Yet another differential shape lemma

Joris van der Hoeven

Joint work with Gleb Pogudin

CNRS, École polytechnique, France

Funded by
the European Union

European Research Council
Established by the European Commission

Online Kolchin seminar

November 7, 2025

Classical shape lemma, general position

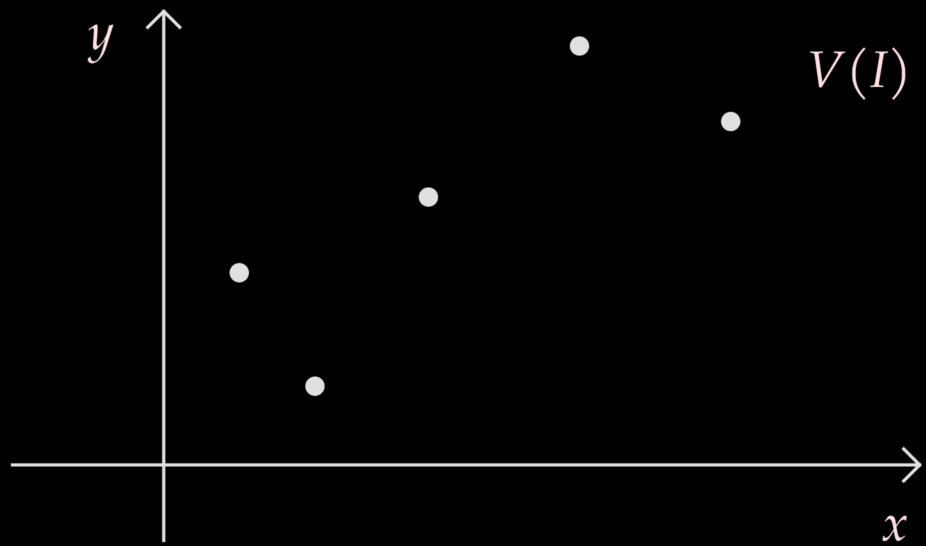
$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x, y]$: radical zero-dimensional ideal

Classical shape lemma, general position

$\mathbb{k}$: field of characteristic zero

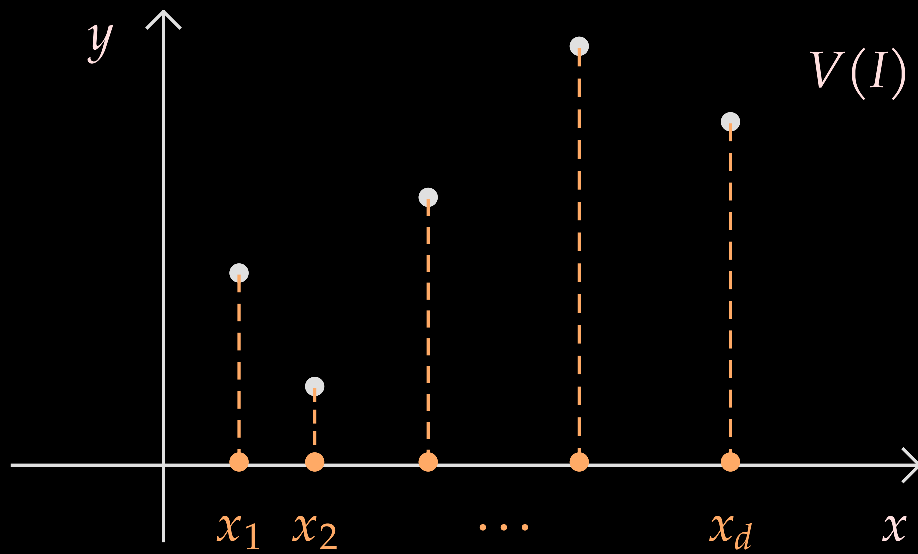
$I \subseteq \mathbb{k}[x,y]$: radical zero-dimensional ideal



Classical shape lemma, general position

$\mathbb{k}$: field of characteristic zero

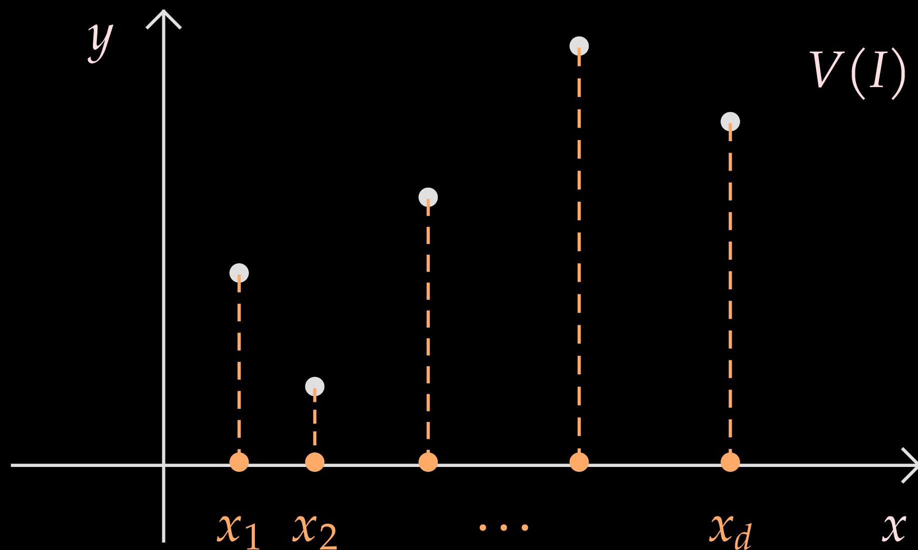
$I \subseteq \mathbb{k}[x, y]$: radical zero-dimensional ideal



Classical shape lemma, general position

$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x,y]$: radical zero-dimensional ideal

