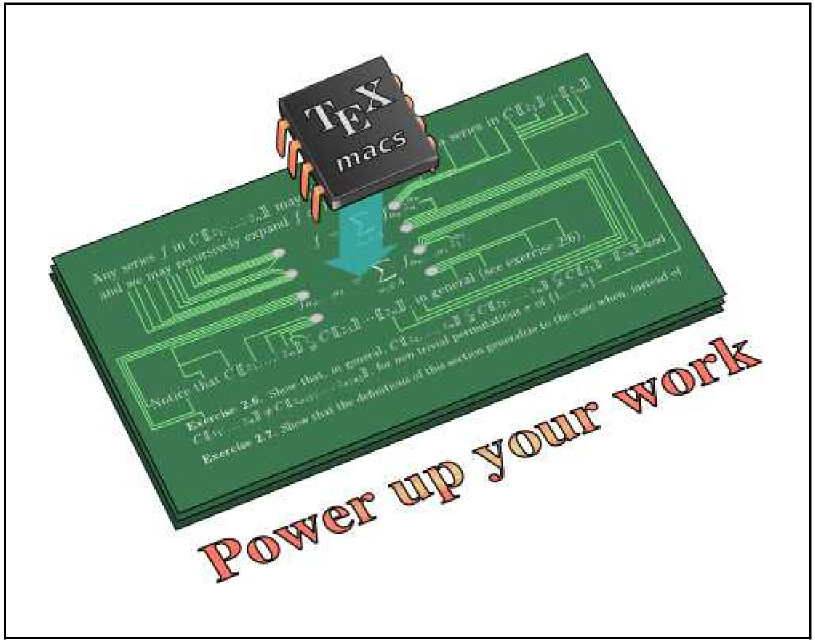
Tutorial: Model Theory of Transseries

Lecture 1: introduction to transseries



Matthias Aschenbrenner, Lou van den Dries, **Joris van der Hoeven**

Toronto 2016

http://www.TEXMACS.org

$$\begin{array}{ccc}
\text{Algebraic geometry} & \longrightarrow & \begin{array}{c}\text{Real algebraic geometry}\\ \textbf{+}\\ \text{Valuation theory}\end{array} \\
\big\downarrow & & \big\downarrow \\
\text{Differential algebra} & \longrightarrow & \color{red}{?}
\end{array}$$

$$
\begin{tikzcd}
\text{Algebraic geometry} \arrow[r] \arrow[d] & \begin{array}{c}\text{Real algebraic geometry}\\ \textbf{+}\\ \text{Valuation theory}\end{array} \arrow[d] \\
\text{Differential algebra} \arrow[r] & \begin{array}{c}\text{Real differential algebra}\\ \textbf{+}\\ \text{Asymptotic differential algebra}\end{array}
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Let $\mathcal{G}$ be the ring of germs of real differentiable functions at infinity.

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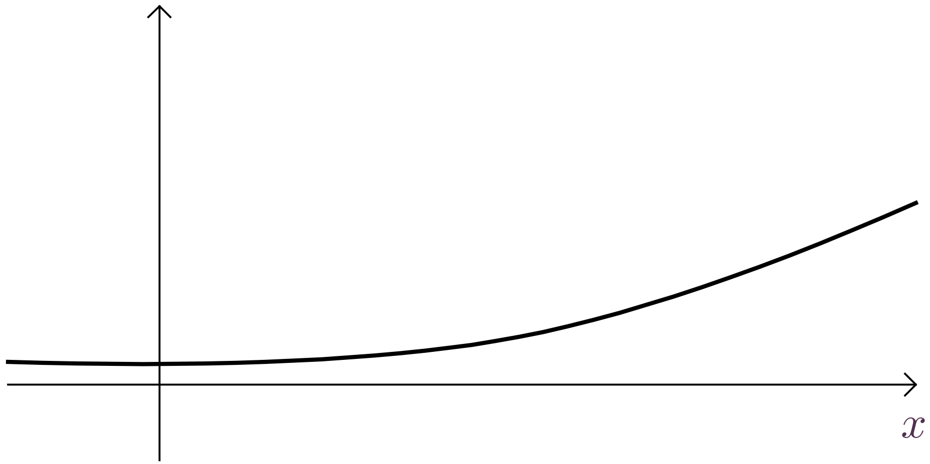

