Factoring multivariate sparse polynomials

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Sparse polynomials — example 1

Symbolic expressions $\longrightarrow$ sparse polynomials or rational functions

Sparse polynomials — example 1

Symbolic expressions ⟶ sparse polynomials or rational functions

```
Caas] derive (x^x^x, x, x, x, x)
```

```
Caas]
```

Sparse polynomials — example 1

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```

$$\Bigg(6\,\frac{(\log(x)+1)^2 x^x \log(x)}{x} - 4\,\frac{(\log(x)+1)\,x^x \log(x)}{x^2} - 6\,\frac{(\log(x)+1)^2 x^x}{x^2} + (\log(x)+$$

$$1)^4\,x^x\log(x) + 4\,\frac{(\log(x)+1)^3 x^x}{x} + 8\,\frac{(\log(x)+1)\,x^x}{x^3} + 12\,\frac{(\log(x)+1)\,x^x}{x^2} + 2\,\frac{x^x\log(x)}{x^3} +$$

$$3\,\frac{x^x\log(x)}{x^2} - 6\,\frac{x^x}{x^4} - 10\,\frac{x^x}{x^3}\Bigg)\,x^{x^x} + 4\,\Bigg((\log(x)+1)^3\,x^x\log(x) + 3\,\frac{(\log(x)+1)\,x^x\log(x)}{x} +$$

$$3\,\frac{(\log(x)+1)^2 x^x}{x} - 3\,\frac{(\log(x)+1)\,x^x}{x^2} - \frac{x^x\log(x)}{x^2} + 2\,\frac{x^x}{x^3} + 3\,\frac{x^x}{x^2}\Bigg)\Bigg((\log(x)+1)\,x^x\log(x) +$$

$$\frac{x^x}{x}\Bigg)\,x^{x^x} + 6\,\Bigg((\log(x)+1)^2 x^x\log(x) + 2\,\frac{(\log(x)+1)\,x^x}{x} + \frac{x^x\log(x)}{x} - \frac{x^x}{x^2}\Bigg)\Bigg((\log(x)+$$

$$1)\,x^x\log(x) + \frac{x^x}{x}\Bigg)^2 x^{x^x} + 3\,\Bigg((\log(x)+1)^2 x^x\log(x) + 2\,\frac{(\log(x)+1)\,x^x}{x} + \frac{x^x\log(x)}{x} - \frac{x^x}{x^2}\Bigg)^2 x^{x^x} +$$

$$\Bigg((\log(x)+1)\,x^x\log(x) + \frac{x^x}{x}\Bigg)^4 x^{x^x}$$

Caas]

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$$\left((\log(x)+1)^3 x^x \log(x) + 3\,\frac{(\log(x)+1)\,x^x \log(x)}{x} + 3\,\frac{(\log(x)+1)^2 x^x}{x} - 3\,\frac{(\log(x)+1)\,x^x}{x^2} - \frac{x^x \log(x)}{x^2} + 2\,\frac{x^x}{x^3} + 3\,\frac{x^x}{x^2}\right) x^{x^x} + 3\left((\log(x)+1)^2 x^x \log(x) + 2\,\frac{(\log(x)+1)\,x^x}{x} + \frac{x^x \log(x)}{x} - \frac{x^x}{x^2}\right)\left((\log(x)+1)\,x^x \log(x) + \frac{x^x}{x}\right) x^{x^x} + \left((\log(x)+1)\,x^x \log(x) + \frac{x^x}{x}\right)^3 x^{x^x}$$

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$$\left((\log(x)+1)^3 x^x \log(x) + 3\frac{(\log(x)+1)\,x^x \log(x)}{x} + 3\frac{(\log(x)+1)^2 x^x}{x} - 3\frac{(\log(x)+1)\,x^x}{x^2} - \frac{x^x \log(x)}{x^2} + 2\frac{x^x}{x^3} + 3\frac{x^x}{x^2}\right) x^{x^x} + 3\left((\log(x)+1)^2 x^x \log(x) + 2\frac{(\log(x)+1)\,x^x}{x} + \frac{x^x \log(x)}{x} - \frac{x^x}{x^2}\right)\left((\log(x)+1)\,x^x \log(x) + \frac{x^x}{x}\right) x^{x^x} + \left((\log(x)+1)\,x^x \log(x) + \frac{x^x}{x}\right)^3 x^{x^x}$$

$$f = x^{x^x}$$

$$f', f'', f''', \ldots \in \mathbb{Q}\left[\log x, \tfrac{1}{x}, x^x, x^{x^x}\right]$$

