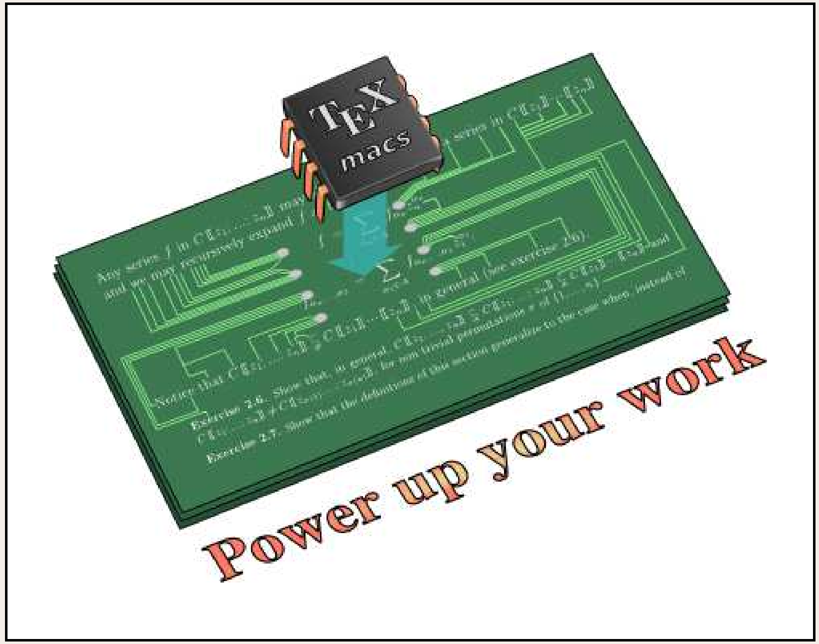
Asymptotic differential equations

Lecture 2: transserial Hardy fields



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Cuts in the transseries

Definition: a cut of $\mathbb{T}$ is an open interval $I \subseteq \mathbb{T}$ with $x < y \in I \Rightarrow x \in I$

Special cuts

- $\hat{\mho} = \mathbb{T}$
- $\hat{m} = \{f \in \mathbb{T} : \exists g \in \mathbb{R}, f \preccurlyeq g\}$
- $\hat{\varkappa}_l = \{f \in \mathbb{T} : \forall k, \exp_k \circ f \circ \log_k < \exp_l x\}$, for each $l \in \mathbb{Z}$
- Cuts in $\mathbb{R}$ (don't exist)
- Serial cuts $\hat{f} \in \mathbb{T}^{\mathrm{wb}}$, $\forall g \lhd \hat{f}, g \in \mathbb{T}$.

Proposition. *Each $\hat{f} \in \hat{\mathbb{T}}$ admits a unique nested expansion of one and only one of the following forms:*

$$
\begin{aligned}
\hat{f} &\in \mathbb{T}; \\
\hat{f} &= \pm\hat{\mho}; \\
\hat{f} &= \varphi_0 + \epsilon_0 \mathrm{e}^{\varphi_1 + \epsilon_1 \mathrm{e}^{\cdot^{\cdot^{\cdot \varphi_{l-1} + \epsilon_{l-1} \mathrm{e}^{\hat{\varkappa}_l}}}}} && (l \in \mathbb{Z}); \\
\hat{f} &= \varphi_0 + \epsilon_0 \mathrm{e}^{\varphi_1 + \epsilon_1 \mathrm{e}^{\cdot^{\cdot^{\cdot \varphi_{l-1} + \epsilon_{l-1} \mathrm{e}^{\hat{c}}}}}} && (\hat{c} \in \hat{C} \setminus C); \\
\hat{f} &= \varphi_0 + \epsilon_0 \mathrm{e}^{\varphi_1 + \epsilon_1 \mathrm{e}^{\cdot^{\cdot^{\cdot \varphi_{l-1} + \epsilon_{l-1} \mathrm{e}^{\hat{g}}}}}} && (\hat{g} \textit{ serial cut}); \\
\hat{f} &= \varphi_0 + \epsilon_0 \mathrm{e}^{\varphi_1 + \epsilon_1 \mathrm{e}^{\varphi_2 + \epsilon_2 \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}},
\end{aligned}
$$

with $\epsilon_0, \epsilon_1, \ldots \in \{-1, 1\}$.

