Fast modular composition using spiroids

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Joint work with Grégoire Lecerf

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$\longrightarrow$ Fundamental: cost of "polynomial evaluation" in algebraic extensions $\mathbb{A}[x]/(R)$

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$\longrightarrow$ If $\mathbb{A} = \mathbb{F}_q$ and R irreducible of degree $d_1 d_2$, then R factors over $\mathbb{F}_{q^{d_1}}$

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$$\hat{P}\Big(\underbrace{Q_1(\alpha_k), Q_2(\alpha_k), \ldots, Q_n(\alpha_k)}_{\xi_k} \Big) \text{ for } k = 1, \ldots, D d n \quad \longrightarrow \quad \hat{R}$$

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  - vdH–Lecerf 2021: *On the complexity exponent of polynomial system solving*

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Special ring $\mathbb{A}=\mathbb{Z}/r\mathbb{Z}$

Complexity bounds

General rings $\mathbb{A}$

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Amortized complexity

- $n = O(1)$, fixed points ξ, vdH–Lecerf 2023: $\tilde{O}(N)$

New approach for modular composition

Idea: reduce to the computation of $P \circ (Q_1, \ldots, Q_n) \in \mathbb{Z}/r\mathbb{Z}[x]$,

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Example for $n=2$, $r=10$, $d=4$

$$\begin{aligned} P &= 3z_1^2+5z_2^2+2z_1+7 & & & \check{P} &= 3z_1^2+5z_2^2+2z_1+7 \\ Q_1 &= x^{14}+7x^8+2x^4+6x^2+8 & &\xrightarrow{\text{lift } \mathbb{Z}} & \check{Q}_1 &= x^{14}+7x^8+1002x^4+6x^2+8 \\ Q_2 &= x^{11}+3x^{10}+2x^2+5x+1 & & & \check{Q}_2 &= x^{11}+3x^{10}+2x^2+1005x+1 \end{aligned}$$

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Required precision: $\approx n\,d \log(Mr)$, where $1000 = Mr$

$$|\tilde{P}| < 10 \cdot 1000^2, \qquad |\tilde{Q}_1 - x^4| < \frac{10}{1000}, \qquad |\tilde{Q}_2 - x| < \frac{10}{1000}$$

Regular spiroids

$$\delta_1, \ldots, \delta_n \in 2^{\mathbb{N}}, \qquad \Delta := \delta_1 \cdots \delta_n, \qquad \omega = \mathrm{e}^{2\pi \mathrm{i}/\Delta}$$

Regular spiroid: tuple of points $\boldsymbol{\omega} := (\omega_k)_{0 \leqslant k < \Delta} \in (\mathbb{C}^n)^\Delta$ with

$$\omega_k := (\omega^{k\Delta/\delta_1}, \omega^{k\Delta/(\delta_1\delta_2)}, \ldots, \omega^k) \in \mathbb{C}^n$$

Associated ideal: $I_\omega \subseteq \mathbb{C}[\boldsymbol{z}]$ generated by the grevlex Gröbner basis

$$z_1^{\delta_1} - 1,\ z_2^{\delta_2} - z_1,\ z_3^{\delta_3} - z_2,\ \ldots, z_n^{\delta_n} - z_{n-1}$$

