Dendromorphic functions

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Generating function of the partition numbers

$$p(z) := \prod_{k \geqslant 1} \frac{1}{1 - z^k}$$

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p is differentially algebraic over $\mathbb{C}(z)$.

Question (Bostan–Jiménez-Pastor, 2020)

Does p lie in the Picard-Vessiot closure of $\mathbb{C}(z)$?

Computing with special functions

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What does this mean?

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Differential algebra perspective

$$\begin{aligned}
\mathbb{K}_0 &:= \mathbb{C}(z)\\
\mathbb{K}_1 &:= \mathbb{K}_0\{E\}/\{E' - E\}\\
\mathbb{K}_2 &:= \mathbb{K}_1\{L\}/\left\{L' - \frac{E}{E+1}\right\} \ni L + E^2 - zE =: f
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Limitation

$$\mathrm{tp}^{\mathbb{K}_0}(2E) = \mathrm{tp}^{\mathbb{K}_0}(E)$$

In other words: $E =: \mathrm{e}^z$ is only defined up to a non-zero constant

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Local differential algebra perspective

$$\begin{aligned} E'(z) &= E(z) & E(0) &= 1 & \mathrm{e}^z &\equiv E(z) \\ L'(z) &= \frac{E}{E+1} & L(0) &= \log 2 & \log(\mathrm{e}^z+1) &\equiv L(z) \\ & & & & f &\equiv L + E^2 - zE \end{aligned}$$

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$$f \equiv L + E^2 - z\,E$$

Limitation

Only defines f locally near $z=0$, no insight into global behavior.

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Complex analytic perspective

f can be continued analytically

$\longrightarrow$ our take: local differential perspective + analytic continuation

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Subtleties

- f becomes multivalued: $\log(\mathrm{e}^z+1)$ or $\log(\mathrm{e}^z+1)+2\pi\mathrm{i}$?
- What is the meaning of “composition” ?
- What can we say about “the Riemann surface of definition” ?

What about singularities ?

Question: can expressions like f give rise to natural boundaries?

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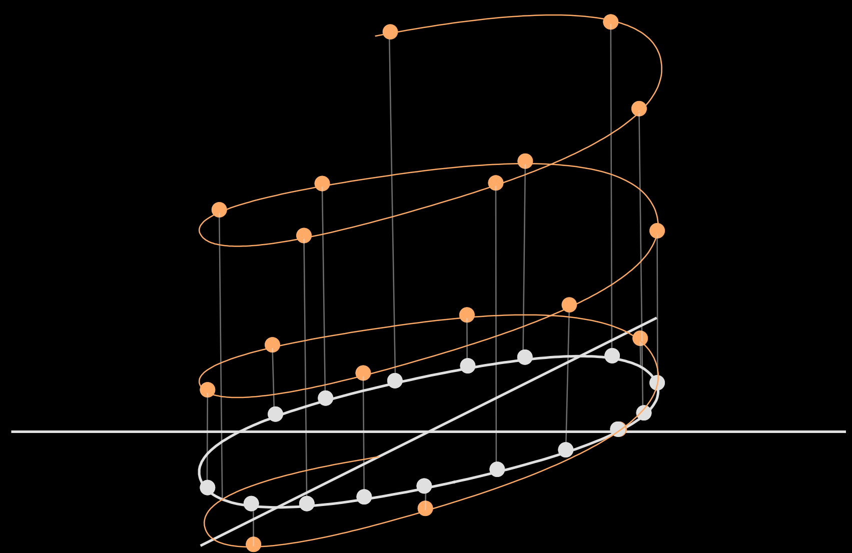
In projection: $$f(z) := \frac{1}{e^{2\pi \log z} - 1}$$



Tree-like singularity structure

$\tan z$

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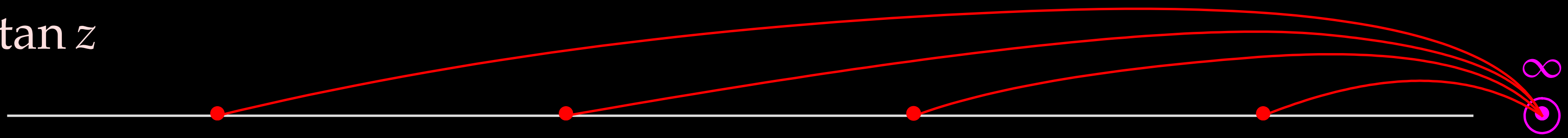

