Fast integer multiplication

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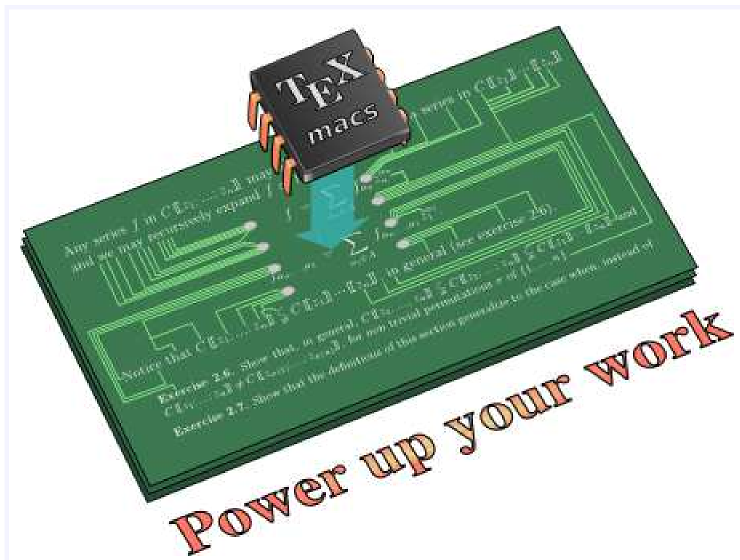

LIX, December 15, 2016

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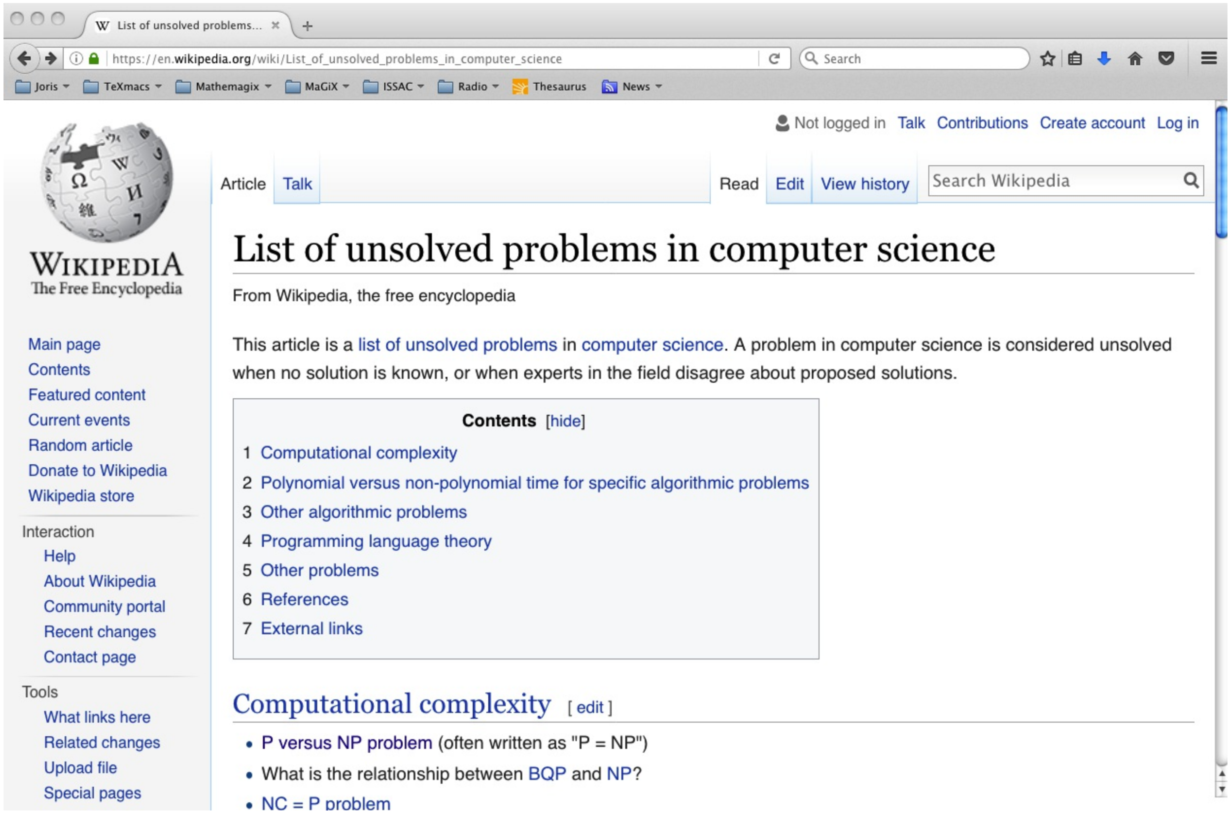
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List of unsolved problems in computer science
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This article is a list of unsolved problems in computer science. A problem in computer science is considered unsolved when no solution is known, or when experts in the field disagree about proposed solutions.
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2 Polynomial versus non-polynomial time for specific algorithmic problems
3 Other algorithmic problems
4 Programming language theory
5 Other problems
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Computational complexity [edit]
• P versus NP problem (often written as "P = NP")
• What is the relationship between BQP and NP?
• NC = P problem

Computational complexity

- P versus NP problem (often written as “P = NP”)
- What is the relationship between BQP and NP?
- NC = P problem ...

Polynomial vs non-polynomial time for specific algorithmic problems

- Can integer factorization be done in polynomial time?
- Is integer factorization NP-complete?
- Can clustered planar drawings be found in polynomial time? ...

Other algorithmic problems

- What is the fastest algorithm for multiplication of two n-digit numbers?
- What is the fastest algorithm for matrix multiplication?
- Can the Schwartz–Zippel lemma for polynomial identity testing be derandomized?
- Can a depth-first search tree be constructed in NC?
- Does linear programming admit a strongly polynomial-time algorithm?
- What is the lower bound on the complexity of fast Fourier transform algorithms? ...

Programming language theory ...

Other problems ...