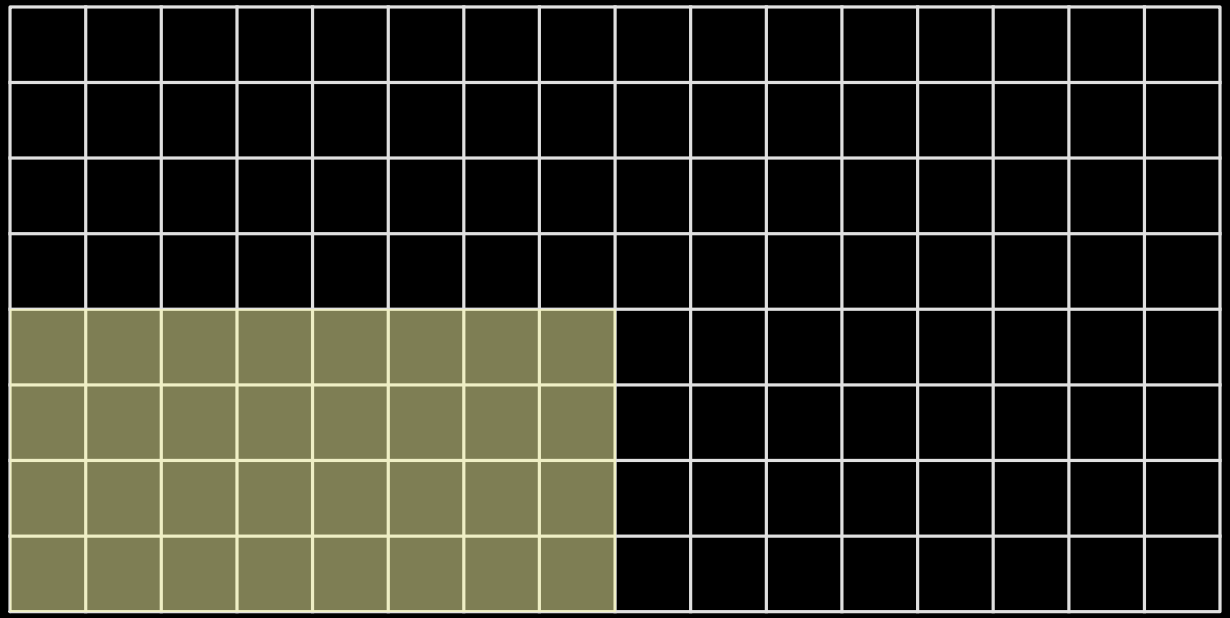
Reduced elements: $\mathbb{C}[\boldsymbol{z}]_{\delta_1,\ldots,\delta_n} := \{P \in \mathbb{C}[\boldsymbol{z}] : \forall k, \deg_{z_k} P < \delta_k\}$

Quotient algebra: $\mathbb{S}_\omega := \mathbb{C}[\boldsymbol{z}] / I_\omega$ with $\dim_{\mathbb{C}} \mathbb{S}_\omega = \Delta$

Regular spiroid arithmetic

Product: $P, Q \in \mathbb{C}[z]_{\delta_1,\ldots,\delta_n} \implies PQ \in \mathbb{C}[z]_{2\delta_1,\ldots,2\delta_n}$

Reduction: $S \in \mathbb{C}[z]_{2\delta_1,\ldots,2\delta_n} \xrightarrow{2^n\Delta \text{ additions}} \operatorname{red} S \in \mathbb{C}[z]_{\delta_1,\ldots,\delta_n}$



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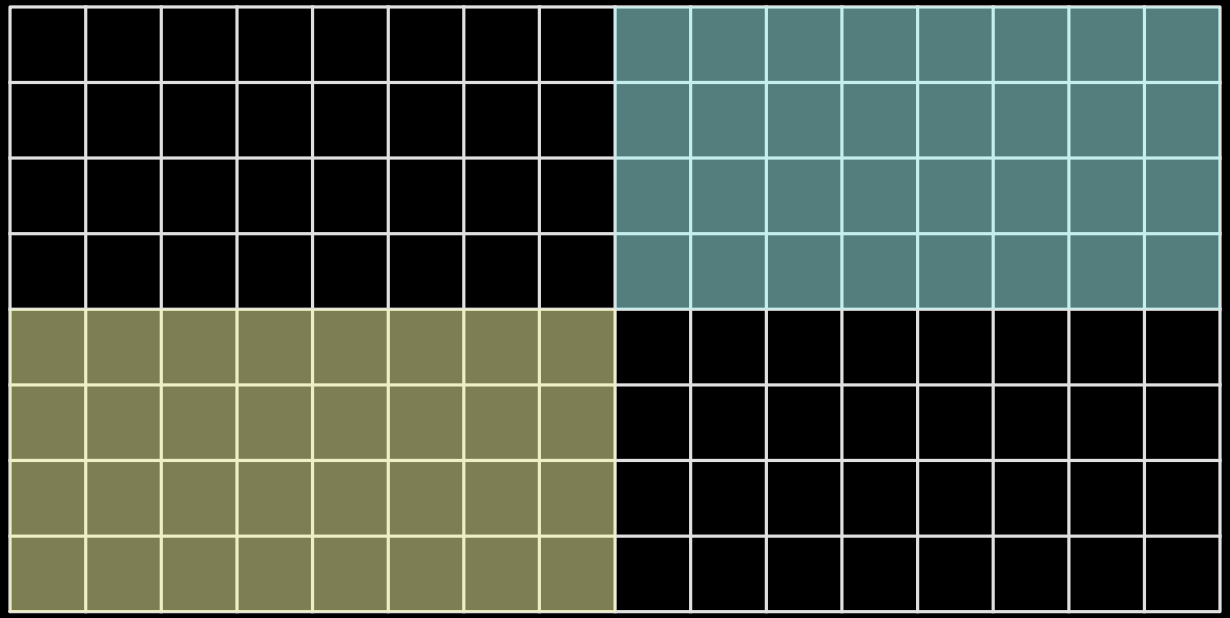

