Integer multiplication: classical tools

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Joint work with David Harvey (UNSW, Sydney)

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Motivation

$M(n)$: the complexity of multiplying two n-bit integers (Turing machine model)

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$$\mathsf{M}(n) = \text{speed of basic arithmetic}$$

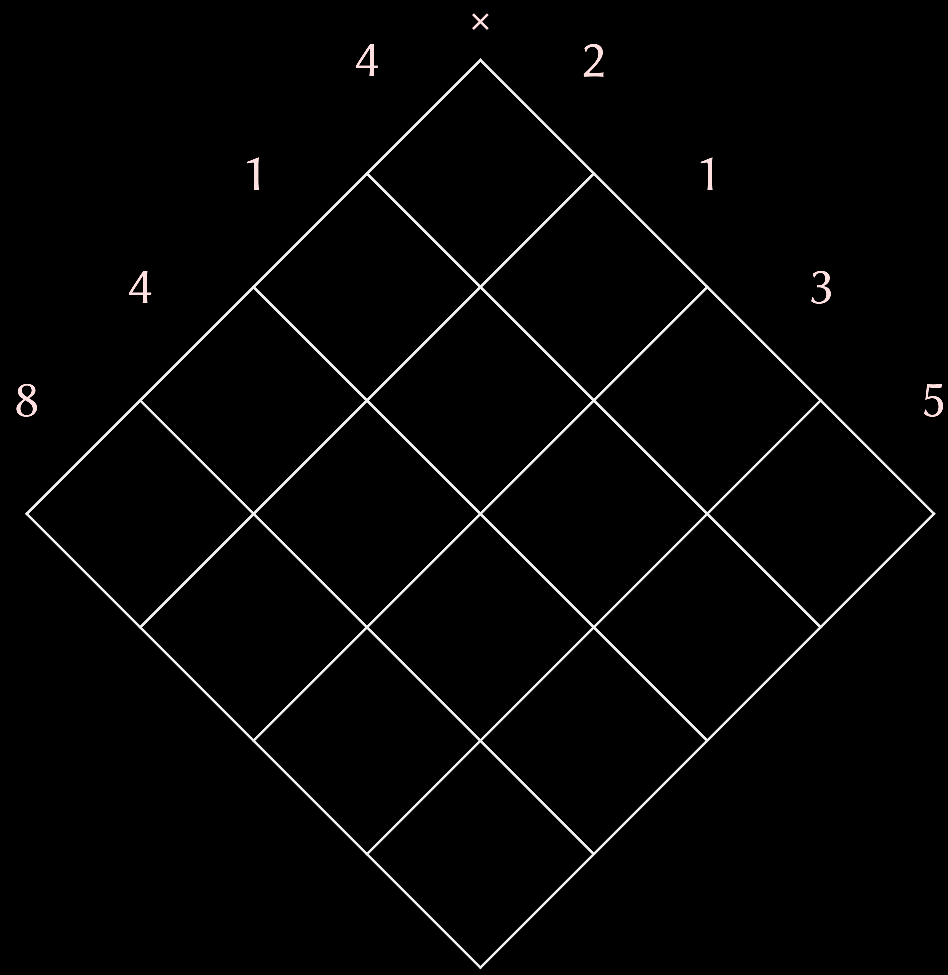
Also

- Asymptotic complexity abstracts from concrete machines
- Better theoretical techniques $\xrightarrow{\text{often}}$ faster practical implementations

