Fast multiple precision $\exp(x)$ with precomputations

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Main challenges

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Given $x \in [0,1)$ and fixed $n \in \mathbb{N}$, compute $\exp x$ with a precision of n binary digits

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- Complexity as a function of n ?
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Focus on $\exp$, but similar approach applies to $\log$, $\sin$, $\cos$, etc.

Complexity measures

$n \longrightarrow$ bit precision

$\mathsf{A}(n)$	cost to add two n-bit numbers
$\mathsf{M}(n)$	cost to multiply two n-bit numbers
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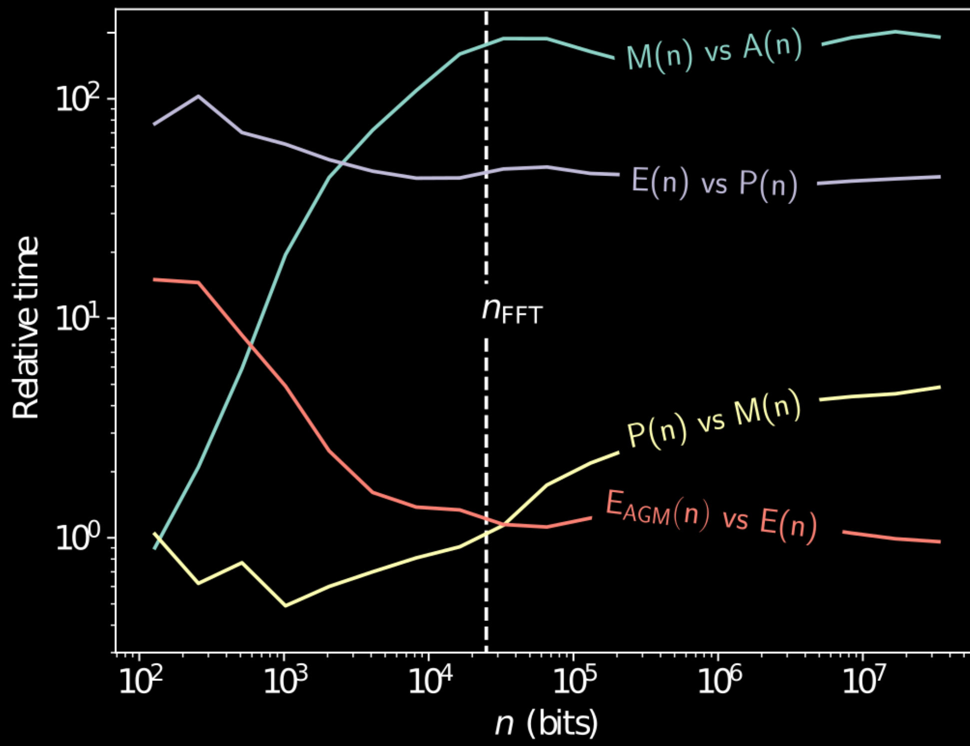
In theory:

$$\begin{aligned} \mathsf{A}(n) &= O(n) \\ \mathsf{M}(n) &= O(n \log n) \\ \mathsf{P}(n) &= O(\mathsf{M}(n) \log n) \\ \mathsf{E}(n) &= O(\mathsf{M}(n) \log n) \end{aligned}$$

Practical complexity, currently in FLINT



General ideas for exponentiation

Argument reduction

$$\begin{aligned} x &= x_1 + \cdots + x_k + \varepsilon \\ e^x &= e^{x_1} \times \cdots \times e^{x_k} \times e^{\varepsilon} \end{aligned}$$

$\longrightarrow$ multiple decomposition strategies

Taylor series expansion

$$e^{\varepsilon} \approx 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots + \frac{1}{(N-1)!}\varepsilon^{N-1}$$

$\longrightarrow$ multiple strategies to evaluate the Taylor series

Bitwise reduction (naive)

$$x = 0.\ 0101\ 1001\ 0011\ 1011\ \cdots$$

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$$\begin{aligned}
x &= 0.\ 0101\ 1001\ 0011\ 1011\ \cdots \\
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&\ \cdots
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{e}^x &= 1.\ 0101\ 1101\ 1110\ 1001\ \cdots\ \times \\
&\ 1.\ 0000\ 1001\ 0010\ 1000\ \cdots\ \times \\
&\ 1.\ 0000\ 0000\ 0011\ 0000\ \cdots\ \times \\
&\ 1.\ 0000\ 0000\ 0000\ 1011\ \cdots\ \times \\
&\ \cdots
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Bitwise reduction (CORDIC, BKM)

