Tutorial: Model Theory of Transseries

Lecture 2

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Overview

I. Asymptotic Differential Algebra

In the previous lecture, JORIS introduced a variety of interesting differential fields (of transseries, germs of functions, . . .) equipped with asymptotic structure, such as ordering and dominance. We will now introduce an algebraic framework to unify these examples and to study their general properties.

II. The Main Results

We'll give statements of our main theorems, though leaving some definitions as black boxes for JORIS' next lecture.

III. The Next Lectures

I. Asymptotic Differential Algebra

Let K be a differential field (always of characteristic 0), with derivation ∂. As usual

$$f' = \partial(f),\ f'' = \partial^2(f),\ \ldots,\ f^{(n)} = \partial^n(f),\ \ldots$$

The **constant field** of K is $C = C_K = \{f \in K : f' = 0\}$.

For $f \neq 0$ let $f^\dagger := f'/f$ be the **logarithmic derivative** of f. Note

$$(f \cdot g)^\dagger = f^\dagger + g^\dagger \qquad \text{for } f, g \neq 0.$$

The ring of **differential polynomials** (= d-polynomials) in $Y_1, \ldots, Y_n$ with coefficients in K is denoted by $K\{Y_1, \ldots, Y_n\}$.

For $\phi \neq 0$, we denote by K^ϕ the **compositional conjugate** of K by ϕ: the field K equipped with the derivation $\phi^{-1}\partial$.

For every $P \in K\{Y\}$ there is a $P^\phi \in K^\phi\{Y\}$ with $P(y) = P^\phi(y)$ for all y. (This will play an important role later.)

Valued differential fields

A **valued differential field** is a differential field K equipped with a valuation $v\colon K^\times \to \Gamma = \Gamma_K$, extended by $v(0) := \infty > \Gamma$. Put

$$\mathcal{O} := \{f : vf \geqslant 0\}, \quad \mathcal{o} := \{f : vf > 0\}, \quad \operatorname{res}(K) := \mathcal{O}/\mathcal{o}.$$

In our context it is often more natural to encode v in terms of its associated **dominance relation**:

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad vf \geqslant vg \qquad \text{“}g \text{ dominates } f\text{”.}$$

We also use:

$$\begin{aligned}
f \prec g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \;\&\; g \not\preccurlyeq f && \text{“}g \text{ strictly dominates } f\text{”}\\
f \asymp g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \;\&\; g \preccurlyeq f && \\
f \sim g \quad &:\Longleftrightarrow \quad f - g \prec g && \text{“asymptotic equivalence”}
\end{aligned}$$

The derivation of K is **small** if $\partial\mathcal{o} \subseteq \mathcal{o}$. (This implies the continuity of ∂.) Then $\partial\mathcal{O} \subseteq \mathcal{O}$, so we get a derivation on $\operatorname{res}(K)$.

Examples

1. For $K = \mathbb{T}$:

$$(\Gamma, +, \leqslant) \cong \left(\{\text{transmonomials}\}, \cdot, \succcurlyeq\right).$$

2. For $K = \mathbb{R}(\ell_0, \ell_1, \dots) \subseteq \mathbb{T}$:

$$\Gamma = \bigoplus_n \mathbb{Z} e_n, \qquad e_n = v\ell_n, \qquad \ell_n = \underbrace{\log\log\cdots\log}_{n \text{ times}} x,$$

$$e_n < m\, e_{n+1} < 0 \qquad \text{for all } m > 0 \text{ and all } n.$$

In both cases $\mathcal{O} = \mathbb{R} + o$, so $\operatorname{res}(K) \cong \mathbb{R}$.

An **ordered differential field** is a differential field K equipped with an ordering making it an ordered field. We can then turn K into a valued field with dominance relation

