Hyperserial fields

VINCENT BAGAYOKO

Département de Mathématique
Université de Mons, Le Pentagone
20, Place du Parc
B-7000 Mons, Belgique

CNRS, LIX
Campus de l'École polytechnique
1, rue Honoré d'Estienne d'Orves
Bâtiment Alan Turing, CS35003
91120 Palaiseau, France

Email: `vincent.bagayoko@umons.ac.be`

JORIS VAN DER HOEVEN

CNRS, LIX
Campus de l'École polytechnique
1, rue Honoré d'Estienne d'Orves
Bâtiment Alan Turing, CS35003
91120 Palaiseau, France

Email: `vdhoeven@lix.polytechnique.fr`

ELLIOT KAPLAN

Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL 61801
U.S.A.

Email: `eakapla2@illinois.edu`

Abstract

Transseries provide a universal framework for the formal asymptotics of regular solutions to ordinary differential equations at infinity. More general functional equations such as $E_\omega(x+1) = \exp E_\omega(x)$ may have solutions that grow faster than any iterated exponential and thereby faster than any transseries.

In order to develop a truly universal framework for the asymptotics of regular univariate functions at infinity, we therefore need a generalization of transseries: hyperseries. Hyperexponentials and hyperlogarithms play a central role in such a program. The first non-trivial hyperexponential and hyperlogarithm are E_ω and its functional inverse L_ω, where E_ω satisfies the above equation. Formally, such functions E_α and L_α can be introduced for any ordinal α. For instance, $E_1(x) = \mathrm{e}^x$, $E_2(x) = \mathrm{e}^{\mathrm{e}^x}$, $L_{\omega+1}(x) = \log L_\omega(x)$, and E_{ω^2} satisfies $E_{\omega^2}(x+1) = E_\omega(E_{\omega^2}(x))$.

In the present work, we construct a field of hyperseries that is closed under E_α and L_α for all ordinals α. This generalizes previous work by Schmeling [29] in the case when $\alpha < \omega^\omega$, as well as the previous construction of the field of logarithmic hyperseries by van den Dries, van der Hoeven, and Kaplan [12].

1 Introduction

1.1 The quest of a universal framework for asymptotic calculus

In order to get our hands on a complex mathematical expression, we first need to simplify it as much as possible. This can often be achieved by eliminating parts that are asymptotically negligible. For instance, when studying the expression $f(x) = \log \log x + \log(x^2+1)$ for large values of x, we may compute the approximations $x^2+1 \sim x^2$, $\log(x^2+1) \sim 2\log x$, and then $f(x) \sim 2\log x$. Such approximations rely on our ability to determine and compare asymptotic rates of growth.

Is it possible to develop a universal framework for this kind of asymptotic simplification? This sounds like a difficult problem in general, especially for multivariate functions or functions with an irregular growth like $x\sin(x^{x\sin x})$. On the other hand, the problem might become tractable for univariate functions $f(x)$ in a neighborhood of infinity $x \to \infty$, provided that f is constructed using a limited number of well-behaved primitives.

An important first step towards a systematic asymptotic calculus of this kind was made by Hardy in [20, 21], based on earlier ideas by du Bois-Reymond [14, 15, 16]. We say that f is an *L-function* if it is constructed from x and the real numbers $\mathbb{R}$ using the field operations, exponentiation, and logarithms. Given two non-zero germs of L-functions f, g at infinity, Hardy proved that exactly one of the relations $f \prec g$, $f \asymp g$, or $g \prec f$ holds, where

$$f \prec g \iff \lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = 0,$$
$$f \asymp g \iff \lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} \in \mathbb{R}^{\neq 0}.$$

Hardy also observed [20, p. 22] that "The only scales of infinity that are of any practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions." In other words, Hardy suggested that the framework of L-functions not only allows for the development of a systematic asymptotic calculus, but that this framework is also sufficient for all "practical" purposes.

Hardy went on [20, chapter V] with the examination of possible counterexamples, through the exploration of pathological functions whose asymptotic behavior does not conform to any logarithmico-exponential scale. Here he made a distinction between irregular asymptotic behavior (such as oscillating functions) and regular asymptotic behavior that yet cannot be described in terms of L-functions. Basic examples of the second type were already constructed by du Bois-Reymond and Hardy [20, chapter II]. For instance, let $\exp_k := \exp^{\circ k} = \exp \circ \overset{\times k}{\cdots} \circ \exp$ for $k \in \mathbb{N}$ and let $\mathcal{E}(x) := \exp_{\lfloor x \rfloor} x$ for each $x \in \mathbb{R}^{\geqslant}$. Then $\mathcal{E}$ grows faster than any L-function.

In order to formalize the concept of "regular" growth at infinity, we focus on classes of (germs of) functions that are stable under common calculus operations such as addition, multiplication, differentiation, and composition. The class of L-functions indeed satisfies these conditions, but it is interesting to investigate whether there exist larger classes of functions with similar properties.

Two particular settings that have received a lot of attention are Hardy fields (i.e. fields of germs of real continuously differentiable functions at infinity that are closed under differentiation [7]) and germs of definable functions in o-minimal structures [10]. Each of these settings excludes oscillatory behavior in a strong sense. For instance, although the function $x^2 + \sin x$ does not oscillate for large values of x, its second derivative does, so the germ of this function at infinity does not belong to a Hardy field.

With a more precise definition of regularity at hand, one may re-examine the existence of regular functions whose asymptotic growth falls outside any scale of L-functions. For instance, the function $\mathcal{E}$ from above is not even continuous and thereby not sufficiently regular. Nevertheless, it was shown by Kneser [28] that the functional equation

$$E_\omega(x+1) \;=\; \exp E_\omega(x) \tag{1.1}$$

has a real analytic solution on $\mathbb{R}^{\geqslant}$. This provides us with a more natural candidate for a regular function that grows faster than any L-function. Indeed, it was shown by Boshernitzan [6] that Kneser's solution belongs to a Hardy field. The functional inverse L_ω of E_ω frequently occurs in the complexity analysis of recursive algorithms that use exponential size reductions. For instance, the fastest known algorithm [22] for multiplying two polynomials of degree $<n$ in $\mathbb{F}_2[t]$ runs in time $O(n \log n 4^{L_\omega(n)})$. This shows that Hardy's framework of L-functions is insufficient, even for practical purposes.

Another example of a regular function that is not asymptotic to any L-function is the functional inverse of $\log x \log \log x$. This fact was actually raised as a question by Hardy and only proved in [23] and [13]. More explicit examples of such functions, like $\exp \int \mathrm{e}^{x^2}$, were given in [23]. It turns out that the class of L-functions lacks several important closure properties (e.g. functional inversion and integration), which makes it unsuitable as a universal framework for asymptotic calculus.

The class of transseries forms a better candidate for such a universal framework. A transseries is a formal object that is constructed from x (with $x \to \infty$) and the real numbers, using exponentiation, logarithms, and *infinite* sums. One example of a transseries is

$$\mathrm{e}^{\mathrm{e}^x + \mathrm{e}^{x/2} + \mathrm{e}^{x/3} + \cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^\pi + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + 24\,x^{-4} + \cdots + \mathrm{e}^{-x}.$$

Depending on conditions satisfied by their supports, there are different types of transseries. The first constructions of fields of transseries are due to Dahn and Göring [9] and Écalle [17]. More general constructions were proposed subsequently by van der Hoeven and his student Schmeling [23, 29].

Transseries form a natural "infinitary" extension of the concept of an L-function. The transseries are closed under integration and functional inversion [23, 13]. They also satisfy a differential intermediate value property [26, Chapter 9]. However, transseries are only defined formally, so their analytic meaning is not necessarily obvious. One technique for associating an analytic meaning to certain divergent transseries is accelero-summation [17], a generalization of Borel summation [5]. An alternative technique is based on differential algebra and model theory [27, 1]. In this paper, we focus on formal asymptotic computations, without worrying about analytic counterparts.

Despite the excellent closure properties of transseries for the resolution of differential equations, the functional equation (1.1) still does not have a transseries solution. In order to establish a universal formal framework for asymptotic calculus, we therefore need to incorporate extremely fast growing functions such as E_ω, as well as formal solutions E_{ω^2}, E_{ω^3}, etc. to the following equations:

$$\begin{aligned} E_{\omega^2}(x+1) &= E_\omega(E_{\omega^2}(x)) && (1.2)\\ E_{\omega^3}(x+1) &= E_{\omega^2}(E_{\omega^3}(x)) && (1.3)\\ &\vdots \end{aligned}$$

The fast growing functions E_ω, E_{ω^2}, ... are called *hyperexponentials*. Their functional inverses L_ω, L_{ω^2}, ... are called *hyperlogarithms* and they grow extremely slowly. The first construction of a field of generalized transseries that is closed under E_{ω^n} and L_{ω^n} for all $n \in \mathbb{N}$ was accomplished in [29]. Here we understand that $E_1 = \exp$ and $L_1 = \log$.

The hyperlogarithms L_ω, L_{ω^2}, etc. obviously satisfy the functional equations

$$\begin{aligned} L_\omega(L_1(x)) &= L_\omega(x)-1 \\ L_{\omega^2}(L_\omega(x)) &= L_{\omega^2}(x)-1 \\ L_{\omega^3}(L_{\omega^2}(x)) &= L_{\omega^3}(x)-1 \\ &\vdots \end{aligned}$$

In addition, we have a simple formula for their derivatives

$$L_\alpha(x)' = \prod_{\beta<\alpha} \frac{1}{L_\alpha(x)}, \tag{1.4}$$

where $\alpha \in \{1, \omega, \omega^2, \ldots\}$ and

$$L_{\omega^n k_n + \cdots + \omega k_1 + k_0}(x) := L_1^{\circ k_0}(L_\omega^{\circ k_1}(\cdots(L_{\omega^n}^{\circ k_n}(x))\cdots))$$

for all $n \in \mathbb{N}$ and $k_0, \ldots, k_n \in \mathbb{N}$. The formula (1.4) is eligible for generalization to arbitrary ordinals α. Taking $\alpha = \omega^\omega$, we note that the function L_{ω^ω} does not satisfy any functional equation. Yet any truly universal formal framework for asymptotic calculus should accommodate functions such as L_{ω^ω} for the simple reason that it is possible to construct models with good properties in which they exist. For instance, by [6], there exist Hardy fields with infinitely large functions that grow more slowly than L_{ω^n} for all $n \in \mathbb{N}$.

The construction of the field $\mathbb{L}$ of *logarithmic hyperseries* in [12] was the first step towards the incorporation of hyperlogarithms L_α with arbitrary α. The field $\mathbb{L}$ is the smallest non-trivial field of generalized power series over $\mathbb{R}$ that is closed under all hyperlogarithms L_α and infinite real power products. It turns out that $\mathbb{L}$ is a proper class and that $\mathbb{L}$ is closed under differentiation, integration, and composition.

The main purpose of the present paper is the construction of a field of general hyperseries that is also closed under the functional inverse E_α of L_α for every ordinal α. Our construction strongly relies on properties of the field $\mathbb{L}$ of logarithmic hyperseries. Intuitively speaking, the reason is that the derivative of E_α can be expressed as the composition of a logarithmic hyperseries with E_α:

$$E_\alpha(x)' = \frac{1}{L'_\alpha(E_\alpha(x))}$$

and similarly for all higher derivatives. One key aspect of our approach is therefore to construct increasingly large fields $\mathbb{T}$ of hyperseries simultaneously with compositions

$$\circ : \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ},$$

where $\mathbb{T}^{>,\succ}$ denotes the class of positive infinitely large elements of $\mathbb{T}$.

The main result of this paper is the construction of a field $\mathbb{H} \supseteq \mathbb{L}$ of hyperseries that is closed under all hyperlogarithms L_α and all hyperexponentials E_α. Does this end our quest for a universal formal framework for asymptotic calculus? Not quite yet. First of all, it remains to be shown that $\mathbb{H}$ is closed under all common calculus operations, such as differentiation and composition. Secondly, the field $\mathbb{H}$ does not contain any solution to the functional equation

$$f(x) = \sqrt{x} + \mathrm{e}^{f(\log x)}, \qquad f(x) \sim \sqrt{x}.$$

Fortunately, it is possible to construct fields of transseries with "nested" solutions

$$f(x) = \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log\log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$$

to such equations [29, Section 2.5]. Something similar is possible for hyperseries; although this is beyond the scope of the present paper, we introduce the fundamental concepts that we expect to use for this generalization.

One may also wonder whether there exist natural models for hyperexponential functions and hyperseries. We already noted that Kneser constructed a real analytic solution E_ω of the equation (1.1). Schmeling also constructed real analytic solutions $E_{\omega^2}, E_{\omega^3}, \ldots$ of (1.2), (1.3), etc. Écalle introduced a systematic technique for the construction of quasi-analytic solutions to these and more general iteration equations [17]. In general, it seems unlikely that there exist any "privileged" regular solutions at infinity.

Another interesting model for hyperexponentiation is Conway's field **No** of *surreal numbers* [8]. The field **No** is a non-standard extension of the field $\mathbb{R}$ which contains the class **On** of ordinal numbers. The arithmetic operations are defined in a surprisingly "simple" way, using transfinite induction. Nevertheless, the field **No** has a remarkably rich structure; e.g. it is real-closed. The exponential function on the reals has been extended to **No** by Gonshor [18] and this extension preserves the first order properties of exp [11].

A "simplest" surreal solution to (1.1) with good properties has been constructed in [3]. This solution is only defined on $\mathbf{No}^{>,\succ} = \{x \in \mathbf{No} : x > \mathbb{R}\}$, but in view of our previous remarks on real solutions of (1.1), is interesting to note that we may indeed consider it as "the" privileged solution on $\mathbf{No}^{>,\succ}$. Constructing each hyperlogarithm and hyperexponential $L_\alpha, E_\alpha, \alpha \in \mathbf{On}$ on surreal numbers involves overcoming many technical difficulties. Our results from this paper reduce this to the simpler task of defining partial hyperlogarithms on **No**, which satisfy a short list of axioms.

We finally note that Berarducci and Mantova also defined a derivation ∂_{BM} with respect to ω on **No** [4]. This derivation again has good model theoretic properties [2]. However, although the derivation ∂_{BM} satisfies $\partial_{\mathrm{BM}} \exp x = (\partial_{\mathrm{BM}} x) \exp x$ for all $x \in \mathbf{No}$, it was pointed out in [1] that it does *not* satisfy $\partial_{\mathrm{BM}} E_\omega(x) = (\partial_{\mathrm{BM}} x)\, E_\omega'(x)$ for all $x \in \mathbf{No}^{>,\succ}$, for "reasonable" definitions of E_ω. Indeed, for the definition of E_ω from [3], we have $\partial_{\mathrm{BM}}(E_\omega(E_\omega(\omega))) = E_\omega'(E_\omega(\omega)) \neq (\partial_{\mathrm{BM}} E_\omega(\omega))\, E_\omega'(E_\omega(\omega))$.