Intermediate value theorem

Theorem. *(2000) Given* $P \in \mathbb{T}\{F\}$ *and* $f < g \in \mathbb{T}$ *with* $P(f)\, P(g) < 0$*. Then there exists an* $h \in \mathbb{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

1. Calculus with cuts $\hat{f} \in \hat{\mathbb{T}}$.
2. Classification of cuts and behaviour of $P(f)$ near a cut.
3. Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

Corollary. *Any* $P \in \mathbb{T}\{F\}$ *of odd degree admits a root in* $\mathbb{T}$*.*

Intermediate value theorem

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1. Calculus with cuts $\hat{f} \in \hat{\mathbb{T}}$.
2. Classification of cuts and behaviour of $P(f)$ near a cut.
3. Newton polygon method for shrinking interval on which a sign change occurs and whose end-points are cuts.

Corollary. *Any monic $L \in \mathbb{T}[\partial]$ admits a factorization with factors*

$$\partial - a \quad \textit{or}$$

$$\partial^2 - (2\,a + b^{\dagger})\,\partial + (a^2 + b^2 - a' + a\,b^{\dagger}) = (\partial - (a - b\,\mathrm{i} + b^{\dagger}))\,(\partial - (a + b\,\mathrm{i}))$$

Complex transseries

Main problem: define an ordering on $\tilde{\mathbb{T}} = \mathbb{C} [\![\mathfrak{T}]\!] = \mathbb{C} [\![\![z]\!]\!]$.

Idea: $f > 0 \Longleftrightarrow c_f \in P_{\mathfrak{d}_f}$, with a set

$$P_{\mathfrak{m}} = \{c \in \mathbb{C} | (\mathrm{Re}\,(c\,e^{-\mathrm{i}\theta_{\mathfrak{m}}}) > 0) \vee (\mathrm{Re}\,(c\,e^{-\mathrm{i}\theta_{\mathfrak{m}}}) = 0 \wedge \mathrm{Im}(\epsilon_{\mathfrak{m}}\,c\,e^{-\mathrm{i}\theta_{\mathfrak{m}}}) > 0)\}$$

for each $\mathfrak{m} \in \mathfrak{T} \longrightarrow$ unique $\tilde{\mathbb{T}}$ as strong field (see also: Bouffet).

Closure properties

Theorem. *(2001) Every asymptotic differential equation over* $\tilde{\mathbb{T}}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

Warning. $\tilde{\mathbb{T}}$ is not differentially algebraically closed

$$\begin{aligned} f^3 + (f')^2 + f &= 0 \\ f^3 + f &\neq 0 \end{aligned}$$

Rather desingularize vector fields? Cano, Panazzolo, etc.

Closure properties

Theorem. *(2001) Every asymptotic differential equation over* $\tilde{\mathbb{T}}$ *of Newton degree* d *admits at least* d *solutions (when counting with multiplicities). Moreover, it suffices to add iterated logarithms to the asymptotic scale.*

Corollary. $\tilde{\mathbb{T}}$ *is Picard-Vessiot closed.*

Remark. No Grigoriev & Singer type undecidability results.

Remark. Zero-test algorithm for polynomials in power series solutions to algebraic differential equations.

Model theory

with Matthias Aschenbrenner & Lou van den Dries

Question: generalizations to H-fields and asymptotic fields?

Model theory

Warning. Fields $\mathcal{K}$ with a "gap" of the form $\hat{\gamma} = \frac{1}{x \log x \log_2 x \cdots}$ admit two Liouvillian extensions

$$\begin{aligned} \mathcal{K}_1 &= \mathcal{K}[\textstyle\int \hat{\gamma}], \qquad \int \hat{\gamma} \succ 1 \\ \mathcal{K}_2 &= \mathcal{K}[\textstyle\int \hat{\gamma}], \qquad \int \hat{\gamma} \prec 1 \end{aligned}$$

Notation. $\hat{\lambda} = -\hat{\gamma}^{\dagger} = \frac{1}{x} + \frac{1}{x \log x} + \cdots$, $\hat{\rho} = 2\,\hat{\lambda}' - \hat{\lambda}^2 = \frac{1}{x^2} + \frac{1}{x^2 \log^2 x} + \cdots$.