Sparse polynomials — example 1

Symbolic expressions $\longrightarrow$ sparse polynomials or rational functions

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$$\left((\log(x)+1)^3 x^x \log(x) + 3\frac{(\log(x)+1)\,x^x\log(x)}{x} + 3\frac{(\log(x)+1)^2 x^x}{x} - 3\frac{(\log(x)+1)\,x^x}{x^2} - \frac{x^x\log(x)}{x^2} + 2\frac{x^x}{x^3} + 3\frac{x^x}{x^2}\right) x^{x^x} + 3\left((\log(x)+1)^2 x^x\log(x) + 2\frac{(\log(x)+1)\,x^x}{x} + \frac{x^x\log(x)}{x} - \frac{x^x}{x^2}\right)\left((\log(x)+1)\,x^x\log(x) + \frac{x^x}{x}\right) x^{x^x} + \left((\log(x)+1)\,x^x\log(x) + \frac{x^x}{x}\right)^3 x^{x^x}$$

$$\begin{aligned} f &= x^{x^x} \\ f', f'', f''', \ldots &\in \mathbb{Q}\left[\log x, \tfrac{1}{x}, x^x, x^{x^x}\right] \\ &\subseteq \mathbb{Q}\left(\log x, x, x^x, x^{x^x}\right) \end{aligned}$$

Sparse polynomials — example 2


```
Caas] M == [ a, b, c; d, e, f; g, h, i ]
Caas] det M
Caas] simplify invert M
Caas] simplify transpose invert transpose invert M
Caas]
```

Sparse polynomials — example 2


```
Caas] M == [ a, b, c; d, e, f; g, h, i ]
```

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

```
Caas] det M
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$$(cg+fh)(-a-e)+(ac+bf)g+(cd+ef)h+(ae-bd)i$$

```
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$$\left[\begin{matrix} \frac{ei-fh}{(dh-eg)c+(ei-fh)a+(fg-di)b} & \frac{ch-bi}{(dh-eg)c+(ei-fh)a+(fg-di)b} & \frac{bf-c}{(dh-eg)c+(ei-f} \\ \frac{fg-di}{(dh-eg)c+(ei-fh)a+(fg-di)b} & \frac{ai-cg}{(dh-eg)c+(ei-fh)a+(fg-di)b} & \frac{cd-a}{(dh-eg)c+(ei-f} \\ \frac{dh-eg}{(dh-eg)c+(ei-fh)a+(fg-di)b} & \frac{bg-ah}{(dh-eg)c+(ei-fh)a+(fg-di)b} & \frac{ae-b}{(dh-eg)c+(ei-f} \end{matrix}\right.$$

```
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```
Caas]
```

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Computations with parameters ⟶ expression swell

Sparse polynomials — example 2


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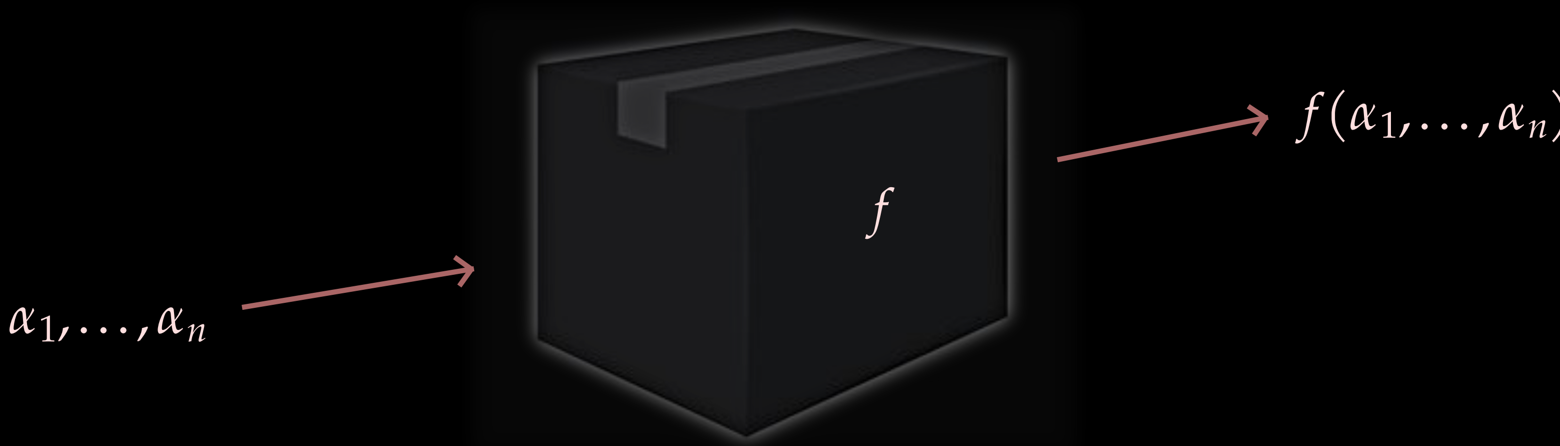
Computations with parameters ⟶ expression swell

How to make running times depend only on the input & output size ?