Sequential *vs.* parallel

We will consider sequential algorithms

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Turing machines

Turing machines with a finite number of tapes [Papadimitriou 94]

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Other bit complexity models

- Operations on $\log n$-bit numbers in time $O(1)$
- Random access machine (RAM)

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« Straight Line Programs » (SLPs)

DAGs, non branching programs [Bürgisser–Clausen–Shokrollahi 97]

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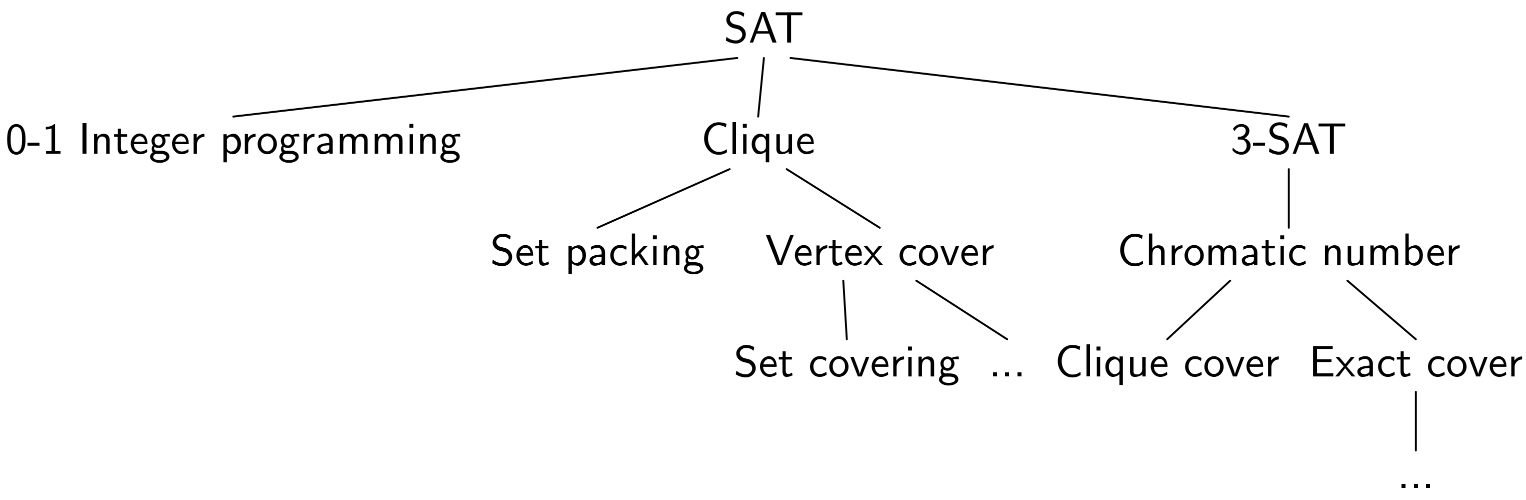
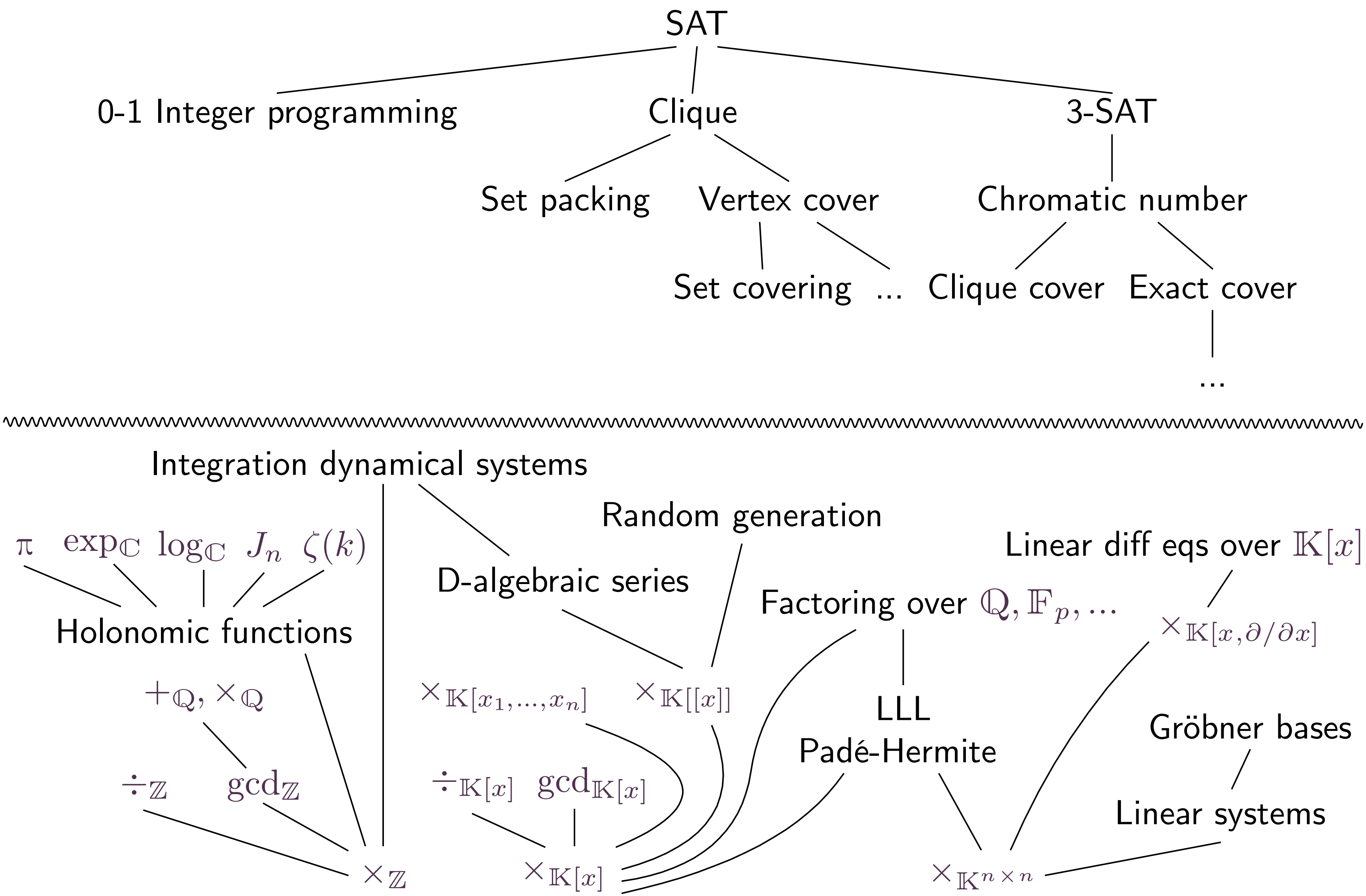
Other algebraic complexity models

- Turing machines with entries in model-theoretic structures $\mathfrak{S}$ [Friedman 69]
- BSS machines [Blum–Shub–Smale 89]


SAT
0-1 Integer programming
Clique
3-SAT
Set packing
Vertex cover
Chromatic number
Set covering
...
Clique cover
Exact cover
...