Theorem. *(2003) There exists a field of well-based transseries $\mathbb{T}$, such that $\hat{\rho} \in \mathbb{T}$, but $\hat{\lambda} \notin \mathbb{T}$.*

Theorem. *(2006) N_P well-defined for asymptotic fields $\mathcal{K} \not\ni \hat{\rho}$.*

On the special status of $\hat{\rho}$

Statement. *(Écalle, 1992) For any $P \in \mathbb{R}\{F\}$, the first ω terms of $P(\hat{\lambda})$ are either "similar" to $\hat{\lambda}$ or to $\hat{\rho}$.*

Proof. Recent proof by AvdDvdH. □

Corollary. *For any $P \in \mathbb{R}\{F\}$ such that $P(\hat{\lambda}) = \frac{1}{x^k} + \frac{1}{x^k \log^k x} + \cdots$, we have either $k = 1$ or $k = 2$.*

Meta-theorem. $\hat{\rho}$-free H-fields and asymptotic fields have a nice model theory.

Real transseries $\longrightarrow$ analytic germs

1: Accelero-summation

$$
\begin{array}{ccccccccc}
\tilde{f} & & & & & & & & f \\
\tilde{\mathcal{B}}_{z_1}\big\downarrow & & & & & & & & \big\uparrow\hat{\mathcal{L}}^{\theta_p}_{z_p} \\
\hat{f}_1 & \underset{\hat{\mathcal{A}}^{\theta_1}_{z_1\to z_2}}{\longrightarrow} & \hat{f}_2 & \longrightarrow & \cdots & \longrightarrow & \hat{f}_{p-1} & \underset{\hat{\mathcal{A}}^{\theta_{p-1}}_{z_{p-1}\to z_p}}{\longrightarrow} & \hat{f}_p
\end{array}
$$

2: Transserial Hardy fields

$$\mathbb{T} \supseteq \mathcal{T} \overset{\rho}{\hookrightarrow} \mathcal{G}$$

- $\mathcal{G}$: ring of infinitely differentiable real germs at $+\infty$.

Real transseries → analytic germs

1: Accelero-summation

Advantages	Disadvantages
Canonical after choosing average	Requires many different tools
Preserves composition	Not yet written down
Classification local vector fields	
Differential Galois theory	

2: Transserial Hardy fields

Advantages	Disadvantages
Less hypotheses on coefficients	Not canonical
Might generalize to other models	No preservation of composition
Written down	

Transserial Hardy fields

A **transserial Hardy** field is a differential subfield $\mathcal{T}$ of $\mathbb{T}$, together with a monomorphism $\rho: \mathcal{T} \rightarrow \mathcal{G}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. $\forall f \in \mathcal{T}: \quad \operatorname{supp} f \subseteq \mathcal{T}$.

TH2. $\forall f \in \mathcal{T}: \quad f_{\prec} \in \mathcal{T}$. $\qquad f_{\prec} = \sum_{\mathfrak{m} \prec 1} f_{\mathfrak{m}} \mathfrak{m}$

TH3. $\exists d \in \mathbb{Z}: \quad \forall \mathfrak{m} \in \mathfrak{T} \cap \mathcal{T}: \quad \log \mathfrak{m} \in \mathcal{T} + \mathbb{R} \log_d x$.

TH4. $\mathfrak{T} \cap \mathcal{T}$ is stable under taking real powers.

TH5. $\forall f \in \mathcal{T}^{>}: \quad \log f \in \mathcal{T} \Rightarrow \rho(\log f) = \log \rho(f)$.

Example. $\mathcal{T} = \mathbb{R}\{\{x^{-\mathbb{R}}\}\}$.

