Integer multiplication is at least as hard as matrix transposition

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$$\log^* N := \min\{\ell \in \mathbb{N} : \log \circ \overset{\ell \times}{\cdots} \circ \log N \leqslant 1\}$$

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Question: is $N \log N$ also a lower bound for the complexity of multiplication?

Matrix transposition on a Turing machine

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{pmatrix}$$

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a_0	b_0	c_0	d_0	a_1	b_1	c_1	d_1	a_2	b_2	c_2	d_2	a_3	b_3	c_3	d_3	

a_0	a_1	b_0	b_1	c_0	c_1	d_0	d_1	a_2	a_3	b_2	b_3	c_2	c_3	d_2	d_3	

In general: $r \times c$ transposition $\longrightarrow O(rc \lg \min(r,c))$ operations

Folklore algorithm. $O(n^2 \lg n)$

Complexity of $n \times n$ matrix transposition

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Warning. We implicitly assumed 1-bit entries.
For b-bit entries, the complexity becomes $O(b n^2 \lg n)$.

The main theorem

$\mathsf{M}(N)$ complexity of integer multiplication
$\mathsf{T}(N)$ complexity of $r \times c$ matrix transposition with b-bit entries and $brc \leqslant N$.

Theorem

If the conjectured lower bound

$$\mathsf{T}(N) = \Omega(N \log N)$$

holds for transposition, then so does the one for multiplication:

$$\mathsf{M}(N) = \Omega(N \log N).$$

Bluestein: DFT $\longrightarrow$ convolution

$\zeta_n := \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}, \quad X \in \mathbb{C}^n$

$$Y_t \;:=\; \mathrm{DFT}_n(X)_t \;:=\; \frac{1}{n}\sum_{0\leqslant s<n} \zeta_n^{-st} X_s$$

$$X_s \;=\; \mathrm{DFT}_n^{-1}(Y)_s \;=\; \sum_{0\leqslant t<n} \zeta_n^{st} Y_t$$

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Setting $\omega_k := \zeta_n^{\binom{k}{2}}$, we have

$$Y_t \;=\; \frac{1}{n} \bar{\omega}_t \sum_{0 \leqslant s < n} (\bar{\omega}_{-s} X_s)\, \omega_{t-s}$$

DFT of length n $\xrightarrow{\text{reduces}}$ product of $\tilde{X}, \Omega \in \mathbb{C}[u]$ of degrees $< n$, with Ω fixed

Kronecker: convolution $\longrightarrow$ integer multiplication

DFT: approximated using b-bit fixed point arithmetic

Reduces to arithmetic on b-bit integer coefficients

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$$\tilde{X} = 12u^2 + 85u + 27, \qquad \Omega = 33u^2 + 42u + 11$$

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$$\tilde{X}(10^5) = 120008500027, \qquad \Omega(10^5) = 330004200011$$

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$$\tilde{X} = 12\,u^2 + 85\,u + 27, \qquad \Omega = 33\,u^2 + 42\,u + 11$$

$$\tilde{X}(10^5) = 1200008500023, \qquad \Omega(10^5) = 3300004200011$$

$$(\tilde{X}\Omega)(10^5) = 396033090446101901000253$$

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Lemma

If $\lg n = O(b)$, $\lg^2 b = O(n)$, *then computing a b-bit DFT of length n takes time*

$$O(\mathsf{M}(nb) + n\,\mathsf{M}(b)).$$

Sliced DFTs

$$\begin{array}{ccc} X_0^{[0]},\ldots,X_0^{[l-1]}, & \cdots & ,X_{n-1}^{[0]},\ldots,X_{n-1}^{[l-1]} \\ & \downarrow & \\ Y_0^{[0]},\ldots,Y_0^{[l-1]}, & \cdots & ,Y_{n-1}^{[0]},\ldots,Y_{n-1}^{[l-1]} \end{array}$$

$$Y^{[i]} = \mathrm{DFT}(X^{[i]})$$

Sliced DFTs

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& \downarrow & \\
Y_0^{[0]}, \ldots, Y_0^{[l-1]}, & \ldots & , Y_{n-1}^{[0]}, \ldots, Y_{n-1}^{[l-1]}
\end{array}
\qquad Y^{[i]} = \mathrm{DFT}(X^{[i]})
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$$Y^{[i]}(u) = \tilde{X}^{[i]}(u)\,\Omega(u)$$

Sliced DFTs

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$$\downarrow$$

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$$Y^{[i]} = \mathrm{DFT}(X^{[i]})$$

$$Y^{[i]}(u) = \tilde{X}^{[i]}(u)\,\Omega(u)$$

$$Y^{[0]}(u) + \cdots + Y^{[l-1]}(u)\,u^{2n(l-1)} = (\tilde{X}^{[0]}(u) + \cdots + \tilde{X}^{[l-1]}(u)\,u^{2n(l-1)})\,\Omega(u)$$

