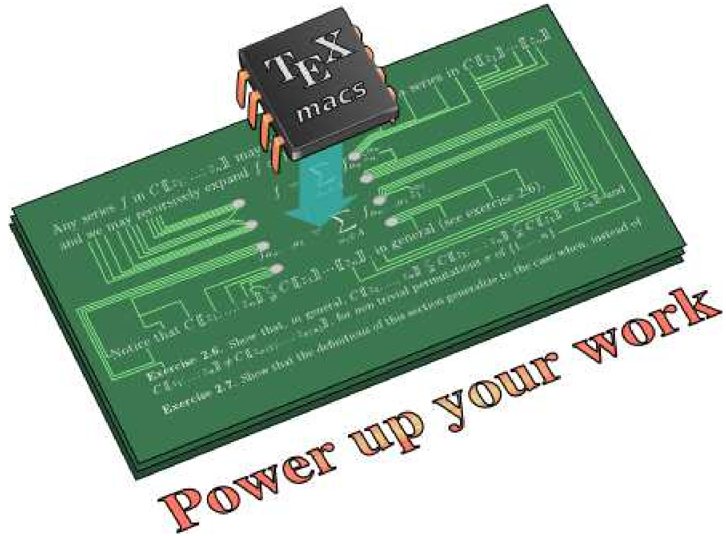
Tutorial: Model Theory of Transseries

Lecture 3: the differential Newton polygon method



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Toronto 2016

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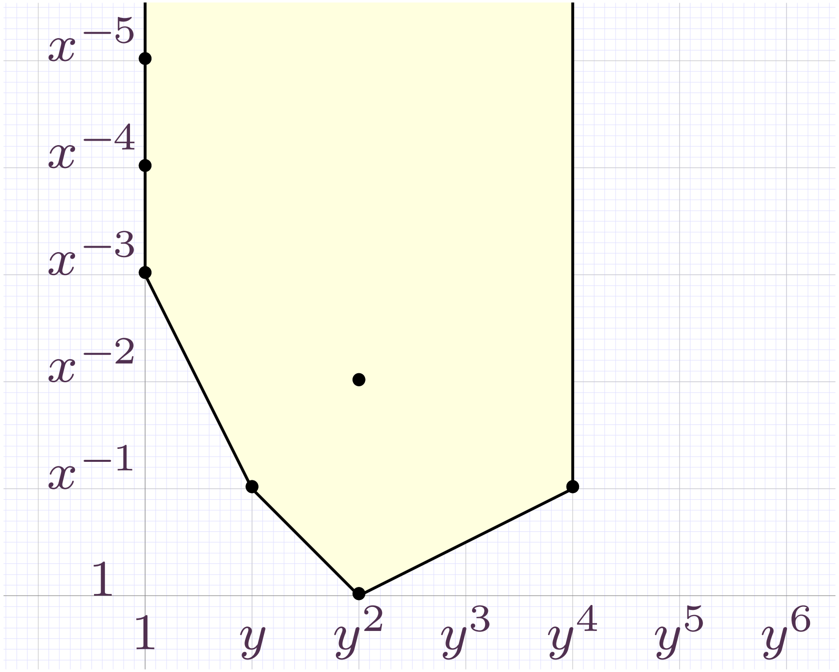
$$\frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x}\, y + \frac{6}{x^3 - x^4} = 0$$



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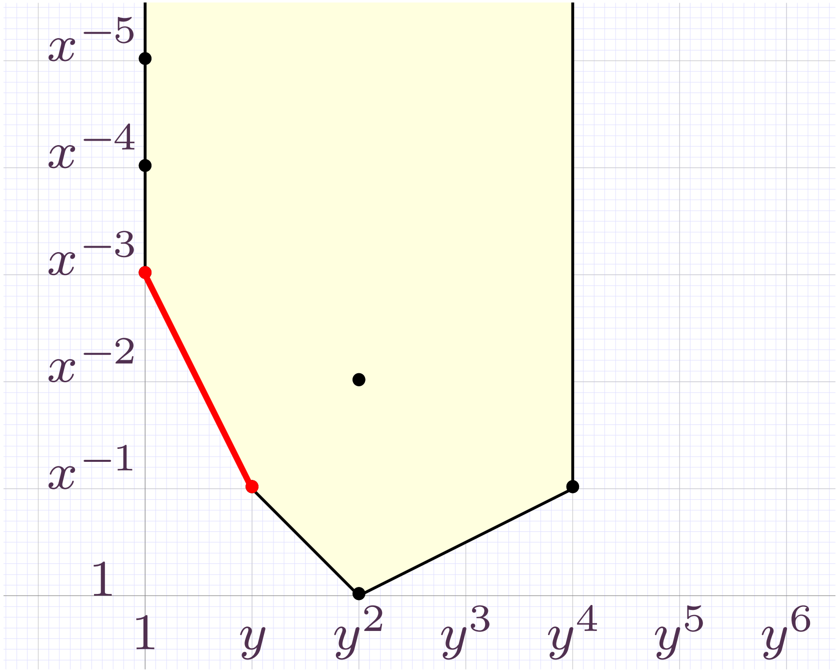
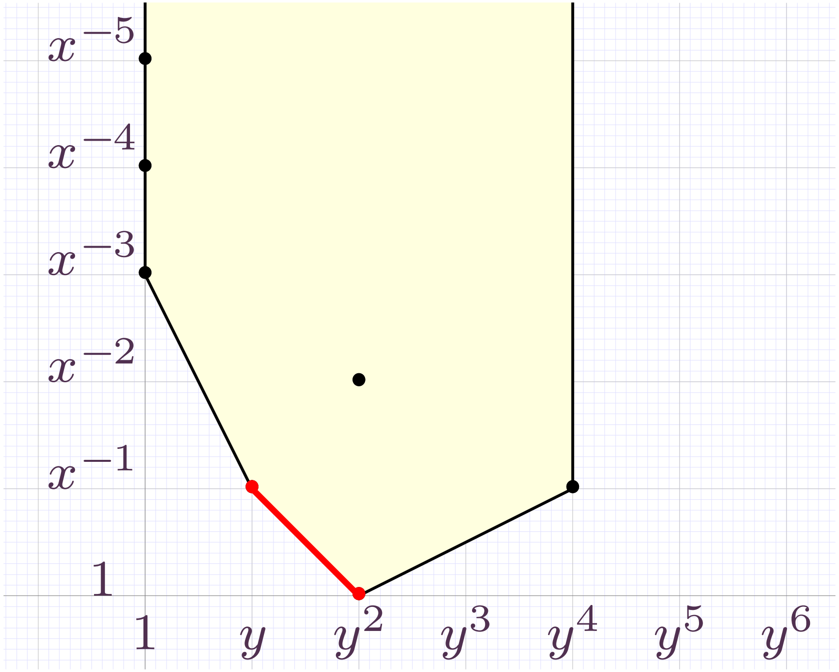


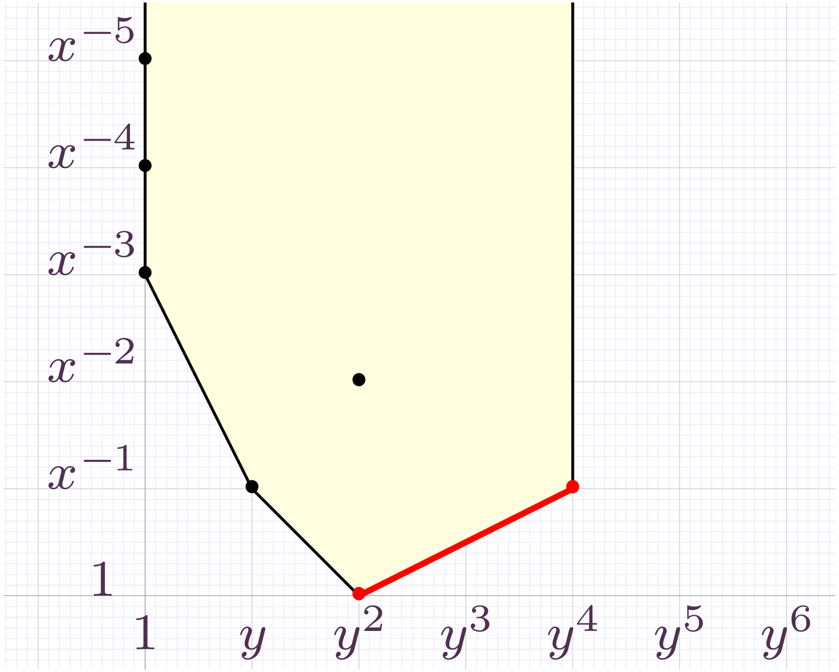
$$y = \frac{2}{x^2} + \tilde{y}, \qquad \tilde{y} \prec \frac{1}{x^2}$$

$$\frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x} y + \frac{6}{x^3 - x^4} = 0$$