Stated differently, the hyperexponential structure on **No** reveals that ∂_{BM} is not *the* ultimate derivation on **No** with respect to ω that we might hope for. One major motivation behind the work in this paper is precisely the construction of a better derivation on **No**, as well as a composition. The plan, which has been detailed in [1], is to construct an isomorphism between **No** and a suitable field of hyperseries with a natural derivation and composition with respect to x. The present paper can be regarded as one important step in this direction.

1.2 Strategy and outline of the main results

In order to construct a field of hyperseries that is closed under all hyperexponentials E_α and all hyperlogarithms L_α, we follow the common approach of starting with an arbitrary field of hyperseries and then closing it off via a transfinite sequence of extensions.

Now closing off under hyperlogarithms turns out to be much easier than closing off under hyperexponentiation. For this reason, and following [23, 29, 12], it is actually convenient to do this once and for all and only work with fields of hyperseries that are already closed under all hyperlogarithms. In particular, the smallest field of hyperseries of this type with an element $x > \mathbb{R}$ is the field of logarithmic hyperseries $\mathbb{L}$ from [12], where $\ell_\alpha := L_\alpha(x)$ for all α.

The next step is to work out the technical definition of a "field of hyperseries" that will be suitable for the hyperexponential extension process. Quite naturally, such a field should be a Hahn field $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of generalized power series, where $(\mathfrak{M}, \preccurlyeq)$ is a totally ordered monomial group: see Subsection 2.2 for basic definitions and reminders. For reasons mentioned in the previous subsection, we also require the existence of a composition law $\circ: \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$, where $\mathbb{T}^{>,\succ} := \{s \in \mathbb{T} : s > \mathbb{R}\}$. For each $\alpha \in \mathbf{On}$, this allows us to define a function $L_\alpha: \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ by setting $L_\alpha(s) := \ell_\alpha \circ s$ for $s \in \mathbb{T}^{>,\succ}$.

We say that $(\mathbb{T}, \circ)$ is a *hyperserial field* if the composition $\circ$ satisfies a list of natural axioms such as associativity and restricted Taylor expansions; see Section 6 for the full list of axioms. The only non-obvious axiom for traditional fields of transseries states that a transseries $\mathfrak{m} \in \mathbb{T}^{>,\succ}$ is a monomial if and only if $\operatorname{supp} \log \mathfrak{m} \succ 1$ (i.e. the support of $\log \mathfrak{m}$ only contains infinitely large elements). The only non-obvious axiom **HF7** for hyperserial fields is a generalization of this axiom: if $\mu \geqslant 1$, then we require that the support of $L_{\omega^\mu}(\mathfrak{a})$ satisfies $\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ (\ell_{<\omega^\mu} \circ \mathfrak{a})^{-1}$ for any $L_{<\omega^\mu}$-atomic element $\mathfrak{a}$. Here $\mathfrak{a} \in \mathbb{T}^{>,\succ}$ is said to be *$L_{<\omega^\mu}$-atomic* if $L_\beta(\mathfrak{a})$ is a monomial for all $\beta < \omega^\mu$.

Our definition of hyperserial fields is similar to the definition of fields of transseries from [24, 29], with a few differences. The old definition includes an additional axiom of well-nestedness **T4** which is important for the definition of derivations and compositions, but which is not required for the purposes of the present paper. Of course, our current presentation is based on the composition law $\circ$. Finally, we use a slightly different technical notion of confluence. We say that $(\mathbb{T}, \circ)$ is *confluent* if, for all ordinals μ and all $s \in \mathbb{T}^{>,\succ}$, there exist a $L_{<\omega^\mu}$-atomic element $\mathfrak{a}$ and $\gamma < \omega^\mu$ with $\ell_\gamma \circ s \asymp \ell_\gamma \circ \mathfrak{a}$. We refer to Remark 3.6 for a discussion of the differences with the definition from [29].

Given an ordinal $\alpha = \omega^\mu$, it turns out that the hyperlogarithm L_α is entirely determined by its restriction to the set of $L_{<\alpha}$-atomic elements. The field $\mathbb{T}$ together with these restricted hyperlogarithms is called the *hyperserial skeleton* of $\mathbb{T}$. The fact that the logarithm can be recovered from its restriction to $\mathfrak{M}^{\succ}$ is a well-known fact. Indeed, we first recover log on $\mathfrak{M}$, since $\log 1 = 0$ and $\log \mathfrak{m}^{-1} = -\log \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}^{\succ}$. For all $c \in \mathbb{R}^{>}$, $\mathfrak{m} \in \mathfrak{M}$, and infinitesimal $\delta \in \mathbb{T}^{\prec}$, we then have

$$\log(c\,\mathfrak{m}\,(1+\delta)) = \log \mathfrak{m} + \log c + \sum_{k>0} \frac{(-1)^{k+1}}{k} \delta^k.$$

Hyperserial skeletons can also be defined in an abstract manner, i.e. without knowledge of a hyperserial field of which it is the skeleton. Precise definitions will be given in Section 3; for now it suffices to know that an abstract hyperserial skeleton is a field of generalized power series $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ together with partially defined functions $L_{\omega^\mu}: \mathfrak{M}^{\succ} \longrightarrow \mathbb{T}$ that satisfy suitable axioms. The first key result of this paper is the construction of an exact correspondence between abstract hyperserial skeletons and hyperserial fields. This correspondence preserves confluence for a suitable analogue of the confluence axiom for hyperserial skeletons.

Sections 4 and 5 contain the core of this construction. In Section 4, we first show how to extend the partial functions L_{ω^μ} of a confluent hyperserial skeleton $\mathbb{T} = (\mathbb{T}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ to all of $\mathbb{T}^{>,\succ}$. In Section 5, we prove that any confluent hyperserial skeleton $\mathbb{T}$ can be endowed with a well-behaved composition law $\circ: \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$. In Section 6, we complete our construction of a correspondence between hyperserial skeletons and hyperserial fields. More precisely, we prove:

Theorem 1.1. *If* $(\mathbb{T},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ *is a confluent hyperserial skeleton, then there is a unique function* $\circ$ *such that* $(\mathbb{T},\circ)$ *is a confluent hyperserial field with*

$$\begin{aligned}\ell_0^r\circ\mathfrak{m} &= \mathfrak{m}^r \text{ for each } \mathfrak{m}\in\mathfrak{M}^{\succ} \text{ and } r\in\mathbb{R}, \text{ and}\\ \ell_{\omega^\mu}\circ\mathfrak{a} &= L_{\omega^\mu}(\mathfrak{a}) \text{ for each } \mu\in\mathbf{On} \text{ and } \mathfrak{a}\in\operatorname{dom} L_{\omega^\mu}.\end{aligned}$$

Theorem 1.2. *Let* $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ *be a hyperserial field of force* **On**. *Then the skeleton* $(\mathbb{T},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ *of* $\mathbb{T}$ *is a hyperserial skeleton. Moreover, if* $(\mathbb{T},\circ)$ *is confluent, then so is its skeleton and* $\circ$ *coincides with the unique composition from Theorem 1.1.*

Sections 7 and 8 are devoted to the closure of a confluent hyperserial field under hyperexponentiation. In view of Theorem 1.1, it suffices to operate on the level of hyperserial skeletons instead of hyperserial fields. In Section 7, we investigate when the hyperexponential of an element in $\mathbb{T}^{>,\succ}$ already exists in $\mathbb{T}^{>,\succ}$. This gives us a criterion under which the extended hyperlogarithms $L_{\omega^\mu}\colon \mathbb{T}^{>,\succ}\longrightarrow\mathbb{T}^{>,\succ}$ are bijective. In Section 8, we prove our main theorem that every confluent hyperserial skeleton has a minimal extension whose extended hyperlogarithm functions are bijective:

Theorem 1.3. *Let* $(\mathbb{T},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ *be a confluent hyperserial skeleton. Then* $\mathbb{T}$ *has a confluent extension* $\mathbb{T}_{(<\mathbf{On})}$ *such that the function*

$$L_{\omega^\mu}\colon \mathbb{T}^{>,\succ}_{(<\mathbf{On})}\longrightarrow\mathbb{T}^{>,\succ}_{(<\mathbf{On})}$$

is bijective for each ordinal μ. *Moreover, if* $\mathbb{U}\supseteq\mathbb{T}$ *is another confluent extension and if the extended function* $L_{\omega^\mu}\colon\mathbb{U}^{>,\succ}\longrightarrow\mathbb{U}^{>,\succ}$ *is bijective for each* μ, *then there is a unique embedding of* $\mathbb{T}_{(<\mathbf{On})}$ *into* $\mathbb{U}$ *over* $\mathbb{T}$.

Corollary. *There exists a minimal hyperserial extension of* $\mathbb{L}$ *that is closed under* E_{ω^μ} *for all ordinals* μ.

2 Preliminaries

2.1 Set-theoretic notations and conventions

We work in von Neumann-Gödel-Bernays set theory with Global Choice (NBG), which is a conservative extension of ZFC. In this set theory, all proper classes are in bijective correspondence with the class **On** of all ordinal numbers. We will sometimes write $\boldsymbol{\mu},\boldsymbol{\nu}$,... for elements that are either ordinals or equal to the class **On** of ordinals. In that case, we write $\boldsymbol{\mu},\boldsymbol{\nu}\leqslant\mathbf{On}$ instead of $\mu,\nu\in\mathbf{On}$. We make the convention that $\omega^{\mathbf{On}}=\mathbf{On}$. If μ is a successor ordinal, then we define μ_* to be the unique ordinal with $\mu=\mu_*+1$. If μ is a limit ordinal, then we define $\mu_*:=\mu$.

Recall that every ordinal γ has a unique Cantor normal form

$$\gamma=\omega^{\eta_1}n_1+\cdots+\omega^{\eta_r}n_r,$$

where $r\in\mathbb{N}$, $n_1,\ldots,n_r\in\mathbb{N}^{>0}$ and $\eta_1,\ldots,\eta_r\in\mathbf{On}$ with $\eta_1>\cdots>\eta_r$. The ordinals η_i are called the *exponents* of the Cantor normal form and the integers n_i its *coefficients*. We write $\rho\ll\sigma$ (resp. $\rho\lesssim\sigma$) if each exponent η_i of the Cantor normal form of σ satisfies $\rho<\omega^{\eta_i}$ (resp. $\rho<\omega^{\eta_i+1}$). We also define $\gamma_{\gtrsim\omega^\eta}$ to be the unique ordinal with $\omega^\eta\lesssim\gamma_{\gtrsim\omega^\eta}$ and with $\gamma=\gamma_{\gtrsim\omega^\eta}+\iota$ for some $\iota<\omega^\eta$. Note that $\gamma_{\gtrsim\omega^\eta}=0$ if and only if $\gamma<\omega^\eta$.

2.2 Fields of well-based series

Well-based series. Let $(\mathfrak{M}, \times, \prec)$ be a linearly ordered abelian group (which may be a proper class). We let $\mathbb{T} := \mathbb{R}[[\mathfrak{M}]]$ denote the class of functions $f: \mathfrak{M} \longrightarrow \mathbb{R}$ whose support

$$\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\}$$

is a *well-based* set, i.e. a set which is well-ordered for the reverse order $(\mathfrak{M}, \succ)$.

We regard elements f of $\mathbb{T}$ as *well-based series* $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where $f_{\mathfrak{m}} := f(\mathfrak{m}) \in \mathbb{R}$ for each $\mathfrak{m} \in \mathfrak{M}$. By [19], the class $\mathbb{T}$ is a field for the operations

$$\begin{aligned} f + g &:= \sum_{\mathfrak{m}} (f_{\mathfrak{m}} + g_{\mathfrak{m}})\, \mathfrak{m} \\ fg &:= \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{u}\mathfrak{v} = \mathfrak{m}} f_{\mathfrak{u}} g_{\mathfrak{v}} \right) \mathfrak{m}. \end{aligned}$$

Note that each sum $\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} f_{\mathfrak{u}} g_{\mathfrak{v}}$ has finite support. We say that $\mathbb{T}$ is a *field of well-based series* and that $\mathfrak{M}$ is the *monomial group* of $\mathbb{T}$. An element $\mathfrak{m} \in \mathfrak{M}$ is called a *monomial*.

An increasing union of fields of well-based series is not, in general, a field of well-based series. However, this is always true if the union is indexed over **On**:

Lemma 2.1. *Let $(\mathfrak{M}_\mu)_{\mu \in \mathbf{On}}$ be a family of linearly ordered abelian groups such that $\mathfrak{M}_\mu \subseteq \mathfrak{M}_\nu$ whenever $\mu < \nu$. Set $\mathbb{T}_\mu := \mathbb{R}[[\mathfrak{M}_\mu]]$ for each μ, so $\mathbb{T}_\mu \subseteq \mathbb{T}_\nu$ for $\mu < \nu$. Set $\mathfrak{M} := \bigcup_{\mu \in \mathbf{On}} \mathfrak{M}_\mu$. Then*

$$\bigcup_{\mu \in \mathbf{On}} \mathbb{T}_\mu = \mathbb{R}[[\mathfrak{M}]].$$

Proof. Set $\mathbb{T} := \bigcup_{\mu \in \mathbf{On}} \mathbb{T}_\mu$. Clearly, $\mathbb{T} \subseteq \mathbb{R}[[\mathfrak{M}]]$, so it remains to show the other inclusion. Let $f \in \mathbb{R}[[\mathfrak{M}]]$. For each $\mathfrak{m} \in \operatorname{supp} f$, let $\mu_{\mathfrak{m}}$ be the least $\mu \in \mathbf{On}$ with $\mathfrak{m} \in \mathfrak{M}_\mu$. Set

$$\mu_f := \sup \{\mu_{\mathfrak{m}} : \mathfrak{m} \in \operatorname{supp} f\}.$$

Then $f \in \mathbb{T}_{\mu_f} \subseteq \mathbb{T}$. □

If $\operatorname{supp} f \neq \emptyset$, then we define

$$\mathfrak{d}_f := \max \operatorname{supp} f \in \mathfrak{M}$$

to be the *dominant monomial* of f. We give $\mathbb{T}$ the structure of a totally ordered field by setting

$$f > 0 \iff f \neq 0 \text{ and } f_{\mathfrak{d}_f} > 0.$$

We define the asymptotic relations $\prec$, $\preccurlyeq$, $\asymp$, and $\sim$ on $\mathbb{T}$ by

$$\begin{aligned} f \prec g &\iff (\forall r \in \mathbb{R}^>, \ r|f| < |g|) \\ f \preccurlyeq g &\iff (\exists r \in \mathbb{R}^>, \ |f| \leqslant r|g|) \\ f \asymp g &\iff f \preccurlyeq g \preccurlyeq f \\ f \sim g &\iff f - g \prec f \iff g - f \prec g. \end{aligned}$$

The monomial group $\mathfrak{M}$ is naturally embedded in $\mathbb{T}^>$ as an ordered group and

$$\begin{aligned} f \prec g &\iff \mathfrak{d}_f \prec \mathfrak{d}_g & \qquad f \asymp g &\iff \mathfrak{d}_f = \mathfrak{d}_g \\ f \preccurlyeq g &\iff \mathfrak{d}_f \preccurlyeq \mathfrak{d}_g & \qquad f \sim g &\iff f_{\mathfrak{d}_f} \mathfrak{d}_f = g_{\mathfrak{d}_g} \mathfrak{d}_g \end{aligned}$$

for all non-zero $f, g \in \mathbb{T}$.

