Effective elimination for D-algebraic power series equations

Joris van der Hoeven

CNRS, École polytechnique, supported by the ERC ODELIX project

DART 2025, Chinese Academy of Sciences, Beijing **May 26, 2025**

Goal

Polynomial system solving

$$\begin{aligned} x^3 + y^3 + z^3 - 5\,x\,y + 7\,z - 11 &= 0 \\ x^{26} - y^5 + 2025 &= 0 \end{aligned}$$

$$\xrightarrow{?}$$

More general system solving

$$\begin{aligned} \sin \sin x^3 - \log(1 - y^5 z^7) &= 0 \\ \operatorname{erf}(x^5 e^{y^5 \arctan z}) + J_3(x)\, y^2 z^3 &= 0 \end{aligned}$$

Goal

Polynomial system solving

$$\begin{aligned} x^3 + y^3 + z^3 - 5\,x\,y + 7\,z - 11 &= 0 \\ x^{26} - y^5 + 2025 &= 0 \end{aligned}$$

$$\xrightarrow{?}$$

More general system solving

$$\begin{aligned} \sin\sin x^3 - \log(1 - y^5 z^7) &= 0 \\ \operatorname{erf}(x^5 \mathrm{e}^{y^5 \arctan z}) + J_3(x)\, y^2 z^3 &= 0 \end{aligned}$$

Question: what do $\log(1 - y^5 z^7)$ and $\arctan z$ mean ?

Polynomial system solving

$$\begin{aligned} x^3 + y^3 + z^3 - 5xy + 7z - 11 &= 0 \\ x^{26} - y^5 + 2025 &= 0 \end{aligned}$$

$$\xrightarrow{?}$$

More general system solving

$$\begin{aligned} \sin \sin x^3 - \log(1 - y^5 z^7) &= 0 \\ \operatorname{erf}(x^5 e^{y^5 \arctan z}) + J_3(x)\, y^2 z^3 &= 0 \end{aligned}$$

Question: what do $\log(1 - y^5 z^7)$ and $\arctan z$ mean ?

Approach in this talk

- Work with power series at $x = y = z = 0$

Goal

Polynomial system solving

$$\begin{aligned} x^3+y^3+z^3-5xy+7z-11 &= 0\\ x^{26}-y^5+2025 &= 0 \end{aligned}$$

$$\xrightarrow{?}$$

More general system solving

$$\begin{aligned} \sin\sin x^3-\log(1-y^5z^7) &= 0\\ \operatorname{erf}(x^5 e^{y^5 \arctan z})+J_3(x)\,y^2z^3 &= 0 \end{aligned}$$

Question: what do $\log(1-y^5z^7)$ and $\arctan z$ mean ?

Approach in this talk

- Work with power series at $x=y=z=0$
- Restrict attention to *differentially algebraic power series*

Goal

Polynomial system solving $\xrightarrow{?}$ **More general system solving**

$$\begin{aligned} x^3+y^3+z^3-5xy+7z-11 &= 0\\ x^{26}-y^5+2025 &= 0 \end{aligned} \quad\xrightarrow{?}\quad \begin{aligned} \sin\sin x^3-\log(1-y^5 z^7) &= 0\\ \operatorname{erf}(x^5 e^{y^5 \arctan z})+J_3(x)\, y^2 z^3 &= 0 \end{aligned}$$

Question: what do $\log(1-y^5 z^7)$ and $\arctan z$ mean ?

Approach in this talk

- Work with power series at $x=y=z=0$
- Restrict attention to *differentially algebraic power series*
- Note: other “base points” through analytic continuation

D-algebraic power series

$\mathbb{K}$: effective field of characteristic zero

D-algebraic power series

$\mathbb{K}$: effective field of characteristic zero

$y \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **D-algebraic** (over $\mathbb{K}$) $\iff$

$$\forall i \in \{1, \ldots, k\}, \quad \exists P_i \in \mathbb{K}\left[Y, \frac{\partial Y}{\partial z_i}, \frac{\partial^2 Y}{\partial z_i^2}, \ldots\right] \setminus \mathbb{K}, \quad P_i(y) = 0$$

D-algebraic power series

$\mathbb{K}$: effective field of characteristic zero

$y \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **D-algebraic** (over $\mathbb{K}$) $\Longleftrightarrow$

$$\forall i \in \{1, \ldots, k\}, \quad \exists P_i \in \mathbb{K}\left[Y, \frac{\partial Y}{\partial z_i}, \frac{\partial^2 Y}{\partial z_i^2}, \ldots\right] \setminus \mathbb{K}, \quad P_i(y) = 0$$

$y = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k} z_1^{i_1} \cdots z_k^{i_k} \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **computable** $\Longleftrightarrow$

$\exists$ algorithm for $(i_1, \ldots, i_k) \mapsto f_{i_1, \ldots, i_k}$

D-algebraic power series

$\mathbb{K}$: effective field of characteristic zero

$y \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **D-algebraic** (over $\mathbb{K}$) $\Longleftrightarrow$
$\forall i \in \{1, \ldots, k\}, \quad \exists P_i \in \mathbb{K}\left[Y, \frac{\partial Y}{\partial z_i}, \frac{\partial^2 Y}{\partial z_i^2}, \ldots\right] \setminus \mathbb{K}, \quad P_i(y) = 0$

$y = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k} z_1^{i_1} \cdots z_k^{i_k} \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **computable** $\Longleftrightarrow$
$\exists$ algorithm for $(i_1, \ldots, i_k) \mapsto f_{i_1, \ldots, i_k}$

DART 2024

- Representation of computable D-algebraic power series through non-degenerate annihilators and finite number of initial conditions.

D-algebraic power series

$\mathbb{K}$: effective field of characteristic zero

$y \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **D-algebraic** (over $\mathbb{K}$) $\Longleftrightarrow$

$\forall i \in \{1, \ldots, k\}, \quad \exists P_i \in \mathbb{K}\left[Y, \frac{\partial Y}{\partial z_i}, \frac{\partial^2 Y}{\partial z_i^2}, \ldots\right] \setminus \mathbb{K}, \quad P_i(y) = 0$

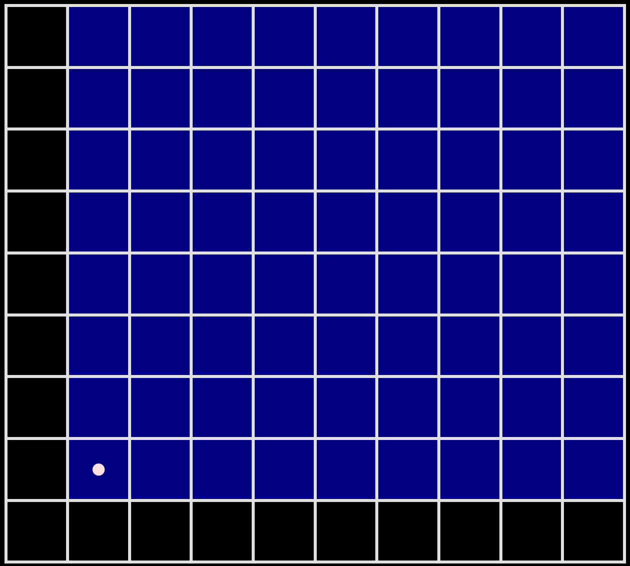
$y = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k} z_1^{i_1} \cdots z_k^{i_k} \in \mathbb{K}[[z_1, \ldots, z_k]]$ is **computable** $\Longleftrightarrow$

$\exists$ algorithm for $(i_1, \ldots, i_k) \mapsto f_{i_1, \ldots, i_k}$

DART 2024

- Representation of computable D-algebraic power series through non-degenerate annihilators and finite number of initial conditions.
- Zero-tests for D-algebraic power series.

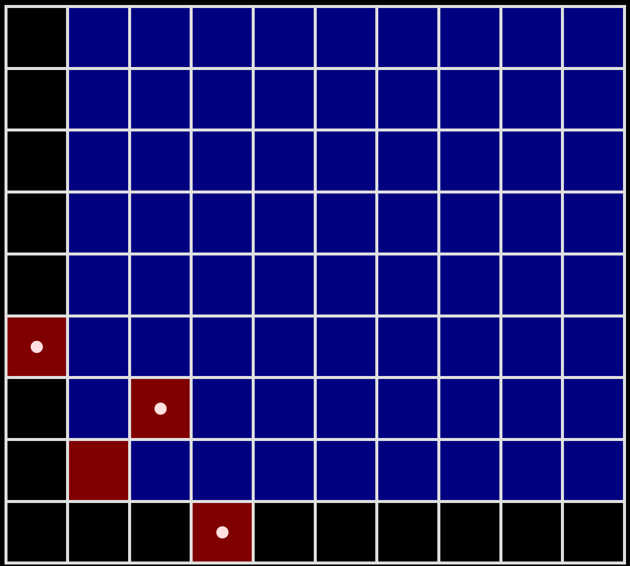
Use standard bases and Hironaka division ?



$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

Use standard bases and Hironaka division ?