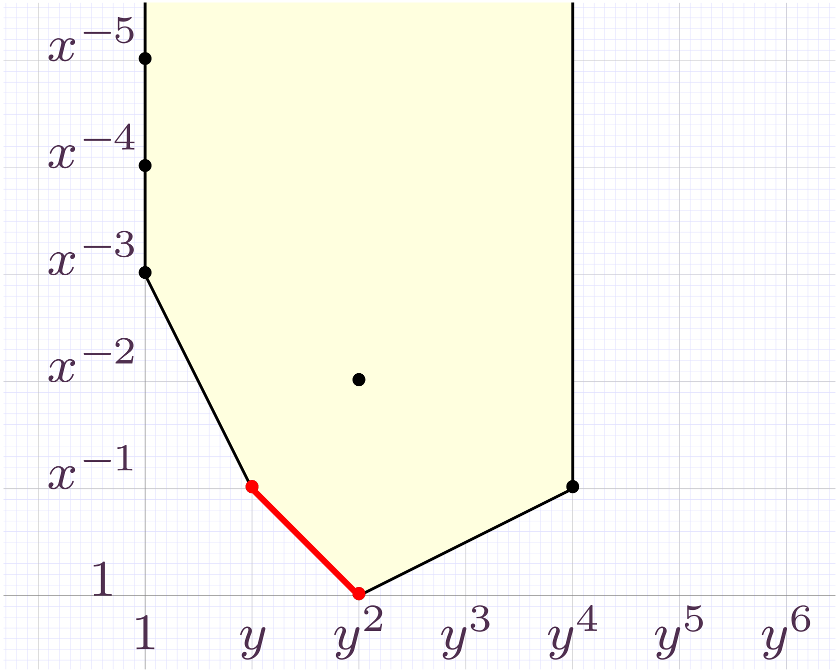
$$y = \frac{-3}{x} + \tilde{y}, \qquad \tilde{y} \prec \frac{1}{x}$$

$$\frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x} y + \frac{6}{x^3 - x^4} = 0$$



$$y = \pm\sqrt{x} + \tilde{y}, \qquad \tilde{y} \prec \sqrt{x}$$

$$\frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x} y + \frac{6}{x^3 - x^4} = 0$$



$$y = \frac{-3}{x} + \tilde{y}, \qquad \tilde{y} \prec \frac{1}{x}$$

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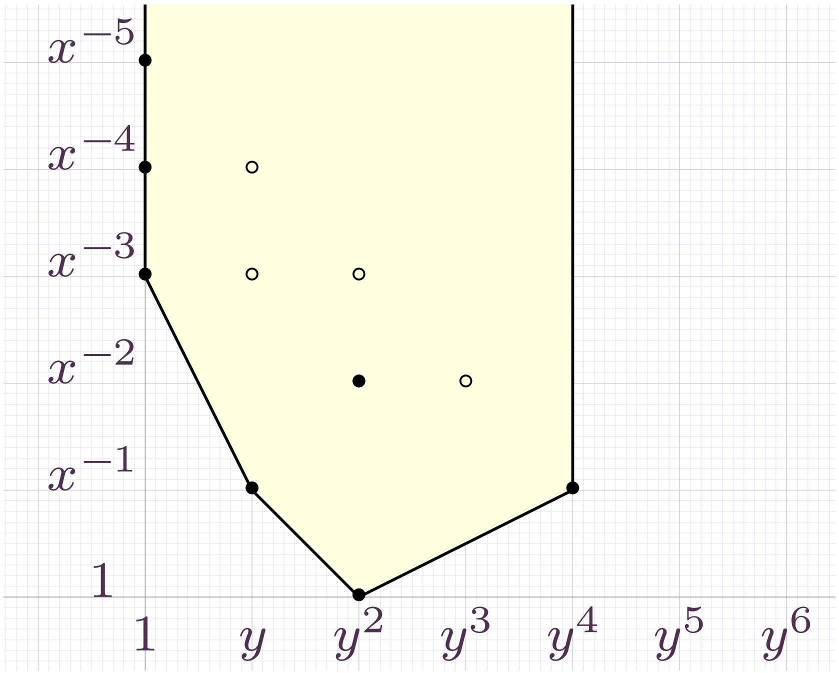
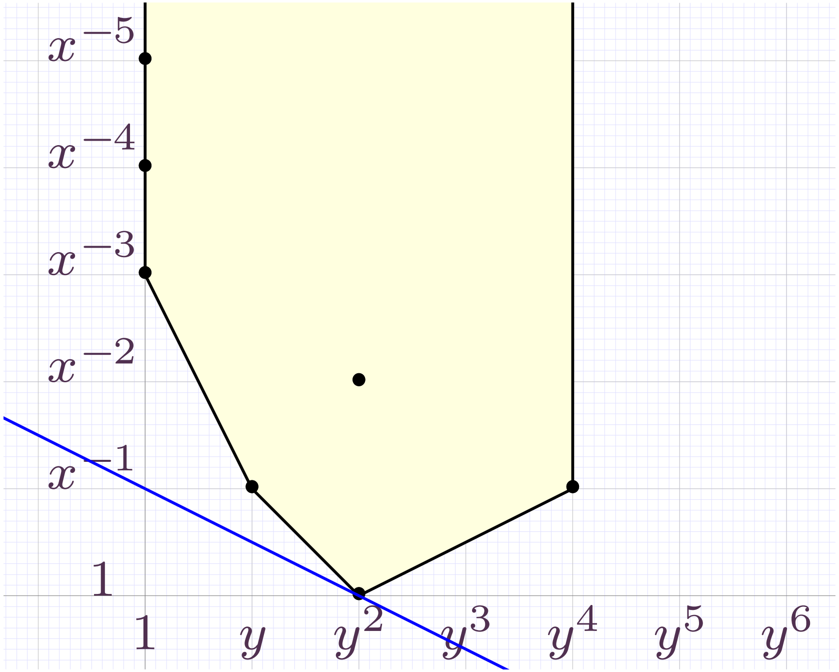


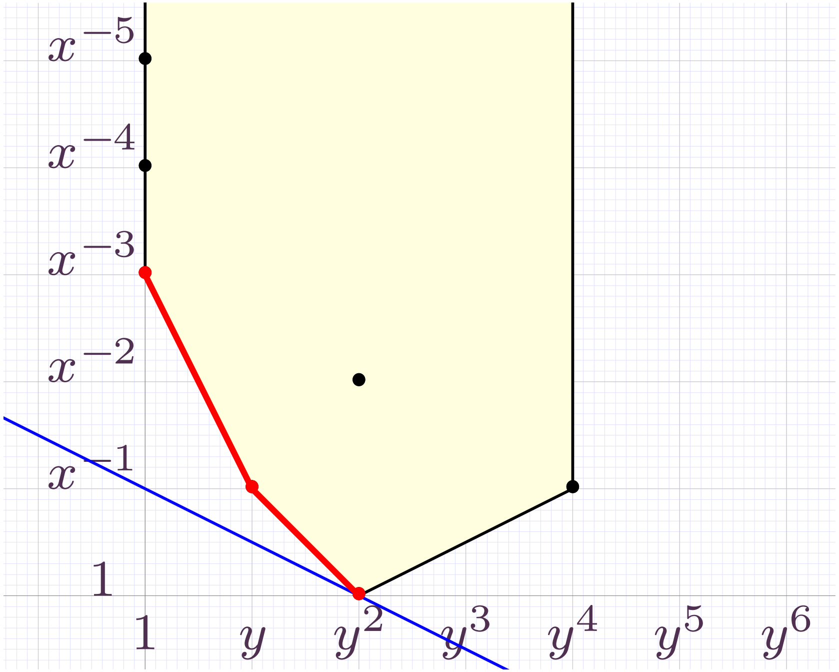
$$y = \frac{-3}{x} + \tilde{y}, \qquad \tilde{y} \prec \frac{1}{x}$$

$$\frac{\tilde{y}^4}{x} - \frac{12}{x^2}\,\tilde{y}^3 - \left(1 + \frac{1}{x^2} - \frac{54}{x^3}\right)\tilde{y}^2 - \left(\frac{3}{x} + \frac{6}{x^3} - \frac{108}{x^4}\right)\tilde{y} + \frac{6}{x^3 - x^4} + \frac{9}{x^3} + \frac{81}{x^4} = 0, \quad \tilde{y} \prec \frac{1}{x}$$

$$P(y) = \frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x}\, y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec \frac{1}{\sqrt{x}}$$

