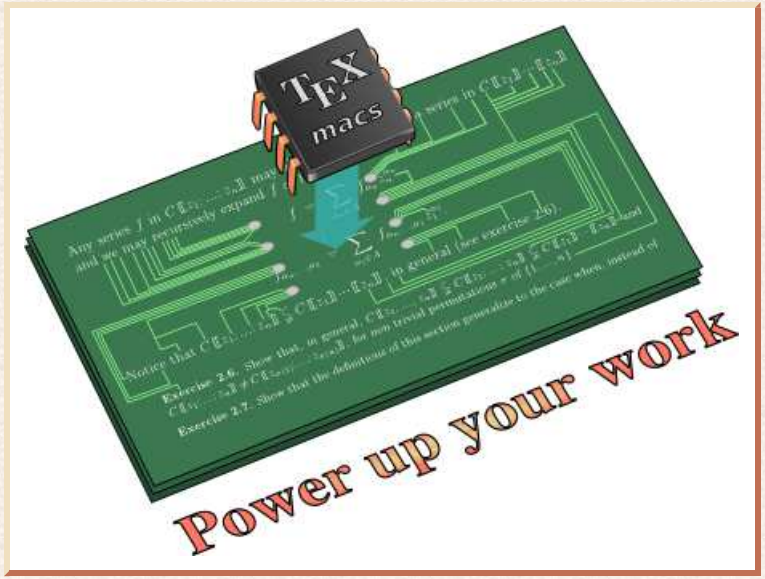
On the practical complexity of computations with real numbers

Joris van der Hoeven

CNRS, École polytechnique



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http://www.TEXMACS.org

Computer algebra *and* analysis system

Computer algebra

Exact computations with algebraic, arithmetic or combinatorial objects

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Numerical analysis

Approximate numerical computations with (generally) manual error estimates

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Computer analysis

Mathematically exact computations with analytic objects.

Example 1: certified integration

Pendulum

$$\ddot{y} = -\alpha \sin y,$$

For $\alpha = 1$, $y(0) = 0$, $\dot{y}(0) = 1$, compute $\tilde{y} \in \mathbb{Q}$ (where $\tilde{y} \in \mathbb{D} = \mathbb{Z}\,2^{\mathbb{Z}}$) with

$$|\tilde{y} - y(2013)| < 2^{-2013}.$$

Example 1: certified integration

Dynamical systems

Let $P = (P_1, ..., P_d) \in \mathbb{Q}[Y]^d = \mathbb{Q}[Y_1, ..., Y_d]^d$ and consider

$$\dot{Y} = P(Y).$$

For $Y(0) \in \mathbb{Q}^d$, $t \in \mathbb{Q}^>$ and $\varepsilon \in \mathbb{Q}^>$, compute (when possible) $\tilde{Y} \in \mathbb{Q}^d$ with

$$\|\tilde{Y} - Y(t)\| < \varepsilon.$$

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Special functions

- Elementary functions $\exp, \log, \cos, \sin, \tan$, etc.
- Holonomic functions: $L(f) = L_r f^{(r)} + \cdots + L_0 f = 0$, $L \in \mathbb{Q}[z][\partial]$.
- Other : $\wp$, Γ, ζ, theta fonctions, Matthieu functions, etc.

Example 2: polynomial system solving

Given a zero dimensional system

$$\begin{cases} P_1(z_1, \ldots, z_n) = 0 \\ \vdots \\ P_n(z_1, \ldots, z_n) = 0 \end{cases}$$

with $P_1, \ldots, P_n \in \mathbb{Q}[z_1, \ldots, z_n]$, find the zeros "in $\mathbb{C}^n$".

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$\mathbb{E}$ = "field of exp-log constants", constructed from $\mathbb{Q}$ using $+, -, \times, \exp, \log$.

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with $P_1, \ldots, P_n \in \mathbb{E}[z_1, \ldots, z_n]$, find the zeros "in $\mathbb{R}^n$".

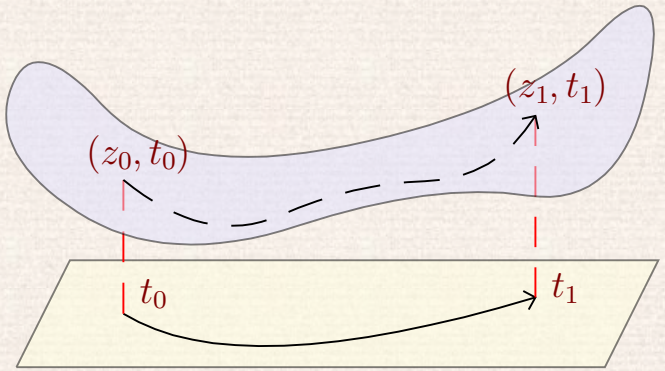
$\mathbb{E}$ = "field of exp-log constants", constructed from $\mathbb{Q}$ using $+, -, \times, \exp, \log$.

Example 3: path following

$$H(z,t)=0 \qquad \begin{cases} H_1(z_1,\ldots,z_n,t)=0 \\ \vdots \\ H_n(z_1,\ldots,z_n,t)=0 \end{cases},$$

with $H(z,t)\in\mathbb{Q}[z,t]^n$ zero-dimensional in z for $t\in\mathbb{C}\backslash\Sigma$ and finite Σ.

Given (z_0,t_0) avec $H(z_0,t_0)=0$ and a path $t_0\rightsquigarrow t_1$ which avoids Σ, compute the path $z_0\rightsquigarrow z_1$ with $H(z_\lambda,t_\lambda)=0$ for all $\lambda\in[0,1]$.



Example 3: path following, application

Polynomial system solving using numerical homotopies

$$P(x,y) \quad \begin{cases} x^2+2\,x\,y-y^2-3\,x+5\,y+8=0 \\ 3\,x^2-x\,y+y^2+8\,x-2\,y+7=0 \end{cases}$$

$$\text{Easy}(x,y) \quad \begin{cases} x^2-1=0 \\ y^2-1=0 \end{cases}$$

$$H(x,y,t) \quad \begin{cases} (x^2-1)\,t+(x^2+2\,x\,y-y^2-3\,x+5\,y+8)\,(1-t)=0 \\ (y^2-1)\,t+(3\,x^2-x\,y+y^2+8\,x-2\,y+7)\,(1-t)=0 \end{cases}$$

Example 4: Bell numbers

$$B(z) \;=\; \sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n \;=\; \mathrm{e}^{\mathrm{e}^z - 1}.$$

Problem 1: compute B_{10000} with a relative error 10^{-10}

Problem 2: automatically find/guess asymptotic expansion

$$\log\left(\frac{B_n}{n!}\right) \;=\; n\left(-\log\log n + \frac{1}{\log n} + \frac{\log\log n}{\log n} + \mathcal{O}\left(\frac{1}{\log^2 n}\right)\right)$$

Problem 3: for $n \geqslant 10^{10}$, find an explicit constant for the $\mathcal{O}\left(\frac{1}{\log^2 n}\right)$ term

Challenges

- Which problems can be solved, at least theoretically?
- Automatic computation of error bounds for numerical computations
- Redevelop numerical analysis for multiple precision computations

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  Computable analysis, BSS computability, ...
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  Interval analysis, ball arithmetic
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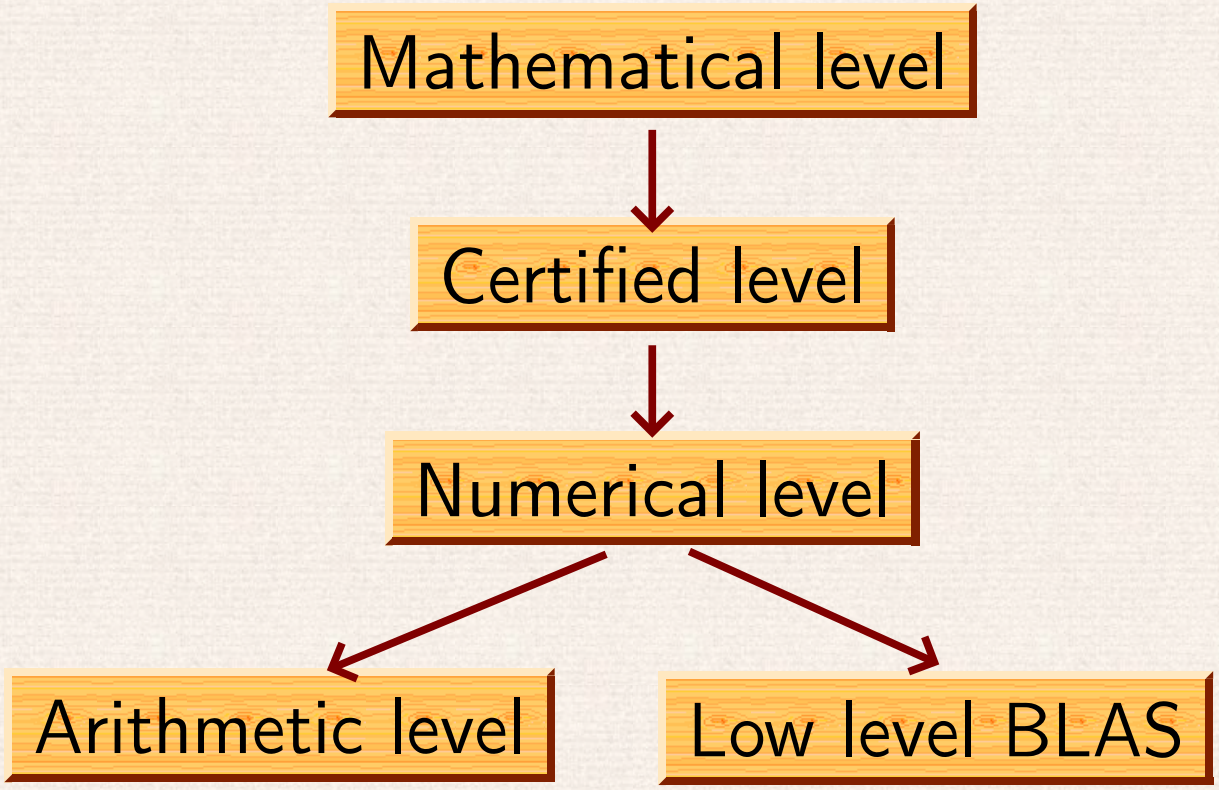
- Redevelop numerical analysis for multiple precision computations

  Runge Kutta $\rightsquigarrow$ Taylor methods

  Complexity, conditioning, ...