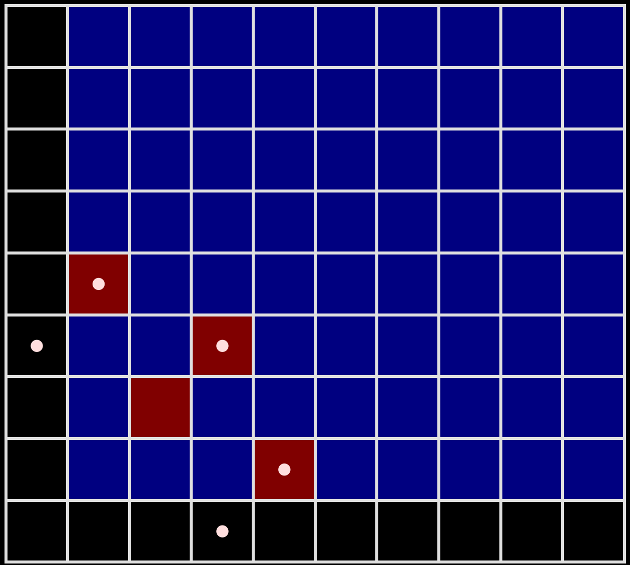


$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$

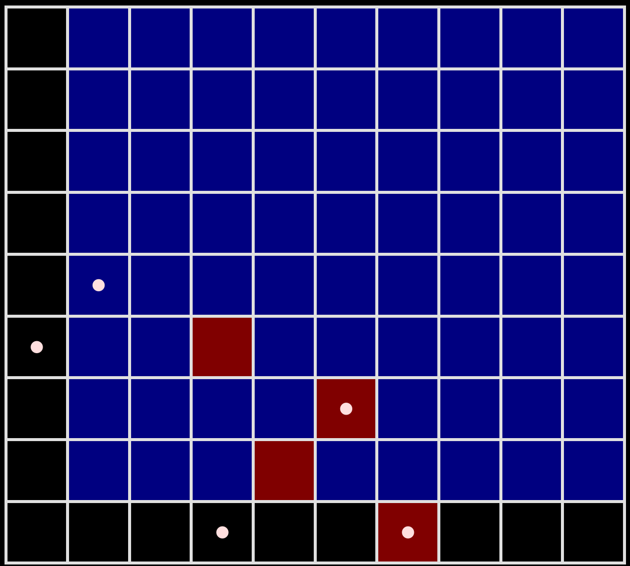
Use standard bases and Hironaka division ?



$$f = z_1 z_2$$
$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$

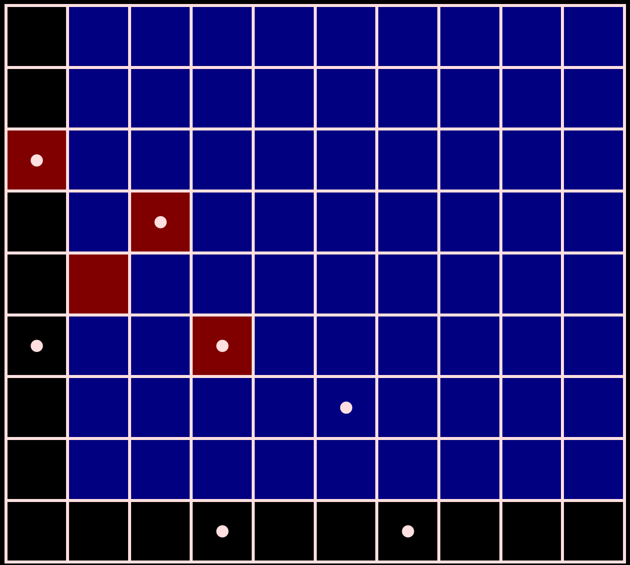
Use standard bases and Hironaka division ?



$$f = z_1 z_2$$
$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 - z_1^3 z_2^3 + z_1^3 z_2^3 + z_1^5 z_2^2$$

Use standard bases and Hironaka division ?



$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

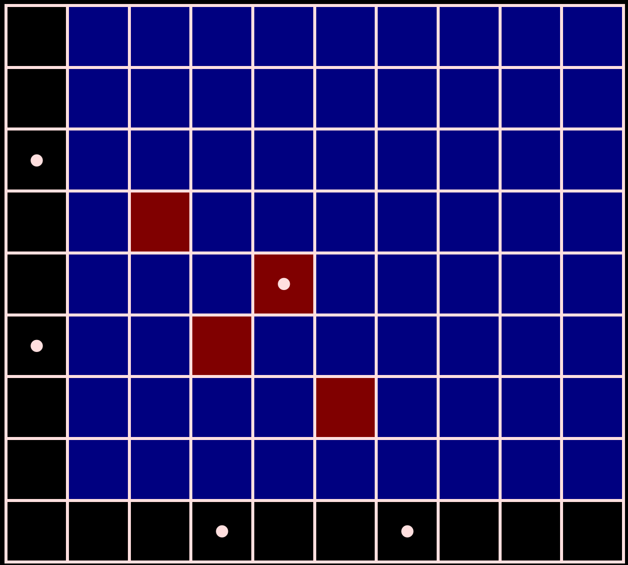
$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^3 z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5$$

Use standard bases and Hironaka division ?



$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$

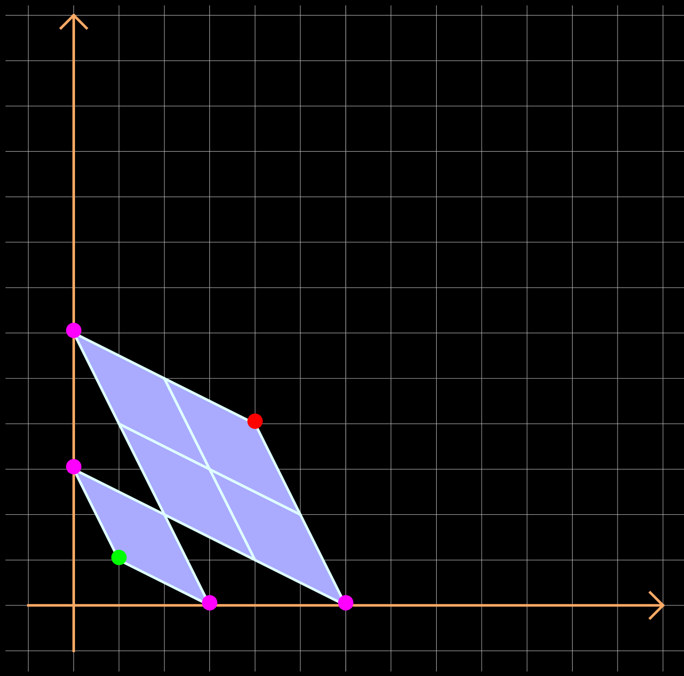
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^3 z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^4 z_2^4$$

$$+ z_1^5 z_2^2 + z_1^2 z_2^5 - z_1^5 z_2^2 - z_1^2 z_2^5$$

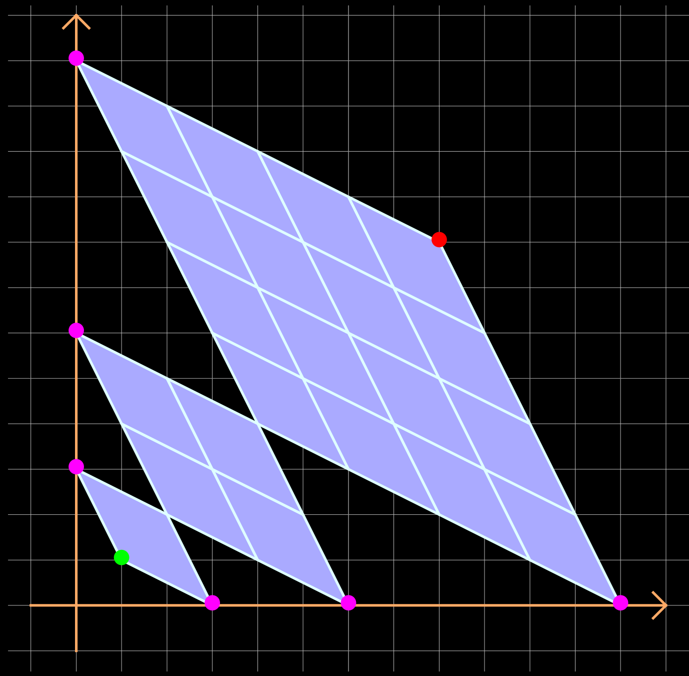
Use standard bases and Hironaka division ?



$$f = z_1 z_2$$
$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^5 z_2^2$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^4 z_2^4$$

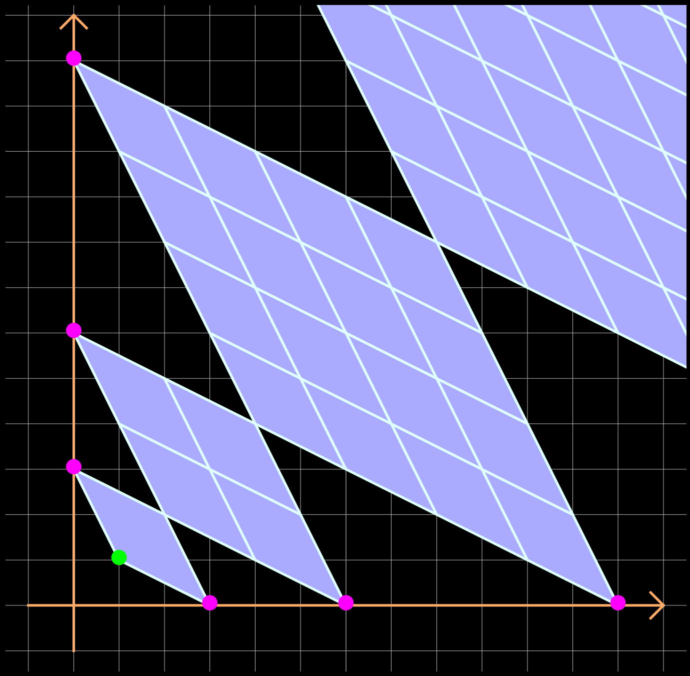
Use standard bases and Hironaka division ?



$$f = z_1 z_2$$
$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^5 z_2^2$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^4 z_2^4$$
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^8 z_2^8$$

Use standard bases and Hironaka division ?

