

Asymptotic Differential Algebra

a Tale of Infinities

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Based on joint work with M. Aschenbrenner, L. van den Dries



September 15, 2026

Colloquia Seminar, 北京

Georg Cantor

Georg Cantor

Cardinal numbers

Georg Cantor

Cardinal numbers

Ordinal numbers

$0, 1, 2, \dots$

Georg Cantor

Cardinal numbers

Ordinal numbers

$0, 1, 2, \dots, \omega$

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Cardinal numbers

Ordinal numbers

$0, 1, 2, \dots, \omega, \omega + 1, \dots$

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$$0, 1, 2, \dots, \omega, \omega + 1, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots$$

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$$0, 1, 2, \dots, \omega, \omega + 1, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots, \omega^2$$

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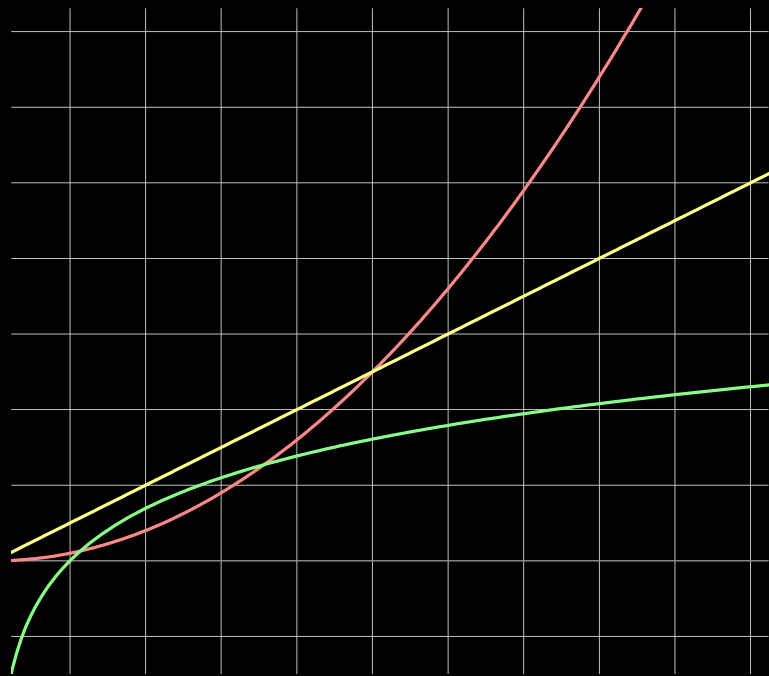
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Cantor normal form

$$\omega^{\omega^{\omega+2} \cdot 3 + \omega^8 \cdot 7 + \omega \cdot 3 + 2} \cdot 9 + \omega^{\omega^{\omega+1}} \cdot 3 + \omega^{\omega \cdot 7} \cdot 5 + \omega^8 + \omega^2 \cdot 111 + 2020$$

Paul du Bois-Reymond

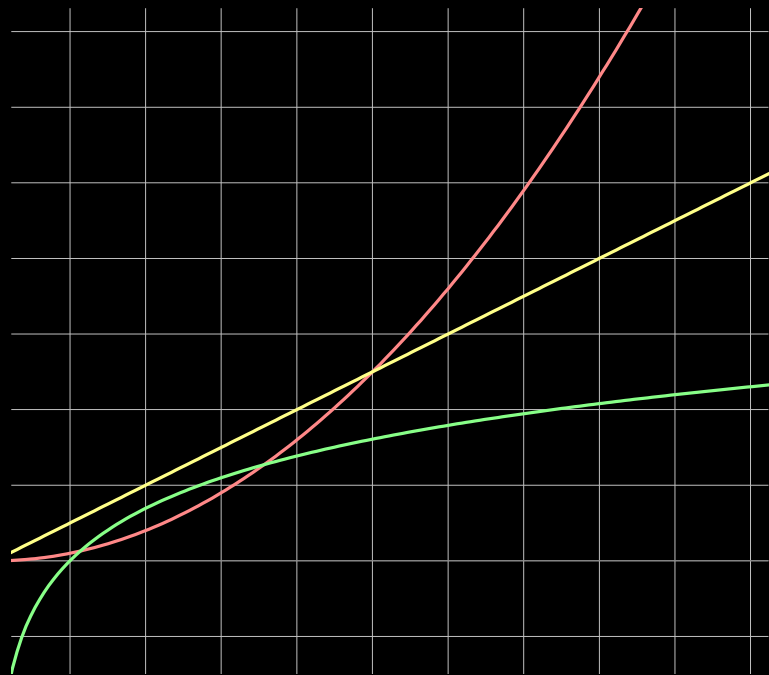
Paul du Bois-Reymond



Precursor of asymptotic calculus

$$\log x < \frac{x}{2} < \frac{x^2}{10} \quad (x \rightarrow \infty)$$

Paul du Bois-Reymond



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Diagonal argument

$$\exists f, \quad x < e^x < e^{e^x} < e^{e^{e^x}} < \dots < f$$

Three intimately related topics...

(surreal)
Numbers

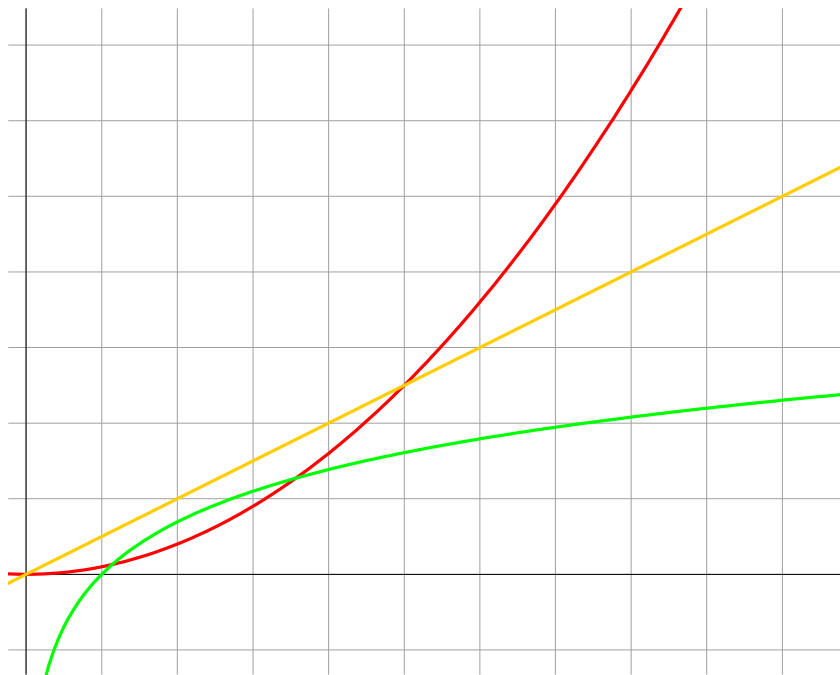
Germs
(in HARDY
fields)

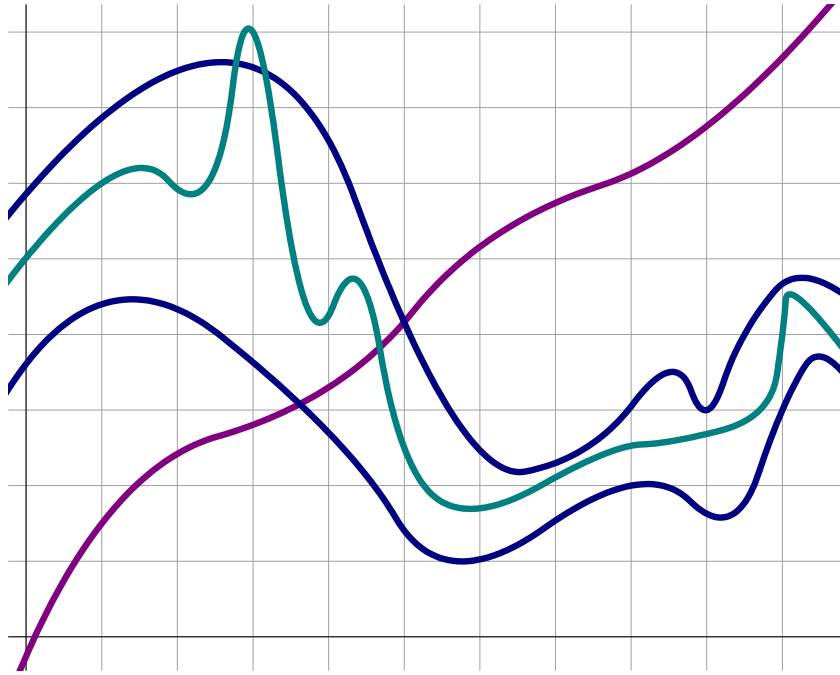
Transseries



Germs

(in HARDY
fields)





Let $\mathcal{C}^1$ be the ring of germs at $+\infty$ of continuously differentiable functions $(a, \infty) \rightarrow \mathbb{R}$ ($a \in \mathbb{R}$).

We denote the germ at $+\infty$ of a function f also by f , relying on context.

Definition

A **Hardy field** is a subring of $\mathcal{C}^1$ which is a field that contains with each germ of a function f also the germ of its derivative f' (where f' might be defined on a smaller interval than f).