Sparse interpolation

Input



Output

$$f(x_1,\ldots,x_n) \;=\; c_1 x_1^{e_{1,1}} \cdots x_n^{e_{1,n}} + \cdots + c_t x_1^{e_{t,1}} \cdots x_n^{e_{t,n}}$$

Complexity of sparse interpolation (see previous talk by Grégoire)

Sparse interpolation

Complexity of sparse interpolation (see previous talk by Grégoire)

INPUT: an SLP f of size L
OUTPUT: sparse interpolation $f = c_1 \boldsymbol{x}^{e_1} + \cdots + c_t \boldsymbol{x}^{e_t}$

Sparse interpolation

Complexity of sparse interpolation (see previous talk by Grégoire)

INPUT: an SLP f of size L

OUTPUT: sparse interpolation $f = c_1 \boldsymbol{x}^{e_1} + \cdots + c_t \boldsymbol{x}^{e_t}$

$$\mathsf{T}(t) = O(L t) + \mathsf{S}(t), \qquad \mathsf{S}(t) = \tilde{O}(t)$$

Sparse interpolation

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CONSTRAINT: evaluation points → roots of unity or geometric progression

Sparse interpolation

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Other operations ?

Sparse interpolation

Complexity of sparse interpolation (see previous talk by Grégoire)

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CONSTRAINT: evaluation points → roots of unity or geometric progression

Other operations ?

- Greatest common divisors

Sparse interpolation

Complexity of sparse interpolation (see previous talk by Grégoire)

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- Sparse interpolation of rational functions

Sparse interpolation

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Long history ⟶ paper with Alexander Demin

Sparse interpolation

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INPUT: an SLP f of size L
OUTPUT: sparse interpolation $f = c_1 \boldsymbol{x}^{e_1} + \cdots + c_t \boldsymbol{x}^{e_t}$

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CONSTRAINT: evaluation points → roots of unity or geometric progression

Other operations ?

- Greatest common divisors
- Sparse interpolation of rational functions
- Factorization

Long history ⟶ paper with Alexander Demin

We will focus on the case where the total degree d is “modest”

Sparse gcd computations

$$G(x_1,\ldots,x_n) \;=\; \gcd\left(A(x_1,\ldots,x_n), B(x_1,\ldots,x_n)\right)$$

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U, V coprime

Sparse gcd computations

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U, V coprime

$$\begin{aligned} A(\alpha_1 t,\ldots,\alpha_n t) &= U(\alpha_1 t,\ldots,\alpha_n t)\,G(\alpha_1 t,\ldots,\alpha_n t) \\ B(\alpha_1 t,\ldots,\alpha_n t) &= V(\alpha_1 t,\ldots,\alpha_n t)\,G(\alpha_1 t,\ldots,\alpha_n t) \end{aligned}$$

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U, V coprime

$$\begin{aligned} A_\alpha(t) &= A(\alpha_1 t,\ldots,\alpha_n t) = U(\alpha_1 t,\ldots,\alpha_n t)\,G(\alpha_1 t,\ldots,\alpha_n t) = U_\alpha(t)\,G_\alpha(t) \\ B_\alpha(t) &= B(\alpha_1 t,\ldots,\alpha_n t) = V(\alpha_1 t,\ldots,\alpha_n t)\,G(\alpha_1 t,\ldots,\alpha_n t) = V_\alpha(t)\,G_\alpha(t) \end{aligned}$$

Sparse gcd computations

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$$\begin{aligned} A_\alpha(t) &= A(\alpha_1 t,\dots,\alpha_n t) = U(\alpha_1 t,\dots,\alpha_n t)\, G(\alpha_1 t,\dots,\alpha_n t) = U_\alpha(t)\, G_\alpha(t) \\ B_\alpha(t) &= B(\alpha_1 t,\dots,\alpha_n t) = V(\alpha_1 t,\dots,\alpha_n t)\, G(\alpha_1 t,\dots,\alpha_n t) = V_\alpha(t)\, G_\alpha(t) \end{aligned}$$

With high probability

$$G_\alpha(t) \overset{\text{HP}}{=\!=\!=} \gcd\big(A_\alpha(t), B_\alpha(t)\big)$$

Sparse gcd computations

$$G(x_1,\ldots,x_n) = \gcd\left(A(x_1,\ldots,x_n), B(x_1,\ldots,x_n)\right)$$

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$$G_\alpha(t) \overset{\text{HP}}{=\!=\!=} \gcd(A_\alpha(t), B_\alpha(t))$$

Why?