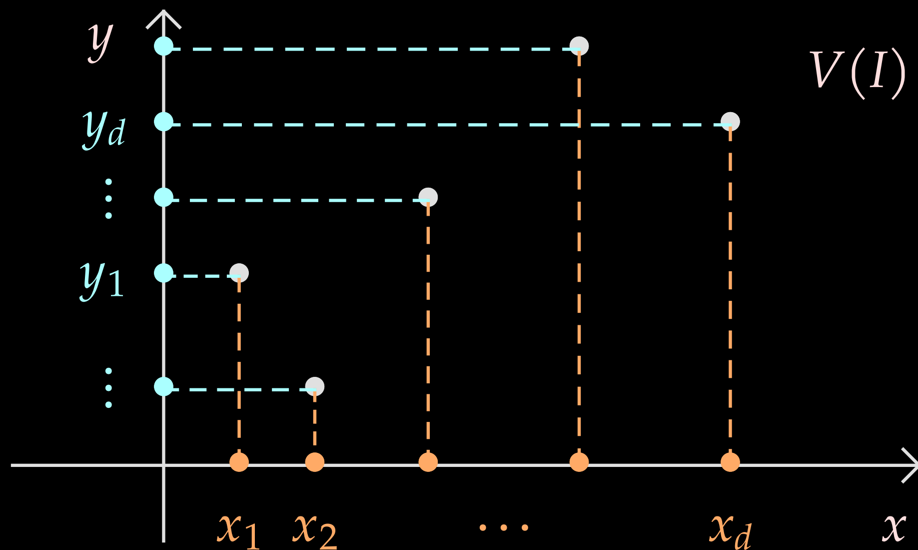


$P(x) = (x - x_1)\,(x - x_2) \cdots (x - x_d) \in I$

Classical shape lemma, general position

$\mathbb{k}$: field of characteristic zero

$I \subseteq \mathbb{k}[x,y]$: radical zero-dimensional ideal

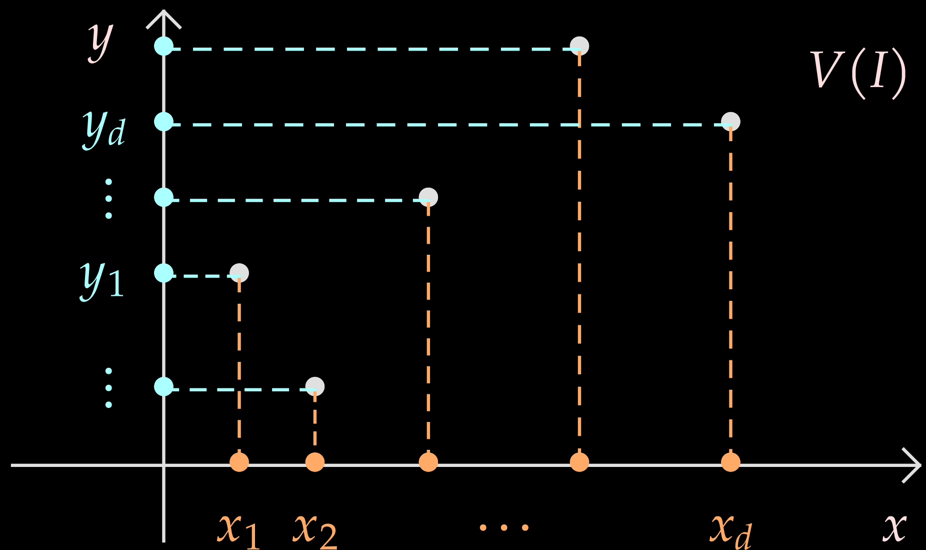


$$P(x) = (x - x_1)\,(x - x_2)\cdots(x - x_d) \in I$$

Classical shape lemma, general position

$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x,y]$: radical zero-dimensional ideal



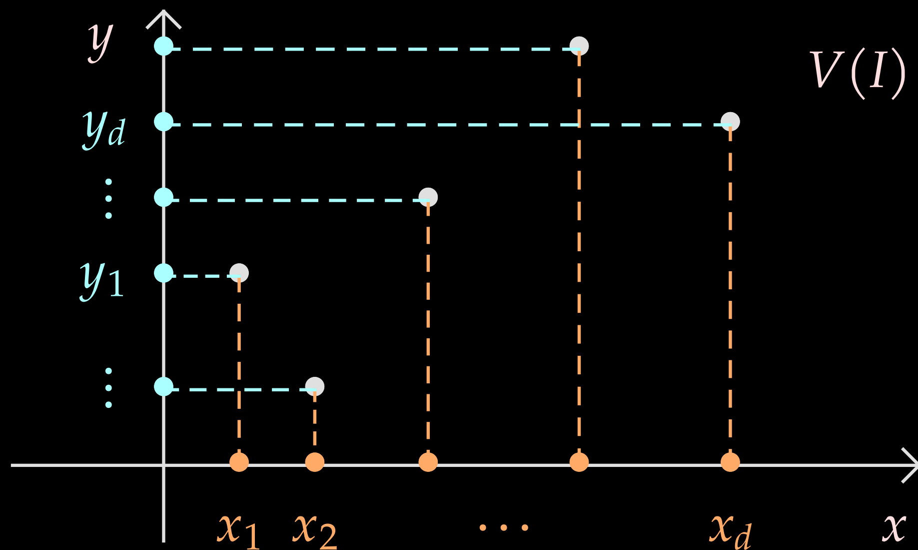
$P(x) = (x - x_1)\,(x - x_2) \cdots (x - x_d) \in I$

$y - Q(x) \in I$, where $y_1 = Q(x_1), \ldots, y_d = Q(x_d)$ and $\deg Q < d$

Classical shape lemma, general position

$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x,y]$: radical zero-dimensional ideal



$$I = (P(x), y - Q(x))$$

$$P(x) = (x - x_1)(x - x_2)\cdots(x - x_d), \qquad y_1 = Q(x_1), \ldots, y_d = Q(x_d)$$

Classical shape lemma, degenerate case

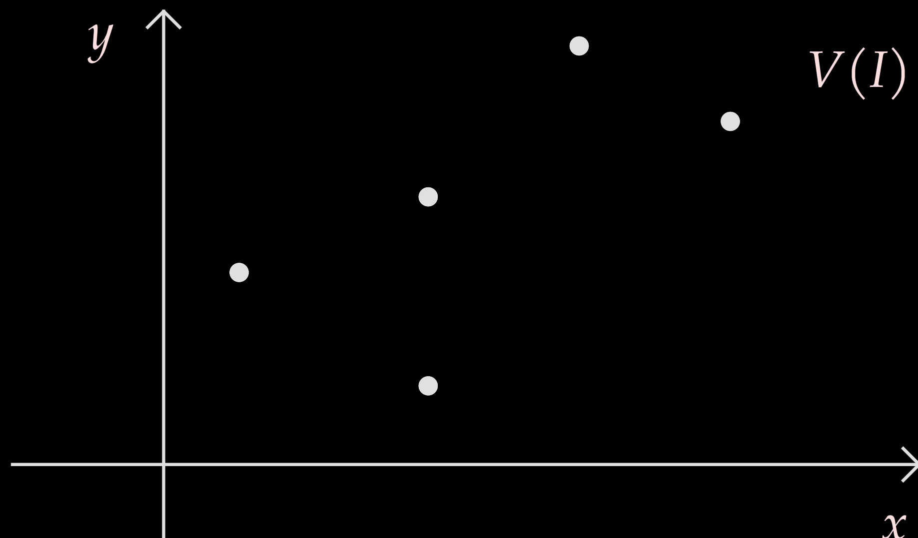
$\mathbb{k}$: field of characteristic zero

$I \subseteq \mathbb{k}[x,y]$: radical zero-dimensional ideal

Classical shape lemma, degenerate case

$\Bbbk$: field of characteristic zero

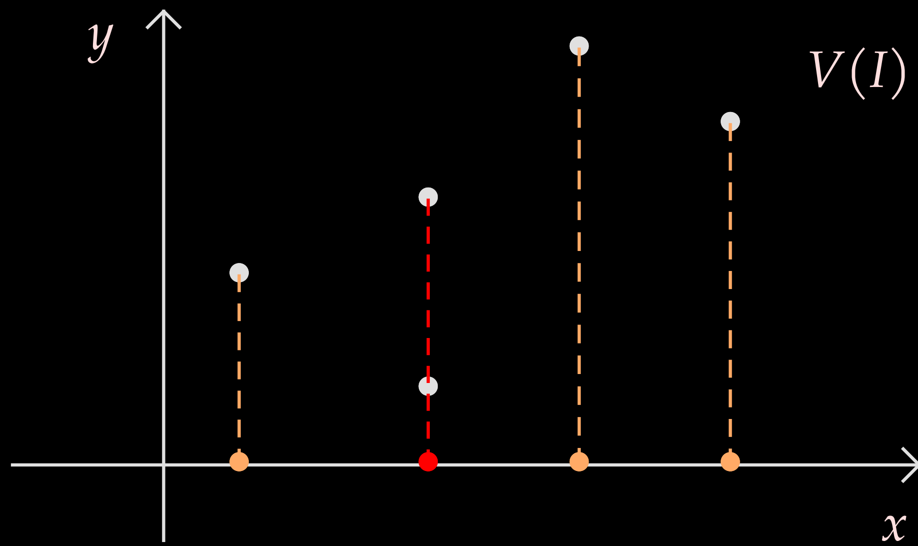
$I \subseteq \Bbbk[x,y]$: radical zero-dimensional ideal