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some } c \in C.$$

Example

Let K be a HARDY field. Then K becomes an ordered field via

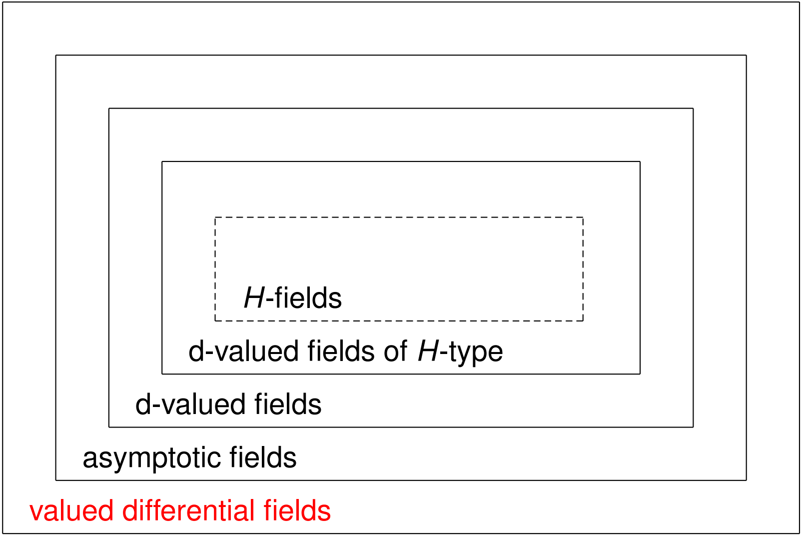
$$f > 0 \quad :\Longleftrightarrow \quad f(t) > 0, \text{ eventually.}$$

For $g \neq 0$, we have:

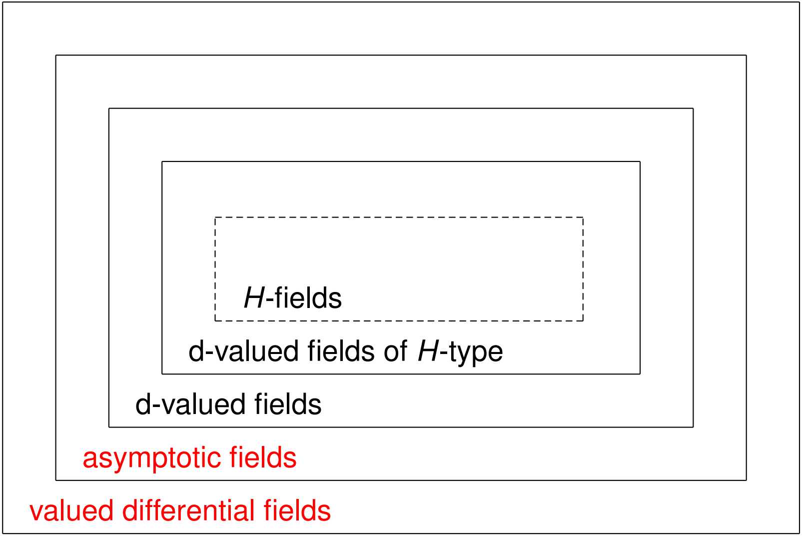
$$f \preccurlyeq g \iff \lim_{t\to+\infty} \frac{f(t)}{g(t)} \in \mathbb{R}, \qquad f \prec g \iff \lim_{t\to+\infty} \frac{f(t)}{g(t)} = 0,$$

$$f \asymp g \iff \lim_{t\to+\infty} \frac{f(t)}{g(t)} \in \mathbb{R}^{\times}, \qquad f \sim g \iff \lim_{t\to+\infty} \frac{f(t)}{g(t)} = 1.$$

Valued differential fields



Valued differential fields



A valued differential field K is an **asymptotic field** if

$$\textit{for all nonzero } f, g \not\asymp 1, \qquad f \preccurlyeq g \quad \Longleftrightarrow \quad f' \preccurlyeq g'.$$

We say that K is of H-**type** (or H-**asymptotic**) if in addition

$$\textit{for all nonzero } f, g \prec 1, \qquad f \preccurlyeq g \quad \Longrightarrow \quad f^\dagger \succcurlyeq g^\dagger.$$

Examples

- Let K be a HARDY field. Then for nonzero $f, g \not\asymp 1$:

$$f \preccurlyeq g \Longleftrightarrow \lim_{t\to+\infty} \frac{f(t)}{g(t)} \in \mathbb{R} \underset{\text{L'HÔPITAL}}{\Longleftrightarrow} \lim_{t\to+\infty} \frac{f'(t)}{g'(t)} \in \mathbb{R} \Longleftrightarrow f' \preccurlyeq g'.$$

  So K is asymptotic. One can check that K is of H-type.
- Every valued differential subfield of $\mathbb{T}$ is H-asymptotic.

