Tutorial: Model Theory of Transseries

Lecture 4

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Overview

I. Newtonization

II. Strategy for the Proof of the Main Results

III. Applications

I. Newtonization

Reminders from the last lecture

Let K be a d-valued field of H-type with asymptotic integration. Recall that $C \cong \operatorname{res}(K)$. Suppose for simplicity that Γ is divisible and K is equipped with a "monomial group" $\mathfrak{M}$.

From JORIS' last lecture recall the definition of the NEWTON *polynomial* $N_P \in C\{Y\}^{\neq}$ of $P \in K\{Y\}^{\neq}$: eventually

$$P^{\phi} = \mathfrak{d} \cdot N_P + R_P \qquad \text{where } \mathfrak{d} = \mathfrak{d}_{\phi} \in \mathfrak{M} \text{ and } R_P \prec_{\phi}^{\flat} \mathfrak{d}.$$

If K is ω-free, then $N_P \in C[Y](Y')^{\mathbb{N}}$, and N_P doesn't change if we pass from K to an extension (of d-valued fields of H-type).

We put $\operatorname{ndeg} P := \deg N_P$ (the NEWTON degree of P).

We say that K is *newtonian* if every $P \in K\{Y\}$ with $\operatorname{ndeg} P = 1$ has a zero in $\mathcal{O}$. (Mostly useful in combination with ω-freeness.)

Constructing immediate extensions

Some reminders from general valuation theory

Let (a_ρ) be an ordinal-indexed sequence in K. Then

1. (a_ρ) **pseudoconverges to** $a \in K$ if $v(a - a_\rho)$ is eventually strictly increasing; notation: $a_\rho \rightsquigarrow a$;
2. (a_ρ) is **divergent** if it has no pseudolimit in K;
3. (a_ρ) is a **pseudocauchy sequence** in K if eventually

$$\tau > \sigma > \rho \quad \Longrightarrow \quad a_\tau - a_\sigma \prec a_\sigma - a_\rho;$$

equivalently: (a_ρ) has a pseudolimit in an extension of K.

Declare $(a_\rho) \sim (b_\sigma)$ if (a_ρ), (b_σ) have the same pseudolimits in all extensions of K, and set

$$\boldsymbol{a} := c_K(a_\rho) = \text{equivalence class of } (a_\rho).$$

Constructing immediate extensions

Let L be an extension of K. Then $C \subseteq C_L$, and naturally $\Gamma \hookrightarrow \Gamma_L$.

If $C_L = C$ and $\Gamma_L = \Gamma$, then L is an **immediate** extension of K. In this case, every $a \in L \setminus K$ is a pseudolimit of a divergent pc-sequence in K.

Conversely, we can always adjoin pseudolimits in immediate extensions, as we now explain.

We introduce a classification of pc-sequences (a_ρ) in K:

1. d-*algebraic type* over K: $P(b_\lambda) \rightsquigarrow 0$ for some $P \in K\{Y\}$ and pc-sequence $(b_\lambda) \sim (a_\rho)$ in K;
2. d-*transcendental type* over K: not of d-algebraic type.

Any P as in 1, chosen so that $Q(b_\lambda) \not\rightsquigarrow 0$ whenever $Q \in K\{Y\}$ has lower complexity than P and $(b_\lambda) \sim (a_\rho)$, is a **minimal** d-**polynomial** of (a_ρ) over K.

Constructing immediate extensions

Theorem (d-analogues of Kaplansky's theorems)

Let (a_ρ) be a divergent pc-sequence in K.

1. *Suppose (a_ρ) is of* d*-algebraic type over K with minimal* d*-polynomial P over K.*

   There is some a in an immediate extension of K with $a_\rho \rightsquigarrow a$ and $P(a) = 0$, and for each b in an extension of K with $a_\rho \rightsquigarrow b$ and $P(b) = 0$ there is a K-isomorphism $K\langle a\rangle \to K\langle b\rangle$ with $a \mapsto b$.

2. *Suppose (a_ρ) is of* d*-transcendental type over K.*

   There is some a in an immediate extension of K with $a_\rho \rightsquigarrow a$, and for each b in an extension of K with $a_\rho \rightsquigarrow b$ there is a K-isomorphism $K\langle a\rangle \to K\langle b\rangle$ with $a \mapsto b$.

A consequence: if K is ω-free and has no proper immediate d-algebraic extension, then K is newtonian.