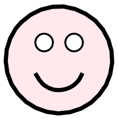

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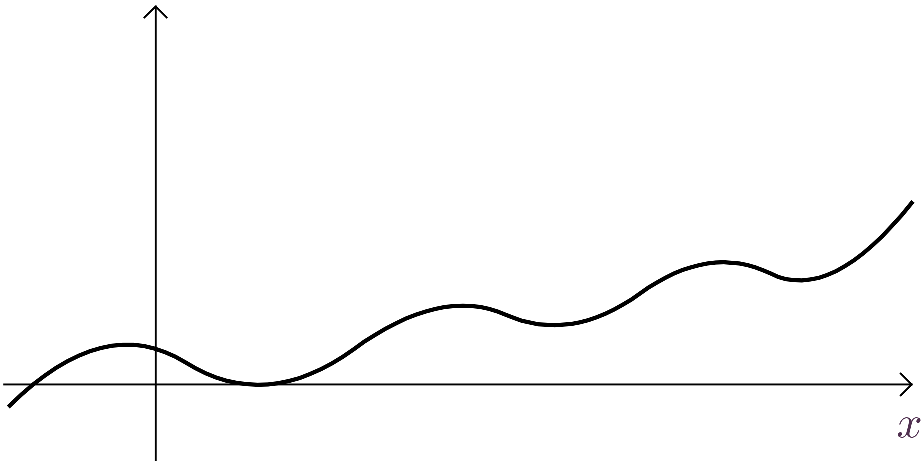

Theorem. (Hardy, Robinson, Bourbaki)

Let K be a Hardy field.

- *The real closure of K is again a Hardy field.*
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- $\vdots$
- $K = \mathbb{R}^{\mathrm{Liouville}}$

Orders at infinity

$$
\begin{aligned}
f \prec g &\iff f = \mathcal{o}(g)\\
&\iff \forall \varepsilon \in \mathbb{R}^{>}, |f| < \varepsilon\, |g|\\
&\iff \forall \varepsilon > 0, \exists x_0, \forall x > x_0, |f(x)| < \varepsilon\, |g(x)|
\end{aligned}
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f \preccurlyeq g &\iff f = \mathcal{O}(g)\\
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The big dream

Use L-functions ($\approx$ elements of $\mathbb{R}^{\mathrm{Liouville}}$) or a suitable generalization as a universal framework for the asymptotic analysis of real functions at infinity.

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For any solution y of

$$y'' + y = \mathrm{e}^{x^2},$$

there exists a Hardy field that contains y. But for two such Hardy fields K_1 and K_2 there is generally no Hardy field K that contains both K_1 and K_2.

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How to go beyond first order equations?

$(x \to +\infty)$

$$\mathrm{e}^{\mathrm{e}^{x} + \cdots} + \cdots$$

$(x \to +\infty)$

$$\mathrm{e}^{\mathrm{e}^{x}+\frac{\mathrm{e}^{x}}{x}+\cdots}+\cdots$$

$(x \to +\infty)$

$$\mathrm{e}^{\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{x} + \frac{\mathrm{e}^{x}}{x^2} + \cdots} + \cdots$$

$(x \to +\infty)$

$$\mathrm{e}^{\mathrm{e}^x + \frac{\mathrm{e}^x}{x} + \frac{\mathrm{e}^x}{x^2} + \cdots} + \frac{2}{\log x}\,\mathrm{e}^{\mathrm{e}^x + \cdots} + \cdots$$

