Dendromorphic functions*†

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We study analytic properties of the Picard-Vessiot closure of $\mathbb{C}(z)$. In particular, we show that analytic functions in this closure do not admit natural boundaries in a strong sense. As a consequence, certain differentially algebraic equations over $\mathbb{C}$ like the generating series of the partition function do not lie in the Picard-Vessiot closure of $\mathbb{C}(z)$.

1. Introduction

Let K be a differential field of characteristic zero whose field of constants is algebraically closed. We say that K is *Picard-Vessiot closed* if any differential equation

$$L_r f^{(r)} + \cdots + L_0 f = 0$$

with $L_0, \ldots, L_r \in K$ and $L_r \notin 0$ has a fundamental system of r linearly independent solutions over the constant field of K. For any differential field K, and up to isomorphism, there exists a smallest Picard-Vessiot closed extension $K^{\mathrm{pv}} \supseteq K$ that contains K and that has the same constant field as K. We refer to [7] for the algebraic theory of Picard-Vessiot extensions. In [4], elements of $\mathrm{D}^\infty(K) := K^{\mathrm{pv}}$ are called differentially definable functions and algorithms are presented to compute with such functions.

Seidenberg's embedding theorem states that any countably generated differential field can be embedded into a field of meromorphic functions on some domain [9, 6]. However, this domain can be quite small. If $K = \mathbb{C}$ or $K = \mathbb{C}(z)$, then the main goal of this note is to show that elements of K^{pv} can be materialized as analytic functions on suitable, much larger, Riemann surfaces and to investigate analytic properties of these functions.

We first explore the kind of Riemann surfaces on which elements of K^{pv} are defined. Intuitively, the only singularities that can occur are isolated ones, or accumulation points of isolated singularities, or accumulation points of accumulation points of isolated singularities, and so on. This leads to the notion of “recursive discrete ramifications” that will be formally defined in section 2. For simplicity, we will restrict our attention to simply

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connected Riemann surfaces, but we note that it should not be hard to extend the theory to arbitrary connected Riemann surfaces.

Let $\mathscr{R}$ be a simply connected Riemann surface. The recursively discretely ramified Riemann surfaces above $\mathscr{R}$ give rise to an inductive system of analytic function spaces on these surfaces. The inductive limit $\mathscr{D}(\mathscr{R})$ of these spaces is the space of *dendromorphic functions* on $\mathscr{R}$. If $\mathscr{R} \subseteq \mathbb{C}$, then we will show in section 3 that $\mathscr{D}(\mathscr{R})$ is a Picard-Vessiot closed field. In particular, $\mathbb{C}^{\mathrm{PV}} \subseteq \mathscr{D}(\mathbb{C})$.

In section 4, we will derive some consequences of the fact that the only singularities of dendromorphic functions arise as recursive accumulation points of isolated singularities. In particular, such singularities cannot give rise to natural boundaries in a strong sense. As a corollary, we shall see that the generating series of the number of partitions of an integer does not belong to $\mathbb{C}^{\mathrm{PV}}$. This answers open question 4 from [1].

The set $\mathrm{D}^{\infty}(\mathbb{C})$ is closed under composition [4]. In section 5, we shall show that this also holds for $\mathscr{D}(\mathbb{C})$ and a suitable class of "boundaryless" functions. We no not know whether $\mathscr{D}(\mathbb{C})$ is closed under functional inversion. Weierstrass $\wp$ functions are examples of differentially algebraic dendromorphic functions that are not in $\mathrm{D}^{\infty}(\mathbb{C})$.