Classical shape lemma, degenerate case

$\mathbb{k}$: field of characteristic zero

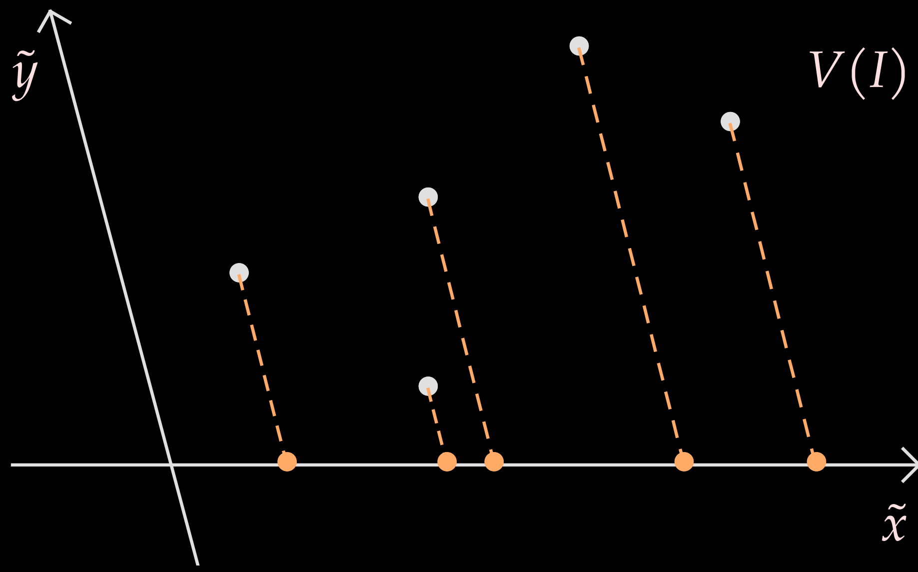
$I \subseteq \mathbb{k}[x,y]$: radical zero-dimensional ideal



Classical shape lemma, degenerate case

$\mathbb{k}$: field of characteristic zero

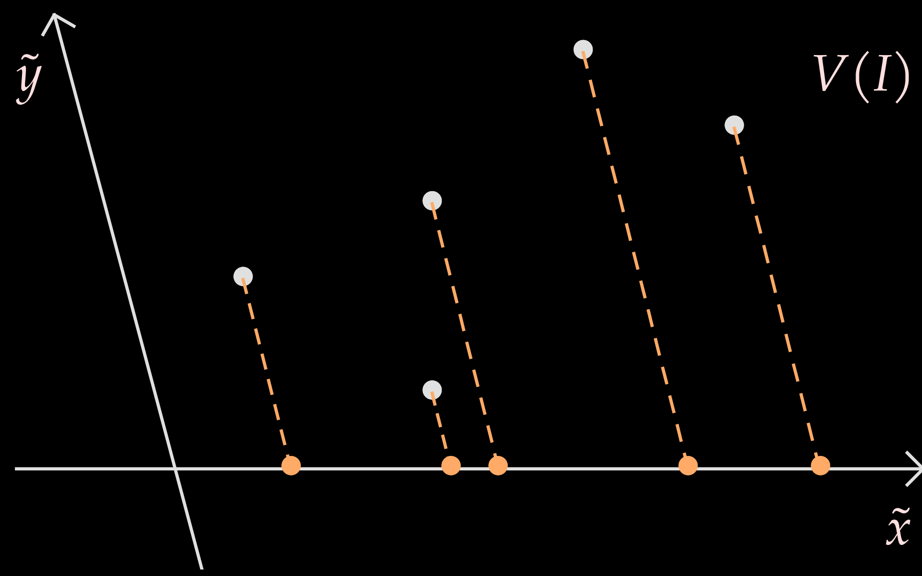
$I \subseteq \mathbb{k}[x,y]$: radical zero-dimensional ideal



Classical shape lemma, degenerate case

$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x,y]$: radical zero-dimensional ideal

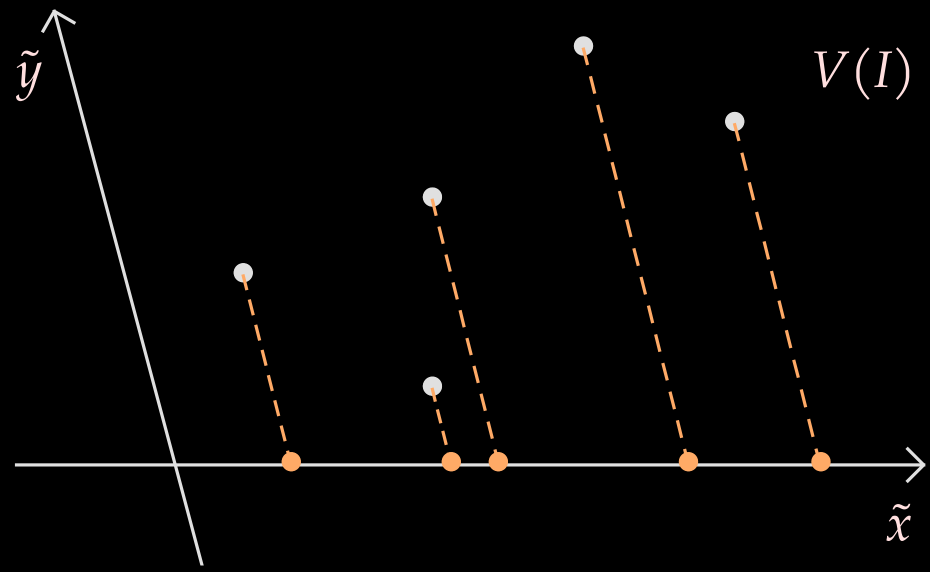


$\tilde{x} = x + cy, \quad \tilde{y} = y$

Classical shape lemma, degenerate case

$\mathbb{k}$: field of characteristic zero

$I \subseteq \mathbb{k}[x,y]$: radical zero-dimensional ideal



$\tilde{x} = x + cy, \quad \tilde{y} = y$

Classical shape lemma $\longrightarrow$ Gianni–Mora 1989

General zero-dimensional ideals

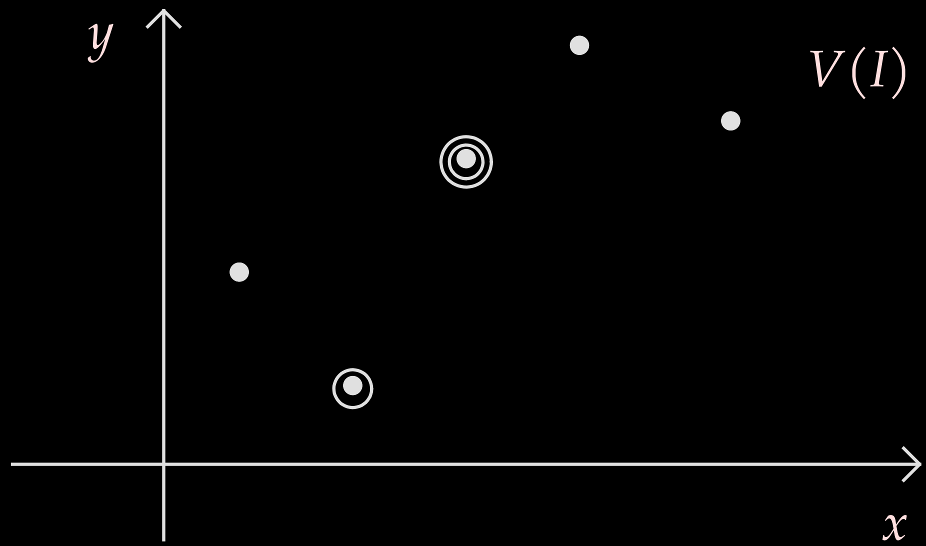
$\mathbb{k}$: field of characteristic zero