Examples. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $\mathbb{R}(x, e^x)$, $\mathbb{R}(x, e^x, \log x)$, $\mathbb{R}(x, e^{x^2}, \operatorname{erf} x)$

Hardy fields capture the somewhat vague notion of functions with “**regular growth**” at infinity (DU BOIS-REYMOND, BOREL, ...):

Let H be a Hardy field and $f \in H$. Then

$$f \neq 0 \implies \frac{1}{f} \in H \implies \begin{cases} f(x) > 0, \text{ eventually, or} \\ f(x) < 0, \text{ eventually.} \end{cases}$$

Consequently,

- H carries an ordering making H an **ordered field**:

$$f > 0 \iff f(x) > 0 \text{ eventually;}$$

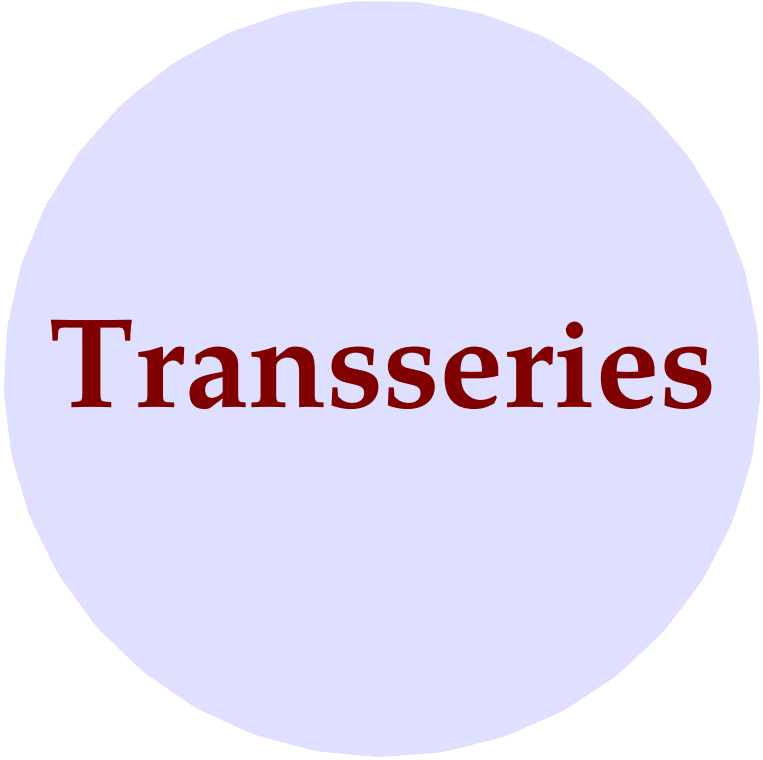
- f is **eventually monotonic**, and

$$\lim_{x \rightarrow +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}.$$

(surreal)
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Transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$e^{e^x + e^{x/2} + e^{x/3} + \dots} - 3e^{x^2} + 5(\log x)^\pi + 42 + x^{-1} + 2x^{-2} + 6x^{-3} + \dots + e^{-x}$$

$\mathbb{T} = \mathbb{R}[[\mathfrak{M}]] :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$\sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} = e^{e^x + e^{x/2} + \dots} - 3e^{x^2} + 5(\log x)^\pi + 42 + x^{-1} + 2x^{-2} + 6x^{-3} + \dots + e^{-x}$$

x : positive infinite indeterminate $f_{\mathfrak{m}}$: coefficient $\mathfrak{m}$: transmonomial

$\text{supp } f$: well-based subset of $\mathfrak{M}$

disallow $x + \log x + \log \log x + \dots$ and $e^{-x} + e^{-e^x} + e^{-e^{e^x}} + \dots$

- With the natural ordering of transseries (via the leading coefficient), $\mathbb{T}$ is a *real closed ordered field* extension of $\mathbb{R}$.
- Each $f \in \mathbb{T}$ can be *differentiated* term by term (with $x' = 1$):

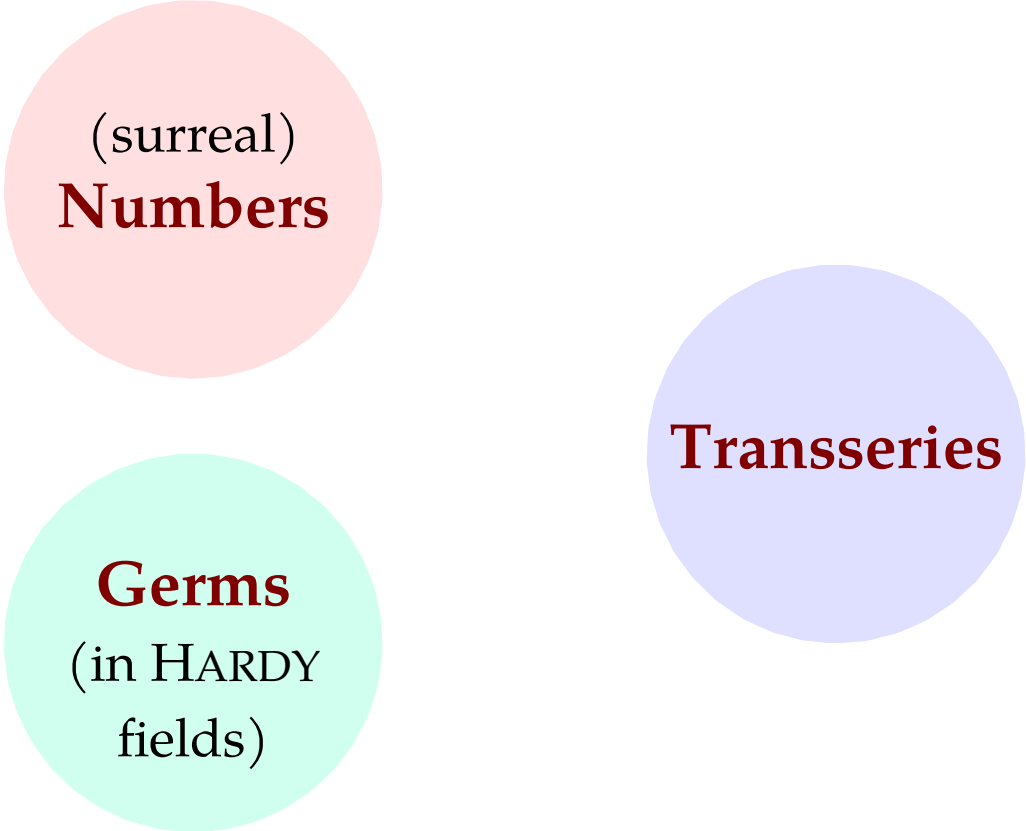
$$\left(\sum_{n=0}^{\infty} n! \frac{e^x}{x^n} \right)' = \sum_{n=0}^{\infty} n! \left(\frac{e^x}{x^n} \right)' = \sum_{n=0}^{\infty} n! \left(\frac{e^x}{x^n} - n \frac{e^x}{x^{n+1}} \right) = \frac{e^x}{x}$$

- This yields a *derivation* $f \mapsto f'$ on the field $\mathbb{T}$:

$$(f + g)' = f' + g', \quad (f \cdot g)' = f' \cdot g + f \cdot g'$$

Its constant field is $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$.

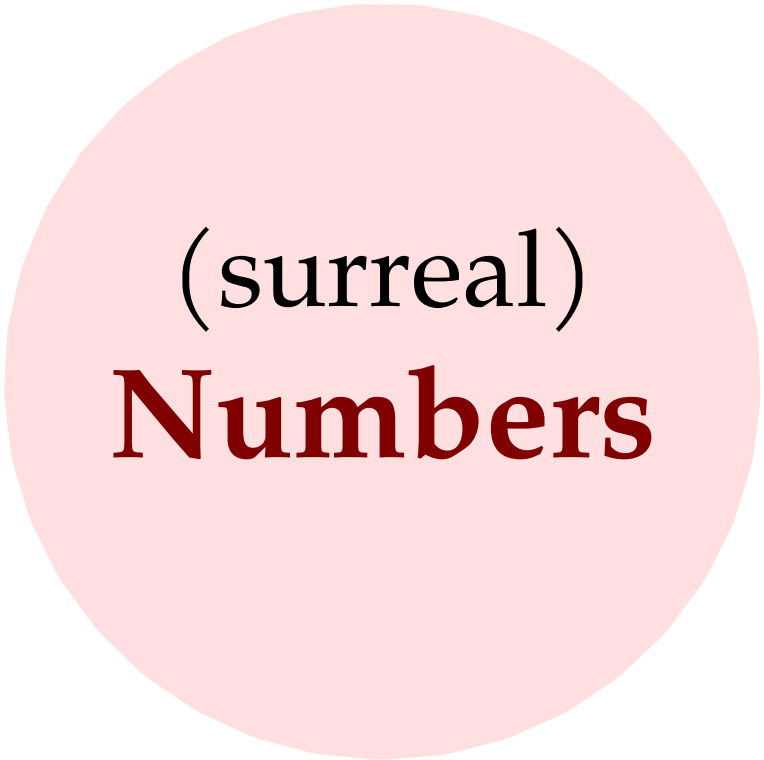
- Given $f, g \in \mathbb{T}$, the equation $y' + fy = g$ admits a solution $y \neq 0$ in $\mathbb{T}$.



(surreal)
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Numbers

Class On of ordinal numbers

For any set L of ordinal numbers, there is a smallest ordinal number $\alpha > L$

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Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a **simplest** surreal number $\{L|R\}$ such that $L < \{L|R\} < R$.

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We have $\mathbf{On} \subseteq \mathbf{No}$ by taking $R = \emptyset$:

$$0 = \{|\}$$

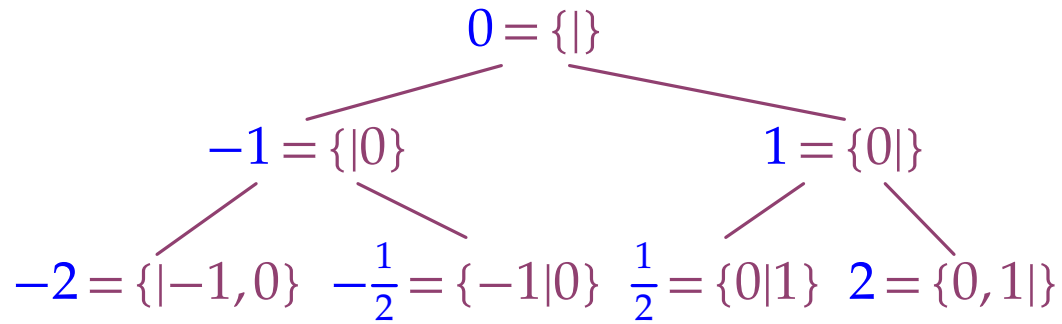
$$1 = \{0|\}$$

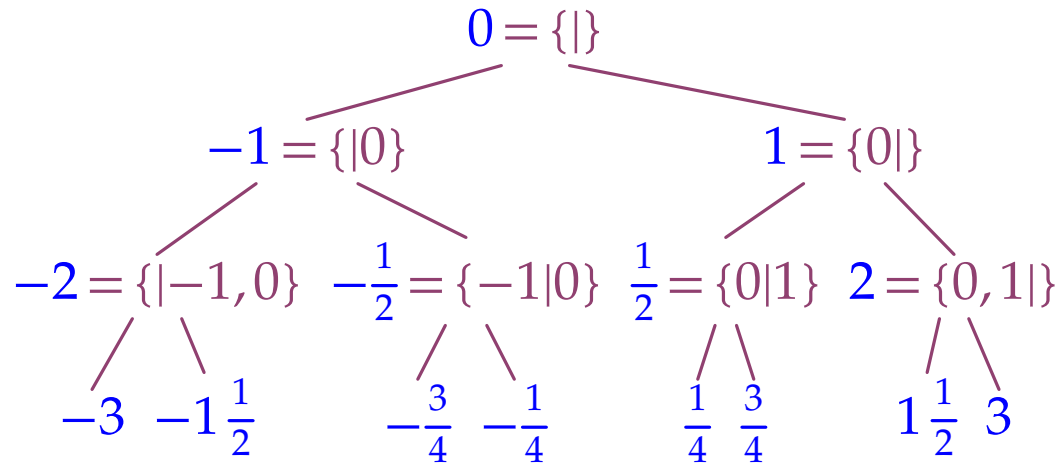
$$2 = \{0, 1|\}$$

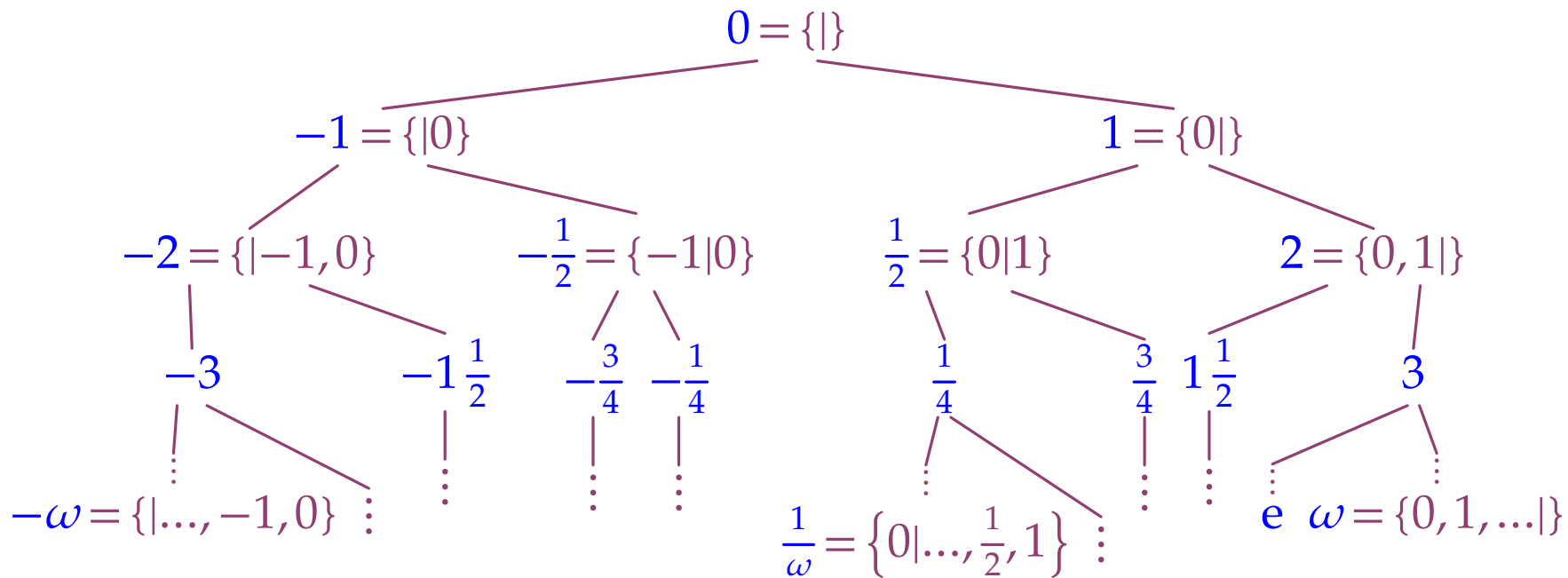
$$\omega = \{0, 1, 2, \dots|\}$$

$$0 = \{|\}$$

$$\begin{array}{c} 0 = \{|\} \\ \swarrow \quad \searrow \\ -1 = \{ | 0 \} \quad 1 = \{ 0 | \} \end{array}$$

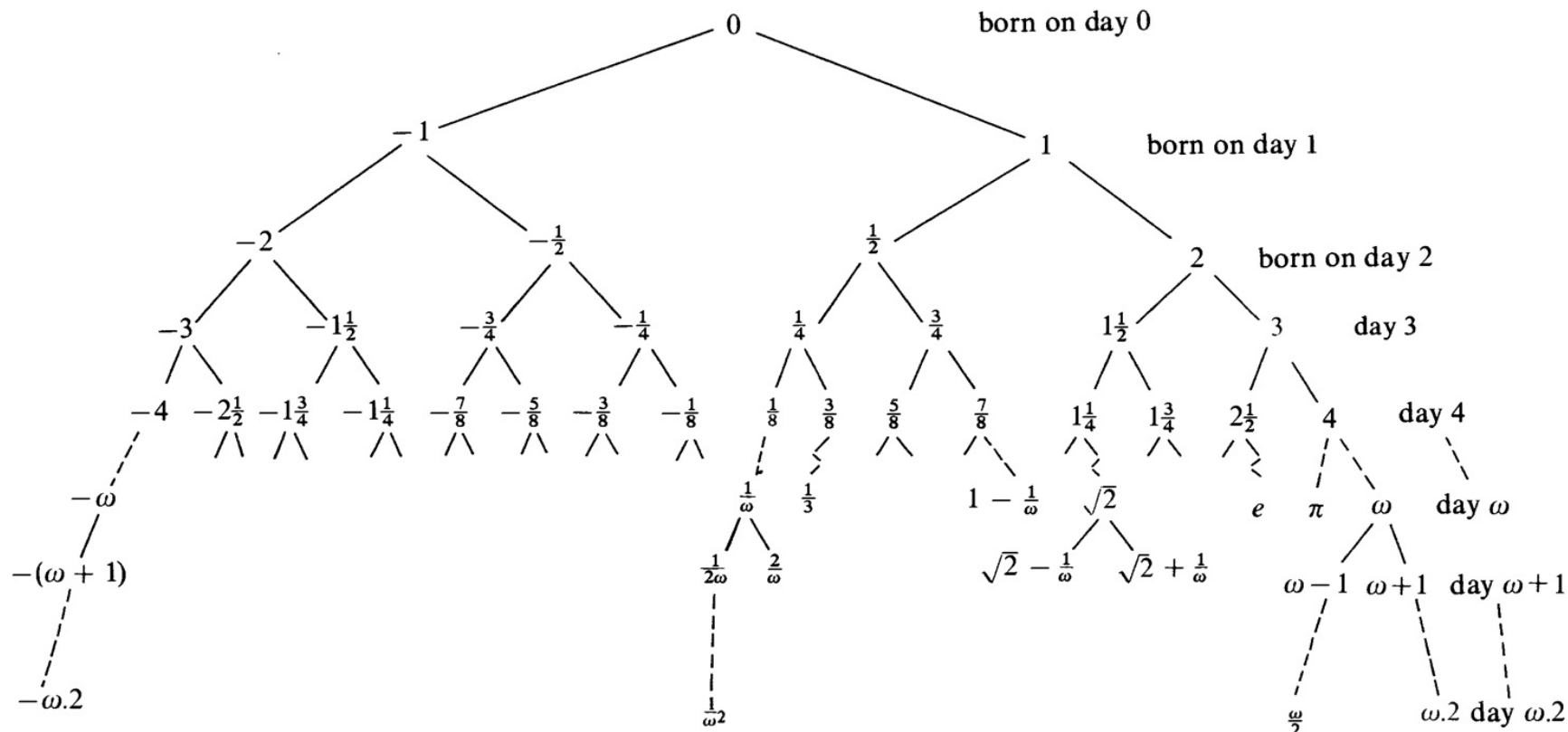






Surreal numbers

13/32



Definition

If $x = \{x^L | x^R\}$ and $y = \{y^L | y^R\}$, then

$$x + y := \{x^L + y, x + y^L | x^R + y, x + y^R\}$$

(Idea: we want $x^L + y < x + y < x^R + y, \dots$)

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where $x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R$

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Theorem (CONWAY)

No is a real closed field.

- In the 1980s, GONSHOR (expanding on ideas of KRUSKAL) defined an exponential function $\exp: \mathbf{No} \rightarrow \mathbf{No}^{>0}$ that extends $x \mapsto e^x$ on $\mathbb{R}$.
- In 2006, BERARDUCCI and MANTOVA (expanding on ideas of VDH and SCHMELING) defined a derivation ∂_{BM} on $\mathbf{No}$ with

$$\ker \partial_{\text{BM}} = \mathbb{R}, \quad \partial_{\text{BM}}(\omega) = 1, \quad \partial_{\text{BM}}(\exp(f)) = \partial_{\text{BM}}(f) \cdot \exp(f) \text{ for } f \in \mathbf{No}.$$

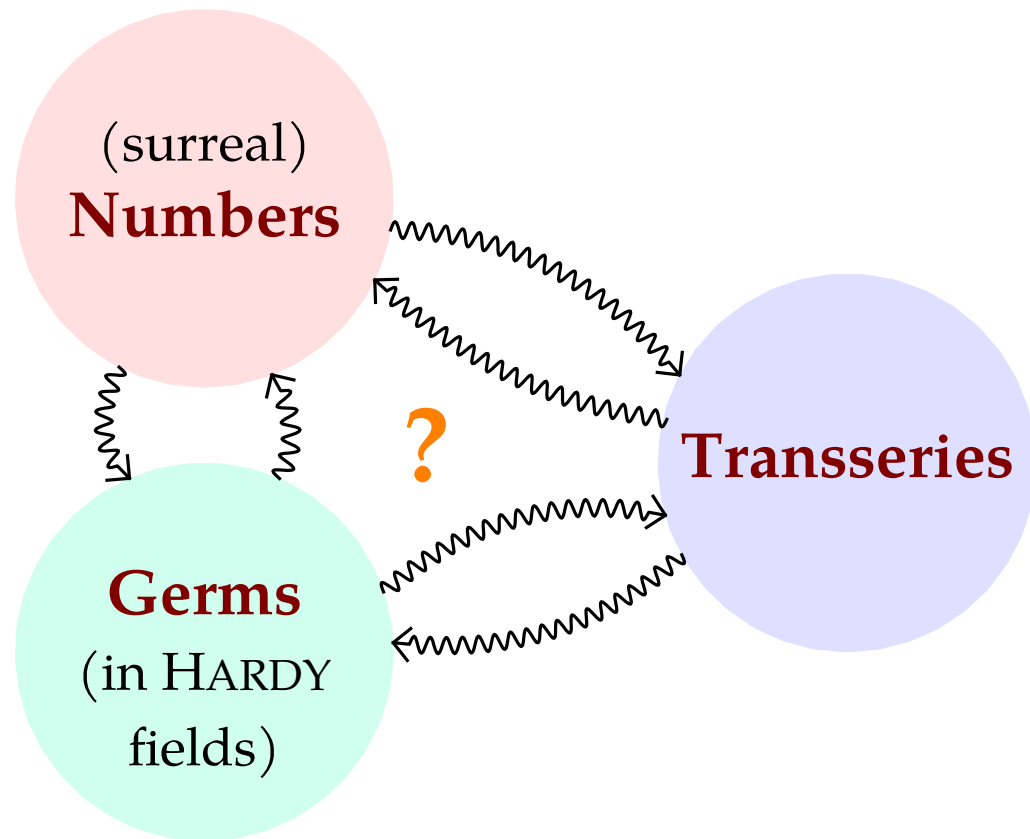
In a certain technical sense, it is the simplest such derivation that satisfies some natural further conditions.