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$$x \approx 1.\,0110 \times 1.\,0000\,0111 \times 1.\,0000\,0000\,1100 \times 1.\,0000\,0000\,0000\,1101$$

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$\longrightarrow$ “Shift and add”

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$\longrightarrow$ “Shift and add” $\longrightarrow$ ”Shift and FMA”

Complexity analysis

Reduction $x \longrightarrow \varepsilon$ with $\varepsilon < 2^{-r}$

m-bitwise

	Naive	**Shift and FMA**
Time	$\frac{r}{m}\mathsf{M}(n)$	$\frac{3r}{m}\mathsf{A}(n)$
Space	$\frac{2^m}{m} r n$	$\frac{2^m}{m} r n$

$$x = 0.11062024$$

Diophantine approximation (Shönhage, Johansson)

$$\begin{aligned} x &= 0.11062024 \\ x &\approx -33\log 2 - 120\log 3 + 177\log 5 + 49\log 7 - 94\log 11 \end{aligned}$$

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\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
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-3 & 4 & -2 & -2 & 2 \\
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$$
x := 0.11062024 \approx 4\,\ell_3
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$$x - 4\,\ell_3 \approx 0.005351 \approx \ell_4$$

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using m prime numbers $p_1, \ldots, p_m$

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$$\text{Time} \quad 1.4\, r\, \mathsf{A}(n) + \mathsf{P}(\overbrace{\mathrm{e}^3 m^2 \log_2 m}^{\leqslant n}) + \mathsf{M}(n)$$