For $f\in\mathbb{T}$ and $\mathfrak{m}\in\mathfrak{M}$, we set $f_{\succ\mathfrak{m}}:=\sum_{\mathfrak{n}\succ\mathfrak{m}}f_{\mathfrak{n}}\,\mathfrak{n}$. We say that a series $g\in\mathbb{T}$ is a *truncation* of f, denoted $g\trianglelefteq f$, if there is $\mathfrak{m}\in\mathfrak{M}$ with $g=f_{\succ\mathfrak{m}}$. We have $g\trianglelefteq f$ if and only if $f-g\prec\operatorname{supp} g$ (which holds vacuously when $g=0$). We finally define

$$\begin{aligned}\mathbb{T}^{\prec} &:= \{f\in\mathbb{T}:\operatorname{supp} f\subseteq\mathfrak{M}^{\prec}\}=\{f\in\mathbb{T}:f\prec 1\}\\ \mathbb{T}^{>,\succ} &:= \{f\in\mathbb{T}:f>\mathbb{R}\}=\{f\in\mathbb{T}:f>0\text{ and }f\succ 1\}.\end{aligned}$$

Series in $\mathbb{T}^{\prec}$ are called *infinitesimal* and series in $\mathbb{T}^{>,\succ}$ are called *positive infinite*. Each $f\in\mathbb{T}^{\neq}$ can be decomposed uniquely as $f=c\,\mathfrak{m}\,(1+\varepsilon)$, where $c\in\mathbb{R}^{\neq}$, $\mathfrak{m}:=\mathfrak{d}_f\in\mathfrak{M}$, and ε is infinitesimal.

Well-based families. If $(f_i)_{i\in I}$ is a family in $\mathbb{T}$, then we say that $(f_i)_{i\in I}$ is *well-based* if

- $\bigcup_{i\in I}\operatorname{supp} f_i$ is a well-based set, and
- $\{i\in I:\mathfrak{m}\in\operatorname{supp} f_i\}$ is finite for all $\mathfrak{m}\in\mathfrak{M}$.

Then we may define the sum $\sum_{i\in I}f_i$ of $(f_i)_{i\in I}$ as the series

$$\sum_{i\in I}f_i:=\sum_{\mathfrak{m}}\left(\sum_{i\in I}(f_i)_{\mathfrak{m}}\right)\mathfrak{m}.$$

If $(f_i)_{i\in I}$ and $(g_j)_{j\in J}$ are families, then we define their product as the family $(f_i g_j)_{(i,j)\in I\times J}$. By [25, Proposition 3.3], if $(f_i)_{i\in I}$ and $(g_j)_{j\in J}$ are well-based, then so is their product, and we have

$$\sum_{(i,j)\in I\times J}f_i g_j=\left(\sum_{i\in I}f_i\right)\left(\sum_{j\in J}g_j\right).$$

We will frequently use the following facts regarding families $(f_n\,\delta^n)_{n\in\mathbb{N}}$ for $(f_n)_{n\in\mathbb{N}}\in\mathbb{T}^{\mathbb{N}}$ and $\delta\in\mathbb{T}$.

Lemma 2.2. *Consider a field of well-based series $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$.*

a) [29, Corollary 1.5.6] *For $\varepsilon\in\mathbb{T}^{\prec}$, the family $(\varepsilon^n)_{n\in\mathbb{N}}$ is well-based.*

b) [29, Corollary 1.5.8] *For $(f_n)_{n\in\mathbb{N}}\in\mathbb{T}^{\mathbb{N}}$ and $\delta\in\mathbb{T}$ such that $(f_n\,\delta^n)_{n\in\mathbb{N}}$ is well-based, the family $(f_n\,\varepsilon^n)_{n\in\mathbb{N}}$ is well-based whenever $\varepsilon\preccurlyeq\delta$.*

Let $\mathbb{U}=\mathbb{R}[[\mathfrak{N}]]$ be another field of well-based series. If $\Psi:\mathbb{T}\longrightarrow\mathbb{U}$ is $\mathbb{R}$-linear, then we say that Ψ is *strongly linear* if for every well-based family $(f_i)_{i\in I}$ in $\mathbb{T}$, the family $(\Psi(f_i))_{i\in I}$ in $\mathbb{U}$ is well-based, and

$$\Psi\left(\sum_{i\in I}f_i\right)=\sum_{i\in I}\Psi(f_i).$$

If $\Phi:\mathfrak{M}\longrightarrow\mathbb{U}$ is a function, then we say that it is *well-based* if for any well-based family $(\mathfrak{m}_i)_{i\in I}$ in $\mathfrak{M}$, the family $(\Phi(\mathfrak{m}_i))_{i\in I}$ in $\mathbb{U}$ is well-based. Then Φ extends uniquely into a strongly linear map $\hat{\Phi}:\mathbb{T}\longrightarrow\mathbb{U}$ [25, Proposition 3.5]. Moreover, $\hat{\Phi}$ is strictly increasing whenever Φ is strictly increasing and it is a ring morphism whenever $\Phi(\mathfrak{m}\,\mathfrak{n})=\Phi(\mathfrak{m})\,\Phi(\mathfrak{n})$ for all $\mathfrak{m},\mathfrak{n}\in\mathfrak{M}$ [25, Corollary 3.8]. In particular, if $\Phi(\mathfrak{m})\in\mathfrak{N}$ for all $\mathfrak{m}\in\mathfrak{M}$ and Φ is strictly increasing, then Φ is well-based. Hence:

Proposition 2.3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be totally ordered by $\prec$ and consider an order-preserving map $\Psi: \mathfrak{M} \longrightarrow \mathfrak{N}$. Then there is a unique strongly linear function $\overline{\Psi}: \mathbb{T} \longrightarrow \mathbb{U}$ that extends Ψ. Moreover, if Ψ is a group morphism, then $\overline{\Psi}$ is an ordered field embedding.* □

If $\mathfrak{N}$ extends $\mathfrak{M}$ (so $\mathbb{U}$ extends $\mathbb{T}$), then the *operator support* of a function $\Phi: \mathfrak{M} \longrightarrow \mathbb{U}$ is the set $\operatorname{supp}_* \Phi := \bigcup_{\mathfrak{m} \in \mathfrak{M}} \operatorname{supp}(\Phi(\mathfrak{m}) / \mathfrak{m})$. If $\operatorname{supp}_* \Phi$ is a well-based set, then Φ is well-based; see [12, Lemma 2.9].

Definition 2.4. *We define a function $\Phi: \mathfrak{M} \longrightarrow \mathbb{U}$ to be* **relatively well-based** *if*

$$\operatorname{supp}_{\odot} \Phi := \bigcup_{\mathfrak{m} \in \mathfrak{M}, \Phi(\mathfrak{m}) \neq 0} \frac{\operatorname{supp} \Phi(\mathfrak{m})}{\mathfrak{d}_{\Phi(\mathfrak{m})}}$$

is well-based.

Proposition 2.5. *Let $\Phi: \mathfrak{M} \longrightarrow \mathbb{U}$ be relatively well-based. Assume that $0 \notin \Phi(\mathfrak{M})$ and that $\mathfrak{d} \circ \Phi: \mathfrak{M} \longrightarrow \mathfrak{N}$ is strictly increasing. Then Φ is well-based and its strongly linear extension $\hat{\Phi}$ is injective.*

Proof. Given a well-based subset $\mathfrak{S} \subseteq \mathfrak{M}$, we have to show that $(\Phi(\mathfrak{m}))_{\mathfrak{m} \in \mathfrak{S}}$ is a well-based family. We have

$$\bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m}) \subseteq \{\mathfrak{d}_{\Phi(\mathfrak{m})} : \mathfrak{m} \in \mathfrak{S}\} \cdot \operatorname{supp}_{\odot} \Phi,$$

so $\bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m})$ is a well-based subset of $\mathfrak{N}$. For any $\mathfrak{n} \in \bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m})$, the set of pairs $(\mathfrak{m}, \mathfrak{u}) \in \mathfrak{S} \times \operatorname{supp}_{\odot} \Phi$ with $\mathfrak{d}_{\Phi(\mathfrak{m})} \mathfrak{u} = \mathfrak{n}$ forms a finite antichain. Since any $\mathfrak{m} \in \mathfrak{S}$ with $\mathfrak{n} \in \operatorname{supp} \Phi(\mathfrak{m})$ induces such a pair $(\mathfrak{m}, \mathfrak{n} / \mathfrak{d}_{\Phi(\mathfrak{m})})$, it follows that the set of all such $\mathfrak{m}$ is also finite. This completes the proof that Φ is well-based. To see that $\hat{\Phi}$ is injective, let $s \in \mathbb{T}^{\neq}$ and take $c \in \mathbb{R}^{\neq}$ with $s \sim c\, \mathfrak{d}_s$. The assumption that $\mathfrak{d} \circ \Phi$ is strictly increasing gives $\hat{\Phi}(s - c\, \mathfrak{d}_s) \prec \hat{\Phi}(c\, \mathfrak{d}_s) = c\, \Phi(\mathfrak{d}_s) \neq 0$, whence $\hat{\Phi}(s) \neq 0$. □

Real powers. We say that $\mathfrak{M}$ *has real powers*, if it comes with a real power operation $\mathbb{R} \times \mathfrak{M} \to \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$ such that $\mathfrak{M}$ is a multiplicative ordered $\mathbb{R}$-vector space, i.e. an ordered $\mathbb{R}$-vector space with multiplication and real powering in the roles of addition and scalar multiplication. Any real power operation on $\mathfrak{M}$ extends to $\mathbb{T}^>$ as follows: for $\varepsilon \in \mathbb{T}^{\prec}$, we set

$$(1 + \varepsilon)^r := \sum_{k \in \mathbb{N}} \binom{r}{k} \varepsilon^k \tag{2.1}$$

and for $s = c\, \mathfrak{m}\, (1 + \varepsilon) \in \mathbb{T}^>$ where $c \in \mathbb{R}^>$, $\mathfrak{m} \in \mathfrak{M}$, and $\varepsilon \in \mathbb{T}^{\prec}$, we set $s^r := c^r\, \mathfrak{m}^r\, (1 + \varepsilon)^r$.

Proposition 2.6. *For $r, r' \in \mathbb{R}$ and $s, t \in \mathbb{T}^>$ we have*

$$\begin{aligned} (s^r)^{r'} &= s^{r r'} \\ (s\, t)^r &= s^r\, t^r \\ s < t, r > 0 &\Rightarrow s^r < t^r. \end{aligned}$$

Proof. For $s, t \sim 1$, the first two relations follow from basic power series manipulations; see [25, Corollary 16]. The extension to the general case when $s, t \in \mathbb{T}^>$ is straightforward and left to the reader.

Assume now that $s<t$ and $r>0$. Since $(s/t)^r=s^r/t^r$, it suffices to show that $(s/t)^r<1$. Write $s/t=c\,\mathfrak{m}\,(1+\varepsilon)$ where $c\in\mathbb{R}^>$, $\mathfrak{m}\in\mathfrak{M}$, and $\varepsilon\in\mathbb{T}^\prec$. Since $0<s<t$, we have $s/t<1$, so either $\mathfrak{m}\prec 1$, or $\mathfrak{m}=1$ and $c<1$, or $\mathfrak{m}=c=1$ and $\varepsilon<0$. If $\mathfrak{m}\prec 1$, then $\mathfrak{m}^r\prec 1$, so $(s/t)^r\prec 1$. If $\mathfrak{m}=1$ and $c<1$, then $c^r<1$ and $(s/t)^r=c^r(1+\varepsilon)^r=c^r+o(1)<1$. If $\mathfrak{m}=c=1$ and $\varepsilon<0$, then $(s/t)^r-1=(1+\varepsilon)^r-1\sim r\,\varepsilon<0$, so $(s/t)^r<1$. □

Thus, the extended real power operation $\mathbb{R}\times\mathbb{T}^>\to\mathbb{T}^>;(r,s)\mapsto s^r$ gives $\mathbb{T}^>$ the structure of a multiplicative ordered $\mathbb{R}$-vector space. Accordingly, we say that $\mathbb{T}$ *has real powers*.

Power series operations. Given a power series

$$F(X_1,\ldots,X_n)=\sum_{\alpha\in\mathbb{N}^n}F_\alpha X_1^{\alpha_1}\cdots X_n^{\alpha_n}\in\mathbb{T}[[X_1,\ldots,X_n]],$$

and elements $s_1,\ldots,s_n\in\mathbb{T}$, we say that F is defined at $(s_1,\ldots,s_n)$ if the family $(F_\alpha s_1^{\alpha_1}\cdots s_n^{\alpha_n})_{\alpha\in\mathbb{N}^n}$ is well-based.

Lemma 2.7. *Suppose that $\mathfrak{M}$ is uncountable and let $F(X_1,\ldots,X_n)\in\mathbb{T}[[X_1,\ldots,X_n]]$ be a power series which is defined on $(\mathbb{T}^\prec)^n$. If $F\neq 0$, then $F(s_1,\ldots,s_n)\neq 0$ for some $s_1,\ldots,s_n\in\mathbb{T}^\prec$.*

Proof. We prove this by induction on n. If $n=1$, then set $X:=X_1$ and write $F=\sum_{k\in\mathbb{N}}F_kX^k$. Suppose that $F\neq 0$ and let $\mathcal{Z}\subseteq\mathbb{T}^\prec$ be the set of $s\neq 0$ with $F(s)=0$. Fix $s\in\mathcal{Z}$ and let m be such that $0\neq F_m s^m\succcurlyeq F_k s^k$ for all k. Since $F(s)=0$, there exists an index $k\neq m$ with $F_m s^m\asymp F_k s^k$. Then $0\neq s\asymp(F_m^{-1}F_k)^{1/(m-k)}$, whence

$$\{\mathfrak{d}_s:s\in\mathcal{Z}^{\neq}\}\subseteq\left\{\left(\frac{\mathfrak{d}_{F_k}}{\mathfrak{d}_{F_m}}\right)^q:k,m\in\mathbb{N},q\in\mathbb{Q},F_k,F_m\neq 0\right\}.$$

In particular, $\{\mathfrak{d}_s:s\in\mathcal{Z}^{\neq}\}$ is countable, whereas $\{\mathfrak{d}_s:s\in\mathbb{T}^\prec,s\neq 0\}=\mathfrak{M}^\prec$ is uncountable, so $\mathcal{Z}\neq\mathbb{T}^\prec$.