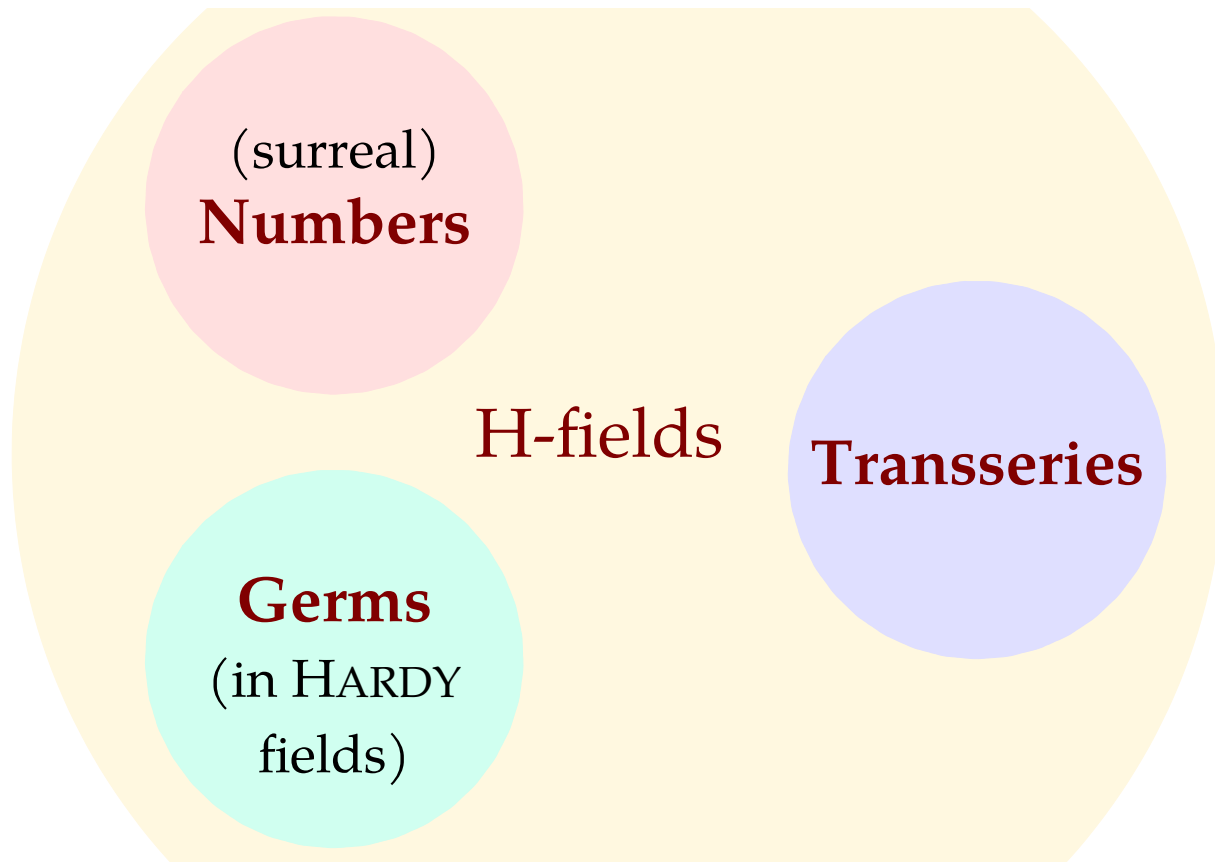
- The BM-derivation on $\mathbf{No}$ behaves in many ways like the derivation on $\mathbb{T}$, with $\omega > \mathbb{R}$ in the role of $x > \mathbb{R}$. For instance, $\partial_{\text{BM}}(\log \omega) = \frac{1}{\omega}$.

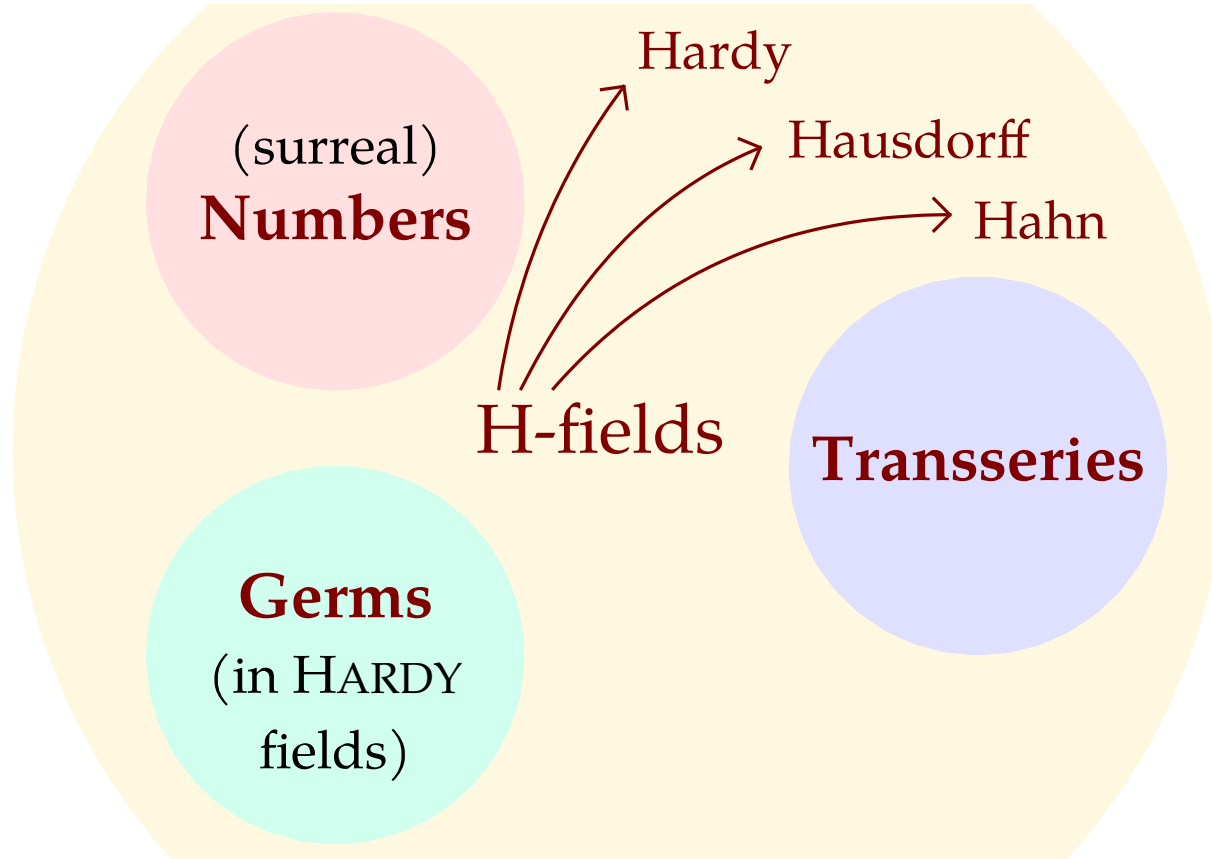
(surreal)
Numbers

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Let K be an ordered differential field with constant field

$$C = \{f \in K : f' = 0\}.$$

We define

$$f \leqslant g : \Longleftrightarrow |f| \leqslant c |g| \text{ for some } c \in C^{>0} \quad (f \text{ is dominated by } g)$$

$$f < g : \Longleftrightarrow |f| \leqslant c |g| \text{ for all } c \in C^{>0} \quad (f \text{ is negligible w.r.t. } g)$$

$$f \asymp g : \Longleftrightarrow f \leqslant g \leqslant f \quad (f \text{ is asymptotic to } g)$$

$$f \sim g : \Longleftrightarrow f - g < g \quad (f \text{ is equivalent to } g)$$

Example. In $\mathbb{T}$:

$$0 < e^{-x} < x^{-10} < 1 \asymp 100 < \log x < x^{1/10} < e^x \sim e^x + x < e^{e^x}$$

Definition

We call K an *H-field* if

H1. $f > C \implies f' > 0$;

H2. $f \asymp 1 \implies f \sim c$ for some $c \in C$.

Examples. Hardy fields containing $\mathbb{R}$; ordered differential subfields of $\mathbb{T}$ or **No** that contain $\mathbb{R}$.

$\mathbb{T}$ admits further elementary properties in addition to being an H-field. It

- has **small derivation**, that is, $f < 1 \implies f' < 1$; and
- is **Liouville closed**, that is, it is real closed and for all f, g , there is some $y \neq 0$ with $y' + fy = g$.

Definition

An H -field K is said to be **H-closed** if

- ① K is a Liouville closed H -field;
- ② K satisfies the intermediate value property for differential polynomials:
Given $P \in K[Y, Y', \dots, Y^{(r)}]$ and $u < v$ in K with $P(u)P(v) < 0$, there exists a $y \in K$ with $u < y < v$ and $P(y) = 0$.

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The following equation has a solution $y \in \mathbb{T}$:

$$e^x y^7 - (y'')^4 y''' + \Gamma(\log \log x) y^5 + x^{-x} y' - e^x = 0$$

We view $\mathbb{T}$ model-theoretically as a structure with the primitives

$0, 1, +, \times, \partial$ (derivation), $\leq$ (ordering).

Theorem (Ann. of Math. Stud. vol. 195 + afterthought)

The elementary theory of $\mathbb{T}$ is completely axiomatized by $\mathbb{T}$ being an H -closed field with small derivation. In particular: the theory of $\mathbb{T}$ is decidable.

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- Quantifier elimination in a “natural expansion” of the above language.