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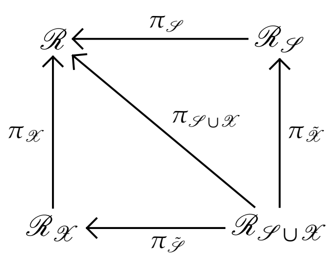
2. RECURSIVE DISCRETE RAMIFICATIONS

Let $\mathscr{R}$ be a connected Riemann surface. A *Riemann surface above* $\mathscr{R}$ is a pair $(\mathscr{R}', \pi)$, where $\mathscr{R}'$ is another Riemann surface $\mathscr{R}'$ and $\pi: \mathscr{R}' \longrightarrow \mathscr{R}$ a holomorphic covering: for every $z \in \mathscr{R}$, there exists an open neighborhood $\mathscr{U} \subseteq \mathscr{R}$ and a countable set Σ such that $\pi^{-1}(\mathscr{U}) = \coprod_{\sigma \in \Sigma} \mathscr{V}_\sigma$ and $\pi_{|\mathscr{V}_\sigma}: \mathscr{V}_\sigma \longrightarrow \mathscr{U}$ is a homeomorphism for every $\sigma \in \Sigma$.

We recall that there exists a Riemann surface $(\hat{\mathscr{R}}, \hat{\pi})$ above $\mathscr{R}$ with the property that for any other Riemann surface $(\mathscr{R}', \pi')$ above $\mathscr{R}$, there exists a unique $\pi: \hat{\mathscr{R}} \longrightarrow \mathscr{R}'$ with $\hat{\pi} = \pi' \circ \pi$ and such that $(\hat{\mathscr{R}}, \pi')$ is a Riemann surface above $\mathscr{R}'$. In particular, if $(\mathscr{R}', \pi')$ has the same universal property as $(\hat{\mathscr{R}}, \hat{\pi})$, then π' is a homeomorphism. In other words, the space $(\hat{\mathscr{R}}, \hat{\pi})$ is unique up to such a homeomorphism and we call it the *covering space* of $\mathscr{R}$. We also recall that the covering space of $\hat{\mathscr{R}}$ is $(\hat{\mathscr{R}}, \mathrm{Id})$.

Assume now that $\mathscr{R}$ is a simply connected Riemann surface and consider a discrete subset $\mathscr{S} \subseteq \mathscr{R}$, i.e. every $z \in \mathscr{S}$ has an open neighborhood $\mathscr{U} \subseteq \mathscr{R}$ with $\mathscr{U} \cap \mathscr{S} = \{z\}$. Let $\mathscr{U} := \mathscr{R} \setminus \mathscr{S}$ and consider the covering space $(\hat{\mathscr{U}}, \hat{\pi})$ of $\mathscr{U}$. We define $\mathscr{R}_{\mathscr{S}} := \hat{\mathscr{U}}$ and $\pi_{\mathscr{S}} := \iota \circ \hat{\pi}$, where $\iota: \mathscr{U} \longrightarrow \mathscr{R}$ is the inclusion map, and call $(\mathscr{R}_{\mathscr{S}}, \pi_{\mathscr{S}})$ a *discrete ramification* of $\mathscr{R}$. We will also say that the map $\pi_{\mathscr{S}}$ is a discrete ramification.

LEMMA 1. *Consider two discrete ramifications* $(\mathscr{R}_{\mathscr{S}}, \pi_{\mathscr{S}})$ *and* $(\mathscr{R}_{\mathscr{X}}, \pi_{\mathscr{X}})$ *of* $\mathscr{R}$. *Let* $\tilde{\mathscr{S}} = \pi_{\mathscr{X}}^{-1}(\mathscr{S} \setminus \mathscr{X}) \subseteq \mathscr{R}_{\mathscr{X}}$ *and* $\tilde{\mathscr{X}} = \pi_{\mathscr{S}}^{-1}(\mathscr{X} \setminus \mathscr{S}) \subseteq \mathscr{R}_{\mathscr{S}}$. *Then* $\mathscr{R}_{\mathscr{S} \cup \mathscr{X}} \cong (\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}} \cong (\mathscr{R}_{\mathscr{X}})_{\tilde{\mathscr{S}}}$ *and, after identification of* $\mathscr{R}_{\mathscr{S} \cup \mathscr{X}}$, $(\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}}$, *and* $(\mathscr{R}_{\mathscr{X}})_{\tilde{\mathscr{S}}}$ *via these isomorphisms, the following diagram commutes*