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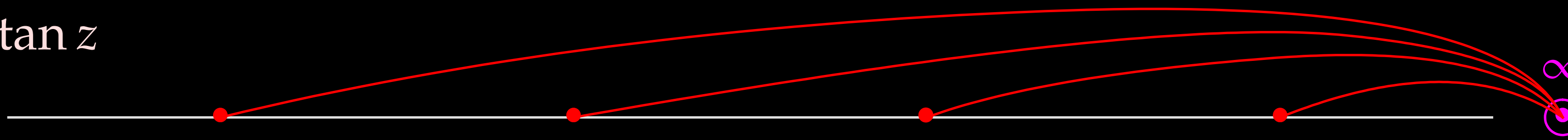

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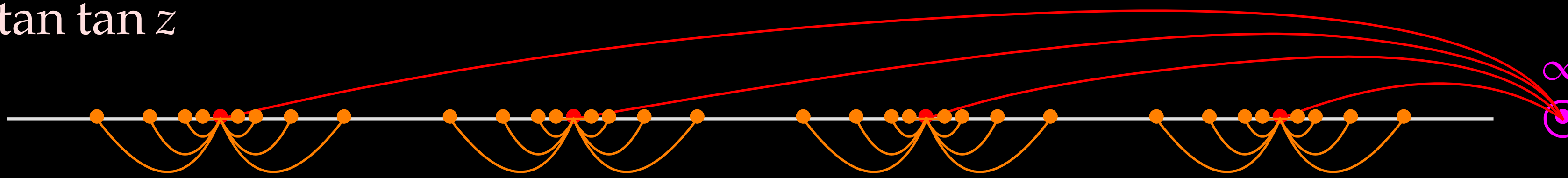

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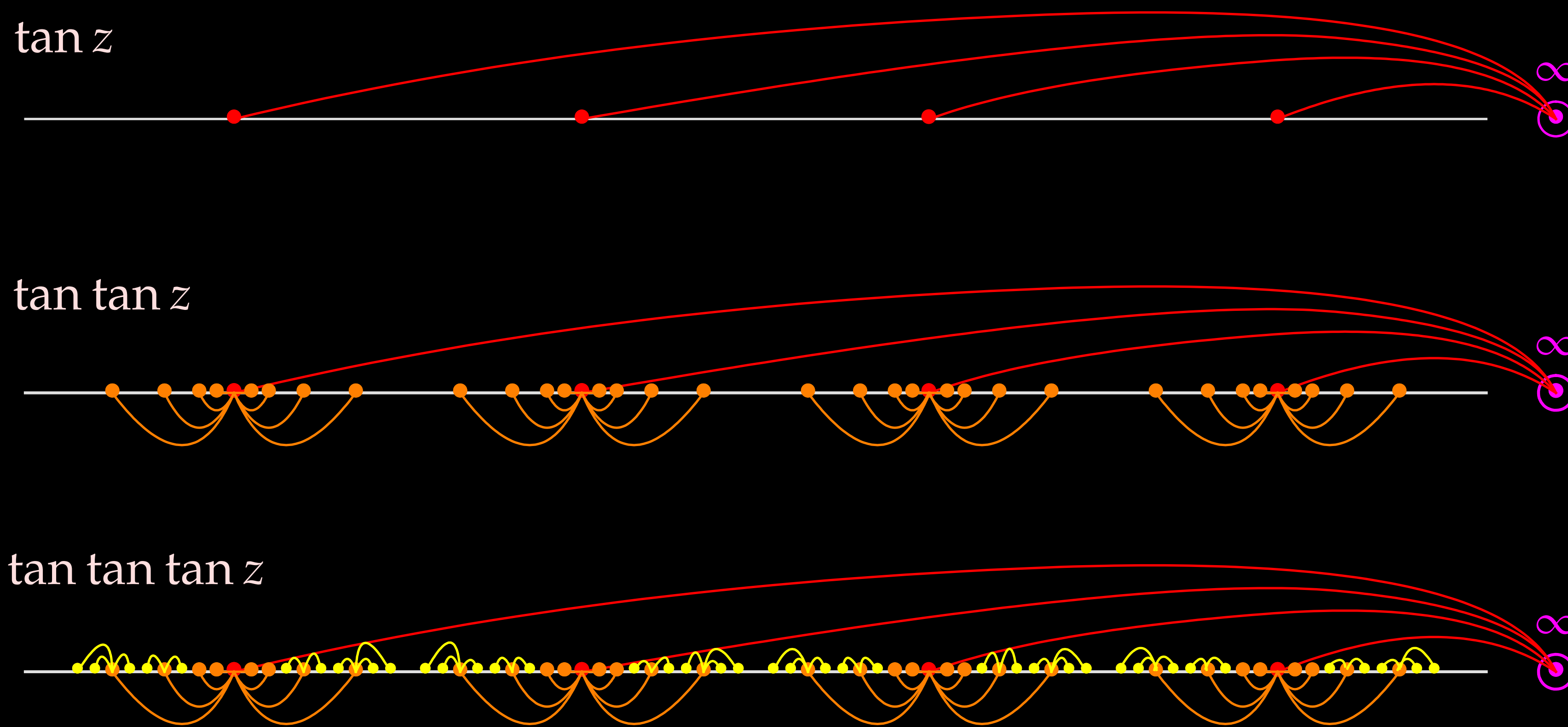

