# Ball arithmetic

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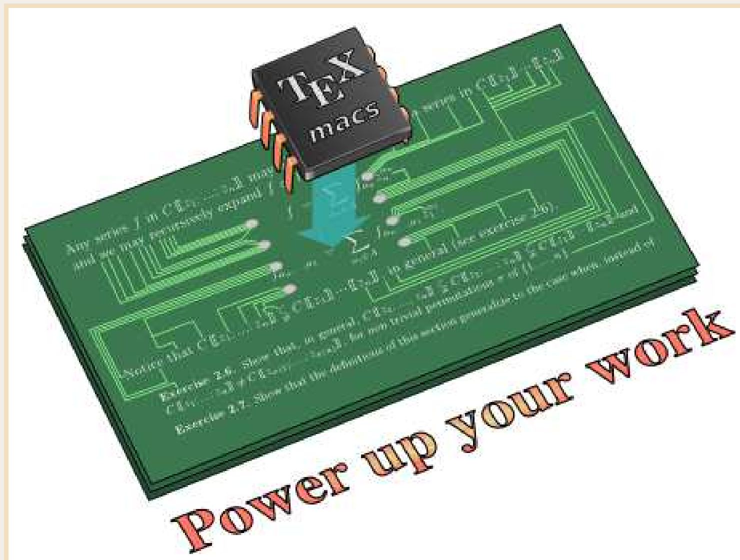


**ACA, Vlora, 2010**

http://www.TeXmacs.org

# Objectives

**Long term program:**

- Understand the practical complexity of computable analysis.
- Design fast algorithms for common analytic computations.
- MATHEMAGIX: a free "computer analysis" system.
- Make computable analysis "competitive" with numerical analysis.

**Existing software:**

- Computable real numbers. Mostly restricted to numbers
- Interval arithmetic. Mostly double precision
- Fast arithmetic. Applications mostly algebraic or discrete
- Other libraries. Mostly specialized, e.g. find roots of $P \in \mathbb{C}[z]$

# The numerical hierarchy

## Mathematical level

$x \in \mathbb{R}^{\text{com}} \leftrightsquigarrow \exists \check{x}: \varepsilon \in \mathbb{Q}^{>} \overset{\text{comp}}{\rightarrow} \tilde{x} \in \mathbb{Q}$, $|\tilde{x} - x| \leqslant \varepsilon$

## Reliable level

$x^{\boxminus} = [x_l, x_r] \in \mathcal{I}(\mathbb{D})$

## Numerical level

$x \in \mathbb{D}_{p,e} = \{-2^{p-1}, \ldots, 0, \ldots, 2^{p-1} - 1\} \, 2^{\{-2^{e-1}, \ldots, 0, \ldots, 2^{e-1}-1\}}$

$\mathbb{D}_{52,12} \leftrightsquigarrow$ "double precision IEEE 784 number"

↯ Correct rounding

## Arithmetic level

$x \in \mathbb{Z}$ or $x \in \{-2^{p-1}, \ldots, 0, \ldots, 2^{p-1} - 1\}$

# The numerical hierarchy

## Mathematical level

$x\in\mathbb{R}^{\text{com}} \leftrightsquigarrow \exists \check{x}\colon \varepsilon\in\mathbb{D}^{>} \xrightarrow{\text{comp}} \tilde{x}\in\mathbb{D},\ |\tilde{x}-x|\leqslant\varepsilon$ $\quad(\mathbb{D}=\mathbb{Z}\,2^{\mathbb{Z}})$

## Reliable level

$x^{\vdash\!\dashv}=[x_l,x_r]\in\mathcal{I}(\mathbb{D})$

## Numerical level

$x\in\mathbb{D}_{p,e}=\{-2^{p-1},\ldots,0,\ldots,2^{p-1}-1\}\,2^{\{-2^{e-1},\ldots,0,\ldots,2^{e-1}-1\}}$