Now suppose that $n>1$ and write $F=\sum_{k\in\mathbb{N}}F_k(X_1,\ldots,X_{n-1})\,X_n^k$. Assume that $F\neq 0$. By the induction hypothesis, we can find $s_1,\ldots,s_{n-1}\in\mathbb{T}^\prec$ and $k\in\mathbb{N}$ such that $F_k(s_1,\ldots,s_{n-1})\neq 0$. Fix such elements $s_1,\ldots,s_{n-1}$ and let $\mathcal{Z}\subseteq\mathbb{T}^\prec$ be the set of $s\in\mathbb{T}^\prec$ such that $F(s_1,\ldots,s_{n-1},s)=0$. By the special case when $n=1$, we see that $\mathcal{Z}\neq\mathbb{T}^\prec$. Thus, $F(s_1,\ldots,s_{n-1},s)\neq 0$ for some $s\in\mathbb{T}$. □

2.3 Logarithmic hyperseries

A central object in our work is the field $\mathbb{L}$ of *logarithmic hyperseries* of [12], equipped with its natural derivation $\partial:\mathbb{L}\longrightarrow\mathbb{L}$ and composition $\circ:\mathbb{L}\times\mathbb{L}^{>,\succ}\longrightarrow\mathbb{L}$. We briefly recall its definition and some of its properties.

Logarithmic hyperseries. For each ordinal γ, there is an element $\ell_\gamma\in\mathbb{L}$ which we call the *γ-th iterated hyperlogarithm*. Intuitively speaking, we have $\ell_0=x$, $\ell_1=\log x$, $\ell_2=\log\log x$, $\ldots$, $\ell_\omega=L_\omega(x)$, $\ell_{\omega+1}=\log L_\omega(x)$, etc. Let α be an ordinal of the form $\alpha=\omega^\nu$. We write $\mathfrak{L}_{<\alpha}$ for the monomial group of all formal products $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}$ with $(\mathfrak{l}_\gamma)_{\gamma<\alpha}\in\mathbb{R}^\alpha$. The group $\mathfrak{L}_{<\alpha}$ is naturally ordered by setting $\mathfrak{l}\succ 1$ if $\mathfrak{l}_\gamma>0$ for some $\gamma<\alpha$ with $\mathfrak{l}_\beta=0$ for all $\beta<\gamma$. We also have a real power operation on $\mathfrak{L}_{<\alpha}$ given by setting $\left(\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}\right)^r:=\prod_{\gamma<\alpha}\ell_\gamma^{r\mathfrak{l}_\gamma}$ for $r\in\mathbb{R}$. This operation extends to all of $\mathbb{L}_{<\alpha}$ as described in the previous subsection.

We call $\mathbb{L}_{<\alpha} := \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ the field of logarithmic hyperseries of *strength* α. If β, γ are ordinals with $\gamma < \beta \leqslant \alpha$, then we let $[\gamma, \beta)$ denote the interval $\{\rho \in \mathbf{On} : \gamma \leqslant \rho < \beta\}$ and we let $\mathfrak{L}_{[\gamma,\beta)}$ denote the subgroup

$$\{\mathfrak{l} \in \mathfrak{L}_{<\alpha} : \mathfrak{l}_\rho = 0 \text{ whenever } \rho \notin [\gamma, \beta)\}.$$

As in [12], we write $\mathbb{L}_{[\gamma,\beta)} := \mathbb{R}[[\mathfrak{L}_{[\gamma,\beta)}]]$, $\mathfrak{L} := \bigcup_{\alpha \in \mathbf{On}} \mathfrak{L}_{<\alpha}$ and

$$\mathbb{L} := \mathbb{R}[[\mathfrak{L}]] = \bigcup_{\alpha \in \mathbf{On}} \mathbb{L}_{<\alpha}.$$

We will sometimes write $\mathfrak{L}_{<\mathbf{On}} = \mathfrak{L}$ and $\mathbb{L}_{<\mathbf{On}} = \mathbb{L}$. We have natural inclusions $\mathfrak{L}_{[\gamma,\beta)} \subseteq \mathfrak{L}_{<\alpha} \subseteq \mathfrak{L}$, which give natural inclusions $\mathbb{L}_{[\gamma,\beta)} \subseteq \mathbb{L}_{<\alpha} \subset \mathbb{L}$.

Derivation on $\mathbb{L}$. The field $\mathbb{L}$ is equipped with a strongly linear derivation $\partial: \mathbb{L} \longrightarrow \mathbb{L}$. Given $\alpha \in \mathbf{On}$ and a logarithmic hypermonomial $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$, we define the derivation of $\mathfrak{l}$ by

$$\partial \mathfrak{l} := \left(\sum_{\gamma < \alpha} \mathfrak{l}_\gamma (\ell_\gamma)^\dagger \right) \mathfrak{l},$$

where $(\ell_\gamma)^\dagger = \prod_{\iota \leqslant \gamma} \ell_\iota^{-1} \in \mathfrak{L}_{<\alpha}$. Note that $\partial \ell_\gamma = (\ell_\gamma)^\dagger \ell_\gamma = \prod_{\iota < \gamma} \ell_\iota^{-1}$. For $f \in \mathbb{L}$ and $k \in \mathbb{N}$, we sometimes write $f^{(k)} := \partial^k f$. Equipped with its derivation, the field $\mathbb{L}$ is an H-field with small derivation, so for $f, g \in \mathbb{L}$, we have

$$f > \mathbb{R} \Longrightarrow f' > 0, \qquad f \prec 1 \Longrightarrow f' \prec 1, \qquad f \prec g \not\asymp 1 \Longrightarrow f' \prec g'.$$

Moreover, $\operatorname{supp}_* \partial \preccurlyeq \ell_0^{-1}$ is well-based, which implies the following variant of [12, Lemma 2.13]:

Lemma 2.8. *Let $\alpha = \omega^\nu$, let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series, and let $\Phi: \mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}$ be a strongly linear field embedding. For $f \in \mathbb{L}_{<\alpha}$ and $s \in \mathbb{T}$ with $s \prec \Phi(\ell_0)$, the family $(\Phi(f^{(n)}) s^n)_{n \in \mathbb{N}}$ is well-based. Moreover, the map $\Psi: \mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}; f \longmapsto \sum_{n \in \mathbb{N}} \frac{\Phi(f^{(n)})}{n!} s^n$ is a strongly linear field embedding.*

Proof. Since $\operatorname{supp}_* \partial \preccurlyeq \ell_0^{-1}$ is well-based and Φ is a strongly linear field embedding, the set $\mathfrak{S} := \bigcup_{\mathfrak{l} \in \operatorname{supp}_* \partial} \operatorname{supp} \Phi(\mathfrak{l}) \preccurlyeq \Phi(\ell_0)^{-1}$ is well-based. Thus $\mathfrak{S} \operatorname{supp} s$ is well-based and $\mathfrak{S} \operatorname{supp} s \prec 1$, since $s \prec \Phi(\ell_0)$. Let $f \in \mathbb{L}$. For each $n \in \mathbb{N}^>$, we have

$$\operatorname{supp} (\Phi(f^{(n)}) s^n) \;\subseteq\; (\operatorname{supp} \Phi(f)) (\mathfrak{S} \operatorname{supp} s)^n.$$

Since $\operatorname{supp} \Phi(f)$ is well-based and $\mathfrak{S} \operatorname{supp} s \prec 1$, it follows that $(\Phi(f^{(n)}) s^n)_{n \in \mathbb{N}}$ is well-based and that the map Ψ is well-defined and strongly linear. For all $f, g \in \mathbb{L}_{<\alpha}$, we also have

$$\sum_{n \in \mathbb{N}} \frac{\Phi((fg)^{(n)})}{n!} s^n = \sum_{n \in \mathbb{N}} \sum_{i+j=n} \frac{\Phi(f^{(i)}) \Phi(g^{(j)})}{i!} s^n = \left(\sum_{i \in \mathbb{N}} \frac{\Phi(f^{(i)})}{i!} s^i \right) \left(\sum_{j \in \mathbb{N}} \frac{\Phi(f^{(j)})}{j!} s^j \right),$$

which shows that Ψ preserves multiplication. $\square$

Composition on $\mathbb{L}$. In addition to its derivation, the field $\mathbb{L}$ comes equipped with a composition law $\circ: \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$ which is unique to satisfy:

- For $g \in \mathbb{L}^{>,\succ}$, the map $\circ_g: \mathbb{L} \to \mathbb{L}; f \mapsto f \circ g$ is a strongly linear field embedding. As a consequence this map preserves the relations $<$ and $\prec$ [12, Lemma 6.6].
- For $f \in \mathbb{L}$ and $g, h \in \mathbb{L}^{>,\succ}$, we have $f \circ (g \circ h) = (f \circ g) \circ h$ [12, Proposition 7.14].
- For $g \in \mathbb{L}^{>,\succ}$ and $r \in \mathbb{R}$, we have $\ell_0^r \circ g = g^r$ [12, Corollary 7.5].
- For $g, h \in \mathbb{L}^{>,\succ}$ and $r \in \mathbb{R}^>$, we have $\ell_1 \circ (g h) = \ell_1 \circ g + \ell_1 \circ h$ and $\ell_1 \circ (r h) = \log r + \ell_1 \circ h$ [12, Section 1.4].
- For ordinals $\sigma \leqslant \rho$, we have $\ell_\sigma \circ \ell_\rho = \ell_{\rho + \sigma}$ [12, Corollary 5.11].
- For any successor ordinal μ, we have $\ell_{\omega^\mu} \circ \ell_{\omega^{\mu_*}} = \ell_{\omega^\mu} - 1$ [12, Lemma 5.8].
- The constant term of $\ell_{\omega^\mu} \circ \ell_{\omega^\gamma}$ vanishes if $\mu > \gamma$ is a limit ordinal [12, Lemma 5.8].
- For $f, h \in \mathbb{L}$ and $g \in \mathbb{L}^{>,\succ}$ with $h \prec g$, the family $((f^{(k)} \circ g) h^k)_{k \in \mathbb{N}}$ is well-based, and
$$f \circ (g + h) \;=\; \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!} h^k. \qquad \text{[12, Proposition 8.1]}$$

The uniqueness follows from [12, Theorem 1.3]. By [12, Proposition 7.8], the derivation also satisfies the chain rule: for all $f \in \mathbb{L}$ and $g \in \mathbb{L}^{>,\succ}$, we have

$$(f \circ g)' = (f' \circ g) g'.$$

As we will see, the field $\mathbb{L}$ equipped with the composition $\circ$ is hyperserial.

For $\alpha = \omega^\nu$, the unique composition $\circ$ from above restricts to a composition $\mathbb{L}_{<\alpha} \times \mathbb{L}_{<\alpha}^{>,\succ} \longrightarrow \mathbb{L}_{<\alpha}$. For $\gamma < \alpha$, the map $\circ_{\ell_\gamma}: \mathbb{L}_{<\alpha} \longrightarrow \mathbb{L}_{<\alpha}$ defined by $\circ_{\ell_\gamma}(f) := f \circ \ell_\gamma$ is a strongly linear field embedding with image $\mathbb{L}_{[\gamma, \alpha)}$ by [12, Lemma 5.13]. Accordingly, for $g \in \mathbb{L}_{[\gamma, \alpha)}$, we let $g^{\uparrow \gamma}$ denote the unique series in $\mathbb{L}_{<\alpha}$ with $g^{\uparrow \gamma} \circ \ell_\gamma = g$. Note that $\ell_{\omega^{\mu+1}}^{\uparrow \omega^\mu} = \ell_{\omega^{\mu+1}} + 1$ for all μ and that, more generally, $\ell_{\omega^{\mu+1}}^{\uparrow \omega^\mu n + \gamma} = \ell_{\omega^{\mu+1}}^{\uparrow \gamma} + n$ for $\gamma < \omega^{\mu+1}$ and $n \in \mathbb{N}$. For $\mu < \nu$ and $f \in \mathbb{L}_{[\omega^{\mu+1}, \alpha)}$ we have

$$f \circ \ell_{\omega^\mu} \;=\; \sum_{k \in \mathbb{N}} \frac{(-1)^k}{k!} \delta^k(f) \;=\; \mathrm{e}^{-\delta} f \tag{2.2}$$

where δ is the derivation $\frac{1}{\ell'_{\omega^{\mu+1}}} \partial$ on $\mathbb{L}_{<\alpha}$ (see [12, Section 5.1]). Let $R(f) := \sum_{k \in \mathbb{N}^>} \frac{(-1)^k}{k!} \delta^k(f)$. Then $R: \mathbb{L}_{[\omega^{\mu+1}, \alpha)} \longrightarrow \mathbb{L}_{[\omega^{\mu+1}, \alpha)}$ is strongly linear and $R(f) \prec f$, so, by [12, Lemma 2.2],

$$f^{\uparrow \omega^\mu} \;=\; f - R(f) + R^2(f) - \cdots \;=\; \mathrm{e}^{\delta} f. \tag{2.3}$$

In particular,

$$f^{\uparrow \omega^\mu} - f \;\sim\; -R(f) \;\sim\; \frac{1}{\ell'_{\omega^{\mu+1}}} f'. \tag{2.4}$$

Lemma 2.9. *For each $\mu < \nu$, each $\gamma < \beta \leqslant \omega^\mu$, and each $k \in \mathbb{N}^>$, we have $(\ell_\beta^{\uparrow \gamma})^{(k)} \in \mathbb{L}_{<\omega^\mu}^{\prec}$.*

Proof. Since $\mathbb{L}_{<\omega^\mu}$ is closed under taking derivatives and the derivation preserves infinitesimals, it suffices to prove the lemma for $k = 1$. We have $\ell_\beta^{\uparrow \gamma} \circ \ell_\gamma = \ell_\beta$, so

$$(\ell_\beta^{\uparrow \gamma} \circ \ell_\gamma)' = ((\ell_\beta^{\uparrow \gamma})' \circ \ell_\gamma) \, \ell_\gamma' = \ell_\beta'.$$

Since $\ell'_\gamma, \ell'_\beta \in \mathbb{L}_{<\omega^\mu}$ and $\ell'_\beta \prec \ell'_\gamma$, this yields $(\ell_\beta^{\uparrow\gamma})' \circ \ell_\gamma \in \mathbb{L}^{\prec}_{<\omega^\mu}$. Since $(\ell_\beta^{\uparrow\gamma})' \circ \ell_\gamma \in \mathbb{L}_{[\gamma,\alpha)}$ as well, we have $(\ell_\beta^{\uparrow\gamma})' \circ \ell_\gamma \in \mathbb{L}^{\prec}_{[\gamma,\omega^\mu)}$. Since the map $f \longmapsto f^{\uparrow\gamma}$ maps $\mathbb{L}^{\prec}_{[\gamma,\omega^\mu)}$ onto $\mathbb{L}^{\prec}_{<\omega^\mu}$, we conclude that $(\ell_\beta^{\uparrow\gamma})' = ((\ell_\beta^{\uparrow\gamma})' \circ \ell_\gamma)^{\uparrow\gamma} \in \mathbb{L}^{\prec}_{<\omega^\mu}$. □

3 Hyperserial skeletons

3.1 Domain of definition

We let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be an ordered field of well-based series with real powers. Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ be an ordinal with $\boldsymbol{\nu} > 0$. Given a structure $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ where L_{ω^μ} are partial functions on $\mathbb{T}$, we consider the following axioms for $\mu < \boldsymbol{\nu}$:

Domain of definition:

DD₀. $\operatorname{dom} L_1 = \mathfrak{M}^{\succ}$.