Sparse gcd computations

$$G(x_1,\ldots,x_n) = \gcd\left(A(x_1,\ldots,x_n), B(x_1,\ldots,x_n)\right)$$

$$\begin{aligned} A(x_1,\ldots,x_n) &= U(x_1,\ldots,x_n)\,G(x_1,\ldots,x_n) \\ B(x_1,\ldots,x_n) &= V(x_1,\ldots,x_n)\,G(x_1,\ldots,x_n) \end{aligned} \qquad U,V \text{ coprime}$$

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With high probability (over Laurent polynomials)

$$G_\alpha(t) \overset{\text{HP}}{=\!=} \gcd(A_\alpha(t), B_\alpha(t))$$

Indeed,

$$U_\alpha, V_\alpha \text{ not coprime} \iff \mathrm{Res}_t(U_\alpha(t), V_\alpha(t)) = 0$$

Sparse gcd computations

$$G(x_1,\ldots,x_n) = \gcd\left(A(x_1,\ldots,x_n), B(x_1,\ldots,x_n)\right)$$

$$\begin{aligned} A(x_1,\ldots,x_n) &= U(x_1,\ldots,x_n)\,G(x_1,\ldots,x_n) \\ B(x_1,\ldots,x_n) &= V(x_1,\ldots,x_n)\,G(x_1,\ldots,x_n) \end{aligned}$$

U, V coprime

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Sparse gcd computations

$$G(x_1,\ldots,x_n) \;=\; \gcd\big(A(x_1,\ldots,x_n),B(x_1,\ldots,x_n)\big)$$

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With high probability

$$G_\alpha(t) \overset{\text{HP}}{=\!=\!=} \gcd(A_\alpha(t),B_\alpha(t))$$

Normalization problem

$G_\alpha(t)$ defined up to a “constant” c_α

Sparse gcd computations

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With high probability

$$G_\alpha(t) \overset{\mathrm{HP}}{=\!=\!=} \gcd(A_\alpha(t),B_\alpha(t))$$

Normalization problem

$$G_\alpha(t) \quad \text{defined up to a “constant”} \quad c_\alpha$$

⟶ no completely specified SLP to compute the coefficients of $G_\alpha(t)$

A lucky example when normalization is easy

$$A = -2025\,xyz - 14\,x^4y - 11\,xy^3 - xy + 14\,x^3 + 11\,y^2 + 2025\,z + 1$$
$$B = 14\,x^3z + 11\,y^2z + 14\,x^3 + 11\,y^2 + 2025\,z^2 + 2026\,z + 1$$

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We can normalize as follows:

$$A = UG \qquad U = \cdots + 1$$
$$B = VG \qquad V = \cdots + 1$$
$$G = \cdots + 1$$

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We can normalize as follows:

$$A = UG \qquad U = \cdots + 1 \qquad G = \cdots + 1$$
$$B = VG \qquad V = \cdots + 1$$

$$G_\alpha(t) = G_{\alpha,\deg G}\,t^{\deg G} + \cdots + G_{\alpha,1}\,t + 1$$

A lucky example when normalization is easy

$$A = -2025\,xyz - 14\,x^4 y - 11\,xy^3 - xy + 14\,x^3 + 11\,y^2 + 2025\,z + 1$$
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$$B = VG \qquad V = \cdots + 1 \qquad G = \cdots + 1$$

$$G_\alpha(t) = G_{\alpha,\deg G}\, t^{\deg G} + \cdots + G_{\alpha,1}\, t + 1$$

- There is only one gcd of $A_\alpha(t)$ and $B_\alpha(t)$ of this form

A lucky example when normalization is easy

$$\begin{aligned} A &= -2025\,xyz - 14\,x^4y - 11\,xy^3 - xy + 14\,x^3 + 11\,y^2 + 2025\,z + 1 \\ B &= 14\,x^3z + 11\,y^2z + 14\,x^3 + 11\,y^2 + 2025\,z^2 + 2026\,z + 1 \end{aligned}$$

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- $\longrightarrow$  $\longrightarrow$ $G = 14\,x^3 + 11\,y^2 + 2025\,z + 1$

Regularizing weights

Standard projection

$$A(x_1,\ldots,x_n) \longrightarrow A(\alpha_1 t,\ldots,\alpha_n t)$$
$$B(x_1,\ldots,x_n) \longrightarrow B(\alpha_1 t,\ldots,\alpha_n t)$$