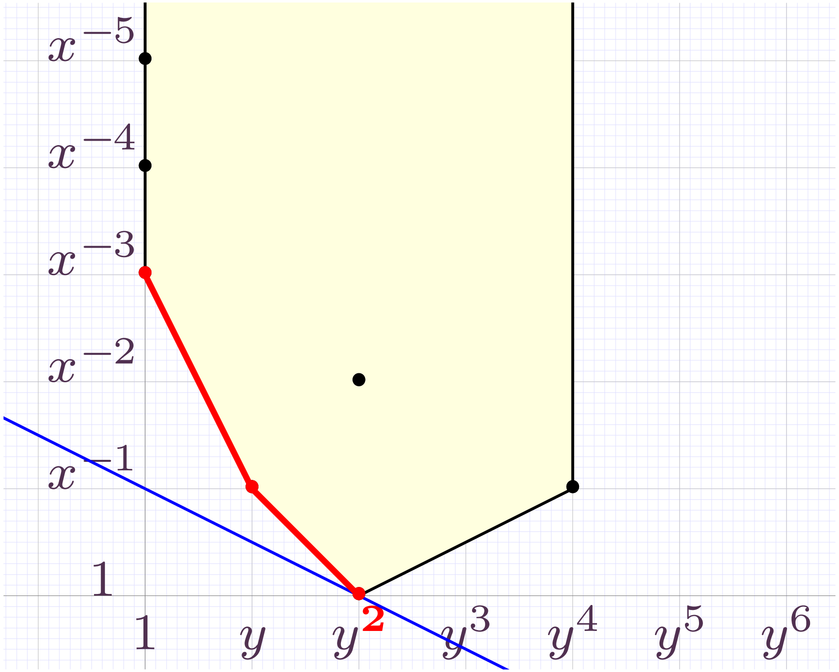


$$P(y) = \frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x} y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec \frac{1}{\sqrt{x}}$$



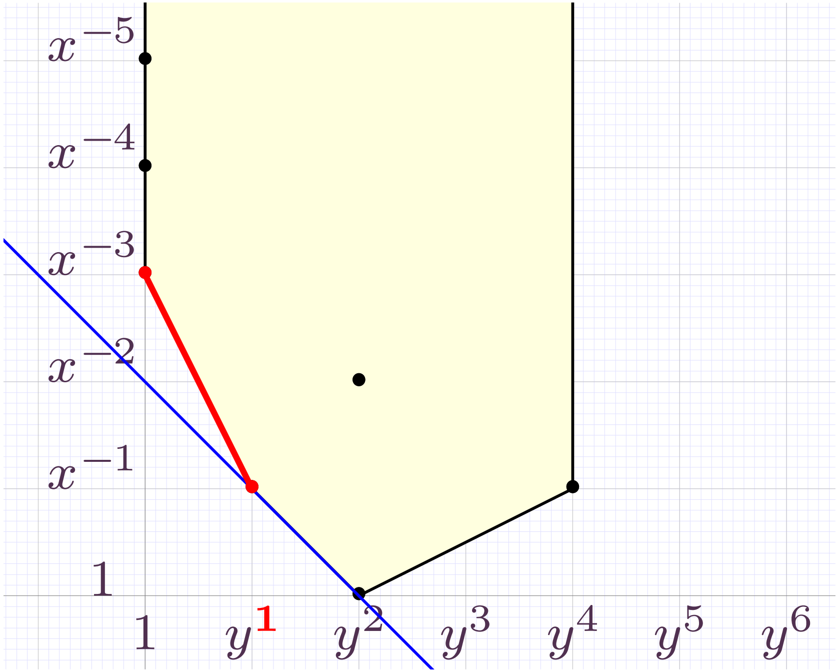
$$\begin{aligned} y &= \frac{-3}{x} + \tilde{y}, & \tilde{y} &\prec \frac{1}{x} \\ y &= \frac{2}{x^2} + \tilde{y}, & \tilde{y} &\prec \frac{1}{x^2} \end{aligned}$$

$$P(y) = \frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x} y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec \frac{1}{\sqrt{x}}$$

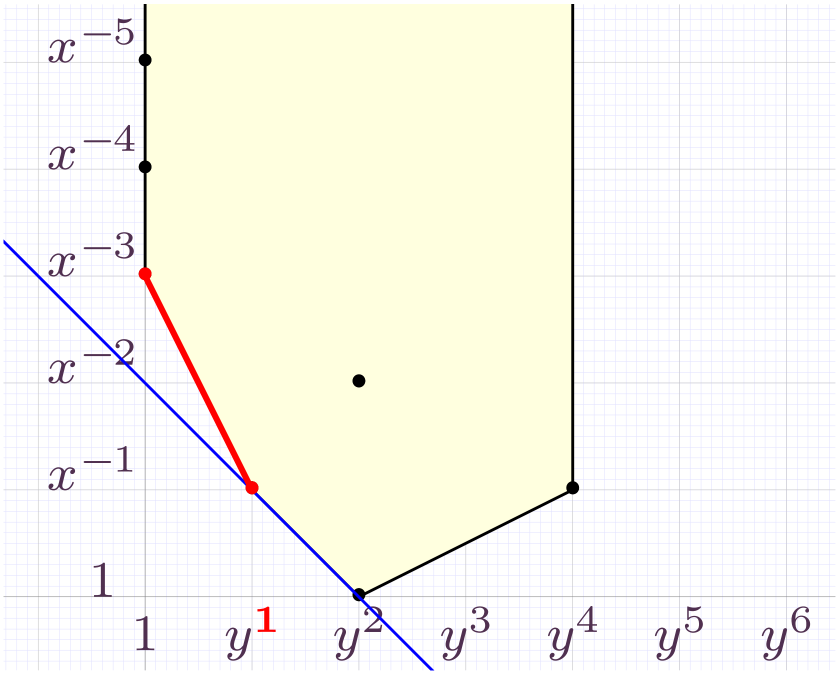


Algebra	Asymptotic algebra
$P(y) = 0$	$P(y) = 0, \quad (y \prec \mathfrak{v})$
$\deg P$	$\operatorname{ndeg}_{\prec \mathfrak{v}} P$

$$P(y) = \frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x}\, y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec \frac{1}{x}$$



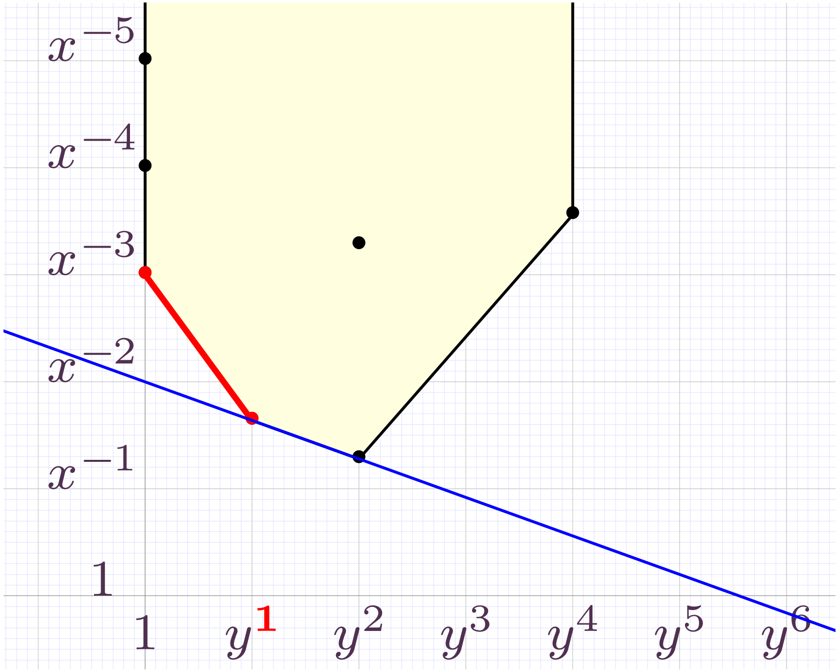
$$P(y) = \frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x}\, y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec \frac{1}{x}$$



$$P \longrightarrow P_{\times x^{-1}}$$

$$y \longrightarrow \frac{y}{x}$$

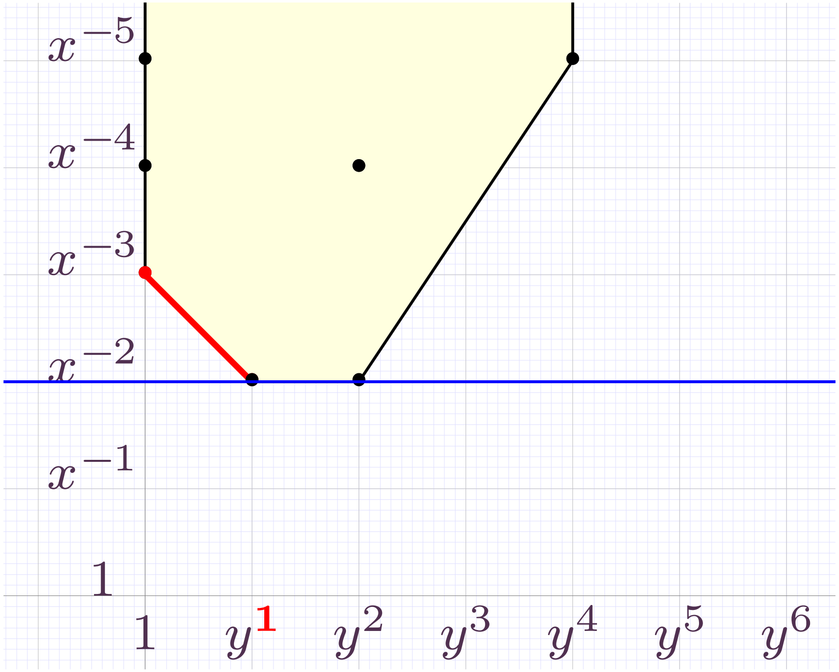
$$P(y) = \frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x}\, y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec \frac{1}{x}$$



$$P \longrightarrow P_{\times x^{-1}}$$

$$y \longrightarrow \frac{y}{x}$$

$$P_{\times x^{-1}}(y) = \frac{y^4}{x^5} - \left(\frac{1}{x^2} + \frac{1}{x^4}\right) y^2 - \frac{3}{x^2}\, y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec 1$$



