The JIL library (Justinline) for computations with SLPs

Joris van der Hoeven

Joint work with Albin Ahlbäck, Ricardo Buring, Grégoire Lecerf

CNRS, École polytechnique, France

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European Research Council
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ODELIX thematic program, Palaiseau **December 8, 2025**

Part I

Introduction and motivation

Example of an SLP (Straight Line Program)

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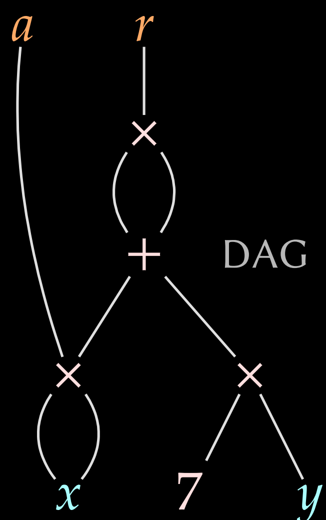

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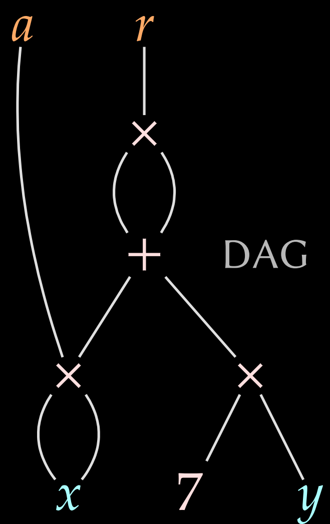

DAG

$$(x^2, (x^2 + 7y)^2)$$

Expression

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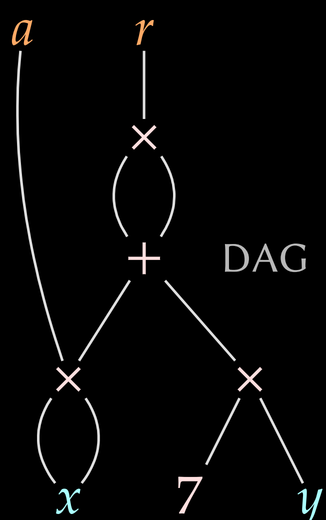

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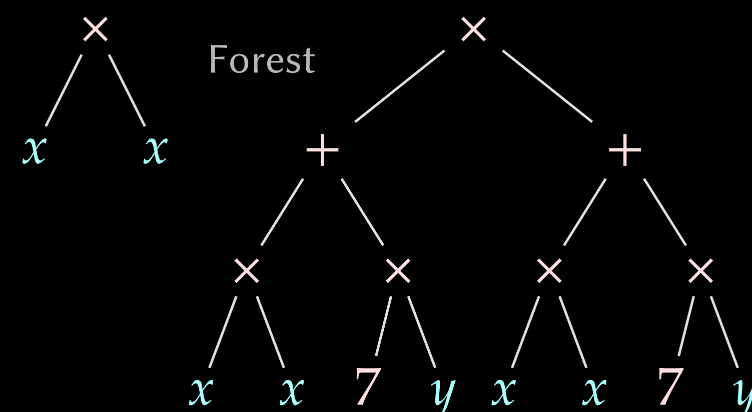

Forest

Why SLPs ?

Theory

- Algebraic complexity analysis

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- Discrete Fourier Transforms, multi-precision arithmetic, other“codelets”
- Solving ODEs via relaxed power series computations

Some references

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- Bürgisser, Clausen, Shokrollahi: *Algebraic complexity theory*, 1997

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Software

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Many *ad hoc* software for specific applications

- Geometric resolution ⟶ Aldaz, Castaño, Llovet, Martìnez, Hägele, Bruno, Heintz, Matera, Giusti, Lecerf, Salvy, Durvye, …, 2000–2008
- Codelets, DFTs ⟶ Frigo-Johnson, *FFTW3*, 1999, Püschel et al., *Spiral*, 2005
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More references

- vdH, Lecerf: *Towards a library for straight-line programs*, AAECC, 2025

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Justinline (in Mathemagix) **(2015–2024)**

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- Beyond SLPs…?