Numerical hierarchy



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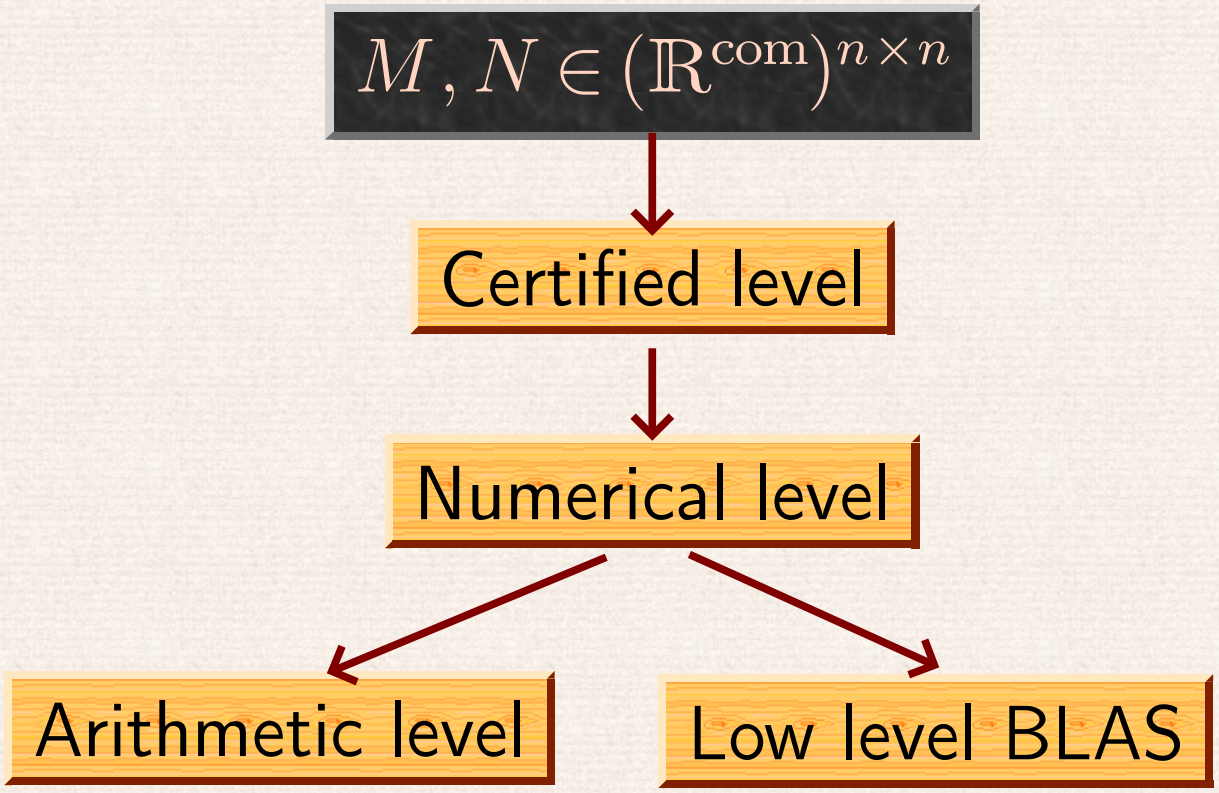

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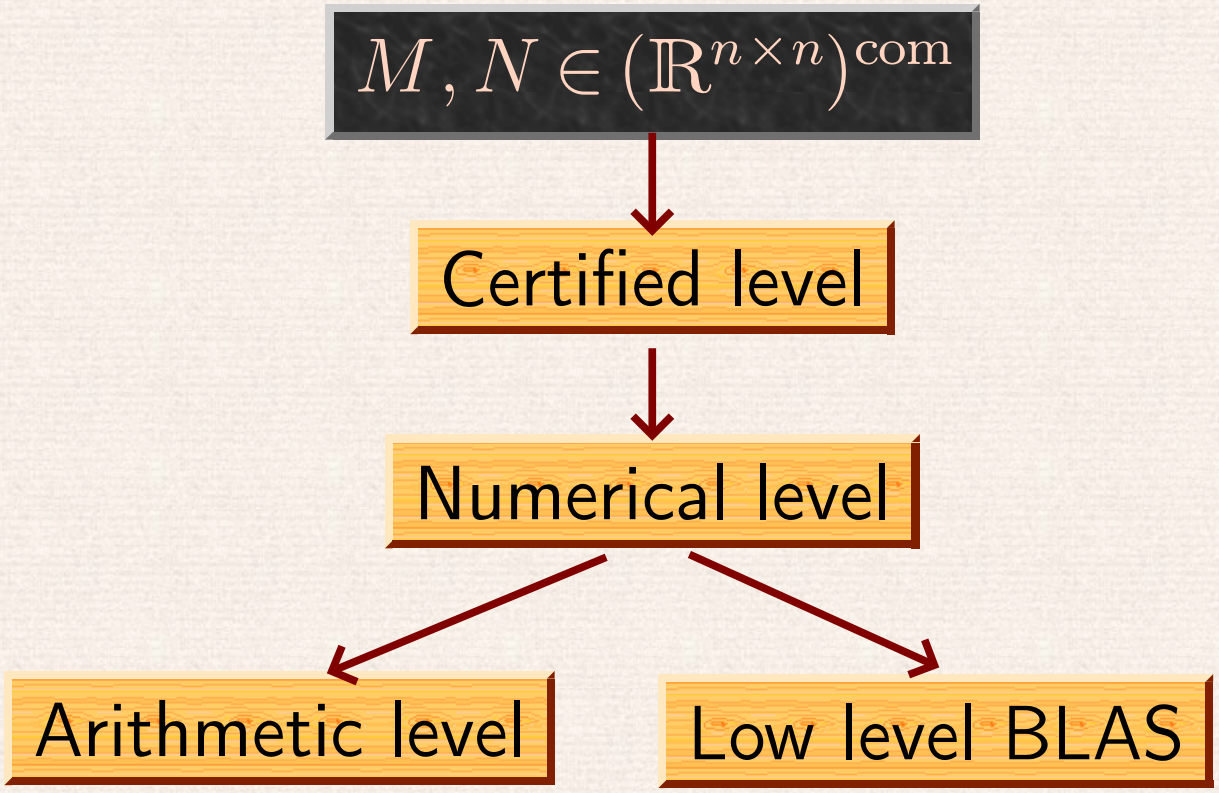

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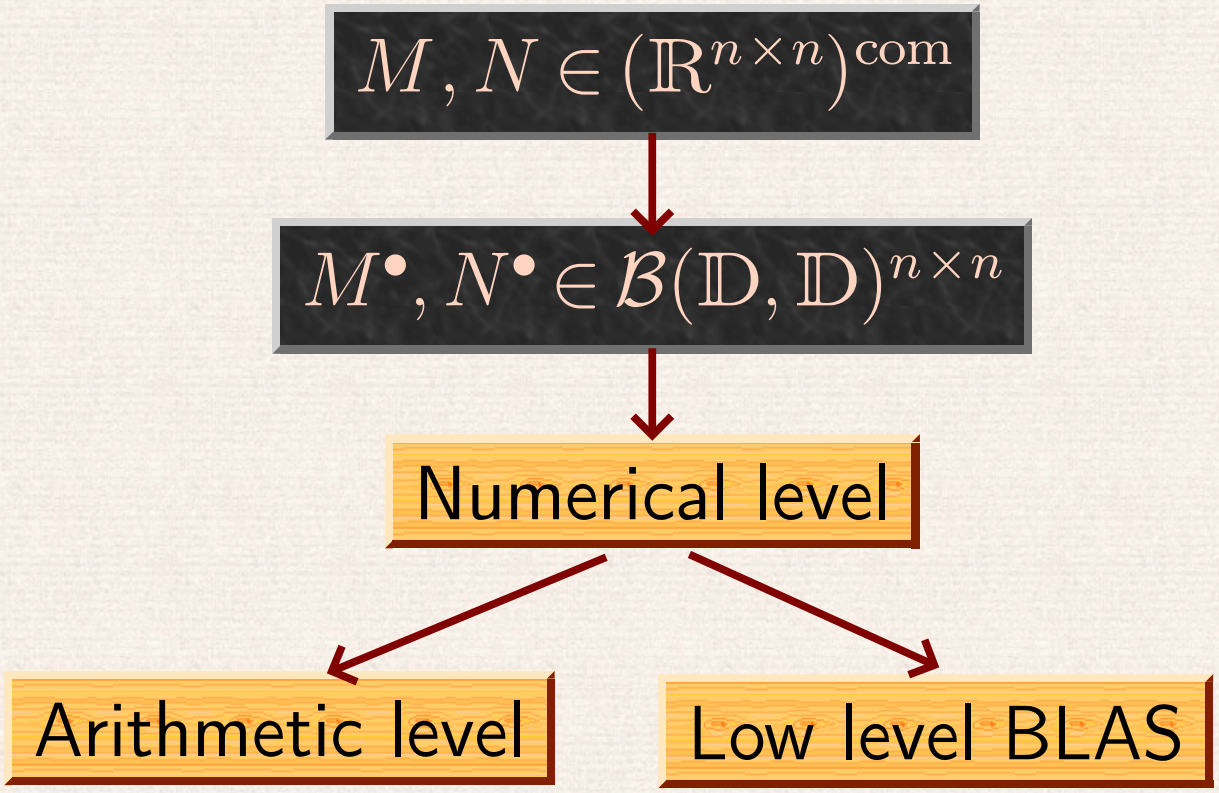