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D-finite towers

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$y \in \mathbb{L}$ is **D-finite** over $\mathbb{K}$ $\overset{\text{def}}{\iff}$ $L_r y^{(r)} + \cdots + L_0 y = 0$ for some $L_0, \ldots, L_r \in \mathbb{K}$

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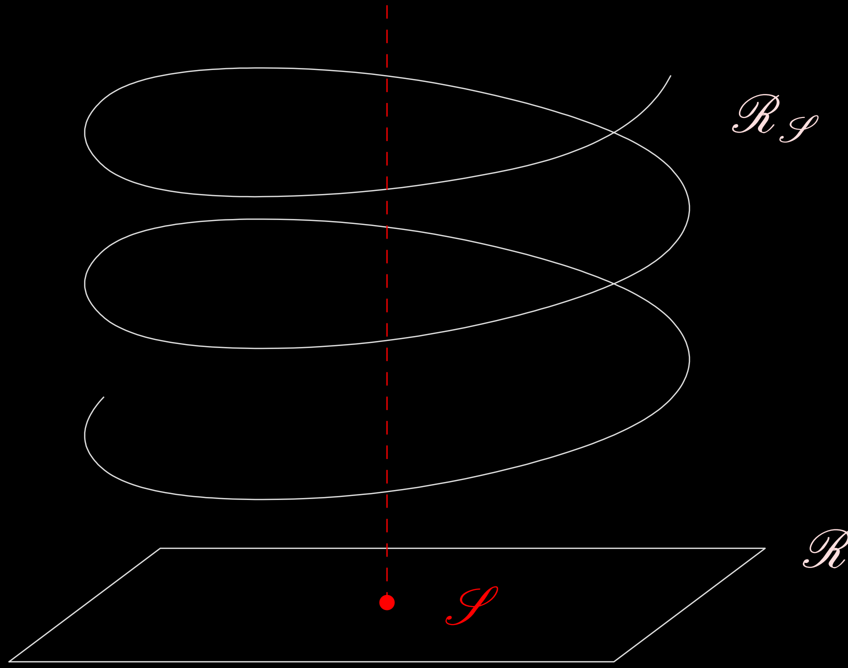
Theorem (Jiménez-Pastor–Pillwein–Singer, 2020)

Assume $f(t)$ is D$^\infty$-finite over $\mathbb{C}(t)$ and $\xi(z)$ is D$^\infty$-finite over $\mathbb{C}(z)$.
Then $(f \circ \xi)(z)$ is D$^\infty$-finite over $\mathbb{C}(z)$.