Let K be an asymptotic field. We can define functions

$$\Gamma^{\neq} := \Gamma \setminus \{0\} \to \Gamma$$

by

$$\gamma = vg \mapsto \gamma' := vg', \qquad \gamma = vg \mapsto \psi(\gamma) := \gamma^\dagger := \gamma' - \gamma = vg^\dagger.$$

The pair (Γ, ψ), with $\psi(0) := \infty$, is an **asymptotic couple**, i.e.,

(AC1) $\psi(\alpha + \beta) \geqslant \min\big(\psi(\alpha), \psi(\beta)\big)$;

(AC2) $\psi(k\alpha) = \psi(\alpha)$ for all $k \in \mathbb{Z}^{\neq}$;

(AC3) $0 < \alpha < \beta \implies \alpha' < \beta'$.

We say that (Γ, ψ) is of *H*-**type**, or *H*-**asymptotic**, if in addition

(HC) $0 < \alpha \leqslant \beta \Longrightarrow \psi(\alpha) \geqslant \psi(\beta)$.

Example

Suppose $K = \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$, so

$$\Gamma = \bigoplus_n \mathbb{Z} e_n \quad \text{where } e_n = v\ell_n < 0.$$

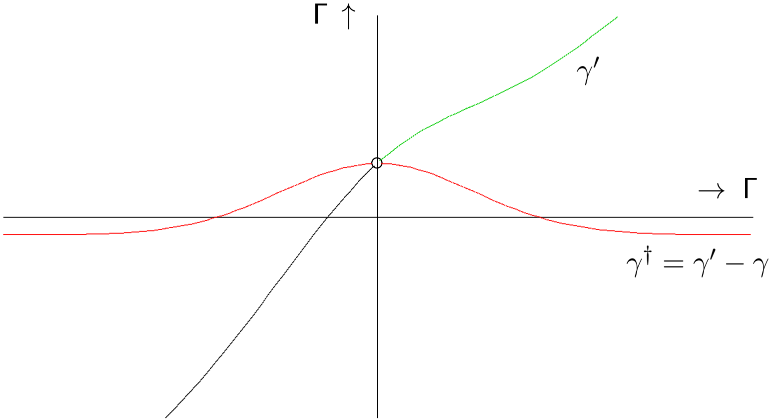
We have

$$\ell_n' = \left(\log \ell_{n-1}\right)' = \frac{\ell_{n-1}'}{\ell_{n-1}} \quad \Longrightarrow \quad \ell_n' = \frac{1}{\ell_0 \cdots \ell_{n-1}}$$
$$\Longrightarrow \quad \ell_n^\dagger = \frac{1}{\ell_0 \cdots \ell_n}$$

Thus

$$(\Gamma^>)^\dagger = \left\{ v\left(\frac{1}{\ell_0}\right), \ v\left(\frac{1}{\ell_0 \ell_1}\right), \ \dots, \ v\left(\frac{1}{\ell_0 \cdots \ell_n}\right), \dots \right\}$$

Here is a picture of a typical *H*-asymptotic couple.



We always have $(\Gamma^>)^\dagger < (\Gamma^>)'$. (Even if (Γ, ψ) is not of *H*-type.)

What happens near the little circle is important.

Let (Γ, ψ) be an H-asymptotic couple. Exactly one of the following statements holds:

1. $(\Gamma^{>})^{\dagger} < \gamma < (\Gamma^{>})'$ *for a (necessarily unique)* γ.
   We call such γ a **gap** in (Γ, ψ).
2. $(\Gamma^{>})^{\dagger}$ *has a largest element.*
   We say that (Γ, ψ) is **grounded**.
3. $(\Gamma^{>})^{\dagger}$ *has no supremum; equivalently:* $\Gamma = (\Gamma^{\neq})'$.
   We say that (Γ, ψ) has **asymptotic integration.**

We use similar terminology for H-asymptotic fields.

Examples

1. $K = \mathbb{R}$ (but there are also more interesting examples);
2. $K = \mathbb{R}(\ell_0, \dots, \ell_n)$: then $\max(\Gamma^{>})^{\dagger} = v\left(\dfrac{1}{\ell_0 \cdots \ell_n}\right)$;
3. $K = \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$, or $K = \mathbb{T}$.

The class of asymptotic fields is very robust, e.g., closed under

- taking substructures, compositional conjugation;
- algebraic extensions;
- coarsening and specialization.

