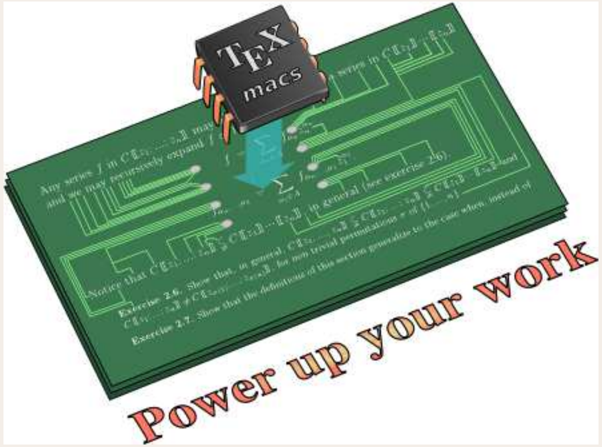
Asymptotic differential equations

In Honor of José Manuel Aroca



Tordesillas 2006

http://www.TeXmacs.org

A missing subject?

Algebraic geometry	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	?

- Hardy fields: Rosenlicht, Boshernitzan, Singer, etc.
- Pfaff systems: Khovanskii, Wilkie, Rolin™, etc.

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A missing subject?

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Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

- LNM 1888: Transseries and Real Differential Algebra
- Other work on `http://www.math.u-psud.fr/~vdhoeven`

Sufficiently closed models

Algebraic geometry	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	Real algebraic geometry + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + Valuation theory
$\downarrow$		$\downarrow$
Differential algebra	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R} \\ + \\ \mathbb{C}[[z^{\mathbb{Q}}]]\end{array} \\ \downarrow & & \downarrow \\ \text{Differential algebra} & \longrightarrow & \begin{array}{c}\text{Real differential algebra} \\ + \\ \text{Asymptotic differential algebra}\end{array} \end{array}$$

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Wild	$\longrightarrow$	Real differential algebra + Asymptotic differential algebra

Sufficiently closed models

$\mathbb{C}$	$\longrightarrow$	$\mathbb{R}$ + $\mathbb{C}[[z^{\mathbb{Q}}]]$
$\downarrow$		$\downarrow$
Wild	$\longrightarrow$	Maximal Hardy field + Asymptotic differential algebra

Sufficiently closed models

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \begin{array}{c} \mathbb{R} \\ + \\ \mathbb{C}[\![z^{\mathbb{Q}}]\!] \end{array} \\ \downarrow & & \downarrow \\ \textit{Wild} & \longrightarrow & \begin{array}{c} \mathbb{R}[\![\![x]\!]\!] \\ + \\ \text{Asymptotic differential algebra} \end{array} \end{array}$$

Sufficiently closed models

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \begin{array}{c}\mathbb{R}\\ + \\ \mathbb{C}[[z^{\mathbb{Q}}]]\end{array}\\
\downarrow & & \downarrow\\
\textit{Wild} & \longrightarrow & \begin{array}{c}\mathbb{R}[[[x]]]\\ + \\ \mathbb{C}[[[z]]]\end{array}
\end{array}
$$

What is a transseries?

$(x \succ 1)$

$$\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}+\frac{2}{\log x}\,\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\frac{\mathrm{e}^{x}}{x^{2}}+\cdots}+\mathrm{e}^{\sqrt{x}+\mathrm{e}^{\sqrt{\log x}+\mathrm{e}^{\sqrt{\log\log x}+\cdots}}}+\cdots$$

What is a transseries?

$(x \succ 1)$

$$\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots} + \frac{2}{\log x}\,\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots} + \mathrm{e}^{\sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \cdots}}} + \cdots$$

- Dahn & Göring
- Écalle

Examples of transseries

$$
\begin{aligned}
\frac{1}{1-x^{-1}-x^{-\mathrm{e}}} &= 1+x^{-1}+x^{-2}+x^{-\mathrm{e}}+x^{-3}+x^{-\mathrm{e}-1}+\cdots\\
\frac{1}{1-x^{-1}+\mathrm{e}^{-x}} &= 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots\\
-\mathrm{e}^{x}\int\frac{\mathrm{e}^{-x}}{x} &= \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots\\
\Gamma(x) &= \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{288\,x^{5/2}}+\cdots\\
\zeta(x) &= 1+2^{-x}+3^{-x}+4^{-x}+\cdots\\
\varphi(x) &= \frac{1}{x}+\varphi(x^{\pi})=\frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots\\
\psi(x) &= \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x})=\frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots
\end{aligned}
$$

