

Tutorial: Model Theory of Transseries

Lecture 4

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- I. Newtonization
- II. Strategy for the Proof of the Main Results
- III. Applications

I. Newtonization

Reminders from the last lecture

Let K be a d -valued field of H -type with asymptotic integration. Recall that $C \cong \text{res}(K)$. Suppose for simplicity that Γ is divisible and K is equipped with a “monomial group” $\mathfrak{M}$.

From JORIS' last lecture recall the definition of the NEWTON polynomial $N_P \in C\{Y\}^\neq$ of $P \in K\{Y\}^\neq$: eventually

$$P^\phi = \mathfrak{d} \cdot N_P + R_P \quad \text{where } \mathfrak{d} = \mathfrak{d}_\phi \in \mathfrak{M} \text{ and } R_P \prec_\phi^\flat \mathfrak{d}.$$

If K is ω -free, then $N_P \in C[Y](Y')^\mathbb{N}$, and N_P doesn't change if we pass from K to an extension (of d -valued fields of H -type).

We put $\text{ndeg } P := \deg N_P$ (the NEWTON degree of P).

We say that K is *newtonian* if every $P \in K\{Y\}$ with $\text{ndeg } P = 1$ has a zero in $\mathcal{O}$. (Mostly useful in combination with ω -freeness.)

Constructing immediate extensions

Some reminders from general valuation theory

Let (a_ρ) be an ordinal-indexed sequence in K . Then

- ① (a_ρ) **pseudoconverges to** $a \in K$ if $v(a - a_\rho)$ is eventually strictly increasing; notation: $a_\rho \rightsquigarrow a$;
- ② (a_ρ) is **divergent** if it has no pseudolimit in K ;
- ③ (a_ρ) is a **pseudocauchy sequence** in K if eventually

$$\tau > \sigma > \rho \implies a_\tau - a_\sigma \prec a_\sigma - a_\rho;$$

equivalently: (a_ρ) has a pseudolimit in an extension of K .

Declare $(a_\rho) \sim (b_\sigma)$ if $(a_\rho), (b_\sigma)$ have the same pseudolimits in all extensions of K , and set

$$\mathbf{a} := c_K(a_\rho) = \text{equivalence class of } (a_\rho).$$

Constructing immediate extensions

Let L be an extension of K . Then $C \subseteq C_L$, and naturally $\Gamma \hookrightarrow \Gamma_L$.

If $C_L = C$ and $\Gamma_L = \Gamma$, then L is an **immediate** extension of K . In this case, every $a \in L \setminus K$ is a pseudolimit of a divergent pc-sequence in K .

Conversely, we can always adjoin pseudolimits in immediate extensions, as we now explain.

We introduce a classification of pc-sequences (a_ρ) in K :

- ① *d-algebraic type* over K : $P(b_\lambda) \rightsquigarrow 0$ for some $P \in K\{Y\}$ and pc-sequence $(b_\lambda) \sim (a_\rho)$ in K ;
- ② *d-transcendental type* over K : not of d-algebraic type.

Any P as in ①, chosen so that $Q(b_\lambda) \not\rightsquigarrow 0$ whenever $Q \in K\{Y\}$ has lower complexity than P and $(b_\lambda) \sim (a_\rho)$, is a **minimal d-polynomial** of (a_ρ) over K .

Constructing immediate extensions

Theorem (d-analogues of KAPLANSKY's theorems)

Let (a_ρ) be a divergent pc-sequence in K .

- 1 *Suppose (a_ρ) is of d-algebraic type over K with minimal d-polynomial P over K .*

There is some a in an immediate extension of K with $a_\rho \rightsquigarrow a$ and $P(a) = 0$, and for each b in an extension of K with $a_\rho \rightsquigarrow b$ and $P(b) = 0$ there is a K -isomorphism $K\langle a \rangle \rightarrow K\langle b \rangle$ with $a \mapsto b$.