SAT
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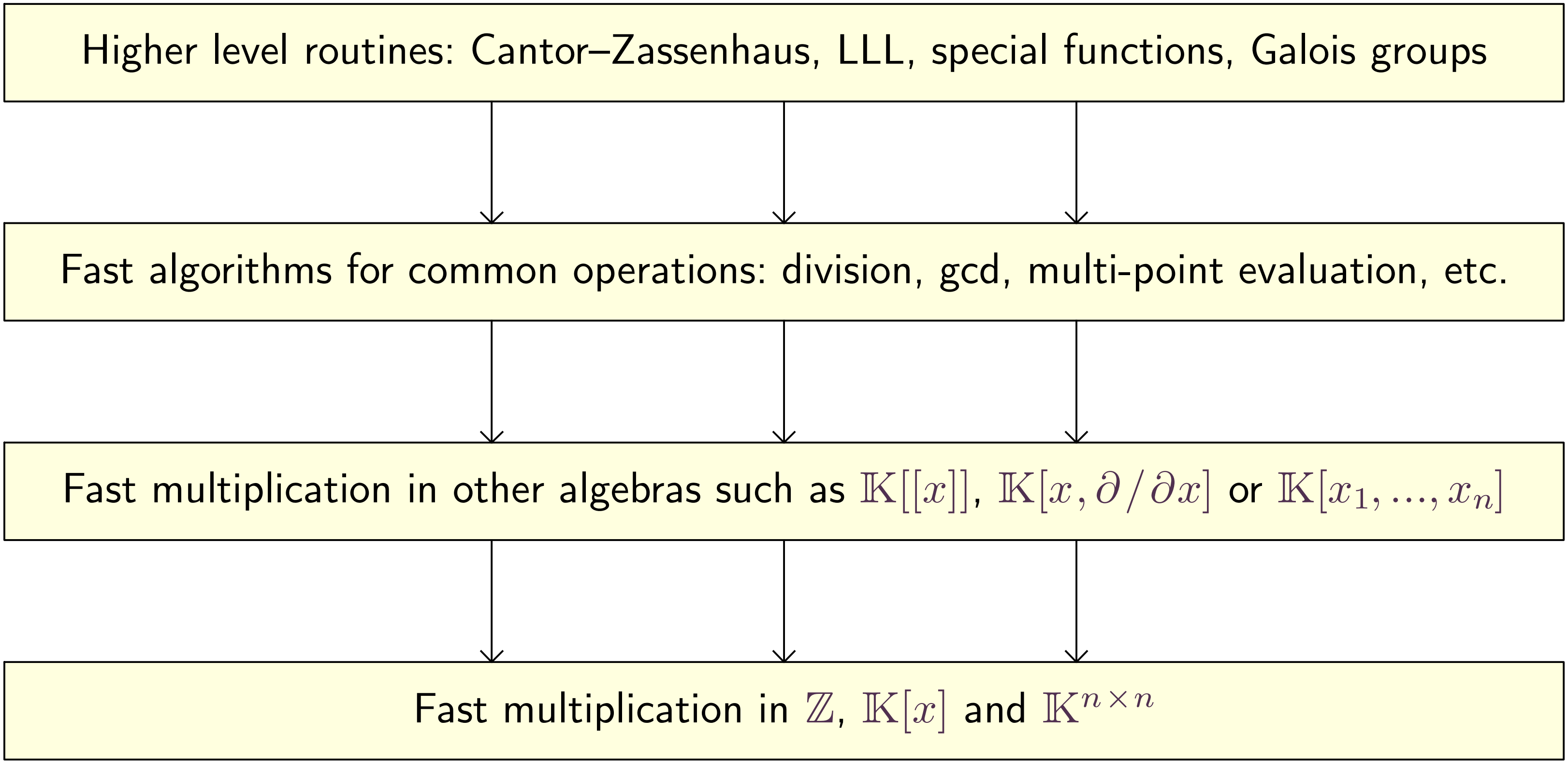
Integration dynamical systems
Random generation
π $\exp_{\mathbb{C}}$ $\log_{\mathbb{C}}$ J_n $\zeta(k)$
D-algebraic series
Linear diff eqs over $\mathbb{K}[x]$
Factoring over $\mathbb{Q}, \mathbb{F}_p, \ldots$
Holonomic functions
$\times_{\mathbb{K}[x,\partial/\partial x]}$
$+_{\mathbb{Q}}, \times_{\mathbb{Q}}$
$\times_{\mathbb{K}[x_1,\ldots,x_n]}$
$\times_{\mathbb{K}[[x]]}$
LLL
Gröbner bases
Padé-Hermite
$\div_{\mathbb{Z}}$
$\gcd_{\mathbb{Z}}$
$\div_{\mathbb{K}[x]}$
$\gcd_{\mathbb{K}[x]}$
Linear systems
$\times_{\mathbb{Z}}$
$\times_{\mathbb{K}[x]}$
$\times_{\mathbb{K}^{n\times n}}$


Higher level routines: Cantor–Zassenhaus, LLL, special functions, Galois groups
Fast algorithms for common operations: division, gcd, multi-point evaluation, etc.
Fast multiplication in other algebras such as $\mathbb{K}[[x]]$, $\mathbb{K}[x, \partial/\partial x]$ or $\mathbb{K}[x_1, \ldots, x_n]$
Fast multiplication in $\mathbb{Z}$, $\mathbb{K}[x]$ and $\mathbb{K}^{n \times n}$

Algebra	Model	Current record	Author(s)
$\mathbb{Z}$	binary	$\mathsf{I}(n) = O(n \log n\, 8^{\log^* n})$	Harvey-vdH-Lecerf 2014
$\mathbb{K}[z]$	algebraic	$\mathsf{M}(d) = O(d \log d \log \log d)$	Cantor-Kaltofen 1981
$\mathbb{F}_p[z]$	binary	$O(t \log t\, 8^{\log^* t})$ $t = d \log p$	Harvey-vdH-Lecerf 2014
$\mathbb{K}[z]$	algebraic char. $p > 0$	$\mathsf{M}(d) = O(d \log d\, 8^{\log^* d})$	Harvey-vdH-Lecerf 2014
$\mathbb{K}^{r \times r}$	algebraic	$O(n^\omega)$, $\omega \leqslant 2.3728639$	Le Gall 2014

$$\begin{aligned} \log^* x &:= \min \{k \in \mathbb{N} : \log^{\circ k} x \leqslant 1\}, \\ \log^{\circ k} &:= \underbrace{\log \circ \cdots \circ \log}_{k \times}. \end{aligned}$$

Algebra	Model	Current record	Author(s)
$\mathbb{K}[x]^{r \times r}$	algebraic	$O(\mathsf{M}(d)\, r^2 + d\, r^{\omega})$	Bostan-Schost 2005
$\mathbb{Z}^{r \times r}$	binary	$O\left(r^2\, \mathsf{M}(n) + r^{\omega} n 2^{\log^* n - \log^* d} \frac{\mathsf{M}(\log r)}{\log r}\right)$	Harvey-vdH 2014
$\mathbb{K}[[z]]$	algebraic relaxed	$O(d \log d\, \mathrm{e}^{\sqrt{2 \log 2} \sqrt{\log \log d}}\, X)$ $X = (\log \log d)^{5/2} \log \log \log d$	vdH 2014
$\mathbb{Z}_p$	binary relaxed	$O(d \log d\, \mathrm{e}^{\sqrt{2 \log 2} \sqrt{\log \log d}}\, X)$ $X = ((\log \log d)^{3/2} \log p)\, \ell \log \ell$ $\ell = \log (\log d + \log p)$	vdH 2014 Bertomieu-vdH-Lecerf 2011
$\mathbb{K}[z_1, \ldots, z_k]$	algebraic dense	$O(\mathsf{M}(s))$	vdH-Schost 2010
$\mathbb{Z}[z_1, \ldots, z_k]$	binary sparse	$O(\mathsf{I}(s\, \nu) \log s + s\, \mathsf{I}(\nu) \log \nu)$ $\nu = n + k \log d$	vdH-Lecerf 2009
$\mathbb{K}[z, \partial]$	algebraic	$O(r^{\omega - 1} d + r\, \mathsf{M}(d) \log d)$ si $d \geqslant r$ $O(d^{\omega - 1} r + d\, \mathsf{M}(r) \log r)$ si $r \geqslant d$	Benoit-Bostan-vdH 2012