Generalized power series

- C: constant field
- $(\mathfrak{M}, \preccurlyeq)$: totally ordered group of monomials
- $C[[\mathfrak{M}]]$: field of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with well-based support.

$$\mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots \quad \text{with} \quad \mathfrak{m}_1, \mathfrak{m}_2, \ldots \in \operatorname{supp} f \quad \text{is impossible}$$

- $C[\![\mathfrak{M}]\!]$: field of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with grid-based support.

$$\operatorname{supp} f \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_m\}^* \mathfrak{n}, \qquad \mathfrak{m}_1, \ldots, \mathfrak{m}_m \prec 1$$

Abstract fields of transseries

Totally ordered field $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ with a logarithm such that

T1. $\operatorname{dom}\log = \mathbb{T}^{>}$.

T2. $\log \mathfrak{m} \in \mathbb{T}_{\succ}$, for all $\mathfrak{m} \in \mathfrak{T}$, i.e. $\operatorname{supp}(\log \mathfrak{m}) \succ 1$.

T3. $\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \cdots$, for all $\varepsilon \in \mathbb{T}_{\prec}$.

Example. $\mathbb{L} = \mathbb{R}[[\mathfrak{L}]] = \mathbb{R}[[x^{\mathbb{R}}(\log x)^{\mathbb{R}}(\log_2 x)^{\mathbb{R}}\cdots]]$ with

$$\begin{aligned}\log(x^{\alpha_0}\cdots(\log_k x)^{\alpha_k}) &= \alpha_0 \log x + \cdots + \alpha_k \log_{k+1} x\\ \log(f) = \log(c_f\, \mathfrak{d}_f\, (1+\delta_f)) &= \log \mathfrak{d}_f + \log c_f + \log(1+\delta_f)\end{aligned}$$

The field of grid-based transseries in x

I) $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$ field of transseries $\Longrightarrow$ $\mathbb{T}_{\exp} = \mathbb{R}[[\mathfrak{T}_{\exp}]] \supseteq \mathbb{T}$ also

$$\mathfrak{T}_{\exp} \;=\; \exp(\mathbb{R}[[\mathfrak{T}]]_{\succ})$$

Example. $\mathrm{e}^{x^2 + \frac{x^2}{\log x} + \frac{x^2}{\log^2 x} + \cdots + x + \log \log x} \in \mathfrak{L}_{\exp}$

II) Increasing limits of fields of transseries are fields of transseries

$$\mathbb{T} = \mathbb{L} \cup \mathbb{L}_{\exp} \cup \mathbb{L}_{\exp,\exp} \cup \cdots$$

Operations on transseries

I) Unique strong exp-log differentiation on $\mathbb{T}$ with $x' = 1$

D5. $f \prec g \Rightarrow f' \prec g'$, for all $f, g \in \mathbb{T}$ with $g \not\asymp 1$.

D6. $f \succ 1 \Rightarrow (f > 0 \Rightarrow f' > 0)$, for all $f \in \mathbb{T}$.

II) Unique strong exp-log postcomposition δ with $g \in \mathbb{T}^{>,\succ}$ with $\delta x = g$

Δ5. $f \prec 1 \Rightarrow \delta(f) \prec 1$, for all $f \in \mathbb{T}$.

Δ6. $f \geqslant 0 \Rightarrow \delta(f) \geqslant 0$, for all $f \in \mathbb{T}$.

III) Each $g \in \mathbb{T}^{>,\succ}$ admits a compositional inverse.

