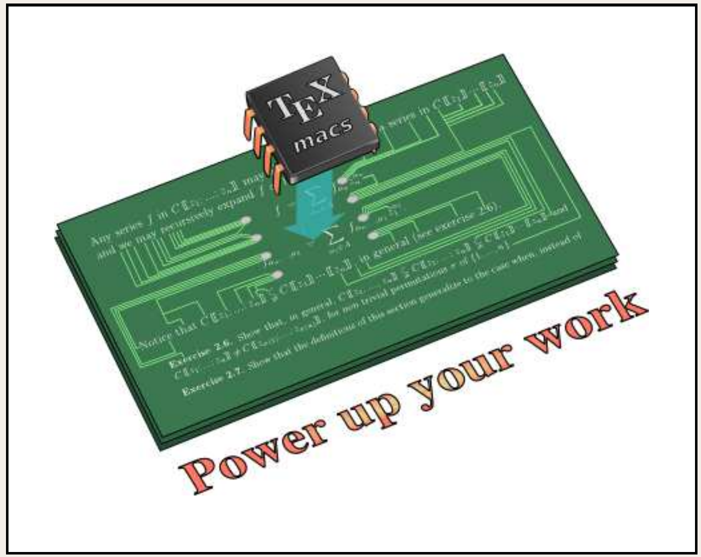
# The MATHEMAGIX type system

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http://www.TeXmacs.org

# Motivation

- Existing computer algebra systems are slow for numerical algorithms

  $\rightsquigarrow$ we need a compiled language

- Low level systems (GMP, MPFR, FLINT) painful for compound objects

  $\rightsquigarrow$ we need a mathematically expressive language

- More and more complex architectures (SIMD, multicore, web)

  $\rightsquigarrow$ general efficient algorithms cannot be designed by hand

- Existing systems lack sound semantics

  $\rightsquigarrow$ we need mathematically clean interfaces

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- More and more complex architectures (SIMD, multicore, web)

  Non standard but efficient numeric types

  ⇝ general efficient algorithms cannot be designed by hand

- Existing computer algebra systems lack sound semantics

  Difficult to connect different systems in a sound way

  ⇝ we need mathematically clean interfaces

# Main design goals

- Strongly typed functional language
- Access to low level details and encapsulation
- Inter-operability with C/C++ and other languages
- Large scale programming *via* intuitive, strongly local writing style

**Guiding principle.**

| | | |
|---|---|---|
| *Prototype* | $\leftrightsquigarrow$ | *Mathematical theorem* |
| *Implementation* | $\leftrightsquigarrow$ | *Formal proof* |

# Example

```
forall (R: Ring) square (x: R) == x * x;
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## Mathemagix

```
category Ring == {
  convert: Int -> This;
  prefix -: This -> This;
  infix +: (This, This) -> This;
  infix -: (This, This) -> This;
  infix *: (This, This) -> This;
}
```

# Example

```
forall (R: Ring) square (x: R) == x * x;
```

## C++

```cpp
template<typename R>
square (const R& x) {
  return x * x;
}
```

# Example

```
forall (R: Ring) square (x: R) == x * x;
```

## C++

```
concept Ring<typename R> {
  R::R (int);
  R::R (const R&);
  R operator - (const R&);
  R operator + (const R&, const R&);
  R operator - (const R&, const R&);
  R operator * (const R&, const R&);
}

template<typename R>
requires Ring<R>
operator * (const R& x) {
  return x * x;
}
```

# Example

```
forall (R: Ring) square (x: R) == x * x;
```

## Axiom, Aldor

```
define Ring: Category == with {
  0: %;
  1: %;
  -: % -> %;
  +: (%, %) -> %;
  -: (%, %) -> %;
  *: (%, %) -> %;
}

Square (R: Ring): with {
  square: R -> R;
} == add {
  square (x: R): R == x * x;
}

import from Square (Integer);
```

# Example

```
forall (R: Ring) square (x: R) == x * x;
```

## Ocaml

```
# let square x = x * x;;
val square: int -> int = <fun>

# let square_float x = x *. x;;
val square_float: float -> float = <fun>
```

# Example

```
forall (R: Ring) square (x: R) == x * x;
```

## Ocaml

```
# module type RING =
    sig
      type t
      val cst : int -> t
      val neg : t -> t
      val add : t -> t -> t
      val sub : t -> t -> t
      val mul : t -> t -> t
    end;;

