Further Thoughts on Transseries and Valued Differential Fields

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Overview

I. Dimension of definable subsets of $\mathbb{T}^n$

II. Dimension zero = fiberable by the constant field

III. Automorphisms of $\mathbb{T}$

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II. Dimension zero $=$ fiberable by the constant field

III. Automorphisms of $\mathbb{T}$

IV. Gehret's work on $\mathbb{T}_{\log}$

V. Camacho's result on truncation

VI. Hakobyan: generalizing Scanlon's AKE-theorem

Part I

Dimension of definable subsets of $\mathbb{T}^n$

This is a coarse notion of dimension with $\dim \mathbb{T}^n = n$, and $\dim \mathbb{R} = 0$. Within dimension 0 we seem to have a finer notion of dimension "with respect to $\mathbb{R}$" to be discussed in part II.

A consequence of QE

From now on $K \equiv \mathbb{T}$. The **dimension** of a definable set $S \subseteq K^n$ is based on the pregeometry of d-algebraic dependence, that is, $\dim S :=$ largest $m \leqslant n$ for which there exist m differential polynomial functions on S that are d-algebraically independent over K.

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The good topological properties of "dimension" are due to a consequence of our QE:

Corollary

A definable set $S \subseteq K^n$ has empty interior in K^n iff $S \subseteq \{y \in K^n : P(y) = 0\}$ for some nonzero d-*polynomial $P \in K\{Y\},\ Y = (Y_1, \dots, Y_n)$.*

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$$\therefore \qquad \dim S \geqslant m \iff \pi(S) \text{ has nonempty interior in } K^m \text{ for some coordinate projection } \pi : K^n \to K^m$$

Further properties of dimension

- $\dim(S_1 \cup S_2) \;=\; \max(\dim S_1, \dim S_2)$;
- if $f : S \to K^m$ is definable, then $\dim S \geqslant \dim f(S)$;
- if $S \subseteq K^{m+n}$ is definable, and $d \leqslant n$, then the set $D := \{a \in K^m : \; \dim S(a) = d\}$ is definable, and $\dim S|_D = \dim D + d$; here $S|_D = \{(a, b) \in S : \; a \in D\}$;
- if $S \subseteq K^n$ is definable and nonempty, then $\dim\big(\mathrm{cl}(S) \setminus S\big) < \dim S$.

Dimension 0 equals Discrete

Local o-minimality of K yields for definable $S \subseteq K$:

$$\dim S = 0 \iff S \text{ has empty interior} \iff S \text{ is discrete}$$

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Proof for direction $\Leftarrow$: can arrange K is **uncountable**, with **countable** basis for the topology.

Suppose $S \subseteq K^n$ is discrete. Then S is countable, so $\pi_i(S) \subseteq K$ is countable, hence discrete, so $\dim \pi_i(S) = 0$, for $i = 1, \ldots, n$, and thus $\dim S = 0$.

Part II

Dimension zero = fiberable by C

Reduction to the zero set of P

A nonempty definable set $S \subseteq K^n$ has dimension 0 iff

$$S \subseteq \text{Zero}(P_1) \times \cdots \times \text{Zero}(P_n)$$

for some nonzero $P_i \in K\{Y\}$. Thus we can focus on sets of the form $\text{Zero}(P)$.

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Theorem

K has no proper d-algebraic H-field extension with the same constants.

Thus for nonzero $P \in K\{Y\}$:

if $K \preccurlyeq L$, $C_K = C_L$, then P has the same zeros in K and L.

This suggests that $\text{Zero}(P)$ is **controlled** by the constant field $C = C_K$. But how?

Internal to C?

Initially we thought $\mathrm{Zero}(P)$ might be **internal** to C, that is, $\mathrm{Zero}(P)$ (if nonempty) is the image of a definable map $C^n \to K$ for some n.

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However, this fails for $K = \mathbb{T}$ and $P = YY'' - (Y')^2$. Its zero set is

$$\{a\, \mathrm{e}^{bx} : \ a, b \in \mathbb{R}\}$$

and for any finite set A of parameters there is an automorphism of $\mathbb{T}$ over A that is not the identity on this zeroset; more on this in Part III.