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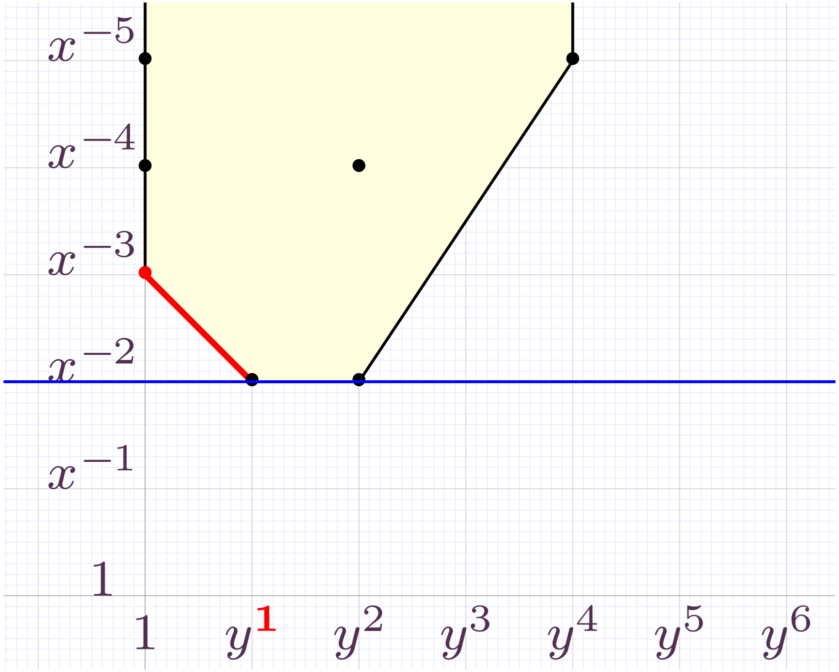
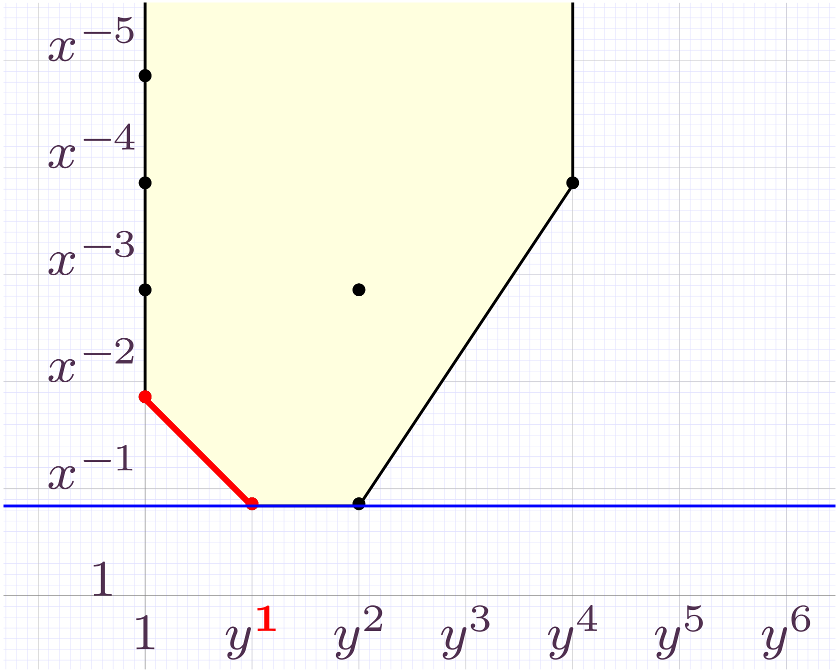


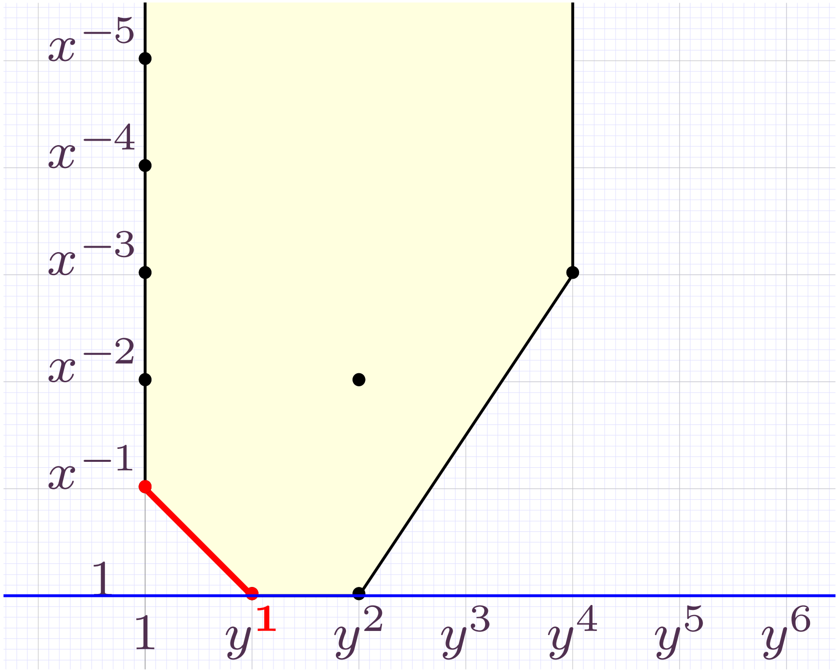
$$P_{\times x^{-1}} \longrightarrow x^2\, P_{\times x^{-1}}$$

$$P_{\times x^{-1}}(y) = \frac{y^4}{x^5} - \left(\frac{1}{x^2} + \frac{1}{x^4}\right) y^2 - \frac{3}{x^2}\, y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec 1$$



$$P_{\times x^{-1}} \longrightarrow x^2\, P_{\times x^{-1}}$$

$$x^2\,P_{\times x^{-1}}(y) = \frac{y^4}{x^3} - \left(1 + \frac{1}{x^2}\right) y^2 - 3\,y + \frac{6}{x^3 - x^4} = 0, \qquad y \prec 1$$



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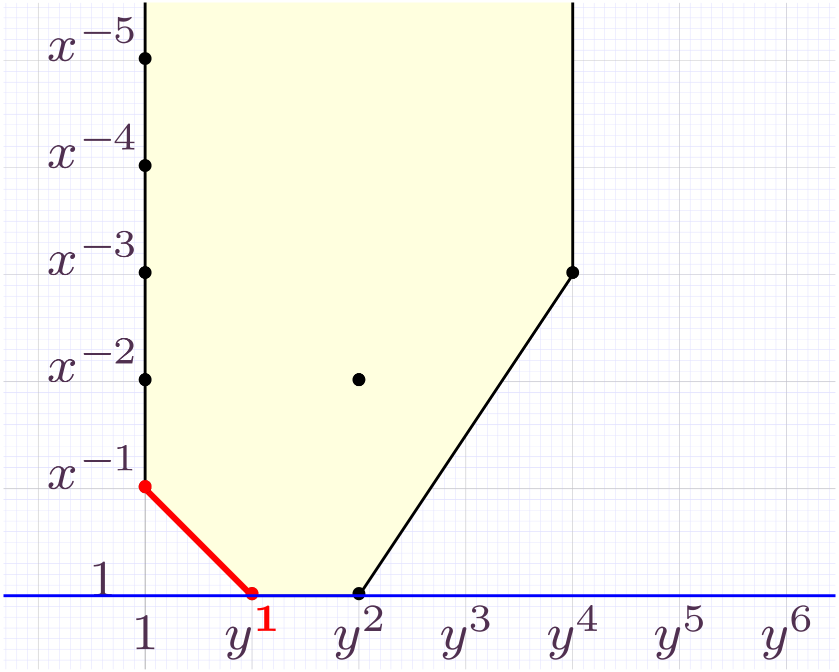

Hensel position $\Longrightarrow$ quasi-linear equation admits a unique solution

$$\begin{aligned} C[[\mathfrak{M}]][Y] &\subseteq C[Y][[\mathfrak{M}]] \\ N_P &= [\text{dominant coefficient of } P] \in C[Y] \end{aligned}$$