Discrete ramifications

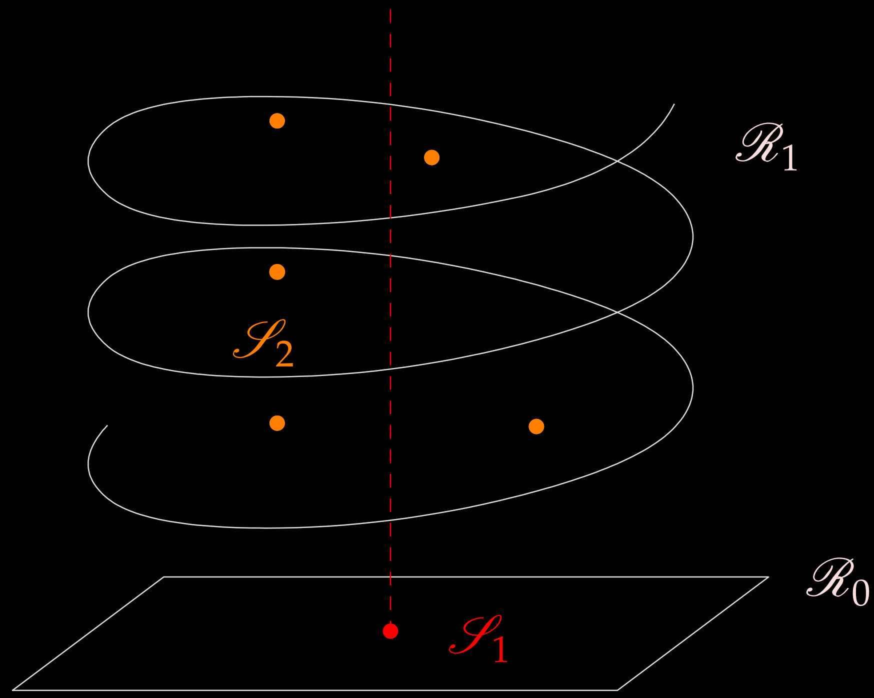
$\mathscr{R}$: simply connected Riemann surfaces
$\mathscr{S}$: discrete subset of $\mathscr{R}$

Discrete ramification: covering space $(\mathscr{R}_{\mathscr{S}}, \pi_{\mathscr{S}})$ of $\mathscr{R} \setminus \mathscr{S}$



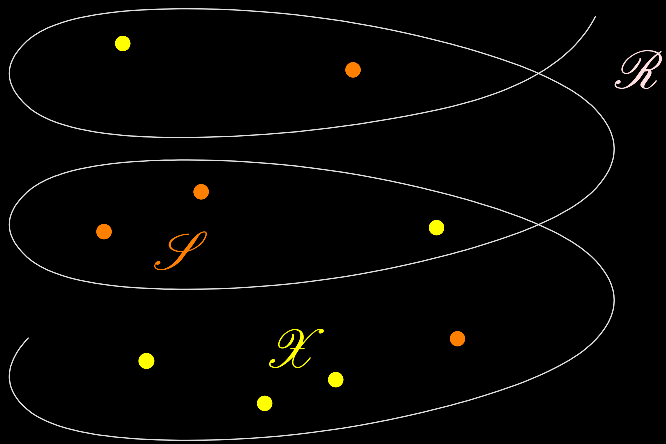
Recursive discrete ramifications

$$\mathscr{R}_0 \xleftarrow{\pi_{\mathscr{S}_1}} (\mathscr{R}_0)_{\mathscr{S}_1} = \mathscr{R}_1 \xleftarrow{\pi_{\mathscr{S}_2}} (\mathscr{R}_1)_{\mathscr{S}_2} = \mathscr{R}_2 \xleftarrow{\pi_{\mathscr{S}_3}} \cdots \xleftarrow{\pi_{\mathscr{S}_\ell}} \mathscr{R}_\ell$$



