

Closed function spaces for differential algebra

Joris van der Hoeven

Partially joint work with ASCHENBRENNER, VDDRIES, POGUDIN, CHEN, FANG
CNRS, École polytechnique, France



**Funded by
the European Union**



European Research Council
Established by the European Commission

Hilbert Nullstellensatz

Radical ideals of $\mathbb{C}[x_1, \dots, x_n]$ $\xleftrightarrow{\text{bijection}}$ Zero-sets in $\mathbb{C}^n$ of polynomial systems

Hilbert Nullstellensatz

Radical ideals of $\mathbb{C}[x_1, \dots, x_n]$ $\xleftrightarrow{\text{bijection}}$ Zero-sets in $\mathbb{C}^n$ of polynomial systems

The theory ACF_0 of algebraically closed fields of characteristic 0

Complete, model complete, elimination of quantifiers, ...

Hilbert Nullstellensatz

Radical ideals of $\mathbb{C}[x_1, \dots, x_n]$ $\xleftrightarrow{\text{bijection}}$ Zero-sets in $\mathbb{C}^n$ of polynomial systems

The theory ACF_0 of algebraically closed fields of characteristic 0

Complete, model complete, elimination of quantifiers, ...

Transfer principle for simple models like $\mathbb{C}$

Any first order statement proved for the concrete model $\mathbb{C}$ holds for all algebraically closed fields.

Hilbert Nullstellensatz

Radical ideals of $\mathbb{C}[x_1, \dots, x_n]$ $\xleftrightarrow{\text{bijection}}$ Zero-sets in $\mathbb{C}^n$ of polynomial systems

The theory ACF_0 of algebraically closed fields of characteristic 0

Complete, model complete, elimination of quantifiers, ...

Transfer principle for simple models like $\mathbb{C}$

Any first order statement proved for the concrete model $\mathbb{C}$ holds for all algebraically closed fields.

What can we say in the case of differential algebra?

Flokllore dictionary

$$\mathbb{C}[x_1, \dots, x_n] \longleftrightarrow \mathbb{C}[\partial_{x_1}, \dots, \partial_{x_n}]$$

Buchberger reduction $\longleftrightarrow$ Ritt reduction

S -polynomials $\longleftrightarrow$ Δ -polynomials

Gröbner basis $\longleftrightarrow$ Coherent autoreduced set

$\vdots$

See also: **Rosenfeld 1959**

Radical ideals

$$\begin{aligned} \{(x, y) : (x-1)^2 = x^2 + y^2 = 0\} &\longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\} \\ \{(1, i), (1, -i)\} &\longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} e^{x-iy} \end{aligned}$$

Radical ideals

$$\{(x, y) : (x-1)^2 = x^2 + y^2 = 0\} \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\{(1, i), (1, -i)\} \longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C})$

Radical ideals

$$\{(x, y) : (x-1)^2 = x^2 + y^2 = 0\} \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\{(1, i), (1, -i)\} \longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C})$

General ideals

$$((x-1)^2, x^2 + y^2) \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} (x+iy) e^{x+iy} + \mathbb{C} e^{x-iy} + \mathbb{C} (x-iy) e^{x-iy}$$

Radical ideals

$$\{(x, y) : (x-1)^2 = x^2 + y^2 = 0\} \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\{(1, i), (1, -i)\} \longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C})$

General ideals

$$((x-1)^2, x^2 + y^2) \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} (x+iy) e^{x+iy} + \mathbb{C} e^{x-iy} + \mathbb{C} (x-iy) e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (x^k y^l e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C}, k, l \in \mathbb{N})$

Radical ideals

$$\{(x, y) : (x-1)^2 = x^2 + y^2 = 0\} \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\{(1, i), (1, -i)\} \longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C})$

General ideals

$$((x-1)^2, x^2 + y^2) \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} (x+iy) e^{x+iy} + \mathbb{C} e^{x-iy} + \mathbb{C} (x-iy) e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (x^k y^l e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C}, k, l \in \mathbb{N})$

Note: the *differential* ideal $[(\partial_x - 1)^2 F, (\partial_x^2 + \partial_y^2) F]$ is radical.