Motivation and spoiler

		JustInline		Jil		GPU	
	sols	jit	exe_8	jit	exe_8	jit	exe_{4096}
$Katsura_6$	64	1.11 s	3 ms	7.13 ms	39.1 ms	1.00 s	68.9 ms
$Katsura_8$	256	2.17 s	10 ms	13.6 ms	45.5 ms	3.21 s	0.42 s
$Katsura_{10}$	1024	3.74 s	37 ms	23.8 ms	0.20 s	7.58 s	0.72 s
$Katsura_{12}$	4096	6.16 s	0.23 s	42.4 ms	0.46 s	15.3 s	1.22 s
$Katsura_{14}$	16384	9.34 s	1.59 s	90.1 ms	1.94 s	29.4 s	5.53 s
$Katsura_{16}$	65536	13.4 s	11.3 s	0.27 s	11.9 s	49.6 s	29.1 s
$Katsura_{18}$	262144	21.3 s	145 s	0.95 s	83.2 s	82.3 s	157 s
$Katsura_{20}$	1048576	45.1 s	824 s	3.95 s	512 s	124 s	911 s
$Posso_{3,3}$	27	0.35 s	<1 ms	3.05 ms	0.26 ms	0.49 s	5.98 ms
$Posso_{4,4}$	256	1.50 s	3 ms	11.9 ms	1.5 ms	1.76 s	0.12 s
$Posso_{5,5}$	3125	8.22 s	78 ms	52.4 ms	0.31 s	19.1 s	0.96 s

Intel Xeon 3,2 GHz, AVX2 (JustInline), AVX512 (JIL), 8 threads

Nvidia GeForce RTX 4070 Super GPU, OpenCL, 4096 threads

2nd order stepper (JustInline), 1st order stepper (JIL)

Modern hardware

CPU

- Out of order execution
- Cache hierarchies
- Up to 8 or 16 wide SIMD parallelism
- Up to 100 cores

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TPU

- Only matrix multiplication, but about 8 times faster than SIMD in theory

Part II

Representing SLPs in JIL

Low level representation of SLPs

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0	1

out

3	5

prg

×	3	0	0	×	4	2	1
+	5	3	4	+	5	5	5

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$\Longrightarrow$ operations $+$, $\times$, … encoded as 32 bit integers

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out(a, r)	out(x_3, x_5)

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$\Longrightarrow$ operations $+$, $\times$, … encoded as 32 bit integers

data: array of elements in some “scalar domain” $\mathbb{K}$

Domains

Hardware domains

- $\mathbb{Z}_8, \mathbb{Z}_{16}, \mathbb{Z}_{32}, \mathbb{Z}_{64}, \mathbb{N}_8, \mathbb{N}_{16}, \mathbb{N}_{32}, \mathbb{N}_{64}, \mathbb{R}_{32}, \mathbb{R}_{64}$

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- SIMD variants, e.g. $\mathbb{Z}_8^{64}, \mathbb{Z}_{16}^{32}, \mathbb{Z}_{32}^{16}, \mathbb{Z}_{64}^{8}, \mathbb{N}_8^{64}, \mathbb{N}_{16}^{32}, \mathbb{N}_{32}^{16}, \mathbb{N}_{64}^{8}, \mathbb{R}_{32}^{16}, \mathbb{R}_{64}^{8}$

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- $\mathbb{Z}_{64}/(m\,\mathbb{Z}_{64})$, …
- $\mathrm{Recorder}(\mathbb{K})$

Operations

Signature Σ: the supported operations $\sigma\colon \mathbb{K}^{|\sigma|} \to \mathbb{K}$

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$$\Sigma_{\text{basic}} \;\supseteq\; \{\text{move}, \text{neg}, \text{add}, \text{sub}, \text{sqr}, \text{mul}, \text{nmul}, \text{fma}, \text{fnma}, \ldots\}$$

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Σ_{ext}: further operations for special domains $\mathbb{K}$ and user extensions

$$\begin{aligned}\Sigma_{\text{ext}} \;\supseteq\; & \{\text{duplicate}, \text{permute}\} \\ & \cup \{\text{shl}, \text{shr}, \ldots\} \\ & \cup \{\text{addc}, \text{subc}, \text{fmac}, \ldots\}\end{aligned}$$

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Embedding Boolean $\hookrightarrow \mathbb{K}$

- Needed for $\mathrm{eq}, \mathrm{neq}, \ldots, \mathrm{not}, \mathrm{or}, \ldots$

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- Needed for $\mathrm{eq}, \mathrm{neq}, \ldots, \mathrm{not}, \mathrm{or}, \ldots$
- Usually, natural implementation: for $\mathbb{K} = \mathbb{Z}_{64}$, take $\mathrm{false} \mapsto 0$, $\mathrm{true} \mapsto -1$.