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## Reliable level

$x^{\circ} = \{\tilde{x} \in \mathbb{R}: |\tilde{x} - x_c| \leqslant x_r\} \in \mathcal{B}(\mathbb{D}, \mathbb{D})$

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$x \in \mathbb{D}_{p,e} = \{-2^{p-1}, \ldots, 0, \ldots, 2^{p-1} - 1\}\, 2^{\{-2^{e-1}, \ldots, 0, \ldots, 2^{e-1} - 1\}}$

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$f: \mathbb{R} \to \mathbb{R} \rightsquigarrow f^{\circ}: \mathcal{B}(\mathbb{R}, \mathbb{R}) \to \mathcal{B}(\mathbb{R}, \mathbb{R})$, $\forall \tilde{x} \in x^{\circ}, f(\tilde{x}) \in f^{\circ}(x^{\circ})$.

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$f : \mathbb{R}^{\text{com}} \to \mathbb{R}^{\text{com}} \rightsquigarrow f^{\circ} : \mathcal{B}(\mathbb{D}, \mathbb{D}) \to \mathcal{B}(\mathbb{D}, \mathbb{D}), \; \forall \tilde{x} \in x^{\circ}, \; f(\tilde{x}) \in f^{\circ}(x^{\circ}).$

$x \in \mathbb{R}^{\text{com}} \leftrightsquigarrow \exists \check{x} : n \in \mathbb{N} \overset{\text{comp}}{\longmapsto} \check{x}_n \in \mathcal{B}(\mathbb{D}, \mathbb{D}), \; x \in \check{x}_n, \; \lim_{n \to \infty} (\check{x}_n)_r = 0$

## Numerical level

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## Arithmetic level

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# Matrix multiplication

## Mathematical level

$M, N \in (\mathbb{R}^{\text{com}})^{n \times n}$, $M N$?

## Reliable level

$M^\circ, N^\circ \in \mathcal{B}(\mathbb{D}, \mathbb{D})^{n \times n}$, $M N$?

## Numerical level

$M, N \in \mathbb{D}_{p,e}^{n \times n}$, $M N$?

- $\mathbb{D}_{p,e} = \mathbb{D}_{52,12} \rightsquigarrow$ Blas
- $p > 52$, $\mathbb{D}^{n \times n} \cong \mathbb{Z}^{n \times n} 2^{\mathbb{Z}}$

## Arithmetic level

$M, N \in \mathbb{Z}^{n \times n}$, $M N$

FFT or Chinese remaindering depending on $\frac{n}{p}$, with $p =$ bit precision

# Matrix multiplication

## Mathematical level

$M, N \in (\mathbb{R}^{\mathrm{com}})^{n \times n} \cong (\mathbb{R}^{n \times n})^{\mathrm{com}}$

$M \in (\mathbb{R}^{\mathrm{com}})^{n \times n} \leftrightsquigarrow \exists \check{M} : \varepsilon \in \mathbb{D}^{>} \overset{\mathrm{comp}}{\longmapsto} \tilde{M} \in \mathbb{D}^{n \times n}$, $|\tilde{M} - M| \leqslant \varepsilon$, sup-norm on $\mathbb{R}^{n \times n}$

## Reliable level

$M^{\circ}, N^{\circ} \in \mathcal{B}(\mathbb{D}, \mathbb{D})^{n \times n}$, $M N$?

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## Reliable level

$M^{\circ}, N^{\circ} \in \mathcal{B}(\mathbb{D}, \mathbb{D})^{n \times n} \cong \mathcal{B}(\mathbb{D}^{n \times n}, \mathbb{D}^{n \times n}) \cong \mathcal{B}(\mathbb{D}^{n \times n}, \mathbb{D})$

## Numerical level

$M, N \in \mathbb{D}_{p,e}^{n \times n}$, $M N$?