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$$\frac{x\, \mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}}$$
$$\|$$
$$x\,\mathrm{e}^x + \mathrm{e}^x + x^{-1}\,\mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1}\,\mathrm{e}^{-x} + \cdots\cdots$$

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$$\left(\frac{x \, \mathrm{e}^x}{1 - x^{-1} - \mathrm{e}^{-x}} \right)_{\prec}$$

$$\|$$

$$x \, \mathrm{e}^x + \mathrm{e}^x + x^{-1} \mathrm{e}^x + \cdots + x + 1 + x^{-1} + \cdots + x \, \mathrm{e}^{-x} + \mathrm{e}^{-x} + x^{-1} \mathrm{e}^{-x} + \cdots\cdots$$

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Elementary extensions

Definitions. $\mathcal{T}$ transserial Hardy field, $f \in \mathbb{T}$, $\hat{f} \in \mathcal{G}$

$$f \sim \hat{f} \quad \Longleftrightarrow \quad (\exists \varphi \in \mathcal{T}\colon\ f \sim_{\mathbb{T}} \varphi \sim_{\mathcal{G}} \hat{f})$$

$$f \textit{ asympt. equiv.} \text{ to } \hat{f} \text{ over } \mathcal{T} \quad \Longleftrightarrow \quad (\forall \varphi \in \mathcal{T}\colon\ f - \varphi \sim \hat{f} - \varphi)$$

$$f \textit{ diff. equiv.} \text{ to } \hat{f} \text{ over } \mathcal{T} \quad \Longleftrightarrow \quad (\forall P \in \mathcal{T}\{F\}\colon\ P(f) = 0 \Leftrightarrow P(\hat{f}) = 0)$$

Lemma. *Let $f \in \mathbb{T} \setminus \mathcal{T}$ and $\hat{f} \in \mathcal{G} \setminus \mathcal{T}$ be such that*

- f *is a serial cut over* $\mathcal{T}$.
- f *and* $\hat{f}$ *are asymptotically equivalent over* $\mathcal{T}$.
- f *and* $\hat{f}$ *are differentially equivalent over* $\mathcal{T}$.

Then $\exists!$ transserial Hardy field extension $\rho\colon \mathcal{T}\langle f \rangle \rightarrow \mathcal{G}$ with $\rho(f) = \hat{f}$.

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Basic extension theorems

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field. Then its real closure $\mathcal{T}^{\mathrm{rcl}}$ admits a unique transserial Hardy field structure which extends the one of $\mathcal{T}$.*

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field and let $\varphi \in \mathcal{T}_{\succ}$ be such that $\mathrm{e}^{\varphi} \notin \mathcal{T}$. Then the set $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho(\mathrm{e}^{\lambda\varphi}) = \mathrm{e}^{\lambda\rho(\varphi)}$ for all $\lambda \in \mathbb{R}$.*

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field of depth $d < \infty$. Then $\mathcal{T}((\log_d x)^{\mathbb{R}})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}((\log_d x)^{\mathbb{R}}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ for all $\lambda \in \mathbb{R}$.*

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - f' + \frac{x}{\mathrm{e}^x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{x}{\mathrm{e}^x} + f^2 \right)$$

Step 3. Integral transseries solution

Differential equations (main ideas)

Step 1. A given algebraic differential equation

$$f^2 - \mathrm{e}^x f' + \frac{\mathrm{e}^{2x}}{x} = 0$$

Step 2. Put equation in integral form

$$f = \int \left(\frac{\mathrm{e}^x}{x} + \frac{f^2}{\mathrm{e}^x} \right)$$

Step 3. Integrate from a fixed point $x_0 < \infty$

$$f = \int_{x_0} \frac{\mathrm{e}^x}{x} + \int_{x_0} \frac{1}{\mathrm{e}^x} \left(\int_{x_0} \frac{\mathrm{e}^x}{x} \right)^2 + 2 \int_{x_0} \frac{1}{\mathrm{e}^{2x}} \left(\int_{x_0} \frac{\mathrm{e}^x}{x} \right) \left(\int_{x_0} \frac{1}{\mathrm{e}^x} \left(\int_{x_0} \frac{\mathrm{e}^x}{x} \right)^2 \right) + \cdots$$

Differential equations (main ideas)

Step 1. A general algebraic differential equation

$$P(f)=0$$

Step 2. Equation in split-normal form

$$(\partial-\varphi_1)\cdots(\partial-\varphi_r)\,f=P(f) \qquad \text{with } P(f) \text{ small}$$

Attention: $\varphi_1,\ldots,\varphi_r\in\mathcal{T}[\mathrm{i}]$, even though $(\partial-\varphi_1)\cdots(\partial-\varphi_r)\in\mathcal{T}[\partial]$.

Step 3. Solve the split-normal equation using the fixed-point technique.