Calculus with transseries

Taylor rule. $f, \delta \in \mathbb{T}$ with $\delta \prec x$ and $\mathfrak{m}^{\dagger} \delta \prec 1$ for all $\mathfrak{m} \in \operatorname{supp} f$. Then

$$f \circ (x + \delta) = f + f' \delta + \frac{1}{2} f'' \delta^2 + \cdots$$

Translagrange (Écalle). Notation:

$$f_{[M, N]} = \langle M \circ f, N \rangle = ((M \circ f)\, g)_{\asymp}$$

Let $M, N, \varepsilon \prec 1$ be exponential transseries, $f = x + \varepsilon$ and $g = f^{\text{inv}}$. Then

$$g_{[M, N']} = -f_{[N, M']}.$$

Asymptotic algebraic equations

Algebra	Asymptotic algebra

Asymptotic algebraic equations

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$P(f) = 0$	

Asymptotic algebraic equations

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$P(f)=0$	$P(f)=0, \quad (f \prec \mathfrak{v})$

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Asymptotic algebraic equations

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Newton polynomials

- $P \in C[[\mathfrak{M}]][F] \subseteq C[F][[\mathfrak{M}]]$
- $N_P = c_P \in C[F]$

Starting terms

- $\mathfrak{w} \prec \mathfrak{v}$ is a "starting monomial" $\Longleftrightarrow N_{P_{\times \mathfrak{w}}} \notin C F^{\mathbb{N}}$
- $c\,\mathfrak{w}$ is a "starting term" $(c \neq 0) \Longleftrightarrow N_{P_{\times \mathfrak{w}}}(c) = 0$

$$\begin{aligned} P_{\times \varphi}(f) &= P(\varphi f) \\ P_{+\varphi}(f) &= P(\varphi + f) \end{aligned}$$

Newton degree

$$
\begin{aligned}
\deg_{\preccurlyeq \mathfrak{v}} P &= \deg N_{P_{\times \mathfrak{v}}} \\
\deg_{\prec \mathfrak{v}} P &= \operatorname{val} N_{P_{\times \mathfrak{v}}}
\end{aligned}
$$

$$
\begin{aligned}
\deg_{\prec \mathfrak{w}} P &\leqslant \deg_{\prec \mathfrak{v}} P, && \mathfrak{w} \prec \mathfrak{v} \\
\deg_{\prec \mathfrak{v}} P_{+\varphi} &= \deg_{\prec \mathfrak{v}} P, && \varphi \prec \mathfrak{v} \\
\deg_{\prec \mathfrak{v}} P_{\times \mathfrak{w}} &= \deg_{\prec \mathfrak{v}\mathfrak{w}} P \\
\deg_{\prec \mathfrak{v}} (PQ) &= \deg_{\prec \mathfrak{v}} P + \deg_{\prec \mathfrak{v}} Q
\end{aligned}
$$

$$
\begin{aligned}
\deg_{\prec \varphi} P_{+\varphi} &= \mu(c_\varphi; N_{P_{\times \mathfrak{d}_\varphi}}) \\
\mu_{\prec \mathfrak{v}}(f; P) &= \deg_{\prec \mathfrak{v}} P_{+f}
\end{aligned}
$$

Newton polygon method

1. $\deg_{\prec \mathfrak{v}} P = d > 0$

   ($P = A_{+g}$ and g root modulo $\prec \mathfrak{v}$ of A)

2. If $d = 1$ then unique solution

3. Determine starting monomial $\mathfrak{m} \prec \mathfrak{v}$

4. Solve $N_{P_{\times \mathfrak{m}}}(c) = 0$ and set $\varphi := c\,\mathfrak{m}$

5. Refine $f = \varphi + \tilde{f}, \tilde{f} \prec \mathfrak{m} \longrightarrow 0 < \deg_{\prec \mathfrak{m}} \tilde{P} \leqslant d$ with $\tilde{P} = P_{+\varphi}$

   ($\tilde{P} = A_{+g+\varphi}$ and $g + \varphi$ root modulo $\prec \mathfrak{m}$ of A)

6. Return to step 1

Newton polygon method

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   $$\left(f - \frac{1}{1-z}\right)^2 = z^{10000}$$

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4. Solve $N_{P_{\times \mathfrak{w}}}(c) = 0$ and set $\varphi := c\,\mathfrak{w}$

   If $\mu_{N_{P_{\times}}}(c) = d$, then $\varphi :=$ unique solution to $\frac{\partial^{d-1} P}{\partial F^{d-1}}(\varphi) = 0, \varphi \prec \mathfrak{v}$