Operation	Model	Current record	Author(s)
Euclidean division in $\mathbb{Z}$	binary	$\lesssim {}^5\!/_3\, \mathsf{I}(n)$	vdH 2010*
Square root in $\mathbb{Z}$	binary	$\lesssim {}^4\!/_3\, \mathsf{I}(n)$	Harvey 2009* vdH
G.c.d. in $\mathbb{Z}$	binary	$O(\mathsf{I}(n) \log n)$	Schönhage 1971 Lehmer
Chinese remaindering	binary	$O(\mathsf{I}(n\, d) \log d)$	Borodin-Moenck 1972
Chinese remaindering Fixed moduli	binary	$O\left(\mathsf{I}(n\, d) \frac{\log d}{\log \log (n\, d)}\right)$	vdH 2016

* Adapted from power series analogue

Date	Authors	Complexity
<3000 aJC	Unknown	$O(n^2)$
1962	Karatsuba	$O(n^{\log 3/\log 2})$
1963 (1965)	Toom (Cook)	$O(n\,2^{5\sqrt{\log n/\log 2}})$
1966	Schönhage	$O(n\,2^{\sqrt{2\log n/\log 2}}\,(\log n)^{3/2})$
1969	Knuth	$O(n\,2^{\sqrt{2\log n/\log 2}}\,\log n)$
1971	Schönhage–Strassen	$O(n\log n\log\log n)$
2007	Fürer	$O(n\log n\,2^{O(\log^* n)})$
2014	Harvey–vdH–Lecerf	$O(n\log n\,8^{\log^* n})$

From integers to polynomials and back

$$\begin{aligned} 971362651726262537182735 &= 971362\,X^3 + 651726\,X^2 + 262537\,X + 182735 \\ X &= 1000000 \end{aligned}$$

$$p = 127622142187 \qquad q = 209836129877$$

$$P = 127622\,X + 142187 \qquad Q = 209836\,X + 129877$$

$Q_1 = 209836$	$P_0\,Q_1$	$P_1\,Q_1$
$Q_0 = 129877$	$P_0\,Q_0$	$P_1\,Q_0$
	$P_0 = 142187$	$P_1 = 127622$

$$PQ = P_0\,Q_0 + (P_0\,Q_1 + P_1\,Q_0)\,X + P_1\,Q_1\,X^2$$

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$$A = (P_0 + P_1)\,(Q_0 + Q_1) = 91657624817$$

$$A - P_0\,Q_0$$

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$$A - P_0\,Q_0 - P_1\,Q_1$$

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$$\begin{aligned} P_0\,Q_0 &= 18466820999 \\ P_1\,Q_1 &= 26779689992 \\ A = (P_0 + P_1)\,(Q_0 + Q_1) &= 91657624817 \\ A - P_0\,Q_0 - P_1\,Q_1 &= P_0\,Q_1 + P_1\,Q_0 \\ &= 46411113826 \end{aligned}$$

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$$P\,Q = 26779689992 + 46411113826\,X + 18466820999\,X^2$$

$$p\,q = \begin{array}{r} 26779689992000000000000 \\ 00000046411113826000000 \\ 00000000000018466820999 \\ \hline 26779736403132292820999 \end{array}$$

Karatsuba

$$P = 127622\,X + 142187$$
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$$P_0\,Q_0 = (PQ)(0) = P(0)\,Q(0)$$
$$P_1\,Q_1\,\infty^2 = (PQ)(\infty) = P(\infty)\,Q(\infty)$$
$$(P_0 + P_1)\,(Q_0 + Q_1) = (PQ)(1) = P(1)\,Q(1)$$

Karatsuba

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\end{aligned}
$$

$$
\begin{aligned}
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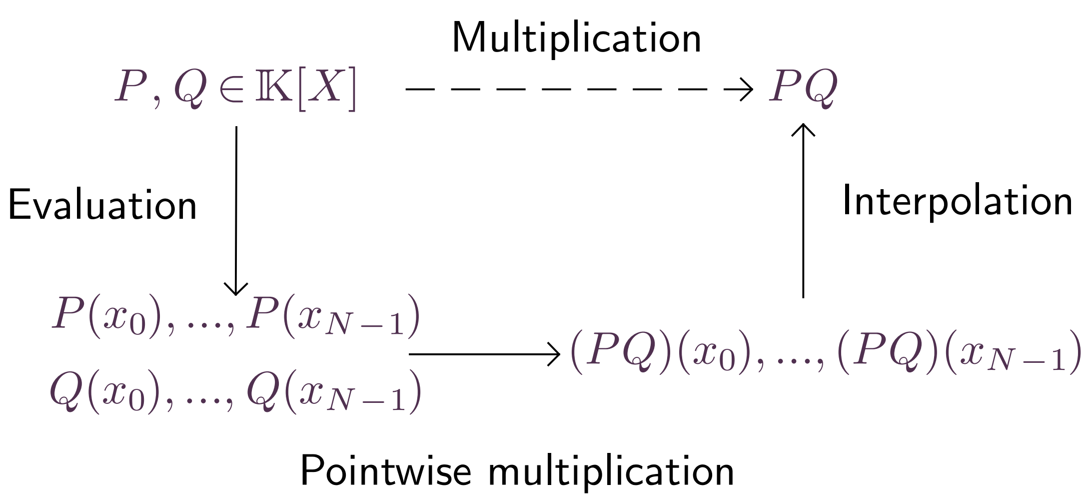
P_1\,Q_1\,\infty^2 &= (PQ)(\infty) = P(\infty)\,Q(\infty)\\
(P_0 + P_1)\,(Q_0 + Q_1) &= (PQ)(1) = P(1)\,Q(1)
\end{aligned}
$$

Evaluation-interpolation



Discrete Fourier transforms

Assume that $\omega \in \mathbb{K}$ is such that $\omega^N = 1$, $N \in 2^{\mathbb{N}}$ and $1, \omega, \omega^2, \ldots, \omega^{N-1}$ all distinct.

$$\mathrm{DFT}_\omega(P_0, \ldots, P_{N-1}) \;=\; (P(1), P(\omega), P(\omega^2), \ldots, P(\omega^{N-1}))$$

