Towards a model theory for transseries

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http://www.TeXmacs.org

$(x \succ 1)$

$$e^{e^{x}+\cdots}+\cdots$$

$(x \succ 1)$

$$e^{e^x + \frac{e^x}{x} + \cdots} + \cdots$$

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$$\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots} + \frac{2}{\log x}\,\mathrm{e}^{\mathrm{e}^x + \cdots} + \cdots$$

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- Dahn & Göring
- Écalle
- Detailed treatment in LNM 1888: "Transseries and Real Differential Algebra"

$$\frac{1}{1-x^{-1}-x^{-\mathrm{e}}} = 1+x^{-1}+x^{-2}+x^{-\mathrm{e}}+x^{-3}+x^{-\mathrm{e}-1}+\cdots$$

$$\frac{1}{1-x^{-1}+\mathrm{e}^{-x}} = 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots$$

$$-\mathrm{e}^{x}\int\frac{\mathrm{e}^{-x}}{x} = \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots$$

$$\Gamma(x) = \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x\,-1)}}{x^{1/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x\,-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x\,-1)}}{288\,x^{5/2}}+\cdots$$

$$\zeta(x) = 1+2^{-x}+3^{-x}+4^{-x}+\cdots$$

$$\varphi(x) = \frac{1}{x}+\varphi(x^{\pi}) = \frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots$$

$$\psi(x) = \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x}) = \frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots$$

Transseries are transfinite series

$$f = \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \mathfrak{m} \qquad (\mathfrak{T}\text{: set of transmonomials, coefficients } f_{\mathfrak{m}} \in \mathbb{R})$$

$$= -3\,e^{e^x} + e^{\frac{e^x}{\log x} + \frac{e^x}{\log^2 x} + \frac{e^x}{\log^3 x} + \cdots} - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x \log x} + \frac{1}{x(\log x)^2} + \cdots$$

$$= f_{\succ} + f_{\asymp} + f_{\prec}$$

$$f_{\succ} = -3\,e^{e^x} + e^{\frac{e^x}{\log x} + \frac{e^x}{\log^2 x} + \frac{e^x}{\log^3 x} + \cdots} - x^{11}$$

$$f_{\asymp} = 7$$

$$f_{\prec} = \frac{\pi}{x} + \frac{1}{x \log x} + \frac{1}{x(\log x)^2} + \cdots$$

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f &= \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}} \, \mathfrak{m} \qquad (\mathfrak{T}\text{: set of transmonomials, coefficients } f_{\mathfrak{m}} \in \mathbb{R}) \\
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f_{\asymp} &= 7 \\
f_{\prec} &= \frac{\pi}{x} + \frac{1}{x \log x} + \frac{1}{x (\log x)^2} + \cdots
\end{aligned}
$$

Transmonomials and exponential structure

$$
\begin{aligned}
\mathfrak{T} &= \exp \mathbb{T}_{\succ} \\
\mathbb{T}_{\succ} &= \{ f \in \mathbb{T} : f_{\succ} = f \} \\
\exp f &= (\exp f_{\succ})\, (\exp f_{\asymp})\, (\exp f_{\prec}) \\
\exp f_{\prec} &= 1 + f_{\prec} + {}^1\!/_2\, f_{\prec}^2 + {}^1\!/_6\, f_{\prec}^3 + \cdots .
\end{aligned}
$$

Differentiation

$$\begin{aligned} x' &= 1 \\ (\mathrm{e}^f)' &= f'\,\mathrm{e}^f \\ \Big(\sum_{\mathfrak{m}\in\mathfrak{T}} f_{\mathfrak{m}}\,\mathfrak{m}\Big)' &= \sum_{\mathfrak{m}\in\mathfrak{T}} f_{\mathfrak{m}}\,\mathfrak{m}' \end{aligned}$$

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Ordering

$$f \geqslant 0 \Longleftrightarrow (\exists g \in \mathbb{T}, g^2 = f) \Longleftrightarrow c(f) \geqslant 0$$

$$c\Bigl(-3\,\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^{\frac{\mathrm{e}^x}{\log x} + \frac{\mathrm{e}^x}{\log^2 x} + \frac{\mathrm{e}^x}{\log^3 x} + \cdots} - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x \log x} + \cdots\Bigr) = -3$$

