Creative telescoping via reductions

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Part I

Introduction to creative telescoping and D-finite functions

Combinatorial identities

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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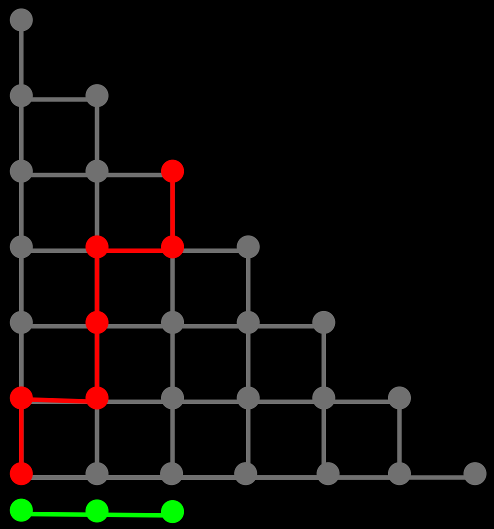

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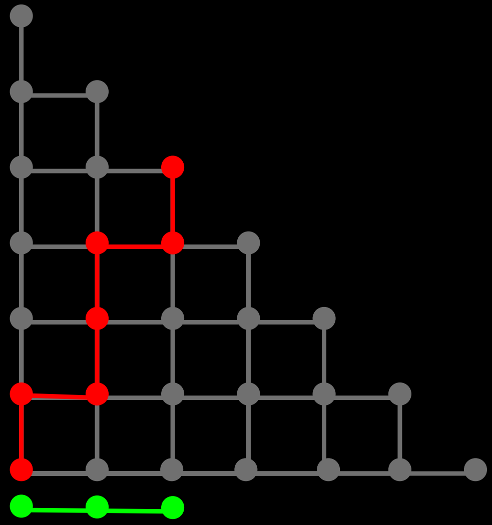