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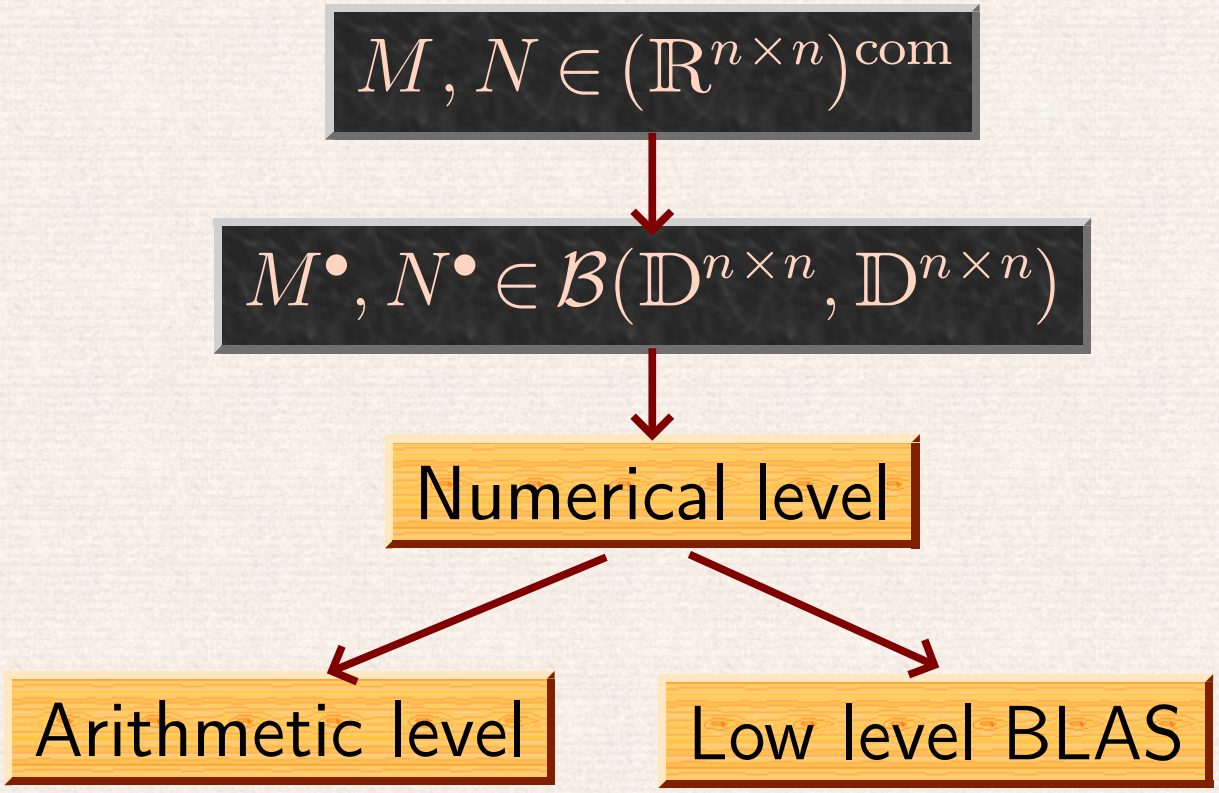

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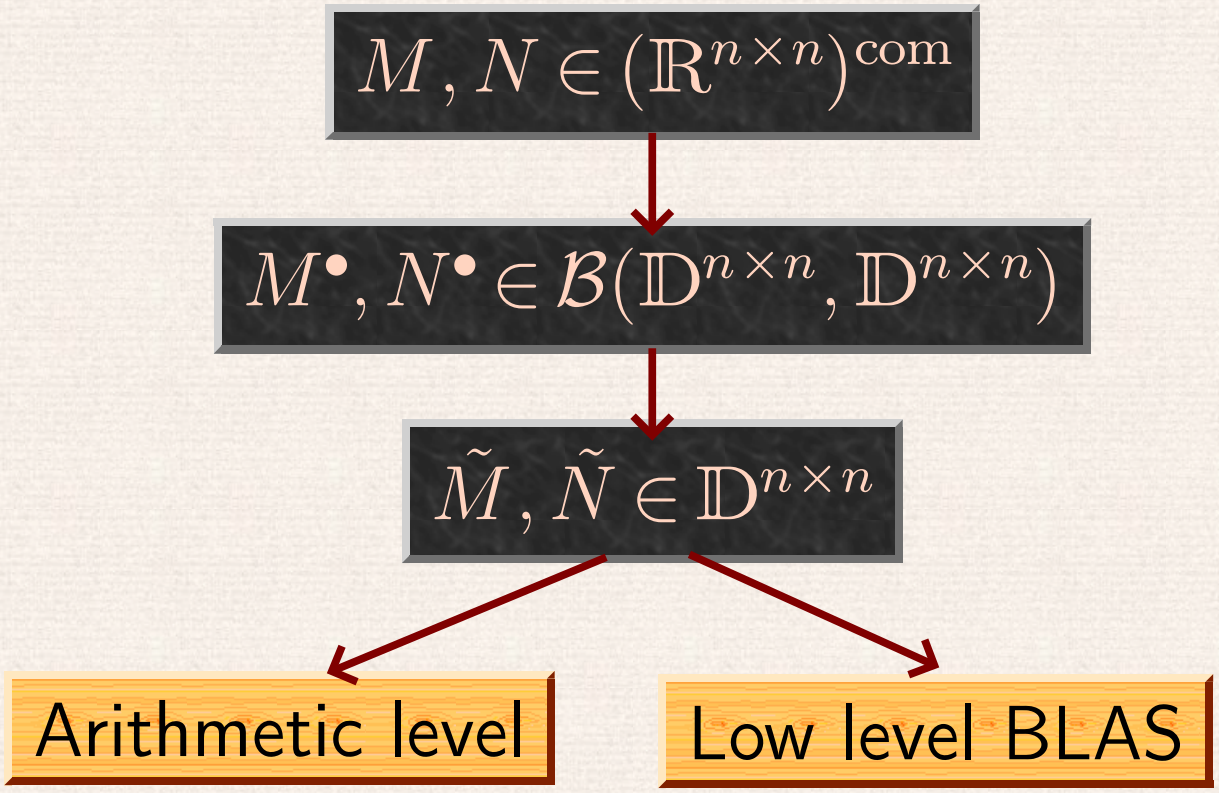

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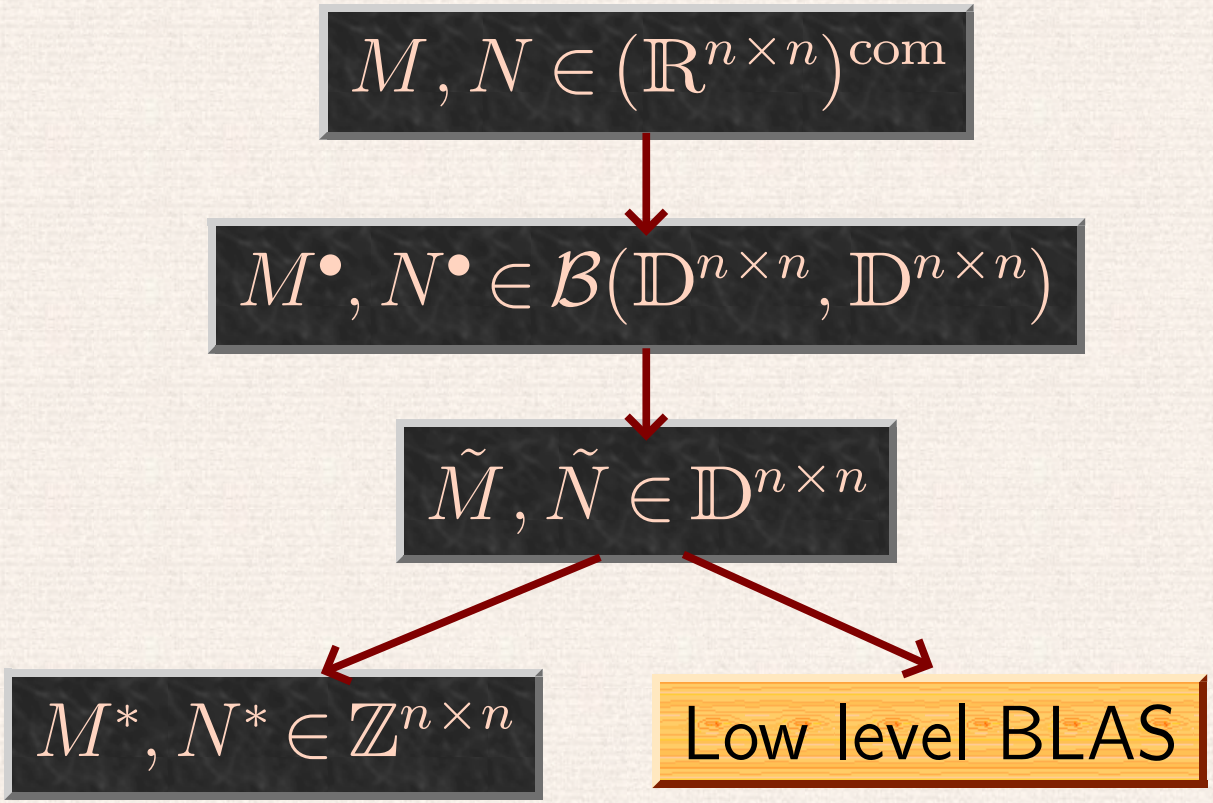

