Making differential closure more explicit*

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Given a differential field K of characteristic zero, we give a short explicit construction of a differentially closed field $\hat{K}$ that contains K.

1. INTRODUCTION

Several well-known first order theories admit natural explicit models. For instance, the reals are the archetype of a real closed field and the complex numbers are a natural example of an algebraically closed field of characteristic zero. More recently, it was shown that both the field of transseries and maximal Hardy fields are models of the theory of closed H-fields [1, 2]. (Here an H-field is a differential field with an ordering that satisfies suitable compatibility conditions with the derivation.)

The first order theories behind the above three examples have excellent model theoretic properties: they are all complete, model complete, and they admit quantifier elimination for appropriate languages. In contrast, even though the theory of differentially closed fields of characteristic zero also enjoys these properties [14, 12, 3], no simple explicit models are known in this case. The goal of this paper is to construct such models.

We recall from [11, 7, 8] that a *differential ring* R is a ring together with a derivation $\partial: R \rightarrow R; y \mapsto y'$. The set $C_R := \{y \in R : y' = 0\}$ is called its *ring of constants*. Given a differential ring R, we define the set of *differential polynomials* in Y over R by $R\{Y\} := \bigcup_{r \in \mathbb{N}} R[Y, \ldots, Y^{(r)}]$. Then $R\{Y\}$ is a differential ring for the unique derivation that extends the derivation on R and such that $\partial Y^{(i)} = Y^{(i+1)}$ for all i. Given $P \in R\{Y\}$, the minimal $r \in \mathbb{N}$ with $P \in R[Y, \ldots, Y^{(r)}]$ is called its *order* and we denote it by $\operatorname{ord} P := r$.

A *differential field* is a differential ring that is also a field. Let K be a differential field of characteristic zero. We say that K is *differentially closed* if it is algebraically closed and for any $P, Q \in K\{Y\}$ with $\operatorname{ord} P > \operatorname{ord} Q$, there exists a $y \in K$ with $P(y) = 0$ and $Q(y) \neq 0$. This notion was first introduced by Robinson [12] and subsequently elaborated by Blum [3]. The definition that we use here is due to Blum.

Although it is known that any differential field K of characteristic zero can be embedded into a differentially closed field $\hat{K}$, the proof is abstract and does not provide us with an explicit description of $\hat{K}$. A more natural—but still fairly lengthy—construction has recently been proposed in [18]. Other ideas were raised in a recent discussion on Mathoverflow[1].

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1. https://mathoverflow.net/questions/417343/why-is-it-so-hard-to-give-examples-of-differentially-closed-fields

There are also partial constructions that go a long way, but fall short of delivering a full differential closure. For instance, Seidenberg's embedding theorem allows for the realization of certain countable differential fields as fields of meromorphic functions [15, 16, 9]. Fields K of complex transseries from [4] have the property that any $P \in K\{Y\} \setminus K$ has a root in K. In particular, such fields are closed under the resolution of linear differential equations. See also [6] for an analytic counterpart.

In this note, we give a short explicit construction for embedding an arbitrary differential field K of characteristic zero into a differentially closed field $\hat{K}$. The key observation is that the ring of formal power series is very close to providing a differential closure: at a non-singular point, formal power series solutions are in sufficient supply to force the required inequality $Q(y) \neq 0$. Through the introduction of generic shifts $\lambda_1, \lambda_2, \ldots$ we will reduce the general case to this favorable situation.

However, the generic shifts $\lambda_1, \lambda_2, \ldots$ also introduce many new constants in our field. An interesting question remains: how to single out a differentially closed subfield of $\hat{K}$ which has the same constant field as K?

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2. CONSTRUCTION

Let K be an algebraically closed field of characteristic zero with the trivial derivation that turns every element of K into a constant. We will write $K((z^{\mathbb{Q}}))$ for the algebraically closed field of formal Puiseux series in z with coefficients in K. For $\ell = 0, 1, 2, \ldots$, let

$$\begin{aligned} K_\ell &:= K((\lambda_1^{\mathbb{Q}})) \cdots ((\lambda_\ell^{\mathbb{Q}})) \\ L_\ell &:= K_\ell((z_\ell^{\mathbb{Q}})). \end{aligned}$$