- 2 *Suppose (a_ρ) is of d-transcendental type over K .*

There is some a in an immediate extension of K with $a_\rho \rightsquigarrow a$, and for each b in an extension of K with $a_\rho \rightsquigarrow b$ there is a K -isomorphism $K\langle a \rangle \rightarrow K\langle b \rangle$ with $a \mapsto b$.

A consequence: if K is ω -free and has no proper immediate d-algebraic extension, then K is newtonian.

The proof of the following important fact uses the full machinery of NEWTON diagrams, including its most complicated part (“unraveling”: differential TSCHIRNHAUS transformations) for dealing with “almost multiple zeros” (only hinted at by JORIS in his last lecture):

Theorem

Suppose K is ω -free. Let (a_ρ) be a divergent pc-sequence in K with minimal d -polynomial P over K . Then $\text{ndeg}_a P = 1$, i.e.,

$$\text{ndeg } P_{+a_\rho, \times(a_{\rho+1}-a_\rho)} = 1 \quad \text{for sufficiently large } \rho.$$

We now discuss how these facts can be used to embed K into a newtonian d -valued field in a “minimal” way.

Definition (an analogue of henselization of valued fields)

A **newtonization** of K is a newtonian extension of K which K -embeds into each newtonian extension of K .

Theorem

Suppose K is ω -free. Then K has a newtonization. Moreover, if L is a newtonization of K , then

- L is an immediate extension of K ;*
- no proper differential subfield of L containing K is newtonian.*

We note the following consequence, which is a key ingredient for the proof of our main results.

Corollary

Suppose K is an ω -free H -field. There is a newtonian Liouville closed H -field extension K^{nl} of K which embeds over K into each newtonian Liouville closed H -field extension of K . Any such K^{nl} is d -algebraic over K . Its constant field is a real closure of C .

We call K^{nl} the NEWTON-LIOUVILLE **closure** of K .

If K is ω -free, then each d -algebraic H -field extension of K is ω -free, and hence K has a unique LIOUVILLE closure up to isomorphism over K .

Thus one can obtain K^{nl} by alternating newtonization with taking LIOUVILLE closures.

Main ingredients for obtaining a newtonization

These are the results on constructing immediate extensions, the theorem on “reduction to $\text{ndeg } 1$ ”, and the following:

Lemma

Suppose K is newtonian. Let (a_ρ) be a pc-sequence in K and $P \in K\{Y\}$ with $\text{ndeg}_a P = 1$:

$$\text{ndeg } P_{+a_\rho, \times(a_{\rho+1}-a_\rho)} = 1 \quad \text{for sufficiently large } \rho.$$

Then there is some $a \in K$ with $P(a) = 0$ and $a_\rho \rightsquigarrow a$.

By our assumptions, for sufficiently large ρ we get $z_\rho \in K$ with

$$P(z_\rho) = 0 \quad \text{and} \quad z_\rho - a_\rho \preccurlyeq a_{\rho+1} - a_\rho.$$

We claim that for large enough ρ we can upgrade this to “ $\asymp$ ” (and so take $a := z_\rho$ for large enough ρ). For this one shows that the zeros of P can’t “accumulate.”

Main ingredients for obtaining a newtonization

These are the results on constructing immediate extensions, the theorem on “reduction to $\text{ndeg } 1$ ”, and the following:

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Then there is some $a \in K$ with $P(a) = 0$ and $a_\rho \rightsquigarrow a$.

With not too much extra work, this lemma also yields:

Corollary (assuming K ω -free)

K is newtonian $\iff K$ has no proper immediate d-algebraic extension.

II. Strategy for the Proof of the Main Results

Recapitulation of Theorem A

Let $\mathcal{L} = \{0, 1, +, \cdot, \partial, \leq, \preceq\}$ and let

T^{nl} = the theory of newtonian LIOUVILLE closed H -fields,

that is, the $\mathcal{L}$ -theory axiomatized by

- the axioms for LIOUVILLE closed H -fields;
- the ω -freeness axiom; and
- the axiom scheme of newtonianity.