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Valuation

$$\begin{aligned} \mathcal{O} &= \{f \in \mathbb{T} : \exists c \in \mathbb{R}^>, |f| \leqslant c\} \\ o &= \{f \in \mathbb{T} : \forall c \in \mathbb{R}^>, |f| < c\} \\ \mathcal{O}/o &\cong \mathbb{R} \end{aligned}$$

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$$\begin{aligned} v(f) \geqslant v(g) &\Longleftrightarrow f \preccurlyeq g \Longleftrightarrow f \in \mathcal{O}\, g \\ v(f) > v(g) &\Longleftrightarrow f \prec g \Longleftrightarrow f \in o\, g \end{aligned}$$

Basic closure properties

$\mathbb{T}$ is Liouville closed (i.e. real closed, stable under integration and exponential integration).

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Given a differential polyonomial $P \in \mathbb{T}\{Y\}$ (i.e. $P = P(Y, Y', \ldots, Y^{(r)})$) and $f < g$ in $\mathbb{T}$ such that $P(f)\,P(g) < 0$, there exists an $y \in \mathbb{T}$ with $f < y < g$ and $P(y) = 0$.

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Newtonianity [vdH], differential analogue of henselian fields

Any *quasi-linear* differential equation with asymptotic side-condition

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Definition. Let K be an ordered differential field and denote

$$\begin{aligned} C &= \{c \in K : c' = 0\} \\ \mathcal{O} &= \{a \in K : |a| \leqslant c \text{ for some } c \in C\} \\ \scriptstyle\mathcal{O} &= \{a \in K : |a| < c \text{ for all } c > 0 \text{ in } C\}. \end{aligned}$$

K is an *H-field* if:

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A *d-valued* field is a valued differential field such that

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Also: pre-H-fields, pre-d-valued fields, asymptotic fields, ...

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Complex transseries

- $\mathbb{T}[\mathrm{i}]$ is algebraically closed.
- Zeros of $L \in \mathbb{T}[\mathrm{i}][\partial]$ of order r form a subspace of $\mathbb{C}[[\mathrm{e}^{\mathbb{T}^{\succ}[\mathrm{i}]}]]$ of dimension r.
- Any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \mathbb{C}$ admits a zero in $\tilde{\mathbb{T}}$.

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Differentially closed fields?

K is *d-closed* if for every $P \in K\{Y\}$ of order r and $Q \in K\{Y\}$ of order $s < r$, there exists an $y \in K$ with $P(y) = 0$ and $Q(y) \neq 0$.

Unfortunately [Rosenlicht]: d-closed d-valued fields do not exist.

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- Generalize this theory to arbitrary H-fields.
- Main obstruction: problem with gaps

$$\gamma = \frac{1}{x \log x \log \log x \cdots}.$$

Indeed: should we have $\int \gamma \succ 1$ or $\int \gamma \prec 1$ in extensions?

Systematic study of *asymptotic d-algebraic equations*

$$P(y) = 0, \quad y \prec v,$$

where $P \in K\{Y\}$ and $\varphi \in K$. For example:

$$\mathrm{e}^{-\mathrm{e}^{x}} y^2 y'' + y^2 - 2\,x\,y\,y' - 7\,\mathrm{e}^{-x} y' - 4 + \frac{1}{\log x} = 0, \quad y \prec x.$$

The slopes correspond to dominant monomials of candidate solutions. Two kinds:

Approximate solutions of algebraic type. $y = \pm 2 + \cdots$

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- Natural analogue of usual Newton polygon method.
- Slopes can be read off from the Newton diagram modulo "adjustments", e.g. $y' = \mathrm{e}^{\mathrm{e}^x}$ implies $y \sim \mathrm{e}^{\mathrm{e}^x} / \mathrm{e}^x$. In general: **equalizer theorem**.

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Approximate solutions of differential type. $y = c \sqrt{x} + \cdots$

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Cancellation in homogeneous component $y^2 - 2 x y y' = \left(1 - 2 x \frac{y'}{y}\right) y^2 \rightsquigarrow$ Riccati equation

$$1 - 2 x y^{\dagger} = 0,$$

whence $y^{\dagger} = 1/2 \, x$ and $y \asymp \sqrt{x}$.