DD$_{\boldsymbol{\mu}}$. $\operatorname{dom} L_{\omega^\mu} = \begin{cases} \bigcap_{\eta<\mu} \operatorname{dom} L_{\omega^\eta} & \text{if } \mu \text{ is a non-zero limit} \\ \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\mu_*}}^{\circ n} & \text{if } \mu = \mu_* + 1. \end{cases}$

Suppose $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ satisfies all axioms **DD$_{\boldsymbol{\mu}}$** for $\mu < \boldsymbol{\nu}$. We set $\mathfrak{M}_{\omega^\mu} := \operatorname{dom} L_{\omega^\mu}$ for all $\mu < \boldsymbol{\nu}$ and we extend this notation to the case when $\mu = \boldsymbol{\nu}$, by setting

$$\mathfrak{M}_{\omega^\nu} = \begin{cases} \bigcap_{\eta<\boldsymbol{\nu}} \operatorname{dom} L_{\omega^\eta} & \text{if } \boldsymbol{\nu} \text{ is a non-zero limit} \\ \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\nu_*}}^{\circ n} & \text{if } \boldsymbol{\nu} = \boldsymbol{\nu}_* + 1. \end{cases} \tag{3.1}$$

For $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$, we call $\mathfrak{M}_{\omega^\mu}$ the class of *$L_{<\omega^\mu}$-atomic* elements. Note that $\mathfrak{M}_{\omega^\mu} \subseteq \mathfrak{M}_{\omega^\eta}$ for all $\boldsymbol{\eta} \leqslant \boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. We let L_0 be the identity function with $\operatorname{dom} L_0 := \mathfrak{M}^{\succ}$ and, for $\beta < \omega^{\nu}$ with Cantor normal form $\beta = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_k} n_k$, we define $L_\beta := L_{\omega^{\gamma_k}}^{\circ n_k} \circ \cdots \circ L_{\omega^{\gamma_1}}^{\circ n_1}$. Here we understand that $x \in \operatorname{dom} L_\beta$ whenever $x \in \operatorname{dom} L_{\omega^{\gamma_1}}^{\circ n_1}$, $L_{\omega^{\gamma_1}}^{\circ n_1} x \in \operatorname{dom} L_{\omega^{\gamma_2}}^{\circ n_2}$, and so on until $L_{\omega^{\gamma_{k-1}}}^{\circ n_{k-1}} \circ \cdots \circ L_{\omega^{\gamma_1}}^{\circ n_1} x \in \operatorname{dom} L_{\omega^{\gamma_k}}^{\circ n_k}$.

Proposition 3.1. *For $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ with $\boldsymbol{\mu} > 0$, we have*

$$\mathfrak{M}_{\omega^\mu} \;=\; \{s \in \mathbb{T}^{>,\succ} : s \in \operatorname{dom} L_\beta \text{ and } L_\beta(s) \in \mathfrak{M}^{\succ}, \text{ for all } \beta < \omega^{\boldsymbol{\mu}}\}.$$

Proof. Given $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and $\beta < \omega^{\boldsymbol{\mu}}$, let us first show by induction on $\boldsymbol{\mu}$ that $L_\beta(\mathfrak{a})$ is defined and in $\mathfrak{M}^{\succ}$. This holds for $\boldsymbol{\mu} = 0$ by definition. Let $0 < \boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ and assume that the assertion holds strictly below $\boldsymbol{\mu}$. If $\beta = 0$, then $L_0(\mathfrak{a}) = \mathfrak{a} \in \mathfrak{M}^{\succ}$. Assume $\beta > 0$ and let $\eta < \boldsymbol{\mu}$, $n \in \mathbb{N}^{>}$ and $\iota < \omega^\eta$ be such that $\beta = \omega^\eta n + \iota$. We have $\mathfrak{a} \in \mathfrak{M}_{\omega^{\eta+1}}$ so $L_{\omega^\eta n}(\mathfrak{a}) \in \mathfrak{M}_{\omega^{\eta+1}}$ by definition. In particular $L_{\omega^\eta n}(\mathfrak{a}) \in \mathfrak{M}_{\omega^\eta}$, so our inductive hypothesis on $\boldsymbol{\mu}$ applied to η gives that $L_\iota(L_{\omega^\eta n}(\mathfrak{a})) = L_\beta(\mathfrak{a})$ is a monomial.

Given $\mathfrak{a} \in \mathbb{T}^{>,\succ}$ such that $\mathfrak{a} \in \operatorname{dom} L_\beta$ and $L_\beta(\mathfrak{a}) \in \mathfrak{M}^{\succ}$ for all $\beta < \omega^{\boldsymbol{\mu}}$, let us next show by induction that $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$. This is clear if $\boldsymbol{\mu} = 0$. Let $1 < \boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ be such that the statement holds strictly below $\boldsymbol{\mu}$. If $\boldsymbol{\mu}$ is a successor, then for $\iota < \omega^{\boldsymbol{\mu}_*}$ and $n \in \mathbb{N}$, we have $L_{\omega^{\mu_*} n + \iota}(\mathfrak{a}) = L_\iota(L_{\omega^{\mu_*}}(\mathfrak{a})) \in \mathfrak{M}^{\succ}$ so for all $n \in \mathbb{N}, L_{\omega^{\mu_*} n}(\mathfrak{a}) \in \mathfrak{M}_{\omega^{\mu_*}}$, whence $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$. Assume now that μ is a limit and let $\eta < \boldsymbol{\mu}$. Then $L_\beta(\mathfrak{a}) \in \mathfrak{M}^{\succ}$ for all $\beta < \omega^\eta$, so the induction hypothesis yields $\mathfrak{a} \in \mathfrak{M}_{\omega^\eta}$. We again conclude that $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$. □

3.2 Axioms for the hyperlogarithms

Let $\mathbb{T}$ be an ordered field of well-based series with real powers, let $\boldsymbol{\nu} \leqslant \mathbf{On}$, and let $(L_{\omega^\mu})_{\mu<\nu}$ be partial functions $(L_{\omega^\mu})_{\mu<\nu}$ on $\mathbb{T}$ which satisfy the axioms $\mathbf{DD}_\mu$ for all $\mu < \boldsymbol{\nu}$. We consider the following axioms, where μ is an ordinal with $0 < \mu < \boldsymbol{\nu}$.

Functional equations:

FE$_0$. $L_1(\mathfrak{m}^r) = r L_1(\mathfrak{m})$ and $L_1(\mathfrak{m}\mathfrak{n}) = L_1(\mathfrak{m}) + L_1(\mathfrak{n})$ for all $r \in \mathbb{R}^>$ and all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_1$.

FE$_\mu$. For $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$, we have $L_{\omega^\mu}(L_{\omega^{\mu_*}}(\mathfrak{a})) = L_{\omega^\mu}(\mathfrak{a}) - 1$ if μ is a successor (**FE$_\mu$** holds trivially if μ is a limit).

Asymptotics:

A$_0$. $L_1(\mathfrak{m}) \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}_1$.

A$_\mu$. $L_{\omega^\mu}(\mathfrak{a}) < L_{\omega^\eta}(\mathfrak{a})$ for all $\eta < \mu$ and all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

Monotonicity:

M$_0$. $L_1(\mathfrak{m}) > 0$ for all $\mathfrak{m} \in \mathfrak{M}_1$.

M$_\mu$. $L_{\omega^\mu}(\mathfrak{a}) + L_{\omega^\eta n}(\mathfrak{a})^{-1} < L_{\omega^\mu}(\mathfrak{b}) - L_{\omega^\eta n}(\mathfrak{b})^{-1}$ for all $\eta < \mu$, $n \in \mathbb{N}$ and $\mathfrak{a} < \mathfrak{b}$ in $\mathfrak{M}_{\omega^\mu}$.

Regularity:

R$_0$. $\operatorname{supp} L_1(\mathfrak{m}) \succ 1$ for all $\mathfrak{m} \in \mathfrak{M}_1$.

R$_\mu$. $\operatorname{supp} L_{\omega^\mu}(\mathfrak{a}) \succ L_{\omega^\eta n}(\mathfrak{a})^{-1}$ for all $\eta < \mu$, $n \in \mathbb{N}$, and $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

We define a logarithm $\log: \mathfrak{M} \longrightarrow \mathbb{T}$ as follows:

$$\log \mathfrak{m} = \begin{cases} L_1(\mathfrak{m}) & \text{if } \mathfrak{m} \in \mathfrak{M}^{\succ} \\ -L_1(\mathfrak{m}^{-1}) & \text{if } \mathfrak{m} \in \mathfrak{M}^{\prec} \\ 0 & \text{if } \mathfrak{m} = 1. \end{cases} \tag{3.2}$$

Then $\log \mathfrak{M}$ is an ordered $\mathbb{R}$-vector subspace of $\mathbb{T}$. For $\mu \in \mathbf{On}$ with $0 \leqslant \mu \leqslant \boldsymbol{\nu}$, we consider the following axiom:

Infinite products:

P$_\mu$. $\sum_{\gamma<\omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{a}) \in \log \mathfrak{M}$ for all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and all sequences $(r_\gamma)_{\gamma<\omega^\mu}$ of real numbers.

Remark 3.2. The axiom $\mathbf{P}_\mu$ allows us to define the infinite product $\prod_{\gamma<\omega^\mu} L_{\gamma+1}(\mathfrak{a})^{r_\gamma}$ for $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ to be the unique monomial $\mathfrak{m} \in \mathfrak{M}$ with $\log \mathfrak{m} = \sum_{\gamma<\omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{a})$, hence the name. Note that the axiom $\mathbf{P}_0$ is a consequence of $\mathbf{FE}_0$: if $\mathbf{FE}_0$ holds, then for $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}^{\succ}$, we have $r L_1(\mathfrak{m}) = \log \mathfrak{m}^r$.

Definition 3.3. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$. A* **hyperserial skeleton of force $\boldsymbol{\nu}$** *is a structure $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\nu})$ where $\mathbb{T}$ is an ordered field of well-based series with real powers and $(L_{\omega^\mu})_{\mu<\nu}$ are partial functions on $\mathbb{T}$ which satisfy $\mathbf{DD}_\mu$, $\mathbf{FE}_\mu$, $\mathbf{A}_\mu$, $\mathbf{M}_\mu$, and $\mathbf{R}_\mu$ for all $\mu < \boldsymbol{\nu}$, as well as $\mathbf{P}_\mu$ for all $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$.*

Note that a hyperserial skeleton of force 0 is just a field of well-based series with real powers and that $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\mathbf{On}})$ is a hyperserial skeleton of force $\mathbf{On}$ if and only if $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\nu})$ is a hyperserial skeleton of force ν for each ordinal ν. We will often write $\mathbb{T}$ to denote a hyperserial skeleton (of force $\boldsymbol{\nu} \leqslant \mathbf{On}$), where it is implied that for $\mu < \boldsymbol{\nu}$, the term L_{ω^μ} refers to the ω^μ-th hyperlogarithm on $\mathbb{T}$.

Definition 3.4. *Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ and $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ be hyperserial skeletons of force $\boldsymbol{\nu} \leqslant \mathbf{On}$. We say that a function $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ is an* **embedding** *of force $\boldsymbol{\nu}$ if it is a strongly linear strictly increasing ring morphism with $\Phi(\mathfrak{M}_{\omega^\mu}) \subseteq \mathfrak{N}_{\omega^\mu}$ for each $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ such that*

$$\Phi(\mathfrak{m}^r) = \Phi(\mathfrak{m})^r \text{ for all } \mathfrak{m} \in \mathfrak{M} \text{ and } r \in \mathbb{R},$$

and such that

$$\Phi(L_{\omega^\mu}(\mathfrak{a})) = L_{\omega^\mu}(\Phi(\mathfrak{a})) \text{ for all } \mu < \boldsymbol{\nu} \text{ and } \mathfrak{a} \in \mathfrak{M}_{\omega^\mu}.$$

If $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ is an embedding of force $\boldsymbol{\nu}$, then we say that $\mathbb{U}$ is an **extension** *of $\mathbb{T}$ of force $\boldsymbol{\nu}$.*

3.3 Confluence

In this subsection, let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$. We inductively define the notion of ***μ-confluence*** in conjunction with functions $\mathfrak{d}_{\omega^\mu}: \mathbb{T}^{>,\succ} \longrightarrow \mathfrak{M}_{\omega^\mu}$ and the classes $\mathcal{E}_{\omega^\mu}[s] \subseteq \mathbb{T}^{>,\succ}$, as follows:

Definition 3.5. *The field $\mathbb{T}$ is said to be 0-**confluent** if $\mathfrak{M}$ is non-trivial. The function $\mathfrak{d}_1$ maps each $s \in \mathbb{T}^{>,\succ}$ to its dominant monomial $\mathfrak{d}_s$. For each $s \in \mathbb{T}^{>,\succ}$, we set*

$$\mathcal{E}_1[s] := \{t \in \mathbb{T}^{>,\succ} : t \asymp s\}.$$

Let $\mu \in \mathbf{On}$ with $0 < \mu \leqslant \boldsymbol{\nu}$, let $s \in \mathbb{T}^{>,\succ}$, and suppose $\mathbb{T}$ is η-confluent for all $\eta < \mu$.

- *If μ is a successor, then we define $\mathcal{E}_{\omega^\mu}[s]$ to be the class of series t with*

$$(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(s) \asymp (L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(t)$$

  for some $n \in \mathbb{N}$.

- *If μ is a limit, then we define $\mathcal{E}_{\omega^\mu}[s]$ to be the class of series t with*

$$L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(s)) \asymp L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t))$$

  for some $\eta < \mu$.

*If each class $\mathcal{E}_{\omega^\mu}[s]$ contains an $L_{<\omega^\mu}$-atomic element, then we say that $\mathbb{T}$ is μ-**confluent**. We will see that each class $\mathcal{E}_{\omega^\mu}[s]$ contains at most one $L_{<\omega^\mu}$-atomic element, which we denote by $\mathfrak{d}_{\omega^\mu}(s)$.*

Remark 3.6. We note that μ-confluence is somewhat stronger than the similar notion of $\log_{\omega^\mu}$-confluence from [29], due to the extra requirement that we have maps $\mathfrak{d}_{\omega^\mu}$.

Lemma 3.7. *Let $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$ and suppose $\mathbb{T}$ is μ-confluent. Then the function $\mathfrak{d}_{\omega^\mu}$ is well-defined. Moreover, we have $\mathcal{E}_{\omega^\eta}[\mathfrak{a}] \subseteq \mathcal{E}_{\omega^\mu}[\mathfrak{a}]$ for all $\eta \leqslant \mu$ and $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.*

Proof. We prove this by induction on μ, noticing that the case $\mu = 0$ is trivial. Assume that this is the case for all ordinals $\eta < \mu$ and let $s \in \mathbb{T}^{>,\succ}$. To see that $\mathfrak{d}_{\omega^\mu}$ is well-defined, let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_{\omega^\mu}$ with $\mathfrak{b} \in \mathcal{E}_{\omega^\mu}[\mathfrak{a}]$. We need to show $\mathfrak{a} = \mathfrak{b}$.