Corresponds to evaluating $P = P_0 + \cdots + P_{N-1} X^{N-1}$ at $x_i = \omega^i$

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Cyclic polynomials

Elements of $\mathbb{K}[x] / (X^N - 1)$. We have

$$X^N - 1 = (X - 1)(X - \omega) \cdots (X - \omega^{N-1})$$

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$$\begin{aligned} X^N - 1 &= (X-1)(X-\omega)\cdots(X-\omega^{N-1}) \\ \mathbb{K}[X]/(X^N-1) &\cong K[X]/(X-1) \oplus K[X]/(X-\omega) \oplus \cdots \oplus K[X]/(X-\omega^{N-1}) \end{aligned}$$

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$$\begin{aligned} X^N - 1 &= (X-1)(X-\omega)\cdots(X-\omega^{N-1}) \\ \mathbb{K}[X]/(X^N-1) &\cong K[X]/(X-1) \oplus K[X]/(X-\omega) \oplus \cdots \oplus K[X]/(X-\omega^{N-1}) \\ P &\rightsquigarrow (P(1), P(\omega), \ldots, P(\omega^{N-1})) \end{aligned}$$

Discrete Fourier transforms

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Cyclic convolution $\rightsquigarrow$ DFT

Product of $P, Q \in \mathbb{K}[X] / (X^N - 1)$ also called cyclic convolution

$$((PQ)_0, \ldots, (PQ)_{n-1}) = \mathrm{DFT}_\omega^{-1}(\mathrm{DFT}_\omega(P_0, \ldots, P_{n-1})\, \mathrm{DFT}_\omega(Q_0, \ldots, Q_{n-1}))$$

DFT $\rightsquigarrow$ Cyclic convolution [Bluestein 70]

Assume $\eta \in \mathbb{K}$ given with $\eta^2 = \omega$.

$$f_i := \eta^{i^2}, \qquad g_i := \eta^{-i^2}$$

$$f_{i+n} = \eta^{(i+n)^2} = \eta^{i^2+n^2+2ni} = \eta^{i^2} \omega^{(\frac{n}{2}+i)n} = f_i, \qquad g_{i+n} = g_i$$

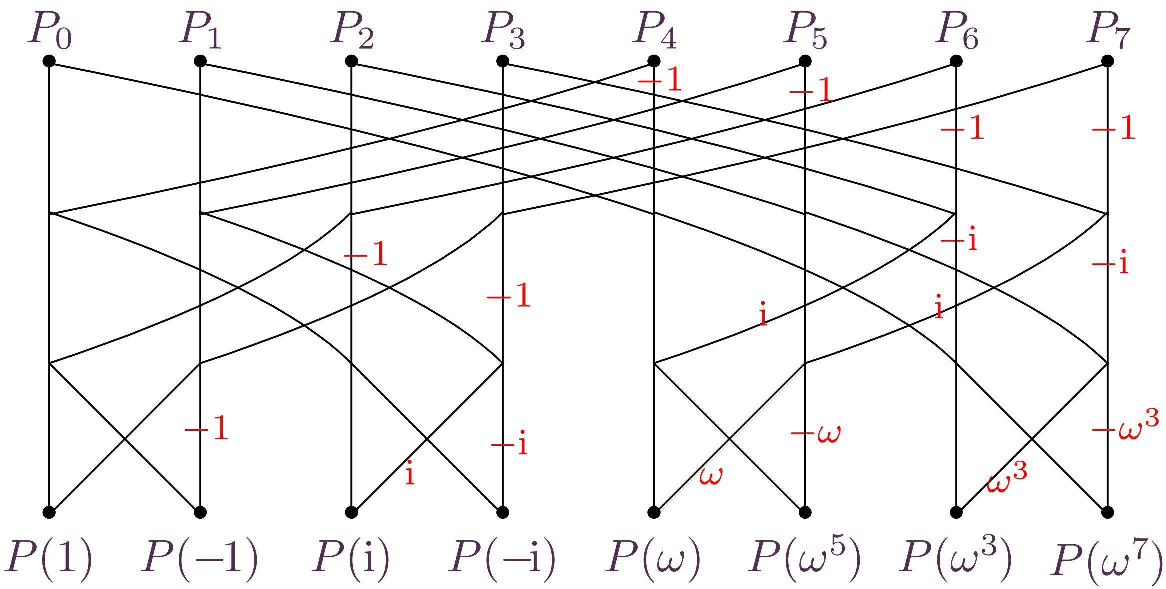
Then $\omega^{ij} = f_i f_j g_{i-j}$, so for all $a \in \mathbb{K}^n$:

$$\hat{a}_i = \mathrm{DFT}_\omega(a)_i = \sum_{j=0}^{n-1} a_j \omega^{ij} = f_i \sum_{j=0}^{n-1} (a_j f_j) \, g_{i-j}$$

One recognizes a cyclic convolution

Cooley–Tuckey method (essentially known to Gauß)

$$
\begin{aligned}
P(X) &= A(X^2) + B(X^2)\,X \\
P(\omega) &= A(\omega^2) + B(\omega^2)\,X \\
(P(1), P(\omega), \ldots, P(\omega^{N-1})) &= \Big(A(1)+B(1), \ldots, A((\omega^2)^{\frac{N}{2}-1}) + B((\omega^2)^{\frac{N}{2}-1})\,\omega^{\frac{N}{2}-1}, \\
&\quad A(1)-B(1), \ldots, A((\omega^2)^{\frac{N}{2}-1}) - B((\omega^2)^{\frac{N}{2}-1})\,\omega^{\frac{N}{2}-1} \Big)
\end{aligned}
$$



Direct transform

$$\mathrm{DFT}_\omega(P_0, \ldots, P_{N-1}) \;=\; (P(1), P(\omega), P(\omega^2), \ldots, P(\omega^{N-1}))$$

Direct transform

$$\mathrm{DFT}_\omega(P_0, \ldots, P_{N-1}) = (P(1), P(\omega), P(\omega^2), \ldots, P(\omega^{N-1}))$$

Inverse transform

$$\mathrm{DFT}_\omega^{-1} = \frac{1}{N}\mathrm{DFT}_{\omega^{-1}}$$

Interpolation $\leftrightsquigarrow$ Evaluation

Direct transform

$$\mathrm{DFT}_\omega(P_0, \ldots, P_{N-1}) \;=\; (P(1), P(\omega), P(\omega^2), \ldots, P(\omega^{N-1}))$$