Fiberability by C

Jim Freitag: take a look at the model-theoretic notion of "co-analyzability" (in a paper by Herwig, Hrushovski, Macpherson). This turns out to fit exactly our situation:

Corollary

For nonempty definable $S \subseteq K^n$,

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Fiberable by C is a convenient minor variant of *co-analyzable by C*.
For ω-saturated K and definable $S \subseteq K^n$ it is defined recursively as follows:

S is fiberable by C in 0 steps iff S is finite;

S is fiberable by C in $r+1$ steps iff there is a definable $f : S \to C$ such that every fiber $f^{-1}(c)$ is fiberable by C in r steps.

Applications

- If $S \subseteq K^n$ is definable and infinite of dimension 0, then $|S| = |C|$;
- If $S \subseteq K^{m+n}$ is definable, then for some $e \in \mathbb{N}$ we have $|S(a)| \leqslant e$ for all $a \in K^m$ for which $S(a)$ is finite.

Part III

Automorphisms of $\mathbb{T}$

Note: an automorphism of the differential field $\mathbb{T}$ preserves the ordering and the valuation ring. It is also the identity on $\mathbb{R}$.

Strong automorphisms

We restrict attention to **strong** automorphisms of $\mathbb{T}$, which respect infinite summation. For any real c we have a strong automorphism $f(x) \mapsto f(x+c) : \mathbb{T} \to \mathbb{T}$, and if $c \neq 0$, then

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Corollary

$\mathbb{R}$ is definably closed in $\mathbb{T}$. Hence C is definably closed in K.

Many natural differential subfields of $\mathbb{T}$ can be shown to be definably closed in $\mathbb{T}$ by exhibiting them as the fixpoint set of a set of strong automorphisms of $\mathbb{T}$.

The group of strong automorphisms

Theorem

Any additive $\alpha : \mathbb{T} \to \mathbb{R}$ *with* $\alpha(\mathcal{O}) = \{0\}$ *determines a strong automorphism* σ *of* $\mathbb{T}$ *by*

$$\sigma(x) \;=\; x, \qquad \sigma(\mathrm{e}^f) \;=\; \mathrm{e}^{\alpha(f)+\sigma(f)} \text{ for all } f \in \mathbb{T}.$$

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Any strong automorphism of $\mathbb{T}$ *fixing* x *has this form for a unique* α.

The strong automorphisms fixing x form a normal subgroup $\mathcal{G}_x$ of the group $\mathcal{G}$ of all strong automorphisms; $\mathcal{G}$ is a semidirect product of $\mathcal{G}_x$ with the group of automorphisms

$$f(x) \mapsto f(x+c) \qquad (c \in \mathbb{R})$$

Part IV

Gehret's work on $\mathbb{T}_{\log}$

Allen Gehret's work on $\mathbb{T}_{\log}$

Set $\ell_0 := x, \ell_1 := \log x, \ldots, \ell_{n+1} = \log \ell_n$. Define

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]].$$

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$\mathbb{T}_{\log}$ is a particularly transparent H-subfield of $\mathbb{T}$. It is ω-free and newtonian.

But $\mathbb{T}_{\log}$ is **not** Liouville closed. Much of the ADH-work concerns arbitrary ω-free newtonian H-fields and does not use Liouville closedness, and this gives hope that $\mathbb{T}_{\log}$ also has a reasonable model theory.

Allen Gehret's work on $\mathbb{T}_{\log}$

Gehret did the following:

1. he identified the complete theory of the asymptotic couple of $\mathbb{T}_{\log}$, and showed it has a good model theory;
2. found an interesting new axiom satisfied by $\mathbb{T}_{\log}$.

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2. found an interesting new axiom satisfied by $\mathbb{T}_{\log}$.

Gehret's Program is to show that the following axiomatizes a complete and model complete theory in a natural 2-sorted language:

- H-field with real closed constant field;
- asymptotic couple $\models$ theory in (1) above;
- axiom from (2) above;
- ω-free;
- newtonian.