$(x \to +\infty)$

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$$\begin{aligned}
\frac{1}{1-x^{-1}-x^{-\mathrm{e}}} &= 1+x^{-1}+x^{-2}+x^{-\mathrm{e}}+x^{-3}+x^{-\mathrm{e}-1}+\cdots \\
\frac{1}{1-x^{-1}+\mathrm{e}^{-x}} &= 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots \\
-\mathrm{e}^x\int\frac{\mathrm{e}^{-x}}{x} &= \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots \\
\Gamma(x) &= \frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\,\pi}\,\mathrm{e}^{x\,(\log x-1)}}{288\,x^{5/2}}+\cdots \\
\zeta(x) &= 1+2^{-x}+3^{-x}+4^{-x}+\cdots \\
\varphi(x) &= \frac{1}{x}+\varphi(x^{\pi})=\frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots \\
\psi(x) &= \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x})=\frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots
\end{aligned}$$

```
Mmx] use "asymptotix";
Mmx] x == infinity ('x);
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Mmx] $\exp\left(\frac{x^4}{x+1}\right)$

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$$\frac{\mathrm{e}^{x^3 - x^2 + x}}{\mathrm{e}} + \frac{\mathrm{e}^{x^3 - x^2 + x}}{\mathrm{e}\,x} - \frac{\mathrm{e}^{x^3 - x^2 + x}}{2\,\mathrm{e}\,x^2} + O\left(\frac{\mathrm{e}^{x^3 - x^2 + x}}{x^7}\right)$$

Mmx] $\int \exp\left(x^2\right) \mathrm{d}x$

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Mmx] fixed_point_transseries$\left(f \mapsto \frac{1}{x} + f \circ x^2 + f \circ \exp x\right)$

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$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^8}+O\left(\frac{1}{x^{16}}\right)+\frac{1}{\mathrm{e}^x}+\frac{1}{\mathrm{e}^{2x}}+\frac{1}{\mathrm{e}^{4x}}+O\left(\frac{1}{\mathrm{e}^{8x}}\right)+\frac{1}{\mathrm{e}^{x^2}}+\frac{1}{\mathrm{e}^{2x^2}}+O\left(\frac{1}{\mathrm{e}^{4x^2}}\right)+\frac{1}{\mathrm{e}^{x^4}}+O\left(\frac{1}{\mathrm{e}^{2x^4}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^x}}+\frac{1}{\mathrm{e}^{2\mathrm{e}^x}}+O\left(\frac{1}{\mathrm{e}^{4\mathrm{e}^x}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{2x}}}+O\left(\frac{1}{\mathrm{e}^{2\mathrm{e}^{2x}}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{x^2}}}+O\left(\frac{1}{\mathrm{e}^{2\mathrm{e}^{x^2}}}\right)+\frac{1}{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}}+O\left(\frac{1}{\mathrm{e}^{2\mathrm{e}^{\mathrm{e}^x}}}\right)$$

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$$\frac{\mathrm{e}^{x\log(x)-x}}{\mathrm{sqrt}(x)}+\frac{\mathrm{e}^{x\log(x)-x}}{12\,x^{\frac{3}{2}}}+\frac{\mathrm{e}^{x\log(x)-x}}{288\,x^{\frac{5}{2}}}-\frac{139\,\mathrm{e}^{x\log(x)-x}}{51840\,x^{\frac{7}{2}}}+O\left(\frac{\mathrm{e}^{x\log(x)-x}}{x^{\frac{9}{2}}}\right)$$

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$$\mathrm{e}^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}+\frac{\mathrm{e}^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{12\,x\log(x)}+\frac{\mathrm{e}^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{288\,x^2\log(x)^2}-\frac{\mathrm{e}^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{360\log(x)\,x^3}-\frac{\mathrm{e}^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{240\,x^3\log(x)^2}+O\left(\frac{\mathrm{e}^{x\log(\log(x))-\frac{x}{\log(x)}-\frac{x}{\log(x)^2}-\frac{2x}{\log(x)^3}+O\left(\frac{x}{\log(x)^4}\right)-\frac{\log(\log(x))}{2}}}{x^3\log(x)^3}\right)$$