Proof. Since $((\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}}, \pi_{\mathscr{S}} \circ \pi_{\tilde{\mathscr{X}}})$ is a Riemann surface above $\mathscr{R} \setminus (\mathscr{S} \cup \mathscr{X})$, there exists a unique $\pi: \mathscr{R}_{\mathscr{S} \cup \mathscr{X}} \longrightarrow (\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}}$ with $\pi_{\mathscr{S} \cup \mathscr{X}} = \pi_{\mathscr{S}} \circ \pi_{\tilde{\mathscr{X}}} \circ \pi$. Conversely, $(\mathscr{R}_{\mathscr{S} \cup \mathscr{X}}, \pi_{\tilde{\mathscr{X}}} \circ \pi)$ is a Riemann surface above $\mathscr{R}_{\mathscr{S}}$, so there exists a unique $\pi': (\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}} \longrightarrow \mathscr{R}_{\mathscr{S} \cup \mathscr{X}}$ with $\pi_{\tilde{\mathscr{X}}} = \pi_{\tilde{\mathscr{X}}} \circ \pi \circ \pi'$. Next $(\mathscr{R}_{\mathscr{S} \cup \mathscr{X}}, \pi' \circ \pi)$ is a Riemann surface above $\mathscr{R}_{\mathscr{S} \cup \mathscr{X}}$, so there exists a unique map $\varphi: \mathscr{R}_{\mathscr{S} \cup \mathscr{X}} \longrightarrow \mathscr{R}_{\mathscr{S} \cup \mathscr{X}}$ with $\pi' \circ \pi = \mathrm{Id}$. This shows that π' and π are mutual inverses and $\mathscr{R}_{\mathscr{S} \cup \mathscr{X}} \cong (\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}}$. Identifying $\mathscr{R}_{\mathscr{S} \cup \mathscr{X}}$ and $(\mathscr{R}_{\mathscr{S}})_{\tilde{\mathscr{X}}}$ via this isomorphism, we have $\pi = \pi' = \mathrm{Id}$ and we already showed above that $\pi_{\mathscr{S} \cup \mathscr{X}} = \pi_{\mathscr{S}} \circ \pi_{\tilde{\mathscr{X}}} \circ \pi = \pi_{\mathscr{S}} \circ \pi_{\tilde{\mathscr{X}}}$. We conclude that the top triangle in the above diagram commutes. The isomorphism $\mathscr{R}_{\mathscr{S} \cup \mathscr{X}} \cong (\mathscr{R}_{\mathscr{X}})_{\tilde{\mathscr{S}}}$ and the commutation of the bottom triangle are proved similarly. □

Now consider a sequence $(\mathscr{R}_k, \pi_k)_{1 \leqslant k \leqslant h}$ with $\mathscr{R}_0 := \mathscr{R}$ and $(\mathscr{R}_k, \pi_k) = ((\mathscr{R}_{k-1})_{\mathscr{S}_k}, \pi_{\mathscr{S}_k})$ for $k = 1, \ldots, h$, where $\mathscr{S}_k$ is a discrete subset of $\mathscr{R}_{k-1}$. Then we call $(\mathscr{R}_h, \pi_1 \circ \cdots \circ \pi_h)$ a *recursive discrete ramification* of $\mathscr{R}$ of height h. We will also say that the map $\pi_1 \circ \cdots \circ \pi_h$ is a recursive discrete ramification. If π, π' are two recursive discrete ramifications, then we will say that π' *lies over* π if there exists a recursive discrete ramification π'' with $\pi' = \pi \circ \pi''$.