$I \subseteq \mathbb{k}[x, y]$: ~~radical~~ zero-dimensional ideal

General zero-dimensional ideals

$\Bbbk$: field of characteristic zero

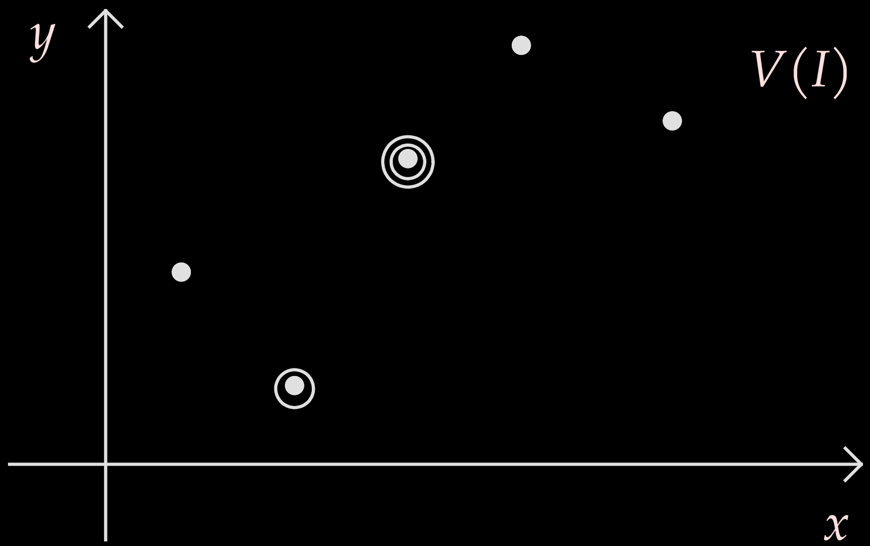
$I \subseteq \Bbbk[x,y]$: ~~radical~~ zero-dimensional ideal

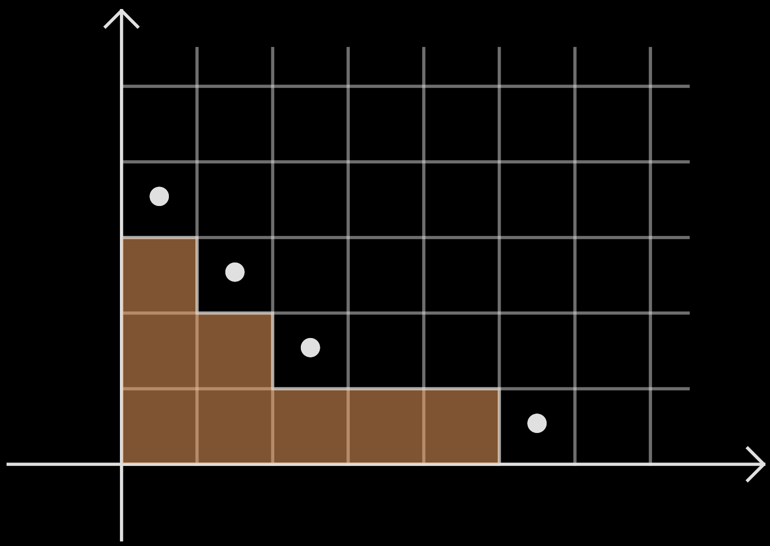


General zero-dimensional ideals

$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x,y]$: ~~radical~~ zero-dimensional ideal

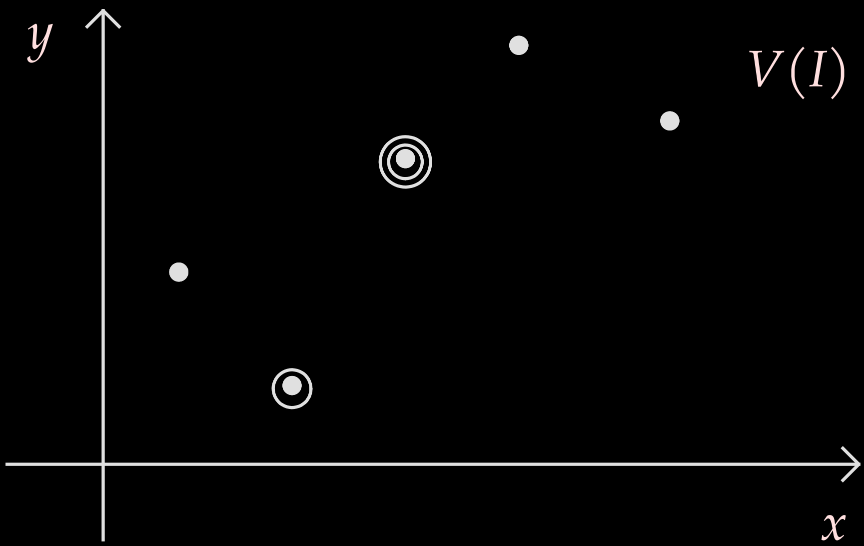


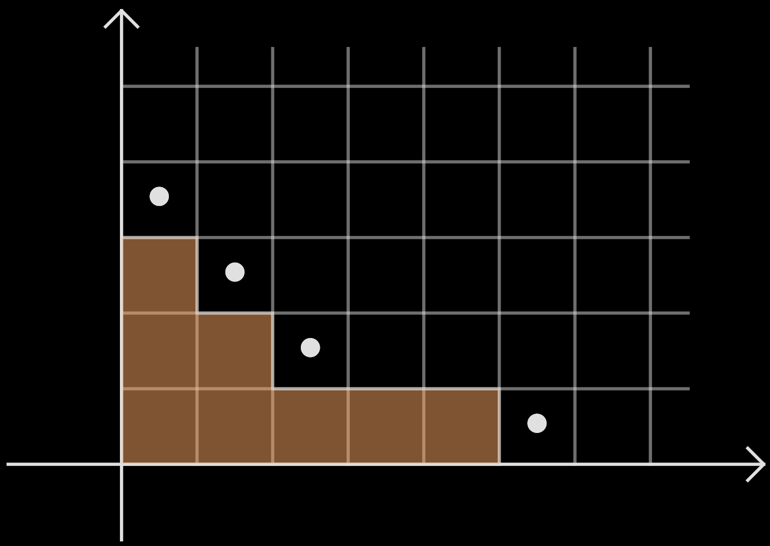


General zero-dimensional ideals

$\Bbbk$: field of characteristic zero

$I \subseteq \Bbbk[x,y]$: ~~radical~~ zero-dimensional ideal





Shape of lexicographical Gröbner basis ⟶ Lazard 1985

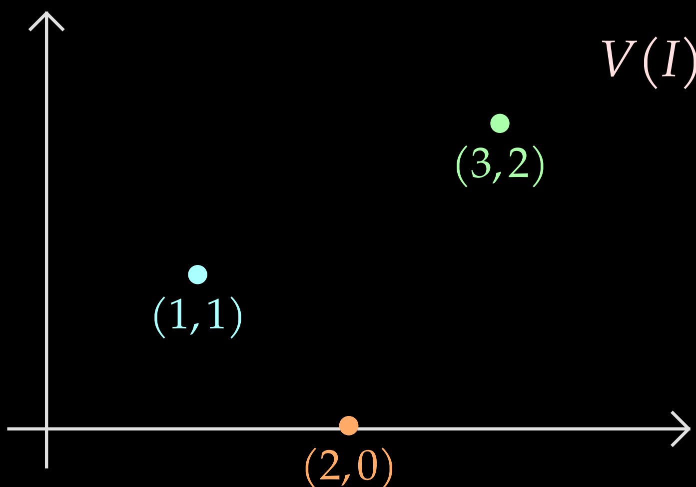
$$\mathbb{k}[x,y] \qquad \cong \qquad \mathbb{k}[D_x, D_y]$$

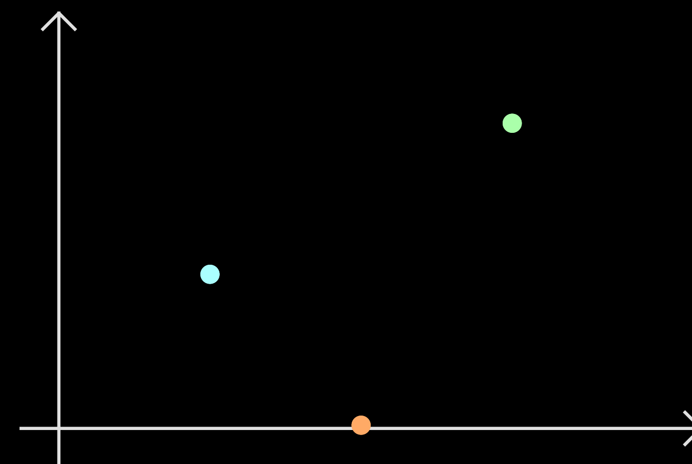
Linear partial differential equations

$$\begin{array}{ccc} \mathbb{k}[x,y] & \cong & \mathbb{k}[D_x, D_y] \\ \cup\!\mid & & \cup\!\mid \\ I & & I \end{array}$$