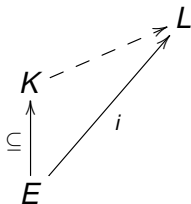
Every H -field extends to a model of T^{nl} , and in JORIS' lectures we heard that $\mathbb{T} \models T^{\text{nl}}$.

Theorem A

T^{nl} is model complete. (Hence T^{nl} is the model companion of the theory of H -fields.)

Strategy for the proof of Theorem A

By the familiar model completeness test of A. ROBINSON, it suffices to solve the following embedding problem:



Let E be an ω -free H -subfield of some $K \models T^{\text{nl}}$, and let i be an embedding of E into a very saturated $L \models T^{\text{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

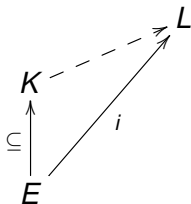
We first make some preliminary reductions. First,

$$\begin{aligned} C_L \text{ is real closed, very saturated} &\Rightarrow i|_{C_E} \text{ extends to } j: C \hookrightarrow C_L \\ &\Rightarrow j \text{ to } E(C) \hookrightarrow L. \end{aligned}$$

Since $E(C)$ is d-algebraic over E , it remains ω -free.

Strategy for the proof of Theorem A

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Next, suppose $\Gamma_E^<$ is not cofinal in $\Gamma^<$. Take $y \in K^>$, $y^* \in L^>$ with

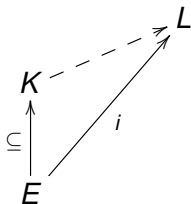
$$\Gamma_E^< < \nu y < 0, \quad \Gamma_{iE}^< < \nu y^* < 0.$$

Now $E\langle y \rangle$ is grounded, but it extends to an ω -free H -field “ $E\langle y \rangle_\omega = E\langle y, \log y, \log \log y, \dots \rangle$ ” in a canonical way.

So i extends to an embedding $E\langle y \rangle_\omega \hookrightarrow L$ with $y \mapsto y^*$.

Strategy for the proof of Theorem A

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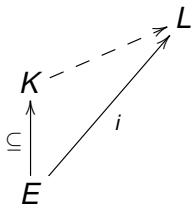


Let E be an ω -free H -subfield of some $K \models T^{\text{nl}}$ such that $C_E = C$ and $\Gamma_E^<$ is cofinal in $\Gamma^<$, and let i be an embedding of E into a very saturated $L \models T^{\text{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

This has the nice consequence that now we don't need to worry about preserving ω -freeness anymore: every differential subfield of K containing E is an ω -free H -subfield of K .

Strategy for the proof of Theorem A

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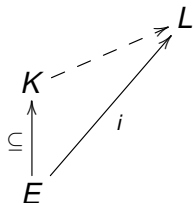


Let E be an ω -free H -subfield of some $K \models T^{\text{nl}}$ such that $C_E = C$ and $\Gamma_E^<$ is cofinal in $\Gamma^<$, and let i be an embedding of E into a very saturated $L \models T^{\text{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

Now we have the following three cases:

- ① E is not newtonian and LIOUVILLE closed;
- ② E is newtonian and LIOUVILLE closed, and there is some $y \in K \setminus E$ such that $E\langle y \rangle | E$ is immediate;
- ③ E is newtonian and LIOUVILLE closed, but there is no $y \in K \setminus E$ such that $E\langle y \rangle | E$ is immediate.

Strategy for the proof of Theorem A

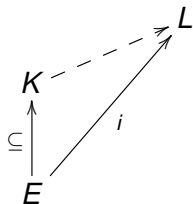


Case 1

E is not newtonian and LIOUVILLE closed.

Then we can extend i to an embedding $E^{\text{nl}} \hookrightarrow L$ of the NEWTON-LIOUVILLE closure E^{nl} of E inside K .