5. Refine $f = \varphi + \tilde{f}, \tilde{f} \prec \mathfrak{w} \longrightarrow 0 < \deg_{\prec \mathfrak{w}} \tilde{P} \leqslant d$ with $\tilde{P} = P_{+\varphi}$

   ($\tilde{P} = A_{+g+\varphi}$ and $g + \varphi$ root modulo $\prec \mathfrak{w}$ of A)

6. Return to step 1

Differential Newton polygon method

$$P(f)=p(f,f',\ldots,f^{(r)})=0,\qquad f\prec\mathfrak{v}$$

Starting monomials cannot directly be read of from "Newton polygon"

$$P=P_0+\cdots+P_d$$

Upward shifting

$P\uparrow$ unique differential polynomial with

$$(P\uparrow)(f \circ \mathrm{e}^x) = P(f) \circ \mathrm{e}^x$$

For instance:

$$\begin{aligned} F'\uparrow &= \frac{F'}{\mathrm{e}^x} \\ F''\uparrow &= \frac{F'' - F'}{\mathrm{e}^{2x}} \\ F'''\uparrow &= \frac{F''' - 3\,F'' + 2\,F'}{\mathrm{e}^{3x}} \\ &\vdots \end{aligned}$$

Differential Newton polynomial

Theorem. *There exists a unique* $N_P \in \mathbb{R}\{F\}$, *such that*

$$c_{P \uparrow_l} = N_P$$

for all sufficiently large l *and*

$$N_P \in \mathbb{R}[F]\,(F')^{\mathbb{N}}.$$

Definition. $\mathfrak{m} \prec \mathfrak{v}$ *is a starting monomial* $\Longleftrightarrow N_{P_{\times \mathfrak{m}}} \notin \mathbb{R}\,F^{\mathbb{N}}$

Example

$$\begin{aligned} P &= (F')^2 - F F'' \\ P{\uparrow} &= \frac{(F')^2 - F F'' + F F'}{\mathrm{e}^{2x}} \\ P{\uparrow\uparrow} &= \frac{F F'}{\mathrm{e}^{x}\,\mathrm{e}^{2\mathrm{e}^{x}}} + \frac{(F')^2 - F F'' + F F'}{\mathrm{e}^{2x}\,\mathrm{e}^{2\mathrm{e}^{x}}} \\ &\vdots \\ N_P &= F F' \end{aligned}$$

Consequence:

$$1 \prec L \prec \log_n x \Longrightarrow P(L) \sim \frac{L\,L'}{x}$$

Starting monomials

Lemma. *Given $i < j$ with $P_i \neq 0$, $P_j \neq 0$, there exists a unique (i, j)-equalizer $\mathfrak{e} \in \mathfrak{T}$ such that $N_{(P_i + P_j)\times\mathfrak{e}}$ is not homogeneous.*

Starting monomials

Lemma. *Given i with $P_i \neq 0$, we have*

$$\mathfrak{m} \textit{ is a starting monomial for } P_i(f) = 0$$

$$\Updownarrow$$

$$\mathfrak{m}^{\dagger} = \frac{\mathfrak{m}'}{\mathfrak{m}} \textit{ is a solution to } R_{P_i}(g) = 0 \textit{ modulo } \frac{1}{x \log x \log_2 x \cdots}$$

Solving asymptotic differential equations

Lemma. $\deg_{\prec \mathfrak{v}} P = 1 \Longrightarrow P(f) = 0,\ f \preccurlyeq \mathfrak{v}$ *admits at least one solution.*

Warning. Problem with almost multiple solutions

$$\begin{aligned}
f^2 - 2\, f' + \frac{1}{x^2} + \cdots + \frac{1}{(x \log x \cdots \log_l x)^2} &= 0, \quad (f \prec 1)\\
f^2 - 2\, \mathrm{e}^{-x} f' + \frac{1}{\mathrm{e}^{2x}} + \cdots + \frac{1}{(\mathrm{e}^x\, x \cdots \log_{l-1} x)^2} &= 0, \quad (f \prec 1)\\
f^2 - 2\, f' - 2\, f + 1 + \frac{1}{x^2} + \cdots + \frac{1}{(x \log x \cdots \log_{l-1} x)^2} &= 0, \quad (f \prec 1)\\
f^2 - 2\, f' + \frac{1}{x^2} + \cdots + \frac{1}{(x \log x \cdots \log_{l-1} x)^2} &= 0, \quad (f \prec 1)
\end{aligned}$$