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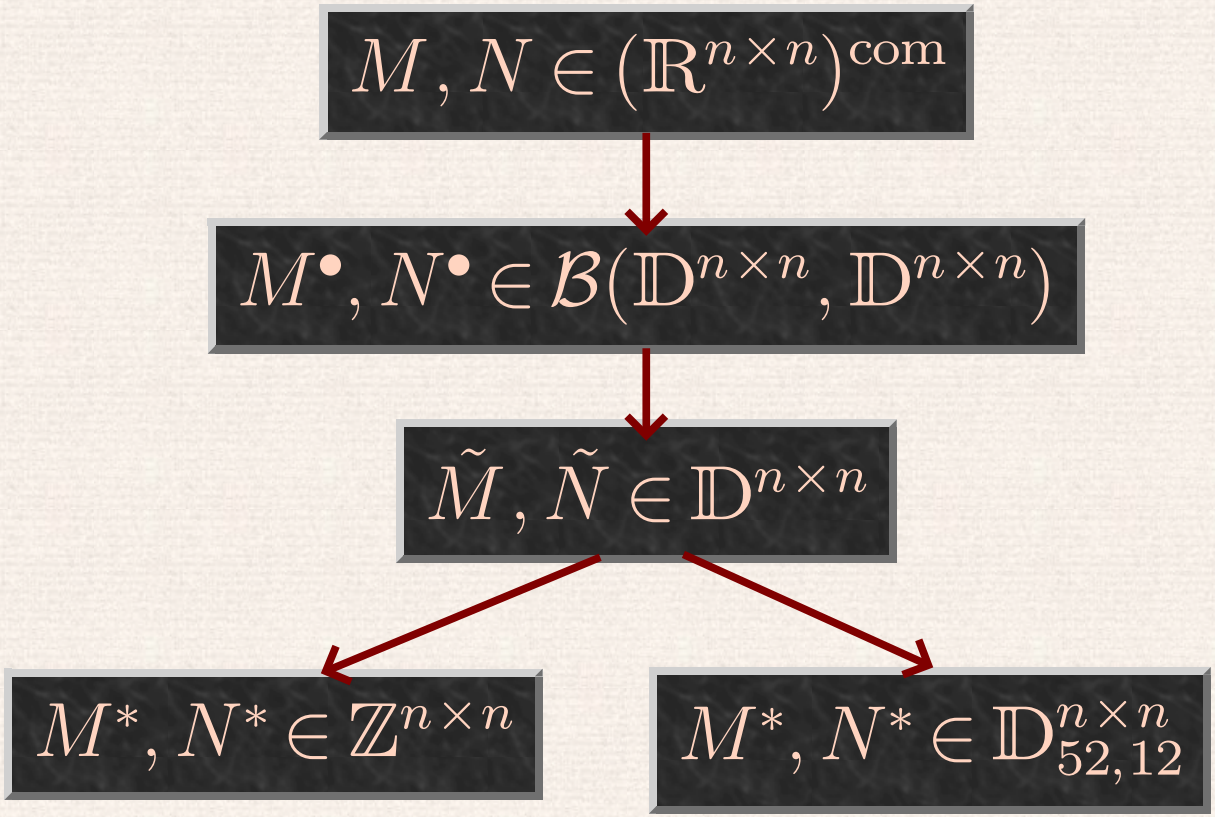

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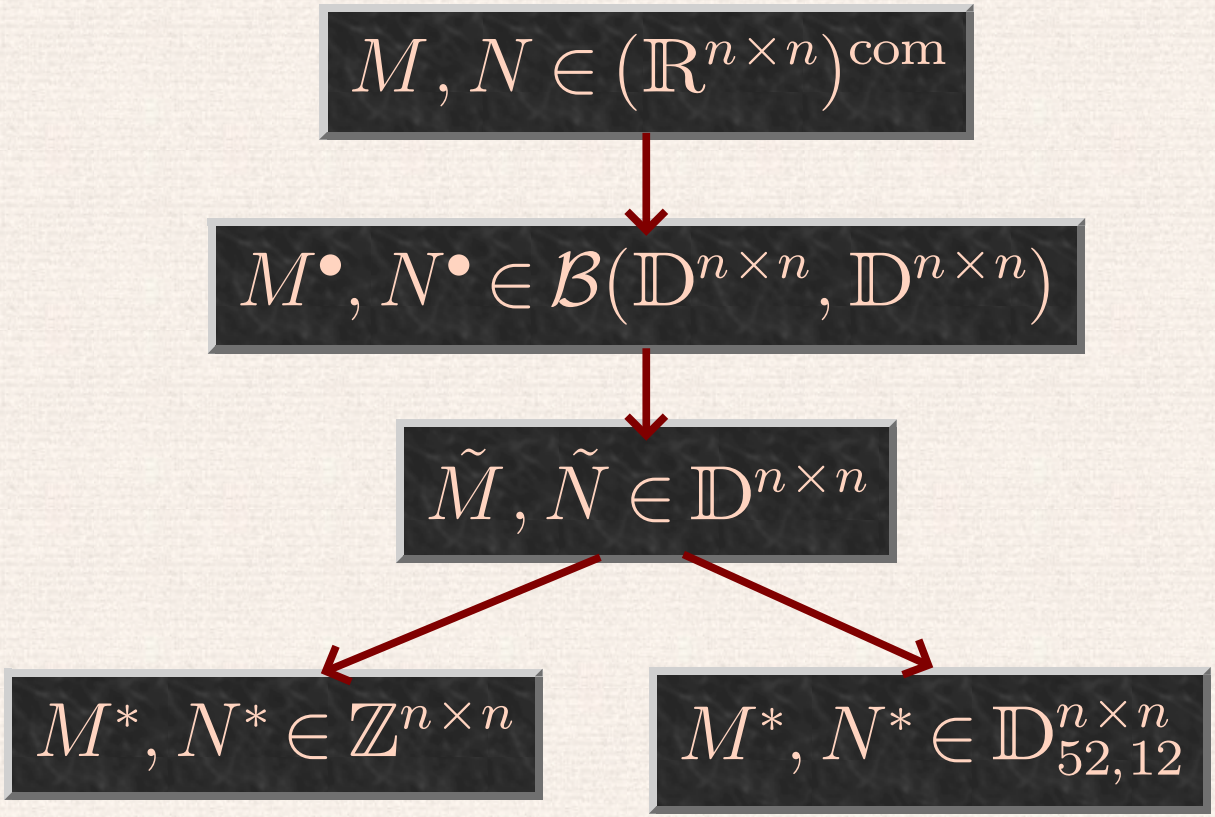

TTE computable analysis

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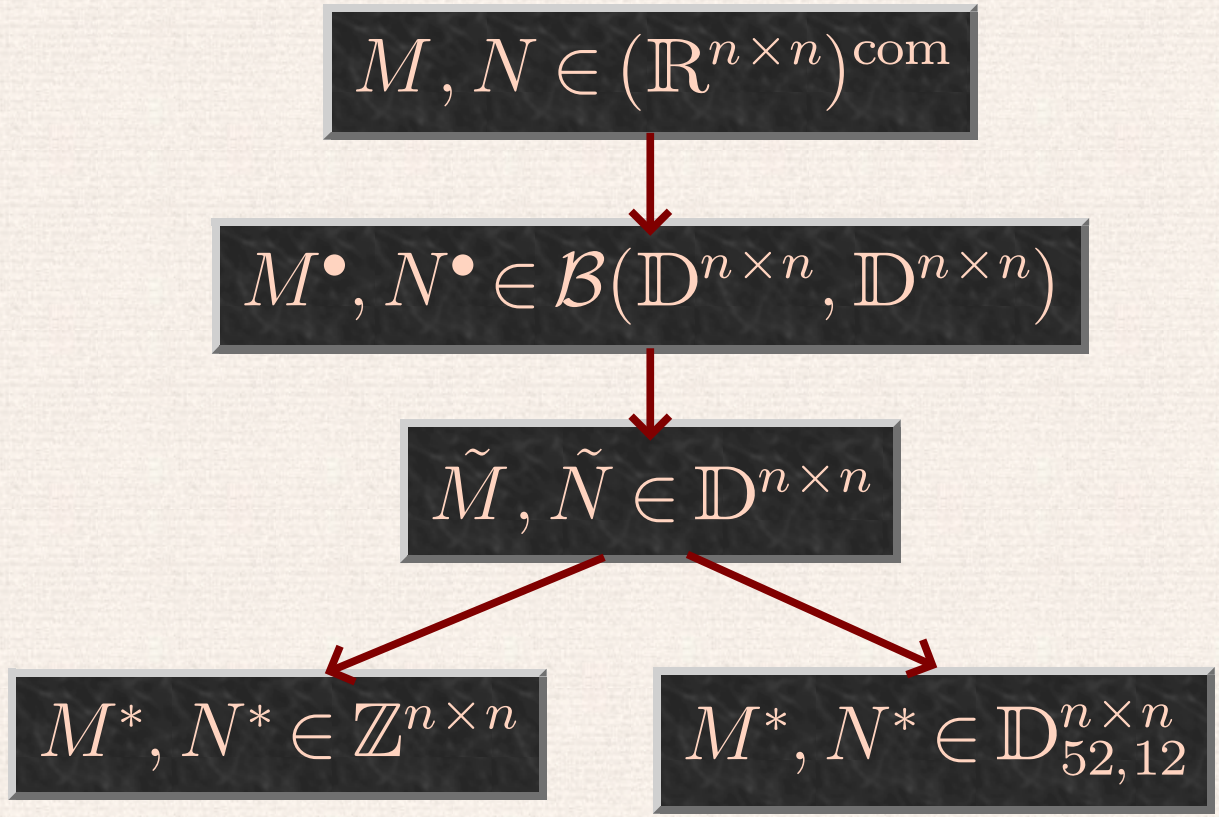

TTE computable analysis

BSS computability

Computable real number

Definition. $x \in \mathbb{R}$ *is* ***computable*** *if there exists an algorithm which takes* $\varepsilon \in \mathbb{D}^{>}$ *on input and produces an* ε*-approximation* $\tilde{x} \in \mathbb{D}$ *of* x *with* $|\tilde{x} - x| \leqslant \varepsilon$*. We denote* $\mathbb{R}^{\mathrm{com}}$ *to be the set of computable real numbers.*

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Mmx] use "asymptotix"; type_mode? := true;
Mmx] one: Approximator (Floating, Floating) == 1;
Mmx] e == exp one
Mmx] approximate (e, 1.0e-1000)
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$$624748477619370179475169176526172040868676631541156088557319107233592458655068693596197197036539240046478541880130367251305801992731793116433134463929399319082714196961652168013602803041772182448739785880587721883539643696750170168607678730423675712080858884633763266376601388$$: Rational

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Principle

Systematic approximation of real numbers $x \in \mathbb{R}$ by closed **balls**

$$x^\bullet = \mathcal{B}(c,r) = c + \mathcal{B}(r) \;=\; \{x \in \mathbb{R} : |x-c| \leqslant r\},$$

with $c \in \mathbb{R}$ and $r \in \mathbb{R}^{\geqslant}$ (or $c \in \mathbb{D} \cup \{\pm\infty\}$ and $r \in \mathbb{D}^{\geqslant} \cup \{\infty\}$)

$\longrightarrow$ while guaranteeing that the « genuine result » x lies in $x^\bullet$