Radical ideals

$$\{(x, y) : (x-1)^2 = x^2 + y^2 = 0\} \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\{(1, i), (1, -i)\} \longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C})$

General ideals

$$((x-1)^2, x^2 + y^2) \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\longleftrightarrow \mathbb{C} e^{x+iy} + \mathbb{C} (x+iy) e^{x+iy} + \mathbb{C} e^{x-iy} + \mathbb{C} (x-iy) e^{x-iy}$$

→ solutions in $\text{Vect}_{\mathbb{C}} (x^k y^l e^{\lambda x + \mu y} : \lambda, \mu \in \mathbb{C}, k, l \in \mathbb{N})$

Note: the *differential ideal* $[(\partial_x - 1)^2 F, (\partial_x^2 + \partial_y^2) F]$ is radical.

Also: *differential shape lemmas*, **Kauers–Koutchan–Verron, vdH–Pogudin**

Local solutions of $L \in \mathbb{C}[x][\partial_x]$ at $x \rightarrow \infty$ (Fabry, Poincaré, Birkhoff, ...)

$$f = (f_d \log^d x + \cdots + f_0) x^\alpha e^{P(\sqrt[r]{x})}, \quad f_k = \sum_{i \geq 0} f_{k,i} x^{-i/r} \in \mathbb{C}[[x^{1/r}]]$$

Local solutions of $L \in \mathbb{C}[x][\partial_x]$ at $x \rightarrow \infty$ (Fabry, Poincaré, Birkhoff, ...)

$$f = (f_d \log^d x + \cdots + f_0) x^\alpha e^{P(\sqrt[r]{x})}, \quad f_k = \sum_{i \geq 0} f_{k,i} x^{-i/r} \in \mathbb{C}[[x^{1/r}]]$$

Partial differential equations

System $Lf = g$, $L \in \mathbb{C}[x_1, \dots, x_n][\partial_{x_1}, \dots, \partial_{x_n}]^m$, $g \in \mathbb{S}^m$

$$\mathbb{S} := \mathbb{C}[\log x_1, \dots, \log x_n]((x_1^{\mathbb{C}} \cdots x_n^{\mathbb{C}}))_{\text{gb}}$$

vdH 2007: consistent system admits distinguished (computable) solution in $\mathbb{S}$

Local solutions of $L \in \mathbb{C}[x][\partial_x]$ at $x \rightarrow \infty$ (Fabry, Poincaré, Birkhoff, ...)

$$f = (f_d \log^d x + \cdots + f_0) x^\alpha e^{P(\sqrt[r]{x})}, \quad f_k = \sum_{i \geq 0} f_{k,i} x^{-i/r} \in \mathbb{C}[[x^{1/r}]]$$

Partial differential equations

System $Lf = g$, $L \in \mathbb{C}[x_1, \dots, x_n][\partial_{x_1}, \dots, \partial_{x_n}]^m$, $g \in \mathbb{S}^m$

$$\mathbb{S} := \mathbb{C}[\log x_1, \dots, \log x_n]((x_1^{\mathbb{C}} \cdots x_n^{\mathbb{C}}))_{\text{gb}}$$

vdH 2007: consistent system admits distinguished (computable) solution in $\mathbb{S}$

$$\mathbb{S}_{\text{exp}} := \text{Vect}_{\mathbb{C}}(\mathbb{S} e^{\mathbb{S}}) \quad \text{or} \quad \text{Vect}_{\mathbb{C}}(\mathbb{S} e^{\mathbb{C}[x_1, \dots, x_n]^{\text{ac}}})$$

Open: is $\mathbb{S}_{\text{exp}}$ “closed” for solving homogeneous systems $Lf = 0$?

$$(e^x)' = e^x, \quad (e^{e^x})' = e^x e^{e^x}, \quad (e^{e^{e^x}})' = e^x e^{e^x} e^{e^{e^x}}, \quad \dots$$

$$(e^x)' = e^x, \quad (e^{e^x})' = e^x e^{e^x}, \quad (e^{e^{e^x}})' = e^x e^{e^x} e^{e^{e^x}}, \quad \dots$$

Real transseries $\mathbb{T} = \mathbb{R}[[[x]]]$ at infinity $x \rightarrow \infty$

$$e^{e^x + 2\frac{e^x}{x} + 6\frac{e^x}{x^2} + 24\frac{e^x}{x^3} + \dots} + e^{(\log x)^3 + (\log x)^2} + \pi - \frac{1}{\log x} + \frac{1}{e^x} - \frac{1}{e^{2x}} + \dots$$

vdH 1997: any $L \in \mathbb{T}[\partial_x]$ of odd order has a non-zero root in $\mathbb{T}$

$$(e^x)' = e^x, \quad (e^{e^x})' = e^x e^{e^x}, \quad (e^{e^{e^x}})' = e^x e^{e^x} e^{e^{e^x}}, \quad \dots$$