Continuous right inverses (first order)

Lemma. *The operator* $J=(\partial-\varphi)_{x_0}^{-1}$, *defined by*

$$(Jf)(x)=\begin{cases} \mathrm{e}^{\Phi(x)}\int_{\infty}^{x}\mathrm{e}^{-\Phi(t)}f(t)\,\mathrm{d}t & \textit{(repulsive case)}\\ \mathrm{e}^{\Phi(x)}\int_{x_0}^{x}\mathrm{e}^{-\Phi(t)}f(t)\,\mathrm{d}t & \textit{(attractive case)}\end{cases}$$

and

$$\Phi(x)=\begin{cases}\int_{\infty}^{x}\varphi(t)\,\mathrm{d}t & \textit{(repulsive case)}\\ \int_{x_0}^{x}\varphi(t)\,\mathrm{d}t & \textit{(attractive case)}\end{cases}$$

is a continuous right-inverse of $L=\partial-\varphi$ *on* $\mathcal{G}_{x_0}^{\preccurlyeq}[\mathrm{i}]$, *with*

$$|||J|||_{x_0}\leqslant\left\|\frac{1}{\operatorname{Re}\varphi}\right\|_{x_0}.$$

Continuous right-inverses (higher order)

Lemma. *Given a split-normal operator*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r), \tag{1}$$

with a factorwise right-inverse $L^{-1} = J_r \cdots J_1$*, the operator*

$$\mathfrak{v}^{\nu} J_r \cdots J_1 \colon \mathcal{G}_{x_0}^{\preccurlyeq}[\mathrm{i}] \rightarrow \mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$$

is a continuous operator for every $\nu > r\,\sigma_L$*. Here* $\mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$ *carries the norm*

$$\|f\|_{x_0;r} = \max\{\|f\|_{x_0}, \ldots, \|f^{(r)}\|_{x_0}\}.$$

Lemma. *If* $L \in \mathcal{T}[\partial]$ *and the splitting* (1) *(formally) preserves realness, then* $J_r \cdots J_1$ *preserves realness in the sense that it maps* $\mathcal{G}_{x_0}^{\preccurlyeq}$ *into itself.*

Non-linear equations

Theorem. *Consider a split-monic equation*

$$L f = P(f), \quad f \prec 1,$$

and let ν *be such that* $r\,\sigma_L < \nu < v_P$. *Then for any sufficiently large* x_0, *there exists a continuous factorwise right-inverse* $J_{r,\ltimes\mathfrak{v}^\nu} \cdots J_{1,\ltimes\mathfrak{v}^\nu}$ *of* $L_{\ltimes\mathfrak{v}^\nu}$, *such that the operator*

$$\Xi\colon f \longmapsto (J_r \cdots J_1)(P(f))$$

admits a unique fixed point

$$f \;=\; \lim_{n\to\infty} \Xi^{(n)}(0) \;\in\; \mathcal{B}(\mathcal{G}^{\preccurlyeq}_{x_0;r}, \tfrac{1}{2}).$$

Preservation of asymptotics

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq \mathrm{e}^x$. Consider a monic split-normal quasi-linear equation*

$$L f = P(f), \quad f \prec 1, \tag{2}$$

over $\mathcal{T}$ without solutions in $\mathcal{T}$. Assume that one of the following holds:

- *$\mathcal{T}$ is $(1,1,1)$-differentially closed in $\mathbb{T}_{\preccurlyeq \mathfrak{v}}$ and (2) is first order.*

  i.e. $\mathcal{T}$ is closed under the resolution of linear first order equations.

- *$\mathcal{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]_{\preccurlyeq \mathfrak{v}}$.*

Then there exist solutions $f \in \mathcal{G}$ and $\tilde{f} \in \hat{\mathcal{T}}$ to (2), such that f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$.