# module Squarer =
  functor (El: RING) ->
    struct
      let square x = El.mul x x
    end;;

# module IntRing =
    struct
      type t = int
      let cst x = x
      let neg x = - x
      let add x y = x + y
      let sub x y = x - y
      let mul x y = x * y
    end;;

# module IntSquarer = Squarer(IntRing);;
```

# Functional programming

```
shift (x: Int) (y: Int): Int == x + y;

v: Vector Int == map (shift 123, [ 1 to 100 ]);

test (i: Int): (Int -> Int) == {
  f (): (Int -> Int) == g;
  g (j: Int): Int == i * j;
  return f ();
}
```

# Classes

```
class Point == {
  mutable x: Int;
  mutable y: Int;

  constructor point (a: Int, b: Int) == {
    x == a; y == b; }

  mutable method translate (dx: Int, dy: Int): Void == {
    x := x + dx; y := y + dy; }
}

flatten (p: Point): Syntactic ==
  'point (flatten p.x, flatten p.y);

infix + (p: Point, q: Point): Point ==
  point (p.x + q.x, p.y + q.y);
```

# Overloading

```
category Type == {}

forall (T: Type) f (x: T): T == x;
f (x: Int): Int == x * x;
f (x: Double): Double == x * x * x * x;

mmout << f ("Hallo") << "\n";
mmout << f (11111) << "\n";
mmout << f (1.1) << "\n";
```

```
Castafiore:basic vdhoeven$ ./overload_test
Hallo
123454321
1.4641
Castafiore:basic vdhoeven$
```

# Categories

```
category Ring == {
  convert: Int -> This;
  prefix -: This -> This;
  infix +: (This, This) -> This;
  infix -: (This, This) -> This;
  infix *: (This, This) -> This;
}

category Module (R: Ring) == {
  prefix -: This -> This;
  infix +: (This, This) -> This;
  infix -: (This, This) -> This;
  infix *: (R, This) -> This;
}

forall (R: Ring, M: Module R)
square_multiply (x: R, y: M): M == (x * x) * y;

mmout << square_multiply (3, 4) << "\n";
```

# Implicit conversions

```
convert (x: Double): Floating == mpfr_as_floating x;

forall (R: Ring) {
  infix * (v: Vector R, w: Vector R): Vector R == [ ... ];
  forall (K: To R)
  infix * (c : K, v: Vector R): Vector R ==
    [ (c :> R) * x | x: R in v ];
  infix * (v: Vector R, c :> R): Vector R ==
    [ x*c | x: R in v ];
}

forall (R: Ring)
convert (x :> R): Complex R == complex (x, 0);
// allows for conversion Double --> Complex Floating

convert (p: Point): Vector Int == [ p.x, p.y ];
downgrade (p: Colored_Point): Point == point (p.x, p.y);
// allows for conversion Colored_Point --> Vector Int
// abstract way to implement class inheritance
```

# Value parameters for containers

```
class Vec (R: Ring, n: Int) == {
  private mutable rep: Vector R;

  constructor vec (v: Vector R) == {
    rep == v; }
  constructor vec (c: R) == {
    rep == [ c | i: Int in 0..n ]; }
}

forall (R: Ring, n: Int) {
  flatten (v: Vec (R, n)): Syntactic == flatten v.rep;
  postfix [] (v: Vec (R, n), i: Int): R == v.rep[i];
  postfix [] (v: Alias Vec (R, n), i: Int): Alias R ==
    v.rep[i];
  infix + (v1: Vec (R, n), v2: Vec (R, n)): Vec (R, n) ==
    vec ([ v1[i] + v2[i] | i: Int in 0..n ]);

  assume (R: Ordered)
  infix <= (v1: Vec (R, n), v2: Vec (R, n)): Boolean ==
    big_and (v1[i] <= v2[i] | i: Int in 0..n);
}
```

# Abstract data types