Embedding $\mathbb{Z} \hookrightarrow \mathbb{K}$ and $\mathbb{Z}^w \hookrightarrow \mathbb{K}^w$

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- On $\mathbb{Z}/p\mathbb{Z}$ via $\mathbb{R}_{64}$, better implementation of $\text{fma}(a,b,c)$ than $ab+c$:

$\text{reduce}(a)$

$$\begin{aligned} q &:= a \cdot u \\ q &:= \text{round}(q) \\ r &:= \text{fnma}(p,q,r) \end{aligned}$$

$ab+c$

$$\begin{aligned} h &:= a \cdot b \\ l &:= \text{fms}(a,b,h) \\ r &:= \text{reduce}(h) \\ r &:= l+r \\ r &:= r+c \end{aligned}$$

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Part III

Using JIL

Typical workflow

$$\xrightarrow{\text{initiate}} \text{SLP } f_1 \xrightarrow{\text{transforms}} \text{SLP } f_2 \xrightarrow{\text{optimize}} \text{SLP } f_3 \xrightarrow{\text{backend}} \text{machine code}$$

Example: fast code for the cofactor matrix of a 4×4 matrix

- **Record** a program to compute a generic 4×4 determinant $\longrightarrow$ Initial SLP f_1
- Compute gradient of $f_1 \longrightarrow$ **algebraically transformed** SLP f_2
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template<typename C> C
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We may record an SLP for f as follows:

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slp
record_f (const domain& tp) {
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Idea: an instance of `slp_variable` specifies a data field on which to operate
operations on `slp_variable` are recorded instead of being executed

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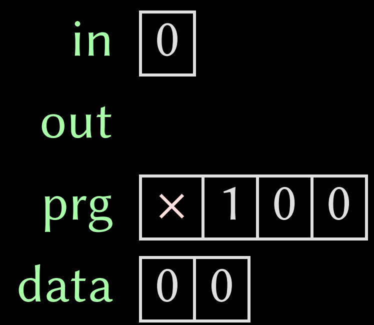

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out								
prg	×	1	0	0	+	2	1	0
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Note: clean execution trace more important than efficient execution of f

Traditional transformations

Important: by design, all transformations of SLP take linear time in JIL

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Simplification

- Common subexpression elimination
- CSE + algebraic simplifications ($0+x \to x$, etc.)
- Dead code elimination
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Backend

- Emulate missing instructions
- Optimizations for immediate arguments
- Rescheduling
- Register allocation

Algebraic transformations

General transformations

- Forward and backward differentiation: $f \longmapsto \mathrm{Jac}(f)$
- If f is linear (w.r.t. some of its inputs), then compute the transposed map
- $P \in \mathbb{K}[x_1, \ldots, x_n] \longmapsto$ homogeneous $\tilde{P} \in \mathbb{K}[x_1, \ldots, x_n, t]$ with $P(\boldsymbol{x}) = \tilde{P}(\boldsymbol{x}, 1)$
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Lifting and related transformations

- Lift SLP over $\mathbb{K}$ to SLP over $\mathbb{A}$ for $\mathbb{K}$-algebra $\mathbb{A}$
- Reinterpret SLP over $\mathbb{A}$ as SLP over $\mathbb{K}$
- Specific vector and ball lifts
- Reduce the number of divisions in SLPs