LEMMA 2. *Given two recursive discrete ramifications π and π' of respective heights h and h', there exists a recursive discrete ramification π'' of height* $\max(h, h)$ *that lies above both π and π'.*

Proof. Setting $\mathscr{R}_{0,0} := \mathscr{R}$, there exist sequences of discrete ramifications

$$\begin{array}{ccccccc} \mathscr{R}_{0,0} & \longleftarrow & \mathscr{R}_{1,0} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{h,0} \\ \mathscr{R}_{0,0} & \longleftarrow & \mathscr{R}_{0,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{0,h'} \end{array}$$

whose compositions are π and π', respectively. Using $h h'$ applications of Lemma 1, we can extend these sequences into a commutative diagram

$$\begin{array}{ccccccc} \mathscr{R}_{0,0} & \longleftarrow & \mathscr{R}_{0,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{0,h'} \\ \uparrow & \nwarrow & \uparrow & \nwarrow & & \nwarrow & \uparrow \\ \mathscr{R}_{1,0} & \longleftarrow & \mathscr{R}_{1,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{1,h'} \\ \uparrow & \nwarrow & \uparrow & \nwarrow & & \nwarrow & \uparrow \\ \vdots & & \vdots & & & & \vdots \\ \uparrow & \nwarrow & \uparrow & \nwarrow & & \nwarrow & \uparrow \\ \mathscr{R}_{h,0} & \longleftarrow & \mathscr{R}_{h,1} & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_{h,h'} \end{array}$$

where all the arrows are discrete ramifications. The composite map $\mathscr{R}_{h,h'} \longrightarrow \mathscr{R}_{0,0}$ is the desired recursive ramification π''. Since we can obtain it by following $\min(h, h')$ diagonal arrows and $\max(h, h') - \min(h, h')$ horizontal or vertical arrows, the height of π'' is $\max(h, h')$, as claimed. □

3. DENDROMORPHIC FUNCTIONS

Given a simply connected Riemann surface $\mathscr{R}$, let $\mathscr{A}(\mathscr{R})$ denote the space of analytic functions on $\mathscr{R}$. For any discrete ramification $(\mathscr{R}_{\mathscr{S}}, \pi_{\mathscr{S}})$, we have a natural injection of $\mathscr{A}(\mathscr{R})$ into $\mathscr{A}(\mathscr{R} \setminus \mathscr{S})$ into $\mathscr{R}_{\mathscr{S}}$. Consequently, any recursive discrete ramification $\pi: \mathscr{R}' \longrightarrow \mathscr{R}$ induces a natural injection $\mathscr{A}(\mathscr{R}) \longrightarrow \mathscr{A}(\mathscr{R}')$. By Lemma 2, we know that these injections form an inductive system. We denote the inductive limit of these injections by $\mathscr{D}(\mathscr{R})$ and call the elements of $\mathscr{D}(\mathscr{R})$ *dendromorphic functions* on $\mathscr{R}$.

Any dendromorphic function can concretely be represented as an analytic function $f:\mathscr{R}' \longrightarrow \mathbb{C}$ for some recursive discrete ramification $\pi:\mathscr{R}' \longrightarrow \mathscr{R}$. Now consider two such representations $f_1:\mathscr{R}_1 \longrightarrow \mathbb{C}$ and $f_2:\mathscr{R}_2 \longrightarrow \mathbb{C}$ for recursive discrete ramifications $\pi_1:\mathscr{R}_1 \longrightarrow \mathscr{R}$ and $\pi_2:\mathscr{R}_2 \longrightarrow \mathscr{R}$. Let $\pi:\mathscr{R}' \longrightarrow \mathscr{R}$ be a recursive discrete ramification over both π_1 and π_2, and let $\iota_1:\mathscr{A}(\mathscr{R}_1) \longrightarrow \mathscr{A}(\mathscr{R}')$ and $\iota_2:\mathscr{A}(\mathscr{R}_2) \longrightarrow \mathscr{A}(\mathscr{R}')$ be the natural induced injections as above. Then f_1 and f_2 represent the same dendromorphic function whenever $\iota_1(f_1) = \iota_2(f_2)$.