- C: constant field
- $(\mathfrak{M}, \preccurlyeq)$: totally ordered group of monomials

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- $C[[\mathfrak{M}]]_{\mathscr{S}}$: field of $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with **$\mathscr{S}$-based support**.

  $\mathscr{S} : \mathfrak{M} \to \mathscr{S}_{\mathfrak{M}} \in \mathscr{P}(\mathfrak{M})$ functor of **well behaved supporters** with
  - $\{\mathfrak{m}\} \in \mathscr{S}_{\mathfrak{M}}$ for all $\mathfrak{m} \in \mathfrak{M}$
  - $\mathscr{S}_{\mathfrak{M}}$ closed under taking subsets, unions and multiplication
  - $\mathscr{S}_{\mathfrak{M}}$ closed under power products of infinitesimal sets

Strong summability

We say that $(f_i)_{i \in I} \in C[[\mathfrak{M}]]_{\mathscr{S}}^I$ is (strongly) summable if

1. $\bigcup_{i \in I} \operatorname{supp} f_i \in \mathscr{S}$
2. $\{i \in I : \mathfrak{m} \in \operatorname{supp} f_i\}$ is finite for each $\mathfrak{m} \in \mathfrak{M}$

Then $g = \sum f \in C[[\mathfrak{M}]]_{\mathscr{S}}$ with $g_{\mathfrak{m}} = \sum_{i \in I} f_{i,\mathfrak{m}}$ is well-defined.

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Example

If $\operatorname{supp} f \prec 1$, then $(f^n)_{n\in\mathbb{N}}$ is summable, allowing to define $\frac{1}{1-f} := \sum_{n\in\mathbb{N}} f^n$

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Properties

- $\sum (f_{\sigma(i)})_{i \in I} = \sum (f_i)_{i \in I}$
- $\sum F \amalg G = \sum F + \sum G$
- For $F = \coprod_{j \in J} G_j$, we have $\sum_{j \in J} \sum G_j = \sum F$ $\cdots$

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…

Strongly linear map

Linear map $\Phi : C[\![\mathfrak{M}]\!] \to C[\![\mathfrak{N}]\!]$ that preserves strong summation

Example

$$\mathbb{R}[[x^{\mathbb{R}}\,\mathrm{e}^{\mathbb{R}x}]] \;\cong\; \mathbb{R}[[x^{\mathbb{R}}]][[\mathrm{e}^{\mathbb{R}x}]]$$

$$\sum_{\alpha,\beta\in\mathbb{R}} f_{\alpha,\beta}\,x^{\alpha}\,\mathrm{e}^{\beta x} \;=\; \sum_{\beta\in\mathbb{R}}\left[\sum_{\alpha\in\mathbb{R}} f_{\alpha,\beta}\,x^{\alpha}\right]\mathrm{e}^{\beta x}$$

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More generally

Let $\mathfrak{M}^{\flat}$ a convex subgroup of a monomial group $\mathfrak{M}^{\sharp}$ and assume that we may decompose

$$\mathfrak{M} = \mathfrak{M}^{\flat}\, \mathfrak{M}^{\sharp}$$

Then

$$C[[\mathfrak{M}]] \cong C[[\mathfrak{M}^{\flat}]][[\mathfrak{M}^{\sharp}]]$$

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Adding logarithms

$$\begin{aligned}\mathbb{L}_n &= \mathbb{R}[[\mathfrak{L}_n]]_{\mathscr{S}} = \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]_{\mathscr{S}}\\ \ell_k &= \big(\log \circ \overset{k\times}{\cdots} \circ \log\big)(x)\end{aligned}$$

For $\mathfrak{m} = \ell_0^{\alpha_0} \cdots \ell_n^{\alpha_n}$, $\log \mathfrak{m} = \alpha_0 \, \ell_1 + \cdots + \alpha_n \, \ell_{n+1} \in \mathbb{L}_{n+1}$. For $f = c \, \mathfrak{m} \, (1 + \delta) \in \mathbb{L}_n^{\prec}$,