Reducing to NEWTON degree 1

The proof of the following important fact uses the full machinery of NEWTON diagrams, including its most complicated part ("unraveling": differential TSCHIRNHAUS transformations) for dealing with "almost multiple zeros" (only hinted at by JORIS in his last lecture):

Theorem

Suppose K is ω-free. Let (a_ρ) be a divergent pc-sequence in K with minimal d*-polynomial P over K. Then* $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$*, i.e.,*

$$\operatorname{ndeg} P_{+a_\rho,\,\times(a_{\rho+1}-a_\rho)} = 1 \qquad \textit{for sufficiently large } \rho.$$

We now discuss how these facts can be used to embed K into a newtonian d-valued field in a "minimal" way.

Definition (an analogue of henselization of valued fields)

A **newtonization** of K is a newtonian extension of K which K-embeds into each newtonian extension of K.

Theorem

Suppose K is ω-free. Then K has a newtonization. Moreover, if L is a newtonization of K, then

- *L is an immediate extension of K;*
- *no proper differential subfield of L containing K is newtonian.*

We note the following consequence, which is a key ingredient for the proof of our main results.

Corollary

Suppose K is an ω-free H-field. There is a newtonian Liouville closed H-field extension K^{nl} of K which embeds over K into each newtonian Liouville closed H-field extension of K. Any such K^{nl} is d-algebraic over K. Its constant field is a real closure of C.

We call K^{nl} the NEWTON-LIOUVILLE **closure** of K.

If K is ω-free, then each d-algebraic H-field extension of K is ω-free, and hence K has a unique LIOUVILLE closure up to isomorphism over K.

Thus one can obtain K^{nl} by alternating newtonization with taking LIOUVILLE closures.

Main ingredients for obtaining a newtonization

These are the results on constructing immediate extensions, the theorem on "reduction to ndeg 1", and the following:

Lemma

Suppose K is newtonian. Let (a_ρ) be a pc-sequence in K and $P \in K\{Y\}$ with $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$:

$$\operatorname{ndeg} P_{+a_\rho,\, \times(a_{\rho+1}-a_\rho)} = 1 \qquad \textit{for sufficiently large } \rho.$$

Then there is some $a \in K$ with $P(a) = 0$ and $a_\rho \rightsquigarrow a$.

By our assumptions, for sufficiently large ρ we get $z_\rho \in K$ with

$$P(z_\rho) = 0 \quad \text{and} \quad z_\rho - a_\rho \preccurlyeq a_{\rho+1} - a_\rho.$$

We claim that for large enough ρ we can upgrade this to "$\asymp$" (and so take $a := z_\rho$ for large enough ρ). For this one shows that the zeros of P can't "accumulate."

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Lemma

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$$\operatorname{ndeg} P_{+a_\rho,\, \times(a_{\rho+1}-a_\rho)} = 1 \qquad \text{for sufficiently large } \rho.$$

Then there is some $a \in K$ with $P(a) = 0$ and $a_\rho \rightsquigarrow a$.

With not too much extra work, this lemma also yields:

Corollary (assuming K ω-free)

K *is newtonian* $\iff$ K *has no proper immediate* d*-algebraic extension.*

II. Strategy for the Proof of the Main Results

Let $\mathcal{L} = \{0, 1, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ and let

$$T^{\mathrm{nl}} = \text{the theory of newtonian L\textsc{iouville} closed } H\text{-fields},$$

that is, the $\mathcal{L}$-theory axiomatized by

- the axioms for LIOUVILLE closed H-fields;
- the ω-freeness axiom; and
- the axiom scheme of newtonianity.

Every H-field extends to a model of T^{nl}, and in JORIS' lectures we heard that $\mathbb{T} \models T^{\mathrm{nl}}$.

Theorem A

T^{nl} is model complete. (Hence T^{nl} is the model companion of the theory of H-fields.)

Strategy for the proof of Theorem A

By the familiar model completeness test of A. ROBINSON, it suffices to solve the following embedding problem:

$$\begin{tikzcd} & L \\ K \arrow[ur, dashed] & \\ E \arrow[u, "\subseteq"] \arrow[uur, "i"'] & \end{tikzcd}$$

Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

We first make some preliminary reductions. First,

$$\begin{aligned} C_L \text{ is real closed, very saturated } &\Rightarrow\ i|C_E \text{ extends to } j\colon C \hookrightarrow C_L \\ &\Rightarrow\ j \text{ to } E(C) \hookrightarrow L. \end{aligned}$$

Since $E(C)$ is d-algebraic over E, it remains ω-free.