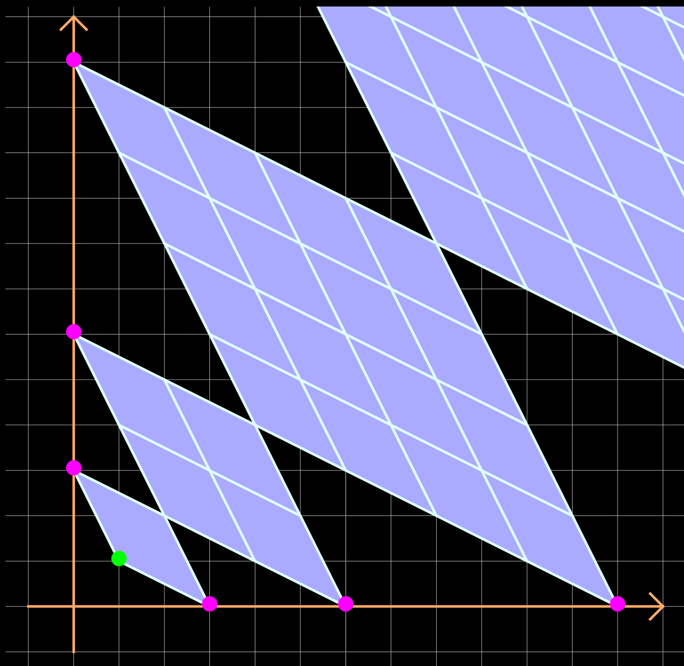


$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$\begin{aligned}
f &\xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2 \\
&\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3 \\
&\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^5 z_2^2 \\
&\xrightarrow{g} z_1^3 + z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5 \\
&\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^4 z_2^4 \\
&\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^8 z_2^8 \\
&\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^{24} - z_2^{24} + \cdots
\end{aligned}$$

Use standard bases and Hironaka division ?



$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^5 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5$$

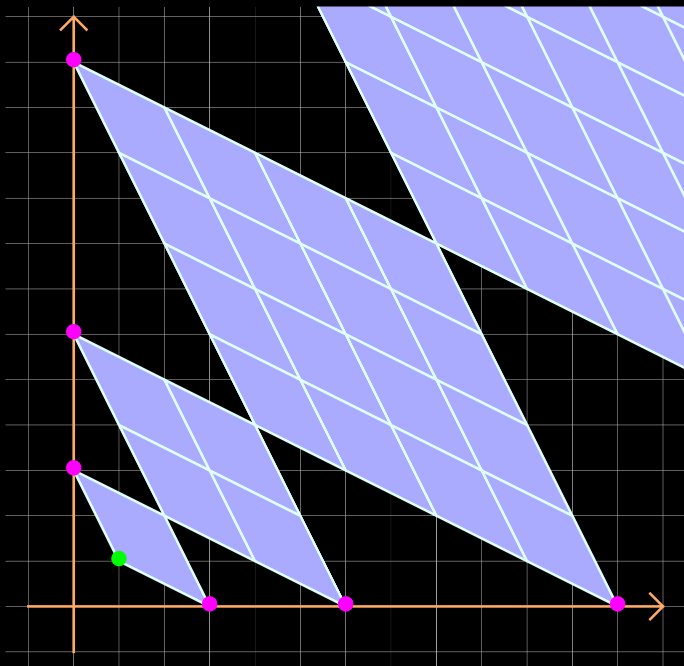
$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^4 z_2^4$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^8 z_2^8$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^{24} - z_2^{24} + \cdots$$

Problem: the remainder $f \operatorname{rem} g$ is not algebraic

Use standard bases and Hironaka division ?



$$f = z_1 z_2$$

$$g = (z_1 - z_2^2)(z_2 - z_1^2) = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$f \xrightarrow{g} f - g = z_1^3 + z_2^3 - z_1^2 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^4 z_2 - z_1 z_2^4 + z_1^3 z_2^3$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1 z_2^4 - z_1^6 + z_1^5 z_2^2$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^3 z_2^3 + z_1^5 z_2^2 + z_1^2 z_2^5$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^4 z_2^4$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^8 z_2^8$$

$$\xrightarrow{g} z_1^3 + z_2^3 - z_1^6 - z_2^6 + z_1^{12} + z_2^{12} - z_1^{24} - z_2^{24} + \cdots$$

Problem: the remainder $f \operatorname{rem} g$ is not even D-algebraic !

Closure properties for D-algebraic power series

$\mathbb{L}_k \subseteq \mathbb{K}[[z_1, \ldots, z_k]]$ set of D-algebraic power series in $z_1, \ldots, z_k$

1. $\mathbb{L}_k$ is a **differential $\mathbb{K}$-subalgebra**.

Closure properties for D-algebraic power series

$\mathbb{L}_k \subseteq \mathbb{K}[[z_1, \ldots, z_k]]$ set of D-algebraic power series in $z_1, \ldots, z_k$

1. $\mathbb{L}_k$ is a **differential $\mathbb{K}$-subalgebra**.

2. $\mathbb{L}_k$ is closed under **restricted division**.
   E.g., $\sin z_1 \in \mathbb{L}_1$ and $\frac{\sin z_1}{z_1} \in \mathbb{K}[[z_1]] \Longrightarrow \frac{\sin z_1}{z_1} \in \mathbb{L}_1$.

Closure properties for D-algebraic power series

$\mathbb{L}_k \subseteq \mathbb{K}[[z_1,\ldots,z_k]]$ set of D-algebraic power series in $z_1,\ldots,z_k$

1. $\mathbb{L}_k$ is a **differential $\mathbb{K}$-subalgebra**.
2. $\mathbb{L}_k$ is closed under **restricted division**.
   E.g., $\sin z_1 \in \mathbb{L}_1$ and $\frac{\sin z_1}{z_1} \in \mathbb{K}[[z_1]] \implies \frac{\sin z_1}{z_1} \in \mathbb{L}_1$.
3. $(\mathbb{L}_k)_{k\in\mathbb{N}}$ is closed under **composition**:
   if $f_1 \in \mathbb{L}_m$ and $g_1,\ldots,g_m \in \mathbb{L}_k$, then $f \circ (g_1,\ldots,g_m) \in \mathbb{L}_k$.

Closure properties for D-algebraic power series

$\mathbb{L}_k \subseteq \mathbb{K}[[z_1, \ldots, z_k]]$ set of D-algebraic power series in $z_1, \ldots, z_k$

1. $\mathbb{L}_k$ is a **differential $\mathbb{K}$-subalgebra**.
2. $\mathbb{L}_k$ is closed under **restricted division**.
   E.g., $\sin z_1 \in \mathbb{L}_1$ and $\frac{\sin z_1}{z_1} \in \mathbb{K}[[z_1]] \implies \frac{\sin z_1}{z_1} \in \mathbb{L}_1$.
3. $(\mathbb{L}_k)_{k \in \mathbb{N}}$ is closed under **composition**:
   if $f_1 \in \mathbb{L}_m$ and $g_1, \ldots, g_m \in \mathbb{L}_k$, then $f \circ (g_1, \ldots, g_m) \in \mathbb{L}_k$.
4. $(\mathbb{L}_k)_{k \in \mathbb{N}}$ satisfies the **implicit function theorem**.

Closure properties for D-algebraic power series

$\mathbb{L}_k \subseteq \mathbb{K}[[z_1,\dots,z_k]]$ set of D-algebraic power series in $z_1,\dots,z_k$

1. $\mathbb{L}_k$ is a **differential $\mathbb{K}$-subalgebra**.
2. $\mathbb{L}_k$ is closed under **restricted division**.
   E.g., $\sin z_1 \in \mathbb{L}_1$ and $\frac{\sin z_1}{z_1} \in \mathbb{K}[[z_1]] \Longrightarrow \frac{\sin z_1}{z_1} \in \mathbb{L}_1$.
3. $(\mathbb{L}_k)_{k\in\mathbb{N}}$ is closed under **composition**:
   if $f_1 \in \mathbb{L}_m$ and $g_1,\dots,g_m \in \mathbb{L}_k$, then $f \circ (g_1,\dots,g_m) \in \mathbb{L}_k$.
4. $(\mathbb{L}_k)_{k\in\mathbb{N}}$ satisfies the **implicit function theorem**.
5. $(\mathbb{L}_k)_{k\in\mathbb{N}}$ is closed under **restricted monomial transformations**.
   E.g., $f := \mathrm{e}^{z_2^2}\sin(z_1 z_2) \in \mathbb{L}_2$ and $g := f\left(\frac{z_1}{\sqrt{z_2}}, \sqrt{z_2}\right) \in \mathbb{K}[[z_1,z_2]] \Longrightarrow g \in \mathbb{L}_2$.