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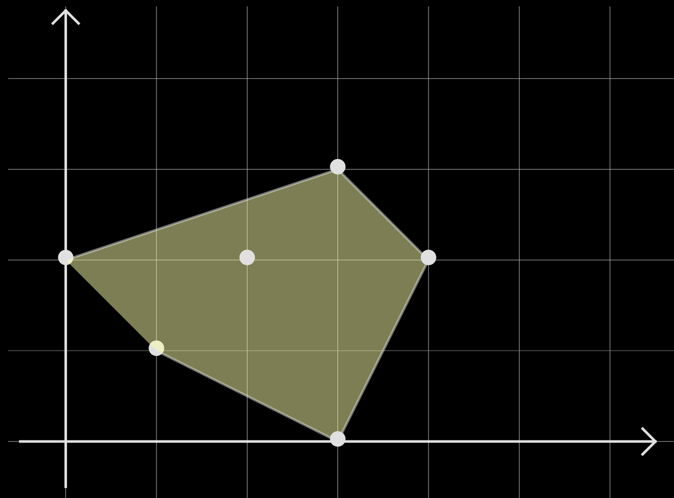
Newton polytopes



$$A(x,y) = x^4y^2 + x^3y^3 - x^2y^2 - x^3 + xy - 2y^2$$

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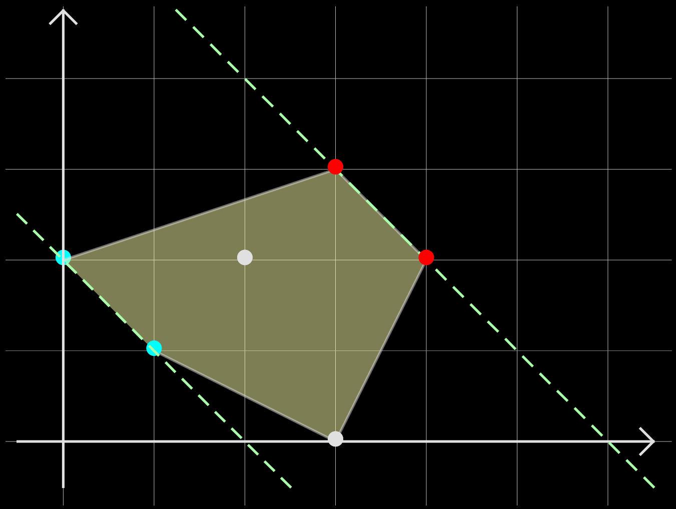

$$A(x,y) = x^4y^2 + x^3y^3 - x^2y^2 - x^3 + xy - 2y^2$$

$$A_\alpha(t) = \square t^6 + \square t^4 + \square t^3 + \square t^2$$

Weighted projection with integer weights

$$A(x_1,\ldots,x_n) \longrightarrow A(\alpha_1 t^{w_1},\ldots,\alpha_n t^{w_n})$$
$$B(x_1,\ldots,x_n) \longrightarrow B(\alpha_1 t^{w_1},\ldots,\alpha_n t^{w_n})$$

Regularizing weights

Weighted projection with integer weights

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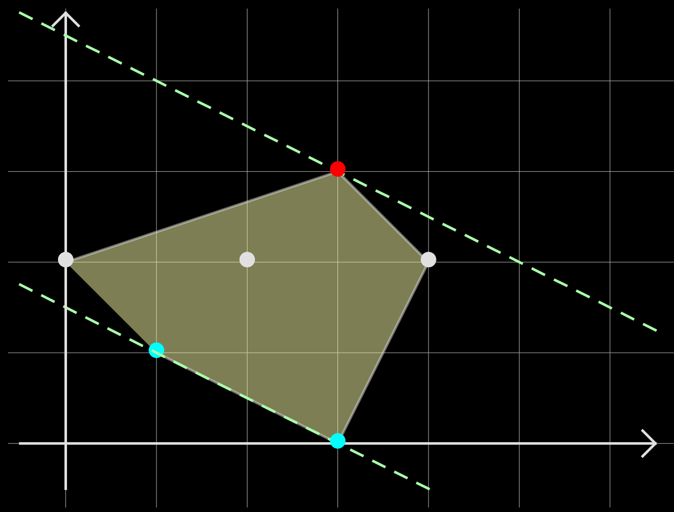