Joins

$$
\begin{tikzcd}
\mathscr{R} & & \mathscr{R}_{\mathscr{S}} \arrow[ll, "\pi_{\mathscr{S}}"'] \\
& & \\
\mathscr{R}_{\mathscr{X}} \arrow[uu, "\pi_{\mathscr{X}}"] & & \mathscr{R}_{\mathscr{S} \cup \mathscr{X}} \arrow[ll, "\pi_{\tilde{\mathscr{S}}}"] \arrow[uu, "\pi_{\tilde{\mathscr{X}}}"'] \arrow[uull, "\pi_{\mathscr{S} \cup \mathscr{X}}"']
\end{tikzcd}
$$



Recursive joins

Given sequences of discrete ramifications

$$\begin{array}{ccccccc} \mathscr{R}_{0,0} & \longleftarrow & \mathscr{R}_{1,0} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{h,0} \\ \mathscr{R}_{0,0} & \longleftarrow & \mathscr{R}_{0,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{0,h'} \end{array}$$

we may form their join:

$$\begin{array}{ccccccc} \mathscr{R}_{0,0} & \longleftarrow & \mathscr{R}_{0,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{0,h'} \\ \big\uparrow & \nwarrow & \big\uparrow & \nwarrow & & \nwarrow & \big\uparrow \\ \mathscr{R}_{1,0} & \longleftarrow & \mathscr{R}_{1,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{1,h'} \\ \big\uparrow & \nwarrow & \big\uparrow & \nwarrow & & \nwarrow & \big\uparrow \\ \vdots & & \vdots & & & & \vdots \\ \big\uparrow & \nwarrow & \big\uparrow & \nwarrow & & \nwarrow & \big\uparrow \\ \mathscr{R}_{h,0} & \longleftarrow & \mathscr{R}_{h,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{h,h'} \end{array}$$

Hence: recursive discrete ramifications above fixed $\mathscr{R}$ form an inductive family

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$\mathscr{R}$: Riemann surface

$\mathscr{A}(\mathscr{R})$: analytic functions on $\mathscr{R}$

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Definition

$\mathscr{D}(\mathscr{R})$ inductive limit of $\mathscr{A}(\mathscr{R}')$ with $\mathscr{R} \xleftarrow{\pi} \mathscr{R}'$ recursive ramification.
Dendromorphic function *on $\mathscr{R}$:= element of $\mathscr{D}(\mathscr{R})$.*

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Representation: $f \in \mathscr{A}(\mathscr{R}')$ with $\mathscr{R} \xleftarrow{\pi} \mathscr{R}'$ recursive ramification.

Equality: $f_1 \in \mathscr{A}(\mathscr{R}_1)$, $f_2 \in \mathscr{A}(\mathscr{R}_2)$
$\mathscr{R}'$ join of $\mathscr{R}_1$ and $\mathscr{R}_2$ with $\iota_k : \mathscr{A}(\mathscr{R}_k) \hookrightarrow \mathscr{A}(\mathscr{R}')$, $k = 1,2$
Then $f_1 \underset{\mathscr{D}(\mathscr{R})}{=\!=} f_2 \quad \Longleftrightarrow \quad \iota_1(f_1) = \iota_2(f_2)$

Picard–Vessiot closure

Theorem

$\mathscr{D}(\mathscr{R})$ *is a Picard–Vessiot closed field.*

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Field structure. Let $f \in \mathscr{A}(\mathscr{R}')$ with $\mathscr{R} \xleftarrow{\pi} \mathscr{R}'$ recursive ramification.

Then $\mathscr{S} := \{z \in \mathscr{R}' : f(z) = 0\}$ is a discrete subset of $\mathscr{R}'$

Now $f^{-1} \in \mathscr{R}'_{\mathscr{S}}$ and $\mathscr{R} \xleftarrow{\pi} \mathscr{R}' \xleftarrow{\pi_{\mathscr{S}}} \mathscr{R}'_{\mathscr{S}}$ is a recursive ramification.

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PV closed. Let $L_0, \ldots, L_r \in \mathscr{A}(\mathscr{R}')$ with $\mathscr{R} \xleftarrow{\pi} \mathscr{R}'$ recursive ramification.