Naive multiplication

$$8 \quad 4 \quad 1 \quad 4 \quad \times \quad 2 \quad 1 \quad 3 \quad 5$$

Naive multiplication



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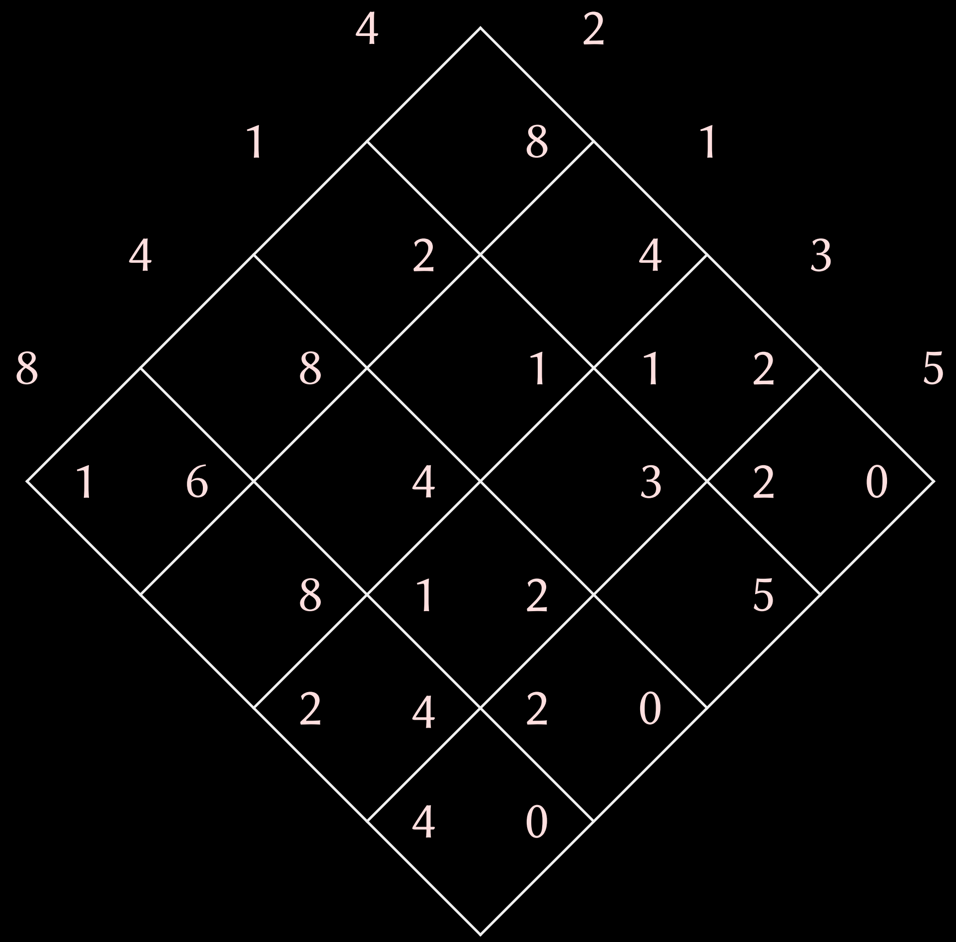

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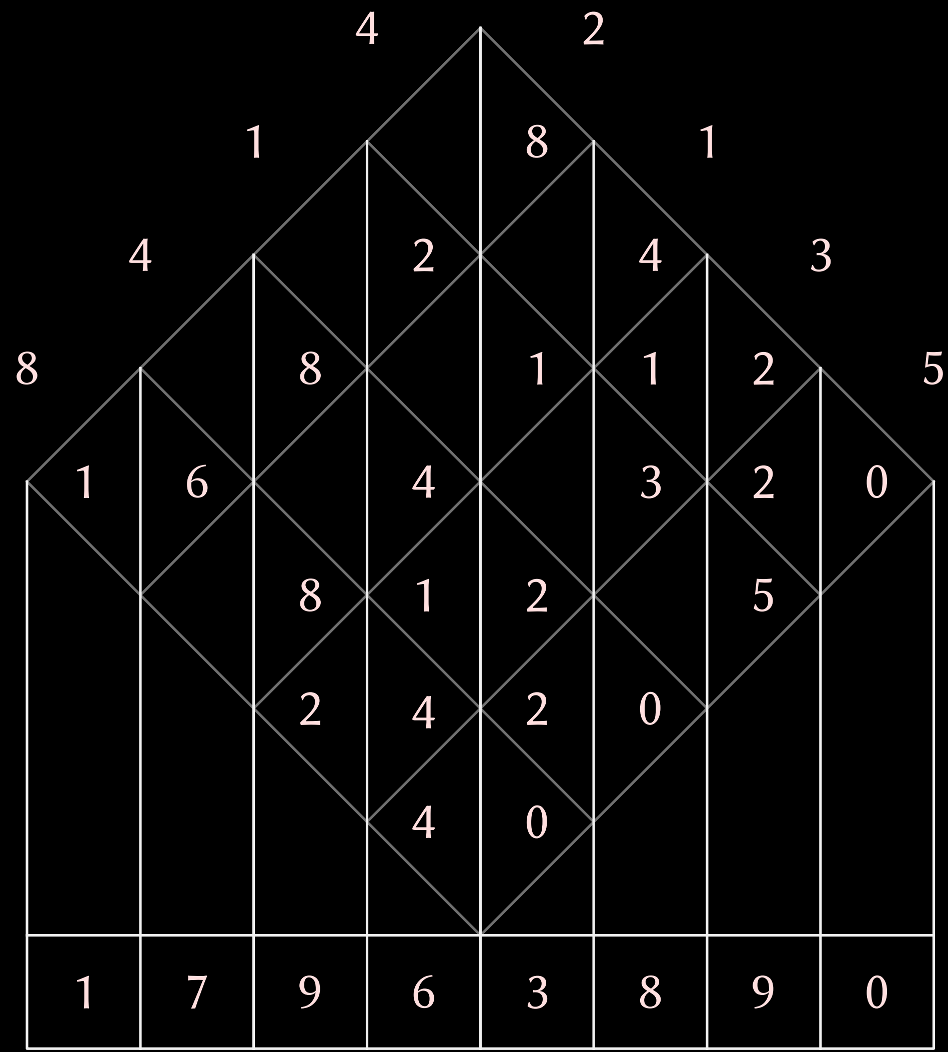