Assume that μ is a successor. Take $n \in \mathbb{N}$ with $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(\mathfrak{a}) \asymp (L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(\mathfrak{b})$ Since $L_{\omega^{\mu_*} k}(\mathfrak{a})$ is $L_{<\omega^\mu}$-atomic for each k and since $\mathfrak{d}_{\omega^{\mu_*}}$ is well-defined by our induction hypothesis, we have $\mathfrak{d}_{\omega^{\mu_*}}(L_{\omega^{\mu_*} k}(\mathfrak{a})) = L_{\omega^{\mu_*} k}(\mathfrak{a})$ for each k. It follows by induction on k that $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ k}(\mathfrak{a}) = L_{\omega^{\mu_*} k}(\mathfrak{a})$ for each k and, likewise, $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ k}(\mathfrak{b}) = L_{\omega^{\mu_*} k}(\mathfrak{b})$ for each k, so $L_{\omega^{\mu_*} n}(\mathfrak{a}) \asymp L_{\omega^{\mu_*} n}(\mathfrak{b})$. As both $L_{\omega^{\mu_*} n}(\mathfrak{a})$ and $L_{\omega^{\mu_*} n}(\mathfrak{b})$ are monomials, we have $L_{\omega^{\mu_*} n}(\mathfrak{a}) = L_{\omega^{\mu_*} n}(\mathfrak{b})$. The axiom $\mathbf{M}_{\mu_*}$ implies that $L_{\omega^{\mu_*}}$ is injective and thus $\mathfrak{a} = \mathfrak{b}$.

Assume now that μ is a limit and take $\eta<\mu$ with $L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{a}))\asymp L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{b}))$. Since $\mathfrak{a}$ is $L_{<\omega^\mu}$-atomic and since $\mathfrak{d}_{\omega^\eta}$ is well-defined by our induction hypothesis, we have $\mathfrak{d}_{\omega^\eta}(\mathfrak{a})=\mathfrak{a}$. Likewise, $\mathfrak{d}_{\omega^\eta}(\mathfrak{b})=\mathfrak{b}$, so $L_{\omega^\eta}(\mathfrak{a})\asymp L_{\omega^\eta}(\mathfrak{b})$. As both $L_{\omega^\eta}(\mathfrak{a})$ and $L_{\omega^\eta}(\mathfrak{b})$ are monomials, we have $L_{\omega^\eta}(\mathfrak{a})=L_{\omega^\eta}(\mathfrak{b})$. Since L_{ω^η} is injective by $\mathbf{M}_\eta$, we conclude that $\mathfrak{a}=\mathfrak{b}$.

As to our second assertion, consider $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$ and $t\in\mathcal{E}_{\omega^\eta}[\mathfrak{a}]$ with $\eta<\mu$. If μ is a successor, then the inductive hypothesis $\mathcal{E}_{\omega^\eta}[\mathfrak{a}]\subseteq\mathcal{E}_{\omega^{\mu_*}}[\mathfrak{a}]$ implies $\mathfrak{d}_{\omega^{\mu_*}}(t)=\mathfrak{a}$, so $L_{\omega^{\mu_*}}(\mathfrak{d}_{\omega^{\mu_*}}(t))=L_{\omega^{\mu_*}}(\mathfrak{a})=L_{\omega^{\mu_*}}(\mathfrak{d}_{\omega^{\mu_*}}(\mathfrak{a}))$ and $t\in\mathcal{E}_{\omega^\mu}[\mathfrak{a}]$. If μ is a limit, then $\mathfrak{d}_{\omega^\eta}(t)=\mathfrak{a}$, so $L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t))=L_{\omega^\eta}(\mathfrak{a})=L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{a}))$ and $t\in\mathcal{E}_{\omega^\mu}[\mathfrak{a}]$. □

Corollary 3.8. *Let $\mu,\eta\in\mathbf{On}$ with $\eta\leqslant\mu\leqslant\boldsymbol{\nu}$. If $\mathbb{T}$ is μ-confluent, then $\mathfrak{d}_{\omega^\mu}(s)=\mathfrak{d}_{\omega^\mu}(\mathfrak{d}_{\omega^\eta}(s))$ for all $s\in\mathbb{T}^{>,\succ}$.*

Proposition 3.9. *Let $\mu\in\mathbf{On}$ with $\mu\leqslant\boldsymbol{\nu}$. If $\mathbb{T}$ is η-confluent for all $\eta<\mu$, then the class $\mathcal{E}_{\omega^\mu}[s]$ is convex for each $s\in\mathbb{T}^{>,\succ}$. Moreover, if $\mathbb{T}$ is μ-confluent, then $\mathfrak{d}_{\omega^\mu}\colon\mathbb{T}^{>,\succ}\longrightarrow\mathfrak{M}_{\omega^\mu}$ is non-decreasing.*

Proof. We prove this by induction on μ. Let $s\in\mathbb{T}^{>,\succ}$. It is clear that $\mathcal{E}_1[s]$ is convex and that $\mathfrak{d}_1$ is increasing. Let $\mu>0$ and assume that the result holds for all $\eta<\mu$. By the monotonicity axioms, each function L_{ω^η} is strictly increasing on $\mathfrak{M}_{\omega^\eta}$ (when $\eta=0$, one also needs to use $\mathbf{FE}_0$ to see that $L_1(\mathfrak{m}/\mathfrak{n})=L_1(\mathfrak{m})-L_1(\mathfrak{n})>0$ for $\mathfrak{m}\succ\mathfrak{n}\in\mathfrak{M}_1$). As the composition of non-decreasing functions is non-decreasing, the function $(L_{\omega^\eta}\circ\mathfrak{d}_{\omega^\eta})^{\circ n}$ is non-decreasing for each $\eta<\mu$ and each $n\in\mathbb{N}$. We deduce that $\mathfrak{d}_{\omega^\mu}$ is non-decreasing and that the classes $\mathcal{E}_{\omega^\mu}[s],s\in\mathbb{T}^{>,\succ}$ are convex. □

If $\mathbb{T}$ is η-confluent for all $\eta<\mu$, then the proposition implies that the classes $\mathcal{E}_{\omega^\eta}[s]$ with $s\in\mathbb{T}^{>,\succ}$ form a partition of $\mathbb{T}^{>,\succ}$ into convex subclasses. If $\mathbb{T}$ is also μ-confluent, then we have the following explicit decomposition for all $\eta\leqslant\mu$:

$$\mathbb{T}^{>,\succ}=\bigsqcup_{\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}}\mathcal{E}_{\omega^\eta}[\mathfrak{a}].$$

Definition 3.10. *$\mathbb{T}$ is said to be* **confluent** *if it is μ-confluent for each $\mu\in\mathbf{On}$ with $\mu\leqslant\boldsymbol{\nu}$. An extension/embedding $\Psi\colon\mathbb{T}\longrightarrow\mathbb{U}$ of force $\boldsymbol{\nu}$ is* ***confluent*** *if $\mathbb{U}$ is confluent.*

Note that if $\boldsymbol{\nu}\in\mathbf{On}$, then $\mathbb{T}$ is confluent if and only if it is $\boldsymbol{\nu}$-confluent.

3.4 The case of logarithmic hyperseries

Let ν be an ordinal and set $\alpha:=\omega^\nu$. The goal of this section is to check that $\mathbb{L}_{<\alpha}$ is a confluent hyperserial skeleton of force ν. This is immediate for $\nu=0$, so we assume that $\nu>0$.

Definition 3.11. *Let $\operatorname{dom}L_1:=\mathfrak{L}_{<\alpha}^{\succ}$ and for $0<\mu<\nu$, let $\operatorname{dom}L_{\omega^\mu}:=\{\ell_\sigma:\omega^{\mu_*}\leqslant\sigma<\alpha\}$. Given $\mathfrak{l}\in\operatorname{dom}L_{\omega^\mu}$, set*

$$L_{\omega^\mu}(\mathfrak{l}):=\ell_{\omega^\mu}\circ\mathfrak{l}.$$

We will show that $(\mathbb{L}_{<\alpha},(L_{\omega^\mu})_{\mu<\nu})$ is a hyperserial skeleton by checking that the axioms are satisfied. We begin with the domain of definition axioms.

Lemma 3.12. *$(\mathbb{L}_{<\alpha},(L_{\omega^\mu})_{\mu<\nu})$ satisfies $\mathbf{DD}_\mu$ and $(\mathfrak{L}_{<\alpha})_{\omega^\mu}=\{\ell_\sigma:\omega^{\mu_*}\leqslant\sigma<\alpha\}$, for all $\mu\leqslant\nu$.*

Proof. We prove this by induction on μ. The case when $\mu = 0$ is immediate. For $\mu = 1$, consider an infinite monomial $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$. We have $L_1(\mathfrak{l}) = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma \ell_{\gamma+1}$, which is a monomial if and only $\mathfrak{l} = \ell_\gamma$ for some $\gamma < \alpha$. Conversely, for each $\gamma < \alpha$ we have $L_n(\ell_\gamma) = \ell_{\gamma+n} \in \mathfrak{L}_{<\alpha}$. Now let $1 < \mu \leqslant \nu$ and suppose that the lemma holds for all non-zero ordinals less than μ. Assume that μ is a limit. We have $\mu_* = \mu$, whence

$$\bigcap_{\eta < \mu} \operatorname{dom} L_{\omega^\eta} = \bigcap_{\eta < \mu_*} \operatorname{dom} L_{\omega^{\eta+1}} = \bigcap_{\eta < \mu_*} \{\ell_\gamma : \omega^\eta \leqq \gamma < \alpha\} = \{\ell_\gamma : \omega^{\mu_*} \leqq \gamma < \alpha\} = \operatorname{dom} L_{\omega^\mu}.$$

Assume now that μ is a successor. If $\mathfrak{l} \in \operatorname{dom} L_{\omega^\mu}$, then $\mathfrak{l} = \ell_\sigma$ where $\omega^{\mu_*} \leqq \sigma < \alpha$, and we clearly have $L_{\omega^{\mu_*}}^{\circ n}(\mathfrak{l}) = \ell_{\sigma + \omega^{\mu_*} n} \in \operatorname{dom} L_{\omega^{\mu_*}}$ for all $n \in \mathbb{N}$, whence $\mathfrak{l} \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_{\omega^{\mu_*}}^{\circ n}$. Conversely, let $\mathfrak{l} \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_{\omega^{\mu_*}}^{\circ n}$. Then $\mathfrak{l} = \ell_\sigma$ where $\omega^{\mu_{**}} \leqq \sigma < \alpha$. If μ_* is a limit, then $\mu_{**} = \mu_*$, whence $\omega^{\mu_*} \leqq \sigma < \alpha$ and $\mathfrak{l} \in \operatorname{dom} L_{\omega^\mu}$. If μ_* is a successor, then $\sigma = \gamma + \omega^{\mu_{**}} m$ for some $\gamma \geqq \omega^{\mu_*}$ and some $m \in \mathbb{N}$, so

$$\ell_\sigma = \ell_{\gamma + \omega^{\mu_{**}} m} = \ell_{\omega^{\mu_{**}} m} \circ \ell_\gamma.$$

Since $L_{\omega^{\mu_*}}(\ell_{\omega^{\mu_{**}} m}) = \ell_{\omega^{\mu_*}} - m$, we see that

$$L_{\omega^{\mu_*}}(\mathfrak{l}) = L_{\omega^{\mu_*}}(\ell_\sigma) = (\ell_{\omega^{\mu_*}} - m) \circ \ell_\gamma = \ell_{\gamma + \omega^{\mu_*}} - m.$$

Since $L_{\omega^{\mu_*}}(\mathfrak{l}) \in \operatorname{dom} L_{\omega^{\mu_*}}$, we must have $m = 0$, so $\mathfrak{l} = \ell_\sigma \in \operatorname{dom} L_{\omega^\mu}$. □

For $\beta < \omega^\nu$ and $\mathfrak{l} \in \operatorname{dom} L_\beta$, note that $L_\beta(\mathfrak{l}) = \ell_\beta \circ \mathfrak{l}$. Note also that the notions of $L_{<\omega^{\mu+1}}$-atomicity and $L_{<\omega^\mu}$-atomicity coincide in $\mathbb{L}_{<\alpha}$ whenever μ is a limit with $\mu + 1 \leqslant \nu$. This will not be the case in general.

Proposition 3.13. *The field $\mathbb{L}_{<\alpha}$ satisfies $\mathbf{P}_\mu$ for all $\mu \leqslant \nu$.*

Proof. Let $\mu \leqslant \nu$ and let $\mathfrak{l} \in (\mathfrak{L}_{<\alpha})_{\omega^\mu}$. By Remark 3.2, we may assume $\mu > 0$. We have $\mathfrak{l} = \ell_\sigma$ for some $\omega^{\mu_*} \leqq \sigma < \alpha$. Let $(r_\gamma)_{\gamma < \omega^\mu}$ be a sequence of real numbers. We have

$$\sum_{\gamma < \omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{l}) = \sum_{\gamma < \omega^\mu} r_\gamma \ell_{\gamma+1} \circ \ell_\sigma = \sum_{\gamma < \omega^\mu} r_\gamma \ell_{\sigma+\gamma+1}.$$

This sum coincides with $\log \mathfrak{m}$ where $\mathfrak{m} := \prod_{\gamma < \omega^\mu} \ell_{\sigma+\gamma}^{r_\gamma} \in \mathfrak{L}_{<\alpha}$. □

Proposition 3.14. *The field $\mathbb{L}_{<\alpha}$ satisfies $\mathbf{R}_\mu$, $\mathbf{A}_\mu$, and $\mathbf{M}_\mu$, for all $0 < \mu < \nu$.*

Proof. Let $0 < \mu < \nu$ and let $\mathfrak{l} \in (\mathfrak{L}_{<\alpha})_{\omega^\mu}$. We have $\mathfrak{l} = \ell_\sigma$ for some $\omega^{\mu_*} \leqq \sigma < \alpha$. Write $\sigma = \gamma + \omega^{\mu_*} n$ where $\gamma = \sigma_{\geqq \omega^\mu}$, $n \in \mathbb{N}$, and $n = 0$ if μ is a limit. We claim $L_{\omega^\mu}(\mathfrak{l}) = \ell_{\gamma + \omega^\mu} - n$. If μ is a successor, then since $\ell_{\omega^\mu} \circ \ell_{\omega^{\mu_*} n} = \ell_{\omega^\mu} - n$, we have

$$L_{\omega^\mu}(\mathfrak{l}) = \ell_{\omega^\mu} \circ \ell_\sigma = \ell_{\omega^\mu} \circ \ell_{\gamma + \omega^{\mu_*} n} = \ell_{\omega^\mu} \circ (\ell_{\omega^{\mu_*} n} \circ \ell_\gamma) = (\ell_{\omega^\mu} - n) \circ \ell_\gamma = \ell_{\gamma + \omega^\mu} - n.$$

If μ is a limit, then $\mathfrak{l} = \ell_\gamma$, so

$$L_{\omega^\mu}(\mathfrak{l}) = \ell_{\omega^\mu} \circ \ell_\gamma = \ell_{\gamma + \omega^\mu}.$$

Now we move on to verification of $\mathbf{R}_\mu$, $\mathbf{A}_\mu$, and $\mathbf{M}_\mu$. The only elements in $\operatorname{supp} L_{\omega^\mu}(\mathfrak{l})$ are $\ell_{\gamma + \omega^\mu}$ and possibly 1 (if $n \neq 0$), so $\operatorname{supp} L_{\omega^\mu}(\mathfrak{l}) \succcurlyeq 1 \succ L_{\omega^\eta n}(\mathfrak{l})^{-1}$ for all $\eta < \mu$ and $n \in \mathbb{N}$, which proves $\mathbf{R}_\mu$. For $\eta < \mu$, we have

$$L_{\omega^\eta}(\mathfrak{l}) = \ell_{\omega^\eta} \circ \ell_\sigma = \ell_{\sigma + \omega^\eta} \prec \ell_{\gamma + \omega^\mu} \asymp L_{\omega^\mu}(\mathfrak{l}),$$

so $\mathbf{A}_{\mu}$ holds as well.