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Operations

$f^\bullet : \mathcal{B}(\mathbb{R}, \mathbb{R}^>)^d \to \mathcal{B}(\mathbb{R}, \mathbb{R}^>)$ **extension** of $f : \mathbb{R}^d \to \mathbb{R}$ if

$$f(x_1^\bullet, \ldots, x_d^\bullet) = \{f(x_1, \ldots, x_d) : x_1 \in x_1^\bullet, \ldots, x_n \in x_n^\bullet\} \;\subseteq\; f^\bullet(x_1^\bullet, \ldots, x_n^\bullet),$$

for all $x_1^\bullet, \ldots, x_d^\bullet \in \mathcal{B}(\mathbb{R}, \mathbb{R}^>)$

Computability revisited

Computability = Ultimate approximability using ball arithmetic

Definition. *$x \in \mathbb{R}$ is **computable** if there exists an algorithm which takes $p \in \mathbb{N}$ on input and which produces a ball $x_p^\bullet \in \mathcal{B}(\mathbb{D}, \mathbb{D})$ with $\{x\} = \bigcap_p x_p^\bullet$.*

Definition. *a function $f\colon \mathbb{R} \to \mathbb{R}$ is **computable** if there exists an algorithm which takes $p \in \mathbb{N}$ on input and which produces a computable function $f_p^\bullet\colon \mathcal{B}(\mathbb{D}, \mathbb{D}) \to \mathcal{B}(\mathbb{D}, \mathbb{D})$ with $f(x^\bullet) \subseteq f_p^\bullet(x^\bullet)$ for all $x^\bullet \in \mathcal{B}(\mathbb{D}, \mathbb{D})$ and such that for each sequence $(x_p^\bullet)_{p\in\mathbb{N}} \in \mathcal{B}(\mathbb{D}, \mathbb{D})^{\mathbb{N}}$ with $\{x\} = \bigcap_p x_p^\bullet$, we have $\{f(x)\} = \bigcap_p f_p^\bullet(x_p^\bullet)$.*

Remark. Equivalent to TTE computability,
but relying on ordinary Turing machines

Exact formulas

$$\begin{aligned}
\mathcal{B}(x,r) +^\bullet \mathcal{B}(y,s) &= \mathcal{B}(x+y, r+s) \\
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\mathcal{B}(x,r) \times^\bullet \mathcal{B}(y,s) &= \mathcal{B}(x \times y, r \times (|y|+s) + |x| \times s)
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Elementary operations

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Approximate formulas

$x, y, r, s \in \mathbb{D}_{p,e} = \pm\{0, \ldots, 2^p - 1\}\, 2^{\pm\{0,\ldots,2^e-1\}} \cup \{\pm\infty\}$

$$\begin{aligned}
\mathcal{B}(x,r) +^\bullet \mathcal{B}(y,s) &= \mathcal{B}(x +^{\text{near}} y, r +^\uparrow s +^\uparrow \text{error}_+(x,y)) \\
\mathcal{B}(x,r) -^\bullet \mathcal{B}(y,s) &= \mathcal{B}(x -^{\text{near}} y, r +^\uparrow s +^\uparrow \text{error}_-(x,y)) \\
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\end{aligned}$$

Interval arithmetic

Interval arithmetic	Ball arithmetic
Good hardware support	Poor hardware support
Natural standardization	Non canonical standardization
Natural for subdivision algorithms	Not natural for subdivision algorithms
Loses factor 2 for multiple precision	Efficient for multiple precision
Less natural for vectorization	Easy to vectorize
Requires total ordering	Natural extension to $\mathbb{C}$ and beyond
Not natural under perturbations	Natural for perturbation algorithms
ad hoc theoretical foundations (IEEE)	Well understood ε-δ calculus

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Alefeld & Herzberger (1983): Introduction to interval analysis
Rump (2010): Verification methods: rigourous results using FP arithmetic

Digression: *a priori* error estimates

Example: computing an ε-approximation for $z = x + y$

Compute $\frac{\varepsilon}{2}$-approximations $\tilde{x}$ and $\tilde{y}$ for x and y
Return $\tilde{x} + \tilde{y}$

Digression: *a priori* error estimates

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Compute $\frac{\varepsilon}{2}$-approximations $\tilde{x}$ and $\tilde{y}$ for x and y
Return $\tilde{x} + \tilde{y}$

Advantage: adaptive precision in $1_{\text{cheap}} + 10^{-100}_{\text{expensive}}$

Digression: *a priori* error estimates

Example: computing an ε-approximation for $z = x + y$

Compute $\frac{\varepsilon}{2}$-approximations $\tilde{x}$ and $\tilde{y}$ for x and y
Return $\tilde{x} + \tilde{y}$

Advantage: adaptive precision in $1_{\text{cheap}} + 10^{-100}_{\text{expensive}}$

Problem 1: Horner evaluation of $P_d\, x^d + \cdots + P_0$ at $x = 1$

$\longrightarrow$ computes $\frac{\varepsilon}{2^{d-k}}$-approximation of P_k for $k = 0, \ldots, d-1$

$\longrightarrow$ small tolerances and bad complexity

Digression: *a priori* error estimates

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Problem 2: necessitates « dags » for the intermediate results

The enemy

Overestimation of the error

$$\mathcal{B}(0,1) - \mathcal{B}(0,1) \;=\; \mathcal{B}(0,2)$$

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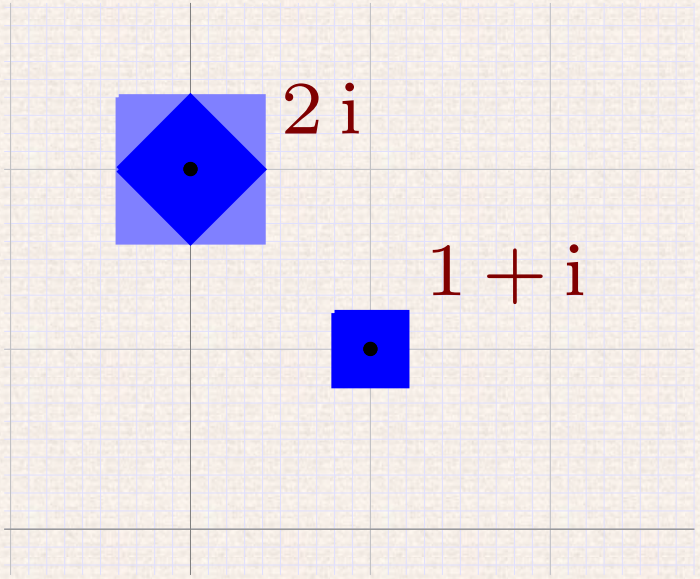
Wrapping effect

$$([1-\varepsilon, 1+\varepsilon] + [1-\varepsilon, 1+\varepsilon]\,\mathrm{i})^2 \;=\; [-2\,\varepsilon, 2\,\varepsilon] + [2-2\,\varepsilon, 2+2\,\varepsilon]\,\mathrm{i}$$



Example

```
Mmx] use "algebramix";
Mmx] power (z, n) == if n = 1 then z else z * power (z, n-1);
Mmx] z == complex (interval (1.0, 1.0000000001),
                   interval (1.0, 1.0000000001))
Mmx] [ power (z, 4*n) || n in 1 to 10 ]
Mmx]
```

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```

$$\begin{bmatrix} -4.00000000 \\ 16.0000000 \\ -64.000000 \\ 256.0000 \\ -1024.000 \\ 4096.00 \\ -16384.0 \\ 6.5536e4 \\ -2.62e5 \\ 1.05e6 \end{bmatrix}$$

```
Mmx]
```

Changing the kind of enclosures

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Mmx] use "algebramix";
Mmx] power (z, n) == if n = 1 then z else z * power (z, n-1);
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```
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```

Minimizing the depth of the computation

```
Mmx] use "algebramix";
```

```
Mmx] power (z, n) ==
       if n = 1 then z
       else power (z, n quo 2) * power (z, n - (n quo 2));
```

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```
Mmx]
```

Perturbation method (Hansen)

Algorithm: invert the matrix $M \in \mathcal{B}(\mathbb{D}, \mathbb{D})^{n \times n}$

- Write $M = \mathcal{B}(C, R)$, with $C \in \mathbb{D}^{n \times n}$ et $R \in (\mathbb{D}^{>})^{n \times n}$
- Numerically invert $U :\approx C^{-1}$
- Compute $E := C\,\mathcal{B}(U, 0) - \mathcal{B}(1, 0)$, so that

$$M^{-1} \;=\; U\,(MU)^{-1} = U\,(1+E)^{-1}$$

- With $\varepsilon := \|E\|_\infty = \max_{1 \leqslant i \leqslant n} (|E_{i1}| + \cdots + |E_{in}|)$, we have

$$\|(1+E)^{-1} - 1\|_\infty = \|-E + E^2 - E^3 + \cdots\|_\infty \leqslant \frac{\varepsilon}{1-\varepsilon}$$

- Let $\mathbb{1}$ be the matrix with entries $\mathbb{1}_{ij} = 1$
- Return $V := U\,\mathcal{B}(1, \frac{\varepsilon}{1-\varepsilon}\,\mathbb{1})$

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More generally $\rightsquigarrow$

Phase of **guessing** or **prospection**, next phase of **validation**

Example

```
Mmx] use "analyziz"
```

```
Mmx] rnd () == {
       x == uniform_deviate (0.0, 1.0);
       return ball (x, 0.00000001);
     };
```

```
Mmx] M == [ rnd () | j in 1 to 7 || i in 1 to 7 ]
```

```
Mmx] row_echelon M
```

```
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$$\begin{bmatrix} 0.6003643 & 0.3710189 & 0.4791752 & 0.1559513 & 0.4159951 & 0.1407053 & 0.0044911 \\ 0.5120348 & 0.7247114 & 0.1947940 & 0.655922 & 0.9355430 & 0.251486 & 0.2468534 \\ 0.462639 & 0.5648943 & 0.744350 & 0.4275757 & 0.7804102 & 0.9920186 & 0.7571225 \\ 0.2508314 & 0.0131642 & 0.8669620 & 0.6802075 & 0.7552889 & 0.3166660 & 0.3160914 \\ 0.2876469 & 0.988199 & 0.6639689 & 0.303261 & 0.498002 & 0.0277941 & 0.9222224 \\ 0.1304958 & 0.7945297 & 0.7988684 & 0.5890020 & 0.3421886 & 0.0285099 & 0.9893513 \\ 0.3882938 & 0.7500550 & 0.9237840 & 0.594590 & 0.8730709 & 0.5362681 & 0.134547 \end{bmatrix}$$