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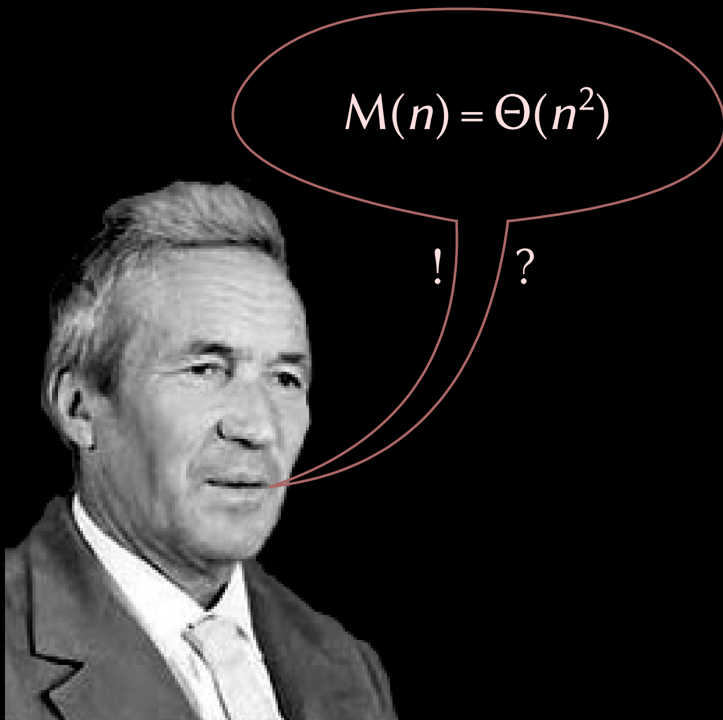
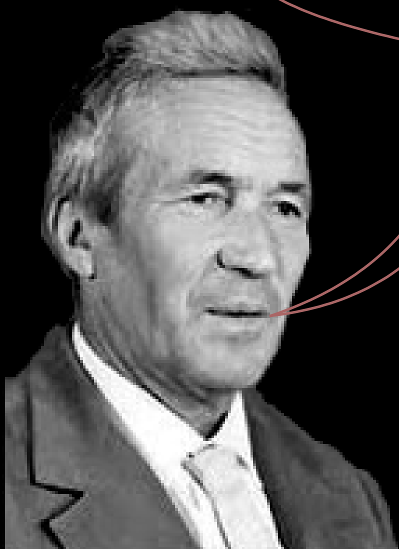


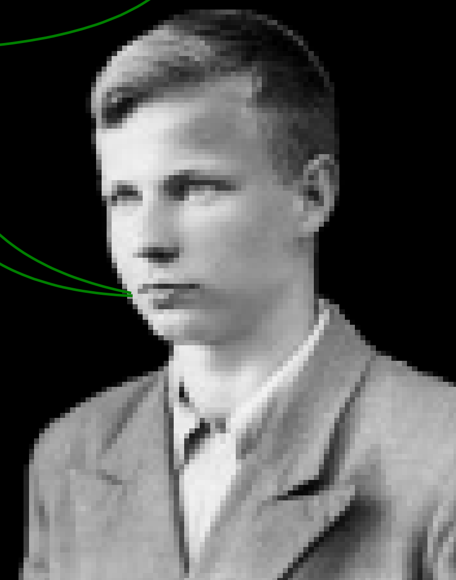
Can we do better?

$M(n) = \Theta(n^2)$

! ?