Real transseries $\mathbb{T} = \mathbb{R}[[[x]]]$ at infinity $x \rightarrow \infty$

$$e^{e^x + 2\frac{e^x}{x} + 6\frac{e^x}{x^2} + 24\frac{e^x}{x^3} + \dots} + e^{(\log x)^3 + (\log x)^2} + \pi - \frac{1}{\log x} + \frac{1}{e^x} - \frac{1}{e^{2x}} + \dots$$

vdH 1997: any $L \in \mathbb{T}[\partial_x]$ of odd order has a non-zero root in $\mathbb{T}$

Complex transseries $\tilde{\mathbb{T}} = \mathbb{C}[[[z]]]^{\text{osc}}$

Construction requires choosing a region: $e^{iz} \succ 1$ or $e^{iz} \prec 1$, $e^{z^2} \succ 1$ or $e^{z^2} \prec 1$?

vdH 2001: $\tilde{\mathbb{T}}$ is Picard-Vessiot closed for any choice of a region

***H*-fields (Aschenbrenner–vdDries 2002):**

Differential field K with ordering $\leq$ (whence valuation) and constants C such that

HF1. If $f > C$, then $f' > 0$.

HF2. If $f \leq 1$, then there exists $c = \lim f \in C$ with $f \sim c$ (i.e. $f - c < 1$).

Often, the derivation is also **small**: $f < 1 \implies f' < 1$

***H*-fields (Aschenbrenner–vdDries 2002):**

Differential field K with ordering $\leq$ (whence valuation) and constants C such that

HF1. If $f > C$, then $f' > 0$.

HF2. If $f \leq 1$, then there exists $c = \lim f \in C$ with $f \sim c$ (i.e. $f - c < 1$).

Often, the derivation is also **small**: $f < 1 \implies f' < 1$

Closed *H*-fields (Aschenbrenner–vdDries–vdH 2017, 2022)

LC. K is Liouville closed: real closed + closed under $\int$ and $\exp \circ \int$.

DIVP. For $P \in K\{Y\}$, if $P(f)P(g) < 0$ for $f < g$ in K , then $\exists h \in (f, g), P(h) = 0$.

***H*-fields (Aschenbrenner–vdDries 2002):**

Differential field K with ordering $\leq$ (whence valuation) and constants C such that

HF1. If $f > C$, then $f' > 0$.

HF2. If $f \leq 1$, then there exists $c = \lim f \in C$ with $f \sim c$ (i.e. $f - c < 1$).

Often, the derivation is also **small**: $f < 1 \implies f' < 1$

Closed *H*-fields (Aschenbrenner–vdDries–vdH 2017, 2022)

LC. K is Liouville closed: real closed + closed under $\int$ and $\exp \circ \int$.

DIVP. For $P \in K\{Y\}$, if $P(f)P(g) < 0$ for $f < g$ in K , then $\exists h \in (f, g), P(h) = 0$.

Examples

- The transseries $\mathbb{T}$ (grid-based, well-based, ...). **(2017)**
- Any maximal Hardy field. **(2023)**
- The surreal numbers. **(2019)**

Aschenbrenner–vdDries–vdH 2017

The theory of H -closed fields with small derivation is complete, model complete, and eliminates quantifiers in a suitable language.

Aschenbrenner–vdDries–vdH 2017

The theory of H -closed fields with small derivation is complete, model complete, and eliminates quantifiers in a suitable language.

Applications

- Any formal d -algebraic transseries can be realized in a Hardy field.

Aschenbrenner–vdDries–vdH 2017

The theory of H -closed fields with small derivation is complete, model complete, and eliminates quantifiers in a suitable language.

Applications

- Any formal d -algebraic transseries can be realized in a Hardy field.
- $\zeta(x)$ is d -transcendental over $\mathbb{R}$ (**Grigoriev–Singer 1991**)

Aschenbrenner–vdDries–vdH 2017

The theory of H -closed fields with small derivation is complete, model complete, and eliminates quantifiers in a suitable language.

Applications

- Any formal d -algebraic transseries can be realized in a Hardy field.
- $\zeta(x)$ is d -transcendental over $\mathbb{R}$ (**Grigoriev–Singer 1991**)
- $\zeta(x)$ and $x + \sqrt{x} + \sqrt{\sqrt{x}} + \cdots$ are mutually d -transcendental over $\mathbb{R}$.