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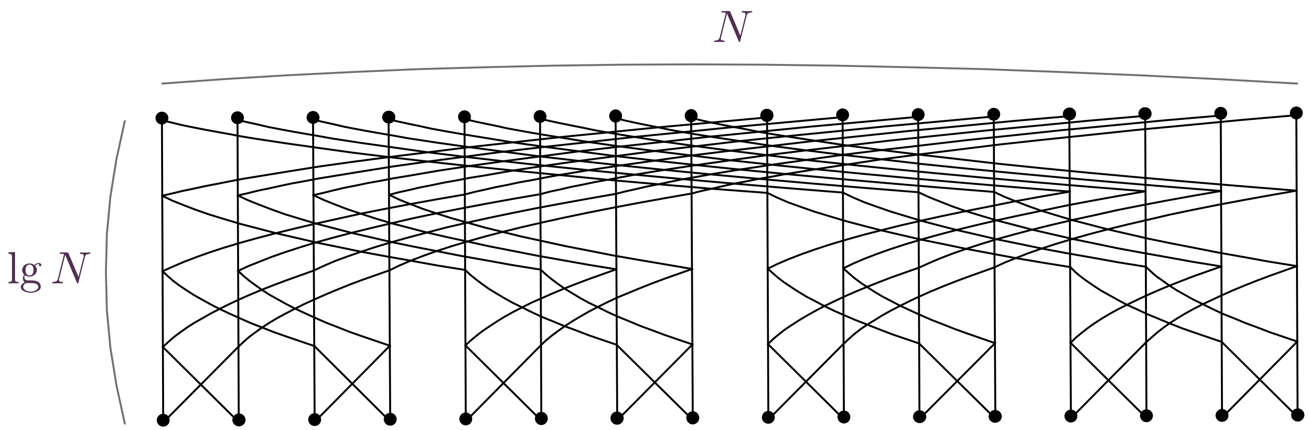
Interpolation $\leftrightsquigarrow$ Evaluation

Variants

- $\mathbb{K} = \mathbb{C}_b$: **complex DFT**, complex fixed-point arithmetic with b-bit precision
- $\mathbb{K} = \mathbb{F}_p$, **modular DFT**, with p prime number of the form $k\,2^N \pm 1$ [Pollard 71]
- $\mathbb{K} = \mathbb{L}[Y]/(Y^{2^N} \pm 1)$, **synthetic DFT**, *à la* Schönhage–Strassen



Cost of one DFT : $\frac{1}{2} N \lg N$ "butterflies" $\rightsquigarrow O(N \lg N)$ operations in $\mathbb{K}$

Complex DFT	$N \asymp n/\lg n$	$\mathsf{M}_{\mathbb{K}}(1) = O(\mathsf{I}(\lg n))$	$\mathsf{I}(n) = O(n \lg n\, \mathsf{I}(\lg n) + n \lg n)$
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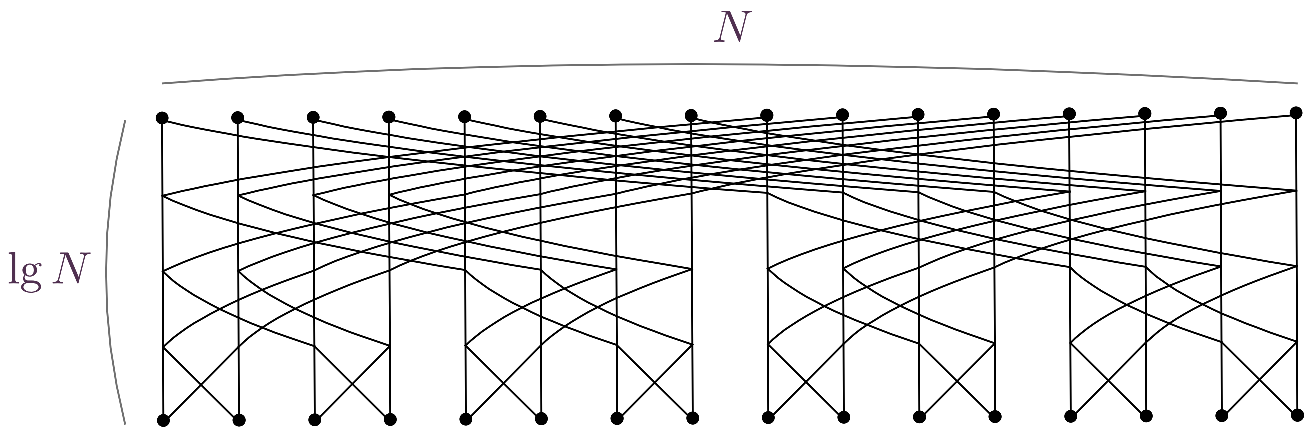

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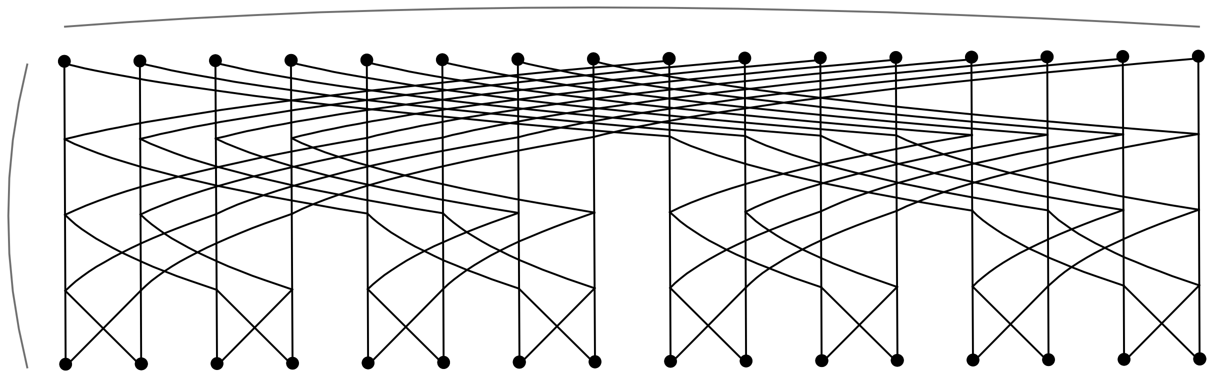

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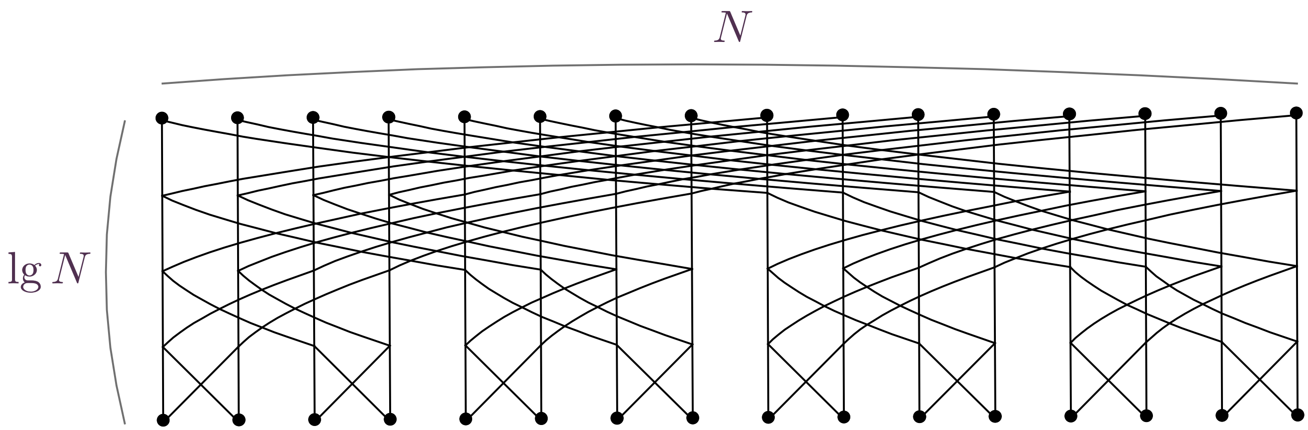