Closure properties for D-algebraic power series

$\mathbb{L}_k \subseteq \mathbb{K}[[z_1, \ldots, z_k]]$ set of D-algebraic power series in $z_1, \ldots, z_k$

1. $\mathbb{L}_k$ is a **differential $\mathbb{K}$-subalgebra**.
2. $\mathbb{L}_k$ is closed under **restricted division**.
   E.g., $\sin z_1 \in \mathbb{L}_1$ and $\frac{\sin z_1}{z_1} \in \mathbb{K}[[z_1]] \Longrightarrow \frac{\sin z_1}{z_1} \in \mathbb{L}_1$.
3. $(\mathbb{L}_k)_{k \in \mathbb{N}}$ is closed under **composition**:
   if $f_1 \in \mathbb{L}_m$ and $g_1, \ldots, g_m \in \mathbb{L}_k$, then $f \circ (g_1, \ldots, g_m) \in \mathbb{L}_k$.
4. $(\mathbb{L}_k)_{k \in \mathbb{N}}$ satisfies the **implicit function theorem**.
5. $(\mathbb{L}_k)_{k \in \mathbb{N}}$ is closed under **restricted monomial transformations**.
   E.g., $f := \mathrm{e}^{z_2^2} \sin(z_1 z_2) \in \mathbb{L}_2$ and $g := f\left(\frac{z_1}{\sqrt{z_2}}, \sqrt{z_2}\right) \in \mathbb{K}[[z_1, z_2]] \Longrightarrow g \in \mathbb{L}_2$.

We say that $(\mathbb{L}_k)_{k \in \mathbb{N}}$ forms a **tribe**. It is **effective** if all these operations can be carried out algorithmically and if we have a zero-test.

What about Weierstrass division ?

Let $(\mathbb{L}_k)_{k\in\mathbb{N}}$ be an effective tribe.

What about Weierstrass division ?

Let $(\mathbb{L}_k)_{k\in\mathbb{N}}$ be an effective tribe.

Weierstrass preparation theorem

Let $f \in \mathbb{L}_k$ be in Weierstrass position of degree d in z_k:

$$f(0) = \frac{\partial f}{\partial z_k}(0) = \cdots = \frac{\partial^{d-1} f}{\partial z_k^{d-1}}(0) = 0, \text{ but } \frac{\partial^d f}{\partial z_k^d}(0) \neq 0.$$

Then we may compute a unit $u \in \mathbb{L}_k$ with $uf = z_k^d + P_{d-1} z_k^{d-1} + \cdots + P_0 \in \mathbb{L}_{k-1}[z]$.

What about Weierstrass division ?

Let $(\mathbb{L}_k)_{k\in\mathbb{N}}$ be an effective tribe.

Weierstrass preparation theorem

Let $f \in \mathbb{L}_k$ be in Weierstrass position of degree d in z_k:

$$f(0) = \frac{\partial f}{\partial z_k}(0) = \cdots = \frac{\partial^{d-1} f}{\partial z_k^{d-1}}(0) = 0, \text{ but } \frac{\partial^d f}{\partial z_k^d}(0) \neq 0.$$

Then we may compute a unit $u \in \mathbb{L}_k$ with $u f = z_k^d + P_{d-1} z_k^{d-1} + \cdots + P_0 \in \mathbb{L}_{k-1}[z]$.

Weierstrass division theorem

In addition, if $g \in \mathbb{L}_k$, then we may compute (the unique) $Q \in \mathbb{L}_k$ and $R \in \mathbb{L}_{k-1}[z_k]$ with $g = Q f + R$ and $\deg_{z_k} R < d$.

Grid-based power series

A **monomial monoid** is $\approx$ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow \exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow$ $\exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow$ $\exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow$ $\exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

$\mathbb{K}[[z^{\mathbb{Z}}]] \cong \mathbb{K}((z))$

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow$ $\exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

$\mathbb{K}[[z^{\mathbb{Z}}]] \cong \mathbb{K}((z))$

$\mathbb{K}[[z^{\mathbb{Q}}]]$ Puiseux series in z

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow$ $\exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

$\mathbb{K}[[z_1^{\mathbb{N}} \times z_2^{\mathbb{N}}]] \cong \mathbb{K}[[z_1, z_2]] \quad z_1, z_2 \prec 1$

$\mathbb{K}[[z^{\mathbb{Z}}]] \cong \mathbb{K}((z))$

$\mathbb{K}[[z^{\mathbb{Q}}]]$ Puiseux series in z

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow \exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

$\mathbb{K}[[z^{\mathbb{Z}}]] \cong \mathbb{K}((z))$

$\mathbb{K}[[z^{\mathbb{Q}}]]$ Puiseux series in z

$\mathbb{K}[[z_1^{\mathbb{N}} \times z_2^{\mathbb{N}}]] \cong \mathbb{K}[[z_1, z_2]] \quad z_1, z_2 \prec 1$

$\mathbb{K}[[z_1^{\mathbb{Z}} \times z_2^{\mathbb{Z}}]] \cong \mathbb{K}[[z_1, z_2]]\, z_1^{\mathbb{Z}} z_2^{\mathbb{Z}}$

Grid-based power series

A **monomial monoid** is $\approx$ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow \exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

$\mathbb{K}[[z^{\mathbb{Z}}]] \cong \mathbb{K}((z))$

$\mathbb{K}[[z^{\mathbb{Q}}]]$ Puiseux series in z

$\mathbb{K}[[z_1^{\mathbb{N}} \times z_2^{\mathbb{N}}]] \cong \mathbb{K}[[z_1, z_2]] \quad z_1, z_2 \prec 1$

$\mathbb{K}[[z_1^{\mathbb{Z}} \times z_2^{\mathbb{Z}}]] \cong \mathbb{K}[[z_1, z_2]] z_1^{\mathbb{Z}} z_2^{\mathbb{Z}}$

$\mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]] \subseteq \mathbb{K}((z_1))((z_2))$

Grid-based power series

A **monomial monoid** is ≈ a multiplicative monoid $\mathfrak{M}$ with a partial order $\preccurlyeq$.

$\mathfrak{S} \subseteq \mathfrak{M}$ is **grid-based** $\Longleftrightarrow$ $\exists \mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and

$$\mathfrak{S} \subseteq \{\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_k^{i_k} \mathfrak{n} : i_1, \ldots, i_k \in \mathbb{N}\}.$$

$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a **grid-based power series** if $\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\}$ is grid-based.

The set $\mathbb{K}[[\mathfrak{M}]]$ of such series is a ring and a field if $\mathfrak{M}$ is a totally ordered group.

$\mathbb{K}[[z^{\mathbb{N}}]] \cong \mathbb{K}[[z]], \quad z \prec 1$

$\mathbb{K}[[z^{\mathbb{Z}}]] \cong \mathbb{K}((z))$

$\mathbb{K}[[z^{\mathbb{Q}}]]$ Puiseux series in z

$\mathbb{K}[[z_1^{\mathbb{N}} \times z_2^{\mathbb{N}}]] \cong \mathbb{K}[[z_1, z_2]] \quad z_1, z_2 \prec 1$

$\mathbb{K}[[z_1^{\mathbb{Z}} \times z_2^{\mathbb{Z}}]] \cong \mathbb{K}[[z_1, z_2]]\, z_1^{\mathbb{Z}} z_2^{\mathbb{Z}}$

$\mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]] \subseteq \mathbb{K}((z_1))((z_2))$

$\mathbb{K}[[z_2^{\mathbb{Q}} \times_{\text{lex}} z_1^{\mathbb{Q}}]] \subseteq \mathbb{K}[[z_1^{\mathbb{Q}}]][[z_2^{\mathbb{Q}}]]$

$\mathbb{L}$-based series

Any $f \in \mathbb{K}[[\mathfrak{M}]]$ has a **Cartesian representation** $f = \check{f}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)\, \mathfrak{n}$, where $\check{f} \in \mathbb{K}[[z_1, \ldots, z_k]]$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$.

$\mathbb{L}$-based series

Any $f \in \mathbb{K}[[\mathfrak{M}]]$ has a **Cartesian representation** $f = \check{f}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)\,\mathfrak{n}$, where $\check{f} \in \mathbb{K}[[z_1, \ldots, z_k]]$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$.

Example. If $f \in \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$, then $f = \check{f}\left(z_1, \frac{z_2}{z_1^n}\right) z_1^{i_1} z_2^{i_2}$ for some $i_1, i_2 \in \mathbb{N}$ and $n > 0$.

$\mathbb{L}$-based series

Any $f \in \mathbb{K}[[\mathfrak{M}]]$ has a **Cartesian representation** $f = \check{f}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)\,\mathfrak{n}$, where $\check{f} \in \mathbb{K}[[z_1, \ldots, z_k]]$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$.

Example. If $f \in \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$, then $f = \check{f}\left(z_1, \frac{z_2}{z_1^n}\right) z_1^{i_1} z_2^{i_2}$ for some $i_1, i_2 \in \mathbb{N}$ and $n > 0$.