Definition

Let Δ be a convex subgroup of Γ, with ordered quotient group $\dot{\Gamma} := \Gamma/\Delta$. Then K with its valuation replaced by

$$K^\times \xrightarrow{v} \Gamma \xrightarrow{\gamma \mapsto \gamma + \Delta} \dot{\Gamma}$$

is an asymptotic field, called the **coarsening** of K by Δ.

The class of asymptotic fields is very robust, e.g., closed under

- taking substructures, compositional conjugation;
- algebraic extensions;
- coarsening and specialization.

Important special case

Suppose K is H-asymptotic. Then

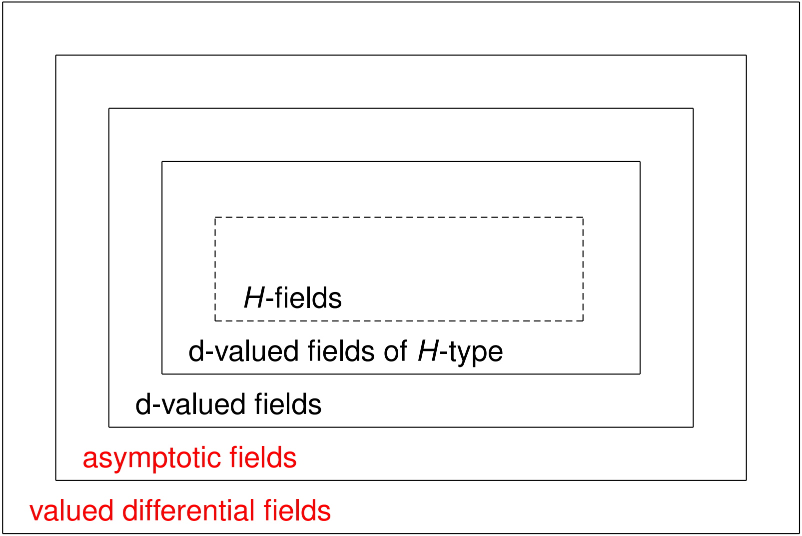
$$\Gamma^\flat := \{\gamma : \gamma^\dagger > 0\}$$

is a convex subgroup of Γ. More generally, so is

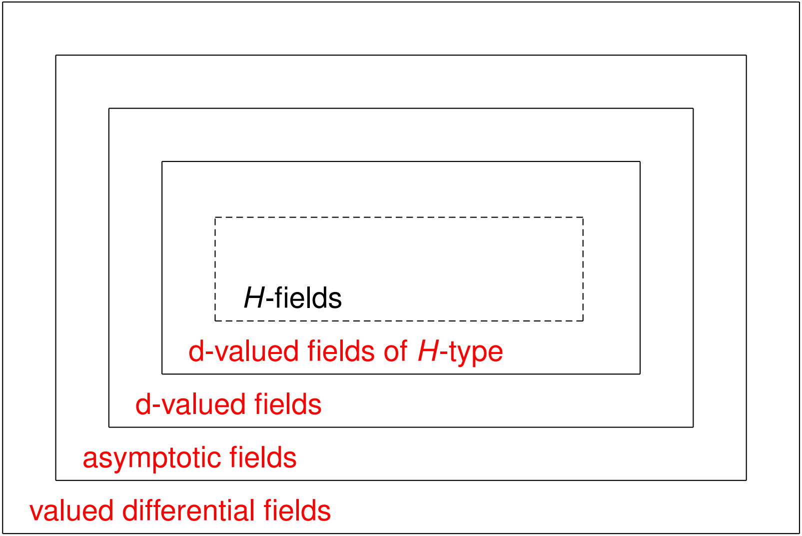
$$\Gamma^\flat_\phi := \{\gamma : \gamma^\dagger > v\phi\} \qquad \text{for } \phi \in K^\times.$$

If $K = \mathbb{T}$, then $\Gamma^\flat = \{vf : f \in \mathbb{T}^\times, f^n \prec \mathrm{e}^x \text{ for all } n\}$.