First order extensions

Lemma. *Let $L = \partial - \varphi \in \mathcal{T}[\partial]$ be a normal operator. Let $\tilde{f} \in \hat{\mathcal{T}}^{\preccurlyeq}$ and $g \in \mathcal{T}^{\preccurlyeq}$ be such that $\tilde{f}$ is transcendental over $\mathcal{T}$ and $L\tilde{f} = g$. Then there exists an $f \in \mathcal{G}^{\preccurlyeq}$ with $Lf = g$, such that f and $\tilde{f}$ are both differentially and asymptotically equivalent over $\mathcal{T}$.*

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field. Let $\mathcal{T}^{\mathrm{fo}} \supseteq \mathcal{T}$ be the smallest differential subfield of $\mathbb{T}$, such that for any $P \in \mathcal{T}^{\mathrm{fo}}\{F\}^{\neq}$ with $r_P \leqslant 1$ and $f \in \mathbb{T}$ we have $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{fo}}$. Then the transserial Hardy field structure of $\mathcal{T}$ can be extended to $\mathcal{T}^{\mathrm{fo}}$.*

Proof. As long as $\mathcal{T}^{\mathrm{fo}} \neq \mathcal{T}$:

- Close off under $\exp$, $\log$ and algebraic equations.
- Choose $P \in \mathcal{T}\{F\}^{\neq}, r_P = 1, f \in \mathbb{T}, P(f) = 0$ such that P has minimal "complexity" (r_P, d_P, t_P) and apply the previous results. □

Higher order extensions

Lemma. *Let* $L = \partial - \varphi \in \mathcal{T}[\mathrm{i}][\partial]$ *be a normal operator. Let* $\tilde{f} \in \hat{\mathcal{T}}[\mathrm{i}]^{\preccurlyeq}$ *and* $g \in \mathcal{T}[\mathrm{i}]^{\preccurlyeq}$ *be such that* $\operatorname{Re} \tilde{f}$ *has order* 2 *over* $\mathcal{T}$ *and* $L\,\tilde{f} = g$*. Then there exists an* $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ *with* $L\,f = g$*, such that* $\operatorname{Re} f$ *and* $\operatorname{Re} \tilde{f}$ *are both differentially and asymptotically equivalent over* $\mathcal{T}$*.*

Theorem. *Let* $\mathcal{T}$ *be a transserial Hardy field. Let* $\mathcal{T}^{\mathrm{dalg}} \supseteq \mathcal{T}$ *be the smallest differential subfield of* $\mathbb{T}$*, such that for any* $P \in \mathcal{T}^{\mathrm{dalg}}\{F\}^{\neq}$ *and* $f \in \mathbb{T}$ *we have* $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{dalg}}$*. Then the transserial Hardy field structure of* $\mathcal{T}$ *can be extended to* $\mathcal{T}^{\mathrm{dalg}}$*.*

Applications

Corollary. *There exists a transserial Hardy field* $\mathcal{T}$*, such that for any* $P \in \mathcal{T}\{F\}$ *and* $f, g \in \mathcal{T}$ *with* $f < g$ *and* $P(f)\,P(g) < 0$*, there exists a* $h \in \mathcal{T}$ *with* $f < h < g$ *and* $P(h) = 0$*.*

Corollary. *There exists a transserial Hardy field* $\mathcal{T}$*, such that* $\mathcal{T}[\mathrm{i}]$ *is weakly differentially closed.*

Corollary. *There exists a differentially Henselian transserial Hardy field* $\mathcal{T}$*, i.e., such that any quasi-linear differential equation over* $\mathcal{T}$ *admits a solution in* $\mathcal{T}$*.*

A partial inverse

Theorem. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian and stable under exponentiation. Then there exists a transserial Hardy field structure on $\mathcal{H}$ which extends the structure on $\mathcal{T}$.*

Corollary. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian. Assume that $\mathcal{H}$ admits no non-trivial algebraically differential Hardy field extensions. Then $\mathcal{H}$ satisfies the differential intermediate value property.*

Theorem. *(Boshernitzan 1987) Any solution of the equation*

$$f'' + f = \mathrm{e}^{x^2}$$

is contained in a Hardy field. However, none of these solutions is contained in the intersection of all maximal Hardy fields.

Open problems

- Embeddability of Hardy fields in differentially Henselian Hardy fields.
- Do maximal Hardy fields satisfy the intermediate value property?
- Restricted analytic (instead of algebraic) differential equations.
- Preservation of composition:
  - $f(x+\varepsilon)$, small ε: expand.
  - $f(q\,x+\varepsilon)$: expand, but more intricate.
  - $f(\varphi(x)), \varphi \succ x$: abstract nonsense.