Strategy for the proof of Theorem A



Case 2

E is newtonian and LIOUVILLE closed, and we have $y \in K \setminus E$ with $E\langle y \rangle | E$ immediate.

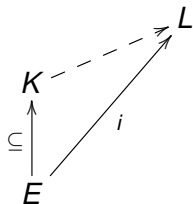
Take a divergent pc-sequence (a_ρ) in E such that $a_\rho \rightsquigarrow y$.

By saturation, take $z \in L$ with $i(a_\rho) \rightsquigarrow z$.

Since E has no proper immediate d-algebraic extension, (a_ρ) is of d-transcendental type over E .

Thus i extends to $E\langle y \rangle \hookrightarrow L$ with $y \mapsto z$.

Strategy for the proof of Theorem A



Case 3

E is newtonian and LIOUVILLE closed, and for no $y \in K \setminus E$ is $E\langle y \rangle|E$ immediate.

In this case it turns out that for each $f \in K \setminus E$, the *cut of f in the ordered set E* uniquely determines the isomorphism type of $E\langle f \rangle$ over E (and so we can again appeal to saturation).

Let's look at this case in some more detail.

Here we are approximating f by iterated exponential integrals.

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω -free newtonian LIOUVILLE closed H -fields with $C_E = C$, and suppose E is maximal in K : for no $y \in K \setminus E$ is $E\langle y \rangle|E$ immediate.

Then no divergent pc-sequence in E has a pseudolimit in K .

Definition

Let $f \in K \setminus E$. Then $v(f - E) \subseteq \Gamma$ has a largest element, and we call $b \in E$ a **best approximation** to f if

$$v(f - b) = \max v(f - E).$$

Note that then $v(f - b) \notin \Gamma_E$ since $C = C_E$.

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω -free newtonian LIOUVILLE closed H -fields with $C_E = C$, and suppose E is maximal in K : for no $y \in K \setminus E$ is $E\langle y \rangle|E$ immediate.

Let $f \in K \setminus E$. Pick a best approximation $b_0 \in E$ to $f_0 := f$. Then $f_1 := (f_0 - b_0)^\dagger \notin E$ since E is LIOUVILLE closed and $C_E = C$. So we can take a best approximation b_1 to f_1 , etc.

We get sequences (f_n) in $K \setminus E$ and $(a_n), (b_n)$ in E such that

- $a_n^\dagger = b_n$ is a best approximation to f_n , and
- $f_{n+1} = (f_n - b_n)^\dagger$.

$$“f = b_0 + e \int f_1 = b_0 + e \int b_1 + e \int f_2 = \dots”.$$

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω -free newtonian LIOUVILLE closed H -fields with $C_E = C$, and suppose E is maximal in K : for no $y \in K \setminus E$ is $E\langle y \rangle | E$ immediate.

Then for each $P \in E\{Y\}$ one can expand $P(f) \in K$ as a polynomial in the “monomials”

$$m_n := (f_n - b_n)/a_{n+1} \in E\langle f \rangle.$$

Using this one gets detailed information about the asymptotic couple of $E\langle f \rangle$: with $\mu_n := v m_n \in \Gamma_{E\langle f \rangle}$,

- $\Gamma_{E\langle f \rangle} = \Gamma_E \oplus \bigoplus_n \mathbb{Z} \mu_n$, and $\Gamma_E^{\leq}$ is cofinal in $\Gamma_{E\langle f \rangle}^{\leq}$;
- $\psi(\Gamma_{E\langle f \rangle}^{\geq}) = \psi(\Gamma_E^{\geq}) \cup \{\mu_0^{\dagger} < \mu_1^{\dagger} < \dots\}$ with $\mu_n^{\dagger} \notin \Gamma_E$.

All this turns out to only depend on the cut of f in E !

Before we move on to Theorem B, we record:

Corollary

The completions of T^{nl} are $T_{\text{small}}^{\text{nl}} = \text{Th}(\mathbb{T})$ and $T_{\text{large}}^{\text{nl}}$.