$$\text{Space} \quad m\, n$$

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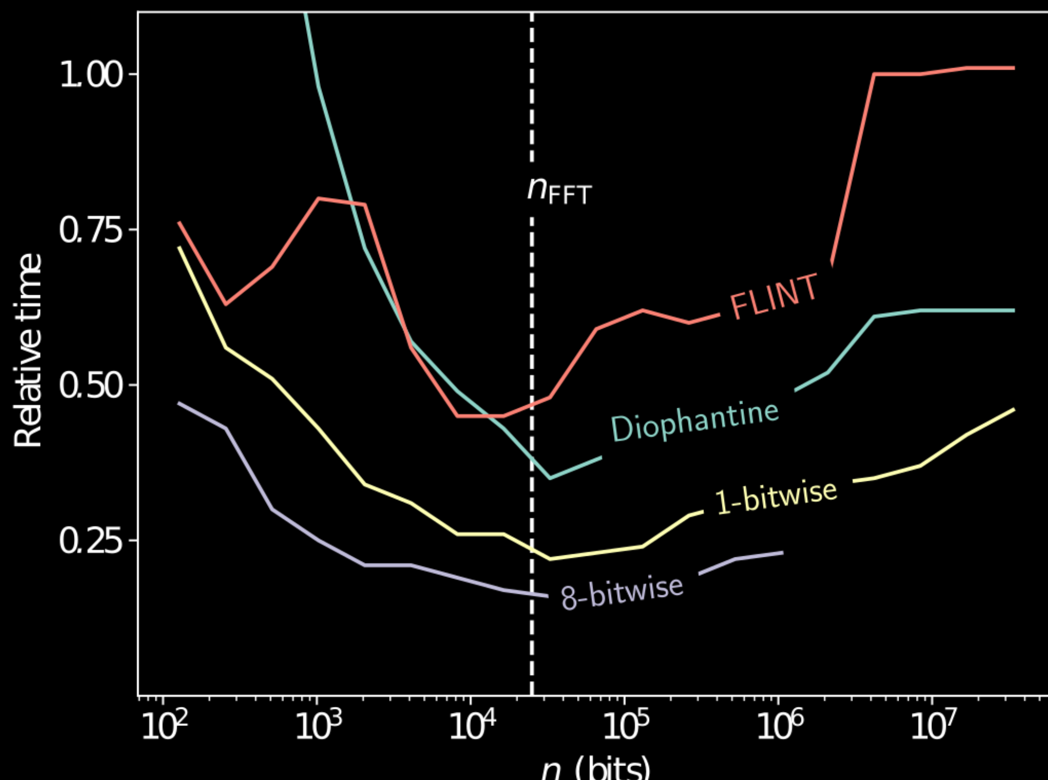
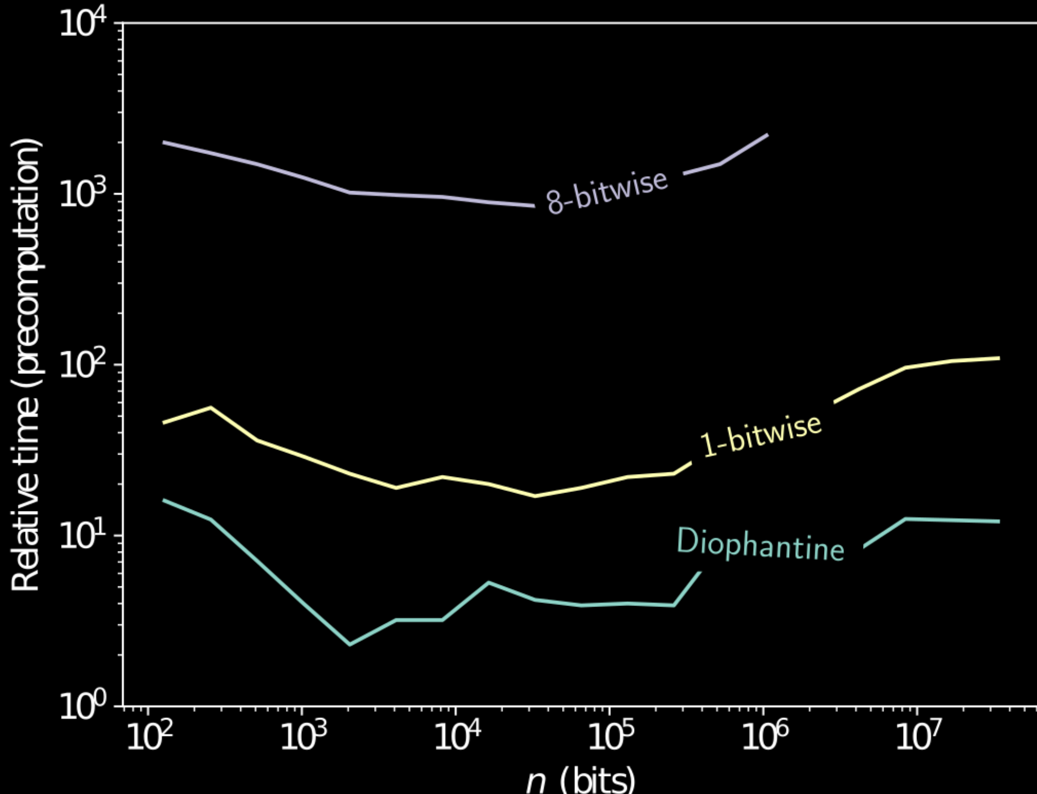
Optimum $\quad r \;\approx\; 2.9\, m \;\approx\; 0.64 \sqrt{\frac{n}{\log_2 n}}$