Allen Gehret's work on $\mathbb{T}_{\log}$

The new axiom in (2) above was suggested by trying to existentially define the **complement** of the existentially definable set $\{f^\dagger : f \in \mathbb{T}_{\log}\}$, an $\mathbb{R}$-linear subspace of $\mathbb{T}_{\log}$.

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Gehret noticed that this is possible in the two-sorted structure consisting of $\mathbb{T}_{\log}$ with its asymptotic couple as second sort:

$y \notin \{f^\dagger : f \in \mathbb{T}_{\log}\} \iff$ there exists $g \neq 0$ such that $v(y - g^\dagger) \in \Psi^\downarrow \setminus \Psi$, where

$$\Psi := \{\gamma^\dagger : 0 \neq \gamma \in \text{value group}\}.$$

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The correct language for model-completeness will presumably include a predicate for $\{f^\dagger : f \in \mathbb{T}_{\log}\}$. Allen has proved several embedding results that reduce the conjecture to one on adjoining solutions to linear differential equations.

Part V

Camacho's result on truncation

Part VI

Hakobyan: generalizing Scanlon's Ax-Kochen-Ersov theorem

This concerns monotone valued differential fields. *Monotone*: $a' \preccurlyeq a$ for all a.

Scanlon's Theorem

Let $\mathbf{k}$ be a differential field and Γ an ordered abelian group. Then $\mathbf{k}((t^\Gamma))$ is naturally a valued field. We extend the derivation ∂ of $\mathbf{k}$ to $\mathbf{k}((t^\Gamma))$ by

$$\partial\Big(\sum_\gamma a_\gamma t^\gamma\Big) = \sum_\gamma \partial(a_\gamma) t^\gamma, \qquad \text{so } (t^\gamma)' = 0 \text{ for all } \gamma.$$

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Theorem

Assume $\mathbf{k}$ *is linearly surjective. Then the theory of* $\mathbf{k}((t^\Gamma))$ *is axiomatized by:*

- *axioms for valued differential fields with small derivation;*
- *many constants:* $v(C^\times) = \Gamma$*;*
- *differential henselianity;*
- $\mathrm{Th}(\mathbf{k})$ *and* $\mathrm{Th}(\Gamma)$.

The first two conditions together imply monotonicity.

Hakobyan's Generalization

Let any additive map $c : \Gamma \to \mathbf{k}$ be given. Then the derivation ∂ of $\mathbf{k}$ can be extended to a derivation ∂_c on $\mathbf{k}((t^\Gamma))$ by

$$\partial_c\Big(\sum_\gamma a_\gamma t^\gamma\Big) \;=\; \sum_\gamma \big(\partial(a_\gamma) + a_\gamma c(\gamma)\big)\, t^\gamma, \qquad \text{so } (t^\gamma)^\dagger = c(\gamma) \text{ for all } \gamma.$$

Let $\mathbf{k}((t^\Gamma))_c$ be the valued differential field $\mathbf{k}((t^\Gamma))$ with derivation ∂_c.
It is monotone, and if $\mathbf{k}$ is linearly surjective, then $\mathbf{k}((t^\Gamma))_c$ is differential henselian.

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It is monotone, and if $\mathbf{k}$ is linearly surjective, then $\mathbf{k}((t^\Gamma))_c$ is differential henselian.

Theorem

Assume $\mathbf{k}$ is linearly surjective. Then the theory of the valued differential field $\mathbf{k}((t^\Gamma))_c$ is completely determined by $\mathrm{Th}(\mathbf{k}, \Gamma; c)$, where $(\mathbf{k}, \Gamma; c)$ is the 2-sorted structure with $\mathbf{k}$ as differential field and Γ as ordered abelain group.

Continuation and an algebraic consequence

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Theorem

Every monotone differential-henselian field is elementarily equivalent to some $\mathbf{k}((t^{\Gamma}))_c$ *as in the previous theorem.*

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Here is an algebraic consequence:

Corollary

If K *is a monotone differential-henselian field, then every algebraic valued differential field extension of* K *is also* (*monotone and*) *differential-henselian.*