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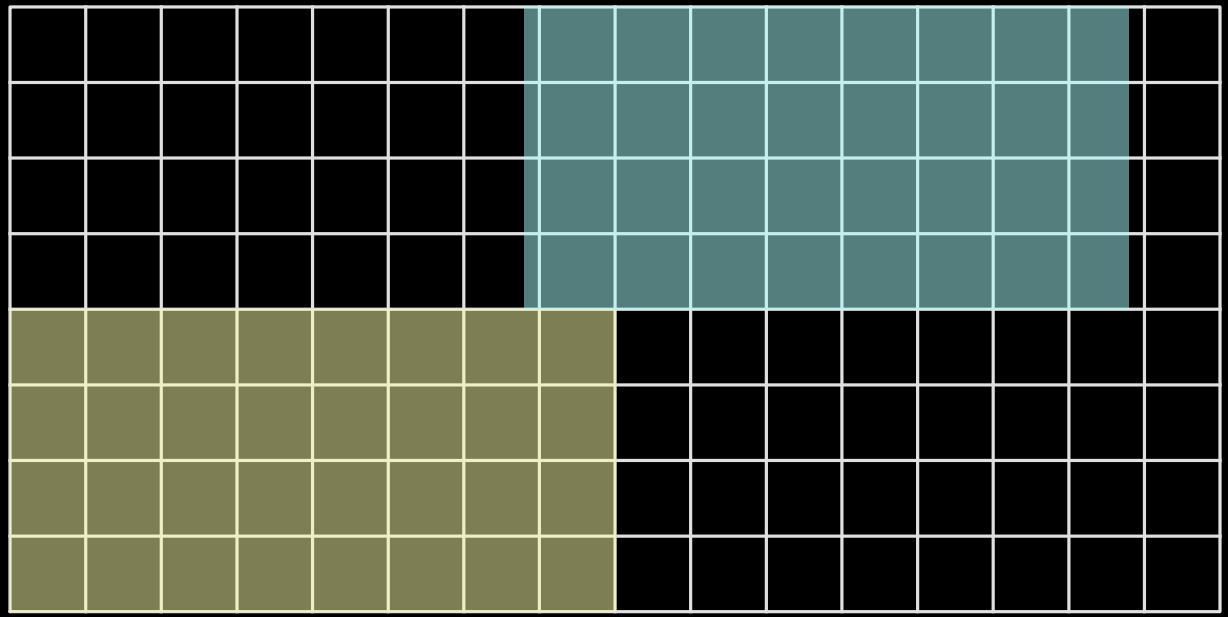