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```
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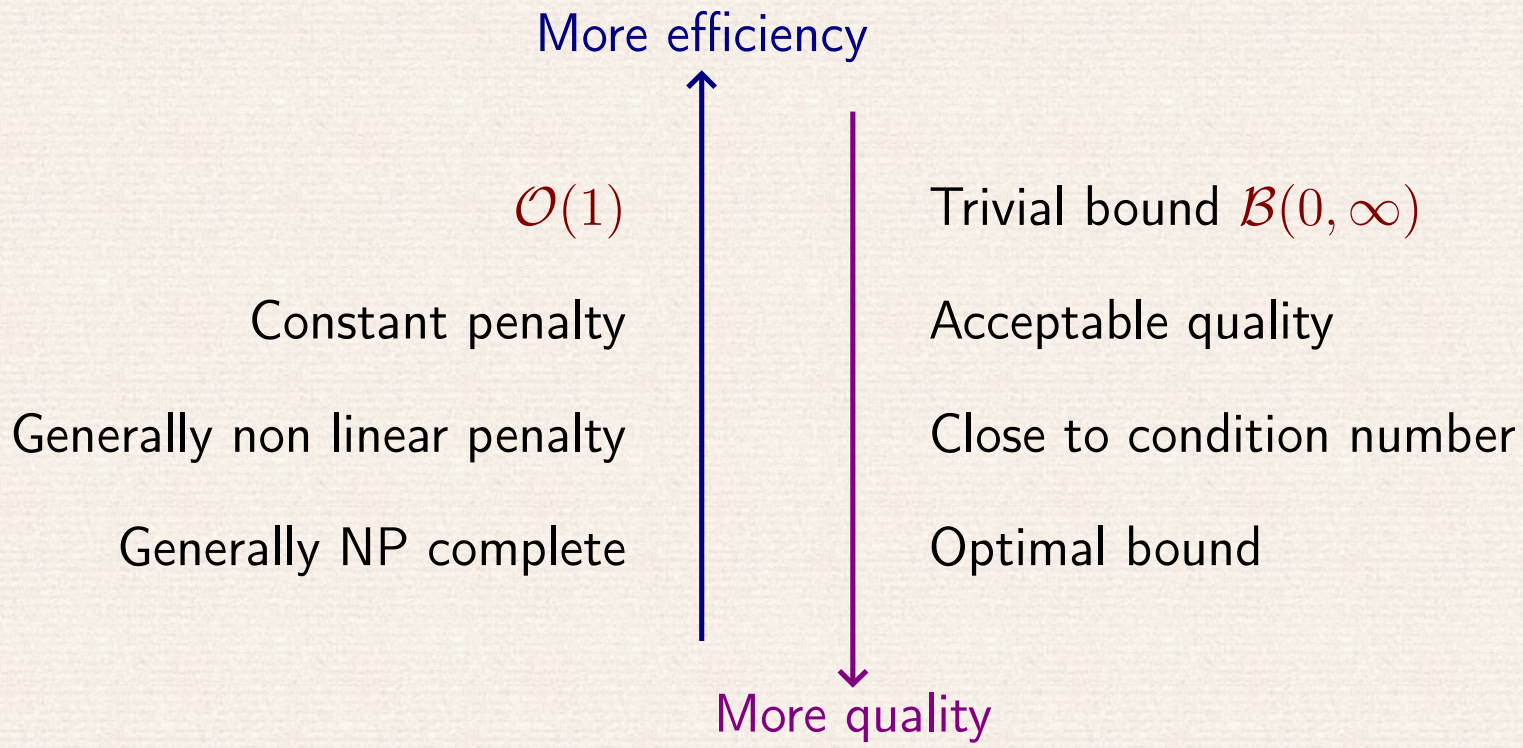
```
Mmx]
```

Graal: efficient *and* tight error bounding



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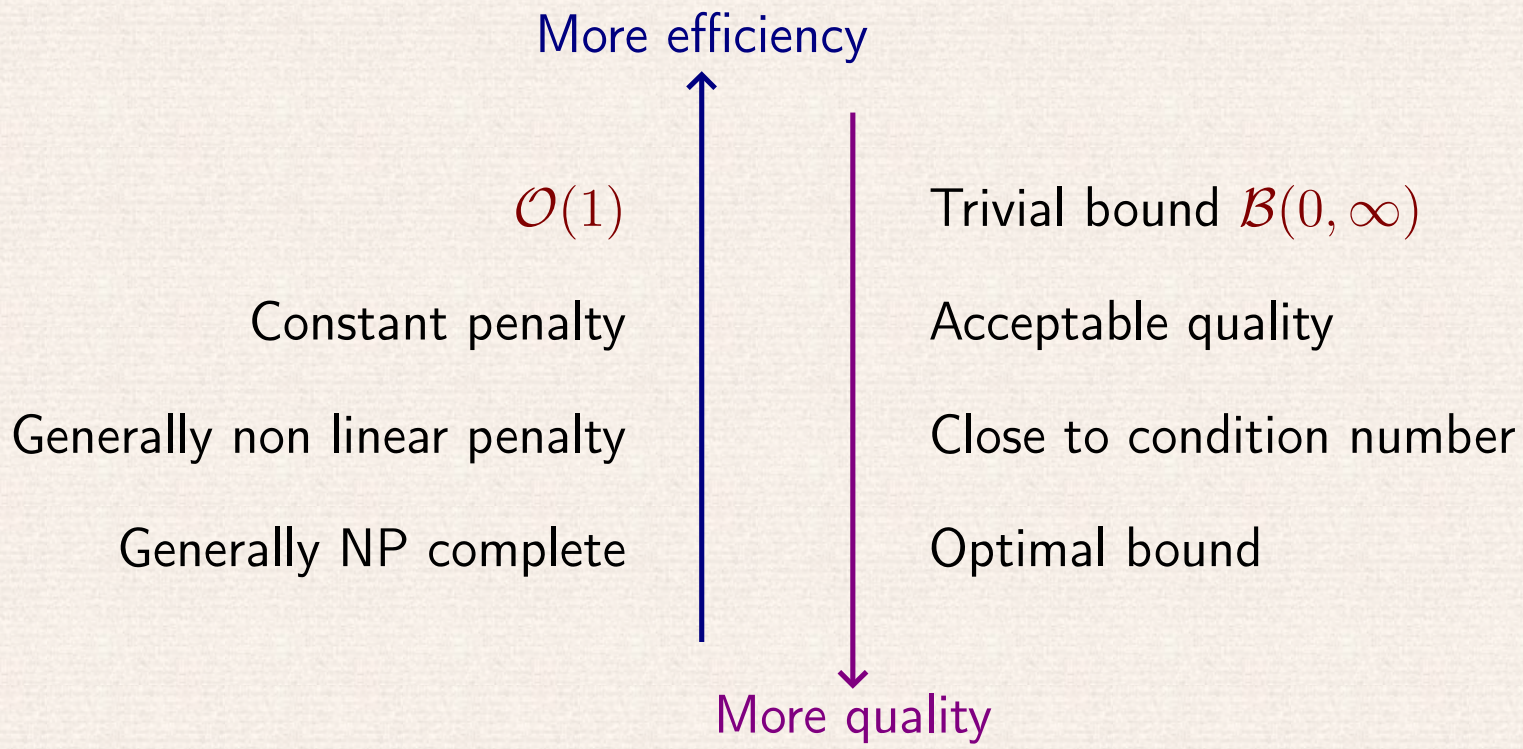

Early termination: start with fast strategy, but *a priori* of poor quality
Use slower algorithms only if this quality is not enough

Example: matrix multiplication

Majoration $|(MN)_{ij}| \leqslant [\max |M_{i\cdot}|]\,[\max |N_{\cdot j}|]$:

1 multiplication in $\mathbb{D}^{n\times n}$ and $\mathcal{O}(n^2)$ majorations, possibly bad quality

Compromise: cut M and N into $k \times k$ matrices for fixed k:

α_k multiplications in $\mathbb{D}^{n\times n}$ with $1 \leqslant \alpha_k \leqslant 3$, usually good quality

Compromise: resort to naive multiplication in case of ill conditioning

Between 1 and 3 multiplications in $\mathbb{D}^{n\times n}$, always good quality

Rewrite $M, N \in \mathcal{B}(\mathbb{D}^{n\times n}, \mathbb{D}^{n\times n})$:

3 multiplications in $\mathbb{D}^{n\times n}$ (with 2 multiplications at single precision)

Naive product M, N with $M, N \in \mathcal{B}(\mathbb{D}, \mathbb{D})^{n\times n}$:

$\mathcal{O}(n^3)$ operations in $\mathcal{B}(\mathbb{D}, \mathbb{D})$, always good quality

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General principle (observed in practice):

Validation cost asymptotically negligible for well conditioned operations

Example: polynomial multiplication

$$P, Q \in \mathcal{B}(\mathbb{D}_p, \mathbb{D}_{52})[x]_{<d}$$

FFT multiplication
Cost $\tilde{\mathcal{O}}(d\,p)$, only good quality if all coefficients of same magnitude

FFT multiplication with scaling $[P(\alpha\,x)\,Q(\alpha\,x)] \circ (x/\alpha)$
Cost $\tilde{\mathcal{O}}(d\,p)$, good quality for truncated power series

FFT multiplication with scaling and cutting into blocks (VdH, Arb)
(vdH) Cost $\tilde{\mathcal{O}}(d\,p)$, good quality in terms of roots of polynomials
(Arb) Cost between $\tilde{\mathcal{O}}(d\,p)$ and $\tilde{\mathcal{O}}(d^2\,p)$, optimal quality

Naive multiplication of $P, Q \in \mathcal{B}(\mathbb{D}_p, \mathbb{D}_{52})[x]_{<d}$
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Cost $\tilde{\mathcal{O}}(d^2\,p)$, optimal quality

Condition number (intrinsic)

$$
\begin{aligned}
f\colon \mathbb{R}^n &\to \mathbb{R}^k \\
\kappa_f(x) &= \limsup_{\varepsilon \to 0} \frac{\|f(x+\varepsilon) - f(x)\|}{\|f(x)\|} \Big/ \frac{\|\varepsilon\|}{\|x\|} \\
\kappa(M) &= \|M\| \, \|M^{-1}\|
\end{aligned}
$$