$$\log(f) = \log(c_f \, \mathfrak{d}_f \, (1 + \delta_f)) \;=\; \log \mathfrak{d}_f + \log c_f + \log (1 + \delta_f).$$

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Closing off

$$\mathbb{L} \;=\; \mathbb{L}_0 \cup \mathbb{L}_1 \cup \mathbb{L}_2 \cup \cdots$$

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Closing off

$$\begin{aligned} \mathbb{L} &= \mathbb{L}_0 \cup \mathbb{L}_1 \cup \mathbb{L}_2 \cup \cdots \\ \hat{\mathbb{L}} &= \mathbb{R}[[\mathfrak{L}]]_{\mathscr{S}}, \quad \mathfrak{L} = \mathfrak{L}_0 \cup \mathfrak{L}_1 \cup \mathfrak{L}_2 \cup \cdots \end{aligned}$$

Grid-based case: $\hat{\mathbb{L}} = \mathbb{L}$
Well-based case: $\ell_0 + \ell_1 + \ell_2 + \cdots \in \hat{\mathbb{L}} \setminus \mathbb{L}$

Field of transseries

$$\begin{aligned} \mathbb{T} &\subseteq \mathbb{R}[[\mathfrak{T}]]_{\mathscr{S}} \\ \log\colon \mathfrak{T} &\to \mathbb{T}_{\succ} = \{f \in \mathbb{T} \colon \forall \mathfrak{m} \in \mathfrak{T}, \mathfrak{m} \succ 1\} \end{aligned}$$

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Exponential extension

$$\begin{aligned}\mathfrak{T}_{\exp} &= \exp(\mathbb{T}_{\succ}) \supseteq \mathfrak{T}\\ \mathbb{T}_{\exp} &= \mathbb{R}[[\mathfrak{T}_{\exp}]]_{\mathscr{S}}\end{aligned}$$

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Example

$$\mathrm{e}^{x^2 + \frac{x^2}{\log x} + \frac{x^2}{\log^2 x} + \cdots + x + \log\log x} \in \mathfrak{L}_{\exp}$$

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Grid-based case: $\hat{\mathbb{L}}_{\exp^{\infty}} = \mathbb{L}_{\exp^{\infty}}$

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First close off under exponentiation

$$\begin{aligned} \mathbb{E}_0 &= \mathbb{R}[[\mathfrak{E}_0]]_{\mathscr{S}} \\ \mathbb{E}_{n+1} &= \mathbb{R}[[\mathfrak{E}_n]]_{\mathscr{S}} \\ \mathbb{E} &= \mathbb{E}_0 \cup \mathbb{E}_1 \cup \mathbb{E}_2 \cup \cdots \end{aligned}$$

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Our book

$\mathbb{T} = \bigcup_{k \in \mathbb{N}} \mathbb{E} \circ \ell_k$ **“standard field of transseries”**

Totally ordered field $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]_{\mathscr{S}}$ with a logarithm such that

T1. $\operatorname{dom}\log = \mathbb{T}^{>}$.

T2. $\log \mathfrak{m} \in \mathbb{T}_{\succ}$, for all $\mathfrak{m} \in \mathfrak{T}$, i.e. $\forall \mathfrak{n} \in \operatorname{supp}(\log \mathfrak{m}), \mathfrak{n} \succ 1$.

T3. $\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \cdots$, for all $\varepsilon \in \mathbb{T}_{\prec}$.

T4. *see Schmeling's PhD.*

Strictly increasing transfinite sequence of fields of transseries

$$\begin{aligned}\mathbb{T}_1 &= \mathbb{L}\\ \mathbb{T}_{\alpha+1} &= (\mathbb{T}_\alpha)_{\exp}\\ \mathbb{T}_\lambda &= \mathbb{R}[[\textstyle\bigcup_{\alpha<\lambda}\mathfrak{T}_\alpha]]\end{aligned}$$

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Transfinite sequence of $f_\alpha \in \mathbb{T}_\alpha$ with supp $f_\alpha \cong \alpha$ and $f_\alpha \notin \bigcup_{\beta<\alpha} \mathbb{T}_\beta$