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$$\begin{aligned} \operatorname{ndeg}_{\preccurlyeq \mathfrak{v}} P &= \deg N_{P_{\times \mathfrak{v}}} \\ \operatorname{ndeg}_{\prec \mathfrak{v}} P &= \operatorname{val} N_{P_{\times \mathfrak{v}}} \end{aligned}$$

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$$\begin{aligned} \operatorname{ndeg}_{\prec \mathfrak{w}} P &\leqslant \operatorname{ndeg}_{\prec \mathfrak{v}} P, & \mathfrak{w} \prec \mathfrak{v} \\ \operatorname{ndeg}_{\prec \mathfrak{v}} P_{+\varphi} &= \operatorname{ndeg}_{\prec \mathfrak{v}} P, & \varphi \prec \mathfrak{v} \\ \operatorname{ndeg}_{\prec \mathfrak{v}} P_{\times \mathfrak{w}} &= \operatorname{ndeg}_{\prec \mathfrak{v}\mathfrak{w}} P & \\ \operatorname{ndeg}_{\prec \mathfrak{v}} (PQ) &= \operatorname{ndeg}_{\prec \mathfrak{v}} P + \operatorname{ndeg}_{\prec \mathfrak{v}} Q & \end{aligned}$$

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\operatorname{ndeg}_{\prec \mathfrak{w}} P &\leqslant \operatorname{ndeg}_{\prec \mathfrak{v}} P, && \mathfrak{w} \prec \mathfrak{v} \\
\operatorname{ndeg}_{\prec \mathfrak{v}} P_{+\varphi} &= \operatorname{ndeg}_{\prec \mathfrak{v}} P, && \varphi \prec \mathfrak{v} \\
\operatorname{ndeg}_{\prec \mathfrak{v}} P_{\times \mathfrak{w}} &= \operatorname{ndeg}_{\prec \mathfrak{v} \mathfrak{w}} P \\
\operatorname{ndeg}_{\prec \mathfrak{v}} (PQ) &= \operatorname{ndeg}_{\prec \mathfrak{v}} P + \operatorname{ndeg}_{\prec \mathfrak{v}} Q
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{ndeg}_{\prec \varphi} P_{+\varphi} &= \mu(c_\varphi; N_{P_{\times \mathfrak{d}_\varphi}}) \\
\mu_{\prec \mathfrak{v}}(f; P) &= \operatorname{ndeg}_{\prec \mathfrak{v}} P_{+f}
\end{aligned}
$$

Solution tree

$$\frac{y^4}{x} - \left(1 + \frac{1}{x^2}\right) y^2 - \frac{3}{x}\, y + \frac{6}{x^3 - x^4} = 0$$

$$y_1 = \frac{2}{x^2} + \tilde{y}_1 \qquad \tilde{y}_1 \prec \frac{1}{x^2}$$

$$y_2 = \frac{-3}{x} + \tilde{y}_2 \qquad \tilde{y}_2 \prec \frac{1}{x}$$

$$y_3 = \sqrt{x} + \tilde{y}_3 \qquad \tilde{y}_3 \prec \sqrt{x}$$

$$y_4 = -\sqrt{x} + \tilde{y}_4 \qquad \tilde{y}_4 \prec \sqrt{x}$$

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$$y_4 = -\sqrt{x} + \tilde{y}_4 \qquad \tilde{y}_4 \prec \sqrt{x}$$

$$\mathrm{ndeg}_{\prec \frac{1}{x^2}} P_{+\frac{2}{x^2}} = 1 \qquad \mathrm{ndeg}_{\prec \frac{1}{x}} P_{-\frac{3}{x}} = 1 \qquad \mathrm{ndeg}_{\prec \sqrt{x}} P_{+\sqrt{x}} = 1 \qquad \mathrm{ndeg}_{\prec \sqrt{x}} P_{-\sqrt{x}} = 1$$

$$P(y) = 0, \quad y \prec \mathfrak{v}$$

$$\begin{array}{ccc} y_1 = \varphi_1 + \tilde{y}_1 & & y_k = \varphi_k + \tilde{y}_k \\ \tilde{y}_1 \prec \tilde{\mathfrak{v}}_1 & \cdots & \tilde{y}_k \prec \tilde{\mathfrak{v}}_k \end{array}$$

$$\operatorname{ndeg}_{\prec \tilde{\mathfrak{v}}_1} P_{+\varphi_1} + \cdots + \operatorname{ndeg}_{\prec \tilde{\mathfrak{v}}_k} P_{+\varphi_k} \leqslant \operatorname{ndeg}_{\prec \mathfrak{v}} P$$

$$P(y)=0, \quad y \prec \mathfrak{v}$$

$$y_1 = \varphi_1 + \tilde{y}_1 \qquad \cdots \qquad y_k = \varphi_k + \tilde{y}_k$$

$$\tilde{y}_1 \prec \tilde{\mathfrak{v}}_1 \qquad\qquad \tilde{y}_k \prec \tilde{\mathfrak{v}}_k$$

$$\tilde{y}_{1,1} = \tilde{\varphi}_{1,1} + \tilde{\tilde{y}}_{1,1} \quad \cdots \quad \tilde{y}_{1,\tilde{k}_1} = \tilde{\varphi}_{1,\tilde{k}_1} + \tilde{\tilde{y}}_{1,\tilde{k}_1} \qquad \tilde{y}_{k,1} = \tilde{\varphi}_{k,1} + \tilde{\tilde{y}}_{k,1} \quad \cdots \quad \tilde{y}_{k,\tilde{k}_k} = \tilde{\varphi}_{k,\tilde{k}_k} + \tilde{\tilde{y}}_{k,\tilde{k}_k}$$

$$\tilde{\tilde{y}}_{1,1} \prec \tilde{\tilde{\mathfrak{v}}}_{1,1} \qquad \tilde{\tilde{y}}_{1,\tilde{k}_1} \prec \tilde{\tilde{\mathfrak{v}}}_{1,\tilde{k}_1} \qquad \tilde{\tilde{y}}_{k,1} \prec \tilde{\tilde{\mathfrak{v}}}_{k,1} \qquad \tilde{\tilde{y}}_{k,\tilde{k}_k} \prec \tilde{\tilde{\mathfrak{v}}}_{k,\tilde{k}_k}$$

$$\operatorname{ndeg}_{\prec \tilde{\mathfrak{v}}_1} P_{+\varphi_1} + \cdots + \operatorname{ndeg}_{\prec \tilde{\mathfrak{v}}_k} P_{+\varphi_k} \leqslant \operatorname{ndeg}_{\prec \mathfrak{v}} P$$

$$\begin{aligned}
\operatorname{ndeg}_{\prec \tilde{\tilde{\mathfrak{v}}}_{1,1}} P_{+\varphi_1+\tilde{\varphi}_{1,1}} + \cdots + \operatorname{ndeg}_{\prec \tilde{\tilde{\mathfrak{v}}}_{1,\tilde{k}_1}} P_{+\varphi_1+\tilde{\varphi}_{1,\tilde{k}_1}} &\leqslant \operatorname{ndeg}_{\prec \tilde{\mathfrak{v}}_1} P_{+\varphi_1}\\
&\vdots\\
\operatorname{ndeg}_{\prec \tilde{\tilde{\mathfrak{v}}}_{k,1}} P_{+\varphi_k+\tilde{\varphi}_{k,1}} + \cdots + \operatorname{ndeg}_{\prec \tilde{\tilde{\mathfrak{v}}}_{k,\tilde{k}_k}} P_{+\varphi_k+\tilde{\varphi}_{k,\tilde{k}_k}} &\leqslant \operatorname{ndeg}_{\prec \tilde{\mathfrak{v}}_k} P_{+\varphi_k}
\end{aligned}$$

$$\sum_{i,j} \operatorname{ndeg}_{\prec \tilde{\tilde{\mathfrak{v}}}_{i,j}} P_{+\varphi_i+\tilde{\varphi}_{i,j}} \leqslant \operatorname{ndeg}_{\prec \mathfrak{v}} P$$

$$y^2 - \frac{2}{1-x^{-1}}\,y + \left(\frac{1}{1-x^{-1}}\right)^2 \;=\; \mathrm{e}^{-x}$$

$$y^2 - \frac{2}{1-x^{-1}}\, y + \left(\frac{1}{1-x^{-1}}\right)^2 \;=\; \mathrm{e}^{-x}$$

$$y \;=\; 1 + \tilde{y}, \quad \tilde{y} \prec 1$$

$$y^2 - \frac{2}{1-x^{-1}}\,y + \left(\frac{1}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}$$

$$y = 1+\tilde{y}, \quad \tilde{y} \prec 1$$

$$\tilde{y}^2 - \frac{2\,x^{-1}}{1-x^{-1}}\,\tilde{y} + \left(\frac{x^{-1}}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}, \quad \tilde{y} \prec 1$$