Linear partial differential equations

$$\begin{array}{ccc} \Bbbk[x,y] & \cong & \Bbbk[D_x,D_y] \\ \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} \\ I & & I \end{array}$$





$$V(I) = \Bbbk\, e^{x+y} \oplus \Bbbk\, e^{2x} \oplus \Bbbk\, e^{3x+2y}$$

The shape lemma revisited

Lemma

Let $I \in \mathbb{k}[x,y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \mathbb{k}$ *and* $P, Q \in \mathbb{k}[\tilde{x}, \tilde{y}]$*, where* $\tilde{x} = x + c\,y$*,* $\tilde{y} = y$*, such that* P *is monic,* $\deg Q < \deg P$*, and* $I = (P, \tilde{y} - Q)$*.*

The shape lemma revisited

Lemma

Let $I \in \mathbb{k}[x,y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \mathbb{k}$ *and* $P, Q \in \mathbb{k}[\tilde{x}, \tilde{y}]$, *where* $\tilde{x} = x + c\,y$, $\tilde{y} = y$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, \tilde{y} - Q)$.

Corollary

Let $I \in \mathbb{k}[D_x, D_y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \mathbb{k}$ *and* $P, Q \in \mathbb{k}[D_{\tilde{x}}, D_{\tilde{y}}]$, *where* $\tilde{x} = x$, $\tilde{y} = y - c\,x$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, D_{\tilde{y}} - Q)$.

The shape lemma revisited

Lemma

Let $I \in \Bbbk[x,y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \Bbbk$ *and* $P, Q \in \Bbbk[\tilde{x},\tilde{y}]$, *where* $\tilde{x} = x + c\,y$, $\tilde{y} = y$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, \tilde{y} - Q)$.

Corollary

Let $I \in \Bbbk[D_x, D_y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \Bbbk$ *and* $P, Q \in \Bbbk[D_{\tilde{x}}, D_{\tilde{y}}]$, *where* $\tilde{x} = x$, $\tilde{y} = y - c\,x$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, D_{\tilde{y}} - Q)$.

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} D_{\tilde{x}} \\ D_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

The shape lemma revisited

Lemma

Let $I \in \Bbbk[x,y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \Bbbk$ *and* $P, Q \in \Bbbk[\tilde{x}, \tilde{y}]$, *where* $\tilde{x} = x + c\,y$, $\tilde{y} = y$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, \tilde{y} - Q)$.

Corollary

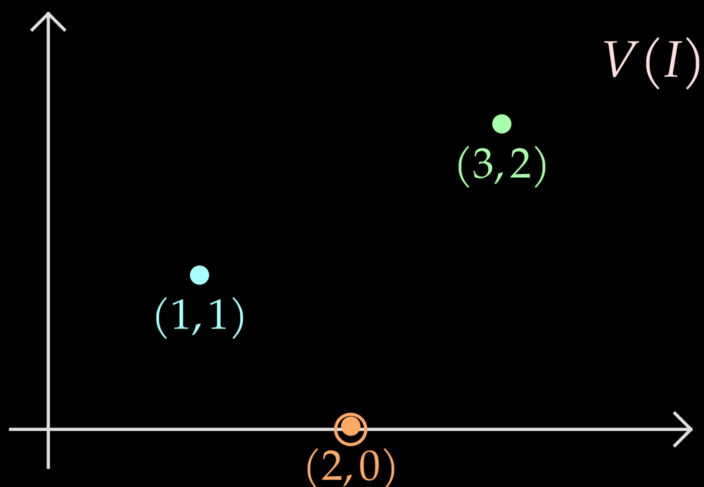
Let $I \in \Bbbk[D_x, D_y]$ *be a radical zero-dimensional ideal. Then there exist* $c \in \Bbbk$ *and* $P, Q \in \Bbbk[D_{\tilde{x}}, D_{\tilde{y}}]$, *where* $\tilde{x} = x$, $\tilde{y} = y - c\,x$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, D_{\tilde{y}} - Q)$.

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} D_{\tilde{x}} \\ D_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

Question: what about left-ideals of $\Bbbk(x,y)[D_x, D_y]$ instead of $\Bbbk[D_x, D_y]$?

$$\mathbb{k}[x,y] \quad \cong \quad \mathbb{k}[D_x, D_y]$$

$$I = ((x-1)(x-2)^2(x-3),\ y - {}^3/_2 x^2 + {}^{11}/_2 x - 5)$$

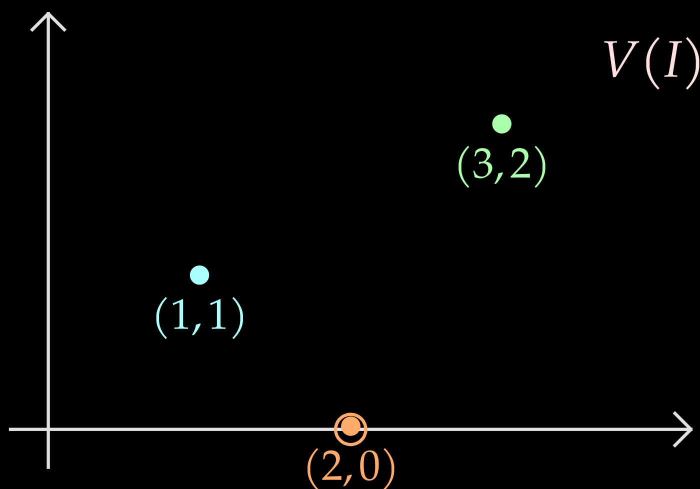


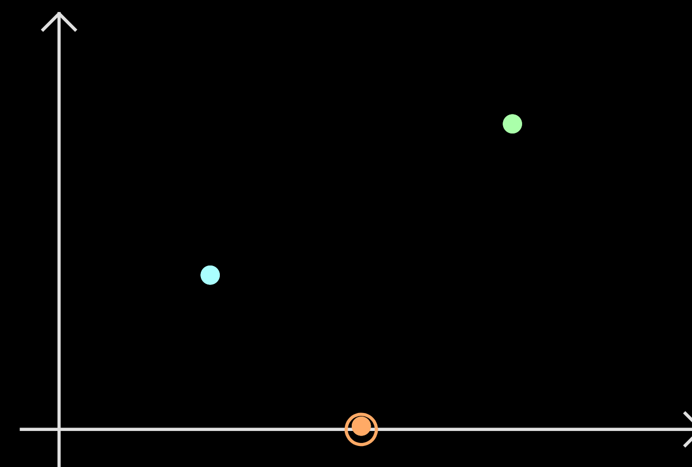
Differential counterpart of “radical ideal”

$$\mathbb{k}[x,y] \quad \cong \quad \mathbb{k}[D_x,D_y]$$

$$I = \left((x-1)\,(x-2)^2\,(x-3),\; y - {}^3\!/_2\, x^2 + {}^{11}\!/_2\, x - 5\right)$$

$$I = \left((D_x-1)\,(D_x-2)^2\,(D_x-3),\; D_y - {}^3\!/_2\, D_x^2 + {}^{11}\!/_2\, D_x - 5\right)$$