Head regularizing weight $w = (1,2)$

$$A(x,y) = x^4y^2 + x^3y^3 - x^2y^2 - x^3 + xy - 2y^2$$

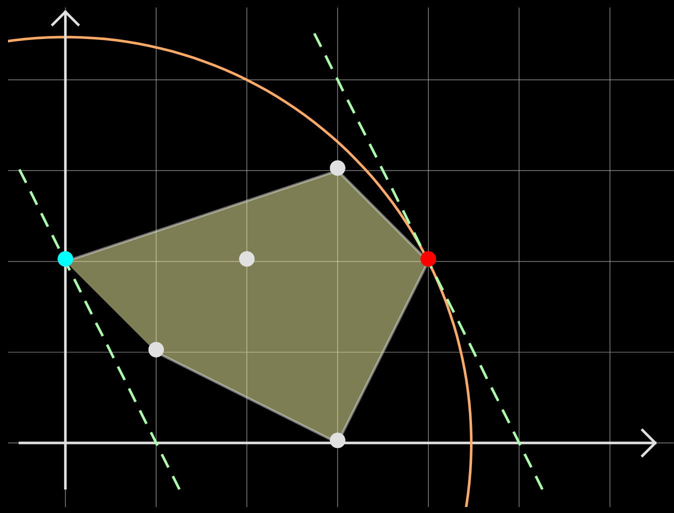
$$A_\alpha(t) = \square t^9 + \square t^8 + \square t^6 + \square t^4 + \square t^3$$

Regularizing weights

Weighted projection with integer weights

$$A(x_1,\dots,x_n) \longrightarrow A(\alpha_1 t^{w_1},\dots,\alpha_n t^{w_n})$$
$$B(x_1,\dots,x_n) \longrightarrow B(\alpha_1 t^{w_1},\dots,\alpha_n t^{w_n})$$

Existence of regularizing weight of small norm $|w| \leqslant d$



Regularizing weight $w = (4,2)$

$$A(x,y) = x^4y^2 + x^3y^3 - x^2y^2 - x^3 + xy - 2y^2$$

$$A_\alpha(t) = \square t^{20} + \square t^{18} + \square t^{12} + \square t^6 + \square t^4$$

Complexity of sparse gcd computations

Notation

s_P: number of terms of $P \in \mathbb{K}[x_1, \ldots, x_n]$
d_P: total degree of P
$\mathrm{ec}_w P = \deg_w P - \mathrm{val}_w P$: weighted écart of P

$\mathsf{S}(s)$ complexity of sparse interpolation (for s terms)
$\mathsf{M}(d)$ complexity of dense polynomial multiplication (for degree d)

Theorem

Let $A, B \in \mathbb{K}[x_1, \ldots, x_n]$, $G = \gcd(A, B)$, and w regularizing for A or B.
Let $s := s_G$, $\bar{s} := s_A + s_B + s_G$, $d := \max(d_A, d_B)$, and $e := \max(\mathrm{ec}_w A, \mathrm{ec}_w B) \leqslant d^2$.
Then there is an algorithm to compute G with high probability, in time

$$O((\bar{s}/s + e)\,\mathsf{S}(s) + s\,\mathsf{M}(e)\log e).$$

An iterative alternative

Goal

$$G(x_1, \ldots, x_{n-1}, x_n) \;=\; \gcd\left(A(x_1, \ldots, x_{n-1}, x_n), B(x_1, \ldots, x_{n-1}, x_n)\right)$$

An iterative alternative

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Fibers for sparse interpolation

$$\underbrace{G(\alpha_1,\ldots,\alpha_{n-1},t)}_{G_\alpha(t)} \;=\; \gcd\left(\underbrace{A(\alpha_1,\ldots,\alpha_{n-1},t)}_{A_\alpha(t)},\underbrace{B(\alpha_1,\ldots,\alpha_{n-1},t)}_{B_\alpha(t)}\right)$$

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Recursively compute

$$G_c(x_1,\dots,x_{n-1}) \;=\; \gcd\left(A(x_1,\dots,x_{n-1},c),B(x_1,\dots,x_{n-1},c)\right)$$

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Iterative normalization

$$G_\alpha(t) = \gcd\left(A_\alpha(t), B_\alpha(t)\right)$$

$$G_\alpha(c) = G_c(x_1,\ldots,x_{n-1})$$

Complexity of the iterative algorithm

Notation

$\delta_P = \max(\deg_{x_1} P, \ldots, \deg_{x_n} P)$: maximum of partial degrees of P

Theorem

Let $A, B \in \mathbb{K}[x_1, \ldots, x_n]$ and $G = \gcd(A, B)$.

Let $s := s_G$, $\bar{s} := s_P + s_Q + s_G$, $d := \max(d_P, d_Q)$, and $\delta := \max(\delta_P, \delta_Q)$.

Then there is an algorithm to compute G with high probability, in time

$$O(n((\bar{s}/s + \delta)\,\mathsf{S}(s) + s\,\mathsf{M}(\delta)\log\delta)).$$

Related operations

Content factorization

$$\mathrm{cont}_z(xz - yz + x^2 - y^2 + x - y) \;=\; \gcd(x - y, x^2 - y^2 + x - y) \;=\; x - y$$

$$xz - yz + x^2 - y^2 + x - y \;=\; (x - y)(1 + x + y + z)$$

Related operations

Content factorization

$$\mathrm{cont}_z(xz - yz + x^2 - y^2 + x - y) = \gcd(x - y, x^2 - y^2 + x - y) = x - y$$

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Root extraction

$$6xy^2z^2 - y^3z^3 - 12x^2yz + 8x^3 = (2x - yz)^3$$

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Square-free factorization

$$F = A_1 A_2^2 \cdots A_k^k, \qquad A_1, \ldots, A_k \text{ pairwise coprime}$$

E.g. repeat $\gcd(F, F')$ and root extraction.