THEOREM 3. *Let $\mathscr{U}$ be a simply connected open subset of $\mathbb{C}$. Then the space $\mathscr{D}(\mathscr{U})$ is a Picard-Vessiot closed field.*

Proof. The space $\mathscr{D}(\mathscr{U})$ is clearly a differential ring, since it is the inductive limit of differential rings $\mathscr{A}(\mathscr{R})$. Consider a non-zero dendromorphic function, represented by an analytic function $f \in \mathscr{A}(\mathscr{R})$. Then the set $\mathscr{S} \subseteq \mathscr{R}$ of points where f vanishes is discrete, so $1/f$ is defined on $\mathscr{R} \setminus \mathscr{S}$, and $1/f \in \mathscr{A}(\mathscr{R}_{\mathscr{S}})$. Since $\mathscr{A}(\mathscr{R}_{\mathscr{S}})$ embeds into $\mathscr{D}(\mathscr{U})$, this shows that $\mathscr{D}(\mathscr{U})$ is a field. Let us next consider a differential equation

$$L_r(z)\, f^{(r)}(z) + \cdots + L_0(z)\, f(z) = 0, \tag{1}$$

where $L_0, \ldots, L_r \in \mathscr{D}(\mathscr{U})$ with $L_r \neq 0$. By Lemma 2, we may represent $L_0, \ldots, L_r$ by analytic functions in $\mathscr{A}(\mathscr{R})$ for some recursive discrete ramification $\pi:\mathscr{R} \longrightarrow \mathscr{U}$. The set $\mathscr{S} \subseteq \mathscr{R}$ of points where L_r vanishes is discrete and it is classical that any solution of (1) at a point $z_0 \in \mathscr{R} \setminus \mathscr{S}$ can be analytically continued along any path on $\mathscr{R}$ that avoids $\mathscr{S}$. Doing this for a fundamental system of solutions at z_0, we obtain a fundamental system of solutions $h_1, \ldots, h_r \in \mathscr{A}(\mathscr{R}_{\mathscr{S}}) \hookrightarrow \mathscr{D}(\mathscr{U})$. We conclude that $\mathscr{D}(\mathscr{U})$ is Picard-Vessiot closed. □

4. BOUNDARYLESS FUNCTIONS

Let $\mathscr{U}$ be a simply connected open subset of $\mathbb{C}$ and consider a dendromorphic function $f \in \mathscr{D}(\mathscr{U})$. We say that f is *boundaryless* if the following property holds for every representation $f \in \mathscr{A}(\mathscr{R})$, where $\pi:\mathscr{R} \longrightarrow \mathscr{U}$ is a recursive discrete ramification: given a continuous path $\varphi:[0,1] \longrightarrow \mathscr{U}$ with $\varphi(0) = \pi(z_0)$ for some $z_0 \in \mathscr{R}$ and $\varepsilon > 0$, there exists a continuous path $\psi:[0,1] \longrightarrow \mathscr{R}$ with $\psi(0) = z_0$ and

$$\|\pi \circ \psi - \varphi\| := \sup_{t \in [0,1]} |\pi(\psi(t)) - \varphi(t)| < \varepsilon.$$

THEOREM 4. *Any dendromorphic function in $\mathscr{D}(\mathscr{U})$ is boundaryless above $\mathscr{U}$.*

Proof. With f, π, z_0, and ε as in the above definition of boundaryless, there exist discrete ramifications $\pi_1:\mathscr{R}_1 \longrightarrow \mathscr{R}_0, \ldots, \pi_h:\mathscr{R}_h \longrightarrow \mathscr{R}_{h-1}$ with $\mathscr{R}_0 = \mathscr{U}$, $\mathscr{R}_h = \mathscr{R}$, and $\pi = \pi_1 \circ \cdots \circ \pi_h$. We proceed by induction over h. The result is clear for $h = 0$, so assume that $h > 0$ and let $\pi' := \pi_1 \circ \cdots \circ \pi_{h-1}$. The induction hypothesis implies the existence of a continuous path $\xi:[0,1] \longrightarrow \mathscr{R}_{h-1}$ with $\xi(0) = \pi_h(z_0)$ and $\|\pi' \circ \xi - \varphi\| < \varepsilon/2$. Let $\mathscr{S}$ be the discrete subset of $\mathscr{R}_{h-1}$ such that $\mathscr{R}_h = (\mathscr{R}_{h-1})_{\mathscr{S}}$.