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$$

$$
\begin{aligned}
f_1 &= x^2 \\
f_2 &= x^2 - \mathrm{e}^{\log^2 x} \\
&\vdots \\
f_\omega &= x^2 - \mathrm{e}^{\log^2 x} - \mathrm{e}^{\log^2 x - \mathrm{e}^{\log\log^2 x}} - \cdots \\
f_{\omega+1} &= x^2 - \mathrm{e}^{\log^2 x} - \mathrm{e}^{\log^2 x - \mathrm{e}^{\log\log^2 x}} - \cdots - \mathrm{e}^{\log^2 x - \mathrm{e}^{\log\log^2 x} - \cdots} \\
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This derivation satisfies:

AD1. $f \prec g \Rightarrow f' \prec g'$, for all $f, g \in \mathbb{T}$ with $g \not\asymp 1$.

AD2. $f \succ 1 \Rightarrow (f > 0 \Rightarrow f' > 0)$, for all $f \in \mathbb{T}$.

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We call δ the *post-composition* operator with g and also write $\delta(f) = f \circ g$. It satisfies:

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$$f \circ (x + \delta) = f + f' \delta + \tfrac{1}{2} f'' \delta^2 + \cdots$$

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- Kneser $\Rightarrow$ $\exists$ real analytic solution to

$$\ell_\omega \circ \ell_1 = \ell_\omega - 1$$

- Differentiate $\Rightarrow$ $\gamma = \ell'_\omega$ indeed a solution to

$$\frac{\gamma \circ \ell_1}{x} = \gamma$$

- ℓ_ω grows slowlier than any iterated logarithm

Transfinite iterators of the logarithm

$$\ell_\alpha = \int \frac{1}{\prod_{\beta<\alpha} \ell_\beta}$$

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Generalized transseries

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Fork in the road

Let $\mathbb{T}$ be a field of transseries with $\gamma \in \mathbb{T}$ but $\int \gamma \notin \mathbb{T}$

- Possible to construct extension $\mathbb{T}\langle \int \gamma \rangle$ in which $\int \gamma \succ 1$
- Also possible to construct extension $\mathbb{T}\langle \int \gamma \rangle$ in which $\int \gamma \prec 1$

Let $\mathbb{T}$ be the standard field of transseries

Theorem. (Écalle / vdD–Macintyre–Marker / vdH) $\mathbb{T}$ *is Liouville closed (real closed and stable under resolution of first order linear differential equations).*

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$$y^{17}\,y''\,y''' + \Gamma(\Gamma(x))\,y^3\,(y')^6 - \frac{y^{(2016)}}{\log\log x} = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\frac{x}{\log\log\log x}}}}$$

Theorem. *Let* $P \in \mathbb{T}_{\mathscr{S}}\{Y\}$ *and* $y \in \mathbb{T}$ *be such that* $P(y) = 0$*. Then* $y \in \mathbb{T}_{\mathscr{S}}$*.*

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Corollary. *Let* $\mathscr{S}_{\mathfrak{M}}$ *be the set of well-ordered supports* $\mathfrak{S}$ *such that there exist a finite number of monomials* $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ *with* $\mathfrak{S} \subseteq \mathfrak{b}_1^{\mathbb{R}} \cdots \mathfrak{b}_n^{\mathbb{R}}$. *Then* $\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^{\mathrm{e}^{x/2}} + \mathrm{e}^{\mathrm{e}^{x/3}} + \cdots$ *is differentially transcendental over* $\mathbb{T}_{\mathscr{S}}$.