Example of complexification

$\mathbb{K}[i]$ is an **SLP algebra**: operations in Σ_{basic} can be implemented using SLPs:

add	subtract	multiply
$\text{in}(x_1, y_1, x_2, y_2)$	$\text{in}(x_1, y_1, x_2, y_2)$	$\text{in}(x_1, y_1, x_2, y_2)$
$x_3 := x_1 + x_2$	$x_3 := x_1 + x_2$	$x_3 := x_1 \cdot x_2$
$y_3 := y_1 + y_2$	$y_3 := y_1 + y_2$	$y_3 := x_1 \cdot y_2$
$\text{out}(x_3, y_3)$	$\text{out}(x_3, y_3)$	$x_3 := \text{fms}(y_1, y_2, x_3)$
		$y_3 := \text{fma}(x_2, y_1, y_3)$
		$\text{out}(x_3, y_3)$

Be careful with *aliasing* like in $z := u \cdot z$

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add

$$\begin{aligned} x_3 &:= x_1 + x_2 \\ y_3 &:= y_1 + y_2 \end{aligned}$$

subtract

$$\begin{aligned} x_3 &:= x_1 + x_2 \\ y_3 &:= y_1 + y_2 \end{aligned}$$

multiply

$$\begin{aligned} x_3 &:= x_1 \cdot x_2 \\ y_3 &:= x_1 \cdot y_2 \\ x_3 &:= \mathrm{fms}(y_1, y_2, x_3) \\ y_3 &:= \mathrm{fma}(x_2, y_1, y_3) \end{aligned}$$

$$\begin{aligned} &\mathrm{in}(z_0) \\ z_1 &:= z_0 \cdot z_0 \\ z_2 &:= z_1 + z_0 \\ z_4 &:= z_2 - 7 \\ &\mathrm{out}(z_4) \end{aligned}$$

$\xrightarrow{\text{lift}}$

$$\begin{aligned} &\mathrm{in}(x_0, y_0) \\ x_1 &:= x_0 \cdot x_0 \\ y_1 &:= x_0 \cdot y_0 \\ x_1 &:= \mathrm{fms}(y_0, y_0, x_1) \\ y_1 &:= \mathrm{fma}(x_0, y_0, y_1) \\ x_2 &:= x_1 + x_0 \\ y_2 &:= y_1 + y_0 \\ x_4 &:= x_2 - 7 \\ y_4 &:= y_2 - 0 \\ &\mathrm{out}(x_4, y_4) \end{aligned}$$

$\xrightarrow{\text{simplify}}$

$$\begin{aligned} &\mathrm{in}(x_0, y_0) \\ x_1 &:= x_0 \cdot x_0 \\ y_1 &:= x_0 \cdot y_0 \\ x_1 &:= \mathrm{fms}(y_0, y_0, x_1) \\ y_1 &:= \mathrm{fma}(x_0, y_0, y_1) \\ x_2 &:= x_1 + x_0 \\ y_2 &:= y_1 + y_0 \\ x_4 &:= x_2 - 7 \\ &\mathrm{out}(x_4, y_2) \end{aligned}$$

Some timings

n	len	cse	sim	∇	lift	reg	jit	exe
2	2	1460	2687	1221	1796	1436	12784	10.00
3	9	543	825	407	614	562	2963	2.222
4	40	229	341	184	262	291	924	0.550
5	205	126	196	125	163	237	452	0.424
6	1236	97	149	112	138	221	455	0.422
7	8659	89	144	127	134	229	447	0.419
8	69280	94	171	125	160	261	470	0.424
9	623529	133	207	188	178	347	472	0.865
10	6235300	158	242	296	245	391	522	3.308

Table. Timings in cycles per instruction on INTEL XEON for (very) naive $n \times n$ determinants.

Part IV
Upcoming features

Integrated benching, selection, and caching

We can efficiently generated multiple implementations of a function and determine which one is best by benching each implementation

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This requires

- Good random sample generators
- Mechanism to list alternative implementations and bench them
- Mechanism to cache results on disk
- Mechanism to predict timings without running any code

Beyond SLPs

Typical use cases:

- Run SLP $f: \mathbb{K}^m \rightarrow \mathbb{K}^n$ on vectors: $\tilde{f}: (\mathbb{K}^m)^N \longrightarrow (\mathbb{K}^n)^N$ with $N \gg 1$

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Need for *shallow* control structures

Dedicated support for various frequent patters…
… or better to use general purpose techniques?