Aschenbrenner–vdDries–vdH 2017

The theory of H -closed fields with small derivation is complete, model complete, and eliminates quantifiers in a suitable language.

Applications

- Any formal d -algebraic transseries can be realized in a Hardy field.
- $\zeta(x)$ is d -transcendental over $\mathbb{R}$ (**Grigoriev–Singer 1991**)
- $\zeta(x)$ and $x + \sqrt{x} + \sqrt{\sqrt{x}} + \cdots$ are mutually d -transcendental over $\mathbb{R}$.
- **To check!** Coherence of ring of integro-differential operators (**Bavula 2013**)

Aschenbrenner–vdDries–vdH 2017

The theory of H -closed fields with small derivation is complete, model complete, and eliminates quantifiers in a suitable language.

Applications

- Any formal d-algebraic transseries can be realized in a Hardy field.
- $\zeta(x)$ is d-transcendental over $\mathbb{R}$ (**Grigoriev–Singer 1991**)
- $\zeta(x)$ and $x + \sqrt{x} + \sqrt{\sqrt{x}} + \cdots$ are mutually d-transcendental over $\mathbb{R}$.
- **To check!** Coherence of ring of integro-differential operators (**Bavula 2013**)
- Definable subsets of $\mathbb{R}^n$ are the semi-algebraic subsets.

What about complex transseries ?

9/14

vdH 2001: any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \tilde{\mathbb{T}}$ has a root in $\tilde{\mathbb{T}}$.

vdH 2001: any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \tilde{\mathbb{T}}$ has a root in $\tilde{\mathbb{T}}$.

Also true for $\mathbb{C}((x^{\mathbb{R}} (\log x)^{\mathbb{R}} (\log \log x)^{\mathbb{R}} \dots))_{\text{gb}}$ instead of $\tilde{\mathbb{T}}$.

vdH 2001: any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \tilde{\mathbb{T}}$ has a root in $\tilde{\mathbb{T}}$.

Also true for $\mathbb{C}((x^{\mathbb{R}} (\log x)^{\mathbb{R}} (\log \log x)^{\mathbb{R}} \dots))_{\text{gb}}$ instead of $\tilde{\mathbb{T}}$.

Applications

- Zero-test for d-algebraic power series

(vdH 2002).

vdH 2001: any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \tilde{\mathbb{T}}$ has a root in $\tilde{\mathbb{T}}$.

Also true for $\mathbb{C}((x^{\mathbb{R}} (\log x)^{\mathbb{R}} (\log \log x)^{\mathbb{R}} \dots))_{\text{gb}}$ instead of $\tilde{\mathbb{T}}$.

Applications

- Zero-test for d-algebraic power series
- Zero-test for d-algebraic transseries

(vdH 2002).

(Chen–Fang–vdH 2026).

vdH 2001: any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \tilde{\mathbb{T}}$ has a root in $\tilde{\mathbb{T}}$.

Also true for $\mathbb{C}((x^{\mathbb{R}} (\log x)^{\mathbb{R}} (\log \log x)^{\mathbb{R}} \dots))_{\text{gb}}$ instead of $\tilde{\mathbb{T}}$.

Applications

- Zero-test for d-algebraic power series
- Zero-test for d-algebraic transseries
- $\tilde{\mathbb{T}}$ is Picard-Vessiot closed.

(vdH 2002).

(Chen–Fang–vdH 2026).

vdH 2001: any $P \in \tilde{\mathbb{T}}\{Y\} \setminus \tilde{\mathbb{T}}$ has a root in $\tilde{\mathbb{T}}$.

Also true for $\mathbb{C}((x^{\mathbb{R}} (\log x)^{\mathbb{R}} (\log \log x)^{\mathbb{R}} \dots))_{\text{gb}}$ instead of $\tilde{\mathbb{T}}$.

Applications

- Zero-test for d-algebraic power series (vdH 2002).
- Zero-test for d-algebraic transseries (Chen–Fang–vdH 2026).
- $\tilde{\mathbb{T}}$ is Picard-Vessiot closed.

However

(Rosenlicht 1980)

$\begin{cases} y^3 + y = (y')^2 \\ y^3 + y \neq 0 \end{cases}$ has no solution at $z \rightarrow \infty$ (so $\tilde{\mathbb{T}}$ is not differentially closed)

Hence, $\wp$ does not lie in the Picard-Vessiot closure of $\mathbb{C}(z)$.