Example. $\sum_{n \in \mathbb{N}} z_1^{-n!} z_2^n \notin \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$.

$\mathbb{L}$-based series

Any $f \in \mathbb{K}[[\mathfrak{M}]]$ has a **Cartesian representation** $f = \check{f}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)\, \mathfrak{n}$, where $\check{f} \in \mathbb{K}[[z_1, \ldots, z_k]]$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$.

Example. If $f \in \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$, then $f = \check{f}\left(z_1, \frac{z_2}{z_1^n}\right) z_1^{i_1} z_2^{i_2}$ for some $i_1, i_2 \in \mathbb{N}$ and $n > 0$.

Example. $\sum_{n \in \mathbb{N}} z_1^{-n!} z_2^n \notin \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$.

We say that f is **$\mathbb{L}$-based** if we can take $\check{f} \in \mathbb{L}_k$.
The set $\mathbb{K}[[\mathfrak{M}]]_{\mathbb{L}}$ of such series is a ring (a field if $\mathfrak{M}$ is a totally ordered group).

$\mathbb{L}$-based series

Any $f \in \mathbb{K}[[\mathfrak{M}]]$ has a **Cartesian representation** $f = \check{f}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)\,\mathfrak{n}$, where $\check{f} \in \mathbb{K}[[z_1, \ldots, z_k]]$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_k, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$.

Example. If $f \in \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$, then $f = \check{f}\left(z_1, \frac{z_2}{z_1^n}\right) z_1^{i_1} z_2^{i_2}$ for some $i_1, i_2 \in \mathbb{N}$ and $n > 0$.

Example. $\sum_{n \in \mathbb{N}} z_1^{-n!} z_2^n \notin \mathbb{K}[[z_2^{\mathbb{Z}} \times_{\text{lex}} z_1^{\mathbb{Z}}]]$.

We say that f is **$\mathbb{L}$-based** if we can take $\check{f} \in \mathbb{L}_k$.
The set $\mathbb{K}[[\mathfrak{M}]]_{\mathbb{L}}$ of such series is a ring (a field if $\mathfrak{M}$ is a totally ordered group).

Newton polygon method

If $f \in \mathbb{L}_k$ is in Weierstrass position of degree d in z_k, then we can compute the d infinitesimal roots $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ of f in z_k.

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ for a unit u, then f and P have the same infinitesimal roots

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$P \;=\; (z_k - \alpha_1) \cdots (z_k - \alpha_p) \;\in\; \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}.$$

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$\begin{aligned} P &= (z_k - \alpha_1) \cdots (z_k - \alpha_p) \in \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}. \\ u &= {}^{f}/_{P} \end{aligned}$$

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$\begin{aligned} P &= (z_k - \alpha_1) \cdots (z_k - \alpha_p) \in \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}. \\ u &= {}^{f}/_{P} \end{aligned}$$

Actually, $P, u \in \mathbb{L}_k$, since $\mathbb{L}$ is closed under restricted monomial transformations.

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$\begin{aligned} P &= (z_k - \alpha_1) \cdots (z_k - \alpha_p) \in \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}. \\ u &= {}^{f}/_{P} \end{aligned}$$

Actually, $P, u \in \mathbb{L}_k$, since $\mathbb{L}$ is closed under restricted monomial transformations.

For a Weierstrass division with remainder $\mathbb{L}_k \ni g = Q\,f + R$,
we have $R(\alpha_i) = g(\alpha_i)$ for $i = 1, \ldots, d$

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\mathrm{lex}} \cdots \times_{\mathrm{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$P = (z_k - \alpha_1) \cdots (z_k - \alpha_p) \in \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\mathrm{lex}} \cdots \times_{\mathrm{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}.$$

$$u = {}^{f}/_{P}$$

Actually, $P, u \in \mathbb{L}_k$, since $\mathbb{L}$ is closed under restricted monomial transformations.

For a Weierstrass division with remainder $\mathbb{L}_k \ni g = Q\,f + R$, we have $R(\alpha_i) = g(\alpha_i)$ for $i = 1, \ldots, d$, so

$$R = g(\alpha_1)\frac{L_1(z_k)}{L_1(\alpha_1)} + \cdots + g(\alpha_d)\frac{L_d(z_k)}{L_d(\alpha_1)}, \qquad L_i(z_k) = \frac{P(z_k)}{z_k - \alpha}$$

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.

If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$\begin{aligned} P &= (z_k - \alpha_1) \cdots (z_k - \alpha_p) \in \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}. \\ u &= {}^{f}/_{P} \end{aligned}$$

Actually, $P, u \in \mathbb{L}_k$, since $\mathbb{L}$ is closed under restricted monomial transformations.

For a Weierstrass division with remainder $\mathbb{L}_k \ni g = Q\,f + R$,
we have $R(\alpha_i) = g(\alpha_i)$ for $i = 1, \ldots, d$, so

$$\begin{aligned} R &= g(\alpha_1)\frac{L_1(z_k)}{L_1(\alpha_1)} + \cdots + g(\alpha_d)\frac{L_d(z_k)}{L_d(\alpha_1)}, \qquad L_i(z_k) = \frac{P(z_k)}{z_k - \alpha} \\ Q &= \frac{g - R}{f} \end{aligned}$$

Weierstrass preparation and division

Let $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[z_{k-1}^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}$ be the infinitesimal roots of f in z_k.
If $f = u\,P$ is the Weierstrass preparation with unit u, then

$$\begin{aligned} P &= (z_k - \alpha_1) \cdots (z_k - \alpha_p) \;\in\; \mathbb{K}[[z_1, \ldots, z_k]] \cap \mathbb{K}[[z_k^{\mathbb{Q}} \times_{\text{lex}} \cdots \times_{\text{lex}} z_1^{\mathbb{Q}}]]_{\mathbb{L}}. \\ u &= {}^{f}/_{P} \end{aligned}$$

Actually, $P, u \in \mathbb{L}_k$, since $\mathbb{L}$ is closed under restricted monomial transformations.

For a Weierstrass division with remainder $\mathbb{L}_k \ni g = Q\,f + R$,
we have $R(\alpha_i) = g(\alpha_i)$ for $i = 1, \ldots, d$, so

$$\begin{aligned} R &= g(\alpha_1) \frac{L_1(z_k)}{L_1(\alpha_1)} + \cdots + g(\alpha_d) \frac{L_d(z_k)}{L_d(\alpha_1)}, \qquad L_i(z_k) = \frac{P(z_k)}{z_k - \alpha} \\ Q &= \frac{g - R}{f} \end{aligned}$$

Again, $Q, R \in \mathbb{L}_k$, for the same reason as above.

Effective elimination theory

Questions

- Given a finite subset $\mathscr{F} \subseteq \mathbb{L}_k$, can we compute a “basis” for $(\mathscr{F})$?
- Given in addition $g \in \mathbb{L}_k$, can we test whether $g \in (\mathscr{F})$?
- Can we compute the Hilbert function of $(\mathscr{F})$?
- Can we compute a standard basis for $(\mathscr{F})$?

Effective elimination theory

Questions

- Given a finite subset $\mathscr{F} \subseteq \mathbb{L}_k$, can we compute a “basis” for $(\mathscr{F})$?
- Given in addition $g \in \mathbb{L}_k$, can we test whether $g \in (\mathscr{F})$?
- Can we compute the Hilbert function of $(\mathscr{F})$?
- Can we compute a standard basis for $(\mathscr{F})$?

Some ingredients

- Recursive approach based on Weierstrass division.
- Also drawing inspiration from Janet-Riquier theory.
- Also use general position using generic linear change of coordinates.

Very special case of divisibility

Question: if $f, g \in \mathbb{L}_k$ and $\mathscr{F} = \{f\}$, can we check whether $g \in (\mathscr{F})$?

Very special case of divisibility

Question: if $f, g \in \mathbb{L}_k$ and $\mathscr{F} = \{f\}$, can we check whether $g \in (\mathscr{F})$?

Consider generic linear forms $\lambda_1, \ldots, \lambda_k \in \mathbb{L}_k$ with $\lambda_i = \lambda_{i,1} z_1 + \cdots + \lambda_{i,k} z_k$.

Very special case of divisibility

Question: if $f, g \in \mathbb{L}_k$ and $\mathscr{F} = \{f\}$, can we check whether $g \in (\mathscr{F})$?

Consider generic linear forms $\lambda_1, \ldots, \lambda_k \in \mathbb{L}_k$ with $\lambda_i = \lambda_{i,1} z_1 + \cdots + \lambda_{i,k} z_k$.

Then we have a computable isomorphism

$$\begin{aligned} \Lambda \colon \mathbb{L}_k &\longrightarrow \mathbb{L}_k \\ \varphi &\longmapsto \varphi \circ (\lambda_1, \ldots, \lambda_k). \end{aligned}$$

Very special case of divisibility

Question: if $f, g \in \mathbb{L}_k$ and $\mathscr{F} = \{f\}$, can we check whether $g \in (\mathscr{F})$?

Consider generic linear forms $\lambda_1, \ldots, \lambda_k \in \mathbb{L}_k$ with $\lambda_i = \lambda_{i,1} z_1 + \cdots + \lambda_{i,k} z_k$.