Example on AVX512 platform

Input: $N \in \mathbb{N}$ and vectors $u, v \in \mathbb{R}_{64}[\mathrm{i}]^N$

Output: $w \in \mathbb{R}_{64}[\mathrm{i}]^N$ with $w_i = u_i v_i$ for all $0 \leqslant i < N$

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- Reinterpret u, v as $\tilde{u}, \tilde{v} \in (\mathbb{R}_{64}^8)^4[\mathrm{i}]$ (this requires SIMD matrix transposition)
- Lift complex multiplication over $\mathbb{R}_{64}$ to multiplication in $(\mathbb{R}_{64}^8)^4[\mathrm{i}]$ over $\mathbb{R}_{64}^8$
- Reinterpret $\tilde{w} := \tilde{u}\,\tilde{v}$ as an element of $\mathbb{R}_{64}[\mathrm{i}]$ (another SIMD transposition)

Create a loop which repeats this code until $N < 32$. Next

- Reduce further to the cases when $N < 16$ and then $N < 8$
- Reduce the case when $N < 8 \longrightarrow$ case $N = 8$ with an appropriate mask

Example on AVX512 platform

Input: $N \in \mathbb{N}$ and vectors $u, v \in \mathbb{R}_{64}[\mathrm{i}]^N$

Output: $w \in \mathbb{R}_{64}[\mathrm{i}]^N$ with $w_i = u_i v_i$ for all $0 \leqslant i < N$

First assume that $N = 32$.

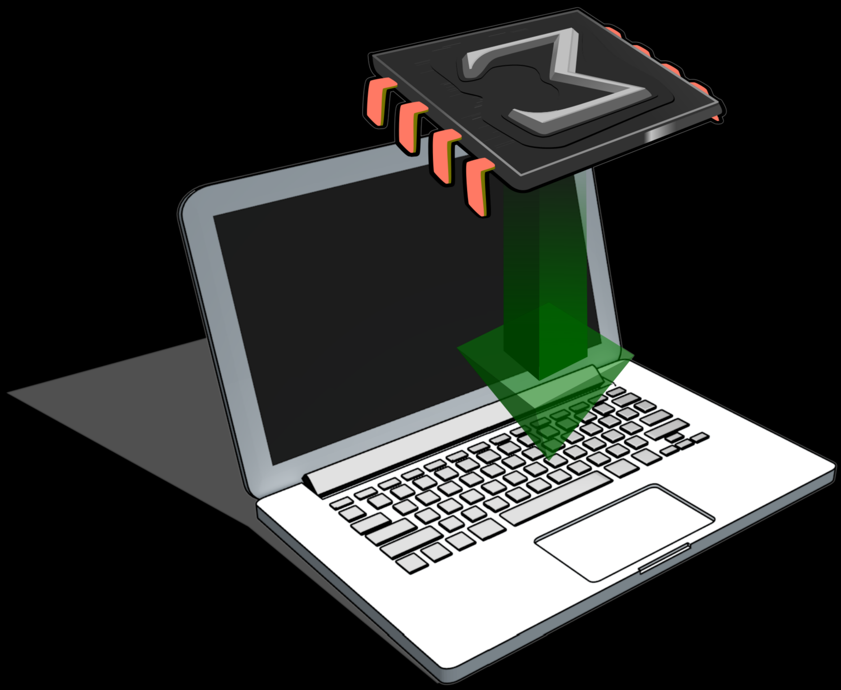
- Reinterpret u, v as $\tilde{u}, \tilde{v} \in (\mathbb{R}_{64}^8)^4[\mathrm{i}]$ (this requires SIMD matrix transposition)
- Lift complex multiplication over $\mathbb{R}_{64}$ to multiplication in $(\mathbb{R}_{64}^8)^4[\mathrm{i}]$ over $\mathbb{R}_{64}^8$
- Reinterpret $\tilde{w} := \tilde{u}\,\tilde{v}$ as an element of $\mathbb{R}_{64}[\mathrm{i}]$ (another SIMD transposition)

Create a loop which repeats this code until $N < 32$. Next

- Reduce further to the cases when $N < 16$ and then $N < 8$
- Reduce the case when $N < 8 \longrightarrow$ case $N = 8$ with an appropriate mask

This whole implementation can be run using only AVX512 vector instructions.

Thank you !



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