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Lemma

If $\lg n = O(b)$ *and* $\lg^2 b = O(n)$*, then computing* l *sliced* b*-bit DFTs of length* n *can be done in time*

$$O(\mathsf{M}(l\,n\,b) + l\,n\,\mathsf{M}(b)).$$

$n_1 \times n_2$ transposition via DFTs

$$\begin{array}{cl} X_0, \ldots, X_{n_1 n_2 - 1} & \\ \downarrow & \mathrm{DFT}_{n_1 n_2} \\ Y_0, \ldots, Y_{n_1 n_2 - 1} & \\ \downarrow & \mathrm{DFT}^{-1}_{n_1, n_2} \\ X_0, \ldots, X_{(n_1-1) n_2}, \ldots, X_{n_2 - 1}, \ldots, X_{n_1 n_2 - 1} & \end{array}$$

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$$\mathrm{DFT}^{-1}_{n_1,n_2} \overset{\text{Cooley-Tukey}}{=\!=\!=\!=} (\mathrm{Id}_{n_2} \otimes \mathrm{DFT}^{-1}_{n_1}) \circ \mathrm{Twiddle} \circ (\mathrm{DFT}^{-1}_{n_2} \otimes \mathrm{Id}_{n_1})$$

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\begin{array}{ccc}
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\end{array}
$$

$$
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Theorem

$n = n_1 n_2$, $\lg n = O(b)$, $\lg^2 b = O(n)$ $\Rightarrow$ $n_1 \times n_2$ *transposition* b*-bit entries take time*

$$O(\mathsf{M}(n\,b) + n\,\mathsf{M}(b)).$$

Exponential reduction of b, the bit-size of entries

$n_1 \times n_2$ transposition, 1-bit entries, $\qquad \log n_1 \approx \log n_2$

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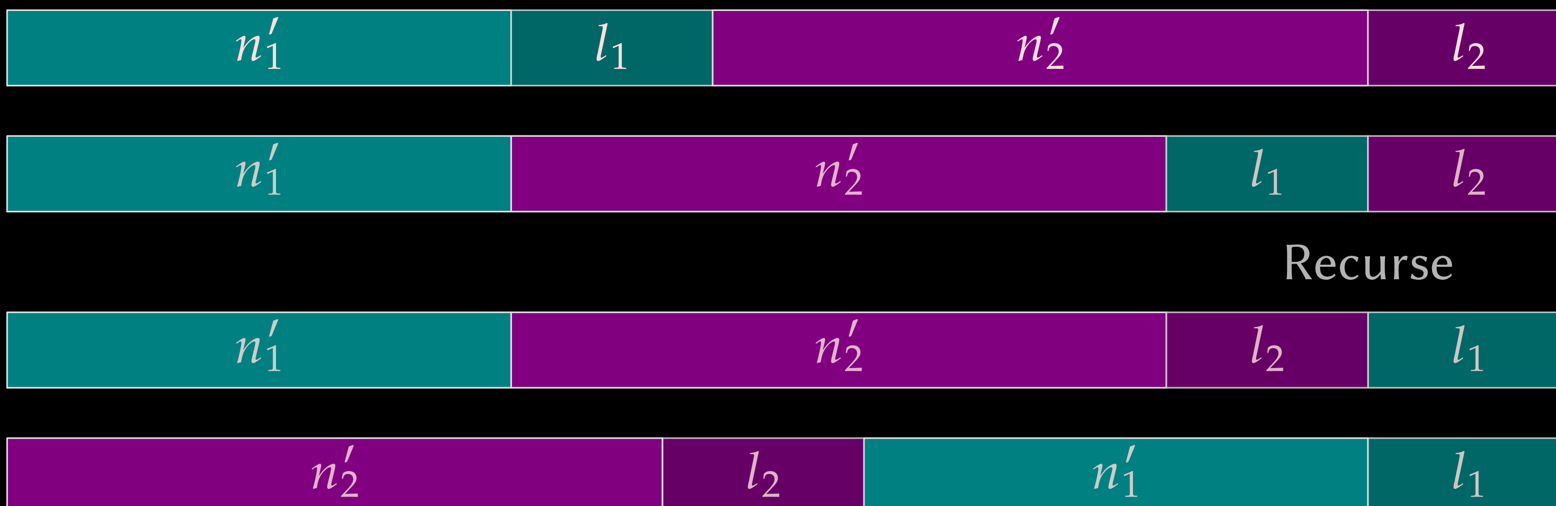

Conclusion

Theorem

Assume $M(N)/N$ non-decreasing. Then $n_1 \times n_2$ transposition can be done in time

$$O(\mathsf{M}(n_1 n_2) + n_1 n_2 \log^* \max(n_1, n_2)).$$

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In general, $n_1 \times n_2$ transposition can be done in time

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Corollary

If $\mathsf{T}(N) = \Omega(N \log N)$, then $\mathsf{M}(N) = \Omega(N \log N)$.

Thank you !

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