We view L_ℓ as a differential field with constant field K_ℓ for the derivation $\partial_\ell := \partial / \partial z_\ell$. Setting

$$\begin{aligned} \Phi_\ell\left(\sum_{i \geqslant i_0} y_i z_\ell^i\right) &:= \sum_{i \geqslant i_0} y_i (\lambda_{\ell+1} + z_{\ell+1})^i \\ &= \sum_{i \geqslant i_0} y_i \lambda_{\ell+1}^i \sum_{k \geqslant 0} \frac{i \cdots (i-k+1)}{k!} \frac{z_{\ell+1}^k}{\lambda_{\ell+1}^k} \\ &= \sum_{k \geqslant 0} \left(\sum_{i \geqslant i_0} \frac{i \cdots (i-k+1)}{k!} y_i \lambda_{\ell+1}^{i-k}\right) z_{\ell+1}^k \end{aligned}$$

we define a differential field embedding

$$L_\ell \xhookrightarrow{\Phi_\ell} L_{\ell+1}.$$

Note that

$$\operatorname{im} \Phi_\ell \subseteq K_{\ell+1}[[z_{\ell+1}]] \tag{1}$$

and that the evaluation at $z_{\ell+1} = 0$ of $\Phi_\ell(y)$ is non-zero for any non-zero $y \in L_\ell$.

Regarding L_ℓ as being naturally included in $L_{\ell+1}$ via the embedding Φ_ℓ, we have an infinite chain of inclusions

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots.$$

Now consider the differential field

$$L_\infty := \bigcup_{\ell \geqslant 0} L_\ell. \tag{2}$$

THEOREM 1. *The field L_∞ is differentially closed.*

Proof. The field L_∞ is algebraically closed as an increasing union of algebraically closed fields.

Consider two non-zero differential polynomials $P, Q \in L_\infty\{Y\}$ of orders $r := \operatorname{ord} P$ and $\operatorname{ord} Q < r$, respectively. We have to prove that there exists a $y \in L_\infty$ with $P(y) = 0$ and $Q(y) \neq 0$. Since P and Q have finitely many coefficients, we have $P, Q \in L_\ell\{Y\}$ for some $\ell \geqslant 0$. Without loss of generality, we may assume that P is square-free as a polynomial $P = P_d\,(Y^{(r)})^d + \cdots + P_0$ in $Y^{(r)}$, where $P_d \neq 0$. So $\operatorname{Res}_{Y^{(r)}}(P, S_P)$ is non-zero, where $S_P := \partial P / \partial Y^{(r)}$ denotes the separant of P. Let $R := Q\,I_P \operatorname{Res}_{Y^{(r)}}(P, S_P)$, where $I_P := P_d$ is the initial of P. Hence R is a non-zero element of $L_\ell[Y, \ldots, Y^{(r-1)}]$.

In view of (1), we may regard P, I_P, S_P, and R as elements of $K_{\ell+1}[[z_{\ell+1}]]\{Y\}$. Let $\bar{P}, \bar{I}_P, \bar{S}_P, \bar{R} \in K_{\ell+1}\{Y\}$ be their evaluations at $z_{\ell+1} = 0$. Note that $\operatorname{ord} \bar{P} = \operatorname{ord} P = r$, so $\bar{P} \in K_{\ell+1}[Y, \ldots, Y^{(r)}]$. Similarly, $\bar{R} \in K_{\ell+1}[Y, \ldots, Y^{(r-1)}]$. Consider $(\alpha_0, \ldots, \alpha_{r-1}) \in K_{\ell+1}$ with $\bar{R}(\alpha_0, \ldots, \alpha_{r-1}) \neq 0$. Since $K_{\ell+1}$ is algebraically closed and $\bar{I}_P(\alpha_0, \ldots, \alpha_{r-1}) \neq 0$, there exists some $\alpha_r \in K_{\ell+1}$ with $\bar{P}(\alpha_0, \ldots, \alpha_r) = 0$. By construction, $\bar{S}_P(\alpha_0, \ldots, \alpha_r) \neq 0$, since otherwise $\bar{R}(\alpha_0, \ldots, \alpha_{r-1})$ would vanish.

Now consider the differential equation $P(y) = 0$ over $K_{\ell+1}[[z_{\ell+1}]]$ with initial conditions $y_0 = \alpha_0, \ldots, y_r = \alpha_r / r!$. Since $S_P(\alpha_0, \ldots, \alpha_r) \neq 0$, this equation admits a unique formal power series solution $y \in K_{\ell+1}[[z_{\ell+1}]] \subseteq L_\infty$. By construction, the valuation of $R(y)$ in $z_{\ell+1}$ is zero, so $Q(y) \neq 0$. This concludes our proof. □