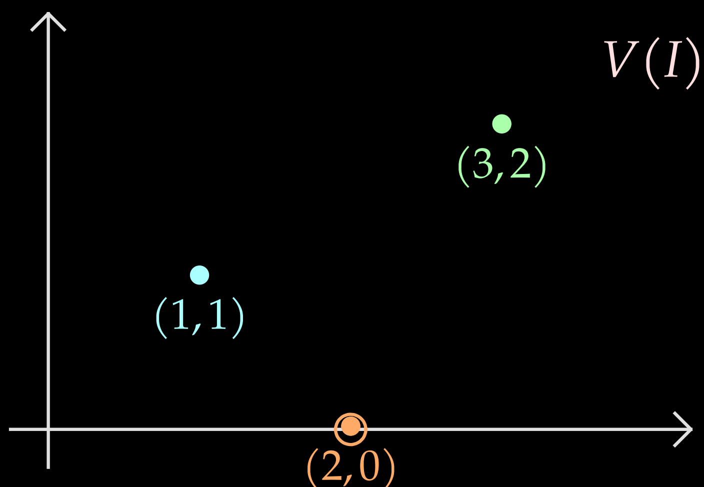
$$\mathbb{k}\,\mathrm{e}^{x+y} \oplus \mathbb{k}\,\mathrm{e}^{2x} \oplus \mathbb{k}\,(2x+y)\,\mathrm{e}^{2x} \oplus \mathbb{k}\,\mathrm{e}^{3x+2y}$$

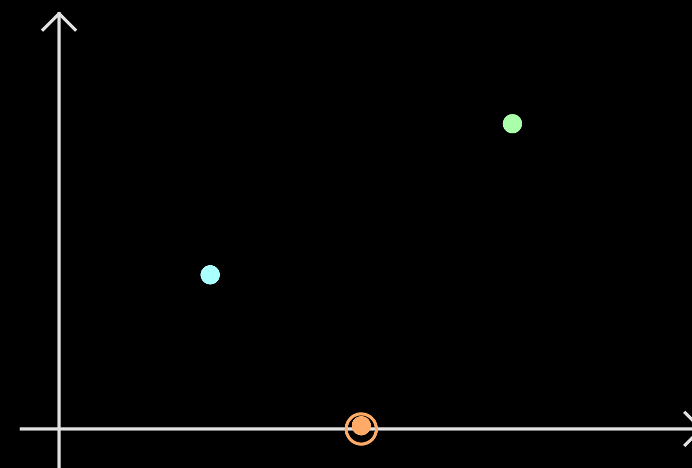
Differential counterpart of “radical ideal”

$$\mathbb{k}[x,y] \cong \mathbb{k}[D_x,D_y]$$

$$I = \big((x-1)(x-2)^2(x-3),\; y - {}^3\!/_2\, x^2 + {}^{11}\!/_2\, x - 5\big)$$

$$I = \big((D_x-1)(D_x-2)^2(D_x-3),\; D_y - {}^3\!/_2\, D_x^2 + {}^{11}\!/_2\, D_x - 5\big)$$





$$\mathbb{k}\,e^{x+y} \oplus \mathbb{k}\,e^{2x} \oplus \mathbb{k}\,(2x+y)\,e^{2x} \oplus \mathbb{k}\,e^{3x+2y}$$

$\{e^{x+y}, e^{2x}, (2x+y)\,e^{2x}, e^{3x+2y}\}$ basis over $\mathbb{k}$ but linearly dependent over $\mathbb{k}(x,y)$.

First differential shape lemma

Definition

We say that a zero-dimensional left ideal $I \subseteq \mathbb{k}(x,y)[D_x, D_y]$ is ***D-radical*** *if its solution space has a $\mathbb{k}$-basis of $\mathbb{k}(x,y)$-linearly independent elements.*

First differential shape lemma

Definition

We say that a zero-dimensional left ideal $I \subseteq \mathbb{k}(x,y)[D_x,D_y]$ *is* ***D-radical*** *if its solution space has a* $\mathbb{k}$*-basis of* $\mathbb{k}(x,y)$*-linearly independent elements.*

Theorem (Kauers–Koutchan–Verron, 2025)

Let $I \in \mathbb{k}(x,y)[D_x,D_y]$ *be a zero-dimensional D-radical ideal. Then there exist* $c \in \mathbb{k}$ *and* $P,Q \in \mathbb{k}(\tilde{x},\tilde{y})[D_{\tilde{x}},D_{\tilde{y}}]$, *where* $\tilde{x} = x$, $\tilde{y} = y - cx$, *such that* P *is monic,* $\deg Q < \deg P$, *and* $I = (P, D_{\tilde{y}} - Q)$.

But wait a minute...

Let $\mathbb{K} := \Bbbk(x, y)$ and let $I \subseteq \mathbb{K}[D_x, D_y]$ be a left ideal.

But wait a minute…

Let $\mathbb{K} := \Bbbk(x,y)$ and let $I \subseteq \mathbb{K}[D_x, D_y]$ be a left ideal.

- The differential ideal $\{LF : L \in I\} \subseteq \mathbb{K}\{F\} = \mathbb{K}[F, D_x F, \dots]$ is **always** radical.

But wait a minute…

Let $\mathbb{K} := \Bbbk(x,y)$ and let $I \subseteq \mathbb{K}[D_x, D_y]$ be a left ideal.

- The differential ideal $\{LF : L \in I\} \subseteq \mathbb{K}\{F\} = \mathbb{K}[F, D_x F, \ldots]$ is **always** radical.
- Why introduce the artificial problem of D-radical ideals ?

But wait a minute…

Let $\mathbb{K} := \Bbbk(x, y)$ and let $I \subseteq \mathbb{K}[D_x, D_y]$ be a left ideal.

- The differential ideal $\{LF : L \in I\} \subseteq \mathbb{K}\{F\} = \mathbb{K}[F, D_x F, \ldots]$ is **always** radical.
- Why introduce the artificial problem of D-radical ideals ?
- Solutions sets of $\Bbbk[D_x, D_y]$ are more "fine-grained" than those of $\Bbbk[x, y]$

But wait a minute…

Let $\mathbb{K} := \Bbbk(x,y)$ and let $I \subseteq \mathbb{K}[D_x, D_y]$ be a left ideal.

- The differential ideal $\{LF : L \in I\} \subseteq \mathbb{K}\{F\} = \mathbb{K}[F, D_x F, \dots]$ is **always** radical.
- Why introduce the artificial problem of D-radical ideals ?
- Solutions sets of $\Bbbk[D_x, D_y]$ are more “fine-grained” than those of $\Bbbk[x,y]$

Double point	Linear PDE counterpart
$I_1 = (x^2, y) \quad V(I_1) = \{(0,0)\}$	$I_1 = (D_x^2, D_y) \quad V(I_1) = \Bbbk \oplus \Bbbk\, x$
$I_2 = (x, y^2) \quad V(I_2) = \{(0,0)\}$	$I_2 = (D_x, D_y^2) \quad V(I_2) = \Bbbk \oplus \Bbbk\, y$

Linear differential Nullstellensatz

$$\mathscr{F} := \bigoplus_{\alpha,\beta \in \mathbb{k}} \mathbb{k}[x,y]\mathrm{e}^{\alpha x + \beta y}$$

Linear differential Nullstellensatz

$$\mathscr{F} \;:=\; \bigoplus_{\alpha,\beta\in\mathbb{k}} \mathbb{k}[x,y]\,\mathrm{e}^{\alpha x+\beta y}$$

$$\mathscr{I} \;:=\; \text{set of ideals of } \mathbb{k}[D_x, D_y]$$

$$\mathscr{Z} \;:=\; \{\{f\in\mathscr{F} : \forall L\in I, Lf=0\} : I\in\mathscr{I}\}$$

Linear differential Nullstellensatz

$$\mathscr{F} := \bigoplus_{\alpha,\beta\in\mathbb{k}} \mathbb{k}[x,y]\mathrm{e}^{\alpha x+\beta y}$$

$$\mathscr{I} := \text{ set of ideals of } \mathbb{k}[D_x,D_y]$$

$$\mathscr{Z} := \{\{f\in\mathscr{F}:\forall L\in I, Lf=0\} : I\in\mathscr{I}\}$$

Theorem (folklore, vdH 2007)