- $\mathbb{D}_{p,e} = \mathbb{D}_{52,12} \rightsquigarrow$ Blas
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## Arithmetic level

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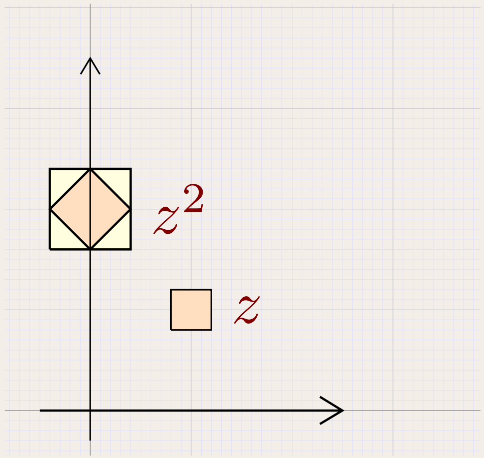
FFT or Chinese remaindering depending on $\frac{n}{p}$, with $p$ = bit precision

# Intervals versus balls

| | Intervals | Balls |
|---|---|---|
| Representation | $[2, \infty]$ | $\{z \in \mathbb{C}: \lvert z \rvert \leqslant 1\}$ |
| Hardware support | IEEE 754<br>$x +^{\downarrow} y = -((-x) +^{\uparrow} (-y))$ | not yet<br>$\Delta(y) = (\lvert y \rvert +^{\uparrow} 2^{-2^{e-1}}) \cdot^{\uparrow} 2^{-p}$ |
| Efficiency | end-points at full precision<br>multiplication $\rightsquigarrow$ branching | radius at single precision<br>completely vectorial |
| Standardization | correct rounding | explicit formulas for operations |
| Standardness | computer science | mathematics |
| Recommendation | algorithms that require subdivision of space | approximation of numbers |
| Predicates | values in $\mathcal{I}_{\{0,1\}}$ | values in $\{0, 1\}$<br>also, $=: (\mathbb{R}^{\text{com}})^2 \rightarrow \{0, 1\}^{\text{rcom}}$ |

# The wrapping effect and the radius type



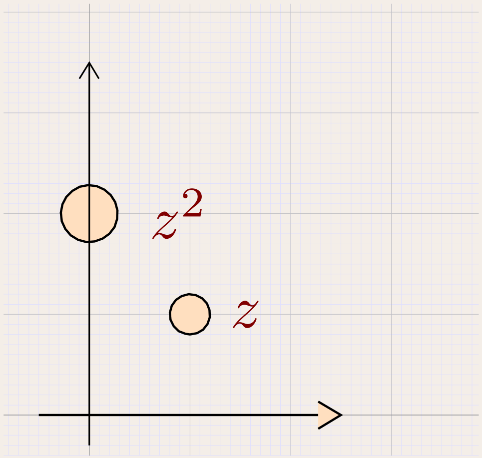




**Figure.** Illustration of the computation of $z^2$ using interval and ball arithmetic, for $z = 1 + \mathrm{i}$.

⇑ **Intervals**

```
Mmx] use "analyziz";
Mmx] one == interval (1.0, next_above 1.0);
Mmx] z1 == complex (one, one)
Mmx] pow (u, n) == if n=1 then u else u * pow (u, n-1);
Mmx] [ pow (z1, 8*i) || i in 1 to 10 ]
Mmx]
```

⇑ **Balls**

```
Mmx] z2 == ball complex (1.0, 1.0)
Mmx] [ pow (z2, 8*i) || i in 1 to 10 ]
Mmx]
```

# Reducing the wrapping effect

## Use divide and conquer algorithms

I.e. keep depth of the arithmetic circuit small

$$z^n = z^{\lfloor \frac{n}{2} \rfloor} z^{\lceil \frac{n}{2} \rceil}$$

```
Mmx] binpow (u, n) ==
       if n = 1 then u else binpow (u, n div 2) * binpow (u, (n+1) div 2);
```

```
Mmx] [ pow (z1, 8*i), binpow (z1, 8*i) || i in 1 to 10 ]
```

```
Mmx]
```