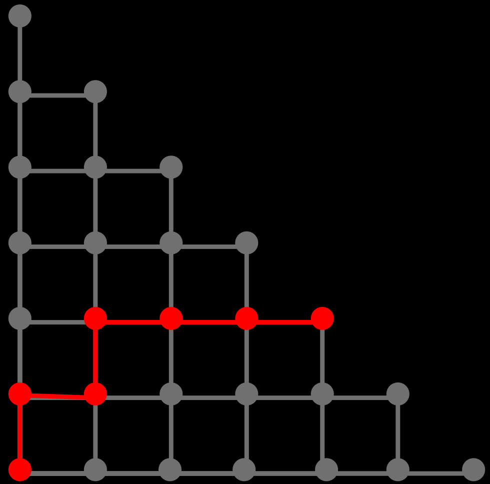

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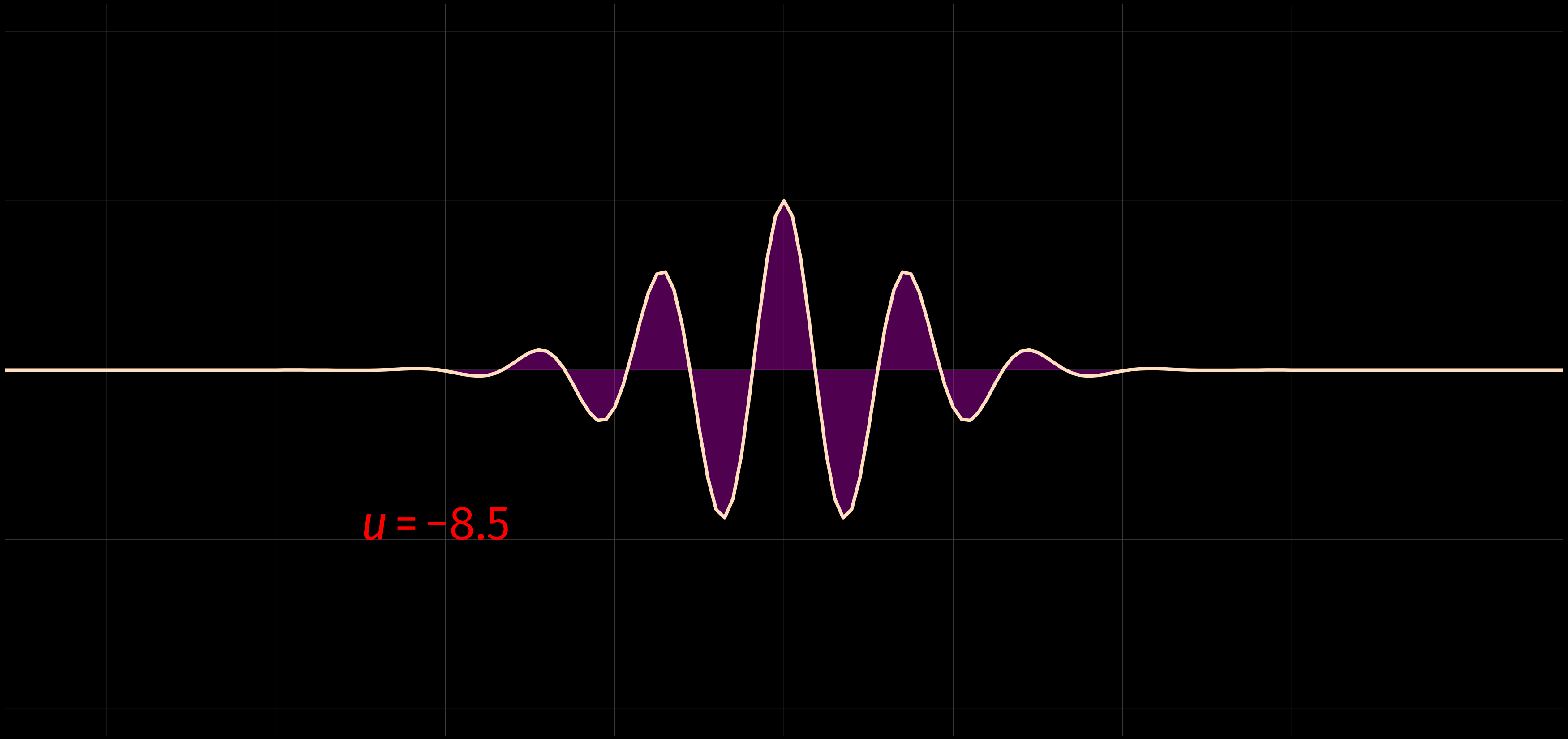
Special functions

$$\int_{-\infty}^{\infty} \cos(x\,u)\, e^{-x^2}\, dx = \sqrt{\pi}\, e^{-\frac{1}{4}u^2}$$



More examples...

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2$$

$$\int_0^\infty x J_1(ax) I_1(ax) Y_0(x) K_0(x) \, \mathrm{d}x = -\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\int_{-1}^{1} x^{\alpha-1} \operatorname{Li}_n(-xy) \, \mathrm{d}x = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(2k^2-k)/2}}{(q;q)_{n-k} (q;q)_{n+k}}$$

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$$S_{n+1} = 2\,S_n$$

$$S_0 = 1$$

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$$\begin{cases} 2^{n+1} = 2 \cdot 2^n \\ 2^0 = 1 \end{cases}$$

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$$\begin{cases} \dfrac{\partial \cos(x\,u)\, e^{-x^2}}{\partial x} = -u \sin(x\,u)\, e^{-x^2} - 2x \cos(x\,u)\, e^{-x^2} \\ \dfrac{\partial \sin(x\,u)\, e^{-x^2}}{\partial x} = u \cos(x\,u)\, e^{-x^2} - 2x \sin(x\,u)\, e^{-x^2} \\ \dfrac{\partial \cos(x\,u)\, e^{-x^2}}{\partial u} = -x \sin(x\,u)\, e^{-x^2} \\ \dfrac{\partial \sin(x\,u)\, e^{-x^2}}{\partial u} = x \cos(x\,u)\, e^{-x^2} \end{cases}$$

$$\cos(0 \cdot 0)\, e^{-0^2} = 1$$

$$\sin(0 \cdot 0)\, e^{-0^2} = 0$$

D-finite sequences

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Definition

A sequence $(a_{k_1,\dots,k_d}) \in \mathbb{k}^{\mathbb{N}^d}$ is ***D-finite*** *if $\mathbb{\Omega} / \operatorname{ann}_{\mathbb{\Omega}} a$ is a finite-dimensional vector space for the ideal*

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Equivalent criteria

Proposition (systems point of view)

A function f is D-finite iff there exist functions $g_1 = f, \dots, g_r$ and matrices $M_1, \dots, M_d \in \mathbb{K}^{r \times r}$ such that, for $k = 1, \dots, d$, we have