Lemma. *"Unravelling process" is finite.*

Results

Theorem. *(1997) There exists a theoretical algorithm to find all solutions to an asymptotic algebraic differential equation.*

Theorem. *(1997) Let P be purely exponential of degree d and order r. There exists a constant $C_{r,d}$ such that any solution to $P(f)=0$ involves at most $C_{r,d}$ levels of iterated logarithms.*

Theorem. *(1997) Any general transseries solution to an algebraic differential equation with grid-based coefficients is again grid-based. Generalization of Grigoriev and Singer (1991).*

Corollary. $\zeta(x)$ *and* $f(x)=\frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\cdots$ *are differentially transcendental over* $\mathbb{R}$.

Intermediate value theorem

Theorem. *(2000) Given* $P \in \mathbb{T}\{F\}$ *and* $f < g \in \mathbb{T}$ *with* $P(f)\,P(g) < 0$. *Then there exists an* $h \in \mathbb{T}$ *with* $f < h < g$ *and* $P(h) = 0$.

1. Calculus with cuts $\hat{f} \in \hat{\mathbb{T}}$.
2. Classification of cuts and behaviour of $P(f)$ near a cut.
3. Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

Corollary. *Any* $P \in \mathbb{T}\{F\}$ *of odd degree admits a root in* $\mathbb{T}$.

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1. Calculus with cuts $\hat{f} \in \hat{\mathbb{T}}$.
2. Classification of cuts and behaviour of $P(f)$ near a cut.
3. Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

Corollary. *Any monic* $L \in \mathbb{T}[\partial]$ *admits a factorization with factors*

$$\partial - a \quad or$$

$$\partial^2 - (2\,a + b^\dagger)\,\partial + (a^2 + b^2 - a' + a\,b^\dagger) = (\partial - (a - b\,\mathrm{i} + b^\dagger))\,(\partial - (a + b\,\mathrm{i}))$$

Complex transseries

Main problem: define an ordering on $\mathbb{T} = \mathbb{C}[[\mathfrak{T}]] = \mathbb{C}[[[z]]]$.

Idea: $f > 0 \Longleftrightarrow c_f \in P_{\mathfrak{d}_f}$, with a set

$$P_{\mathfrak{m}} = \{c \in \mathbb{C} | (\operatorname{Re}(c\, e^{-\mathrm{i}\theta_{\mathfrak{m}}}) > 0) \vee (\operatorname{Re}(c\, e^{-\mathrm{i}\theta_{\mathfrak{m}}}) = 0 \wedge \operatorname{Im}(\epsilon_{\mathfrak{m}}\, c\, e^{-\mathrm{i}\theta_{\mathfrak{m}}}) > 0)\}$$

for each $\mathfrak{m} \in \mathfrak{T}$ $\longrightarrow$ unique $\mathbb{T}$ as strong field (see also: Bouffet).

Closure properties

Theorem. *(2001) Every asymptotic differential equation over* $\mathbb{T}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

Warning. $\mathbb{T}$ is not differentially algebraically closed

$$\begin{aligned} f^3 + (f')^2 + f &= 0 \\ f^3 + f &\neq 0 \end{aligned}$$

Rather desingularize vector fields? Panazzolo, etc.

Closure properties

Theorem. *(2001) Every asymptotic differential equation over* $\mathbb{T}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

Corollary. $\mathbb{T}$ *is Picard-Vessiot closed.*

Remark. No Grigoriev & Singer type undecidability results.

Remark. Zero-test algorithm for polynomials in power series solutions to algebraic differential equations.

Model theory

with Matthias Aschenbrenner & Lou van den Dries

Question: generalizations to H-fields and asymptotic fields?