Refinements. Given an approximate solution $y \sim \varphi$, performing the *refinement*

$$y = \varphi + \tilde{y}, \quad \tilde{y} \prec \varphi$$

leads to a new asymptotic differential equation in $\tilde{y}$. Example: $y = 2 + \tilde{y}, \tilde{y} \prec 1$ transforms

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Newton degree. [analogue of Weierstrass degree] Abscissa of highest slope of Newton diagram which satisfies the asymptotic side condition (e.g. two in our example).

- Newton degree can only decrease during refinements.
- If Newton degree is one, then the equation is said to be *quasi-linear*. In that case, it admits at least one transseries solution.
- Using a suitable generalization of Tschirnhausen transforms, one may reach a quasi-linear equation after only a finite number of refinements.

Multiplicative conjugation

Reduce general asymptotic side condition $y \prec v$ to the case when $v = 1$:

$$(P(y) = 0, y \prec v) \Longleftrightarrow (P_{\times v}(\tilde{y}) = P(\tilde{y} v) = 0, \tilde{y} \prec 1), \qquad \tilde{y} = y / v$$

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Consider $P \in \mathbb{T}\{Y\}$ as a series $P = \sum_{\mathfrak{m} \in \mathfrak{T}} P_{\mathfrak{m}}\, \mathfrak{m}$, with $P_{\mathfrak{m}} \in \mathbb{R}\{Y\}$.
Then $D_P = D(P)$ is the "leading coefficient" of P.

$$D\left(\frac{3}{1 - \mathrm{e}^{-x}}\, Y^2\, Y' - \frac{1}{x}\, Y^2 + (Y')^2 + \mathrm{e}^{-x} \right) \;=\; 3\, Y^2 + (Y')^2.$$

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Requires a cross section of the value group inside K for a general H-field.

Compositional conjugation

Replacing the derivation ∂ by a new derivation $\tilde{\partial} = \phi^{-1}\, \partial$.
Corresponds to a postcomposition $\tilde{y} = y \circ u$.
We typically want to take ϕ as small as possible, while preserving the smallness of $\tilde{\partial}$.
Notation: K^{ϕ}: the field K with derivation $\tilde{\partial}$, P^{ϕ}: the counterpart of $P \in K\{Y\}$ in $K^{\phi}\{Y\}$.

$$\frac{3}{1-e^{-x}} Y^2 Y' - \frac{1}{x} Y^2 + (Y')^2 + e^{-x} \quad \xrightarrow{D} \quad 3 Y^2 Y' + (Y')^2$$

$$\Big\downarrow \; Y = \tilde{Y} \circ \log$$

$$\frac{3}{(1-e^{-x})\, x} (\tilde{Y}^2 \tilde{Y}') \circ \log - \frac{1}{x} \tilde{Y}^2 \circ \log + \frac{(\tilde{Y}')^2 \circ \log}{x^2} + e^{-x} \quad \xrightarrow{D} \quad 3 \tilde{Y}^2 \tilde{Y}' - \tilde{Y}$$

$$\Big\downarrow \; \tilde{Y} = \tilde{\tilde{Y}} \circ \log$$

$$\frac{3}{(1-e^{-x})\, x \log x} (\tilde{\tilde{Y}}^2 \tilde{\tilde{Y}}') \circ \log_2 - \frac{1}{x} \tilde{\tilde{Y}}^2 \circ \log_2 + \frac{(\tilde{\tilde{Y}}')^2 \circ \log_2}{x^2 \log^2 x} + e^{-x} \quad \xrightarrow{D} \quad - \tilde{\tilde{Y}}$$

Theorem. *For any $P \in \mathbb{T}^{\mathrm{gb}}\{Y\} \setminus \{0\}$, there exists an $N \in \mathbb{R}[Y]\,(Y')^{\mathbb{N}}$ with*

$$D_{P^{\phi}} = N,$$

for any $\phi = \frac{1}{x\, \ell_1 \ell_2 \cdots \ell_{l-1} x} = \ell_l'$ with l sufficiently large, where $\ell_k = \log \circ \overset{k\times}{\cdots} \circ \log$.