Differential-valued fields



Differential-valued fields



Let K be an asymptotic field. Then $C \subseteq \mathcal{O}$ and

$$c \mapsto c + \scriptstyle{\mathcal{O}}\colon\ C \to \operatorname{res}(K) = \mathcal{O}/\scriptstyle{\mathcal{O}} \quad \text{is injective.}$$

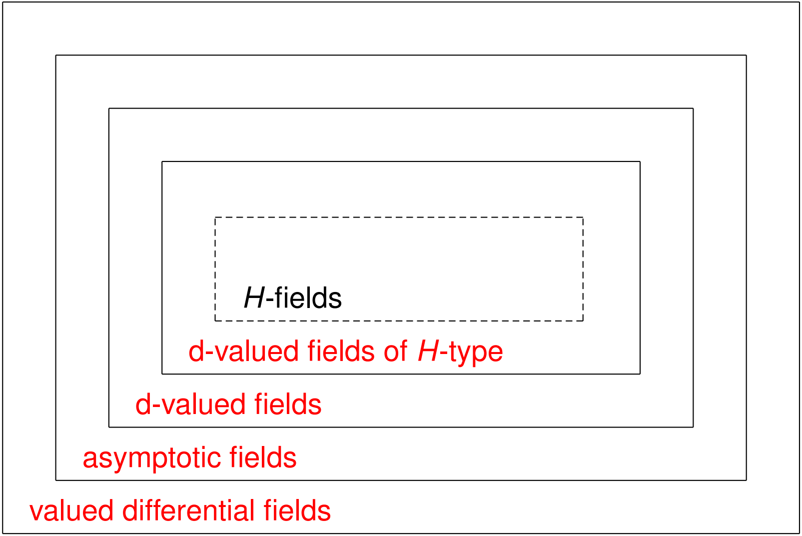
Definition

We say that K is **d-valued** if $\mathcal{O} = C + \scriptstyle{\mathcal{O}}$; equivalently, for each $f \asymp 1$ there is some $c \in C$ with $f \sim c$.

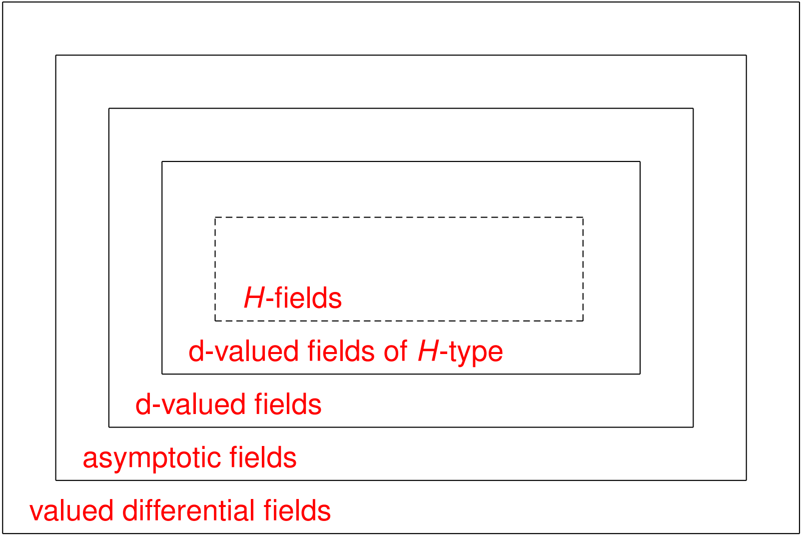
These were defined and first studied by ROSENLICHT (1980s), who in the process also introduced asymptotic couples.

The class of d-valued fields is not as robust as that of asymptotic fields, for example, not closed under taking substructures: consider

$$\mathbb{Q}\left(\sqrt{2 + x^{-1}}\right) \subseteq \mathbb{T}.$$


H-fields
d-valued fields of *H*-type
d-valued fields
asymptotic fields
valued differential fields

H-fields


H-fields
d-valued fields of *H*-type
d-valued fields
asymptotic fields
valued differential fields

Definition

Let K be an *ordered* differential field. Then K is an ***H*-field** if

(H1) $f \succ 1 \;\Rightarrow\; f^{\dagger} > 0$; and

(H2) for each $f \asymp 1$ there is some $c \in C$ with $f \sim c$.

Every H-field, as valued differential field, is H-asymptotic and d-valued (by (H2)).

Each compositional conjugate K^{ϕ} of an H-field K with $\phi \in K$, $\phi > 0$, is an H-field.

Examples

- every ordered differential subfield $K \supseteq \mathbb{R}$ of $\mathbb{T}$;
- every HARDY field $K \supseteq \mathbb{R}$.