As to $\mathbf{M}_{\mu}$, take $\mathfrak{l}' \in (\mathfrak{L}_{<\alpha})_{\omega^{\mu}}$ with $\mathfrak{l}' \succ \mathfrak{l}$. We have $\mathfrak{l}' = \ell_{\sigma'}$ for some σ' with $\omega^{\mu_*} \leqslant\!\!\leqslant \sigma' < \alpha$. Write $\sigma' = \gamma' + \omega^{\mu_*} n'$ where $\gamma' = \sigma'_{\geqslant \omega^{\mu}}$, $n' \in \mathbb{N}$, and $n' = 0$ if μ is a limit. The argument above gives $L_{\omega^{\mu}}(\mathfrak{l}') = \ell_{\gamma' + \omega^{\mu}} - n'$. If $\gamma' < \gamma$, then $L_{\omega^{\mu}}(\mathfrak{l}') > L_{\omega^{\mu}}(\mathfrak{l})$ and if $\gamma' = \gamma$, then $n' < n$ and $L_{\omega^{\mu}}(\mathfrak{l}') - L_{\omega^{\mu}}(\mathfrak{l}) = n - n' \geqslant 1$. In either case, $\mathbf{M}_{\mu}$ is satisfied. □

Recall that for $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$ and $\gamma < \alpha$, we write $\mathfrak{l}_{\gamma}$ for the real exponent of ℓ_{γ} in $\mathfrak{l}$. Given $f \in \mathbb{L}_{<\alpha}^{>,\succ}$, we define λ_f to be the least ordinal with $(\mathfrak{d}_f)_{\lambda_f} \neq 0$; see also [12, p. 23].

Proposition 3.15. *$\mathbb{L}_{<\omega^{\nu}}$ is ν-confluent. More precisely, for $0 < \mu \leqslant \nu$ and $f \in \mathbb{L}_{<\alpha}$, we have*

$$\mathfrak{d}_{\omega^{\mu}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}}. \tag{3.3}$$

Proof. We first note that $\mathbb{L}_{<\alpha}$ is 0-confluent as $\mathfrak{L}_{<\alpha}$ is not trivial. We proceed by induction on $0 < \mu \leqslant \nu$. Take $f \in \mathbb{L}_{<\alpha}^{>,\succ}$. If $\mu = 1$, then we have $L_1(\mathfrak{d}_1(f)) \asymp \ell_{\lambda_f + 1} = L_1(\ell_{\lambda_f})$ where ℓ_{λ_f} is $L_{<\omega}$-atomic, so $\mathfrak{d}_{\omega}(f) = \ell_{\lambda_f}$ and $\mathbb{L}_{<\alpha}$ is 1-confluent. It remains to note that $(\lambda_f)_{\geqslant 1} = \lambda_f$.

Now suppose that $\mu > 1$ and assume that $\mathbb{L}_{<\alpha}$ is η-confluent and satisfies (3.3) for all $\eta < \mu$. Suppose μ is a successor, so $\mathfrak{d}_{\omega^{\mu_*}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_{**}}}}$. Write $(\lambda_f)_{\geqslant \omega^{\mu_{**}}} = (\lambda_f)_{\geqslant \omega^{\mu_*}} + \omega^{\mu_{**}} n$ with $n \in \mathbb{N}$ and with $n = 0$ if μ_* is a limit. We have $\ell_{(\lambda_f)_{\geqslant \omega^{\mu_{**}}}} = \ell_{\omega^{\mu_{**}} n} \circ \ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}}$, so

$$L_{\omega^{\mu_*}}(\mathfrak{d}_{\omega^{\mu_*}}(f)) = (\ell_{\omega^{\mu_*}} \circ \ell_{\omega^{\mu_{**}} n}) \circ \ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}} = L_{\omega^{\mu_*}}(\ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}}) - n \asymp L_{\omega^{\mu_*}}(\ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}})$$

and $\mathfrak{d}_{\omega^{\mu}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}}$.

Now suppose μ is a limit, so there is $\eta < \mu$ with $(\lambda_f)_{\geqslant \omega^{\eta}} = (\lambda_f)_{\geqslant \omega^{\mu}} = (\lambda_f)_{\geqslant \omega^{\mu_*}}$. By hypothesis, we have that $\mathfrak{d}_{\omega^{\eta+1}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\eta}}}$ and so

$$L_{\omega^{\eta+1}}(\mathfrak{d}_{\omega^{\eta+1}}(f)) = L_{\omega^{\eta+1}}(\ell_{(\lambda_f)_{\geqslant \omega^{\eta}}}) = L_{\omega^{\eta+1}}(\ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}}).$$

Again, this yields $\mathfrak{d}_{\omega^{\mu}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_*}}}$. □

Theorem 3.16. *$\mathbb{L}_{<\alpha}$ is a confluent hyperserial skeleton of force ν.*

Proof. Using the identity

$$\ell_1 \circ \mathfrak{l} = \sum_{\gamma < \alpha} \mathfrak{l}_{\gamma} \ell_{\gamma+1}$$

for $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_{\gamma}^{\mathfrak{l}_{\gamma}} \in \mathfrak{L}_{<\alpha}$, the field $\mathbb{L}_{<\alpha}$ is easily seen to satisfy $\mathbf{FE}_0$, $\mathbf{A}_0$, $\mathbf{M}_0$, and $\mathbf{R}_0$. Moreover, $\mathbb{L}_{<\alpha}$ satisfies $\mathbf{FE}_{\mu}$ for all $0 < \mu < \nu$ by [12, Lemma 5.6]. Using Propositions 3.13, 3.14 and 3.15, we conclude that $\mathbb{L}_{<\alpha}$ is a confluent hyperserial skeleton of force ν. □

Corollary 3.17. *$\mathbb{L}$ is a confluent hyperserial skeleton of force* **On**.

4 Extending the partial functions

Let $\boldsymbol{\nu} \leqslant \mathbf{On}$. The purpose of the next two sections is to prove the following theorem:

Theorem 4.1. *Let $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu < \boldsymbol{\nu}})$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu}$. There is a unique function $\circ: \mathbb{L}_{<\omega^{\boldsymbol{\nu}}} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ satisfying:*

C1$_{\boldsymbol{\nu}}$. *$\mathbb{L}_{<\omega^{\boldsymbol{\nu}}} \longrightarrow \mathbb{T}; f \longmapsto f \circ s$ is a strongly $\mathbb{R}$-linear ordered field embedding for each $s \in \mathbb{T}^{>,\succ}$;*

C2$_\nu$. *$\ell_0^r \circ \mathfrak{m} = \mathfrak{m}^r$ for all $\mathfrak{m} \in \mathfrak{M}$ and $r \in \mathbb{R}$;*
$\ell_{\omega^\mu} \circ \mathfrak{a} = L_{\omega^\mu}(\mathfrak{a})$ for all $\mu < \nu$ and $\mathfrak{a} \in \operatorname{dom} L_{\omega^\mu}$;

C3$_\nu$. *$f \circ (g \circ s) = (f \circ g) \circ s$ for all $f \in \mathbb{L}_{<\omega^\nu}$, $g \in \mathbb{L}^{>,\succ}_{<\omega^\nu}$, and $s \in \mathbb{T}^{>,\succ}$;*

C4$_\nu$. *$f \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $f \in \mathbb{L}_{<\omega^\nu}$, $t \in \mathbb{T}^{>,\succ}$, and $\delta \in \mathbb{T}$ with $\delta \prec t$.*

We claim that it suffices to prove the theorem in the case when $\nu < \mathbf{On}$. The case when $\nu = \mathbf{On}$ can then be proved as follows: let $(\mathbb{T}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ be a confluent hyperserial skeleton of force $\mathbf{On}$. Then for every $\nu < \mathbf{On}$, there exists a unique composition $\circ_\nu : \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ that satisfies **C1$_\nu$**, **C2$_\nu$**, **C3$_\nu$**, and **C4$_\nu$**. For $\mu < \nu$, the composition $\circ_\nu$ extends $\circ_\mu$, by uniqueness. For any $f \in \mathbb{L}$ and $s \in \mathbb{T}^{>,\succ}$, we have $f \in \mathbb{L}_{<\nu}$ for some $\nu < \mathbf{On}$, so we may define $f \circ s := f \circ_\nu s$ and this definition does not depend on ν. It is straightforward to check that this defines the unique composition $\circ : \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ which satisfies **C1$_{\mathbf{On}}$**, **C2$_{\mathbf{On}}$**, **C3$_{\mathbf{On}}$**, and **C4$_{\mathbf{On}}$**.

Throughout this section, we fix an ordinal ν and a hyperserial skeleton $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of force ν. We fix also $\mu < \nu$ such that $\mathbb{T}$ is μ-confluent and we set $\beta := \omega^\mu$. We assume that Theorem 4.1 holds for μ, so we have a unique composition $\circ : \mathbb{L}_{<\beta} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ satisfying **C1$_\mu$**, **C2$_\mu$**, **C3$_\mu$**, and **C4$_\mu$**. For $\gamma < \beta$ and $s \in \mathbb{T}^{>,\succ}$, we write $L_\gamma(s) := \ell_\gamma \circ s$. In light of Lemma 2.9, the expression $(\ell_\beta^{\uparrow\gamma})^{(k)} \circ s$ makes sense for each $k > 0$. Our main goal is to prove the following result:

Proposition 4.2. *There is an extension of L_β to $\mathbb{T}^{>,\succ}$ such that for all $s \in \mathbb{T}^{>,\succ}$, $\mathfrak{a} \in \mathfrak{M}_\beta$, and $\gamma < \beta$ with $\varepsilon := L_\gamma(s) - L_\gamma(\mathfrak{a}) \prec 1$, we have*

$$L_\beta(s) = L_\beta(\mathfrak{a}) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_\beta^{\uparrow\gamma})^{(k)} \circ L_\gamma(\mathfrak{a})}{k!} \varepsilon^k.$$

We will also prove that L_β satisfies the extension of **FE$_\mu$** to $\mathbb{T}^{>,\succ}$ (Proposition 4.13), that L_β has Taylor expansions around every point (Proposition 4.15) and that it is strictly increasing on $\mathbb{T}^{>,\succ}$ (Lemma 4.17).

Our extension will heavily depend on Taylor series expansions, so it is convenient to introduce some notation for that. Let $f \in \mathbb{L}_{<\alpha}$ be such that $f^{(k)} \in \mathbb{L}_{<\beta}$ for all $k > 0$. Let $t \in \mathbb{T}^{>,\succ}$ and $\delta \in \mathbb{T}$ with $\delta \prec t$. By Lemma 2.8 with $\alpha = \beta$, f' in place of f, and $\Phi : \mathbb{L}_{<\beta} \longrightarrow \mathbb{T}; g \longmapsto g \circ t$, we have that the family $((f^{(k)} \circ t)\, \delta^k)_{k \in \mathbb{N}^>}$ is well-based. We define

$$\mathcal{T}_f(t, \delta) := \sum_{k \in \mathbb{N}^>} \frac{f^{(k)} \circ t}{k!} \delta^k \in \mathbb{T}.$$

4.1 Extending the logarithm

Here $\mu = 1$. For $\varepsilon \in \mathbb{T}^{\prec}$, we define

$$L(1 + \varepsilon) := \sum_{k \in \mathbb{N}^>} \frac{(-1)^{k-1}}{k} \varepsilon^k$$

and

$$E(\varepsilon) := \sum_{k \in \mathbb{N}} \frac{1}{k!} \varepsilon^k.$$

Note that $L(1+\varepsilon)\in\mathbb{T}^{\prec}$ and $E(\varepsilon)\in 1+\mathbb{T}^{\prec}$. By [25, Corollary 16], we have

$$E(L(1+\varepsilon))=1+\varepsilon, \qquad L(E(\varepsilon))=\varepsilon. \tag{4.1}$$

$$L(1+\varepsilon)\preccurlyeq\varepsilon. \tag{4.2}$$

$$E(rL(1+\varepsilon))=\sum_{k\in\mathbb{N}}\binom{r}{k}\varepsilon^k=(1+\varepsilon)^r \quad \text{for all } r\in\mathbb{R}. \tag{4.3}$$

Proposition 4.3. *There is a unique extension of L_1 into an ordered group embedding*

$$\log\colon(\mathbb{T}^{>},\times,<)\longrightarrow(\mathbb{T},+,<)$$

with

- $\log(1+\varepsilon)=L(1+\varepsilon)$ *for all $\varepsilon\in\mathbb{T}^{\prec}$, and*
- $\log$ *extends the natural logarithm on $\mathbb{R}^{>}$.*

For $s\in\mathbb{T}^{>}$, writing $s=c\,\mathfrak{m}\,(1+\varepsilon)$ for $c\in\mathbb{R}^{>}$, $\mathfrak{m}:=\mathfrak{d}_s$, and $\varepsilon\in\mathbb{T}^{\prec}$, we have

$$\log s=\log\mathfrak{m}+\log c+L(1+\varepsilon).$$

Proof. The uniqueness and the fact that $(\mathbb{T}^{>},\times)\longrightarrow(\mathbb{T},+)$ is a morphism are proven in [29, Example 2.1.3 and Lemma 2.1.4]. To see that log is order-preserving, we need only check that $\log s>0$ for all $s>1$. If $\mathfrak{m}=c=1$, then $\varepsilon>0$ and we have $\log s=L(1+\varepsilon)\sim\varepsilon>0$. If $\mathfrak{m}=1$ and $c>1$, then we have $\log s=\log c+L(1+\varepsilon)\sim\log c>0$, since $L(1+\varepsilon)\prec 1$. If $\mathfrak{m}\succ 1$, we have $\log s\sim\log\mathfrak{m}>\mathbb{R}$, so $\log s>0$. □

For $s\in\mathbb{T}^{>,\succ}$, we often write $L_1(s)$ in place of $\log s$. For $s\in\log\mathbb{T}^{>}$, we define $\exp s$ to be the unique element of $\mathbb{T}$ with $s=\log\exp s$. If $s\in\log\mathbb{T}^{>,\succ}=L_1(\mathbb{T}^{>,\succ})\subseteq\mathbb{T}^{>,\succ}$, then we sometimes use $E_1(s)$ instead of $\exp s$.