Remark 2. The field L_∞ is still far away from being "the" differential closure of K or $K(z_0)$. First of all, L_∞ is not d-algebraic over K. This problem can be addressed by constructing a sequence $\check{L}_0, \check{L}_1, \ldots$ of differential subfields of $L_0, L_1, \ldots$ as follows. We take $\check{L}_0 := K(z_0)$. Assuming that $\check{L}_\ell$ with constant field $\check{K}_\ell \subseteq K_\ell$ has been constructed, let $\check{K}_{\ell+1}$ be the algebraic closure of $\check{L}_\ell$ in which we substituted $\lambda_{\ell+1}$ for z_ℓ. Then we let $\check{L}_{\ell+1}$ be the field of $y \in L_{\ell+1} \cap \check{K}_{\ell+1}((z_{\ell+1}^{\mathbb{Q}}))$ that are d-algebraic over K. The same argument as in the above proof shows that $\check{L}_\infty = \bigcup_{\ell \geqslant 0} \check{L}_\ell$ is differentially closed, but also d-algebraic over K.

A second problem is that the constant fields $K_\infty := \bigcup_{\ell \geqslant 0} K_\ell$ of L_∞ and $\check{K}_\infty := \bigcup_{\ell \geqslant 0} \check{K}_\ell$ of $\check{L}_\infty$ are much larger than K. By construction, we actually have a replica of $\check{L}_\infty$ inside $\check{K}_\infty$, since $\check{K}_{\ell+1}$ is a replica of $\check{L}_\ell$ inside $\check{K}_\infty$ for each ℓ. This suggests that the second problem may be related to the non-minimality of differential closures [13, 17]. Even though K_∞ and $\check{K}_\infty$ are much larger than K, they still admit a very explicit nature. In particular, elements of $\check{K}_\infty$ can recursively be described as solutions of suitable differential equations in $\lambda_1, \lambda_2, \ldots$.

3. EMBEDDING ARBITRARY DIFFERENTIAL FIELDS

For the theorem from the previous section, we worked over an algebraically closed constant field K of characteristic zero. In fact, there is a simple way [10, §44.3] to embed an arbitrary differential field of characteristic zero into a differential field of Puiseux series whose coefficients are constants:

LEMMA 3. *Let K be a differential field of characteristic zero and consider the map*

$$\begin{aligned} \Phi : K &\hookrightarrow K((z^{\mathbb{Q}})) \\ y &\longmapsto \sum_{i \geqslant 0} \frac{1}{i!} y^{(i)} z^i. \end{aligned}$$

We define a derivation on $K((z^{\mathbb{Q}}))$ by $\sum_{i \geqslant i_0} y_i z^i = \sum_{i \geqslant i_0} i\,y_i z^{i-1}$. Then Φ is an embedding of differential fields.

Proof. For $y_1, y_2 \in K$, one verifies that

$$\begin{aligned}
\Phi(y_1+y_2) &= \sum_{i\geqslant 0} \frac{1}{i!}(y_1^{(i)}+y_2^{(i)})\, z^i = \Phi(y_1)+\Phi(y_2)\\
\Phi(y_1 y_2) &= \sum_{i\geqslant 0} \frac{1}{i!}\left(\sum_{0\leqslant k\leqslant i} \frac{i!}{k!\,(i-k)!} y_1^{(k)} y_2^{(i-k)}\right) z^i = \Phi(y_1)\,\Phi(y_2)\\
\Phi(y_1') &= \sum_{i\geqslant 0} \frac{1}{i!} y_1^{(i+1)} z^i = \sum_{i\geqslant 1} \frac{1}{(i-1)!} y_1^{(i)} z^{i-1} = \Phi(y_1)'.
\end{aligned}$$

□

COROLLARY 4. *Let K_0 be a differential field of characteristic zero. Then K_0 can naturally be embedded into a differentially closed field of the form* (2).

Proof. Without loss of generality, we may assume that K_0 is algebraically closed, since any derivation on a field of characteristic zero canonically extends to its algebraic closure. Using the lemma, we may embed K_0 into $L_0 = K_0((z_0^{\mathbb{Q}}))$, where K_0 becomes a field of constants when regarded as a differential subfield of L_0. This allows us to pursue the construction of the previous section and successively embed $K_0 \hookrightarrow L_0 \hookrightarrow L_\infty$. We conclude by Theorem 1. □

Remark 5. Lemma 3 also has the potential of replacing the embeddings $\Phi_\ell: L_\ell \hookrightarrow L_{\ell+1}$ from the previous section by alternative ones. For instance, by generalizing the construction from [4], we expect it to be possible to construct fields of "complex transseries" T_ℓ in z_ℓ over K_ℓ. Such a field T_ℓ would contain L_ℓ and be closed under the resolution of linear differential equations. Thanks to Lemma 3, we might then let T_ℓ play the role of L_ℓ in the construction of the previous section.

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