$$y^2 - \frac{2}{1-x^{-1}}\, y + \left(\frac{1}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}$$

$$y = 1 + \tilde{y}, \quad \tilde{y} \prec 1$$

$$\tilde{y}^2 - \frac{2\, x^{-1}}{1-x^{-1}}\, \tilde{y} + \left(\frac{x^{-1}}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}, \quad \tilde{y} \prec 1$$

$$\tilde{y} = \frac{1}{x} + \tilde{\tilde{y}}, \quad \tilde{\tilde{y}} \prec 1$$

$$
\begin{aligned}
y^2 - \frac{2}{1-x^{-1}}\, y + \left(\frac{1}{1-x^{-1}}\right)^2 &= \mathrm{e}^{-x} \\
y &= 1 + \tilde{y}, \quad \tilde{y} \prec 1 \\
\tilde{y}^2 - \frac{2\,x^{-1}}{1-x^{-1}}\, \tilde{y} + \left(\frac{x^{-1}}{1-x^{-1}}\right)^2 &= \mathrm{e}^{-x}, \quad \tilde{y} \prec 1 \\
\tilde{y} &= \frac{1}{x} + \tilde{\tilde{y}}, \quad \tilde{\tilde{y}} \prec 1 \\
\tilde{\tilde{y}}^2 - \frac{2\,x^{-2}}{1-x^{-1}}\, \tilde{\tilde{y}} + \left(\frac{x^{-2}}{1-x^{-1}}\right)^2 &= \mathrm{e}^{-x}, \quad \tilde{\tilde{y}} \prec \frac{1}{x} \\
&\vdots
\end{aligned}
$$

$$y^2 - \frac{2}{1-x^{-1}}\,y + \left(\frac{1}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}$$

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$$\tilde{y} = \frac{1}{x} + \tilde{\tilde{y}}, \quad \tilde{\tilde{y}} \prec 1$$

$$\tilde{\tilde{y}}^2 - \frac{2\,x^{-2}}{1-x^{-1}}\,\tilde{\tilde{y}} + \left(\frac{x^{-2}}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}, \quad \tilde{\tilde{y}} \prec \frac{1}{x}$$

$$\vdots$$

Idea: differentiate

$$2\,y - \frac{2}{1-x^{-1}} = 0$$

$$y^2 - \frac{2}{1-x^{-1}}\,y + \left(\frac{1}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}$$

$$y = 1+\tilde{y}, \quad \tilde{y} \prec 1$$

$$\tilde{y}^2 - \frac{2\,x^{-1}}{1-x^{-1}}\,\tilde{y} + \left(\frac{x^{-1}}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}, \quad \tilde{y} \prec 1$$

$$\tilde{y} = \frac{1}{x} + \tilde{\tilde{y}}, \quad \tilde{\tilde{y}} \prec 1$$

$$\tilde{\tilde{y}}^2 - \frac{2\,x^{-2}}{1-x^{-1}}\,\tilde{\tilde{y}} + \left(\frac{x^{-2}}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}, \quad \tilde{\tilde{y}} \prec \frac{1}{x}$$

$$\vdots$$

Idea: differentiate

$$\begin{aligned} 2\,y - \frac{2}{1-x^{-1}} &= 0 \\ \varphi &= \frac{1}{1-x^{-1}} \end{aligned}$$

$$y^2 - \frac{2}{1-x^{-1}}\,y + \left(\frac{1}{1-x^{-1}}\right)^2 \;=\; \mathrm{e}^{-x}$$

$$y \;=\; \frac{1}{1-x^{-1}} + \tilde{y}, \quad \tilde{y} \prec 1$$

$$y^2 - \frac{2}{1-x^{-1}}\,y + \left(\frac{1}{1-x^{-1}}\right)^2 = \mathrm{e}^{-x}$$

$$y = \frac{1}{1-x^{-1}} + \tilde{y}, \quad \tilde{y} \prec 1$$

$$\tilde{y}^2 = \mathrm{e}^{-x}, \quad \tilde{y} \prec 1$$

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$$y = \frac{1}{1-x^{-1}} + \tilde{y}, \quad \tilde{y} \prec 1$$

$$\tilde{y}^2 = \mathrm{e}^{-x}, \quad \tilde{y} \prec 1$$

$$\begin{aligned}\tilde{y}_1 &= \mathrm{e}^{-x/2} + \tilde{\tilde{y}}_1, \quad \tilde{\tilde{y}}_1 \prec 1\\ \tilde{y}_2 &= -\mathrm{e}^{-x/2} + \tilde{\tilde{y}}_2, \quad \tilde{\tilde{y}}_2 \prec 1\end{aligned}$$

- C constant field
- $\mathfrak{M}$ divisible monomial group
- $P \in C[[\mathfrak{M}]]_{\mathscr{S}}[Y]$

Lemma. *Assume that* $\operatorname{ndeg}_{\prec \mathfrak{v}} P = d > 1$ *and that* $0 \neq \tau \prec \mathfrak{v}$ *satisfies* $\deg_{\prec \tau} P_{+\tau} = d$.

Then there is a unique $\varphi \sim \tau$ *with* $P^{(d-1)}(\varphi) = 0$, *and we have* $\deg_{\prec \varphi} P_{+\varphi} = d$.

Moreover, for any $\psi \prec \varphi$ *with* $\psi \neq 0$, *we have*

$$\operatorname{ndeg}_{\prec \psi} P_{+\varphi+\psi} \; < \; d.$$

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- $\mathfrak{M}$ divisible monomial group
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Then there is a unique $\varphi \sim \tau$ *with* $P^{(d-1)}(\varphi) = 0$*, and we have* $\deg_{\prec \varphi} P_{+\varphi} = d$.

Moreover, for any $\psi \prec \varphi$ *with* $\psi \neq 0$*, we have*

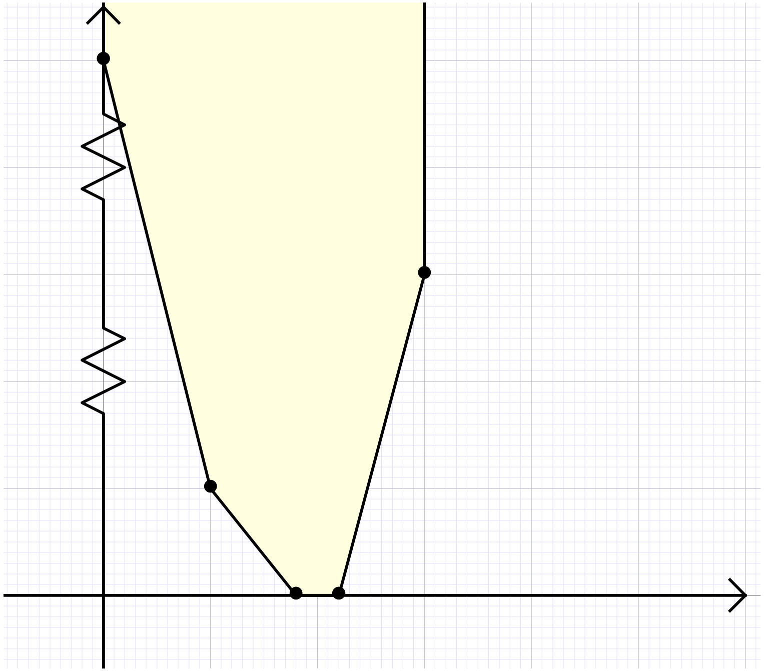
$$\mathrm{ndeg}_{\prec \psi} P_{+\varphi+\psi} \; < \; d.$$

Theorem.

- *If* C *is algebraically closed, then so is* $C[[\mathfrak{M}]]_{\mathscr{S}}$.
- *If* C *is real closed, then so is* $C[[\mathfrak{M}]]_{\mathscr{S}}$.