Recursive discrete ramification

Starting with $\mathcal{R}_0 := \mathbb{C}$, repeat a finite number of times:

- Select a discrete subset $\mathcal{S}_n$ of $\mathcal{R}_n$.
- Let $\mathcal{R}_{n+1}$ be the covering space of $\mathcal{R}_n \setminus \mathcal{S}_n$.

Recursive discrete ramification

Starting with $\mathcal{R}_0 := \mathbb{C}$, repeat a finite number of times:

- Select a discrete subset $\mathcal{S}_n$ of $\mathcal{R}_n$.
- Let $\mathcal{R}_{n+1}$ be the covering space of $\mathcal{R}_n \setminus \mathcal{S}_n$.

Dendromorphic functions

Injective limit of analytic function spaces on recursive discrete ramifications of $\mathbb{C}$.

Recursive discrete ramification

Starting with $\mathcal{R}_0 := \mathbb{C}$, repeat a finite number of times:

- Select a discrete subset $\mathcal{S}_n$ of $\mathcal{R}_n$.
- Let $\mathcal{R}_{n+1}$ be the covering space of $\mathcal{R}_n \setminus \mathcal{S}_n$.

Dendromorphic functions

Injective limit of analytic function spaces on recursive discrete ramifications of $\mathbb{C}$.

- They form a Picard-Vessiot closed space.

Recursive discrete ramification

Starting with $\mathcal{R}_0 := \mathbb{C}$, repeat a finite number of times:

- Select a discrete subset $\mathcal{I}_n$ of $\mathcal{R}_n$.
- Let $\mathcal{R}_{n+1}$ be the covering space of $\mathcal{R}_n \setminus \mathcal{I}_n$.

Dendromorphic functions

Injective limit of analytic function spaces on recursive discrete ramifications of $\mathbb{C}$.

- They form a Picard-Vessiot closed space.
- They are “boundaryless”: for any $\zeta \in \mathcal{R}_n$, $\varepsilon > 0$, and path $\varphi: [0, 1] \rightarrow \mathbb{C}$, there exists an ε -close path that lifts to $\mathcal{R}_n$.

Recursive discrete ramification

Starting with $\mathcal{R}_0 := \mathbb{C}$, repeat a finite number of times:

- Select a discrete subset $\mathcal{S}_n$ of $\mathcal{R}_n$.
- Let $\mathcal{R}_{n+1}$ be the covering space of $\mathcal{R}_n \setminus \mathcal{S}_n$.

Dendromorphic functions

Injective limit of analytic function spaces on recursive discrete ramifications of $\mathbb{C}$.

- They form a Picard-Vessiot closed space.
- They are “boundaryless”: for any $\zeta \in \mathcal{R}_n$, $\varepsilon > 0$, and path $\varphi: [0, 1] \rightarrow \mathbb{C}$, there exists an ε -close path that lifts to $\mathcal{R}_n$.

Application

- The generating function of the partition numbers $\notin$ PV-closure of $\mathbb{C}(z)$.

Differentially closed field

Algebraically closed differential field K such that

$\forall P, Q \in K\{Y\}$ with $\text{ord } Q < \text{ord } P$, $\exists y \in K$ with $P(y) = 0 \neq Q(y)$.

Differentially closed field

Algebraically closed differential field K such that

$\forall P, Q \in K\{Y\}$ with $\text{ord } Q < \text{ord } P$, $\exists y \in K$ with $P(y) = 0 \neq Q(y)$.

Construction over an algebraically closed constant field $K = C_K$

$$K_\ell := K((\lambda_1^{\mathbb{Q}})) \cdots ((\lambda_\ell^{\mathbb{Q}}))$$

$$L_\ell := K_\ell((z_\ell^{\mathbb{Q}}))$$

L_ℓ differential field w.r.t. ∂_{z_ℓ} with constant field $C_{L_\ell} = K_\ell$.

Differentially closed field

Algebraically closed differential field K such that

$\forall P, Q \in K\{Y\}$ with $\text{ord } Q < \text{ord } P$, $\exists y \in K$ with $P(y) = 0 \neq Q(y)$.

Construction over an algebraically closed constant field $K = C_K$

$$K_\ell := K((\lambda_1^{\mathbb{Q}})) \cdots ((\lambda_\ell^{\mathbb{Q}}))$$

$$L_\ell := K_\ell((z_\ell^{\mathbb{Q}}))$$

L_ℓ differential field w.r.t. ∂_{z_ℓ} with constant field $C_{L_\ell} = K_\ell$.