Since $\operatorname{im} \xi$ is compact, there exists an $\eta < \varepsilon/2$ such that for every $t \in [0,1]$, the closed ball $\mathscr{B}(\xi(t), \eta) \subseteq \mathscr{R}_{h-1}$ with center $\xi(t)$ and radius η is an isomorphic lift of the ball $\mathscr{B}(\pi'(\xi(t)), \eta)$. Then the thickened image $K := \bigcup_{t \in [0,1]} (\xi(t), \eta)$ of ξ is also compact, so the set $\mathscr{S} \cap K$ is finite, since $\mathscr{S}$ is discrete. Modulo taking a smaller η, we may therefore assume that $\mathscr{B}(s, \eta) \cap \mathscr{S} = \{s\}$ and $\pi_h(z_0) \notin \mathscr{B}(s, \eta)$ for all $s \in \mathscr{S} \cap K$, and that $\mathscr{B}(s, \eta) \cap \operatorname{im} \xi = \emptyset$ whenever $s \notin \operatorname{im} \xi$.

Let $0<t_1<\cdots<t_l\leqslant 1$ be the values of $t\in[0,1]$ for which $\zeta(t)\in\mathscr{S}$. For some sufficiently small $\delta>0$ with $\delta<t_1$, the intervals $[t_i-\delta,t_i+\delta]$ are pairwise disjoint and $\zeta(t)\in\mathscr{B}(\zeta(t_i),{}^{\eta}\!/_2)$ whenever $t\in[t_i-\delta,t_i+\delta]\cap[t_i,1]$. Now consider any path $\tilde{\zeta}\colon[0,1]\longrightarrow\mathscr{R}_{h-1}$ with the following properties:

- If $t\in[0,1]\setminus\bigcup_{1\leqslant i\leqslant l}[t_i-\delta,t_i+\delta]$, then $\tilde{\zeta}(t)=\zeta(t)$.
- If $t_i+\delta<1$, then $\tilde{\zeta}$ restricted to $[t_i-\delta,t_i+\delta]$ is any path from $\zeta(t_i-\delta)$ to $\zeta(t_i+\delta)$ inside the punctured disk $\mathscr{B}(\zeta(t_i),{}^{\eta}\!/_2)\setminus\{\zeta(t_i)\}\subseteq\mathscr{R}_{h-1}\setminus\mathscr{S}$.
- If $t_l+\delta\geqslant 1$, then $\tilde{\zeta}(t)=\zeta(t_l-\delta)$ for all $t\in[t_l-\delta,1]$.

By construction, we have $\operatorname{im}\tilde{\zeta}\subseteq\mathscr{R}_{h-1}\setminus\mathscr{S}$, $\tilde{\zeta}(0)=\pi_h(z_0)$ and $\|\pi'\circ\tilde{\zeta}-\pi'\circ\zeta\|\leqslant\eta\leqslant{}^{\varepsilon}\!/_2$. Consequently, there exists a unique lift $\psi\colon[0,1]\longrightarrow\mathscr{R}_h$ with $\psi(0)=z_0$ and $\pi_h\circ\psi=\tilde{\zeta}$. This lift satisfies $\|\pi\circ\psi-\pi'\circ\zeta\|\leqslant{}^{\varepsilon}\!/_2$ and $\|\pi\circ\psi-\varphi\|\leqslant\|\pi\circ\psi-\pi'\circ\zeta\|+\|\pi'\circ\zeta-\varphi\|<\varepsilon$. □

Recall that the generating function for the number p_n of partitions of an integer n is given by the explicit formula

$$p(z)=\sum_{n\geqslant 0}p_n z^n=\prod_{k\geqslant 1}\frac{1}{1-z^k}. \tag{2}$$

It is well known [10, 5] that $p(z)$ has a natural boundary on the unit circle and that it satisfies an algebraic differential over $\mathbb{C}$. Consequently, if $\mathscr{U}$ contains the closed unit disk, then $p(z)$ cannot be boundaryless above $\mathscr{U}$.