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Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$ such that $C_E = C$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

Next, suppose $\Gamma_E^{<}$ is not cofinal in $\Gamma^{<}$. Take $y \in K^{>}$, $y^* \in L^{>}$ with

$$\Gamma_E^{<} < vy < 0, \qquad \Gamma_{iE}^{<} < vy^* < 0.$$

Now $E\langle y\rangle$ is grounded, but it extends to an ω-free H-field "$E\langle y\rangle_\omega = E\langle y, \log y, \log\log y, \dots\rangle$" in a canonical way.

So i extends to an embedding $E\langle y\rangle_\omega \hookrightarrow L$ with $y \mapsto y^*$.

Strategy for the proof of Theorem A

By the familiar model completeness test of A. Robinson, it suffices to solve the following embedding problem:

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Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$ such that $C_E = C$ and $\Gamma_E^{<}$ is cofinal in $\Gamma^{<}$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

This has the nice consequence that now we don't need to worry about preserving ω-freeness anymore: *every* differential subfield of K containing E is an ω-free H-subfield of K.

Strategy for the proof of Theorem A

By the familiar model completeness test of A. ROBINSON, it suffices to solve the following embedding problem:

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Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$ such that $C_E = C$ and $\Gamma_E^{<}$ is cofinal in $\Gamma^{<}$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

Now we have the following three cases:

1. E is not newtonian and LIOUVILLE closed;
2. E is newtonian and LIOUVILLE closed, and there is some $y \in K \setminus E$ such that $E\langle y \rangle | E$ is immediate;
3. E is newtonian and LIOUVILLE closed, but there is no $y \in K \setminus E$ such that $E\langle y \rangle | E$ is immediate.

Strategy for the proof of Theorem A

$$\begin{tikzcd} & L \\ K \arrow[ur, dashed] & \\ E \arrow[u, "\subseteq"] \arrow[uur, "i"'] & \end{tikzcd}$$

Case 1

E is not newtonian and LIOUVILLE closed.

Then we can extend i to an embedding $E^{\mathrm{nl}} \hookrightarrow L$ of the NEWTON-LIOUVILLE closure E^{nl} of E inside K.

Strategy for the proof of Theorem A

$$\begin{tikzcd} & L \\ K \arrow[ur, dashed] & \\ E \arrow[u, "\subseteq"] \arrow[uur, "i"'] & \end{tikzcd}$$

Case 2

E is newtonian and LIOUVILLE closed, and we have $y \in K \setminus E$ with $E\langle y\rangle|E$ immediate.

Take a divergent pc-sequence (a_ρ) in E such that $a_\rho \rightsquigarrow y$.

By saturation, take $z \in L$ with $i(a_\rho) \rightsquigarrow z$.

Since E is has no proper immediate d-algebraic extension, (a_ρ) is of d-transcendental type over E.

Thus i extends to $E\langle y\rangle \hookrightarrow L$ with $y \mapsto z$.

Strategy for the proof of Theorem A

$$\begin{tikzcd} & L \\ K \arrow[ur, dashed] & \\ E \arrow[u, "\subseteq"] \arrow[uur, "i"'] & \end{tikzcd}$$

Case 3

E is newtonian and LIOUVILLE closed, and for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

In this case it turns out that for each $f \in K \setminus E$, the *cut of f in the ordered set E* uniquely determines the isomorphism type of $E\langle f\rangle$ over E (and so we can again appeal to saturation).

Let's look at this case in some more detail.

Here we are approximating f by iterated exponential integrals.

Setting

Let $E \subseteq K$ be an extension of ω-free newtonian LIOUVILLE closed H-fields with $C_E = C$, and suppose E is maximal in K: for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

Then no divergent pc-sequence in E has a pseudolimit in K.

Definition

Let $f \in K \setminus E$. Then $v(f - E) \subseteq \Gamma$ has a largest element, and we call $b \in E$ a **best approximation** to f if

$$v(f - b) = \max v(f - E).$$

Note that then $v(f - b) \notin \Gamma_E$ since $C = C_E$.

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω-free newtonian LIOUVILLE closed H-fields with $C_E = C$, and suppose E is maximal in K: for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

Let $f \in K \setminus E$. Pick a best approximation $b_0 \in E$ to $f_0 := f$. Then $f_1 := (f_0 - b_0)^\dagger \notin E$ since E is LIOUVILLE closed and $C_E = C$. So we can take a best approximation b_1 to f_1, etc.