Then we have a computable isomorphism

$$\begin{aligned} \Lambda \colon \mathbb{L}_k &\longrightarrow \mathbb{L}_k \\ \varphi &\longmapsto \varphi \circ (\lambda_1, \ldots, \lambda_k). \end{aligned}$$

In particular: $g \in (f) \Longleftrightarrow \Lambda(g) \in (\Lambda(f))$.

Very special case of divisibility

Question: if $f, g \in \mathbb{L}_k$ and $\mathscr{F} = \{f\}$, can we check whether $g \in (\mathscr{F})$?

Consider generic linear forms $\lambda_1, \ldots, \lambda_k \in \mathbb{L}_k$ with $\lambda_i = \lambda_{i,1} z_1 + \cdots + \lambda_{i,k} z_k$.

Then we have a computable isomorphism

$$\begin{aligned} \Lambda \colon \mathbb{L}_k &\longrightarrow \mathbb{L}_k \\ \varphi &\longmapsto \varphi \circ (\lambda_1, \ldots, \lambda_k). \end{aligned}$$

In particular: $g \in (f) \Longleftrightarrow \Lambda(g) \in (\Lambda(f))$.

$\Lambda(f)$ is in Weierstrass position of some degree $d > 0$ in z_k, by genericity of Λ.

Then $g \in (f) \Longleftrightarrow \Lambda(g) \operatorname{rem} \Lambda(f) = 0$.

Very special case of divisibility

Question: if $f, g \in \mathbb{L}_k$ and $\mathscr{F} = \{f\}$, can we check whether $g \in (\mathscr{F})$?

Consider generic linear forms $\lambda_1, \ldots, \lambda_k \in \mathbb{L}_k$ with $\lambda_i = \lambda_{i,1} z_1 + \cdots + \lambda_{i,k} z_k$.
Then we have a computable isomorphism

$$\begin{aligned} \Lambda \colon \mathbb{L}_k &\longrightarrow \mathbb{L}_k \\ \varphi &\longmapsto \varphi \circ (\lambda_1, \ldots, \lambda_k). \end{aligned}$$

In particular: $g \in (f) \Longleftrightarrow \Lambda(g) \in (\Lambda(f))$.

$\Lambda(f)$ is in Weierstrass position of some degree $d > 0$ in z_k, by genericity of Λ.
Then $g \in (f) \Longleftrightarrow \Lambda(g) \operatorname{rem} \Lambda(f) = 0$.

Let $\alpha_1, \ldots, \alpha_d$ be the roots of $\Lambda(f)$.
Then $g \in (f) \Longleftrightarrow \Lambda(g)(\alpha_i) = 0$ for $i = 1, \ldots, d$.

The bivariate example revisited

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$\Lambda(f_1) = z_1 z_2 + z_2^2$$
$$\Lambda(f_2) = z_1 z_2 + z_2^2 - z_1^3 - 3 z_1^2 z_2 - 3 z_1 z_2^2 - 2 z_2^3 + z_1^2 z_2^2 + 2 z_1 z_2^3 + z_2^4$$

The bivariate example revisited

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$\Lambda(f_1) = z_1 z_2 + z_2^2$$
$$\Lambda(f_2) = z_1 z_2 + z_2^2 - z_1^3 - 3 z_1^2 z_2 - 3 z_1 z_2^2 - 2 z_2^3 + z_1^2 z_2^2 + 2 z_1 z_2^3 + z_2^4$$

$$b_\epsilon := \Lambda(f_1) = z_2^2 + z_1 z_2$$

The bivariate example revisited

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

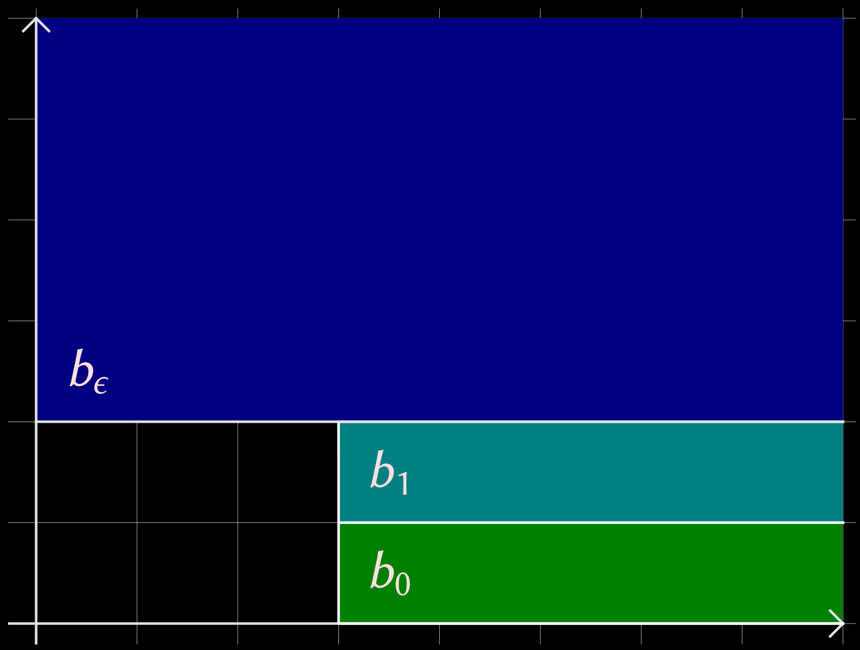
$$\Lambda(f_1) = z_1 z_2 + z_2^2$$
$$\Lambda(f_2) = z_1 z_2 + z_2^2 - z_1^3 - 3 z_1^2 z_2 - 3 z_1 z_2^2 - 2 z_2^3 + z_1^2 z_2^2 + 2 z_1 z_2^3 + z_2^4$$

$$b_\epsilon := \Lambda(f_1) = z_2^2 + z_1 z_2$$
$$b_0 := \Lambda(f_2) \operatorname{rem} b_\epsilon = -z_1^3 - 2 z_1^2 z_2$$

The bivariate example revisited

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$\Lambda(f_1) = z_1 z_2 + z_2^2$$
$$\Lambda(f_2) = z_1 z_2 + z_2^2 - z_1^3 - 3 z_1^2 z_2 - 3 z_1 z_2^2 - 2 z_2^3 + z_1^2 z_2^2 + 2 z_1 z_2^3 + z_2^4$$

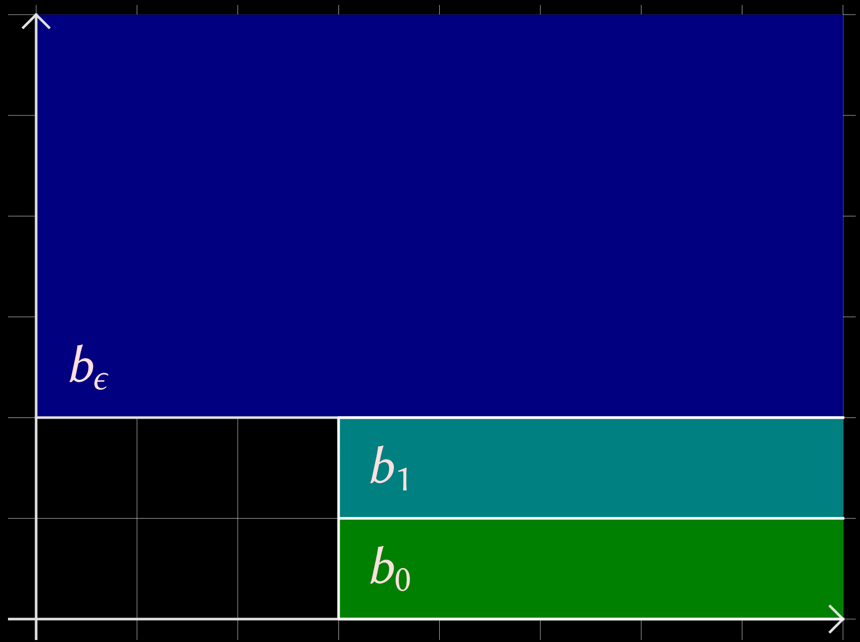


$$b_\epsilon := \Lambda(f_1) = z_2^2 + z_1 z_2$$
$$b_0 := \Lambda(f_2) \operatorname{rem} b_\epsilon = -z_1^3 - 2 z_1^2 z_2$$
$$b_1 := (b_0 z_2) \operatorname{rem} b_\epsilon = -z_1^3 z_2$$