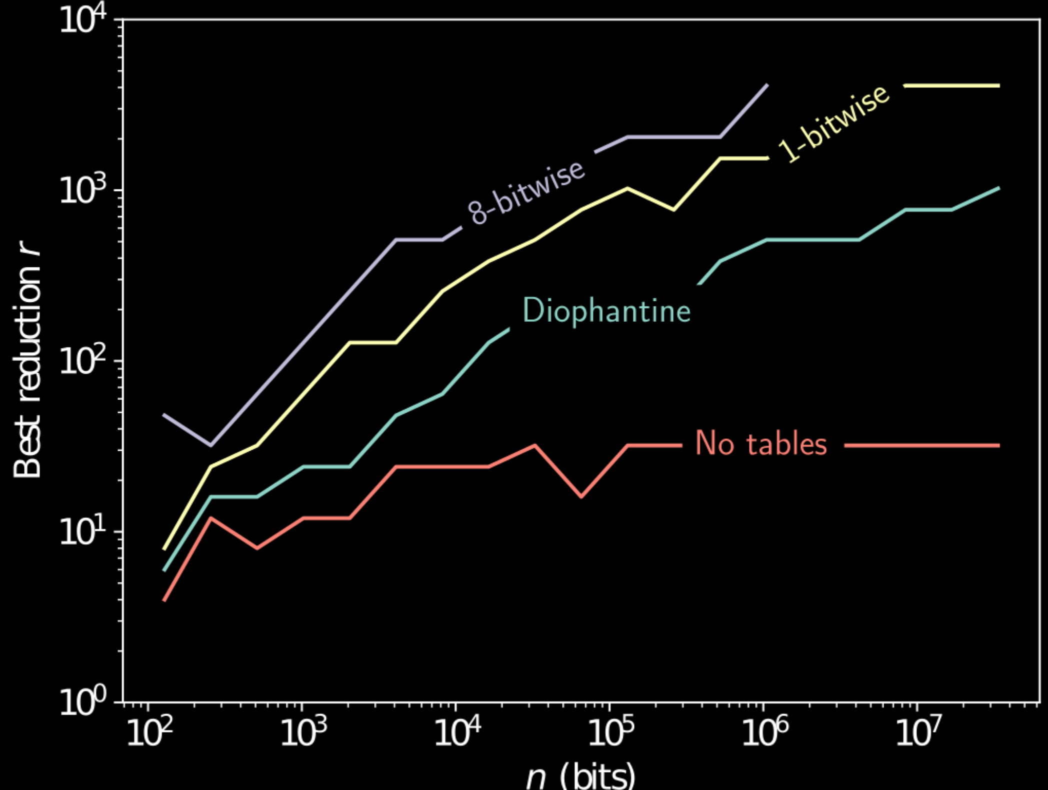
Overall timings w.r.t. squaring + Taylor series



Cost of precomputations



Best reduction threshold r



Taylor series

$$e^{\varepsilon} = 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots + \frac{1}{(N-1)!}\varepsilon^{N-1}, \qquad N \approx \frac{n}{r}$$

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Hyperbolic sines $\frac{n}{2r}\mathrm{M}(n)$

$$\begin{aligned}\operatorname{sh}\varepsilon &= \varepsilon+\tfrac{1}{6}\varepsilon^3+\cdots+\tfrac{1}{(2N-1)!}\varepsilon^{2N-1}, \qquad N \approx \tfrac{n}{2r} \\ &= \frac{e^{\varepsilon}-e^{-\varepsilon}}{2}\end{aligned}$$

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Curiosity: generalization with asymptotic complexity $\frac{n}{3r}\mathsf{M}(n)$

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Rectangular splitting $c\sqrt{\frac{n}{r}}\mathsf{M}(n), \qquad \left(\tfrac{2}{3} \leqslant c \leqslant \tfrac{5}{3}\right)$

Binary splitting, bit burst

$$x = 0.\ 01\ 01\ 1011\ 01001001\ 0000111010111010\ 01001010010100111001010010110010$$

$$\varepsilon = 0.\ 00\ 00\ 0000\ 01001001$$

$$e^{\varepsilon} \approx 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + \frac{1}{24}\varepsilon^4 + \frac{1}{120}\varepsilon^5 + \frac{1}{720}\varepsilon^6 + \frac{1}{5040}\varepsilon^7$$

