Closed function spaces for differential algebra

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Partially joint work with ASCHENBRENNER, VDDRIES, POGUDIN, CHEN, FANG

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Why are the complex numbers nice?

Hilbert Nullstellensatz

Radical ideals of $\mathbb{C}[x_1, \ldots, x_n] \xleftrightarrow{\text{bijection}}$ Zero-sets in $\mathbb{C}^n$ of polynomial systems

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What can we say in the case of differential algebra?

Linear PDEs with constant coefficients

Floklore dictionary

$$
\begin{array}{rcl}
\mathbb{C}[x_1,\dots,x_n] & \longleftrightarrow & \mathbb{C}[\partial_{x_1},\dots,\partial_{x_n}] \\
\text{Buchberger reduction} & \longleftrightarrow & \text{Ritt reduction} \\
S\text{-polynomials} & \longleftrightarrow & \Delta\text{-polynomials} \\
\text{Gröbner basis} & \longleftrightarrow & \text{Coherent autoreduced set} \\
& \vdots &
\end{array}
$$

See also: **Rosenfeld 1959**

Radical ideals

$$\{(x,y):(x-1)^2=x^2+y^2=0\} \longleftrightarrow \{f:(\partial_x-1)^2 f=(\partial_x^2+\partial_y^2)\, f=0\}$$

$$\{(1,\mathrm{i}),(1,-\mathrm{i})\} \longleftrightarrow \mathbb{C}\,\mathrm{e}^{x+\mathrm{i}y}+\mathbb{C}\,\mathrm{e}^{x-\mathrm{i}y}$$

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General ideals

$$((x-1)^2, x^2 + y^2) \longleftrightarrow \{f : (\partial_x - 1)^2 f = (\partial_x^2 + \partial_y^2) f = 0\}$$

$$\longleftrightarrow \mathbb{C}\,\mathrm{e}^{x+\mathrm{i}y} + \mathbb{C}\,(x + \mathrm{i}y)\,\mathrm{e}^{x+\mathrm{i}y} + \mathbb{C}\,\mathrm{e}^{x-\mathrm{i}y} + \mathbb{C}\,(x - \mathrm{i}y)\,\mathrm{e}^{x-\mathrm{i}y}$$

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Note: the *differential* ideal $[(\partial_x-1)^2 F,(\partial_x^2+\partial_y^2)\,F]$ is radical.

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Also: *differential shape lemmas*, **Kauers–Koutchan–Verron**, **vdH–Pogudin**

Polynomial coefficients

Local solutions of $L \in \mathbb{C}[x][\partial_x]$ at $x \to \infty$ **(Fabry, Poincaré, Birkhoff, ...)**

$$f \;=\; (f_d \log^d x + \cdots + f_0)\, x^{\alpha}\, \mathrm{e}^{P(\sqrt[r]{x})}, \qquad f_k \;=\; \sum_{i \geqslant 0} f_{k,i}\, x^{-i/r} \;\in\; \mathbb{C}[[x^{1/r}]]$$

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Partial differential equations

System $Lf = g$, $L \in \mathbb{C}[\boldsymbol{x}_1, \ldots, x_n][\partial_{x_1}, \ldots, \partial_{x_n}]^m$, $g \in \mathbb{S}^m$

$$\mathbb{S} := \mathbb{C}[\log x_1, \ldots, \log x_n]((x_1^{\mathbb{C}} \cdots x_n^{\mathbb{C}}))_{\mathrm{gb}}$$

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$$\mathbb{S}_{\exp} := \mathrm{Vect}_{\mathbb{C}}(\mathbb{S}\, \mathrm{e}^{\mathbb{S}}) \quad \text{or} \quad \mathrm{Vect}_{\mathbb{C}}(\mathbb{S}\, \mathrm{e}^{\mathbb{C}[x_1, \ldots, x_n]^{\mathrm{ac}}})$$

Open: is $\mathbb{S}_{\exp}$ "closed" for solving homogeneous systems $Lf = 0$?

Transseries

$$(\mathrm{e}^x)' = \mathrm{e}^x, \qquad (\mathrm{e}^{\mathrm{e}^x})' = \mathrm{e}^x \mathrm{e}^{\mathrm{e}^x}, \qquad (\mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}})' = \mathrm{e}^x \mathrm{e}^{\mathrm{e}^x} \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}, \qquad \ldots$$

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Real transseries $\mathbb{T} = \mathbb{R}[[[x]]]$ at infinity $x \to \infty$

$$\mathrm{e}^{\mathrm{e}^x + 2\frac{\mathrm{e}^x}{x} + 6\frac{\mathrm{e}^x}{x^2} + 24\frac{\mathrm{e}^x}{x^3} + \cdots} + \mathrm{e}^{(\log x)^3 + (\log x)^2} + \pi - \frac{1}{\log x} + \frac{1}{\mathrm{e}^x} - \frac{1}{\mathrm{e}^{2x}} + \cdots$$