$M(n) = O(n^{\log_2 3})$



1962

Kronecker segmentation

$$4627579679788114 \quad \times \quad 4519170871966234$$

$$\wr$$

$$(4627\,x^3 + 5796\,x^2 + 7978\,x + 8114) \quad \times \quad (4519\,x^3 + 1708\,x^2 + 7196\,x + 6234)$$

Integers versus polynomials

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$$1004003 \times 2001005 \quad = \quad 2009015023015$$

Multiplying by evaluation-interpolation

$\mathbb{K}$: field (or a suitable ring)

$P, Q \in \mathbb{K}[x]$: polynomials of degree $< k$

$R = PQ$ of degree $< 2k - 1$

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$$P \longrightarrow (P(\alpha_1), \ldots, P(\alpha_{2k-1}))$$

$$Q \longrightarrow (Q(\alpha_1), \ldots, Q(\alpha_{2k-1}))$$

$$\searrow \qquad \swarrow$$

$$(P(\alpha_1)\,Q(\alpha_1), \ldots, P(\alpha_{2k-1})\,Q(\alpha_{2k-1})) = (R(\alpha_1), \ldots, R(\alpha_{2k-1})) \longrightarrow R$$

Karatsuba multiplication

$k = 2$

$\alpha_0 = \infty, \ \alpha_2 = 0, \ \alpha_3 = 1$

Karatsuba multiplication

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$\alpha_0 = \infty,\ \alpha_2 = 0,\ \alpha_3 = 1 \quad \longrightarrow \quad \alpha_0 = (1, 0),\ \alpha_1 = (0, 1),\ \alpha_2 = (1, 1)$

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\rightsquigarrow & & \rightsquigarrow \\
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3234 → $5376 - 1764 - 490$

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$$M(2\,n) = 3\,M(n) + O(n)$$

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Complexity analysis

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Can we do better?

Problem

When k gets large, evaluation-interpolation gets expensive

Next goal

Faster evaluation-interpolation for well-chosen points

The Fast Fourier Transform

$\mathbb{K}$: a field (could be a suitable ring)

n : transform length

ω : primitive n-th root of unity in $\mathbb{K}$, say $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$

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Discrete Fourier transform

$$(\hat{c}_0, \ldots, \hat{c}_{n-1}) = \mathrm{DFT}_{\omega;n}(c_0, \ldots, c_{n-1})$$

$$C = c_0 + \cdots + c_{n-1} x^{n-1}$$

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Chinese remainder perspective

$$\mathbb{K}[x]/(x^n - 1) \cong \bigoplus_{0 \leqslant k < n} \mathbb{K}[x]/(x - \omega^k)$$

FFT multiplication

$$P \in \mathbb{K}[x]/(x^n - 1) \qquad\qquad Q \in \mathbb{K}[x]/(x^n - 1)$$

$$\downarrow \text{DFT} \qquad\qquad \downarrow \text{DFT}$$

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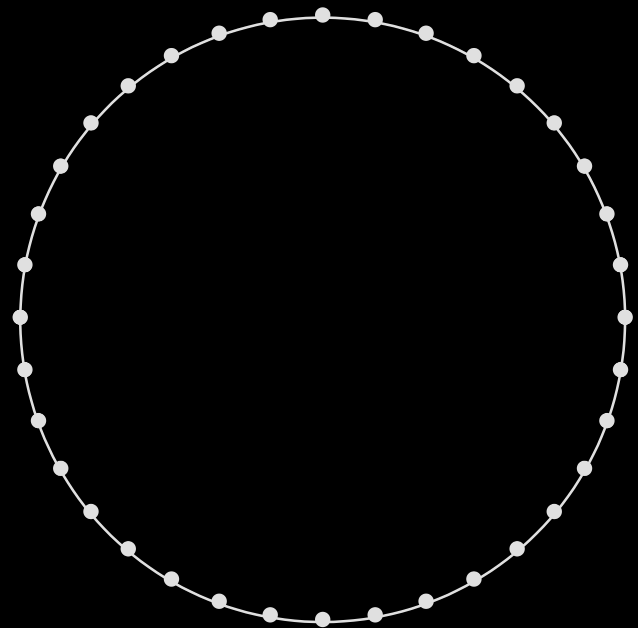
Inverse transforms

$$\begin{aligned}
(\hat{c}_0, \ldots, \hat{c}_{n-1}) &= \text{DFT}_{\omega;n}(c_0, \ldots, c_{n-1}) \\
(c_0, \ldots, c_{n-1}) &= \frac{1}{n}\text{DFT}_{\omega^{-1};n}(\hat{c}_0, \ldots, \hat{c}_{n-1})
\end{aligned}$$