The following maps are mutual inverses:

$$I \in \mathscr{I} \longmapsto Z_L := \{f\in\mathscr{F}:\forall L\in I, Lf=0\} \in \mathscr{Z}$$

$$Z \in \mathscr{Z} \longmapsto I_Z := \{L\in\mathbb{k}[D_x,D_y]:\forall f\in Z, Lf=0\} \in \mathscr{I}$$

Second differential shape lemma

$\mathbb{K} = \Bbbk(x, y)$

D_x, D_y derivations on $\mathbb{K}$ with $\begin{pmatrix} D_x \\ D_y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Second differential shape lemma

$\mathbb{K} = \Bbbk(x,y)$

D_x, D_y derivations on $\mathbb{K}$ with $\begin{pmatrix} D_x \\ D_y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I \subseteq \mathbb{K}[D_x, D_y]$ zero-dimensional left ideal

Second differential shape lemma

$\mathbb{K} = \Bbbk(x, y)$

D_x, D_y derivations on $\mathbb{K}$ with $\begin{pmatrix} D_x \\ D_y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I \subseteq \mathbb{K}[D_x, D_y]$ zero-dimensional left ideal

In other words, $\mathbb{A} := \mathbb{K}[D_x, D_y]/I$ has finite dimension over $\mathbb{K}$

Second differential shape lemma

$\mathbb{K} = \mathbb{k}(x, y)$

D_x, D_y derivations on $\mathbb{K}$ with $\begin{pmatrix} D_x \\ D_y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I \subseteq \mathbb{K}[D_x, D_y]$ zero-dimensional left ideal

In other words, $\mathbb{A} := \mathbb{K}[D_x, D_y] / I$ has finite dimension over $\mathbb{K}$

Theorem (vdH–Pogudin, 2025)

There exists a polynomial $q(y) \in \mathbb{Q}[y]$ with $\mathbb{A} = \mathbb{K}[\bar{D}]$, where $D := q D_x + D_y$.

Second differential shape lemma

$\mathbb{K} = \Bbbk(x,y)$

D_x, D_y derivations on $\mathbb{K}$ with $\begin{pmatrix} D_x \\ D_y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I \subseteq \mathbb{K}[D_x, D_y]$ zero-dimensional left ideal

In other words, $\mathbb{A} := \mathbb{K}[D_x, D_y]/I$ has finite dimension over $\mathbb{K}$

Theorem (vdH–Pogudin, 2025)

There exists a polynomial $q(y) \in \mathbb{Q}[y]$ with $\mathbb{A} = \mathbb{K}[\bar{D}]$, where $D := q\, D_x + D_y$.

Corollary

For certain $P, U, V \in \mathbb{K}[D]$ with P monic and $\deg U, \deg V < \deg P$, we have

$$I \;=\; (P(D),\; D_x - U(D),\; D_y - V(D)).$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$D := y D_x + D_y$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned} D &:= y D_x + D_y \\ D^2 &= y^2 D_x^2 + D_y^2 + 2y D_x D_y + D_x \end{aligned}$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned} D &:= y D_x + D_y \\ D^2 &= y^2 D_x^2 + D_y^2 + 2y D_x D_y + D_x \equiv 2y D_x D_y + D_x \end{aligned}$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned}
D &:= y D_x + D_y \\
D^2 &= y^2 D_x^2 + D_y^2 + 2 y D_x D_y + D_x \equiv 2 y D_x D_y + D_x \\
D^3 &\equiv (y D_x + D_y)(2 y D_x D_y + D_x)
\end{aligned}$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned}
D &:= y D_x + D_y \\
D^2 &= y^2 D_x^2 + D_y^2 + 2y D_x D_y + D_x \equiv 2y D_x D_y + D_x \\
D^3 &\equiv (y D_x + D_y)(2y D_x D_y + D_x) \equiv 3 D_x D_y
\end{aligned}$$

Example

$$I \;=\; (D_x^2, D_y^2) \;\subseteq\; \mathbb{K}[D_x, D_y].$$

$$\begin{aligned}
D &:= y\,D_x + D_y \\
D^2 &= y^2 D_x^2 + D_y^2 + 2\,y\,D_x D_y + D_x \;\equiv\; 2\,y\,D_x D_y + D_x \\
D^3 &\equiv (y\,D_x + D_y)\,(2\,y\,D_x D_y + D_x) \;\equiv\; 3\,D_x D_y \\
D^4 &\equiv (y\,D_x + D_y)\,3\,D_x D_y
\end{aligned}$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned}
D &:= y D_x + D_y \\
D^2 &= y^2 D_x^2 + D_y^2 + 2y D_x D_y + D_x \equiv 2y D_x D_y + D_x \\
D^3 &\equiv (y D_x + D_y)(2y D_x D_y + D_x) \equiv 3 D_x D_y \\
D^4 &\equiv (y D_x + D_y) 3 D_x D_y \equiv 0.
\end{aligned}$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned}
D &:= y D_x + D_y \\
D^2 &= y^2 D_x^2 + D_y^2 + 2y D_x D_y + D_x \equiv 2y D_x D_y + D_x \\
D^3 &\equiv (y D_x + D_y)(2y D_x D_y + D_x) \equiv 3 D_x D_y \\
D^4 &\equiv (y D_x + D_y) 3 D_x D_y \equiv 0.
\end{aligned}$$

$$\begin{aligned}
D_x &\equiv D^2 - \tfrac{2}{3} y D^3 \\
D_y &\equiv D - y D^2 + \tfrac{2}{3} y^2 D^3
\end{aligned}$$

Example

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned}
D &:= y D_x + D_y \\
D^2 &= y^2 D_x^2 + D_y^2 + 2y D_x D_y + D_x \equiv 2y D_x D_y + D_x \\
D^3 &\equiv (y D_x + D_y)(2y D_x D_y + D_x) \equiv 3 D_x D_y \\
D^4 &\equiv (y D_x + D_y) 3 D_x D_y \equiv 0.
\end{aligned}$$

$$\begin{aligned}
D_x &\equiv D^2 - \tfrac{2}{3} y D^3 \\
D_y &\equiv D - y D^2 + \tfrac{2}{3} y^2 D^3
\end{aligned}$$

$$I = \left(D^4,\ D_x + \tfrac{2}{3} y D^3 - D^2,\ D_y - \tfrac{2}{3} y^2 D^3 + y D^2 - D \right)$$

Proof

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Proof

Take $D = q D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z D_x$ for formal $z := z(y)$ (i.e. $z^{(i)} = D_y^i z$, $D_x z^{(i)} = 0$)

Proof

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

Proof

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

Consider

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \; \underset{\textstyle b_{e_0,\ldots,e_{m-1}}}{\overset{\textstyle \bigwedge^{m+1} \mathbb{A}}{\Cup}}$$

Proof

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

Consider

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \;\underbrace{b_{e_0,\ldots,e_{m-1}}}_{\in \bigwedge^{m+1} \mathbb{A}}$$

m minimal $\;\Longrightarrow\;$ $\Omega(p(y)) = 0$ for any polynomial $p(y) \in \mathbb{Q}[y]$

$\Longrightarrow\;$ $\Omega = 0$

$\Longrightarrow\;$ $b_{e_0,\ldots,e_{m-1}} = 0$ for all $e_0,\ldots,e_{m-1}$

Proof

Take $D = q D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z D_x$ for formal $z := z(y)$

Consider

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \; \underset{\displaystyle b_{e_0,\ldots,e_{m-1}}}{\overset{\textstyle \bigwedge^{m+1} \mathbb{A}}{\Cup}}$$

m minimal $\implies$ $\Omega(p(y)) = 0$ for any polynomial $p(y) \in \mathbb{Q}[y]$

$\implies$ $\Omega = 0$

$\implies$ $b_{e_0,\ldots,e_{m-1}} = 0$ for all $e_0, \ldots, e_{m-1}$

We focus on the coefficient of $z^{(m-1)}$ in $\Omega(z)$.