The following special *cuts* will play a crucial role:

$$\begin{aligned} \gamma_{\mathbb{T}} &= \frac{1}{\ell_0 \, \ell_1 \, \ell_2 \cdots} & (\forall n, (1/\ell_n)' \prec \gamma_{\mathbb{T}} \prec \ell_n') \\ \lambda_{\mathbb{T}} &= \frac{1}{\ell_0} + \frac{1}{\ell_0 \, \ell_1} + \frac{1}{\ell_0 \, \ell_1 \, \ell_2} + \cdots = -\gamma_{\mathbb{T}}^{\dagger} \\ \omega_{\mathbb{T}} &= \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2 \, \ell_1^2} + \frac{1}{\ell_0^2 \, \ell_1^2 \, \ell_2^2} + \cdots = -\lambda_{\mathbb{T}}^2 - 2 \lambda_{\mathbb{T}}' \end{aligned}$$

Even though $\gamma_{\mathbb{T}}$, $\lambda_{\mathbb{T}}$ and $\omega_{\mathbb{T}}$ are not in $\mathbb{T}$, the sets

$$\begin{aligned} \Gamma(\mathbb{T}) &= \{a \in \mathbb{T} : a < \gamma_{\mathbb{T}}\} \\ \Lambda(\mathbb{T}) &= \{a \in \mathbb{T} : a < \lambda_{\mathbb{T}}\} \\ \Omega(\mathbb{T}) &= \{a \in \mathbb{T} : a < \omega_{\mathbb{T}}\} \end{aligned}$$

are definable subsets of $\mathbb{T}$. For instance,

$$\begin{aligned} \Gamma(\mathbb{T}) &= \{a \in \mathbb{T} : \forall b \in \mathbb{T}, b \succ 1 \Rightarrow a \neq b^{\dagger}\} \\ &= \{-a' : a \in \mathbb{T}, a \geqslant 0\}. \end{aligned}$$

In other words, $\gamma_{\mathbb{T}}$, $\lambda_{\mathbb{T}}$ and $\omega_{\mathbb{T}}$ are *definable cuts* in $\mathbb{T}$.

Gaps

$\gamma = \gamma_K \in K$, but $\int \gamma \notin K$. In other words, for all $a \in K$ with $a \succ 1$, we have

$$a^\dagger \succ \gamma \succ (1/a)'.$$

Theorem. (AvdD) *If K admits a gap γ, then K admits exactly two "Liouville closures".*

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Indirect gaps

K admits no gap (i.e. K is γ-*free*), but $\lambda \in K$ is such that for all $a \in K$ with $a \succ 1$, we have

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In general

Each of the following cases can occur:

$$\begin{array}{rcl} & & \gamma \in K \\ \gamma \notin K & \wedge & \lambda \in K \\ \lambda \notin K & \wedge & \omega \in K \\ \omega \notin K & & \end{array}$$

Definition

$$\omega = \omega_K \colon K \to K, \qquad \omega(z) := -2\,z' - z^2,$$

$\omega \in K$, if for all $a \succ 1$ in K, we have

$$\omega - \omega(a^{\dagger\dagger}) \prec (a^{\dagger})^2.$$

K is *ω-free* if

$$\forall a, \exists b, [b \succ 1 \wedge a - \omega(b^{\dagger\dagger}) \succcurlyeq (b^{\dagger})^2].$$

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Examples

- $\mathbb{T}$ is ω-free.
- If K has asymptotic integration and K is a union of H-subfields, each of which has a smallest comparability class, then K is ω-free.
- There exist Liouville-closed H-fields that are not ω-free.

Differential Newton polynomials

Theorem. *If K is ω-free, then we can define N_P for any $P \in K\{Y\}$, and $N_P \in C[Y](Y')^{\mathbb{N}}$.*

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Relation with theorem of Écalle

Let $\lambda = \frac{1}{x} + \frac{1}{x\,\ell_1} + \frac{1}{x\,\ell_1\,\ell_2} + \cdots$ and $P \in \mathbb{R}\{Y\} \setminus \mathbb{R}$. Then the first ω terms of $P(\lambda)$ either "behave" like λ or like ω.