Making the FFT fast



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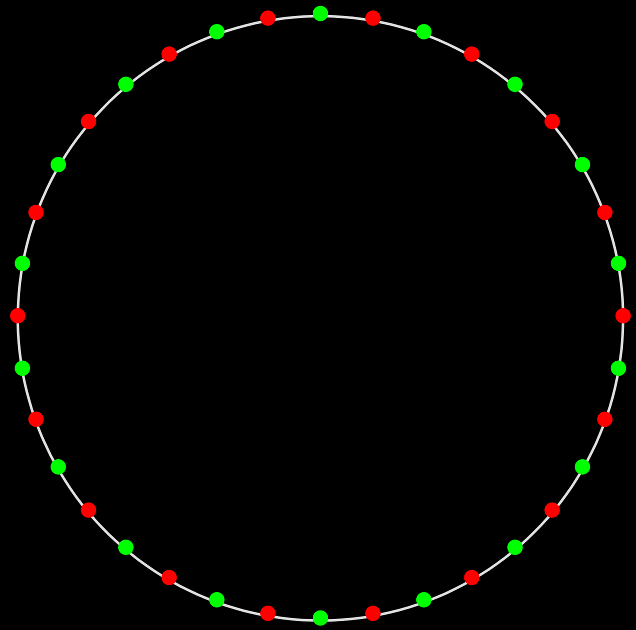

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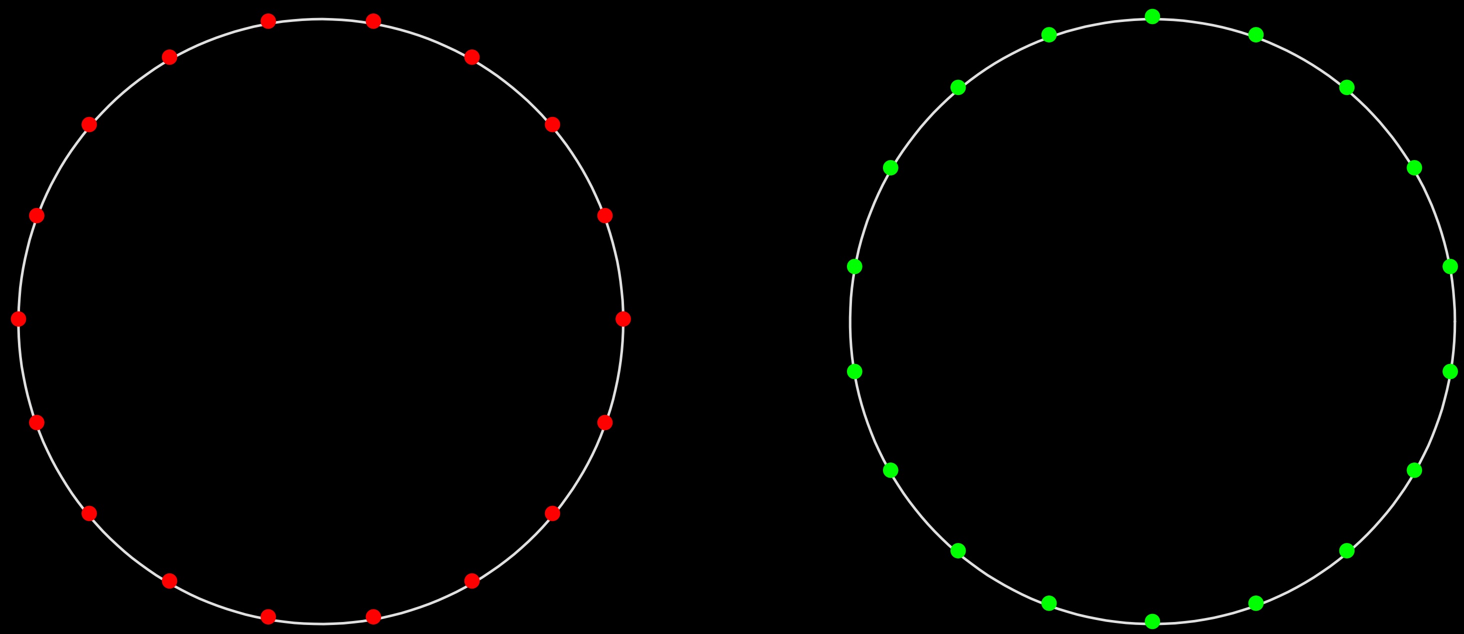