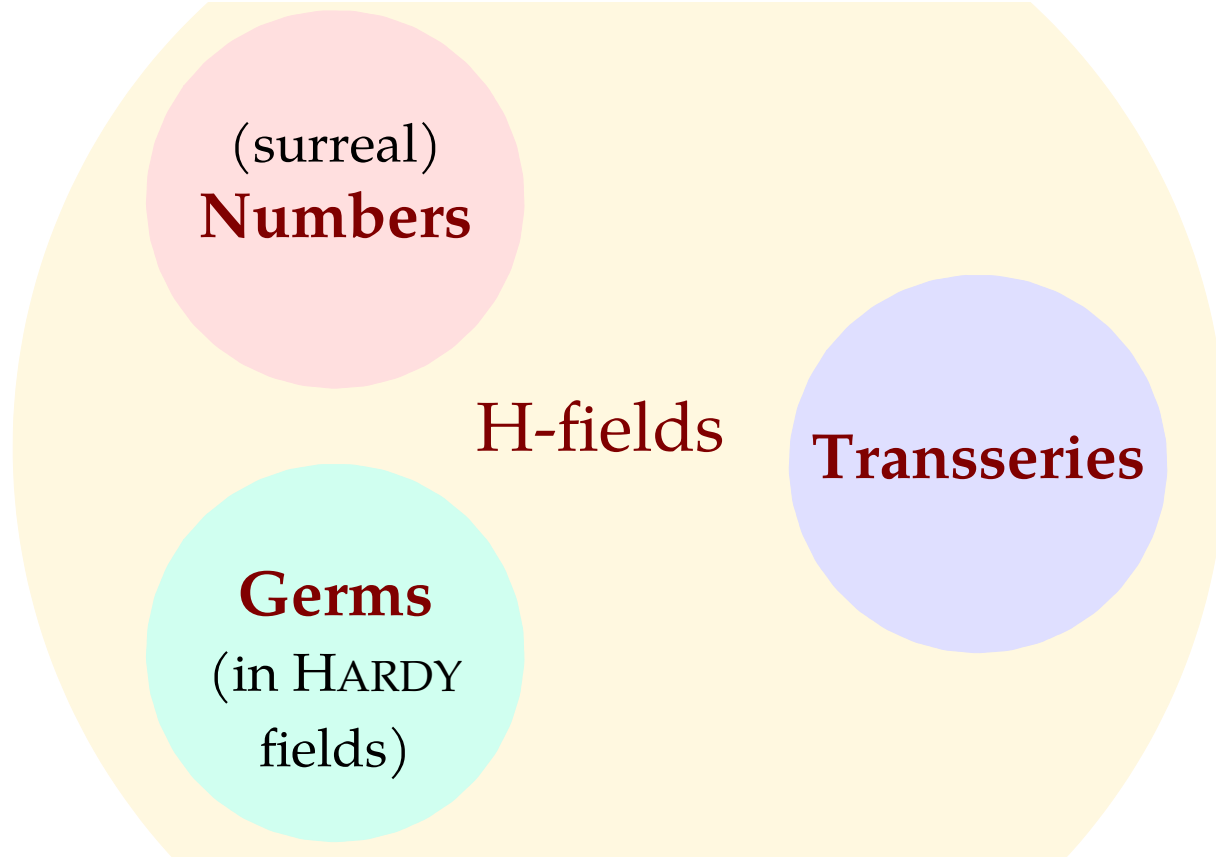
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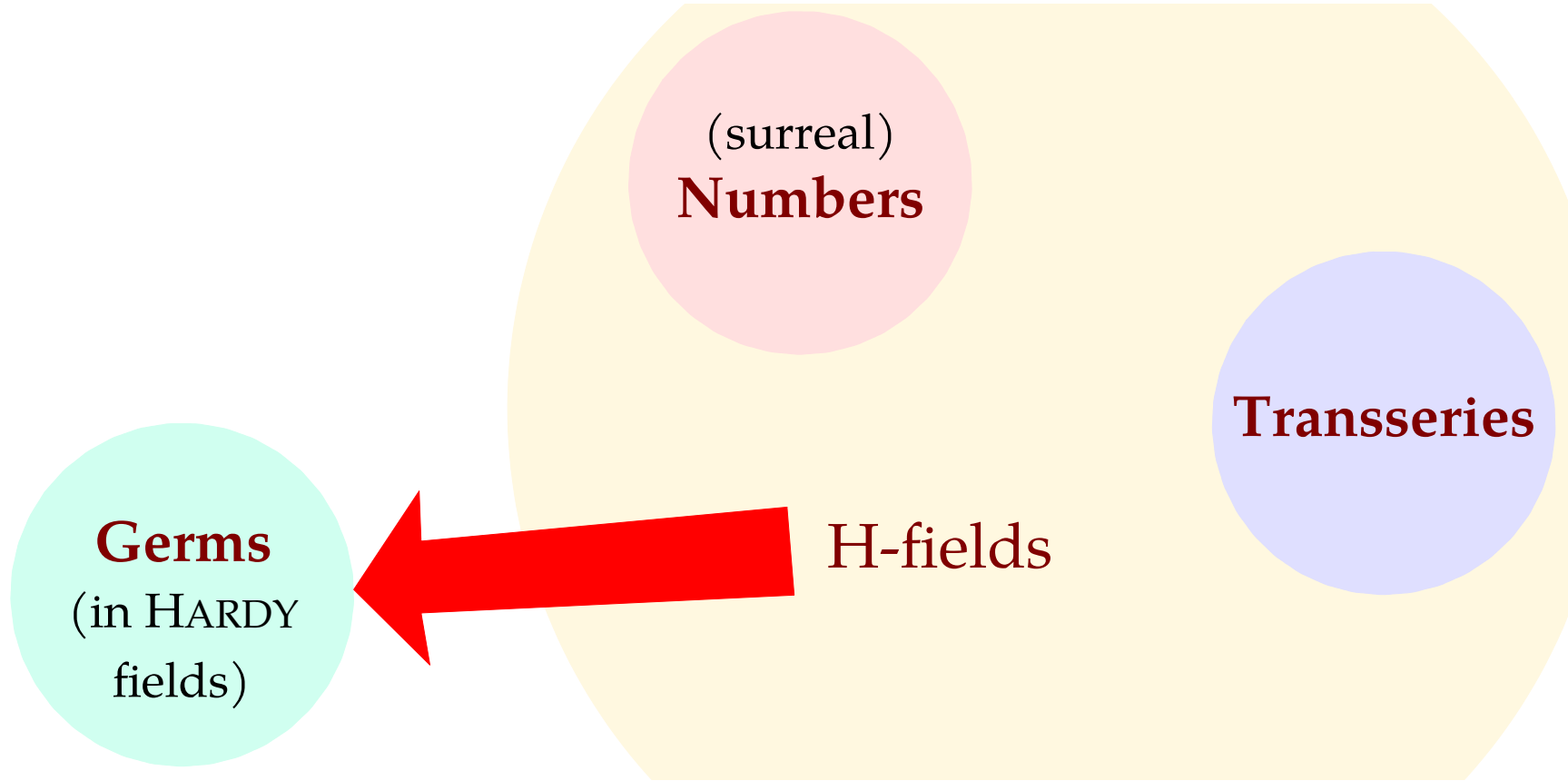
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- Quantifier elimination in a “natural expansion” of the above language.
- In particular, the theory of H -closed fields is model complete.





Theorem (HARDY 1910, BOURBAKI 1951)

Any Hardy field has a smallest Liouville closed Hardy field extension.

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$$\begin{cases} u' = 2uvw - 3v^2 \\ v' = u^2w - 6uv \\ w' = u^2v - 5w^4 \end{cases} \quad \begin{array}{l} \exists \text{ solution } (u, v, w) \in \mathbb{T}^3 \\ \Downarrow \\ \exists \text{ solution in any maximal Hardy field} \end{array}$$

Conjecture (ICM 2018)

*Let $L < R$ be countable subsets of a maximal Hardy field H .
Then there exists an $h \in H$ with $L < h < R$.*

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Corollary (Sjödin 1970)

Special case when $R = \emptyset$. For example: $L = \{x, e^x, e^{e^x}, \dots\}$.

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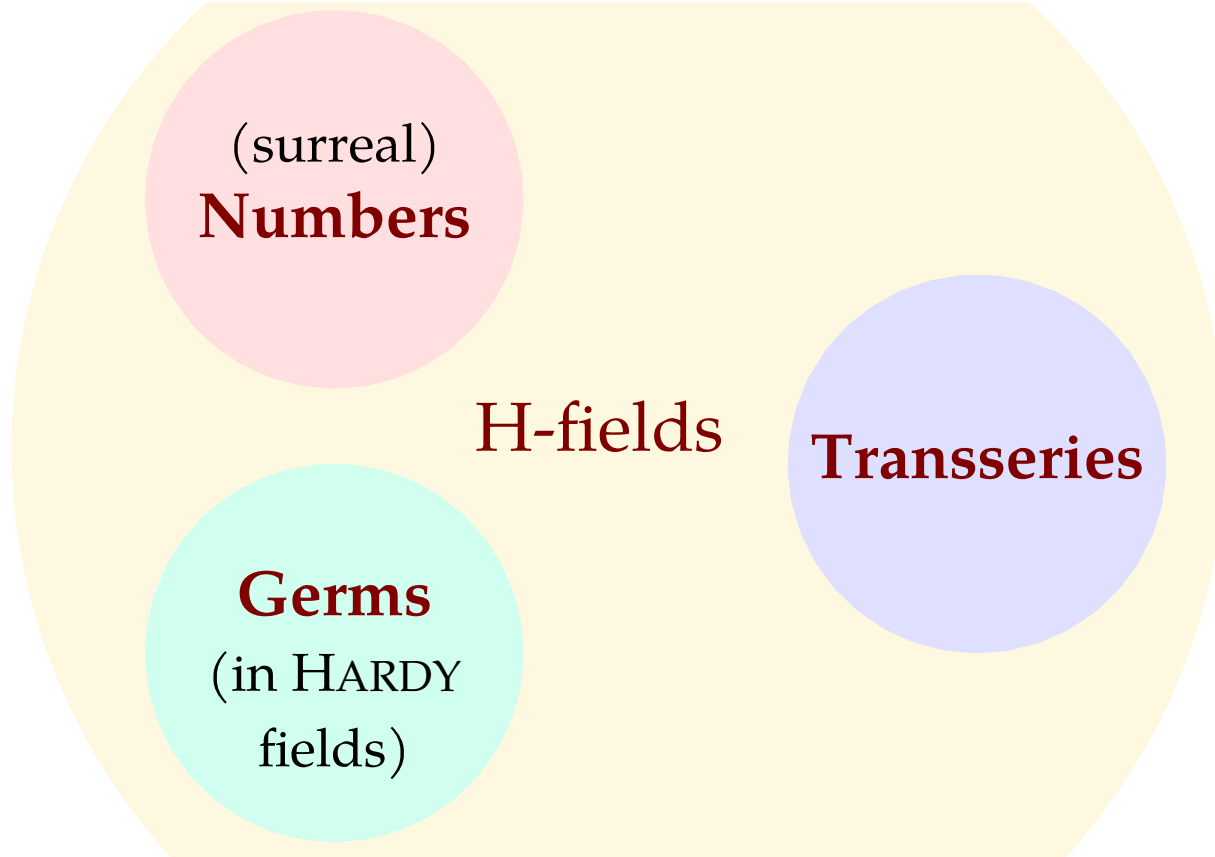
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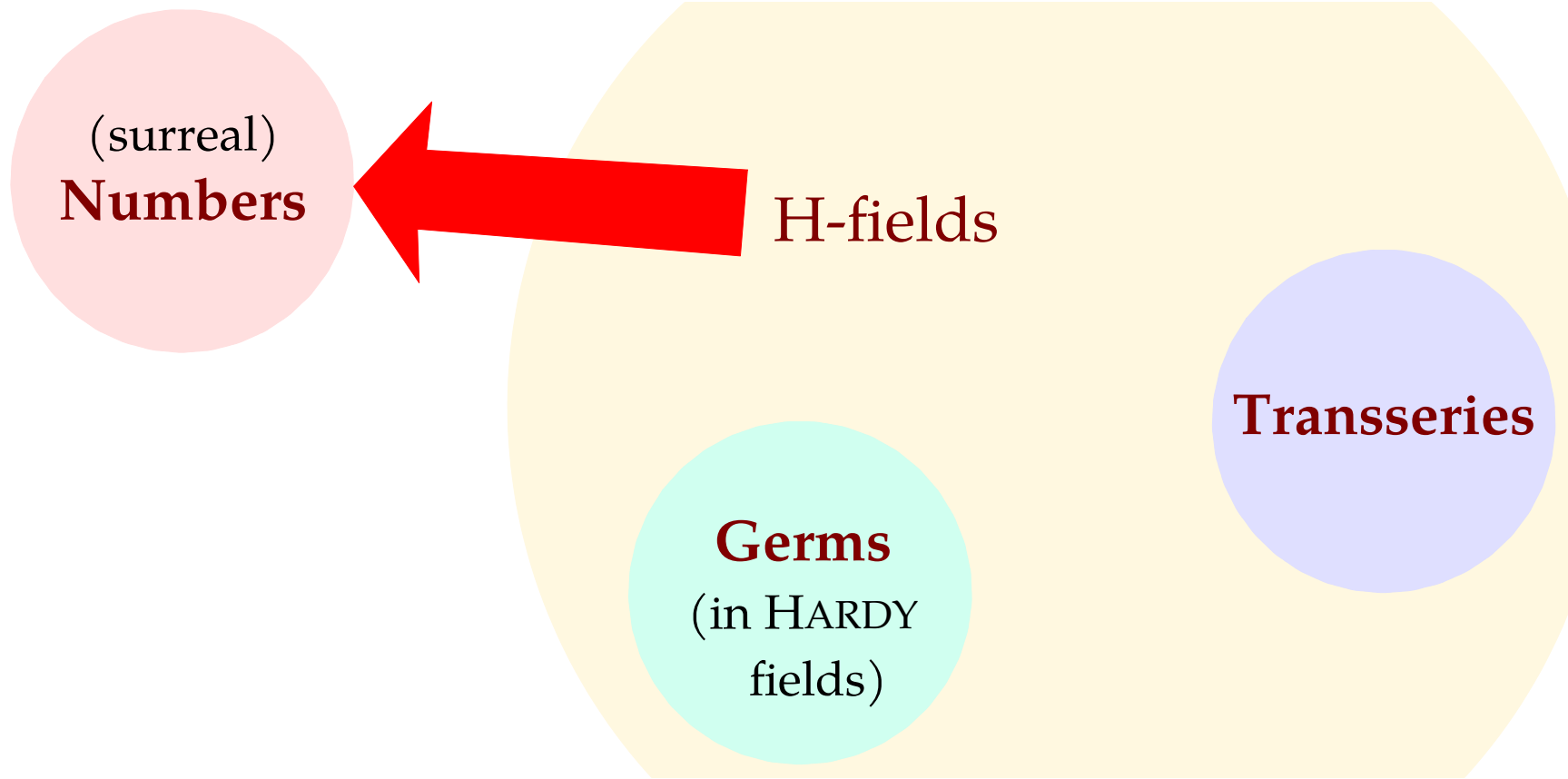
Corollary (Sjödin 1970)