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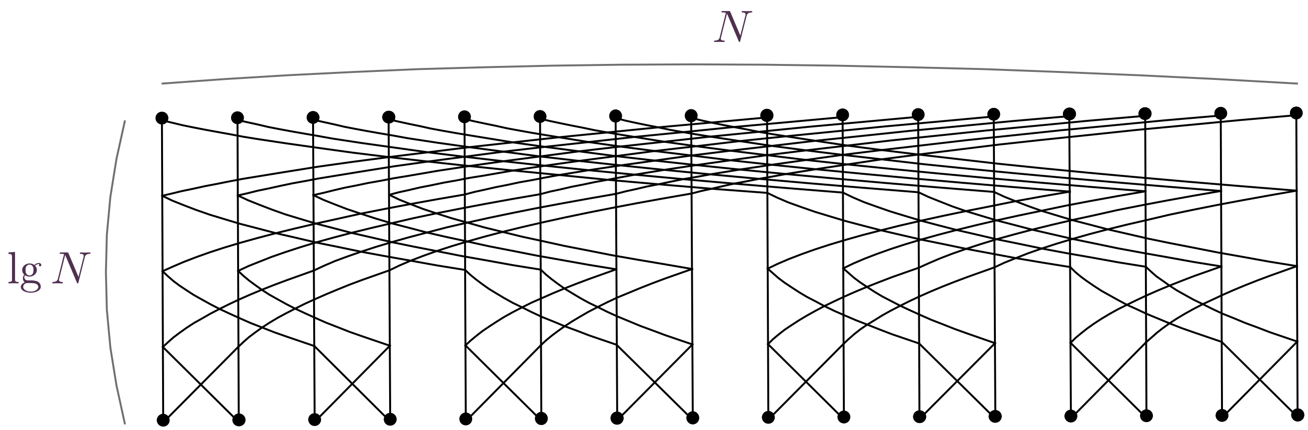

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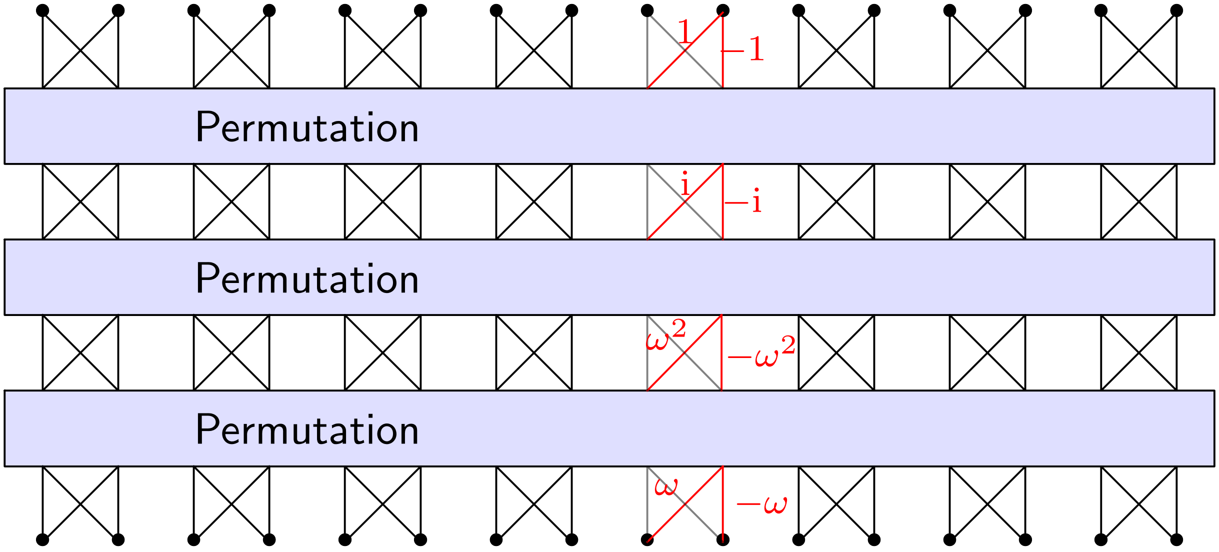
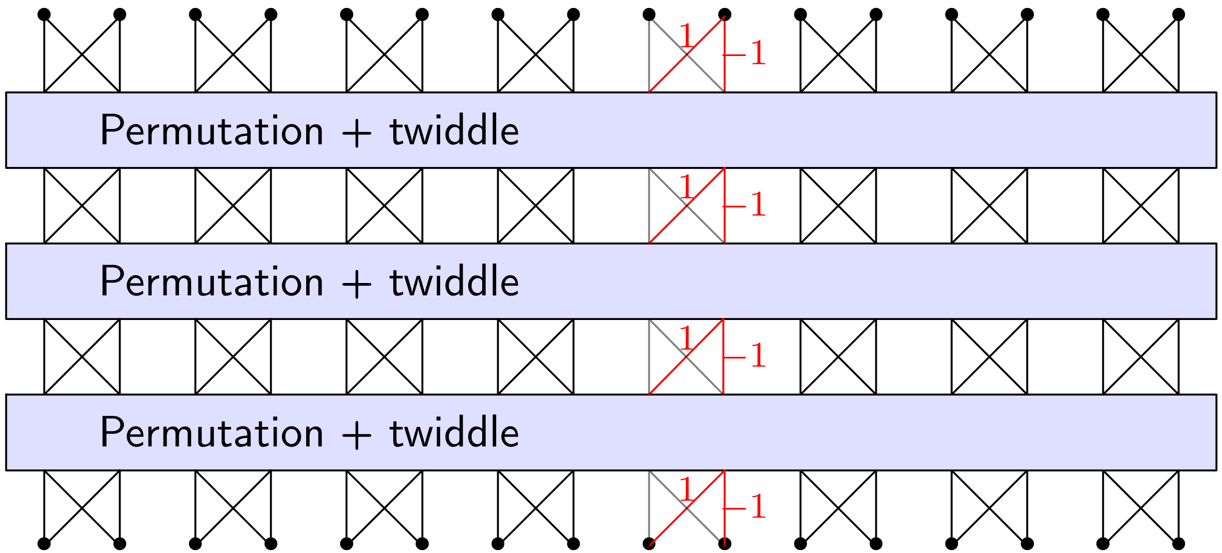