Algebraic and differential starting monomials

$$\frac{y^3}{\mathrm{e}^x} + (y')^2 - y\,y'' - \frac{y'}{x^2} + \mathrm{e}^{-\mathrm{e}^x} = 0$$



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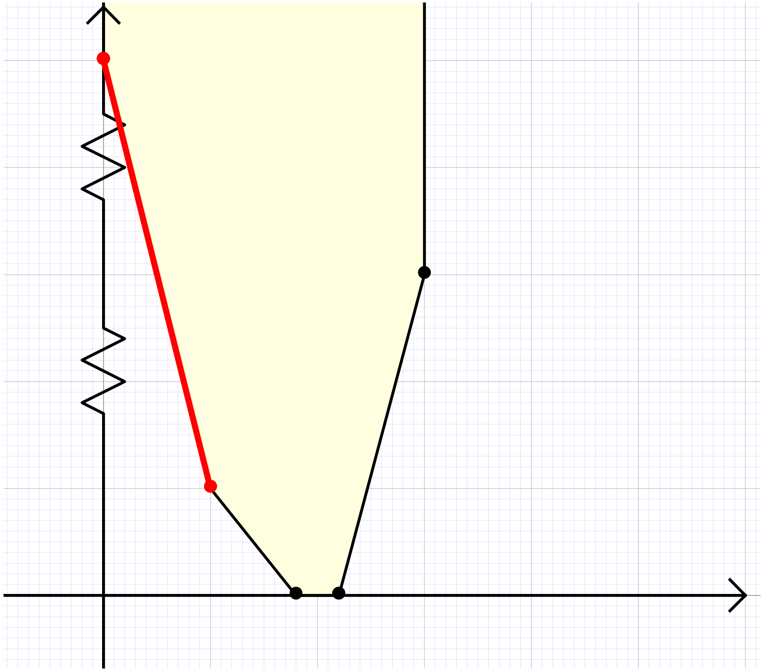

$$y \;=\; \frac{x^2}{\mathrm{e}^x}\,\mathrm{e}^{-\mathrm{e}^x} + \tilde{y}, \qquad \tilde{y} \prec \frac{x^2}{\mathrm{e}^x}\,\mathrm{e}^{-\mathrm{e}^x}.$$

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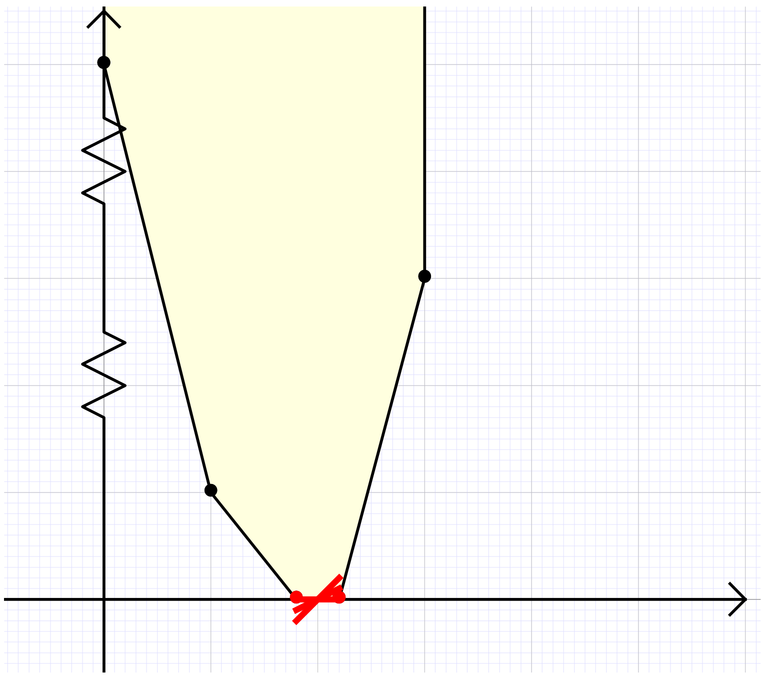

$$y \;=\; \lambda\,\mathrm{e}^{\mu x} + \tilde{y}, \qquad \tilde{y} \prec \mathrm{e}^{\mu x}, \qquad \lambda \neq 0, 0 \leqslant \mu < 1$$

Problems to be tackled

- How to determine the starting monomials?

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- $\mathfrak{m}$ starting monomial for $P(y)=0, y \prec \mathfrak{v} \Longleftrightarrow 1$ starting monomial for $P_{\times \mathfrak{m}}(y)=0, y \prec \mathfrak{v}\,\mathfrak{m}$

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- Suffices to characterize when 1 is a starting monomial
- Algebraic case: 1 is a starting monomial $\iff N_P$ admits a non zero root in $\mathbb{R}$
- **What is the differential analogue of N_P?**

$$P(y) = \frac{1}{x^2} y^3 + \frac{2}{x} y^2 - \left(\frac{4}{x} + \frac{1}{x^2}\right) y + \mathrm{e}^{-x}$$

$$N_P(y) = D_P(y) = 2\,y^2 - 4\,y$$

$$P = \frac{N_P}{x} + R$$

$$y \approx \mathrm{cst} \wedge N_P(y) \neq 0 \implies R(y) \prec \frac{N_P(y)}{x}$$

$$P(\log\log x) = \frac{2\log^2\log x}{x} - \frac{4\log\log x}{x} + \tilde{O}\left(\frac{1}{x^2}\right)$$

$$P(y) = \frac{1}{x^2}\, y^3 + \frac{2}{x}\, (y')^2 - \left(\frac{4}{x} + \frac{1}{x^2}\right) y'' + \mathrm{e}^{-x}$$

$$N_P(y) \stackrel{?}{=} D_P(y) = 2\, (y')^2 - 4\, y''$$

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y' is close to $\gamma = \frac{1}{x \log x \log \log x \cdots}$ near constants instead of being close to 1

y'' is close to $\gamma = \frac{1}{x^2 \log x \log \log x \cdots}$ near constants instead of being close to 1

$$
\begin{aligned}
x &\rightsquigarrow \mathrm{e}^{\tilde{x}} \\
y(x) &\rightsquigarrow \tilde{y}(\tilde{x}) \\
y'(x) &\rightsquigarrow \frac{\tilde{y}'(\tilde{x})}{\mathrm{e}^{\tilde{x}}} \\
y''(x) &\rightsquigarrow \frac{\tilde{y}''(\tilde{x}) - \tilde{y}'(\tilde{x})}{\mathrm{e}^{2\tilde{x}}} \\
y'''(x) &\rightsquigarrow \frac{\tilde{y}'''(\tilde{x}) - 3\,\tilde{y}''(\tilde{x}) + 2\,\tilde{y}'(\tilde{x})}{\mathrm{e}^{3\tilde{x}}} \\
&\vdots
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$$\begin{aligned}
P(y) &= \frac{1}{x^2}\,y^3 + \frac{2}{x}\,(y')^2 - \left(\frac{4}{x} + \frac{1}{x^2}\right) y'' + \mathrm{e}^{-x} \\
&\ \wr\!\!\downarrow \\
\tilde{P}(\tilde{y}) &= \frac{1}{\mathrm{e}^{2\tilde{x}}}\,\tilde{y}^3 + \frac{2}{\mathrm{e}^{3\tilde{x}}}\,(\tilde{y}')^2 - \left(\frac{4}{\mathrm{e}^{3\tilde{x}}} + \frac{1}{\mathrm{e}^{4\tilde{x}}}\right)(\tilde{y}'' - \tilde{y}') + \mathrm{e}^{-\mathrm{e}^{\tilde{x}}}
\end{aligned}$$

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(P\uparrow\uparrow)(y) &= \frac{1}{\mathrm{e}^{2\mathrm{e}^x}}\, y^3 + \frac{2}{\mathrm{e}^{3x}\,\mathrm{e}^{3\mathrm{e}^x}}\,(y')^2 + \left(\frac{4}{\mathrm{e}^{3\mathrm{e}^x}} + \frac{1}{\mathrm{e}^{4\mathrm{e}^x}}\right)\left(\frac{y'}{\mathrm{e}^x} + \frac{y'' - y'}{\mathrm{e}^{2x}}\right) + \mathrm{e}^{-\mathrm{e}^{\mathrm{e}^x}} \\
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Theorem. $\forall\ P\in\mathbb{T}\{Y\}^{\neq}$, $\exists\ n_0\in\mathbb{N}$, $\exists\ Q(Y)\in\mathbb{R}[Y]$, $\exists\ \nu\in\mathbb{N}$, $\forall n\geqslant n_0$

$$D_{P\uparrow^n} \;=\; Q(Y)\,(Y')^{\nu}$$

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\operatorname{ndeg}_{\prec\mathfrak{v}} P &= \operatorname{mul} N_{P_{\times\mathfrak{v}}}\\
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Problematic case ($\omega = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2 \, \ell_1^2} + \frac{1}{\ell_0^2 \, \ell_1^2 \, \ell_2^2} + \cdots$)

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- Obtaining N_P requires an infinite number of upward shiftings, i.e. taking $\phi = \gamma^{-1}$
- But it could be that $\omega \in K$ whereas $\gamma \notin K$

K: d-valued field of H-type with asymptotic integration and small derivation
$\mathfrak{M}$ multiplicative subgroup of K that is isomorphic to Γ_K

Theorem. *Given $P \in K\{Y\}^{\neq}$, there exists an $N \in C\{Y\}$ such that, eventually, $D_{P^\phi} = N$. Moreover, if K is ω-free, then $N \in C[Y]\,(Y')^{\mathbb{N}}$.*

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K: d-valued field of H-type with asymptotic integration and small derivation
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Theorem. *Given $P \in K\{Y\}^{\neq}$, there exists an $N \in C\{Y\}$ such that, eventually, $D_{P^\phi} = N$. Moreover, if K is ω-free, then $N \in C[Y]\,(Y')^{\mathbb{N}}$.*

Theorem. *Let $P, Q \in K\{Y\}^{\neq}$ be homogeneous, of degrees $i < j$. If $\mathfrak{M}$ is divisible if K is ω-free, then there exists a unique* ***equalizer*** *$\mathfrak{e} \in \mathfrak{M}$ for which $N_{(P+Q)_{\times\mathfrak{e}}}$ is not homogeneous.*

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Remark. Solution not necessarily unique (e.g. $y' = 0$). If $P \in \mathbb{T}\{Y\}$ and $\mathrm{ndeg}_{\prec \mathfrak{v}} P = 1$, then $P(y) = 0$, $y \prec \mathfrak{v}$ admits a unique "distinguished" solution y: whenever $P(\tilde{y}) = 0$, $\tilde{y} \prec \mathfrak{v}$ and $\tilde{y} - y \asymp \mathfrak{m}$, then $y_{\mathfrak{m}} = 0$.