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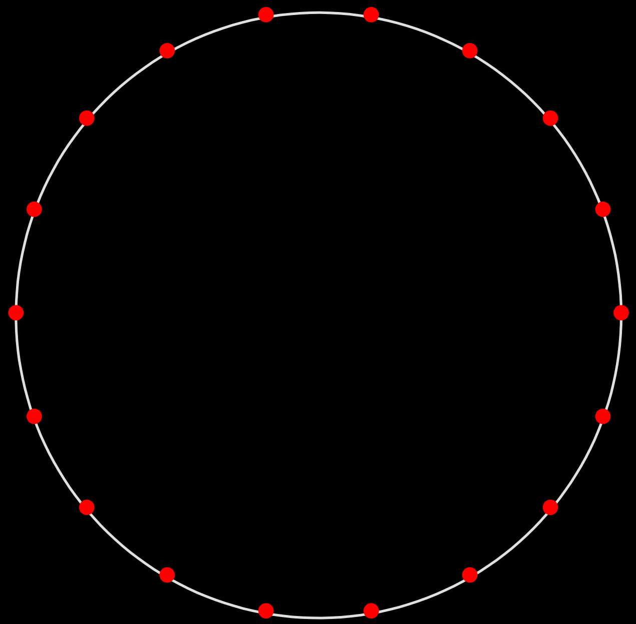

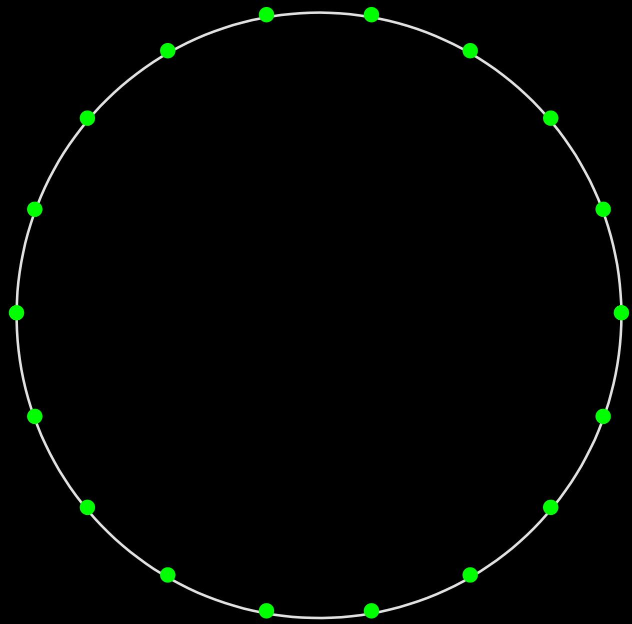

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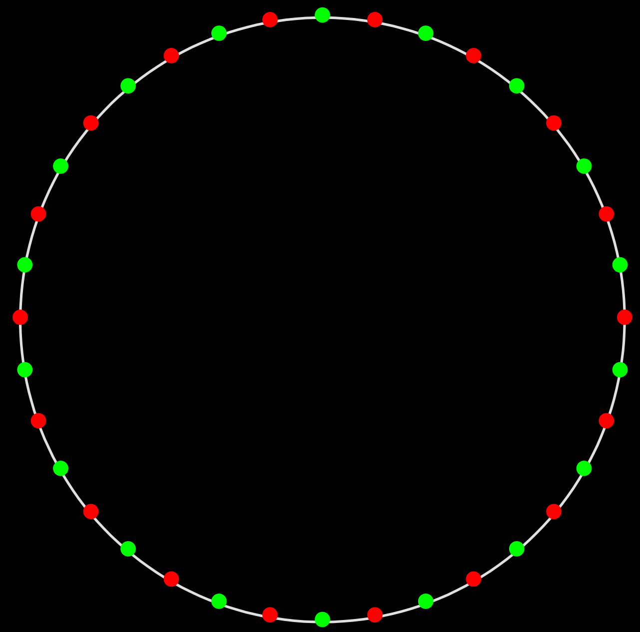
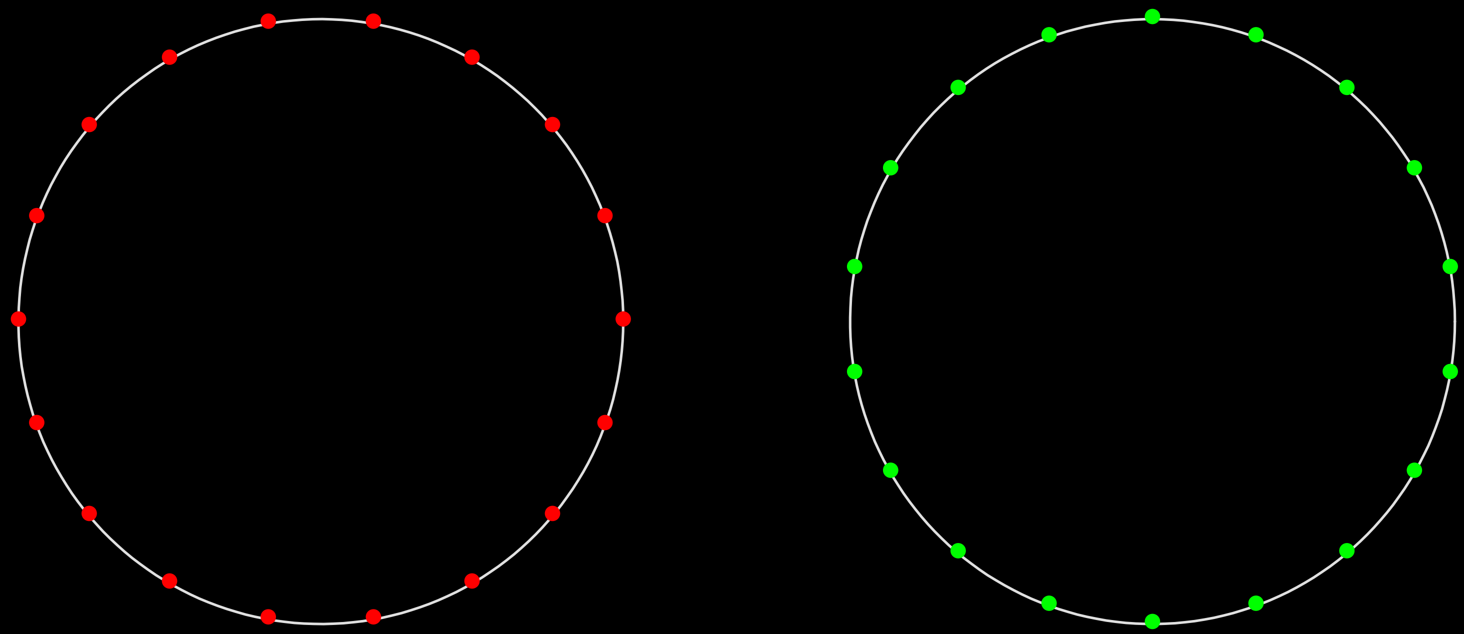