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Proposition (linear operator point of view)

A function f is D-finite iff for each $k = 1, \dots, d$, there exists a linear differential operator $L_k \in \mathbb{K}[\partial_{x_k}]$ with

$$L_k f = 0.$$

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The challenge

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Wanted: a D-finite representation for $F(u) := \int_{\mathscr{C}} f(u,x)\,\mathrm{d}x$

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Telescoper

Certificate

Part II

Introduction to the reduction-based approach

From now on we will focus on the differential setting

Short history creative telescoping

- **Precursors**

  Fasenmyer [1945], Verbaeten ['76], Gosper ['78]: hypergeometric case
  van der Poorten [1979]: coined the name (Apéry's proof that $\zeta(3) \notin \mathbb{Q}$)

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  Zeilberger [1990,...] and collaborators (Wilf, Almkvist, ...)
  Takayama [1992]: holonomic functions
  Chyzak [1998, 2000]: systematic Ore setting
  Koutschan [2009]: faster algorithms

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- **Reduction-based approach**

  Geddes–Le–Li [2004]: suggestion
  Bostan–Chen–Chyzak–Li [2010]: bivariate functions, complexity
  Bostan, Chen, Chyzak, Dumont, Lairez, Li, Xin [2012–2016]: special cases
  Chen, van Hoeij, Kauers, Koutschan [2016]: Fuchsian case
  van der Hoeven [2017]: **general differential case**

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$$\mathbb{k}(x) \ni f = f_{\mathrm{cst}} + \sum_{k \geq 1} f_{(\infty,k)} x^k + \sum_{\sigma \in \mathbb{k}} \sum_{k \geq 1} \frac{f_{(\sigma,k)}}{(x-\sigma)^k}$$

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$$\int_x [f]_{\text{Hermite}} = f_{\text{cst}}\, x + \sum_{k\geq 1} \frac{f_{(\infty,k)}}{k+1} x^{k+1} - \sum_{\sigma\in\mathbb{k}} \sum_{k\geq 2} \frac{f_{(\sigma,k)}}{(k-1)\,(x-\sigma)^{k-1}}$$

From now on we will assume that $\mathbb{k}$ is algebraically closed

Key idea: which “part” of a rational function can we naturally integrate?

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$[f]_{\text{Hermite}}$ → Hermite reduction of f

$\{f\}_{\text{Hermite}}$ → Remainder

Note: the operator $f \longmapsto [f]_{\text{Hermite}}$ is a $\mathbb{k}$-linear projection

- Let $f(u,x) \in \mathbb{Q}(u,x)$
- Take $\mathbb{k} := \mathbb{Q}(u)^{\mathrm{alg}}$
- Regard f as a **univariate** rational function in $\mathbb{k}(x)$

Integration of bivariate rational functions

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Confining the remainders

- Let $\Sigma := \{\sigma_1, \ldots, \sigma_s\} := \mathrm{Sing}(f) \subseteq \mathbb{k}$ be the finite set of singularities of f
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- Both $[\cdot]_{\mathrm{Hermite}}$ and ∂_u map $\mathbb{M}$ into itself
- We have

$$\{\mathbb{M}\}_{\mathrm{Hermite}} = \frac{\mathbb{k}}{x-\sigma_1} + \cdots + \frac{\mathbb{k}}{x-\sigma_s},$$

so $\dim_{\mathbb{k}} \{\mathbb{M}\}_{\mathrm{Hermite}} = |\Sigma| < \infty$; we say that $[\cdot]_{\mathrm{Hermite}}$ is **confined** on $\mathbb{M}$

Finding a linear combination

With $r := \dim_{\mathbb{k}} \{\mathbb{M}\}_{\text{Hermite}}$, compute a non trivial relation

$$c_0 \{f\}_{\text{Hermite}} + c_1 \{\partial_u f\}_{\text{Hermite}} + \cdots + c_r \{\partial_u^r f\}_{\text{Hermite}} = 0,$$

where $c_0, c_1, \ldots, c_r \in \mathbb{k}$.

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Victory!