# Reducing the wrapping effect

## Use divide and conquer algorithms

I.e. keep depth of the arithmetic circuit small

$$z^n \;=\; z^{\lfloor \frac{n}{2} \rfloor} z^{\lceil \frac{n}{2} \rceil}$$

## Changes of coordinates

Enclose vectors by products $M V^{\vdash\!\dashv}$, $M \in \mathbb{D}^{n \times n}$, $V \in \mathcal{I}_{\mathbb{D}}^n$

Useful for differential equations, e.g.

$$f' = \begin{pmatrix} \cos' \\ \sin' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \\ \sin \end{pmatrix} = A\, f$$

Works because rounding errors all accumulate at the same side:

$$\begin{aligned} f(t_0) &= M_0\, V_0^{\vdash\!\dashv} \\ f(t_1) &= \Delta_{t_0,t_1} (M_0\, V_0^{\vdash\!\dashv}) = \overline{\Delta_{t_0,t_1}}\, M_0\, (V_0^{\vdash\!\dashv} + \varepsilon^{\vdash\!\dashv}) \end{aligned}$$

Not suited for general purpose complex ball arithmetic, because of sums

# Quality and condition number

## Condition number

$$
\begin{aligned}
f\colon \mathbb{R}^n &\to \mathbb{R}^m \\
\kappa_f(x) &= \lim_{\varepsilon \to 0} \sup_{\|\delta\| = \varepsilon} \frac{\|f(x+\delta) - f(x)\|}{\|f(x)\|} \Big/ \frac{\|\delta\|}{\|x\|}
\end{aligned}
$$

## Linear algebra

$$
\begin{aligned}
\kappa(M) &= \kappa_{M^{-1}\cdot}(x) \\
&= \|M\| \|M^{-1}\|. \qquad \text{operator norms}
\end{aligned}
$$

## Integration of a dynamical system

$$
\begin{aligned}
f' &= \Phi(f) \\
\kappa(\Phi, f(0), 0, t) &= \max_{0 \leqslant u \leqslant t} \kappa(\Delta_{0,u}(f(0)))
\end{aligned}
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\end{aligned}
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## Integration of a dynamical system

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f' &= \Phi(f) \\
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\end{aligned}
$$

# Efficient ball arithmetic

## Real numbers

$$\mathcal{B}(x_c, x_r) \cdot \mathcal{B}(y_c, y_r) \;=\; \mathcal{B}(x_c \cdot^{\updownarrow} y_c, (|x_c| +^{\uparrow} x_r) \cdot^{\uparrow} y_r +^{\uparrow} x_r \cdot^{\uparrow} |y_c| +^{\uparrow} \Delta(x_c \cdot^{\updownarrow} y_c))$$

## Matrices

$$M^{\circ} N^{\circ} \;=\; \mathcal{B}(M_c N_c, (|M_c| + M_r) N_r + M_r |N_c| + \Delta)$$

## Preconditioning

$$\begin{pmatrix} 1.0000e10 & 2.1000e5 & 6.3333e8 \\ 2.1111e16 & 1.1428e10 & 3.9876e12 \\ 2.2187e7 & 1.2134e2 & 9.8765e5 \end{pmatrix} \begin{pmatrix} 1.2222e10 & 4.3245e6 \\ 1.2345e13 & 2.3456e8 \\ 2.4325e12 & 5.3235e8 \end{pmatrix}$$

## Subdivision

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

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## Matrices

$$M^{\circ} N^{\circ} = \mathcal{B}(M_c N_c, [(\|M_c\| + \|M_r\|) \|N_r\| + \|M_r\| \|N_c\| + \Delta] \, \Omega)$$

$$\Omega = \begin{pmatrix} \mathcal{B}(0,1) & \cdots & \mathcal{B}(0,1) \\ \vdots & & \vdots \\ \mathcal{B}(0,1) & \cdots & \mathcal{B}(0,1) \end{pmatrix}$$

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$$\begin{pmatrix} 1.0000e10 & 2.1000e9 & 6.3333e10 \\ 2.1111e16 & 1.1428e13 & 3.9876e14 \\ 2.2187e7 & 1.2134e6 & 9.8765e7 \end{pmatrix} \begin{pmatrix} 1.2222e10 & 4.3245e6 \\ 1.2345e9 & 2.3456e4 \\ 2.4325e10 & 5.3235e6 \end{pmatrix}$$