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Complex transseries $\tilde{\mathbb{T}} = \mathbb{C}[[[z]]]^{\mathrm{osc}}$

Construction requires chosing a region: $\mathrm{e}^{\mathrm{i}z} \succ 1$ or $\mathrm{e}^{\mathrm{i}z} \prec 1$, $\mathrm{e}^{z^2} \succ 1$ or $\mathrm{e}^{z^2} \prec 1$?

vdH 2001: $\tilde{\mathbb{T}}$ is Picard-Vessiot closed for any choice of a region

Asymptotic differential algebra

H-fields (Aschenbrenner–vdDries 2002):

Differential field K with ordering $\leqslant$ (whence valuation) and constants C such that

HF1. If $f > C$, then $f' > 0$.

HF2. If $f \preccurlyeq 1$, then there exists $c = \lim f \in C$ with $f \sim c$ (i.e. $f - c \prec 1$).

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Examples

- The transseries $\mathbb{T}$ (grid-based, well-based, …). **(2017)**
- Any maximal Hardy field. **(2023)**
- The surreal numbers. **(2019)**

Model theory of H-closed fields

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- **To check!** Coherence of ring of integro-differential operators **(Bavula 2013)**
- Definable subsets of $\mathbb{R}^n$ are the semi-algebraic subsets.

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However **(Rosenlicht 1980)**

$\begin{cases} y^3 + y = (y')^2 \\ y^3 + y \neq 0 \end{cases}$ has no solution at $z \to \infty$ (so $\tilde{\mathbb{T}}$ is not differentially closed)

Hence, $\wp$ does not lie in the Picard-Vessiot closure of $\mathbb{C}(z)$.

Analytic counterparts ?

Recursive discrete ramification

Starting with $\mathscr{R}_0 := \mathbb{C}$, repeat a finite number of times:

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Application

- The generating function of the partition numbers $\notin$ PV-closure of $\mathbb{C}(z)$.

Differentially closed field

Algebraically closed differential field K such that
$\forall\, P, Q \in K\{Y\}$ with $\operatorname{ord} Q < \operatorname{ord} P, \quad \exists y \in K$ with $P(y) = 0 \neq Q(y)$.

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Construction over an algebraically closed constant field $K = C_K$

$$\begin{aligned} K_\ell &:= K((\lambda_1^{\mathbb{Q}}))\cdots((\lambda_\ell^{\mathbb{Q}})) \\ L_\ell &:= K_\ell((z_\ell^{\mathbb{Q}})) \end{aligned}$$

L_ℓ differential field w.r.t. ∂_{z_ℓ} with constant field $C_{L_\ell} = K_\ell$.

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We embed $L_\ell \xhookrightarrow{\Phi_\ell} L_{\ell+1}$ via $z_\ell=\lambda_{\ell+1}+z_{\ell+1}$:

$$\Phi_\ell\left(\sum_{i\geqslant i_0} y_i z_\ell^i\right) := \sum_{k\geqslant 0}\left(\sum_{i\geqslant i_0}\frac{i\cdots(i-k+1)}{k!}y_i\lambda_{\ell+1}^{i-k}\right)z_{\ell+1}^k$$

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- Then $R := Q\, I_P \operatorname{Res}_{Y^{(r)}}(P, S_P) \neq 0$.

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- Then $R := Q\, I_P \operatorname{Res}_{Y^{(r)}}(P, S_P) \neq 0$.
- Rewrite $P, Q, R, \ldots$ as power series in $z_{\ell+1}$.

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- W.l.o.g.: $P, Q \in L_\ell\{Y\}$ with $\operatorname{ord} Q < r := \operatorname{ord} P$ and P square-free.
- Then $R := Q\, I_P \operatorname{Res}_{Y^{(r)}}(P, S_P) \neq 0$.
- Rewrite $P, Q, R, \ldots$ as power series in $z_{\ell+1}$.
- Take initial $\alpha_0, \ldots, \alpha_{r-1} \in K_{\ell+1}$ with $R[z_{\ell+1} = 0](\alpha_0, \ldots, \alpha_{r-1}) \neq 0$.

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- We automatically have $Q(y) \neq 0$, since $Q[z_{\ell+1} = 0](y) \neq 0$.

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Moreover, we have the following differential embedding:

$$\begin{aligned} \Phi : K &\hookrightarrow K((z^{\mathbb{Q}})) \\ y &\longmapsto \sum_{i\geqslant 0} \frac{1}{i!} y^{(i)} z^i. \end{aligned}$$

Thank you !

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