By $\mathbb{k}$-linearity, we have $\{\omega f\}_{\text{Hermite}} = 0$ for

$$\omega = c_0 + c_1 \partial_u + \cdots + c_r \partial_u^r,$$

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$$\omega f = [\omega f]_{\text{Hermite}} \in \partial_x \mathbb{M}.$$

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More parameters

Same argument with respect to each parameter $u_1, \ldots, u_s$

Part III

Confined reductions for the general ∂-finite case

The setup

The system of differential equations

$$\partial_x \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \qquad \partial_u \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = B \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}$$

The setup

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$$\phi \partial_x \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \qquad \phi \partial_u \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = B \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \qquad \phi \in \mathbb{k}[x], \quad A, B \in \mathbb{k}[x]^{r \times r}$$

The system of essentially ordinary differential equations

$$\phi\,\partial_x \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \qquad \phi\,\partial_u \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = B \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \qquad \phi \in \mathbb{k}[x], \quad A, B \in \mathbb{k}[x]^{r\times r}$$

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- $\Sigma := \{\sigma_1, \dots, \sigma_s\} := \{\sigma \in \mathbb{C} : \phi(\sigma) = 0\}$

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Problem

Show that $\dim \mathbb{M} / \operatorname{Im} \partial_x < \infty$ and construct a confined reduction $[\cdot]$ on $\mathbb{M}$

Local problem

Given: $(f_1, \ldots, f_r) \in \mathbb{k}[x]^r$ of sufficiently large degree

Can we find $(g_1, \ldots, g_r) \in \mathbb{k}[x]^r$ with $\partial_x(g_1 y_1 + \cdots + g_r y_r) = f_1 y_1 + \cdots + f_r y_r$?

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Idea

- $y = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \longrightarrow Ty$ for a matrix T and work coefficient by coefficient

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- T: matrix, i: integer parameter, C: constant matrix

$$\begin{aligned}(Cx^i Ty)' &= (Cx^i T)' y + Cx^i T \phi^{-1} y \\ &= Cx^i \underbrace{(\phi^{-1} TA + T' + i x^{-1} T)}_{U} y\end{aligned}$$

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Idea

- $y=\begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \longrightarrow Ty$ for a matrix T and work coefficient by coefficient
- $T\in \mathbb{k}(i)^{r\times r}[x,x^{-1}]^{r\times r}$, i: formal integer parameter, $C\in \mathbb{k}(i)^{1\times r}$

$$\begin{aligned}(Cx^iTy)' &= (Cx^iT)'y+Cx^iT\phi^{-1}y \\ &= Cx^i\underbrace{(\phi^{-1}TA+T'+ix^{-1}T)}_{U}y\end{aligned}$$

- U involves x^{-1} and parameter i, so $U\in \mathbb{k}(i)^{r\times r}[x,x^{-1}]^{r\times r}$

Head choppers

Goal: given $\Lambda = \Lambda_l x^l + \cdots \in \mathbb{k}[x]^{1 \times r}$, find $C x^i \in \mathbb{k}[x]^{1 \times r}$ with

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Writing $U = U_d x^d + \cdots$, we get

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and recursively define

$$[\Lambda y] := [(\Lambda - C x^i U) y]$$

$$(Cx^i T y)' \;=\; Cx^i U y \;=\; Cx^i (\phi^{-1} T A + T' + i x^{-1} T) y$$

Repeatedly use two transformations

$$U_d = \begin{pmatrix} * & * & & * & * \\ & * & & * & \\ * & & * & * & \\ & * & & * & * \\ * & * & * & & * \end{pmatrix} \qquad \begin{matrix} (T, U) \longrightarrow (J\,T, J\,U) \\ J \in \mathrm{GL}_r(\mathbb{k}(i)) \end{matrix} \qquad \tilde{U}_d = \begin{pmatrix} * & * & & * & * \\ & * & & * & * \\ & & * & & * \\ & & & & \\ & & & & \end{pmatrix}$$

$$(T, U) \longrightarrow (\Delta\,T, \Delta\,U)$$

$$U_d = \begin{pmatrix} * & * & & * & * \\ & * & & * & * \\ & & * & & * \\ & & & & \\ & & & & \end{pmatrix} \qquad J = \begin{pmatrix} \Xi^{-1} & & & & \\ & \Xi^{-1} & & & \\ & & \Xi^{-1} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \qquad \tilde{U}_d = \begin{pmatrix} * & * & & * & * \\ & * & & * & * \\ & & * & & * \\ * & * & * & & * \\ * & & * & * & \end{pmatrix}$$

$$(\Xi^{\delta}\,T)(x, i) = x^{\delta}\,T(x, i + \delta)$$

Conclude

Head reduction

We have constructed a confined reduction $[\cdot]_\infty : x\,\mathbb{K}[x]^{1\times r}y \longrightarrow x\,\mathbb{K}[x]^{1\times r}y$