$\mathbb{K}[x]/(x^{2n} - 1)$

Making the FFT fast



$$\mathbb{K}[x]/(x^{2n}-1) \cong \mathbb{K}[x]/(x^n-1) \oplus \mathbb{K}[x]/(x^n+1)$$

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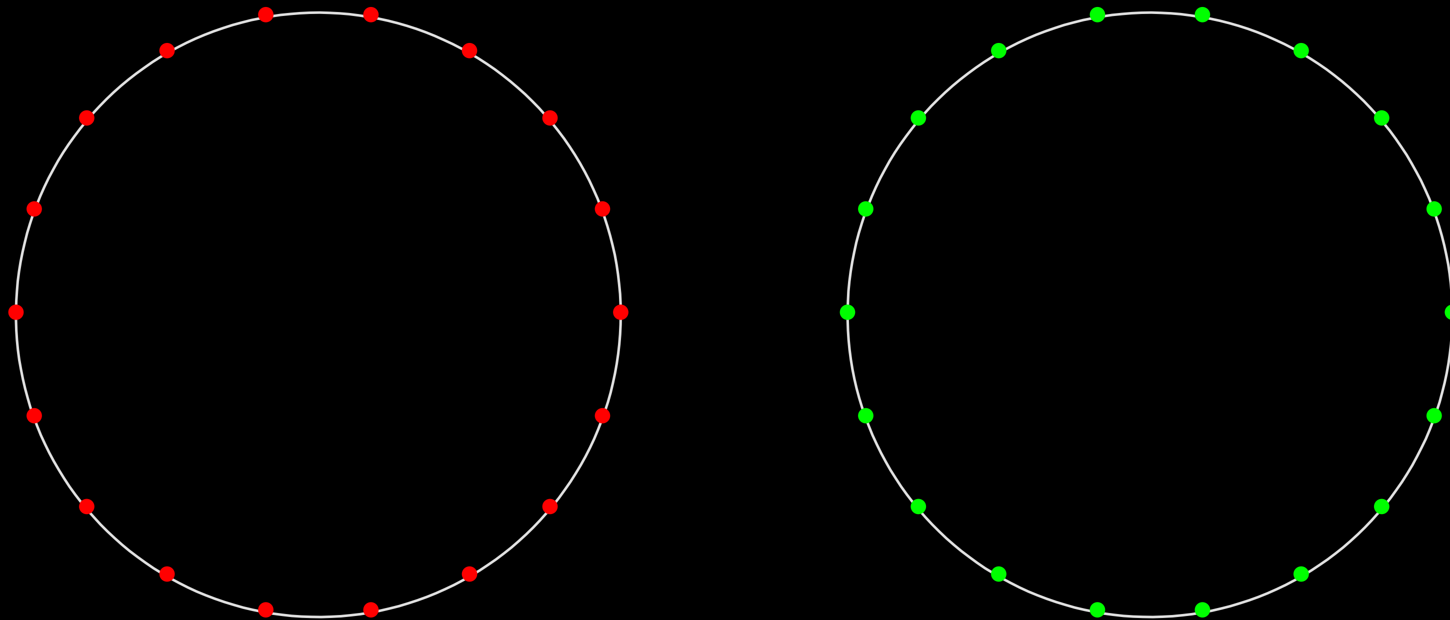