Theorem. *Let K be a d-valued field of H-type that is stable under integration. Assume that K is the directed union of spherically complete grounded d-valued subfields. Then K is newtonian.*

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Theorem. *Let K be an H-asymptotic field with asymptotic integration and divisible value group. Then there exists an immediate asymptotic extension of K that is spherically complete.*

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Theorem. *If K is ω-free, newtonian, and with divisible value group, then we may "unravel" d-fold starts of solutions using a finite number of differential Tschirnhaus refinements.*

$$y^2 + 2\,y' + \omega = 0, \qquad y \prec 1$$

$$\omega = \frac{1}{x^2} + \frac{1}{x^2 \log^2 x} + \frac{1}{x^2 \log^2 x \log^2 \log x} + \cdots$$

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Upward shift

$$y^2 + \frac{2}{\mathrm{e}^x}\,y' + \frac{1}{\mathrm{e}^{2x}}\,(1+\omega), \qquad y \prec 1$$

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Multiplicative conjugation $y \longrightarrow y\,\mathrm{e}^{-x}$

$$y^2 + 2\,y' - 2\,y + 1 + \omega, \qquad y \prec \mathrm{e}^x$$

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Refine $y \longrightarrow 1 + \tilde{y}, \quad \tilde{y} \prec 1$

$$\tilde{y}^2 + 2\,\tilde{y}' + \omega = 0, \qquad \tilde{y} \prec 1$$

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Infinite number of steps $\rightsquigarrow$

$$y = \lambda = \frac{1}{x} + \frac{1}{x \log x} + \frac{1}{x \log x \log \log x} + \cdots$$

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Compositional conjugation $\partial = \gamma\,\tilde{\partial}$

$$y^2 + 2\,\gamma\,\tilde{\partial} y + \omega = 0, \qquad y \prec 1$$

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Multiplicative conjugation $y \longrightarrow \gamma\, y$

$$\gamma^2\, y^2 + 2\, \gamma^2\, \tilde{\partial} y - 2\, \lambda \gamma\, y + \omega = 0, \qquad y \prec \gamma^{-1}$$

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Differentiate

$$2\,\gamma^2\,\phi - 2\,\lambda\gamma = 0, \qquad \phi \prec \gamma^{-1}$$

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Differentiate

$$2\,\gamma^2\, \phi - 2\,\lambda\gamma = 0, \qquad \phi \prec \gamma^{-1}$$

Refine original equation

$$y = \gamma\, \phi + \tilde{y} = \lambda + \tilde{y}, \quad \tilde{y} \prec \lambda$$

Shape of differential Newton polynomials

$$P \rightsquigarrow P{\uparrow} \rightsquigarrow P{\uparrow}{\uparrow} \rightsquigarrow \cdots$$

Potential infinite loops during unravelling

$$P \rightsquigarrow P{\uparrow} \rightsquigarrow P{\uparrow}_{\times \mathrm{e}^{-x}} \rightsquigarrow P{\uparrow}_{\times \mathrm{e}^{-x},+1} \rightsquigarrow P{\uparrow}_{\times \mathrm{e}^{-x},+1}{\uparrow} \rightsquigarrow P{\uparrow}_{\times \mathrm{e}^{-x},+1}{\uparrow}_{\times \mathrm{e}^{-x}} \rightsquigarrow \cdots$$

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Automorphisms

$$\begin{aligned} P &\longmapsto P\!\uparrow \\ P &\longmapsto P_{\times \mathrm{e}^{-x}} \\ P &\longmapsto P_{+1} \end{aligned}$$

$$\begin{pmatrix} Y{\uparrow} \\ Y'{\uparrow} \\ Y''{\uparrow} \\ Y'''{\uparrow} \\ Y''''{\uparrow} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \mathrm{e}^{-x} & & & & \\ & & \mathrm{e}^{-2x} & & & \\ & & & \mathrm{e}^{-3x} & & \\ & & & & \mathrm{e}^{-4x} & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & -1 & 1 & & & \\ & 3 & -2 & 1 & & \\ & -6 & 11 & -6 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} Y \\ Y' \\ Y'' \\ Y''' \\ Y'''' \\ \vdots \end{pmatrix}$$

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$$\nabla = \log \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & -1 & 1 & & & \\ & 3 & -2 & 1 & & \\ & -6 & 11 & -6 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & -1 & 0 & & & \\ & 1/2 & -2 & 0 & & \\ & -1/2 & 2 & -6 & 0 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} Y\uparrow \\ Y'\uparrow \\ Y''\uparrow \\ Y'''\uparrow \\ Y''''\uparrow \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & e^{-x} & & & & \\ & & e^{-2x} & & & \\ & & & e^{-3x} & & \\ & & & & e^{-4x} & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & -1 & 1 & & & \\ & 3 & -2 & 1 & & \\ & -6 & 11 & -6 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} Y \\ Y' \\ Y'' \\ Y''' \\ Y'''' \\ \vdots \end{pmatrix}$$

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$$\nabla = \nabla_1 + \nabla_2 + \cdots = \begin{pmatrix} 0 & & & & \\ & -1 & & & \\ & & -2 & & \\ & & & -6 & \\ & & & & \ddots \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 1/2 & & \\ & & 2 & \\ & & & \ddots \end{pmatrix} + \cdots$$

Note: infinite lower triangular matrices naturally act on $K\{Y\}$

Theorem. *For $P \in K\{Y\}$, we have*

$$\nabla_1 P = \nabla_2 P = 0 \iff P \in K[Y]\,(Y')^{\mathbb{N}}$$

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Decomposition into isobaric parts

$$P \;=\; Y''' + Y^2 Y'' - (Y')^2 + 3\,Y^3 Y' - Y^6 + 3\,Y = \sum_{w \in \mathbb{N}} P_{[w]}$$

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Eventual behaviour

For $\phi = \gamma \rightarrow \gamma$, $\lambda = -\gamma^{\dagger} \rightarrow \lambda$, $\omega = -(\lambda^2 + 2\,\lambda')$ and $P = P_{[w]}$:

$$\begin{aligned}
(P^{\phi})_{[w]} &= \phi^{w}\, P_{[w]}\\
(P^{\phi})_{[w-1]} &= \phi^{w-1}\,(P_{[w-1]} + \lambda\,\nabla_1\, P_{[w]})\\
(P^{\phi})_{[w-2]} &= \phi^{w-2}\,(P_{[w-2]} + \lambda\,\nabla_1\, P_{[w-1]} + (\omega\,\nabla_2 + \tfrac{1}{2}\,\lambda\,\nabla_1^2)\,P_{[w]})\\
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(P^{\phi})_{[w-2]} &= \phi^{w-2}(P_{[w-2]} + \lambda \nabla_1 P_{[w-1]} + (\omega \nabla_2 + \tfrac{1}{2}\lambda \nabla_1^2) P_{[w]}) \\
&\vdots
\end{aligned}$$

Corollary. *If K is ω-free, then $P \in K\{Y\}$ admits a differential polynomial $N_P \in C[Y](Y')^{\mathbb{N}}$.*

Theorem. *Let* $P \in \mathbb{R}\{Y\} \setminus \mathbb{R}$ *and* $\lambda = \frac{1}{\ell_0} + \frac{1}{\ell_0 \, \ell_1} + \frac{1}{\ell_0 \, \ell_1 \, \ell_2} + \cdots$. *Then either*

$$\begin{aligned} P(\lambda) &\approx c \, \mathfrak{m} \, \lambda \\ P(\lambda) &\approx c \, \mathfrak{m} \, \omega \end{aligned}$$

for some $c \in \mathbb{R}^{\neq}$ *and* $\mathfrak{m} \in \mathfrak{L}$.

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Theorem. *If* K *is* ω*-free, newtonian, and with divisible value group, then we may "unravel"* d*-fold starts of solutions using a finite number of differential Tschirnhaus refinements.*

Corollary. *If* K *is* ω*-free, newtonian, and with divisible value group, then* K *is asymptotically d-algebraically maximal.*

Corollary. *In particular, if* C *is also algebraically closed, then any asymptotic differential equation of Newton degree* d *admits at least* d *solutions in* K *(counted with multiplicities).*