Model theory

Warning. Fields $\mathcal{K}$ with a "gap" of the form $\hat{\gamma} = \frac{1}{x \log x \log_2 x \cdots}$ admit two Liouvillian extensions

$$\begin{aligned} \mathcal{K}_1 &= \mathcal{K}[\textstyle\int \hat{\gamma}], \qquad \textstyle\int \hat{\gamma} \succ 1 \\ \mathcal{K}_2 &= \mathcal{K}[\textstyle\int \hat{\gamma}], \qquad \textstyle\int \hat{\gamma} \prec 1 \end{aligned}$$

Notation. $\hat{\lambda} = -\hat{\gamma}^{\dagger} = \frac{1}{x} + \frac{1}{x \log x} + \cdots$, $\hat{\rho} = 2\,\hat{\lambda}' - \hat{\lambda}^2 = \frac{1}{x^2} + \frac{1}{x^2 \log^2 x} + \cdots$.

Theorem. *(2003) There exists a field of well-based transseries* $\mathbb{T}$, *such that* $\hat{\rho} \in \mathbb{T}$, *but* $\hat{\lambda} \notin \mathbb{T}$.

Theorem. *(2006)* N_P *well-defined for asymptotic fields* $\mathcal{K} \not\ni \hat{\rho}$.

On the special status of $\hat{\rho}$

Theorem 1. *For any* $P \in \mathbb{R}\{F\}$, *the first* ω *terms of* $P(\hat{\lambda})$ *are either "similar" to* $\hat{\lambda}$ *or to* $\hat{\rho}$. *(Écalle, 1992)*

Theorem 2. *For any* $P \in \mathbb{R}\{F\}$ *such that* $P(\hat{\lambda}) = \frac{1}{x^k} + \frac{1}{x^k \log^k x} + \cdots$, *we have either* $k = 1$ *or* $k = 2$.

Theorem 3. *Given* $P \in K[F_0, \ldots, F_r]$ *with*

$$P(F_0 - 1, F_1, \ldots, F_r) - P(F_0, F_1 - F_0, F_2 - 3\,F_1 + 2\,F_0, \ldots) \in K,$$

we have $P \in K\,F_0^2 + K\,F_1$.

Theorem 4. *The identity*

$$P(F_0 - 1, F_1, F_2, \ldots) = P(F_0, F_1 - F_0, F_2 - 3\,F_1 + 2\,F_0, \ldots)$$

is verified for

$$P \;=\; F_0^{-1} \sum_{k, i_1, \ldots, i_k} (-1)^{i_1 + \cdots + i_k} \binom{i_1 + \cdots + i_k + k}{i_1 + 1, \ldots, i_k + 1} \frac{F_{i_1}}{F_0^{i_1 + 1}} \cdots \frac{F_{i_k}}{F_0^{i_k + 1}}.$$

Integral transseries

$$f' - f = \frac{1}{z} + f^2$$

$$(\mathrm{e}^{-z} f)' = \frac{\mathrm{e}^{-z}}{z} + \mathrm{e}^{-z} f^2$$

$$f = \mathrm{e}^{z} \int \left(\frac{\mathrm{e}^{-z}}{z} + \mathrm{e}^{-z} f^2 \right)$$

$$f = \mathrm{e}^{z} \int \frac{\mathrm{e}^{-z}}{z} + \mathrm{e}^{z} \int \mathrm{e}^{-z} \left(\mathrm{e}^{z} \int \frac{\mathrm{e}^{-z}}{z}\right)^2 + \cdots$$

$$\frac{1}{z^{-1} \mathrm{e}^{z^2} - 2 \int \mathrm{e}^{z^2}} = \cdots$$

Integral transseries

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$$f = \mathrm{e}^z \int \left(\frac{\mathrm{e}^{-z}}{z} + \mathrm{e}^{-z} f^2 \right)$$

$$f = \mathrm{e}^z \int_\infty^z \frac{\mathrm{e}^{-z'}}{z'} \mathrm{d} z' + \mathrm{e}^z \int_\infty^z \mathrm{e}^{-z'} (\mathrm{e}^{z'} \int_\infty^{z'} \frac{\mathrm{e}^{-z''}}{z''} \mathrm{d} z'')^2 \mathrm{d} z' + \cdots$$

$$\frac{1}{z^{-1} \mathrm{e}^{z^2} - 2 \int \mathrm{e}^{z^2}} = \cdots$$

HAPPY BIRTHDAY