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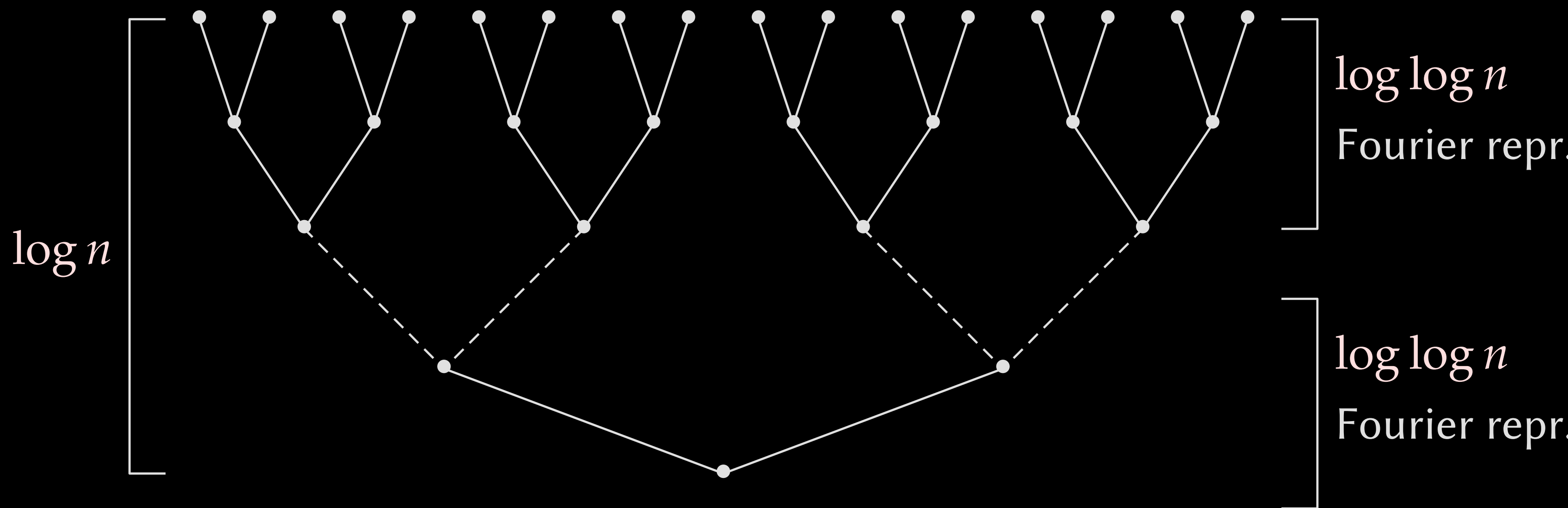
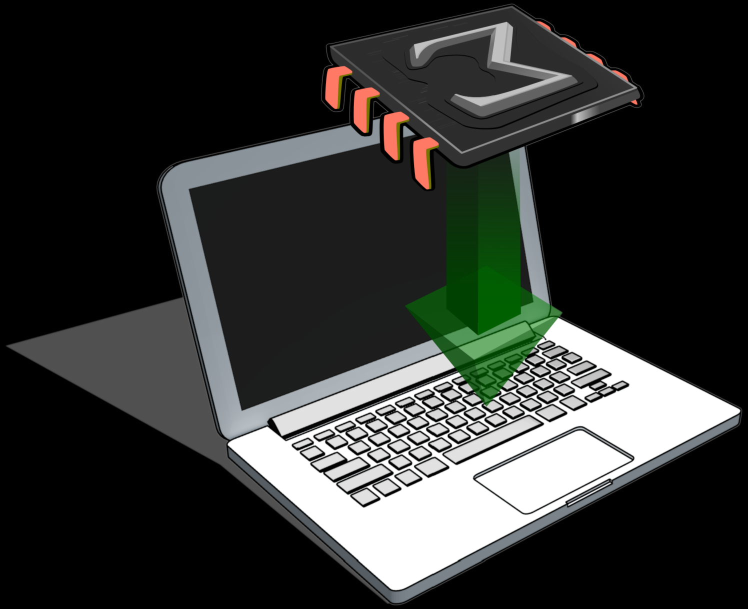
$$\begin{tikzcd}
1+\varepsilon \arrow[dr, "\times 6"] & & 3+\varepsilon \arrow[dl, "\times \varepsilon^2"'] & & 5+\varepsilon \arrow[dr, "\times 42"] & & 7+\varepsilon \arrow[dl, "\times \varepsilon^2"'] & \approx 16 \text{ bits} \\
& 6+6\varepsilon+3\varepsilon^2+\varepsilon^3 \arrow[drr, "\times 840"] & & & & 210+42\varepsilon+7\varepsilon^2+\varepsilon^3 \arrow[dll, "\times \varepsilon^4"'] & & \approx 32 \text{ bits} \\
& & & 5040+5040\varepsilon+2520\varepsilon^2+840\varepsilon^3+210\varepsilon^4+42\varepsilon^5+7\varepsilon^6+\varepsilon^7 & & & & \approx 64 \text{ bits}
\end{tikzcd}$$

FFT trading



Binary splitting exponentiation in time $O\left(\mathsf{M}(n)\,\frac{\log^2 n}{\log\log n}\right)$

Thank you !



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