Then $\mathscr{S} := \{z \in \mathscr{R}' : L_r(z) = 0\}$ is a discrete subset of $\mathscr{R}'$

Now $\mathscr{A}(\mathscr{R}'_{\mathscr{S}})$ contains a fundamental system of solutions of

$$L_r y^{(r)} + \cdots + L_0 y = 0$$

Boundaryless continuation

$\mathscr{U}$: open subset of $\mathbb{C}$

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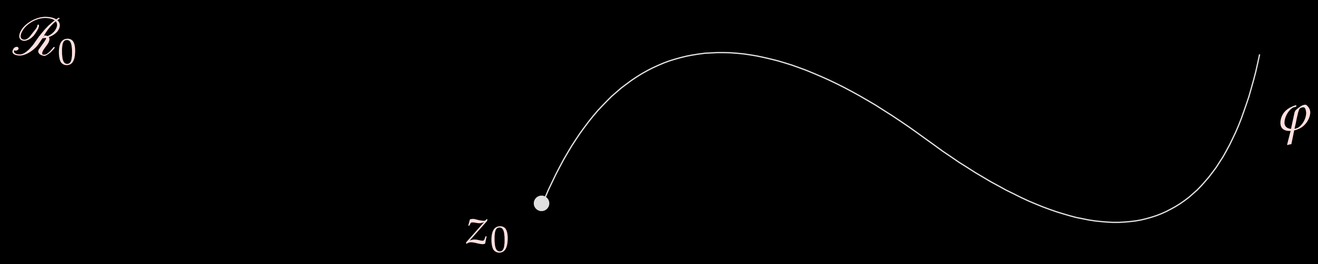

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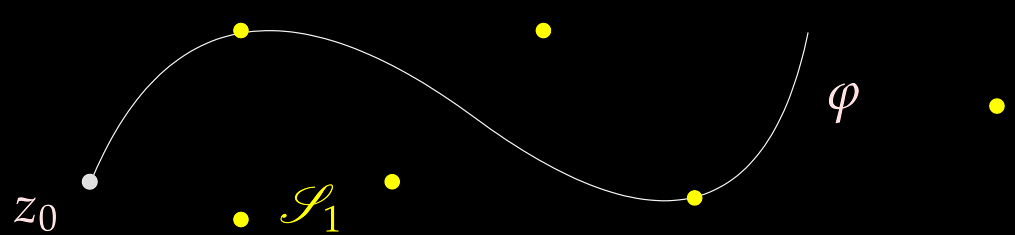

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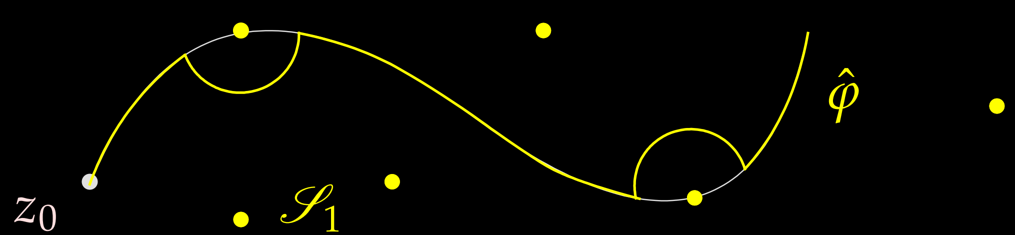

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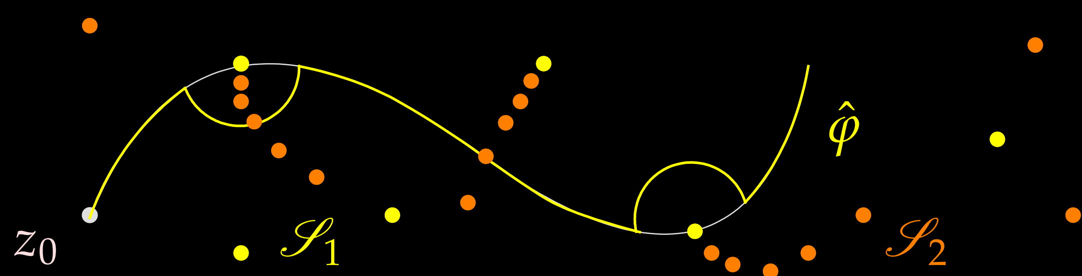