The bivariate example revisited

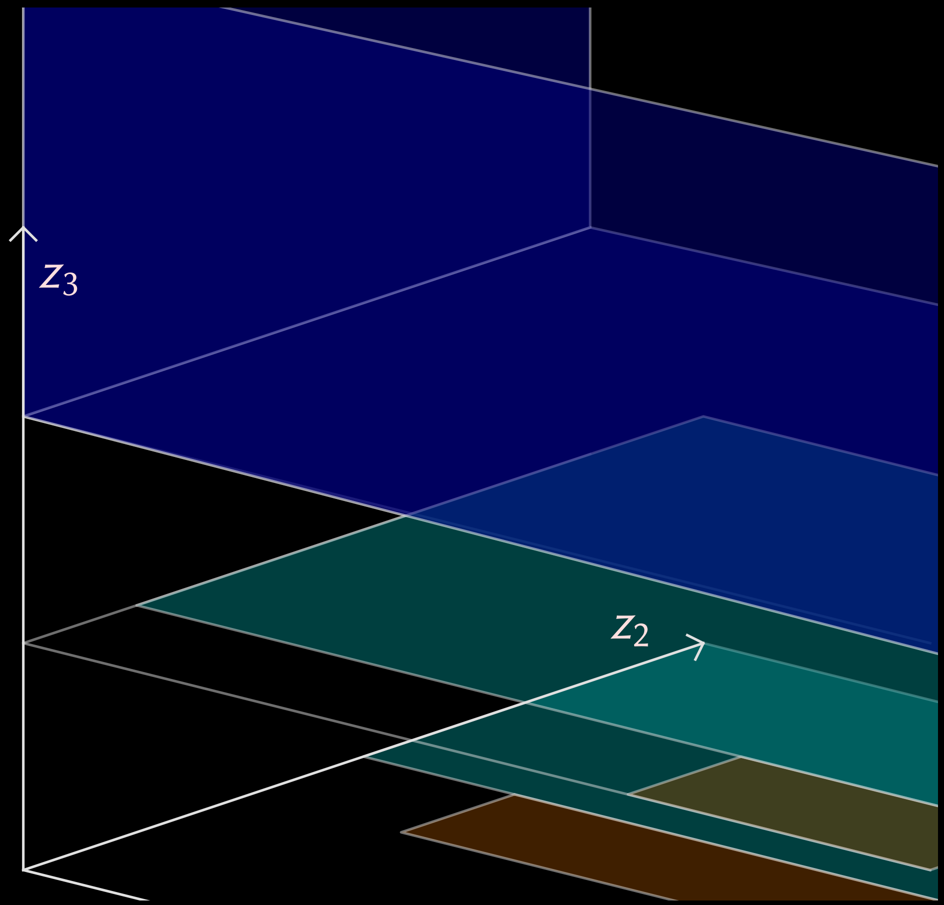
$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$

$$\Lambda(f_1) = z_1 z_2 + z_2^2$$
$$\Lambda(f_2) = z_1 z_2 + z_2^2 - z_1^3 - 3 z_1^2 z_2 - 3 z_1 z_2^2 - 2 z_2^3 + z_1^2 z_2^2 + 2 z_1 z_2^3 + z_2^4$$



$$b_\epsilon := \Lambda(f_1) = z_2^2 + z_1 z_2$$
$$b_0 := \Lambda(f_2) \text{ rem } b_\epsilon = -z_1^3 - 2 z_1^2 z_2$$
$$b_1 := (b_0 z_2) \text{ rem } b_\epsilon = -z_1^3 z_2$$
$$(b_1 z_2) \text{ rem } b_\epsilon = -z_1^4 z_2$$
$$(-z_1^4 z_2) \text{ rem } b_1 = 0$$

Higher dimensions



$$\mathscr{B} = \{b_\epsilon, b_1, b_0, b_{1,0}, b_{0,2}, b_{0,1}\}$$

$$\begin{aligned}
b_\epsilon &= z_3^2 + \cdots & (z_1, z_2, z_3) \\
b_1 &= z_2 z_3 + \cdots & (z_1, z_2) \\
b_0 &= z_2^3 + \cdots & (z_1, z_2) \\
b_{1,0} &= z_1^4 z_3 + \cdots & (z_1) \\
b_{0,2} &= z_1 z_2^2 + \cdots & (z_1) \\
b_{0,1} &= z_1^3 z_2 + \cdots & (z_1)
\end{aligned}$$

$$b_1 z_3, b_0 z_3, b_{1,0} z_3, b_{0,2} z_3, b_{0,1} z_3 \xrightarrow{\mathscr{B}} 0$$

$$b_{1,0} z_2, b_{0,2} z_2, b_{0,1} z_2 \xrightarrow{\mathscr{B}} 0$$

Standard bases

Hilbert function $H_{(\mathscr{F})}$**.** Just count boxes below staircase, as usual!

Standard bases

Hilbert function $H_{(\mathscr{F})}$. Just count boxes below staircase, as usual!

Wanted: standard basis for a graded monomial ordering.
Let $\mathfrak{m} := (z_1, \ldots, z_k)$ be the maximal ideal.

Standard bases

Hilbert function $H_{(\mathscr{F})}$. Just count boxes below staircase, as usual!

Wanted: standard basis for a graded monomial ordering.
Let $\mathfrak{m} := (z_1, \ldots, z_k)$ be the maximal ideal.

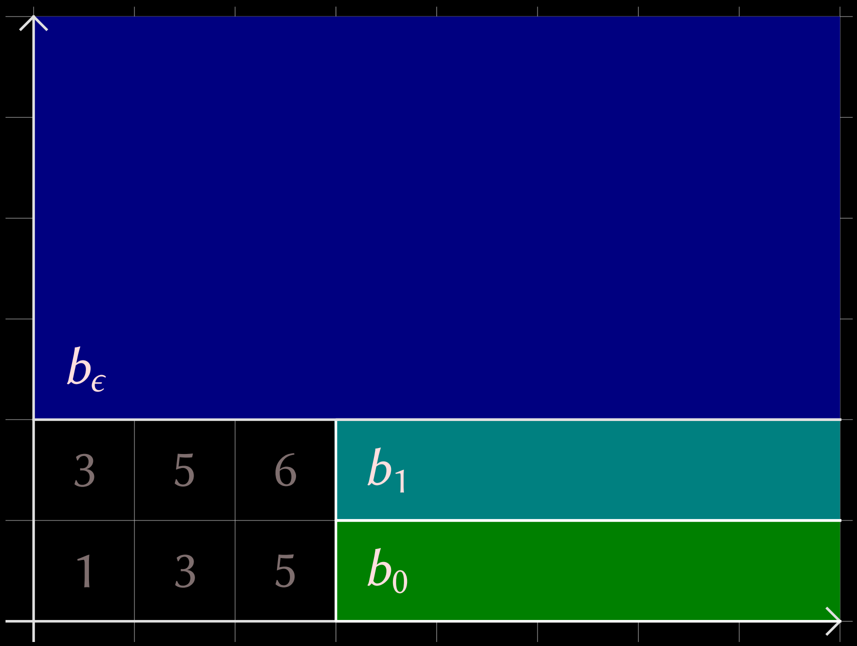
For $\nu := 0, 1, 2, 3, \ldots$ do:

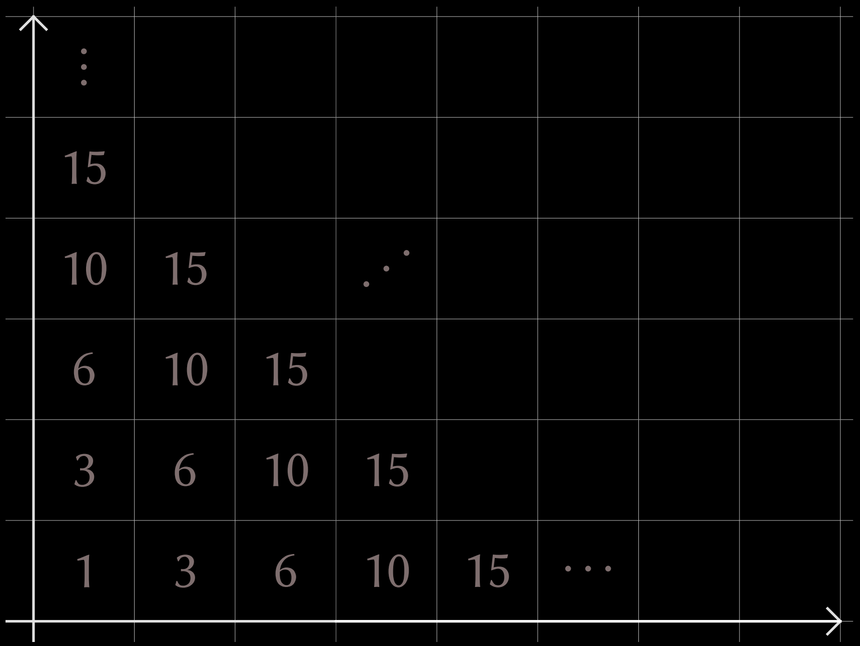
- Compute $B_\nu \subseteq (\mathscr{F})$ such that B_ν is a standard basis for $(\mathscr{F})$ modulo $\mathfrak{m}^{\nu+1}$
- Compute the Hilbert function $H_\nu := H_{(\mathrm{In}(B_\nu))}$ for the initial ideal $(\mathrm{In}(B_\nu))$
- If $H_\nu = H_{(\mathscr{F})}$, then B_ν is the desired standard basis for $(\mathscr{F})$