Condition numbers and precision loss

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$$
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\kappa_f(x) &= \limsup_{\varepsilon \to 0} \frac{\|f(x+\varepsilon) - f(x)\|}{\|f(x)\|} \Big/ \frac{\|\varepsilon\|}{\|x\|} \\
\kappa(M) &= \|M\|\,\|M^{-1}\|
\end{aligned}
$$

Precision loss (of an algorithm)

$$
\begin{aligned}
f_p\colon \mathbb{D}_p^n &\to \mathbb{D}_p^k \\
\chi_{f_p}(x) &= \frac{\|f_p(x) - f(x)\|}{\|f(x)\|} / (2^p\,\kappa_f(x))
\end{aligned}
$$

Optimal extension (intrinsic)

$$
\begin{aligned}
f\colon \mathbb{R}^n &\to \mathbb{R}^k \\
f^*(\mathcal{B}(x,r)) &= \mathcal{B}\left(f(x), \sup_{x' \in \mathcal{B}(x,r)} |f(x') - f(x)| \right)
\end{aligned}
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Optimal extensions and overestimation

Optimal extension (intrinsic)

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Overestimation and punctual overestimation (of an algorithm)

$$\begin{aligned} f^\bullet\colon \mathcal{B}(\mathbb{R},\mathbb{R})^n &\to \mathcal{B}(\mathbb{R},\mathbb{R})^k \\ \chi_{f^\bullet}(\mathcal{B}(x,r)) &= \inf_{\lambda} \{ f^\bullet(\mathcal{B}(x,r)) - f(x) \subseteq \lambda \, (f^*(\mathcal{B}(x,r)) - f(x)) \} \\ \chi_{f^\bullet}(x) &= \limsup_{r \to 0} \chi_{f^\bullet}(\mathcal{B}(x,r)) \end{aligned}$$

Optimal extensions and overestimation

Optimal extension (intrinsic)

$$\begin{aligned} f\colon \mathbb{R}^n &\to \mathbb{R}^k \\ f^*(\mathcal{B}(x,r)) &= \mathcal{B}\Big(f(x), \sup_{x' \in \mathcal{B}(x,r)} |f(x') - f(x)| \Big) \end{aligned}$$

Overestimation and punctual overestimation (of an algorithm)

$$\begin{aligned} f^\bullet\colon \mathcal{B}(\mathbb{R},\mathbb{R})^n &\to \mathcal{B}(\mathbb{R},\mathbb{R})^k \\ \chi_{f^\bullet}(\mathcal{B}(x,r)) &= \inf_{\lambda} \{ f^\bullet(\mathcal{B}(x,r)) - f(x) \subseteq \lambda \, (f^*(\mathcal{B}(x,r)) - f(x)) \} \\ \chi_{f^\bullet}(x) &= \operatorname*{limsup}_{r \to 0} \chi_{f^\bullet}(\mathcal{B}(x,r)) \end{aligned}$$

Question: for $f^\bullet$ constructed using $+^\bullet, -^\bullet, \times^\bullet$, do we have

$$\chi_{f^\bullet}(\mathcal{B}(x,r)) \leqslant \sup_{x \in \mathcal{B}(x,r)} \chi_{f^\bullet}(x) \ ?$$

Overestimation of standard arithmetic

Example

$$
\begin{aligned}
f(x) &= (x-1)^2 \\
f^\bullet(x^\bullet) &= (x^\bullet)^2 -^\bullet 2\, x^\bullet +^\bullet 1
\end{aligned}
$$

$$
\begin{aligned}
f^*(\mathcal{B}(x,\varepsilon)) &= \mathcal{B}(f(x), 2\,|x-1|\,\varepsilon + \mathcal{O}(\varepsilon^2)) \\
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In general

$$\chi_{f^\bullet}(x) = \limsup_{\varepsilon\neq 0} \frac{(\bar\nabla f)(x)\cdot|\varepsilon|}{|(\nabla f)(x)|\cdot|\varepsilon|} \qquad \text{or } (d=1) \qquad \frac{(\bar\nabla f)(x)}{|f'(x)|}$$

$$\begin{aligned} \bar\nabla c &= (0,\ldots,0) && (c\in\mathbb{R}) \\ \bar\nabla X_k &= (0,\overset{k-1}{\ldots},0,1,0,\ldots,0) && (k\in\{1,\ldots,r\}) \\ \bar\nabla(f\pm g) &= \bar\nabla f + \bar\nabla g \\ \bar\nabla(f\,g) &= (\bar\nabla f)\,|g| + |f|\,(\bar\nabla g), \end{aligned}$$

Taylor model of order n on $\mathcal{B}(0,r)$

$$\begin{aligned}\mathcal{B}_r(\varphi,K) &= \{f \text{ analytic on } \mathcal{B}(0,r)\colon \forall|z|\leqslant r, |f(z)-\varphi(z)|\leqslant K\}\\ \mathcal{B}_r(\mathbb{D}[z]_{<n},\mathbb{D}) &= \{\mathcal{B}_r(f_0+\cdots+f_{n-1}\,z^{n-1},K)\colon f_0,\ldots,f_{n-1},K\in\mathbb{D}\}\end{aligned}$$

Taylor models (Berz et Makino)

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\end{aligned}$$

Product

$$\begin{aligned}
& \mathcal{B}_r(f_0 + \cdots + f_{n-1} z^{n-1}, K)\, \mathcal{B}_r(g_0 + \cdots + g_{n-1} z^{n-1}, L) \\
= \; & \mathcal{B}_r(f_0 g_0 + \cdots + (f_0 g_{n-1} + \cdots + f_{n-1} g_0) z^{n-1}, E^{\mathrm{tr}} + E^{\mathrm{rnd}}) \\
E^{\mathrm{tr}} = \; & |f_1|\,|g_{n-1}| + |f_2|\,(|g_{n-1}| + |g_{n-2}|) + \cdots + |f_{n-1}|\,(|g_1| + \cdots + |g_{n-1}|) \quad \text{(up)} \\
E^{\mathrm{rnd}} = \; & [|f_0|\,|g_{n-1}| + \cdots + (|f_0| + \cdots + |f_{n-1}|)\,|g_0|]\, C\, 2^{-p} \quad \text{(up)}
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E^{\mathrm{tr}} = \; & |f_1|\,|g_{n-1}| + |f_2|\,(|g_{n-1}| + |g_{n-2}|) + \cdots + |f_{n-1}|\,(|g_1| + \cdots + |g_{n-1}|) \quad \text{(up)} \\
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\end{aligned}$$

Intégration

$$\int \mathcal{B}_r(f_0 + \cdots + f_{n-1} z^{n-1}, K) = \mathcal{B}_r(f_1 + \cdots + \tfrac{f_{n-2}}{n-1} z^{n-1}, K\, r + \tfrac{|f_{n-1}|}{n} r^n + E^{\mathrm{rnd}})$$

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Idea: use a Taylor model of order $\geqslant 2$

$$\begin{aligned} f^{\text{Taylor}}(\mathcal{B}_r(x,0)) &= \mathcal{B}_r(y_0 + y_1\,t, \varepsilon) \\ f^\bullet(\mathcal{B}(x,r)) &= \mathcal{B}(y_0, |y_1|\,r + \varepsilon) \\ \chi_{f^\bullet}(x) &= 1 \end{aligned}$$

Example challenge

Complexity to solve

$$P(z) \;=\; 0,$$

with $P \in \mathbb{C}[z]$, $\deg P = d$, at bit precision p?

Challenge 1. Optimal complexity, uniformly in p and d

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$p \gg d \rightsquigarrow \tilde{\mathcal{O}}(p\,d)$

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$p \gg d \rightsquigarrow \tilde{\mathcal{O}}(p\,d)$
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$p \ll \sqrt[3]{d} \rightsquigarrow \tilde{\mathcal{O}}(d^{3/2}\,p^{3/2})$

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Challenge 3. Dependence on condition number

Random polynomials (well conditioned)

```
Mmx] use "analyziz"; include "graphix/points.mmx";
Mmx] pi == 4.0 * arctan 1.0;
Mmx] rnd () == exp (complex (0.0, uniform_deviate (0.0, 2*pi)));
Mmx] p == polynomial (rnd () | i in 0 to 100);
Mmx] $points roots p
Mmx]
```

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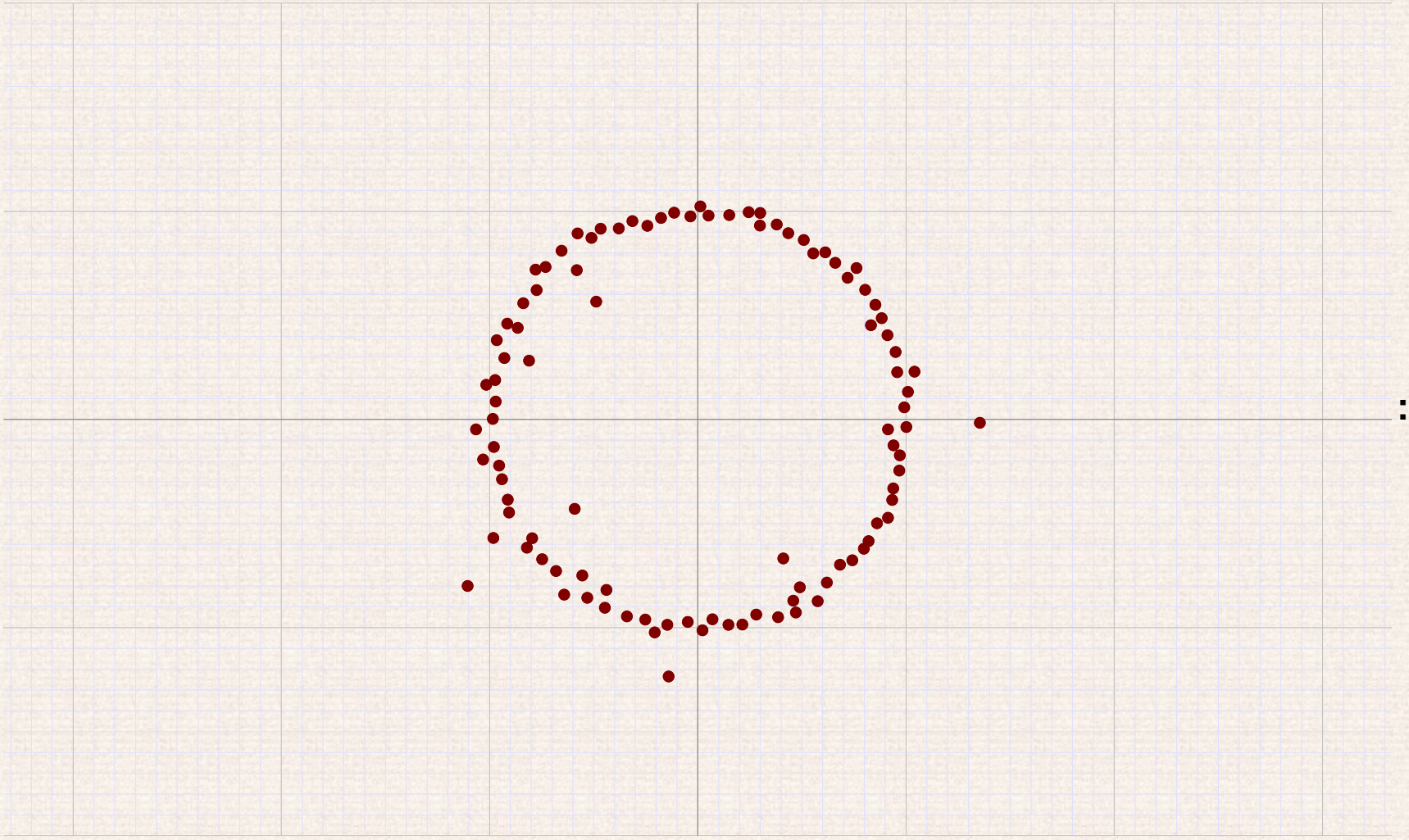