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Sparse interpolation of rational function $f = \frac{A}{B}$

- Guess regularizing weight for A and B (fast to check), next as for gcd.
- Iterative approach (straightforward to adapt).

Lifting factorizations

$$G(\boldsymbol{x},t) = \gcd\left(A(\boldsymbol{x},t),B(\boldsymbol{x},t)\right)$$

$$\Big\Updownarrow \text{HP}$$

$$G(\alpha,t) = \gcd\left(A(\alpha,t),B(\alpha,t)\right)$$

Lifting factorizations

$$F(\boldsymbol{x},t) \quad = \quad A_1(\boldsymbol{x},t)\cdots A_\ell(\boldsymbol{x},t) \quad \text{irreducible factorization}$$

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$$F(x,y,z) \;=\; \Phi(x,y)^2-z$$

$$\Phi(x,y) \;=\; x+y \qquad \text{(or } \Phi \text{ irreducible)}$$

A random $\alpha\in\mathbb{F}_p^*$ is square $\alpha=\beta^2$ with probability $\frac{1}{2}$.

In that case $F(x,y,\alpha)=(\Phi+\beta)\,(\Phi-\beta)$.

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Problem: how to recombine projected factors when lifting back?

Hilbert-Bertini irreducibility theorem

Hilbert-Bertini irreducibility Theorem

Assume $F \in \mathbb{K}[x_1, \ldots, x_n] \setminus \mathbb{K}$ *irreducible. Let* U *be the set of points* $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) \in \mathbb{K}^{3n}$ *for which*

$$F(\alpha_1 t + \beta_1 u + \gamma_1, \ldots, \alpha_n t + \beta_n u + \gamma_n)$$

is irreducible in $\mathbb{K}[t,u]$. *Then* U *is a Zariski open subset of* $\mathbb{K}^{3n}$, *which is dense over the algebraic closure of* $\mathbb{K}$.

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$\longrightarrow$ modulo random shifts, bivariate instead of univariate projections suffice

$\longrightarrow$ in practice a shift is often not necessary

Bivariate factorization

Notation

$2 \leqslant \omega \leqslant 3$: two $n \times n$ matrices can be multiplied in time n^ω

$\mathsf{F}_{\mathbb{K}}(d)$: cost to completely factor a polynomial of degree d in $\mathbb{K}[x]$

Theorem (Lecerf 2010)

Let $F \in \mathbb{K}[x,y]$ of bidegree (d_x, d_y) be square-free and content-free in both x and y. Assume that $\operatorname{char} \mathbb{K} = 0$ *or* $\operatorname{char} \mathbb{K} > d_y(2d_x - 1)$. *Then, with high probability, we can compute the irreducible factorization of F in time*

$$\tilde{O}(d_x^2 d_y + d_x^\omega) + \mathsf{F}_{\mathbb{K}}(d_x).$$

Iterative method via Hensel lifting

Bivariate Hensel lifting

$F \in \mathbb{K}[x,y], \quad F(x,c) = A(x,c)\,B(x,c), \quad A,B$ coprime $\qquad (*)$

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Multivariate case

Assume $\qquad F(x_1,\ldots,x_n) = A(x_1,\ldots,x_n)\,B(x_1,\ldots,x_n)$, but A,B unknown

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$F(x_1,\ldots,x_{n-1},c) = A_c(x_1,\ldots,x_{n-1})\,B_c(x_1,\ldots,x_{n-1})$ coprime fact.

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Iterative method via Hensel lifting

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α →  → $F_\alpha(t,u) = A_\alpha(t,u)\,B_\alpha(t,u)$,

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Complexity of Hensel lifting

Theorem

Let $s := \min(s_A, s_B)$, $s' := \max(s_A, s_B)$ $\bar{s} := \max(s', s_F)$, $d := \deg F$, *and* $\delta := \max(\deg_{x_1} F, \ldots, \deg_{x_n} F)$. *Then Hensel lifting can be done with HP in time*

$$O(n((\bar{s}/s)\,\mathsf{S}(s') + \delta d\,\mathsf{S}(s) + s\,\mathsf{M}(\delta d) + s\,\mathsf{M}(d)\log d))$$