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Question. *Is* $x + \log x + \log\log x + \cdots$ *differentially transcendental over* $\mathbb{T}$*?*

Sufficiently closed models

$$\begin{tikzcd}
\text{Algebraic geometry} \arrow[r] \arrow[d] & \begin{array}{c}\text{Real algebraic geometry}\\ \textbf{+}\\ \text{Valuation theory}\end{array} \arrow[d] \\
\text{Differential algebra} \arrow[r] & \begin{array}{c}\text{Real differential algebra}\\ \textbf{+}\\ \text{Asymptotic differential algebra}\end{array}
\end{tikzcd}$$

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$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{R} \text{ + } \mathbb{C}[[z^{\mathbb{Q}}]] \arrow[d] \\ \text{Differential algebra} \arrow[r] & \text{Real differential algebra + Asymptotic differential algebra} \end{tikzcd}$$

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\mathbb{C} \arrow[r] \arrow[d] & \begin{array}{c}\mathbb{R}\\ \textbf{+}\\ \mathbb{C}[\![z^{\mathbb{Q}}]\!]\end{array} \arrow[d] \\
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\textit{Wild} \arrow[r] & \begin{array}{c} \text{Maximal Hardy field} \\ \textbf{+} \\ \text{Asymptotic differential algebra} \end{array}
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$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{R} + \mathbb{C}[[z^{\mathbb{Q}}]] \arrow[d] \\ \textit{Wild} \arrow[r] & \mathbb{T} + \text{Asymptotic differential algebra} \end{tikzcd}$$

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$$\begin{tikzcd}
\tilde{f}(x) = \displaystyle\sum_{n=0}^{\infty} \frac{n!}{x^{n+1}} \arrow[r, dashed, "\text{Resummation}"] \arrow[d, "\hat{\mathcal{B}}"', "\begin{array}{l} x^{-n-1} \mapsto (-\zeta)^n / n! \\ \times \mapsto * \\ \partial \mapsto -\zeta \end{array}"] & f(x) = \displaystyle\int_0^{\infty} \frac{\mathrm{e}^{-x\zeta}}{1+\zeta} \mathrm{d}\zeta \\
\hat{f}(\zeta) = \displaystyle\sum_{n=0}^{\infty} (-\zeta)^n \arrow[r, "\text{Analytic continuation}"'] & \hat{f}(\zeta) = \dfrac{1}{1+\zeta} \arrow[u, "\mathcal{L}"']
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Accelero-summation (Écalle)

$$\tilde{f}(x) = \left(\sum_{n=0}^{\infty} \frac{n!}{x^{n+1}}\right)\left(\sum_{n=0}^{\infty} \frac{n!}{x^{2n+1}}\right)$$

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Analyzable functions

Let $\mathbb{T}^{\mathrm{accsum}}$ be the subset of $\mathbb{T}$ of accelero-summable transseries

Theorem. $\mathbb{T}^{\mathrm{accsum}}$ *is a Hardy field.*

Conjecture. $\mathbb{T}^{\mathrm{accsum}}$ *contains the field of all differentially algebraic transseries over* $\mathbb{R}$.

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Theorem. *The subfield of* $\mathbb{T}$ *of all differentially algebraic transseries over* $\mathbb{R}$ *can be given the structure of a transserial Hardy field.*

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Corollary. *There exists a Hardy field* K *such that*

- K *is Liouville closed.*
- K *is newtonian.*
- *Operators in* $K[\partial]$ *can be factored in operators of order one or two.*
- K *satisfies the differential intermediate value property.*