Recall that $\mathbb{T}$ is real closed, as well as closed under exponentiation and integration. This motivates the following:

Definition

Let K be an H-field. We say that K is LIOUVILLE **closed** if

1. K is real closed;
2. for each $f \in K$ there is some $y \in K$ with $y \neq 0$, $y^\dagger = f$; and
3. for each $g \in K$ there is some $z \in K$ with $z' = g$.

A LIOUVILLE **closure** of an H-field K is a minimal LIOUVILLE closed H-field extension of K.

Theorem

Every H-field K has exactly one or exactly two LIOUVILLE *closures, up to isomorphism over K.*

What can go wrong when forming LIOUVILLE closures may be seen from the asymptotic couple (Γ, ψ) of K. Recall that exactly one of the following holds:

1. K has a gap γ: $(\Gamma^{>})^{\dagger} < \gamma < (\Gamma^{>})'$
2. K is grounded: $(\Gamma^{>})^{\dagger}$ *has a largest element.*
3. K has asymptotic integration: $(\Gamma^{>})^{\dagger}$ *has no supremum.*

In 1 we have *two* LIOUVILLE closures: if $\gamma = vg$, then we have a choice when adjoining $\int g$: make it $\succ 1$ or $\prec 1$.

In 2 we have *one* LIOUVILLE closure: if $vg = \max(\Gamma^{>})^{\dagger}$, then $\int g \succ 1$ in each LIOUVILLE closure of K.

In 3 we may have *one or two* LIOUVILLE closures.

Order 2 linear differential equations in transseries

Since $\mathbb{T}$ is LIOUVILLE CLOSED, each linear differential equation

$$y' + fy = g \qquad (f, g \in \mathbb{T})$$

has a nonzero solution $y \in \mathbb{T}$. *What other kinds of algebraic differential equations have solutions in $\mathbb{T}$?*

Examples (2nd order linear)

- $y'' = -y$ has *no* solution $y \in \mathbb{T}^{\times}$;
- $y'' = xy$ has *two* $\mathbb{R}$-linearly independent solutions in $\mathbb{T}$:

$$\mathrm{Ai} = \frac{\mathrm{e}^{-\xi}}{2\pi^{1/2}x^{1/4}} \sum_n (-1)^n \frac{a_n}{\xi^n}$$

$$\mathrm{Bi} = \frac{\mathrm{e}^{\xi}}{\pi^{1/2}x^{1/4}} \sum_n (-1)^n \frac{a_n}{\xi^n} \qquad (\xi = \tfrac{2}{3}x^{3/2},\ a_n \in \mathbb{R}).$$

Order 2 linear differential equations in transseries

Let K be a LIOUVILLE closed H-field. For $f \in K$ and $y \in K^\times$,

$$4y'' + fy = 0 \quad \Longleftrightarrow \quad \omega(2y^\dagger) = f$$

$$\text{where } \omega(z) := -(2z' + z^2).$$

Hence

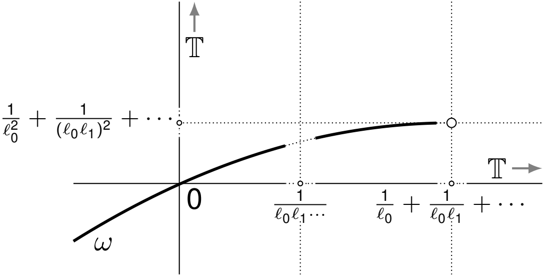
$$\omega(K) = \left\{ f \in K : 4y'' + fy = 0 \text{ for some } y \in K^\times \right\}.$$

Example ($K = \mathbb{T}$)

$$\begin{aligned}
\gamma_n &:= \ell_n^\dagger &&= \frac{1}{\ell_0 \cdots \ell_n} \\
\lambda_n &:= -\gamma_n^\dagger &&= \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} \\
\omega_n &:= \omega(\lambda_n) &&= \frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}
\end{aligned}$$

Order 2 linear differential equations in transseries

One can show that the sequence (ω_n) is cofinal in $\omega(\mathbb{T})$, and that $\omega(\mathbb{T})$ is downward closed in $\mathbb{T}$ (as a consequence of the newtonianitynewtonianity of $\mathbb{T}$).



Definition

Call an H-field K with asymptotic integration **ω-free** if $\omega(K)$ has no supremum in K. (This is not quite the definition of ω-free used in our book, but equivalent to it for LIOUVILLE closed K.)