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In favorable cases, this leads to a HP algorithm to completely factor

$$F = A_1 \cdots A_\ell$$

in time

$$O(n((\bar{s}/s + \delta d)\,\mathsf{S}(s) + s\,\mathsf{M}(\delta d)\log \ell + s\,\mathsf{M}(d)\log d)) + \tilde{O}(\delta^3) + \mathsf{F}_{\mathbb{K}}(\delta)$$

General algorithm

$\alpha_1,\ldots,\alpha_n,\quad \beta_1,\ldots,\beta_n,\quad \gamma_1,\ldots,\gamma_n$ random

$$\begin{aligned}\hat{F}(x_1,\ldots,x_n,t,u,\lambda) \;:=\; F(&(1-\lambda+\alpha_1\lambda)(t+\beta_1 u+\gamma_1)\,x_1,\ldots,\\ &(1-\lambda+\alpha_n\lambda)(t+\beta_n u+\gamma_n)\,x_n).\end{aligned}$$

General algorithm

$\alpha_1,\ldots,\alpha_n, \quad \beta_1,\ldots,\beta_n, \quad \gamma_1,\ldots,\gamma_n$ random

$$\hat{F}(x_1,\ldots,x_n,t,u,\lambda) := F((1-\lambda+\alpha_1\lambda)(t+\beta_1 u+\gamma_1)x_1,\ldots,\\ (1-\lambda+\alpha_n\lambda)(t+\beta_n u+\gamma_n)x_n).$$

We use $\boldsymbol{\alpha}^k = (\alpha_1^k,\ldots,\alpha_n^k)$ as our interpolation points

$$F^{\langle i\rangle}(t,u,\lambda) := \hat{F}(\alpha_1^i,\ldots,\alpha_n^i,t,u,\lambda)$$

General algorithm

$\alpha_1,\ldots,\alpha_n,\quad \beta_1,\ldots,\beta_n,\quad \gamma_1,\ldots,\gamma_n$ random

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By construction

$$F^{\langle i+1\rangle}(t,u,0) \ = \ F^{\langle i\rangle}(t,u,1)$$

General algorithm — continued

By Hilbert-Bertini, with high probability,

we have the following “propagition” of irreducible factorizations:

$$\begin{array}{ccccccl}
F^{\langle i\rangle}(t,u,0) & = & A_1^{\langle i\rangle}(t,u,0) & \cdots & A_\ell^{\langle i\rangle}(t,u,0) & & \\
\downarrow & & \downarrow & & \downarrow & & \text{Hensel} \\
F^{\langle i\rangle}(t,u,\lambda) & = & A_1^{\langle i\rangle}(t,u,\lambda) & \cdots & A_\ell^{\langle i\rangle}(t,u,\lambda) & & \\
\downarrow & & \downarrow & & \downarrow & & \text{evaluate} \\
F^{\langle i\rangle}(t,u,1) & = & A_1^{\langle i\rangle}(t,u,1) & \cdots & A_\ell^{\langle i\rangle}(t,u,1) & & \\
\parallel & & \parallel & & \parallel & & \text{identify} \\
F^{\langle i+1\rangle}(t,u,0) & = & A_1^{\langle i+1\rangle}(t,u,0) & \cdots & A_\ell^{\langle i+1\rangle}(t,u,0) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

General algorithm — continued

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F^{\langle i\rangle}(t,u,\lambda) & = & A_1^{\langle i\rangle}(t,u,\lambda) & \cdots & A_\ell^{\langle i\rangle}(t,u,\lambda) & & \\
\downarrow & & \downarrow & & \downarrow & & \text{evaluate} \\
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\| & & \| & & \| & & \text{identify} \\
F^{\langle i+1\rangle}(t,u,0) & = & A_1^{\langle i+1\rangle}(t,u,0) & \cdots & A_\ell^{\langle i+1\rangle}(t,u,0) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

After one further tweak for normalization $\longrightarrow$ SLP for computing $A_1^{\langle i\rangle},\ldots,A_\ell^{\langle i\rangle}$

General algorithm — complexity

Theorem

Let $s := \max(s_{P_1}, \ldots, s_{P_\ell})$, $\bar{s} := \max(s, s_F)$, $d := \deg F$, *and* $e := \mathrm{ec}_w F \leqslant d^2$ *(for a suitable regularizing weight* w*). Assume* $\mathrm{char}\, \mathbb{K} = 0$ *or* $\mathrm{char}\, \mathbb{K} > 2d^2$*. Then, with high probability, we can compute the irreducible factorization of* F *in time*

$$O(\mathsf{S}(d^3 \bar{s}) + \mathsf{M}(d^3)\, s \log d) + \tilde{O}(e^5) + \mathsf{F}_{\mathbb{K}}(e + 3d)$$

Thank you !

http://www.TeXmacs.org