We get sequences (f_n) in $K \setminus E$ and (a_n), (b_n) in E such that

- $a_n^\dagger = b_n$ is a best approximation to f_n, and
- $f_{n+1} = (f_n - b_n)^\dagger$.

$$\text{“} f \;=\; b_0 + \mathrm{e}^{\int f_1} \;=\; b_0 + \mathrm{e}^{\int b_1 + \mathrm{e}^{\int f_2}} \;=\; \cdots \text{”.}$$

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω-free newtonian LIOUVILLE closed H-fields with $C_E = C$, and suppose E is maximal in K: for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

Then for each $P \in E\{Y\}$ one can expand $P(f) \in K$ as a polynomial in the "monomials"

$$\mathfrak{m}_n := (f_n - b_n)/a_{n+1} \in E\langle f\rangle.$$

Using this one gets detailed information about the asymptotic couple of $E\langle f\rangle$: with $\mu_n := v\mathfrak{m}_n \in \Gamma_{E\langle f\rangle}$,

- $\Gamma_{E\langle f\rangle} = \Gamma_E \oplus \bigoplus_n \mathbb{Z}\mu_n$, and $\Gamma_E^{<}$ is cofinal in $\Gamma_{E\langle f\rangle}^{<}$;
- $\psi\big(\Gamma_{E\langle f\rangle}^{>}\big) = \psi(\Gamma_E^{>}) \cup \{\mu_0^\dagger < \mu_1^\dagger < \cdots\}$ with $\mu_n^\dagger \notin \Gamma_E$.

All this turns out to only depend on the cut of f in E!

Before we move on to Theorem B, we record:

Corollary

The completions of T^{nl} *are* $T^{\text{nl}}_{\text{small}} = \text{Th}(\mathbb{T})$ *and* $T^{\text{nl}}_{\text{large}}$.

To see this, we note that the ω-free H-field $E := \mathbb{Q}(\ell_0, \ell_1, \dots)$ embeds into each LIOUVILLE closed H-field with small derivation, in particular into $\mathbb{T}$.

So the NEWTON-LIOUVILLE closure

$$\mathbb{T}^{\text{da}} := \{f \in \mathbb{T} : f \text{ is d-algebraic}\}$$

of E inside $\mathbb{T}$ is a prime model of $T^{\text{nl}}_{\text{small}}$.

Similarly, the NEWTON-LIOUVILLE closure of E^{ϕ} ($\phi = x^{-2}$), is a prime model of $T^{\text{nl}}_{\text{large}}$.

Let $\mathcal{L}^{\iota}_{\Lambda,\Omega} = \mathcal{L} \cup \{\iota, \Lambda, \Omega\}$ and let

$$T^{\mathrm{nl},\iota}_{\Lambda,\Omega} = T^{\mathrm{nl}} + \text{the universal closures of}$$

$$\big[a \neq 0 \longrightarrow a \cdot \iota(a) = 1\big] \ \&\ \big[a = 0 \longrightarrow \iota(a) = 0\big],$$

$$\Lambda(a) \ \longleftrightarrow\ \exists y\big[y \succ 1 \ \&\ a = -y^{\dagger\dagger}\big],$$

$$\Omega(a) \ \longleftrightarrow\ \exists y\big[y \neq 0 \ \&\ 4y'' + ay = 0\big].$$

We denote $\mathcal{L}^{\iota}_{\Lambda,\Omega}$-structures by boldface letters: $\boldsymbol{K} = (K, \Lambda, \Omega)$.

Theorem B

$T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ admits quantifier elimination.

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:

$$\begin{tikzcd}
 & \boldsymbol{L} \\
\boldsymbol{K} \arrow[ur, dashed] & \\
\boldsymbol{E} \arrow[u, "\subseteq"] \arrow[uur, "i"'] &
\end{tikzcd}$$

Let $\boldsymbol{E}$ be a substructure of some $\boldsymbol{K} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ and let i be an embedding of $\boldsymbol{E}$ into a very saturated $\boldsymbol{L} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$. Then i extends to an embedding $\boldsymbol{K} \hookrightarrow \boldsymbol{L}$.

In order to tackle this, we need to first investigate the substructures of models of $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$.