COROLLARY 5. *If $\mathscr{U}$ contains the closed unit disk, then the function $p(z)$ from (2) is not dendromorphic on $\mathscr{U}$.*

COROLLARY 6. *The function $p(z)$ does not belong to $\mathbb{C}^{\mathrm{pv}}=\mathbb{C}(z)^{\mathrm{pv}}$.*

5. COMPOSITION

Given a simply connected Riemann surface, we say that a local analytic function f defined at $z_0\in\mathscr{R}$ is *dendromorphic* on $\mathscr{R}$ if it lifts and extends into an analytic function on $\mathscr{R}'$ for some recursive discrete ramification $\mathscr{R}'$ of $\mathscr{R}$.

THEOREM 7. *Let f and g be two local analytic functions above 0, which are both dendromorphic. If $f(0)=0$ (after the natural identification of 0 with its lift), then $g\circ f$ is dendromorphic.*

Proof. The result is clear if $f=0$, so assume that f is non-constant. There exist sequences of discrete ramifications

$$\begin{array}{ccccccc}\mathscr{R}_0 & \longleftarrow & \mathscr{R}_1 & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}_h\\ \mathscr{R}'_0 & \longleftarrow & \mathscr{R}'_1 & \longleftarrow & \cdots & \longleftarrow & \mathscr{R}'_{h'}\end{array}$$

of $\mathscr{R}_0=\mathscr{R}'_0=\mathbb{C}$ and distinguished points $0_{\mathscr{R}_h}$ and $0_{\mathscr{R}'_{h'}}$ above 0 that we identify with 0, such that the local functions f and g extend analytically to $\mathscr{R}_h$ and $\mathscr{R}'_{h'}$, respectively. Let us show how to extend the first sequence of discrete ramifications into a sequence

$$\mathscr{R}_0\longleftarrow\mathscr{R}_1\longleftarrow\cdots\longleftarrow\mathscr{R}_{h+h'},$$

where $\mathscr{R}_{h+h'}$ again comes with a distinguished lift of 0, and such that f can be lifted into an analytic map $\mathscr{R}_{h+h'}\longrightarrow\mathscr{R}'_{h'}$ that we will also denote by f. Consequently, the composition $g\circ f$ will be naturally defined on $\mathscr{R}_{h+h'}$.

We proceed by induction and note that the inductive property is trivially satisfied if $h' = 0$. So assume that $h' > 0$ and that $\mathscr{R}_{h+h'-1}$ has been constructed. Let $\mathscr{S}' \subseteq \mathscr{R}'_{h'-1}$ be such that $\mathscr{R}'_{h'} = (\mathscr{R}'_{h'-1})_{\mathscr{S}'}$. By the induction hypothesis, we may regard f as an analytic function from $\mathscr{R}_{h+h'-1}$ into $\mathscr{R}'_{h'-1}$. Since f is non-constant, the set $\mathscr{S} := f^{-1}(\mathscr{S}')$ is discrete. We take $\mathscr{R}_{h+h'} := (\mathscr{R}_{h+h'-1})_{\mathscr{S}}$ and also pick any lift $0_{\mathscr{R}_{h+h'}}$ of $0_{\mathscr{R}_{h+h'-1}}$ to be our distinguished point above zero. By our choice of $\mathscr{S}$, the function $f: \mathscr{R}_{h+h'-1} \setminus \mathscr{S} \longrightarrow \mathscr{R}'_{h'-1} \setminus \mathscr{S}'$ has a unique lift $\tilde{f}: \mathscr{R}_{h+h'} \longrightarrow \mathscr{R}'_{h'}$ with $\tilde{f}(0_{\mathscr{R}_{h+h'}}) = 0_{\mathscr{R}'_{h'}}$. This completes our inductive construction of $\mathscr{R}_{h+h'}$ and thereby also the proof. □

We say that a local analytic function f defined at z_0 is *boundaryless* above an open set $\mathscr{U} \subseteq \mathbb{C}$ if for every path $\varphi: [0,1] \longrightarrow \mathscr{U}$ with $\varphi(0) = z_0$ and every $\varepsilon > 0$, there exists a path $\tilde{\varphi}: [0,1] \longrightarrow \mathscr{U}$ with $\tilde{\varphi}(0) = z_0$ and $\|\tilde{\varphi} - \varphi\| < \varepsilon$, such that f can be continued analytically along $\tilde{\varphi}$. We simply say that f is *boundaryless* if it is boundaryless above $\mathbb{C}$.