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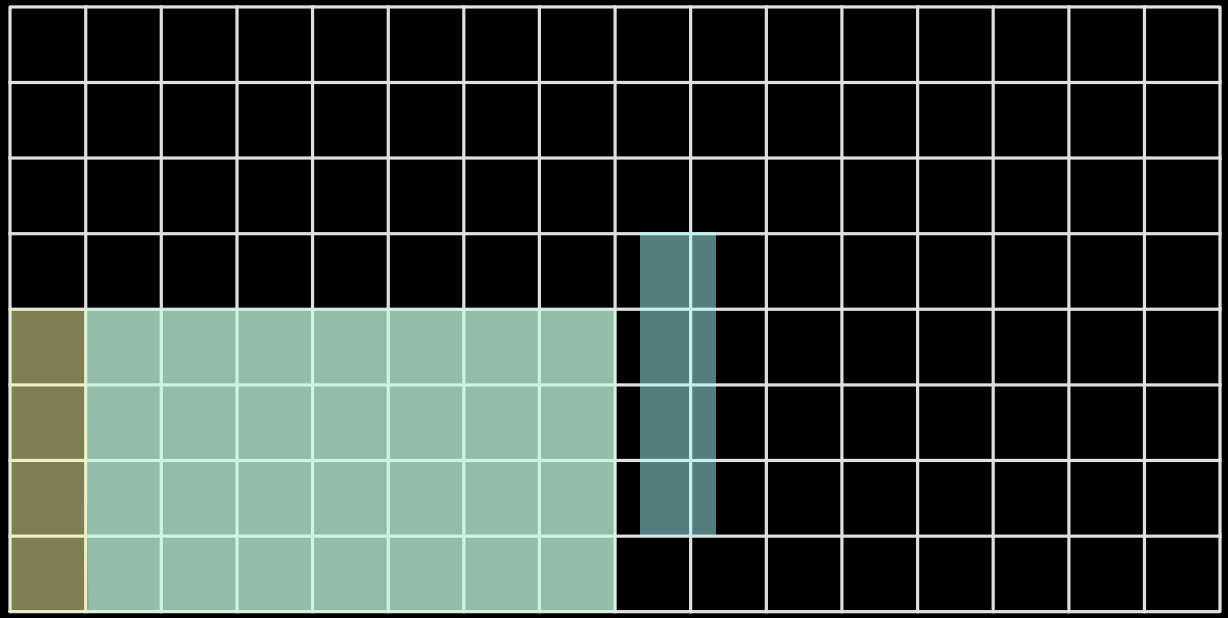

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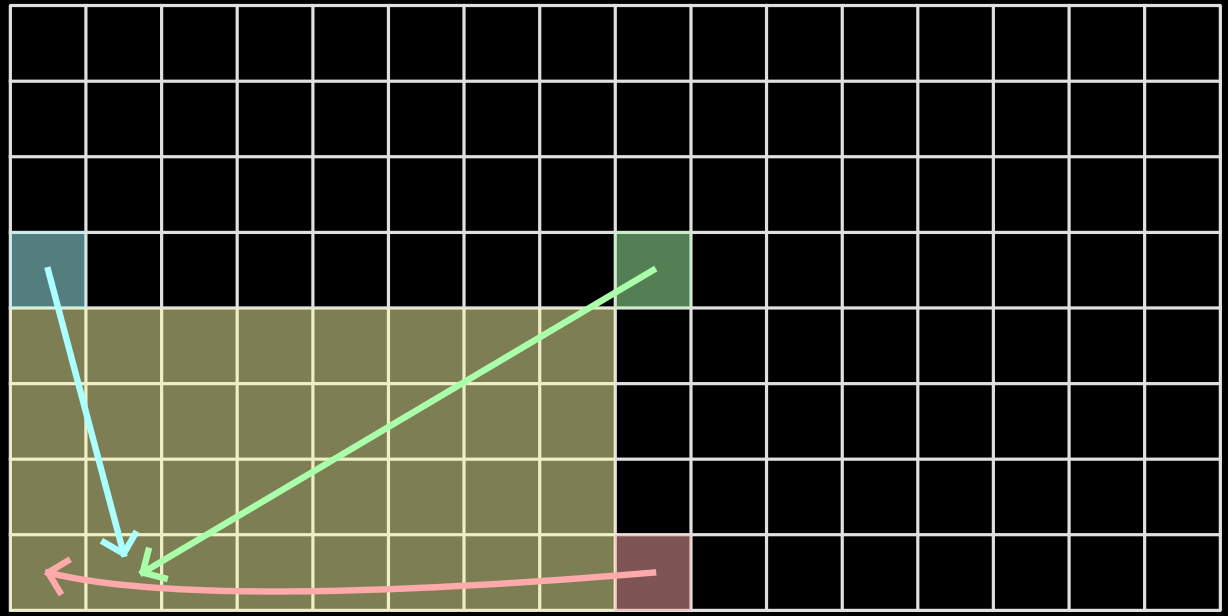
Extended reduction for regular spiroids

