uparrow},$$

for $\varphi, \psi \in \mathbb{R}[\![S]\!]^{conv}$. Hence

$$f \asymp g \Rightarrow (\log f)^{\uparrow} = (\log g)^{\uparrow} = \log g,$$

whence $(\log f)^{\uparrow} = \log g$ and $g = \exp(\log g)$ are L-functions.
Contradiction with Liouville's result.