Newtonian is a version of “d-henselian” satisfied by $\mathbb{T}$, which says that certain kinds of d-polynomials in one variable over K have a zero $y \preccurlyeq 1$ in K. The definition involves compositional conjugation.

It guarantees, for example, that the PAINLEVÉ II equation

$$y'' = 2y^3 + xy + \alpha \qquad (\alpha \in C,\ x' = 1)$$

has a solution in $y \preccurlyeq 1$ in K.

We chose the adjective “newtonian” since it is this property that allows us to develop a NEWTON diagram method for differential polynomials.

ω-freeness and newtonianity will be discussed in more detail in JORIS’ next talk.

II. The Main Results

The main results

From now on, we view each H-field K as a (model-theoretic) structure where we single out the primitives

$$0,\ 1,\ +,\ \cdot,\ \partial\ \text{(derivation)},\ \leqslant\ \text{(ordering)},\ \preccurlyeq\ \text{(dominance)}.$$

Theorem A

The following statements about K axiomatize a model complete theory T^{nl}: K is

1. *a* LIOUVILLE *closed H-field;*
2. *ω-free;*
3. *newtonian.*

Moreover, $\mathbb{T}$ is a model of these axioms.

(The inclusion of $\preccurlyeq$ is necessary.)

The theory T^{nl} is not complete. It has exactly two completions:

- $T^{\mathrm{nl}}_{\mathrm{small}}$: small derivation;
- $T^{\mathrm{nl}}_{\mathrm{large}}$: large derivation.

Thus $T^{\mathrm{nl}}_{\mathrm{small}} = \mathrm{Th}(\mathbb{T})$.

Corollary

$\mathbb{T}$ *is decidable; in particular:* there is an algorithm which, given d-polynomials $P_1, \ldots, P_m \in \mathbb{Q}(x)\{Y_1, \ldots, Y_n\}$, decides whether $P_1(y) = \cdots = P_m(y) = 0$ for some $y \in \mathbb{T}^n$.

There is no such algorithm if $\mathbb{T}$ is replaced by its H-subfield of exponential transseries.

Theorem A is the main step towards a quantifier elimination for $\mathbb{T}$, in a slightly extended language.

Let $\mathcal{L}^{\iota}_{\Lambda,\Omega}$ be our language $\mathcal{L} = \{0, 1, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ augmented by a unary function symbol ι and unary predicates Λ, Ω.

Extend T^{nl} to the $\mathcal{L}^{\iota}_{\Lambda,\Omega}$-theory $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ by adding as defining axioms for these new symbols the universal closures of

$$\begin{aligned}
&\big[a \neq 0 \longrightarrow a \cdot \iota(a) = 1\big] \ \&\ \big[a = 0 \longrightarrow \iota(a) = 0\big],\\
&\Lambda(a) \ \longleftrightarrow\ \exists y\big[y \succ 1 \ \&\ a = -y^{\dagger\dagger}\big],\\
&\Omega(a) \ \longleftrightarrow\ \exists y\big[y \neq 0 \ \&\ 4y'' + ay = 0\big].
\end{aligned}$$

For a model K of T^{nl} this makes both $\Lambda(K)$ and $\Omega(K) = \omega(K)$ downward closed.

Example ($K = \mathbb{T}$)

$$f \in \Lambda(\mathbb{T}) \iff f < \lambda_n = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} \text{ for some } n,$$
$$f \in \Omega(\mathbb{T}) \iff f < \omega_n = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2} \text{ for some } n.$$

Theorem B

$T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ *admits quantifier elimination.*

The predicates Λ and Ω act as switchmen when constructing extensions of K: If an element γ in an H-field extension of K solving $\gamma^\dagger = -\lambda \in K$ is a gap, then $\Lambda(\lambda)$ tells us to choose $\int\gamma \succ 1$, while $\neg\Lambda(\lambda)$ forces $\int\gamma \prec 1$. Likewise, Ω controls what happens when we adjoin λ with $\omega(\lambda) = \omega \in K$.

III. The Next Lectures

Lecture 3 JORIS will discuss the main “machine” behind the proof of our theorems: the NEWTON diagram method.

Lecture 4 I will sketch the main steps in the proofs of Theorems A and B, and give some applications.

Lecture 5 LOU will speak about further developments.