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Corollary

Under CH, all maximal Hardy fields are isomorphic.





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*Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into **No**.*

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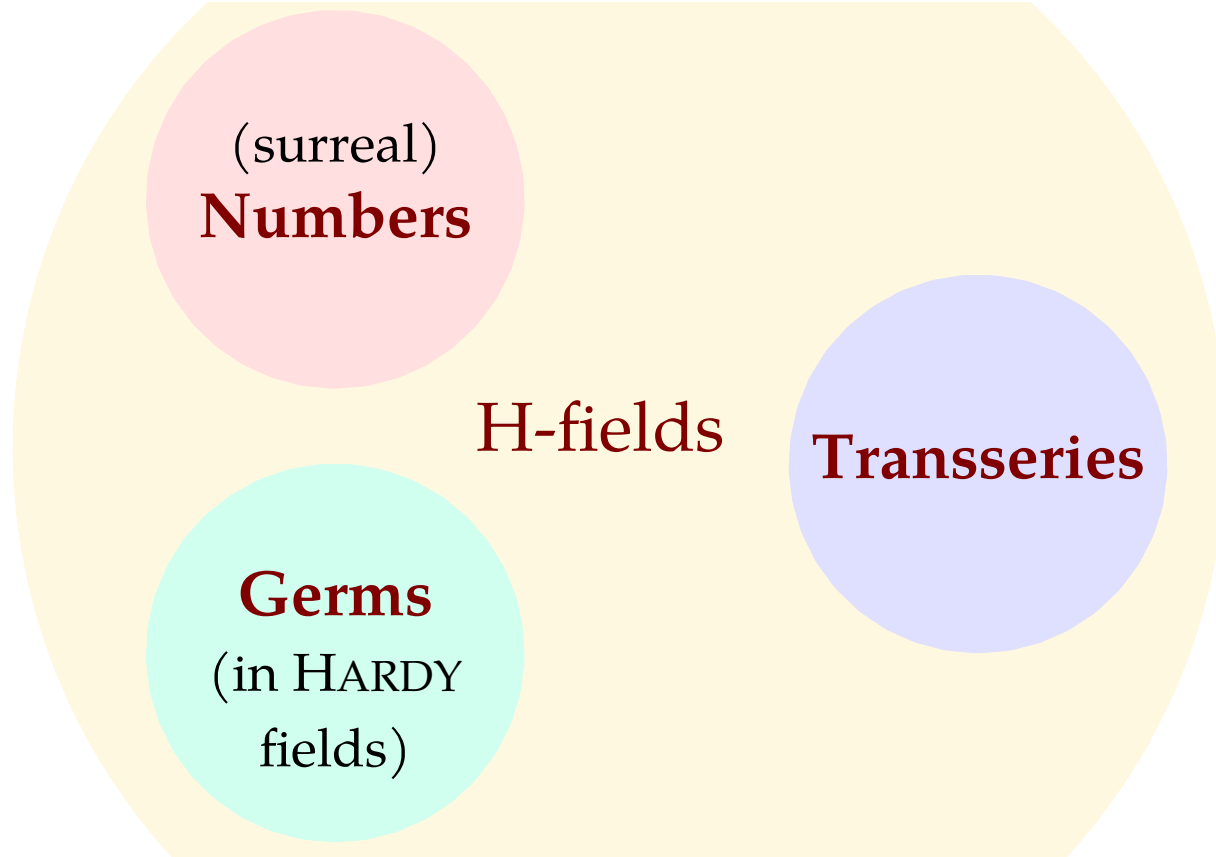
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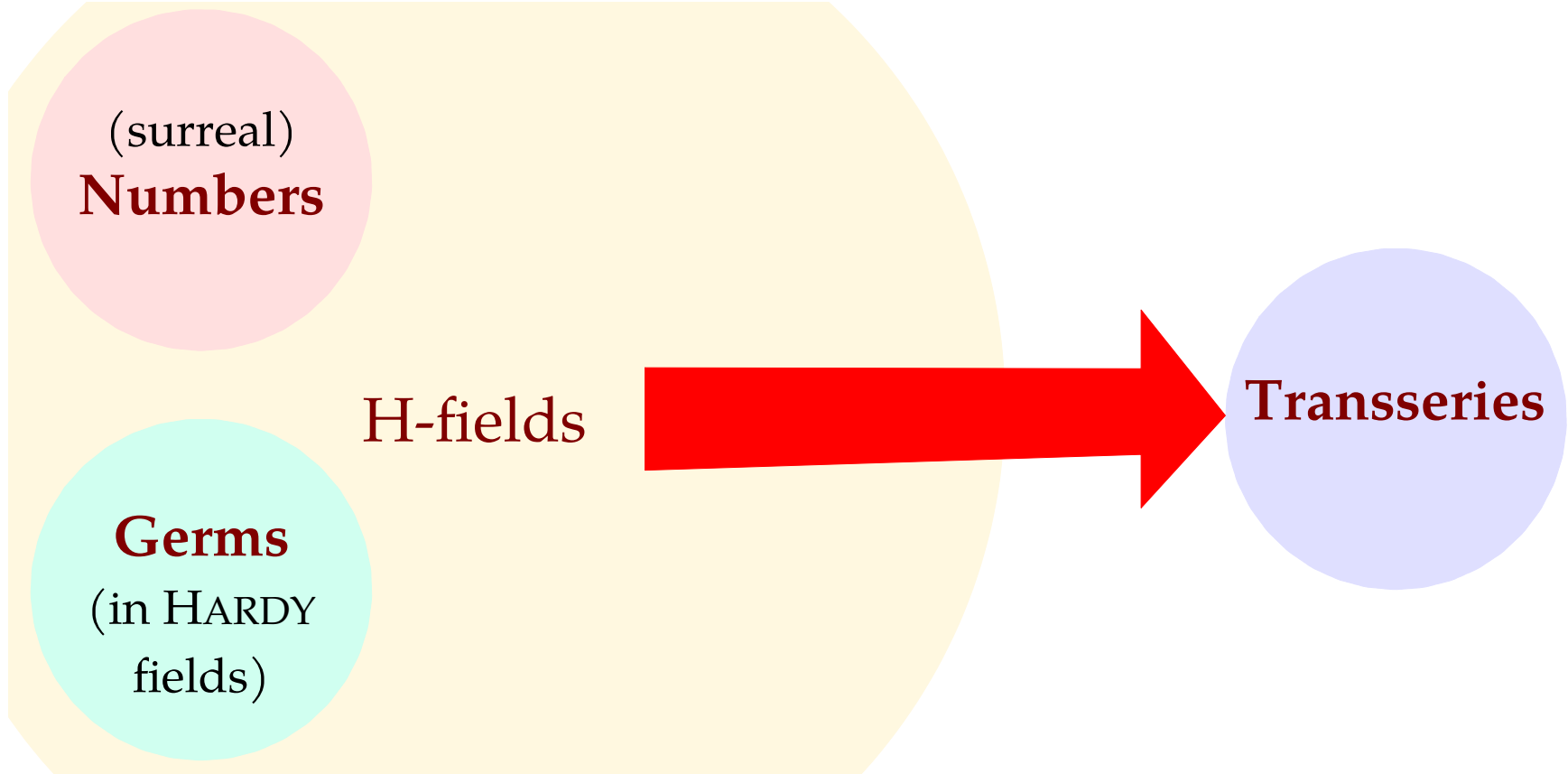
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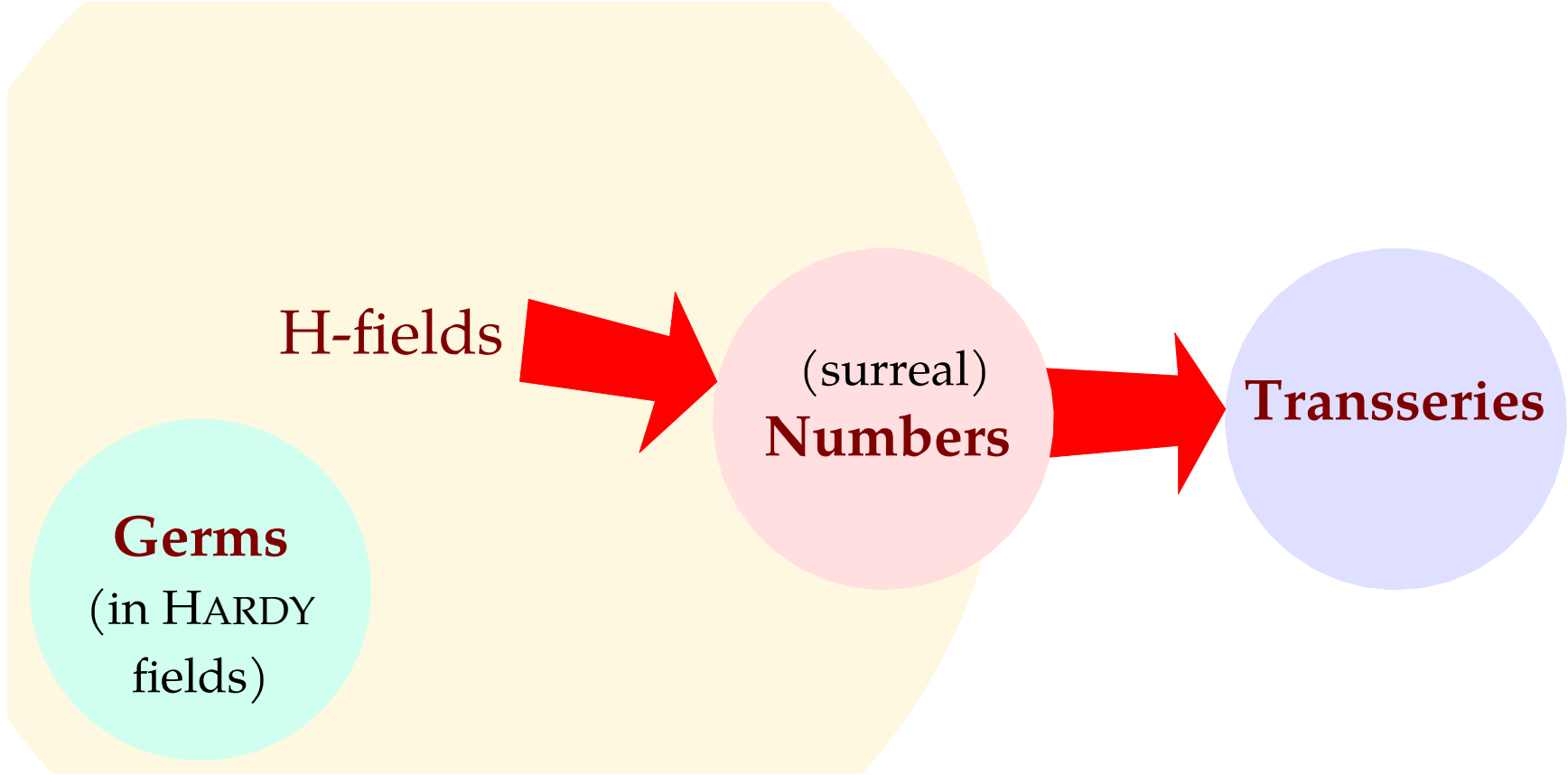
*Let κ be an uncountable cardinal. The field **No**(κ) of surreal numbers of length $< \kappa$ is an elementary submodel of **No**.*