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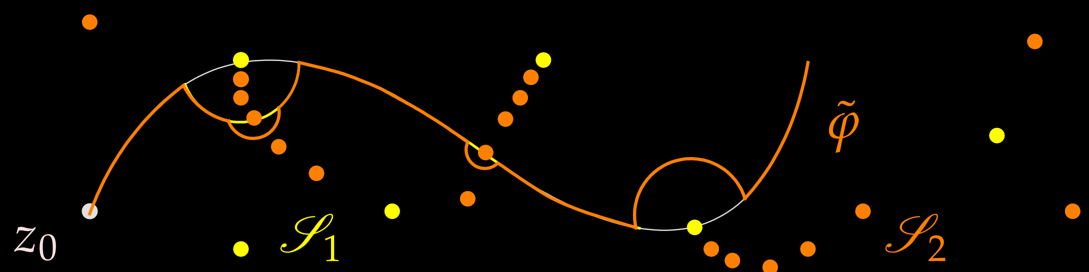

Consequences

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Composition

f and g defined on Riemann surfaces $\mathscr{R}$ and $\mathscr{R}'$.

Distinguished points $\alpha_{\mathscr{R}} \in \mathscr{R}$ and $\beta_{\mathscr{R}'} \in \mathscr{R}'$ above $\alpha, \beta \in \mathbb{C}$ and $f(\alpha_{\mathscr{R}}) = \beta$.

Then $g \circ f$ is locally defined near $\alpha_{\mathscr{R}}$ by lifting $\hat{f}(\alpha_{\mathscr{R}}) = \beta_{\mathscr{R}'}$.

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The composition of two boundaryless functions

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The functional inverse of $p(z)$ is boundaryless, but $p(z)$ is not.

Functional inversion

Proposition

The functional inverse of $\wp(z)$ is D^∞-finite, but not $\wp(z)$ itself.

Proof. Rosenlicht: $\mathbb{C}\{z, \wp(z)\}$ has no valuation that is compatible with ∂.
VdH: but $\mathbb{C}(z)^{\mathrm{pv}}$ does, since it embeds in the complex transseries. □

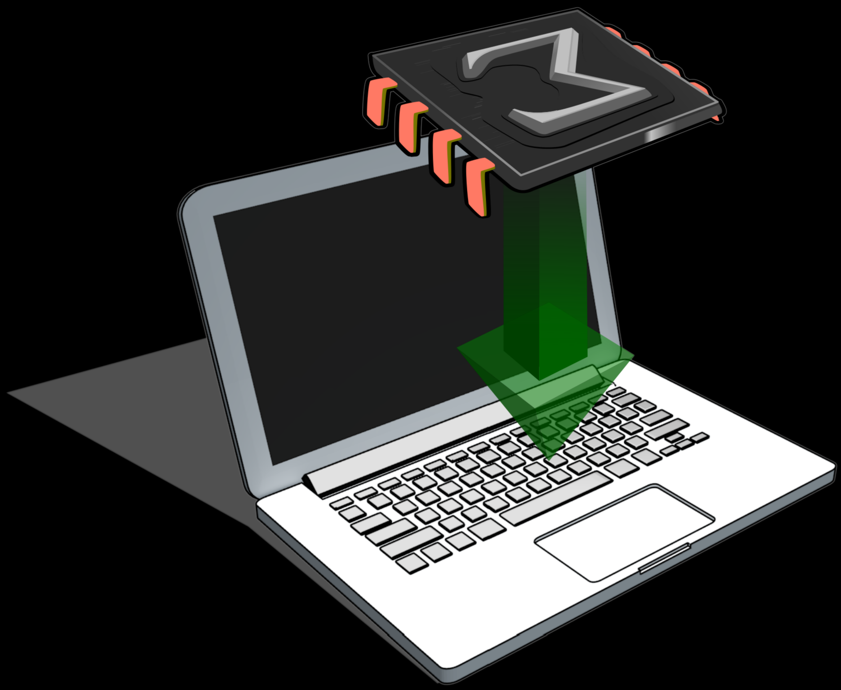
Proposition

The functional inverse of $p(z)$ is boundaryless, but $p(z)$ is not.

Question

Is there a dendromorphic function whose functional inverse is not.

Thank you !



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