$\mathbb{E}_n := \{0,1\}^n, \quad \mathbb{E}_n^* := \mathbb{E}_n \setminus \{\mathbf{0}\}.$

$$F_\epsilon := z_1^{\epsilon_1\delta_1} \cdots z_n^{\epsilon_n\delta_n} - z_1^{\epsilon_2} \cdots z_{n-1}^{\epsilon_n} \in I_\omega \qquad \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{E}_n^*$$



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Lemma

We can compute an extended reduction at precision p in time $O(2^n \Delta (p+n))$.

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Intuitive definition: $\xi \in (\mathbb{C}^n)^\Delta$ sufficiently close to ω

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Assume for now that ξ is a spiroid and that the G_ϵ are known

Extended reduction for general spiroids

Extended reduction of $P \in \mathbb{C}[z]_{2\delta_1,\ldots,2\delta_n}$ with respect to I_ξ:

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Proof. Assume approximation $(\tilde{A}_\epsilon)$ of (A_ϵ) and let $\tilde{P} = P - \sum_{\epsilon \in \mathbb{E}_n} \tilde{A}_\epsilon G_\epsilon$.

Reduce $\tilde{P} = \sum_{\epsilon \in \mathbb{E}_n} B_\epsilon F_\epsilon$. Then we have the better approximation $(\tilde{A}_\epsilon + B_\epsilon)$.

For large precisions p, one may use this in a divide and conquer fashion. □

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Input: given $v = (v_k)_{k<\Delta} \in (\mathbb{C}^n)^\Delta$, compute

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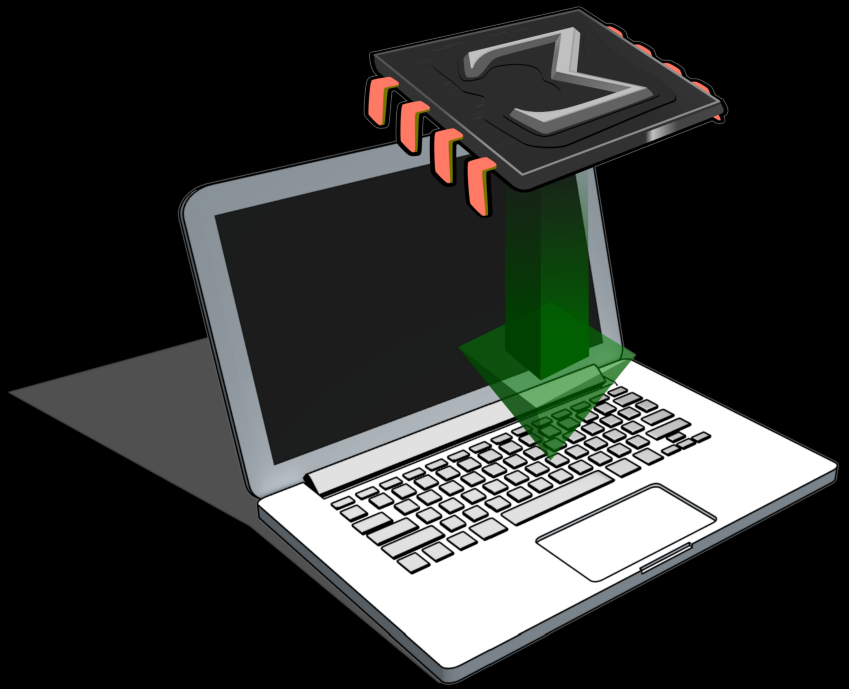
Challenge: does there exist a *practical* faster algorithm ?

Thank you !



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