Proof — continued

Take $D = q D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z D_x$ for formal $z := z(y)$

$$\Omega(z) \; := \; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \; = \; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \; b_{e_0,\ldots,e_{m-1}}$$

By induction on r,

$$D_*^r \; = \; z^{(r-1)} D_x + h_r(z, z', \ldots, z^{(r-2)}).$$

Proof — continued

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \;\; b_{e_0,\ldots,e_{m-1}}$$

By induction on r,

$$D_*^r \;=\; z^{(r-1)}\,D_x + h_r(z, z', \ldots, z^{(r-2)}).$$

$$0 \;=\; [z^{(m-1)}]\,(\bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^{m-1} \wedge \bar{D}_*^m)$$

Proof — continued

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \;\; b_{e_0,\ldots,e_{m-1}}$$

By induction on r,

$$D_*^r \;=\; z^{(r-1)}\,D_x + h_r(z, z', \ldots, z^{(r-2)}).$$

$$\begin{aligned} 0 \;&=\; [z^{(m-1)}]\,(\bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^{m-1} \wedge \bar{D}_*^m) \\ &=\; ([1]\,\bar{1}) \wedge ([1]\,\bar{D}_*) \wedge \cdots \wedge ([1]\,\bar{D}_*{}^{m-1}) \wedge ([z^{(m-1)}]\,\bar{D}_*^m) \end{aligned}$$

Proof — continued

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \;\; b_{e_0,\ldots,e_{m-1}}$$

By induction on r,

$$D_*^r \;=\; z^{(r-1)}\,D_x + h_r(z, z', \ldots, z^{(r-2)}).$$

$$\begin{aligned}
0 \;&=\; [z^{(m-1)}]\,(\bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^{m-1} \wedge \bar{D}_*^m) \\
&=\; ([1]\,\bar{1}) \wedge ([1]\,\bar{D}_*) \wedge \cdots \wedge ([1]\,\bar{D}_*{}^{m-1}) \wedge ([z^{(m-1)}]\,\bar{D}_*^m) \\
&=\; \bar{1} \wedge \bar{D} \wedge \cdots \wedge \bar{D}^{m-1} \wedge \bar{D}_x
\end{aligned}$$

Proof — continued

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \;\; b_{e_0,\ldots,e_{m-1}}$$

By induction on r,

$$D_*^r \;=\; z^{(r-1)}\,D_x + h_r(z, z', \ldots, z^{(r-2)}).$$

$$\begin{aligned} 0 \;&=\; [z^{(m-1)}]\,(\bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^{m-1} \wedge \bar{D}_*^m) \\ &=\; ([1]\,\bar{1}) \wedge ([1]\,\bar{D}_*) \wedge \cdots \wedge ([1]\,\bar{D}_*{}^{m-1}) \wedge ([z^{(m-1)}]\,\bar{D}_*^m) \\ &=\; \bar{1} \wedge \bar{D} \wedge \cdots \wedge \bar{D}^{m-1} \wedge \bar{D}_x \end{aligned}$$

Hence $\bar{D}_x \in \mathbb{K}[\bar{D}]$.

Proof — continued

Take $D = q\,D_x + D_y$ with $m := \dim_{\mathbb{K}} \mathbb{K}[\bar{D}]$ maximal. We want $\mathbb{A} = \mathbb{K}[\bar{D}]$.

Let $D_* := D + z\,D_x$ for formal $z := z(y)$

$$\Omega(z) \;:=\; \bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^m \;=\; \sum z^{e_0} \cdots (z^{(m-1)})^{e_{m-1}} \;\; b_{e_0,\ldots,e_{m-1}}$$

By induction on r,

$$D_*^r \;=\; z^{(r-1)}\,D_x + h_r(z, z', \ldots, z^{(r-2)}).$$

$$\begin{aligned} 0 \;&=\; [z^{(m-1)}]\,(\bar{1} \wedge \bar{D}_* \wedge \cdots \wedge \bar{D}_*^{m-1} \wedge \bar{D}_*^m) \\ &=\; ([1]\,\bar{1}) \wedge ([1]\,\bar{D}_*) \wedge \cdots \wedge ([1]\,\bar{D}_*{}^{m-1}) \wedge ([z^{(m-1)}]\,\bar{D}_*^m) \\ &=\; \bar{1} \wedge \bar{D} \wedge \cdots \wedge \bar{D}^{m-1} \wedge \bar{D}_x \end{aligned}$$

Hence $\bar{D}_x \in \mathbb{K}[\bar{D}]$.

Now D and D_x commute, so $\mathbb{A} = \mathbb{K}[\bar{D}_x, \bar{D}_y] = \mathbb{K}[\bar{D}]$.

Dual interpretation

Corollary

There exists a polynomial $p \in \mathbb{Q}[y]$ *with the following property.*

Consider an invertible polynomial change of variables $\tilde{x} = x + p(y), \tilde{y} = y$ *and denote the partial derivatives with respect to* $\tilde{x}$ *and* $\tilde{y}$ *by* $D_{\tilde{x}}$ *and* $D_{\tilde{y}}$*, respectively.*

Then $\mathbb{A} = \mathbb{K}[\bar{D}_{\tilde{y}}]$.

Proof. For $p := -\int q$, we have

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + p(y) \\ y \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} D_{\tilde{x}} \\ D_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix},$$

so $D = D_{\tilde{y}}$. $\square$

Example — continued

$$I \;=\; (D_x^2, D_y^2) \;\subseteq\; \mathbb{K}[D_x, D_y].$$

$$\begin{aligned} D &:= y\,D_x + D_y \\ D_x &\equiv D^2 - \tfrac{2}{3} y\,D^3 \\ D_y &\equiv D - y\,D^2 + \tfrac{2}{3} y^2 D^3 \end{aligned}$$

$$I \;=\; \left(D^4,\;\; D_x + \tfrac{2}{3} y\,D^3 - D^2,\;\; D_y - \tfrac{2}{3} y^2 D^3 + y\,D^2 - D\right)$$

Example — continued

$$I = (D_x^2, D_y^2) \subseteq \mathbb{K}[D_x, D_y].$$

$$\begin{aligned} D &:= y D_x + D_y \\ D_x &\equiv D^2 - \tfrac{2}{3} y D^3 \\ D_y &\equiv D - y D^2 + \tfrac{2}{3} y^2 D^3 \end{aligned}$$

$$I = \left(D^4,\ D_x + \tfrac{2}{3} y D^3 - D^2,\ D_y - \tfrac{2}{3} y^2 D^3 + y D^2 - D \right)$$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x - \frac{y^2}{2} \\ y \end{pmatrix}, \qquad I = \left(D_{\tilde{y}}^4,\ D_{\tilde{x}} + \tfrac{2}{3} y D_{\tilde{y}}^3 - D_{\tilde{y}}^2 \right).$$

Higher dimensions

$\mathbb{K} = \Bbbk(x_1, \ldots, x_n)$ with derivations $D_{x_1}, \ldots, D_{x_n}$

$\mathbb{A} := \mathbb{K}[D_{x_1}, \ldots, D_{x_n}]/I$ of finite dimension over $\mathbb{K}$

Theorem

There exist univariate polynomials $q_2, \ldots, q_n \in \mathbb{Q}[x_1]$ with $\mathbb{A} = \mathbb{K}[\bar{D}]$, where

$$D \ := \ D_{x_1} + q_2(x_1)\, D_{x_2} + \cdots + q_n(x_1)\, D_{x_n}$$

Corollary

There exist polynomials $p_2, \ldots, p_n \in \mathbb{Q}[x_1]$ with the following property.

Consider an invertible polynomial change of variables

$$\tilde{x}_1 \ = \ x_1,\ \tilde{x}_2 \ = \ x_2 + p_2(x_1),\ \ \ldots,\ \tilde{x}_n \ = \ x_n + p_n(x_1).$$

Then $\mathbb{A} = \mathbb{K}[\bar{D}_{\tilde{x}_1}]$.

Thank you !

http://www.TeXmacs.org