## Subdivision

$$\left(\begin{array}{ccc|ccc} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{array}\right)$$

# Hansen's method

## Inversion of a matrix

$$
\begin{aligned}
M^\circ &\in \mathcal{B}(\mathbb{D},\mathbb{D})^{n\times n}\\
N_c &:= \mathrm{fl}_p(M_c^{-1})\\
E^\circ &:= 1 - M^\circ N_c\\
(1-E^\circ)^{-1} &:= 1 + E^\circ + \frac{\|E^\circ\|^2}{1-\|E^\circ\|}\begin{pmatrix} \mathcal{B}(0,1) & \cdots & \mathcal{B}(0,1)\\ \vdots & & \vdots \\ \mathcal{B}(0,1) & \cdots & \mathcal{B}(0,1)\end{pmatrix}\\
(M^\circ)^{-1} &:= N_c\,(1-E^\circ)^{-1}
\end{aligned}
$$

## Large $p$

$$\|E^\circ\| \approx \kappa(M)\,2^{-p}$$

## Increased quality

$$
M = \begin{pmatrix} 1 & K & & & \\ & 1 & K & & \\ & & \ddots & \ddots & \\ & & & 1 & K \\ & & & & 1 \end{pmatrix}
$$

# Hansen's method

## Inversion of a matrix

$$\begin{aligned} M^\circ &\in \mathcal{B}(\mathbb{D},\mathbb{D})^{n\times n} \\ N_c &:= \mathrm{fl}_p(M_c^{-1}) \\ E^\circ &:= 1 - M^\circ N_c \\ (1-E^\circ)^{-1} &:= 1 + E^\circ + \frac{\|E^\circ\|^2}{1-\|E^\circ\|}\begin{pmatrix} \mathcal{B}(0,1) & \cdots & \mathcal{B}(0,1) \\ \vdots & & \vdots \\ \mathcal{B}(0,1) & \cdots & \mathcal{B}(0,1) \end{pmatrix} \\ (M^\circ)^{-1} &:= N_c\,(1-E^\circ)^{-1} \end{aligned}$$

## Large $p$

$$\|E^\circ\| \approx \kappa(M)\,2^{-p}$$

## Increased quality

$$(1-E^\circ)^{-1} := (1+E^\circ)\,(1+(E^\circ)^2)\cdots(1+(E^\circ)^{2^{p-1}}) + O(\|E^\circ\|^{2^p})$$

# Efficient arithmetic in $\mathcal{B}(\mathbb{D},\mathbb{D})[[z]]^{\mathrm{com}}$

## ⇑ Floating point coefficients at precision $p = 128$

```
Mmx] use "analyziz";
Mmx] bit_precision := 128;
Mmx] time_mode? := true;
Mmx] z1 == series (0.0, 1.0);
Mmx] B1 == exp (exp z1 - 1)
Mmx] B1[5000]
```

## ⇑ Ball coefficients at precision $p = 128$

```
Mmx] z2 == series (ball 0.0, ball 1.0);
Mmx] B2 == exp (exp z2 - 1);
Mmx] B2[5000]
```

## ⇑ Floating point coefficients at precision $p = 256$

```
Mmx] bit_precision := 256;
Mmx] z3 == series (0.0, 1.0);
Mmx] B3 == exp (exp z3 - 1);
Mmx] B3[5000]
```

## ⇑ Ball coefficients at precision $p = 256$

```
Mmx] z4 == series (ball 0.0, ball 1.0);
Mmx] B4 == exp (exp z4 - 1);
Mmx] B4[5000]
```

# Conclusion

- Tradeoff between efficiency and quality; condition number.
- At high precision, certification should be of neglectible cost.
- High precision $\leftrightarrow$ Algebraic complexity

  Low precision $\leftrightarrow$ Geometry, condition number