To see this, we note that the ω -free H -field $E := \mathbb{Q}(\ell_0, \ell_1, \dots)$ embeds into each LIOUVILLE closed H -field with small derivation, in particular into $\mathbb{T}$.

So the NEWTON-LIOUVILLE closure

$$\mathbb{T}^{\text{da}} := \{f \in \mathbb{T} : f \text{ is d-algebraic}\}$$

of E inside $\mathbb{T}$ is a prime model of $T_{\text{small}}^{\text{nl}}$.

Similarly, the NEWTON-LIOUVILLE closure of E^ϕ ($\phi = x^{-2}$), is a prime model of $T_{\text{large}}^{\text{nl}}$.

Recapitulation of Theorem B

Let $\mathcal{L}_{\Lambda, \Omega}^{\iota} = \mathcal{L} \cup \{\iota, \Lambda, \Omega\}$ and let

$T_{\Lambda, \Omega}^{\text{nl}, \iota} = T^{\text{nl}} + \text{the universal closures of}$

$$[a \neq 0 \longrightarrow a \cdot \iota(a) = 1] \ \& \ [a = 0 \longrightarrow \iota(a) = 0],$$

$$\Lambda(a) \longleftrightarrow \exists y [y \succ 1 \ \& \ a = -y^{\dagger\dagger}],$$

$$\Omega(a) \longleftrightarrow \exists y [y \neq 0 \ \& \ 4y'' + ay = 0].$$

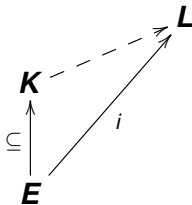
We denote $\mathcal{L}_{\Lambda, \Omega}^{\iota}$ -structures by boldface letters: $\mathbf{K} = (K, \Lambda, \Omega)$.

Theorem B

$T_{\Lambda, \Omega}^{\text{nl}, \iota}$ admits quantifier elimination.

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:



Let $\mathbf{E}$ be a substructure of some $\mathbf{K} \models T_{\Lambda, \Omega}^{\text{nl}, \iota}$ and let i be an embedding of $\mathbf{E}$ into a very saturated $\mathbf{L} \models T_{\Lambda, \Omega}^{\text{nl}, \iota}$. Then i extends to an embedding $\mathbf{K} \hookrightarrow \mathbf{L}$.

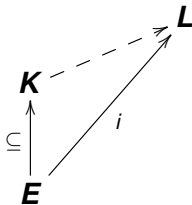
In order to tackle this, we need to first investigate the substructures of models of $T_{\Lambda, \Omega}^{\text{nl}, \iota}$.

Since we included ι in $\mathcal{L}_{\Lambda, \Omega}^{\iota}$, such substructures are valued ordered differential *fields*.

However, they are not automatically *H*-fields $\rightsquigarrow$ **pre-*H*-fields**.

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:



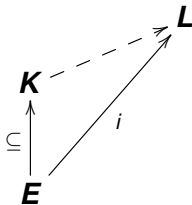
Let E be a substructure of some $K \models T_{\Lambda, \Omega}^{\text{nl}, \ell}$ and let i be an embedding of E into a very saturated $L \models T_{\Lambda, \Omega}^{\text{nl}, \ell}$. Then i extends to an embedding $K \hookrightarrow L$.

The pairs (Λ, Ω) of subsets of a pre- H -field E such that $E = (E, \Lambda, \Omega)$ embeds into a model of $T_{\Lambda, \Omega}^{\text{nl}, \ell}$ are characterized by the axioms for $\Lambda\Omega$ -cuts in E . We show:

- every ω -free pre- H -field has just one $\Lambda\Omega$ -cut;
- E has an extension $E^* = (E^*, \dots)$, where E^* is an ω -free H -field, which embeds over E into any model of $T_{\Lambda, \Omega}^{\text{nl}, \ell}$ extending E .