Permutation
Permutation
Permutation
1
−1
i
−i
ω²
−ω²
ω
−ω


1
−1
Permutation + twiddle
1
−1
Permutation + twiddle
1
−1
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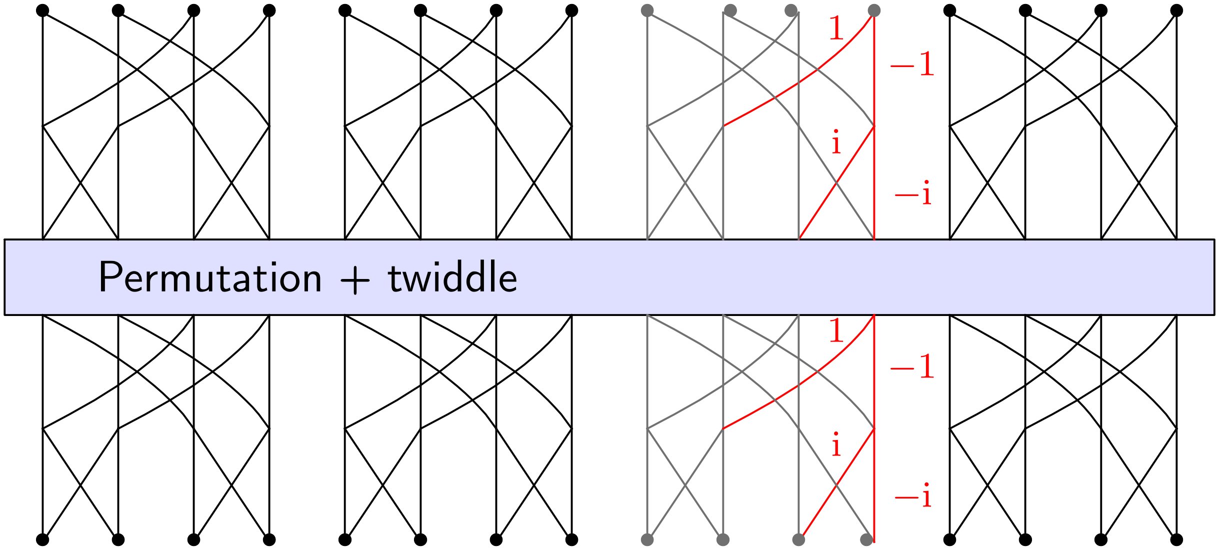
1
−1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1
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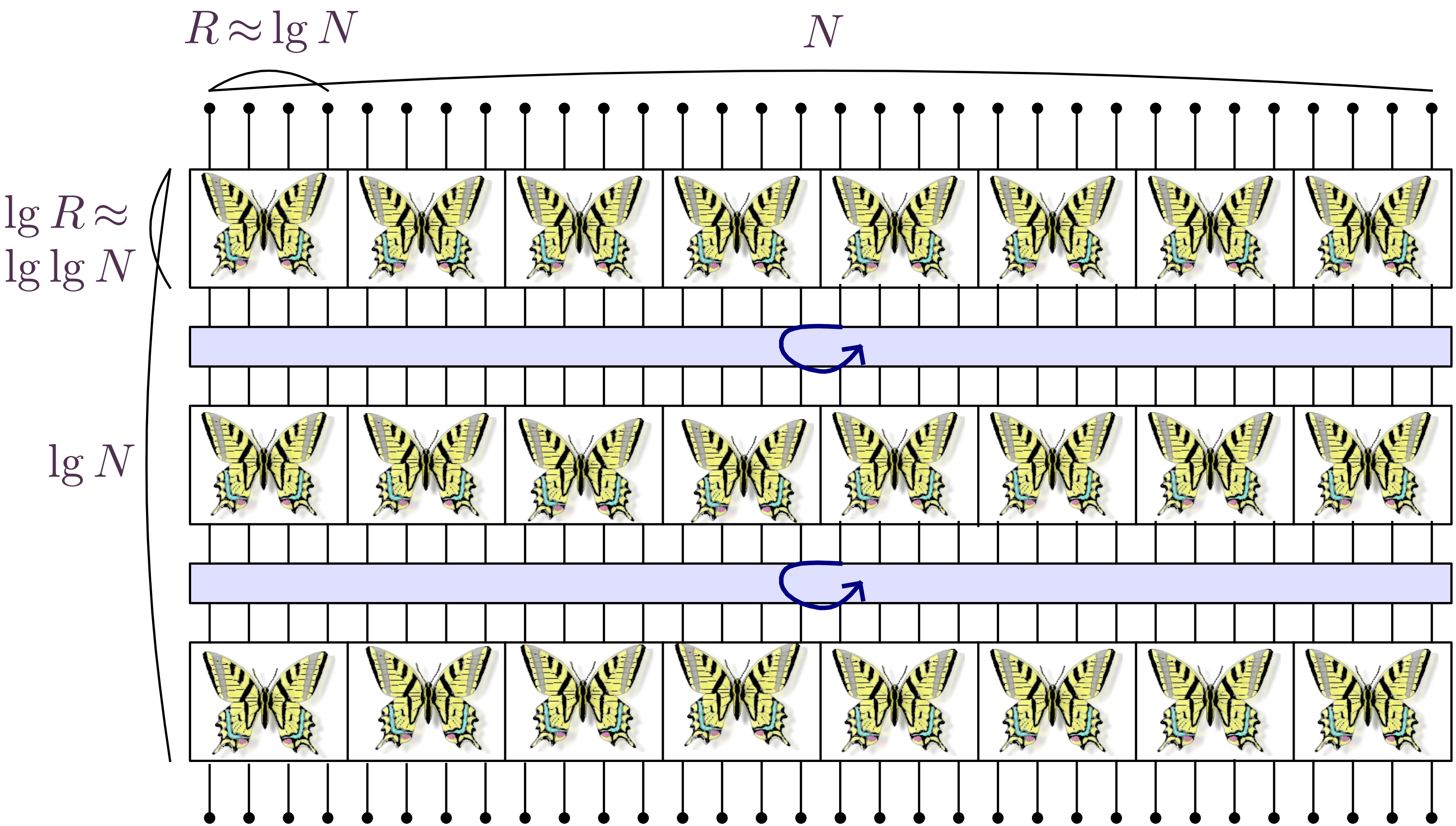

Slightly cheating in picture: we should have used 16 butterflies on every line

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- Conclusion:

$$\mathsf{I}(n) = O(n\lg n\,K^{\log^* n}).$$

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  Cost Bluestein–Kronecker $\gg$ cost twiddling and other.

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Crandall–Fagin algorithm

Multiplication $\mathbb{F}_p[\mathrm{i}][X]/(X^M - 1) \rightsquigarrow \mathbb{F}_{p'}[\mathrm{i}][X, Y]/(X^M - 1, Y^N - 1)$, $p' \lll p$
Conjecture 4 $\Rightarrow$ $K = 4$