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$$\cong \mathbb{K}[x]/(x^n-1) \oplus \mathbb{K}[x]/(\tilde{x}^n-1)$$

$$\tilde{x} = \omega x$$

$$\omega^n = -1$$

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Assume that $n = 2^l$ is a power of two

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$$\begin{aligned}
\mathrm{F}_{\mathbb{K}}(2^l) &\leqslant 2\,\mathrm{F}_{\mathbb{K}}(2^{l-1}) + C\,2^l \\
&\leqslant 2^2\,\mathrm{F}_{\mathbb{K}}(2^{l-2}) + 2\,C\,2^l \\
&\leqslant 2^3\,\mathrm{F}_{\mathbb{K}}(2^{l-3}) + 3\,C\,2^l \\
&\vdots \\
&\leqslant 2^l\,\mathrm{F}_{\mathbb{K}}(1) + l\,C\,2^l
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$$\mathsf{F}_{\mathbb{K}}(n) = O(n \log n)$$

But what is $\mathbb{K}$ for our application to integer multiplication?

Complex number FFTs

$\mathbb{K} = \mathbb{C}$, while working with a finite bit-precision

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$$\overbrace{2872216826762157625376253762153762156735216735126727321653 42}^{N}$$

Complex number FFTs

$\mathbb{K} = \mathbb{C}$, while working with a finite bit-precision

$$\overbrace{28722\ 16826\ 76215\ 76253\ 76253\ 76215\ 37621\ 56735\ 21673\ 51267\ 27321\ 65342}^{N}$$

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$$\overbrace{\underbrace{\underbrace{28722}_{b}\ \underbrace{16826}_{b}\ \underbrace{76215}_{b}\ \underbrace{76253}_{b}\ \underbrace{76253}_{b}\ \underbrace{76215}_{b}\ \underbrace{37621}_{b}\ \underbrace{56735}_{b}\ \underbrace{21673}_{b}\ \underbrace{51267}_{b}\ \underbrace{27321}_{b}\ \underbrace{65342}_{b}}_{n}}^{N}$$

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$$k = \log^* N = \min\left\{ l : \log \circ \overset{l\times}{\cdots} \circ \log N \leqslant 1 \right\}$$

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$\mathbb{K} = \mathbb{F}_p, \quad p = s\,2^l + 1, \quad \text{small } s \qquad p = 3 \cdot 2^{30} + 1$ (Pollard, 1971)

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Cost $\mathsf{M}^{\ominus}(N)$ of multiplication in $\mathbb{Z}/(2^N + 1)\mathbb{Z}$

$$\begin{aligned} \mathsf{M}^{\ominus}(N) &\leqslant 2n\,\mathsf{M}^{\ominus}(m) + O(N \log N) \\ &= \cdots && \text{(log log } N \text{ recursive steps)} \\ &= O(N \log N \log \log N) \end{aligned}$$

Can we do better?

Main problems

- Over $\mathbb{C}$, the multiplications with powers of ω end up being too expensive
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Recent progress

- Fürer 2007 $M(n) = O(n \log n\, 2^{O(\log^* n)})$
- Harvey-vdH-Lecerf 2014 $M(n) = O(n \log n\, 8^{\log^* n})$
- Harvey 2017 $M(n) = O(n \log n\, 6^{\log^* n})$
- Harvey-vdH 2017 $M(n) = O(n \log n\, (4\sqrt{2})^{\log^* n})$
- Harvey-vdH 2018 $M(n) = O(n \log n\, 4^{\log^* n})$

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$$\begin{aligned}
&(\, u^3 + 427\, u^2 + 247\, u + 692)\, v^3 + \\
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Problem: $\gcd(m, n) = 1 \Longrightarrow$ we cannot share primitive roots of unity

Wishful thinking

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Thank you !

http://www.TeXmacs.org