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:



Let $\mathbf{E}$ be a substructure of some $\mathbf{K} \models T_{\Lambda, \Omega}^{\text{nl}, \iota}$ and let i be an embedding of $\mathbf{E}$ into a very saturated $\mathbf{L} \models T_{\Lambda, \Omega}^{\text{nl}, \iota}$. Then i extends to an embedding $\mathbf{K} \hookrightarrow \mathbf{L}$.

These two facts allow us to focus henceforth, for embedding purposes, on ω -free H -fields, and we can forget about $\Lambda\Omega$ -cuts.

Theorem B now follows from the embedding theorem that we used in proving Theorem A. (The embedding theorem is somewhat stronger than model completeness of T^{nl} , since E there is only assumed to be ω -free.)

III. Applications

Corollary

- ① $\mathbb{T}$ is o-minimal at $+\infty$: if $X \subseteq \mathbb{T}$ is definable, then there is some $f \in \mathbb{T}$ with $(f, +\infty) \subseteq X$ or $(f, +\infty) \cap X = \emptyset$.
- ② All definable subsets of $\mathbb{R}^n \subseteq \mathbb{T}^n$ are semialgebraic.
- ③ $\mathbb{T}$ has NIP.

An instance of ①: if P is a one-variable d-polynomial over $\mathbb{T}$, then there is some $f \in \mathbb{T}$ and $\sigma \in \{\pm 1\}$ with $\text{sign } P(y) = \sigma$ for all $y > f$. (Related to old theorems of BOREL, HARDY, ...)

An illustration of ②: the set of $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ such that

$$c_0 y + c_1 y' + \dots + c_n y^{(n)} = 0, \quad 0 \neq y \prec 1$$

has a solution in $\mathbb{T}$ is a semialgebraic subset of $\mathbb{R}^{n+1}$.

One can strengthen ③ to “ $\mathbb{T}$ is distal” (of infinite dp-rank).

Eliminate the primitives $\preccurlyeq$, Λ , Ω , ι using “ideal” elements, thus reducing quantifier-free formulas to a very simple form:

Let $K \models T^{\text{nl}}$. In an immediate H -field extension L of K we find some element λ with

$$\begin{aligned} \Lambda(K) &< \lambda < K \setminus \Lambda(K), \\ \text{so } \Omega(K) &< \omega := \omega(\lambda) < K \setminus \Omega(K). \end{aligned}$$

Next take some c^* in an H -field extension L^* of L with

$$C < c^* < K^{>C}.$$

Then for each 0-definable $X \subseteq K^n$ there is a quantifier-free formula φ in the language $\mathcal{L}_{\text{OR}}$ of ordered rings such that

$$X = \{a \in K^n : L^* \models \varphi(a, a', \dots, a^{(r)}, \lambda, \omega, c^*)\}.$$

We illustrate this by establishing ③ through reduction to NIP for real closed fields: Suppose $R \subseteq K^m \times K^n$ is 0-definable and independent. We just do the case $m = n = 1$. Thus for every $N \geq 1$ there are $a_1, \dots, a_N \in K$ and $b_I \in K$ ($I \subseteq \{1, \dots, N\}$) with

$$R(a_i, b_I) \iff i \in I.$$

Take a quantifier-free $\mathcal{L}_{\text{OR}}$ -formula φ such that for all $a, b \in K$:

$$R(a, b) \iff L^* \models \varphi(a, a', \dots, a^{(r)}, b, b', \dots, b^{(r)}, \lambda, \omega, c^*).$$

Thus the relation $R^* \subseteq (L^*)^{r+1} \times (L^*)^{r+4}$ given by

$$R^*(a_0, \dots, a_r, b_0, \dots, b_{r+3}) \iff L^* \models \varphi(a_0, \dots, a_r, b_0, \dots, b_{r+3})$$

is independent and (q.f.-) definable in the $\mathcal{L}_{\text{OR}}$ -structure L^* . ⚡