Corollary

*Under CH all maximal Hardy fields are isomorphic to **No**(ω_1).*







Definition (VAN DER HOEVEN 2000, SCHMELING 2001)

A field $\mathbf{T} = \mathbb{R}[[\mathfrak{M}]]$ with $\log: \mathbf{T}^> \rightarrow \mathbf{T}$ is a *field of transseries* if ...

A *transserial derivation* on $\mathbf{T}$ is a derivation $\partial: \mathbf{T} \rightarrow \mathbf{T}$ such that ...

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Theorem (BERARDUCCI–MANTOVA, 2015)

No is a field of transseries and ∂_{BM} is a transserial derivation.

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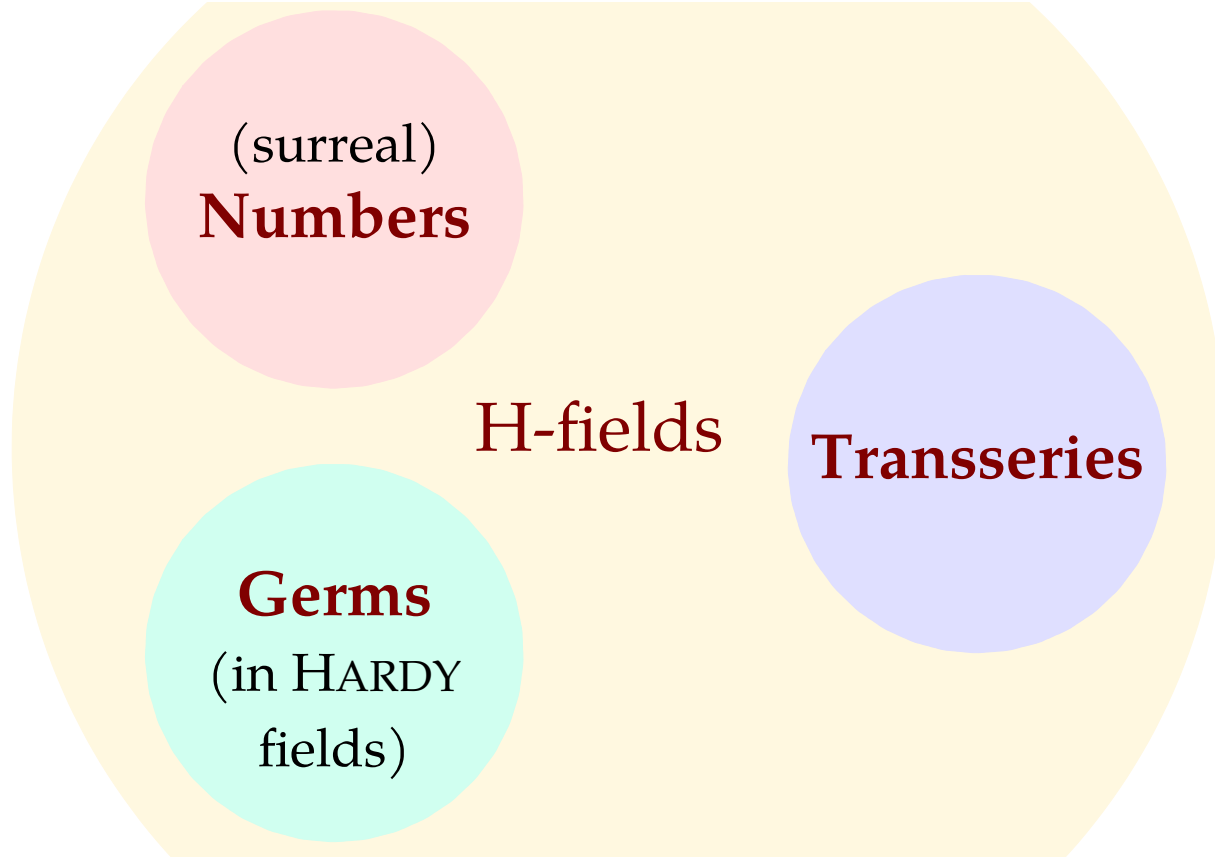
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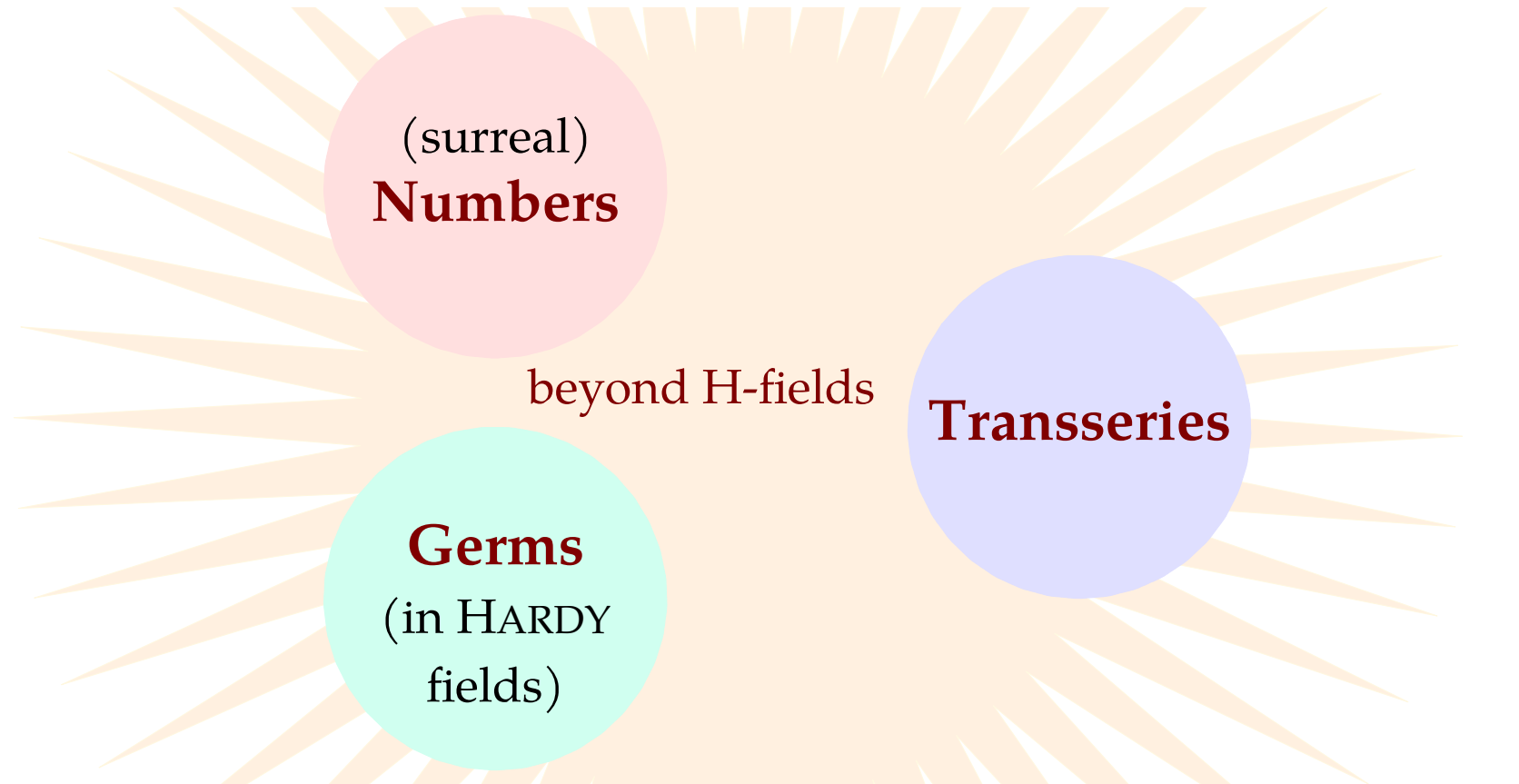
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Corollary

Any H -field with constant field $\mathbb{R}$ can be embedded in a field of transseries with a transserial derivation.