THEOREM 8. *Let f and g be two local analytic functions at $z_0 = 0$, which are both boundaryless. If $f(0) = 0$ (after the natural identification of 0 with its lift), then $g \circ f$ is again boundaryless.*

Proof. The result is clear if $f = 0$, so assume that $f \neq 0$. Modulo replacing f by $f^{1/v(f)}$, where $v(f)$ is the valuation of f at zero, we may assume without loss of generality that $f'(0) \neq 0$.

Let $\varepsilon > 0$ and consider a path $\varphi: [0,1] \longrightarrow \mathbb{C}$ with $\varphi(0) = z_0$. Since f is boundaryless, there exists a path $\tilde{\varphi}: [0,1] \longrightarrow \mathbb{C}$ with $\tilde{\varphi}(0) = 0$ and $\|\tilde{\varphi} - \varphi\| < \varepsilon/2$, such that $f \circ \tilde{\varphi}$ is defined on $[0,1]$. Since the image of $\tilde{\varphi}$ is compact, it contains only a finite number of zeros of f'. By applying Theorem 4 to $1/f'$, we may also arrange that f' does not vanish on $\operatorname{im} \tilde{\varphi}$. Consequently, the local functional inverse f^{inv} of f at zero can be continued analytically on $\operatorname{im} \tilde{\varphi}$. Let $\eta > 0$ with $\eta < \varepsilon/2$ be such that f is defined on the compact set

$$K := \bigcup_{t \in [0,1]} \mathscr{B}(\tilde{\varphi}(t), \eta) \supseteq \operatorname{im} \tilde{\varphi}$$

and such that f^{inv} is defined on $f(K)$. Let

$$\lambda := \sup_{z \in f(K)} |(f^{\mathrm{inv}})'(z)|.$$

Since g is boundaryless, there exists a path $\xi: [0,1] \longrightarrow \mathbb{C}$ with $\xi(0) = 0$ and $\|\xi - f \circ \tilde{\varphi}\| < \eta/\lambda$, such that $g \circ \xi$ is defined on $[0,1]$. But then $(f^{\mathrm{inv}} \circ \xi)(0) = 0$, $\|f^{\mathrm{inv}} \circ \xi - \tilde{\varphi}\| < \eta$, and $(g \circ f) \circ (f^{\mathrm{inv}} \circ \xi)$ is defined on $[0,1]$. We finally observe that $\|f^{\mathrm{inv}} \circ \xi - \varphi\| < \eta + \varepsilon/2 < \varepsilon$. □

It would be interesting to know whether $\mathscr{D}(\mathbb{C})$ is also closed under functional inversion. The functional inverse of any entire function is easily seen to be in $\mathscr{D}(\mathbb{C})$. Any Weierstrass elliptic function $\mathscr{P}$ is also in $\mathscr{D}(\mathbb{C})$, since it is meromorphic on $\mathbb{C}$. The function $\mathscr{P}$ is not holonomic, but it is the functional inverse of a holonomic function. In fact, $\wp$ is differentially algebraic, but not in $\mathbb{C}^{\mathrm{pv}}$. This is due to the fact that $\mathbb{C}^{\mathrm{pv}}$ embeds into any differentially valued field and Picard-Vessiot closed field $\mathbb{T}$ of complex transseries from [2], but such a field $\mathbb{T}$ never contains $\wp$ (see also [8, Example 9]).

The functional inverse of $p(z)$ from (2) might be a candidate for a dendromorphic function $\mathscr{D}(\mathbb{C})$ whose functional inverse is not. We expect this functional inverse to have a very dendromorphic-like structure, although it might necessitate a non-finite number of discrete ramifications in the required global sense. Variations on the concept of dendromorphic functions might therefore be another topic for further investigations.

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