Example

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$
$$H_{(\mathscr{F})} = (1, 3, 5, 6, 6, 6, \ldots)$$

$$v = 0, 1$$
$$B_v = \emptyset$$
$$H_{(\ln(B_v))} = (1, 3, 6, 10, 15, \ldots)$$

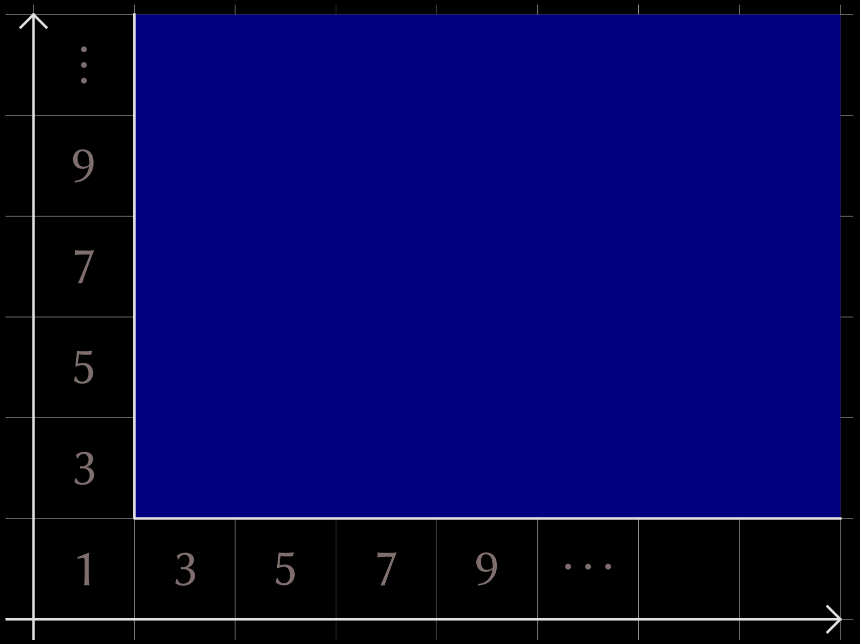




Example

$$
\begin{aligned}
f_1 &= z_1 z_2 \\
f_2 &= z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2 \\
H_{(\mathscr{F})} &= (1, 3, 5, 6, 6, 6, \ldots)
\end{aligned}
$$

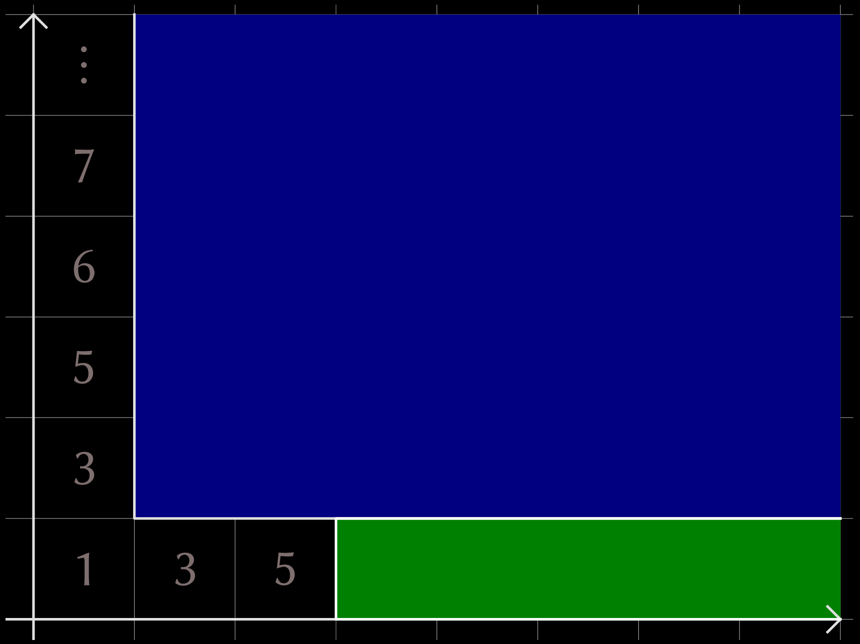
$$
\begin{aligned}
v &= 2 \\
B_2 &= \{z_1 z_2\} \\
H_{(\mathrm{In}(B_2))} &= (1, 3, 5, 7, 9, 11, \ldots)
\end{aligned}
$$



Example

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$
$$H_{(\mathscr{F})} = (1, 3, 5, 6, 6, 6, \dots)$$

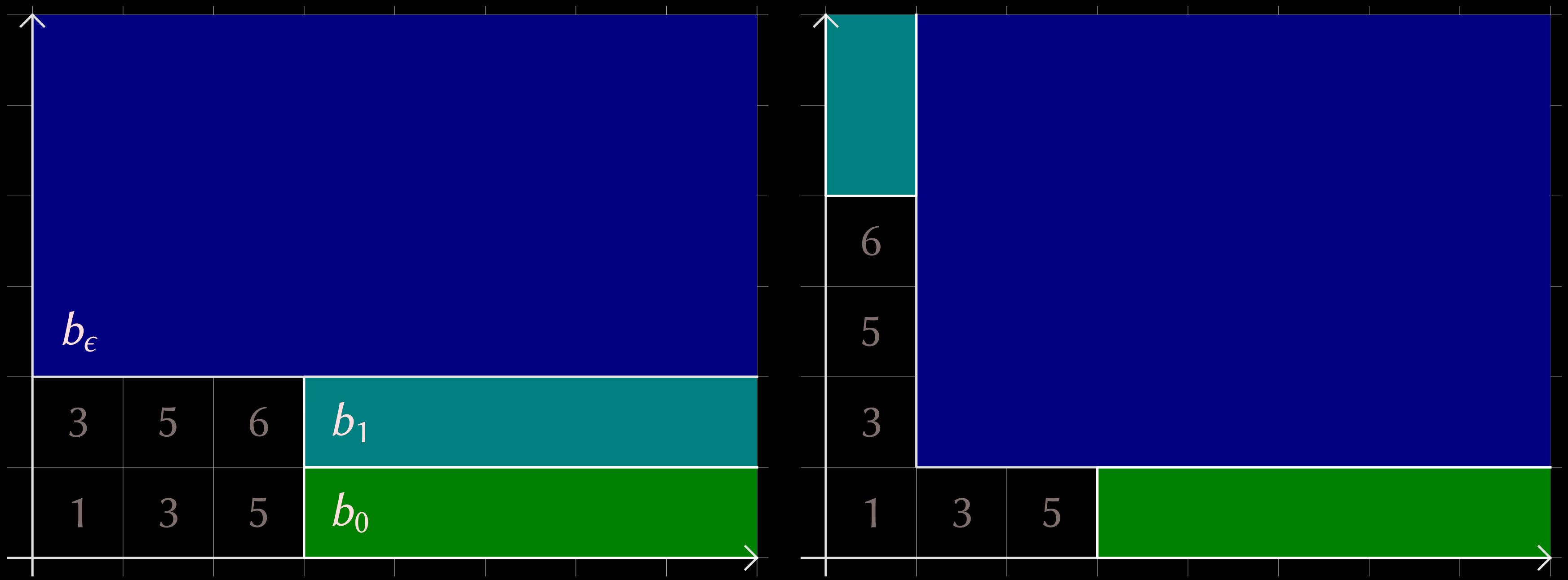
$$v = 3$$
$$B_3 = \{z_1 z_2,\ z_1^3 + z_2^3\}$$
$$H_{(\mathrm{In}(B_3))} = (1, 3, 5, 6, 7, 8, \dots)$$



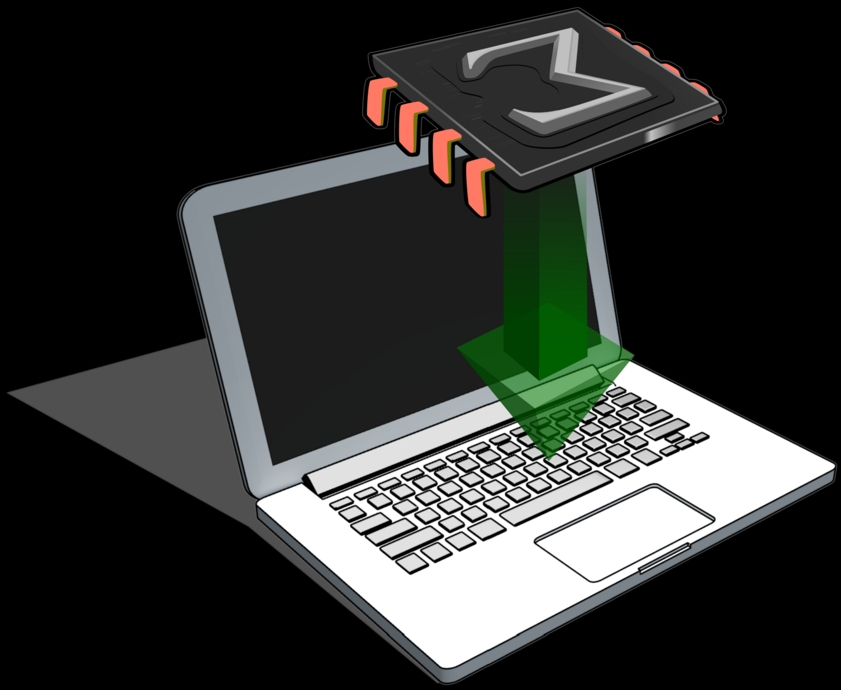
Example

$$f_1 = z_1 z_2$$
$$f_2 = z_1 z_2 - z_1^3 - z_2^3 + z_1^2 z_2^2$$
$$H_{(\mathscr{F})} = (1, 3, 5, 6, 6, 6, \ldots)$$

$$\nu = 4$$
$$B_4 = \{z_1 z_2,\ z_1^3 + z_2^3,\ z_2^4\}$$
$$H_{(\mathrm{In}(B_4))} = (1, 3, 5, 6, 6, 6, \ldots)$$



Thank you !



http://www.TeXmacs.org