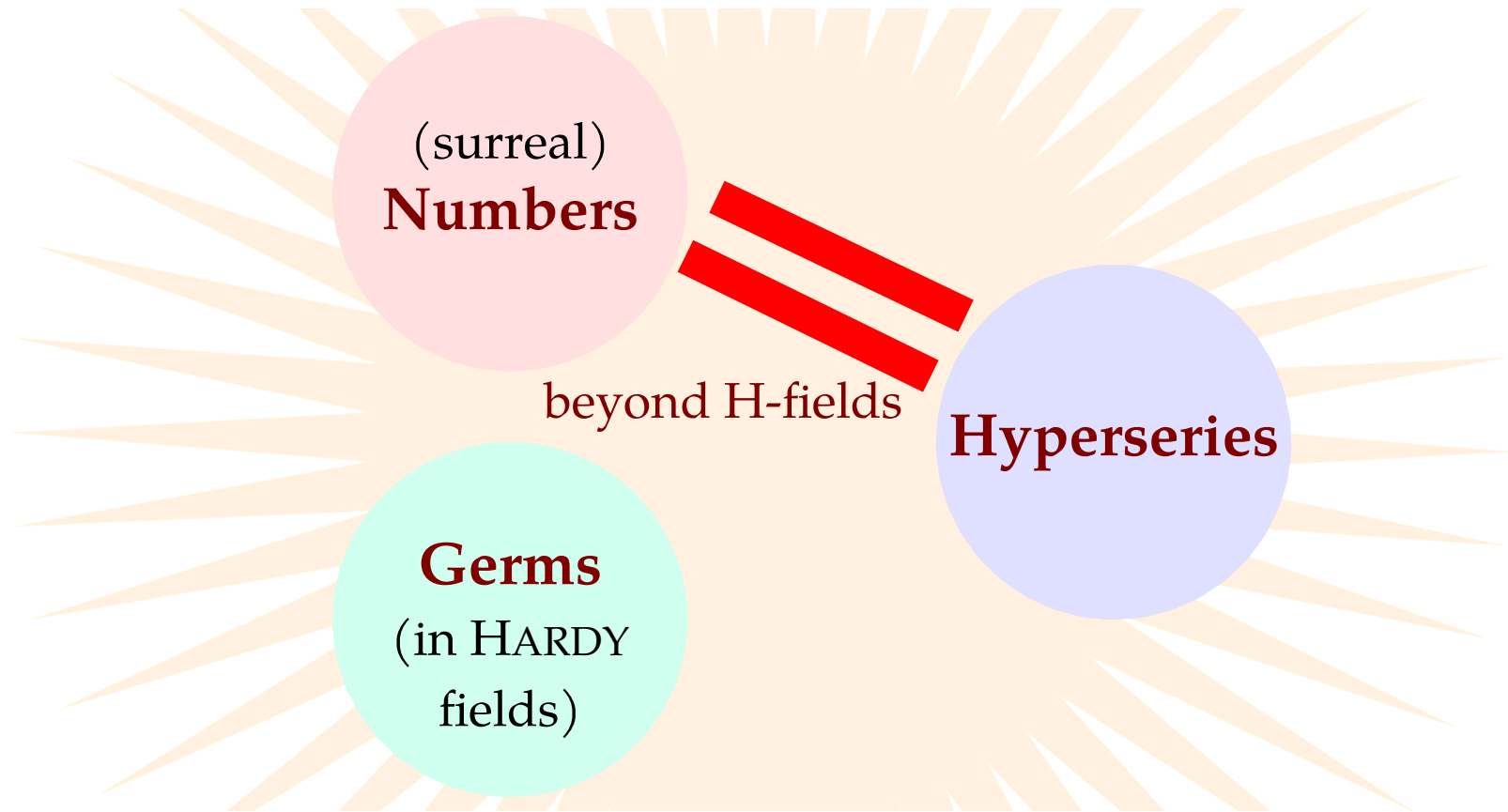


(surreal)
Numbers

beyond H-fields

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Transseries



Iterated exponentials and logarithms

$$\exp_{\omega}(x+1) = \exp \exp_{\omega} x$$

$$\exp_{\omega^2}(x+1) = \exp_{\omega} \exp_{\omega^2} x$$

...

→ stronger growth than $e^x, e^{e^x}, \dots, \exp_{\omega} x, e^{\exp_{\omega} x}, \dots, \exp_{\omega} \exp_{\omega} x, \dots$

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Functional equations

$$f(x) = \sqrt{x} + e^{f(\log x)} = \sqrt{x} + e^{\sqrt{\log x} + e^{\sqrt{\log \log x} + \dots}}$$

Hyperlogarithms and hyperexponentials

$$\begin{aligned}\exp_{\omega}(x+1) &= \exp \exp_{\omega} x \\ \exp_{\omega^2}(x+1) &= \exp_{\omega} \exp_{\omega^2} x \\ &\vdots\end{aligned}$$

$$\begin{aligned}\log_{\omega} \log x &= \log_{\omega} x - 1 \\ \log_{\omega^2} \log_{\omega} x &= \log_{\omega^2} x - 1 \\ &\vdots\end{aligned}$$

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$$\log_{\omega} x = \int \frac{1}{x \log x \log \log x \dots}$$

$$\log_{\alpha} x = \int \prod_{\beta < \alpha} \frac{1}{\log_{\beta} x}$$

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Nested hyperseries

Solutions de $f(x) = \sqrt{x} + e^{f(\log x)}$:

$$f_0(x)$$

Hyperlogarithms and hyperexponentials

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Nested hyperseries

Solutions de $f(x) = \sqrt{x} + e^{f(\log x)}$:

$$f_{-1}(x) < f_0(x) < f_1(x)$$

Hyperlogarithms and hyperexponentials

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Nested hyperseries

Solutions de $f(x) = \sqrt{x} + e^{f(\log x)}$:

$$f_{-2}(x) < f_{-1}(x) < f_{-1/2}(x) < f_0(x) < f_{1/2}(x) < f_1(x) < f_2(x)$$

Hyperlogarithms and hyperexponentials

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Nested hyperseries

Solutions de $f(x) = \sqrt{x} + e^{f(\log x)}$: $\longrightarrow f_{\mathbf{No}}(x)$

$$\dots < f_{-2}(x) < \dots < f_{-1}(x) < \dots < f_0(x) < \dots < f_{1/2}(x) < \dots < f_1(x) < \dots < f_2(x) < \dots$$

Conjecture (vdH 2006)

*For an appropriate definition of the class **Hy** of hyperseries, we have **No** $\cong$ **Hy** for the map $\phi: \mathbf{Hy} \rightarrow \mathbf{No}; f \mapsto f(\omega)$.*

Theorem (Bagayoko–vdH 2019–2022)

*For an appropriate definition of the class **Hy** of hyperseries, we have **No** $\cong$ **Hy** for the map $\phi: \mathbf{Hy} \rightarrow \mathbf{No}; f \mapsto f(\omega)$.*

Theorem (Bagayoko–vdH 2019–2022)

*For an appropriate definition of the class **Hy** of hyperseries, we have $\mathbf{No} \cong \mathbf{Hy}$ for the map $\phi: \mathbf{Hy} \rightarrow \mathbf{No}; f \mapsto f(\omega)$.*

Proof. By constructing a Conway bracket $\{|\}$ on **Hy**.

Theorem (Bagayoko–vdH 2019–2022)

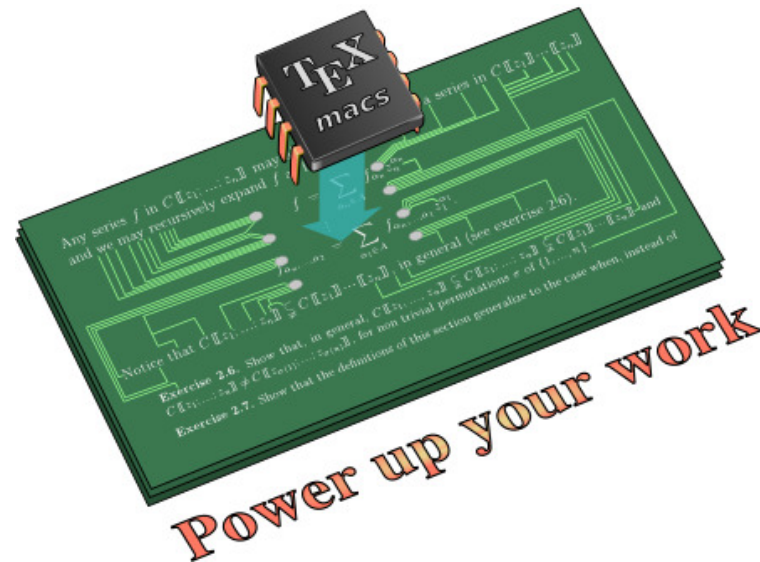
For an appropriate definition of the class **Hy** of hyperseries, we have **No** $\cong$ **Hy** for the map $\phi: \mathbf{Hy} \rightarrow \mathbf{No}; f \mapsto f(\omega)$.

Proof. By constructing a Conway bracket $\{|\}$ on **Hy**.

Examples :

$$\begin{aligned} \{x, e^x, e^{e^x}, \dots | \} &= \exp_{\omega} x \\ \left\{ \sqrt{x}, \sqrt{x} + e^{\sqrt{\log x}}, \dots \middle| \dots, \sqrt{x} + e^{2\sqrt{\log x}}, 2\sqrt{x} \right\} &= f_0(x) \\ \left\{ x^2, e^{\log^2 x}, e^{e^{\log^2 \log x}}, \dots \middle| \dots, e^{e^{e^{\sqrt{\log \log x}}}}, e^{e^{\sqrt{\log x}}}, e^{\sqrt{x}} \right\} &= \exp_{\omega} \left(\log_{\omega} x + \frac{1}{2} \right) \end{aligned}$$

Thank you!



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