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For each $\sigma_i \in \Sigma$, D-finiteness preserved under $x \leftrightarrow \frac{1}{x-\sigma_i}$

$\longrightarrow$ confined reduction $[\cdot]_{\sigma_i} : \frac{1}{x-\sigma_i}\mathbb{K}\left[\frac{1}{x-\sigma_i}\right]^{1\times r} \longrightarrow \frac{1}{x-\sigma_i}\mathbb{K}\left[\frac{1}{x-\sigma_i}\right]^{1\times r}$

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Gluing

$$[\Lambda y] \;=\; \Lambda_{\mathrm{cst}}\,y + [\Lambda_{(\infty)}\,y]_\infty + [\Lambda_{(\sigma_1)}\,y]_{\sigma_1} + \cdots + [\Lambda_{(\sigma_s)}\,y]_{\sigma_s}$$

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Gluing

$$[\Lambda y] = \Lambda_{\mathrm{cst}}\,y + [\Lambda_{(\infty)}\,y]_\infty + [\Lambda_{(\sigma_1)}\,y]_{\sigma_1} + \cdots + [\Lambda_{(\sigma_s)}\,y]_{\sigma_s}$$

Theorem

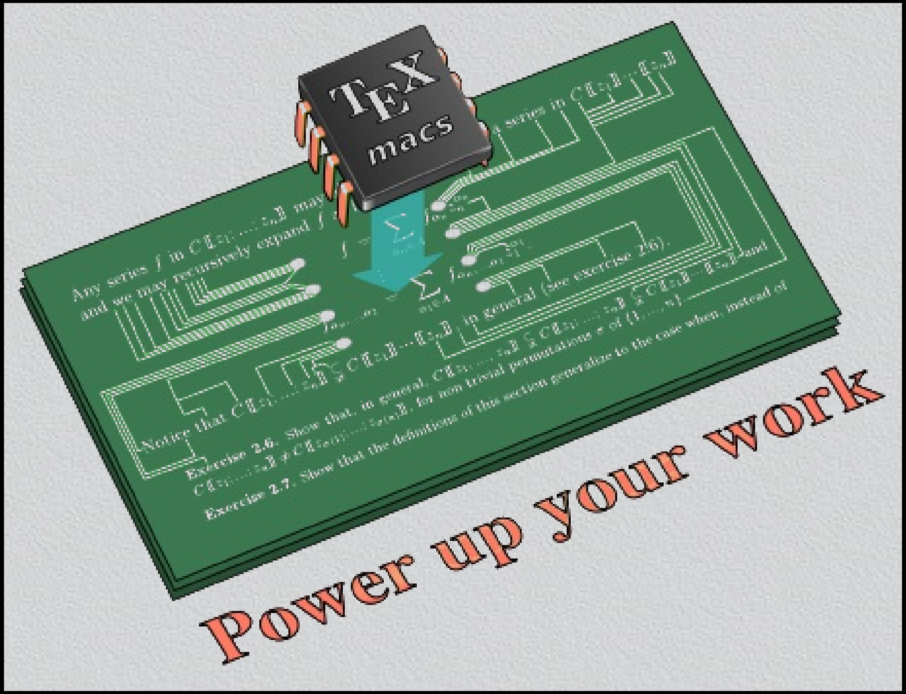
There exists a computable confined reduction $[\cdot] : \mathbb{M} \longrightarrow \mathbb{M}$.

[1] J. VAN DER HOEVEN. Constructing reductions for creative telescoping. Technical Report, HAL, 2017. http://hal.archives-ouvertes.fr/hal-01435877.

[2] A. BOSTAN, F. CHYZAK, P. LAIREZ, AND B. SALVY. Generalized Hermite reduction, creative telescoping and definite integration of differentially finite functions. In *Proc. ISSAC '18*, pages 95–102. New York, 2018. ACM.

[3] J. VAN DER HOEVEN. Creative telescoping using reductions. Technical Report, HAL, 2018. http://hal.archives-ouvertes.fr/hal-01773137.

Thank you !



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