In particular, we cannot have $P(\lambda) = \frac{1}{x^n} + \frac{1}{x^n\,\ell_1^n} + \frac{1}{x^n\,\ell_1^n\,\ell_2^n} + \cdots$ for $n \geqslant 3$.

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Relation with second order linear differential equations

$y'' = -y$ has no non-zero solution $y \in \mathbb{T}$.

$y'' = x\,y$ has two $\mathbb{R}$-linearly independent solutions in $\mathbb{T}$.

In general, $4\,y'' + f y = 0$ has a non-zero solution if and only if $f < \omega$.

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Every $P \in K\{Y\}$ with $\deg N_P = 1$ admits a zero in $\mathcal{O}$.

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Corollary. *Let K be a real closed newtonian H-field. Then*

1. *Each d-polynomial in $K[\mathrm{i}]\{Y\}$ of positive degree has a zero in $K[\mathrm{i}]$.*
2. *Each linear differential operator in $K[\mathrm{i}][\partial]$ of positive order is a composition of such operators of order 1.*
3. *Each linear differential operator in $K[\partial]$ of positive order is a composition of such operators of order 1 and order 2.*

Theorem. *If K is an ω-free H-field with divisible value group, then K has an immediate d-algebraic newtonian H-field extension, and any such extension embeds over K into every ω-free newtonian H-field extension of K.*

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Theorem. *If K is an ω-free H-field, then K has a d-algebraic newtonian Liouville closed H-field extension that embeds over K into every ω-free newtonian Liouville closed H-field extension of K.*

The theory

$$\mathcal{L} = \{0, 1, +, -, \cdot\,, \partial, \leqslant, \preccurlyeq\}$$

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Switchmen predicates

$L^{\iota}_{\mathrm{I},\Lambda,\Omega} = \mathcal{L} \cup \{\iota, I, \Lambda, \Omega\}$ and $T^{\mathrm{nl},\iota}_{\mathrm{I},\Lambda,\Omega}$ is T^{nl} with additional axioms

$$\begin{aligned} a \neq 0 &\implies a\,\iota(a) = 1\\ a = 0 &\implies \iota(a) = 0\\ \mathrm{I}(a) &\iff [\exists y, (a \preccurlyeq y' \wedge y \preccurlyeq 1)] \iff [a = 0 \vee (a \neq 0 \wedge \neg\Lambda(-a^{\dagger}))]\\ \Lambda(a) &\iff \exists y, (y \succ 1 \wedge a = -y^{\dagger\dagger})\\ \Omega(a) &\iff \exists y, (y \neq 0 \wedge 4\,y'' + a\,y = 0)\end{aligned}$$

Assume that K contains a gap γ and that $\Phi \in L \supsetneq K$ such that $\Phi' = \gamma$.

Then we *must* have $\Phi \preccurlyeq 1$ if $\mathrm{I}(\gamma)$ and $\Phi \succ 1$ otherwise.

Λ and Ω control what happens when adjoining γ and λ with $\gamma^{\dagger} = -\lambda$ and $\omega(\lambda) = \omega$.

$L^{\iota}_{\Lambda,\Omega} = \mathcal{L} \cup \{\iota, \Lambda, \Omega\}$ and $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ is T^{nl} with above additional axioms for ι, Λ and Ω.

Theorem. *The theory $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ admits elimination of quantifiers.*

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Theorem. *Let $T^{\mathrm{nl}}_{\mathrm{small}}$ be the L-theory whose models are the newtonian Liouville closed H-fields with small derivation. Then $T^{\mathrm{nl}}_{\mathrm{small}}$ is complete (and thus decidable) and model complete. Every H-field with small derivation can be embedded into some model of $T^{\mathrm{nl}}_{\mathrm{small}}$.*

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Corollary. *Let K be a newtonian Liouville closed H-field. Then:*

1. *K is o-minimal at infinity: if $X \subseteq K$ is definable in K, then for some $a \in K$, either $(a, +\infty) \subseteq K$, or $(a, +\infty) \cap K = \varnothing$.*
2. *If $X \subseteq K^n$ is definable in K, then $X \cap C^n$ is semialgebraic in the sense of the real closed constant field C of K.*
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Theorem. *If K is a newtonian Liouville closed H-field, then K has no proper d-algebraic H-field extension with the same constant field.*