```
structure List (T: Type) == {
  null ();
  cons (head: T, tail: List T);
}

l1: List Int == cons (1, cons (2, null ()));
l2: List Int == cons (1, cons (2, cons (3, null ())));

forall (T: Type)
prefix # (l: List T): Int ==
  if null? l then 0 else #l.tail + 1;
```

# Abstract data types

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structure List (T: Type) == {
  null ();
  cons (head: T, tail: List T);
}

l1: List Int == cons (1, cons (2, null ()));
l2: List Int == cons (1, cons (2, cons (3, null ())));

forall (T: Type)
prefix # (l: List T): Int ==
  match l with {
    case null () do return 0;
    case cons (_, l: List T) do return #l + 1;
  }
```

# Abstract data types

```
structure List (T: Type) == {
  null ();
  cons (head: T, tail: List T);
}

l1: List Int == cons (1, cons (2, null ()));
l2: List Int == cons (1, cons (2, cons (3, null ())));

forall (T: Type) {
  prefix # (l: List T): Int := 0;
  prefix # (cons (_, t: List T)): Int := #t + 1;
}
```

# Symbolic types

```
structure Symbolic := {
  sym_literal (literal: Literal);
  sym_compound (compound: Compound);
}

infix + (x: Symbolic, y: Symbolic): Symbolic :=
  sym_compound ('+ (x :> Generic, y :> Generic));
```

# Symbolic types

```
structure Symbolic := {
  sym_literal (literal: Literal);
  sym_compound (compound: Compound);
}

infix + (x: Symbolic, y: Symbolic): Symbolic :=
  sym_compound ('+ (x :> Generic, y :> Generic));

structure Symbolic += {
  sym_int (int: Int);
  sym_double (double: Double);
}

infix + (sym_double (x: Double),
         sym_double (y: Double)): Symbolic :=
  sym_double (x + y);
```

# Symbolic types

```
structure Symbolic := {
  sym_literal (literal: Literal);
  sym_compound (compound: Compound);
}

infix + (x: Symbolic, y: Symbolic): Symbolic :=
  sym_compound ('+ (x :> Generic, y :> Generic));

structure Symbolic += {
  sym_int (int: Int);
  sym_double (double: Double);
}

pattern sym_as_double (as_double: Double): Symbolic := {
  case sym_double (x: Double) do as_double == x;
  case sym_int (i: Int) do as_double == i;
}

infix + (sym_as_double (x: Double),
         sym_as_double (y: Double)): Symbolic :=
  sym_double (x + y);
```

# Type system: logical types

**Overloading.** Explicit types for overloaded objects

```
forall (T: Type) f (x: T): T == x;
f (x: Int): Int == x * x;
```

Type of `f`: `And (Forall (T: Type, T -> T), Int -> Int)`

Logical types: $f : \mathrm{And}(T, U) \Longleftrightarrow f : T \wedge f : U$

**Preferences in case of ambiguities.**

```
infix +: (Int, Int): Int;
infix +: (Int, Integer): Integer;
infix +: (Integer, Integer): Integer;

prefer infix + :> (Int, Int) -> Int
to     infix + :> (Int, Integer) -> Integer;
```

# Formal theory and compilation

**Level 1.** Source language with syntax constructs for ambiguous notations

$$\text{square}\colon (\forall \mathsf{T}^{\mathsf{Ring}} \to \mathsf{T}) \wedge \mathsf{String} \to \mathsf{String}$$

**Level 2.** Intermediate unambiguous language with
additional constructs for disambiguating the ambiguous notations

$$\text{square} \overset{\text{valid interpretation}}{\rightsquigarrow} \pi_1(\text{square})\#\mathsf{Int}\colon \mathsf{Int} \to \mathsf{Int}$$

Compilation: transform source program in intermediate program.

**Level 3.** Interpretation in traditional $\lambda$-calculus

$$\text{square} \;\equiv\; \text{pair}(\lambda\mathsf{T}.\lambda x.\text{get}_{\times}(\mathsf{T})(x, x), \lambda x.\text{concat}(x, x))$$

Backend: transform intermediate program in object program