Since we included ι in $\mathcal{L}^{\iota}_{\Lambda,\Omega}$, such substructures are valued ordered differential *fields.*

However, they are not automatically H-fields $\rightsquigarrow$ **pre-H-fields.**

Strategy for the proof of Theorem B

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The pairs (Λ, Ω) of subsets of a pre-H-field E such that $\boldsymbol{E} = (E, \Lambda, \Omega)$ embeds into a model of $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ are characterized by the axioms for **$\Lambda\Omega$-cuts** in E. We show:

- every ω-free pre-H-field has just one $\Lambda\Omega$-cut;
- $\boldsymbol{E}$ has an extension $\boldsymbol{E}^* = (E^*, \dots)$, where E^* is an ω-free H-field, which embeds over E into any model of $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ extending $\boldsymbol{E}$.

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:

$$\begin{tikzcd}
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Let $\boldsymbol{E}$ be a substructure of some $\boldsymbol{K} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ and let i be an embedding of $\boldsymbol{E}$ into a very saturated $\boldsymbol{L} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$. Then i extends to an embedding $\boldsymbol{K} \hookrightarrow \boldsymbol{L}$.

These two facts allow us to focus henceforth, for embedding purposes, on ω-free H-fields, and we can forget about $\Lambda\Omega$-cuts.

Theorem B now follows from the embedding theorem that we used in proving Theorem A. (The embedding theorem is somewhat stronger than model completeness of T^{nl}, since E there is only assumed to be ω-free.)

III. Applications

Corollary

1. $\mathbb{T}$ *is* o-minimal at $+\infty$*: if* $X \subseteq \mathbb{T}$ *is definable, then there is some* $f \in \mathbb{T}$ *with* $(f, +\infty) \subseteq X$ *or* $(f, +\infty) \cap X = \emptyset$*.*
2. *All definable subsets of* $\mathbb{R}^n \subseteq \mathbb{T}^n$ *are* semialgebraic.
3. $\mathbb{T}$ *has* NIP*.*

An instance of 1: if P is a one-variable d-polynomial over $\mathbb{T}$, then there is some $f \in \mathbb{T}$ and $\sigma \in \{\pm 1\}$ with sign $P(y) = \sigma$ for all $y > f$. (Related to old theorems of BOREL, HARDY, ...)

An illustration of 2: the set of $(c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ such that

$$c_0 y + c_1 y' + \cdots + c_n y^{(n)} = 0, \qquad 0 \neq y \prec 1$$

has a solution in $\mathbb{T}$ is a semialgebraic subset of $\mathbb{R}^{n+1}$.

One can strengthen 3 to "$\mathbb{T}$ is distal" (of infinite dp-rank).

Eliminate the primitives $\preccurlyeq$, Λ, Ω, ι using "ideal" elements, thus reducing quantifier-free formulas to a very simple form:

Let $K \models T^{\mathrm{nl}}$. In an immediate H-field extension L of K we find some element λ with

$$\begin{aligned} & \Lambda(K) < \lambda && < K \setminus \Lambda(K), \\ \text{so} \quad & \Omega(K) < \omega := \omega(\lambda) && < K \setminus \Omega(K). \end{aligned}$$

Next take some c^* in an H-field extension L^* of L with

$$C < c^* < K^{>C}.$$

Then for each 0-definable $X \subseteq K^n$ there is a quantifier-free formula φ in the language $\mathcal{L}_{\mathrm{OR}}$ of ordered rings such that

$$X = \left\{ a \in K^n : L^* \models \varphi\left(a, a', \ldots, a^{(r)}, \lambda, \omega, c^*\right) \right\}.$$

We illustrate this by establishing ❸ through reduction to NIP for real closed fields: Suppose $R \subseteq K^m \times K^n$ is 0-definable and independent. We just do the case $m = n = 1$. Thus for every $N \geqslant 1$ there are $a_1, \dots, a_N \in K$ and $b_I \in K$ ($I \subseteq \{1, \dots, N\}$) with

$$R(a_i, b_I) \iff i \in I.$$

Take a quantifier-free $\mathcal{L}_{\mathrm{OR}}$-formula φ such that for all $a, b \in K$:

$$R(a, b) \iff L^* \models \varphi\big(a, a', \dots, a^{(r)}, b, b', \dots, b^{(r)}, \lambda, \omega, c^*\big).$$

Thus the relation $R^* \subseteq (L^*)^{r+1} \times (L^*)^{r+4}$ given by

$$R^*(a_0, \dots, a_r, b_0, \dots, b_{r+3}) \iff L^* \models \varphi(a_0, \dots, a_r, b_0, \dots, b_{r+3})$$

is independent and (q.f.-) definable in the $\mathcal{L}_{\mathrm{OR}}$-structure L^*. ↯