We embed $L_\ell \xrightarrow{\Phi_\ell} L_{\ell+1}$ via $z_\ell = \lambda_{\ell+1} + z_{\ell+1}$:

$$\Phi_\ell \left(\sum_{i \geq i_0} y_i z_\ell^i \right) := \sum_{k \geq 0} \left(\sum_{i \geq i_0} \frac{i \cdots (i - k + 1)}{k!} y_i \lambda_{\ell+1}^{i-k} \right) z_{\ell+1}^k$$

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.
- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\text{ord } Q < r := \text{ord } P$ and P square-free.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.
- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\text{ord } Q < r := \text{ord } P$ and P square-free.
- Then $R := QI_P \text{Res}_{Y^{(r)}}(P, S_P) \neq 0$.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.
- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\text{ord } Q < r := \text{ord } P$ and P square-free.
- Then $R := QI_P \text{Res}_{Y^{(r)}}(P, S_P) \neq 0$.
- Rewrite $P, Q, R, \dots$ as power series in $z_{\ell+1}$.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.
- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\text{ord } Q < r := \text{ord } P$ and P square-free.
- Then $R := Q I_P \text{Res}_{Y^{(r)}}(P, S_P) \neq 0$.
- Rewrite $P, Q, R, \dots$ as power series in $z_{\ell+1}$.
- Take initial $\alpha_0, \dots, \alpha_{r-1} \in K_{\ell+1}$ with $R[z_{\ell+1}=0](\alpha_0, \dots, \alpha_{r-1}) \neq 0$.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.
- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\text{ord } Q < r := \text{ord } P$ and P square-free.
- Then $R := QI_P \text{Res}_{Y^{(r)}}(P, S_P) \neq 0$.
- Rewrite $P, Q, R, \dots$ as power series in $z_{\ell+1}$.
- Take initial $\alpha_0, \dots, \alpha_{r-1} \in K_{\ell+1}$ with $R[z_{\ell+1}=0](\alpha_0, \dots, \alpha_{r-1}) \neq 0$.
- Complete: unique power series $y = \alpha_0 + \dots + \frac{\alpha_{r-1}}{(r-1)!} z^{r-1} + \dots$ with $P(y) = 0$.

Theorem. The limit $L_\infty := \bigcup_{\ell \in \mathbb{N}} L_\ell$ is differentially closed.

- Algebraic closedness: automatic as increasing union.
- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\text{ord } Q < r := \text{ord } P$ and P square-free.
- Then $R := QI_P \text{Res}_{Y^{(r)}}(P, S_P) \neq 0$.
- Rewrite $P, Q, R, \dots$ as power series in $z_{\ell+1}$.
- Take initial $\alpha_0, \dots, \alpha_{r-1} \in K_{\ell+1}$ with $R[z_{\ell+1}=0](\alpha_0, \dots, \alpha_{r-1}) \neq 0$.
- Complete: unique power series $y = \alpha_0 + \dots + \frac{\alpha_{r-1}}{(r-1)!} z_{\ell+1}^{r-1} + \dots$ with $P(y) = 0$.
- We automatically have $Q(y) \neq 0$, since $Q[z_{\ell+1}=0](y) \neq 0$.

Folklore lemma (Razmyslov 1994)

Folklore lemma (Razmyslov 1994)

Let K be a differential field of characteristic zero.

Folklore lemma (Razmyslov 1994)

Let K be a differential field of characteristic zero.

Then K becomes a constant field for the following derivation on $K((z^{\mathbb{Q}}))$:

$$\left(\sum_{i \geq i_0} y_i z^i \right)' = \sum_{i \geq i_0} i y_i z^{i-1}$$

Folklore lemma (Razmyslov 1994)

Let K be a differential field of characteristic zero.

Then K becomes a constant field for the following derivation on $K((z^{\mathbb{Q}}))$:

$$\left(\sum_{i \geq i_0} y_i z^i \right)' = \sum_{i \geq i_0} i y_i z^{i-1}$$

Moreover, we have the following differential embedding:

$$\begin{aligned} \Phi: K &\hookrightarrow K((z^{\mathbb{Q}})) \\ y &\mapsto \sum_{i \geq 0} \frac{1}{i!} y^{(i)} z^i. \end{aligned}$$

Thank you !



<http://www.TEXMACS.org>