Compound

Multiple roots

```
Mmx] p == poly (1.0, -1.0)^100;  // (1 - z)^100
Mmx] $points (roots p)
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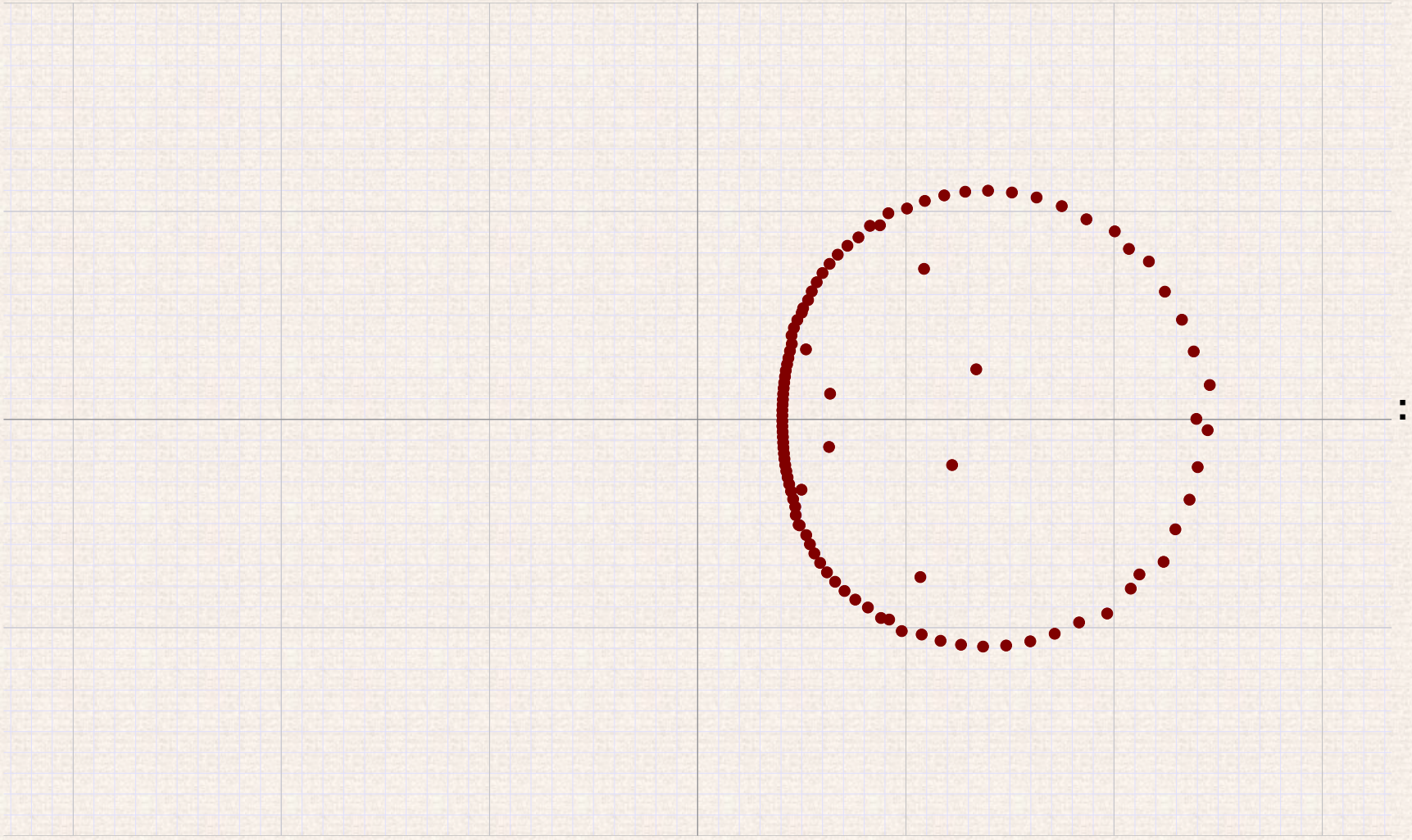

Random roots on half circle

```
Mmx] v == [ rnd () | i in 1 to 100 ];
Mmx] v == [ sqrt rnd () | i in 1 to 100 ];
Mmx] p == annulator v;
Mmx] $points (roots p)
Mmx]
```

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Compound

Truncated power series

```
Mmx] z == series (0.0, 1.0);
Mmx] B == exp (exp z - 1);
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```

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```

```
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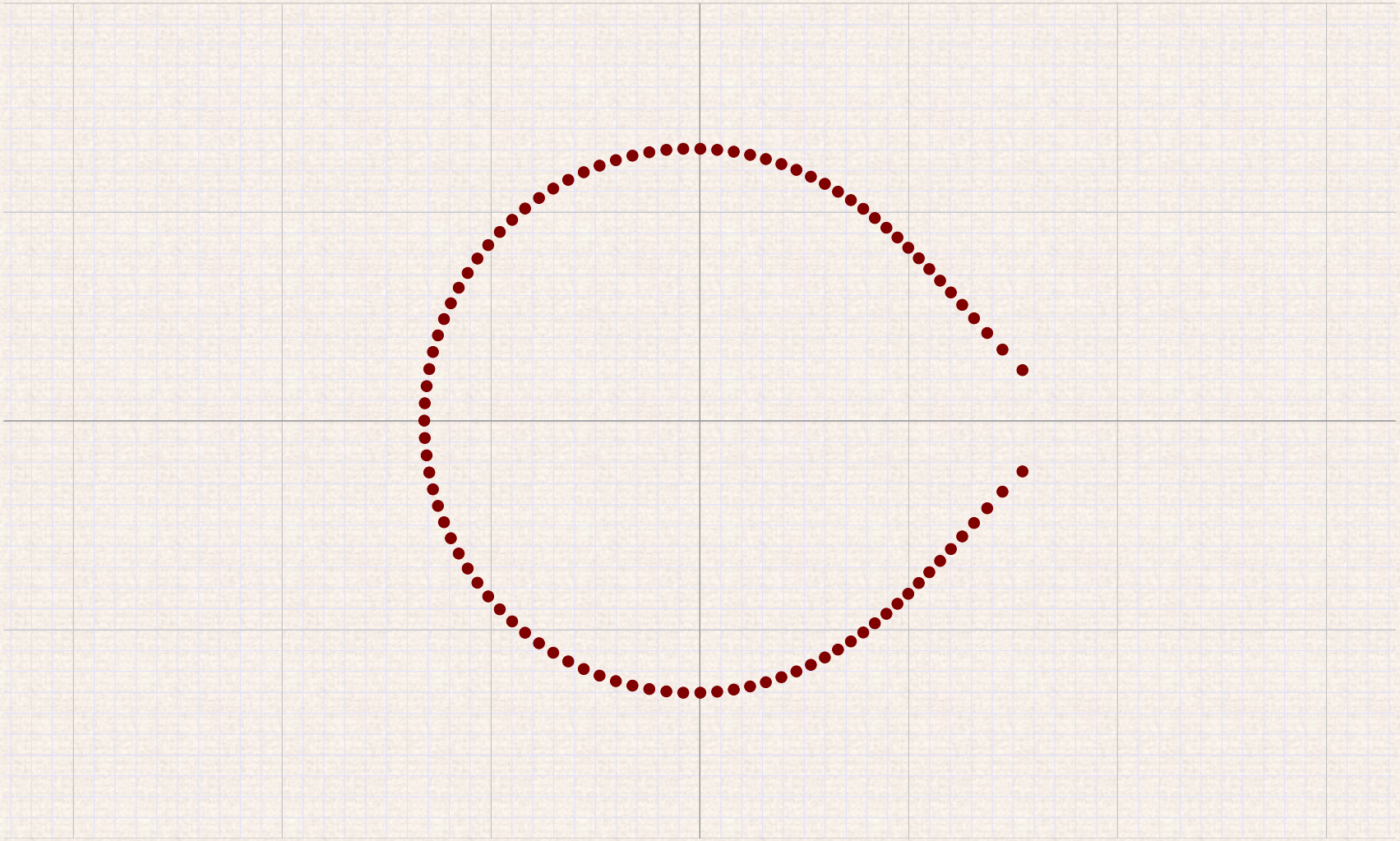

Truncated power series

```
Mmx] z == series (0.0, 1.0);
Mmx] f == integrate exp (-z*z);
Mmx] p == f[0,200];
Mmx] $points (0.2 * roots p)
Mmx]
```

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```
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