Proposition 4.4. *For $s\in\mathbb{T}^{>}$, we have $\log s\leqslant s-1$.*

Proof. Let $s=c\,\mathfrak{m}\,(1+\varepsilon)\in\mathbb{T}^{>}$, where $c\in\mathbb{R}^{>}$, $\mathfrak{m}:=\mathfrak{d}_s$, and $\varepsilon\in\mathbb{T}^{\prec}$. If $\mathfrak{m}=1$, then $\log\mathfrak{m}=0$, so $\log s=\log c+L(1+\varepsilon)$. If $c=1$, then $\log s=L(1+\varepsilon)\leqslant\varepsilon=s-1$. If $c>1$, then $\log c<(c-1)\,(1+\varepsilon)$, since $\log c, c-1\in\mathbb{R}$, $\varepsilon\prec 1$, and $\log c<c-1$. Thus

$$\log s<(c-1)\,(1+\varepsilon)+L(1+\varepsilon)=c\,(1+\varepsilon)-(1+\varepsilon)+L(1+\varepsilon)\leqslant c\,(1+\varepsilon)-1=s-1.$$

If $\mathfrak{m}\succ 1$, then $\log\mathfrak{m}=L_1(\mathfrak{m})\prec\mathfrak{m}$ and $\log s-\log\mathfrak{m}=\log c+L(1+\varepsilon)\preccurlyeq 1$. Hence $\log s\prec s$ and $\log s\leqslant s-1$. If $\mathfrak{m}\prec 1$, then $\log\mathfrak{m}=-L_1(\mathfrak{m}^{-1})$ is negative and infinite, so $\log s<-1\leqslant s-1$. □

Remark 4.5. Proposition 4.4 proves that $(\mathbb{T},\log)$ satisfies the properties of transseries fields in [29, Definition 2.2.1] except possibly for the axiom **T4**.

Proposition 4.6. *For $s\in\mathbb{T}^{>}$, and $r\in\mathbb{R}$, we have $\log s^r=r\log s$.*

Proof. First, note that $\log\mathfrak{m}^r=r\log\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{M}$: if $\mathfrak{m}\succ 1$, then this is just axiom $\mathbf{FE_0}$; if $\mathfrak{m}\prec 1$, then $\log\mathfrak{m}^r=-\log\mathfrak{m}^{-r}=r\log\mathfrak{m}$; if $\mathfrak{m}=1$, then $\log\mathfrak{m}^r=0=r\log\mathfrak{m}$. Now, writing $s=c\,\mathfrak{m}\,(1+\varepsilon)$ with $c\in\mathbb{R}^{>}$, $\mathfrak{m}:=\mathfrak{d}_s$, and $\varepsilon\prec 1$, we have

$$\begin{aligned}
\log(s^r) &= \log(\mathfrak{m}^r)+\log c^r+L((1+\varepsilon)^r) \\
&= \log(\mathfrak{m}^r)+\log c^r+L(E(rL(1+\varepsilon))) && \text{(by (4.3))} \\
&= r\log\mathfrak{m}+r\log c+rL(1+\varepsilon) && \text{(by (4.1))} \\
&= r\log(s).
\end{aligned}$$

□

Proposition 4.7. *For $s \in \mathbb{T}^{>,\succ}$ and $\delta \in \mathbb{T}$ with $\delta \prec s$, the family $((\ell_1^{(k)} \circ s)\, \delta^k)_{k \in \mathbb{N}^>}$ is well-based, with*

$$\log(s+\delta) = \log s + \mathcal{T}_{\ell_1}(s,\delta).$$

Proof. For $k \in \mathbb{N}^>$ and $s \in \mathbb{T}^{>,\succ}$, we have $\ell_1^{(k)} \in \mathbb{L}_{<1}$ and $\ell_1^{(k)} \circ s = (-1)^{k-1}\,(k-1)!\, s^{-k}$. For $\delta \prec s$, we have

$$\frac{(\ell_1^{(k)} \circ s)\, \delta^k}{k!} = \frac{(-1)^{k-1}}{k} \left(\frac{\delta}{s}\right)^k.$$

So the family $((\ell_1^{(k)} \circ s)\, \delta^k)_{k \in \mathbb{N}^>}$ is well-based with $\mathcal{T}_{\ell_1}(s,\delta) = \sum_{k \in \mathbb{N}^>} \frac{(\ell_1^{(k)} \circ s)\, \delta^k}{k!} = L\Big(1 + \frac{\delta}{s}\Big)$. The proposition follows, as

$$\log(s+\delta) = \log\left(s\left(1+\frac{\delta}{s}\right)\right) = \log s + L\left(1+\frac{\delta}{s}\right). \qquad \square$$

4.2 Extending the hyperlogarithms

Assume now that $\mu > 0$. Let us revisit the notion of confluence.

Lemma 4.8. *Let $s,t \in \mathbb{T}^{>,\succ}$ and suppose that $L_\gamma(s) \asymp L_\gamma(t)$ for some $\gamma < \beta$. Then $L_\sigma(s) - L_\sigma(t) \prec 1$ for all $\sigma < \beta$ with $\sigma \geqslant \gamma + 2$.*

Proof. We first show that $L_{\gamma+2}(s) - L_{\gamma+2}(t) \prec 1$. Take $c \in \mathbb{R}^>$ and $\varepsilon \prec 1$ such that $L_\gamma(s) = L_\gamma(t)\,(c+\varepsilon)$. We have

$$L_{\gamma+1}(s) = L_1(L_\gamma(s)) = L_{\gamma+1}(t) + \log(c+\varepsilon),$$

where $\log(c+\varepsilon) \preccurlyeq 1$. Set $\delta := L_{\gamma+1}(t)^{-1} \log(c+\varepsilon) \prec 1$, so $L_{\gamma+1}(s) = L_{\gamma+1}(t)\,(1+\delta)$. We have

$$L_{\gamma+2}(s) = \log L_{\gamma+1}(s) = L_{\gamma+2}(t) + \log(1+\delta),$$

where $\log(1+\delta) \sim \delta \prec 1$. Thus, $L_{\gamma+2}(s) - L_{\gamma+2}(t) \prec 1$.

Now, fix σ with $\gamma + 2 \leqslant \sigma < \beta$ and set $\delta := L_{\gamma+2}(s) - L_{\gamma+2}(t)$. By **C3**$_\mu$ and **C4**$_\mu$, we have

$$L_\sigma(s) = \ell_\sigma^{\uparrow\gamma+2} \circ L_{\gamma+2}(s) = \ell_\sigma^{\uparrow\gamma+2} \circ L_{\gamma+2}(t) + \mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(L_{\gamma+2}(t),\delta) = L_\sigma(t) + \mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(L_{\gamma+2}(t),\delta).$$

Lemma 2.9 in conjunction with the fact that $\delta \prec 1$ gives us that $\mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(L_{\gamma+2}(t),\delta) \prec 1$, so $L_\sigma(s) - L_\sigma(t) \prec 1$. $\square$

Proposition 4.9. *For all $s \in \mathbb{T}^{>,\succ}$, we have*

$$\mathcal{E}_\beta[s] = \{t \in \mathbb{T}^{>,\succ} : L_\gamma(s) - L_\gamma(t) \prec 1 \text{ for some } \gamma < \beta\}.$$

Proof. We fix $s \in \mathbb{T}^{>,\succ}$. Since $\mu > 0$, we know by Lemma 4.8 that it is enough to show that $\mathcal{E}_\beta[s] = \{t \in \mathbb{T}^{>,\succ} : L_\gamma(s) \asymp L_\gamma(t) \text{ for some } \gamma < \beta\}$. We proceed by induction on μ. If $\mu = 1$, then $\beta = \omega$ and

$$\mathcal{E}_\omega[s] = \{t \in \mathbb{T}^{>,\succ} : (L_1 \circ \mathfrak{d}_1)^{\circ n}(t) \asymp (L_1 \circ \mathfrak{d}_1)^{\circ n}(s) \text{ for some } n \in \mathbb{N}\}.$$

An easy induction on n yields $(L_1 \circ \mathfrak{d}_1)^{\circ n}(t) \asymp L_n(t)$ for each $t \in \mathbb{T}^{>,\succ}$, whence the result.

Now suppose that $\mu > 1$. If μ is a successor, then for each $t \in \mathcal{E}_\beta[s]$ there is some $n \in \mathbb{N}$ with $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(t) \asymp L_{\omega^{\mu_*} n}(\mathfrak{d}_\beta(t))$. By our inductive assumption applied to μ_*, we have that $L_\gamma(t) - L_\gamma(\mathfrak{d}_{\omega^{\mu_*}}(t)) \prec 1$ for some $\gamma < \omega^{\mu_*}$. By Lemma 4.8, we have $L_{\omega^{\mu_*}}(t) - L_{\omega^{\mu_*}}(\mathfrak{d}_{\omega^{\mu_*}}(t)) \prec 1$ and an easy induction on n gives us that $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(t) - L_{\omega^{\mu_*} n}(t) \prec 1$. Thus, we have that $L_{\omega^{\mu_*} n}(t) \asymp L_{\omega^{\mu_*} n}(\mathfrak{d}_\beta(t))$ for some $n \in \mathbb{N}$. Likewise, $L_{\omega^{\mu_*} m}(s) \asymp L_{\omega^{\mu_*} m}(\mathfrak{d}_\beta(s))$ for some $m \in \mathbb{N}$. By replacing m and n with $\max\{m, n\}$ and invoking Lemma 4.8, we may assume that $m = n$. Since $\mathfrak{d}_\beta(s) = \mathfrak{d}_\beta(t)$, we have $L_{\omega^{\mu_*} n}(s) \asymp L_{\omega^{\mu_*} n}(t)$. On the other hand, given $t \in \mathbb{T}^{>,\succ}$, if $L_\gamma(s) \asymp L_\gamma(t)$ for some $\gamma < \beta$, then take some $n \in \mathbb{N}$ with $\gamma + 2 \leqslant \omega^{\mu_*} n < \beta$. By Lemma 4.8, we have $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(s) \asymp L_{\omega^{\mu_*} n}(s) \asymp L_{\omega^{\mu_*} n}(t) \asymp (L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(t)$, so $t \in \mathcal{E}_\beta[s]$.

If μ is a limit, then for each $t \in \mathcal{E}_\beta[s]$ there is $\eta < \mu$ with $L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t)) \asymp L_{\omega^\eta}(\mathfrak{d}_\beta(t))$. By our inductive assumption applied to η, we have that $L_\gamma(t) - L_\gamma(\mathfrak{d}_{\omega^\eta}(t)) \prec 1$ for some $\gamma < \omega^\eta$, so $L_{\omega^\eta}(t) - L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t)) \prec 1$ by Lemma 4.8. Thus $L_{\omega^\eta}(t) \asymp L_{\omega^\eta}(\mathfrak{d}_\beta(t))$ and likewise, $L_{\omega^\sigma}(s) \asymp L_{\omega^\sigma}(\mathfrak{d}_\beta(s))$ for some $\sigma < \mu$. By replacing η and σ with $\max\{\eta, \sigma\}$ and invoking Lemma 4.8, we may assume that $\eta = \sigma$. Since $\mathfrak{d}_\beta(s) = \mathfrak{d}_\beta(t)$, we have $L_{\omega^\eta}(s) \asymp L_{\omega^\eta}(t)$. On the other hand, given $t \in \mathbb{T}^{>,\succ}$, if $L_\gamma(s) \asymp L_\gamma(t)$ for some $\gamma < \beta$, then take some η with $\gamma \leqslant \omega^\eta < \beta$. By Lemma 4.8, we have $L_{\omega^\eta}(\mathfrak{d}_\beta(s)) \asymp L_{\omega^\eta}(s) \asymp L_{\omega^\eta}(t) \asymp L_{\omega^\eta}(\mathfrak{d}_\beta(t))$, so $t \in \mathcal{E}_\beta[s]$. □

Proposition 4.9 in conjunction with Lemma 4.8 gives us the following corollary:

Corollary 4.10. *For each $s \in \mathbb{T}^{>,\succ}$ there is $\gamma < \beta$ such that*

$$L_\rho(s) - L_\rho(\mathfrak{d}_\beta(s)) \prec 1,$$

for all $\gamma \leqslant \rho < \beta$. Moreover, if $L_\gamma(s) - L_\gamma(\mathfrak{a}) \prec 1$ for some $\mathfrak{a} \in \mathfrak{M}_\beta$ and some $\gamma < \beta$, then $\mathfrak{a} = \mathfrak{d}_\beta(s)$.

Definition 4.11. *Let $s \in \mathbb{T}^{>,\succ}$ and let $\gamma < \beta$ with $\varepsilon := L_\gamma(s) - L_\gamma(\mathfrak{d}_\beta(s)) \prec 1$. We define*

$$L_\beta(s) := L_\beta(\mathfrak{d}_\beta(s)) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon).$$

As discussed at the beginning of the section, the series $\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon)$ exists in $\mathbb{T}$ by Lemmas 2.8 and 2.9. To prove Proposition 4.2 all that remains is to show:

Lemma 4.12. *The above definition does not depend on the choice of γ.*

Proof. Let s, γ, ε be as in Definition 4.11 and suppose that $L_\sigma(s) - L_\sigma(\mathfrak{d}_\beta(s)) \prec 1$ for some $\sigma < \beta$. Set $\delta := L_\sigma(s) - L_\sigma(\mathfrak{d}_\beta(s))$. We need to show that

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon) = \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta).$$

Without loss of generality, we may assume that $\sigma \leqslant \gamma$. Now

$$\begin{aligned} L_\gamma(\mathfrak{d}_\beta(s)) + \varepsilon \;=\; L_\gamma(s) &= \ell_\gamma^{\uparrow\sigma} \circ L_\sigma(s) \\ &= \ell_\gamma^{\uparrow\sigma} \circ (L_\sigma(\mathfrak{d}_\beta(s)) + \delta) \\ &= \ell_\gamma^{\uparrow\sigma} \circ L_\sigma(\mathfrak{d}_\beta(s)) + \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta). \end{aligned}$$

Since $\ell_\gamma^{\uparrow\sigma} \circ L_\sigma(\mathfrak{d}_\beta(s)) = L_\gamma(\mathfrak{d}_\beta(s))$, this yields $\varepsilon = \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta)$. Let

$$F(X) := \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}\big(\ell_\gamma, \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(\ell_\sigma, X)\big), \quad G(X) := \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(\ell_\sigma, X),$$

considered as formal power series $F(X) = \sum_{i \in \mathbb{N}} F_i X^i$ and $G(X) = \sum_{j \in \mathbb{N}} G_j X^j$ in $\mathbb{L}_{<\alpha}[[X]]$. Then

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon) = \sum_{i \in \mathbb{N}} (F_i \circ \mathfrak{d}_\beta(s))\, \delta^i \quad \text{and} \quad \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta) = \sum_{j \in \mathbb{N}} (G_j \circ \mathfrak{d}_\beta(s))\, \delta^j,$$

so it suffices to show that $F=G$. For each $h\in\mathbb{L}_{<\alpha}^{\prec}$, we have

$$\begin{aligned}F(h) &= \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,\mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(\ell_\sigma,h))=\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,\ell_\gamma^{\uparrow\sigma}\circ(\ell_\sigma+h)-\ell_\gamma)=\ell_\beta^{\uparrow\gamma}\circ(\ell_\gamma+\ell_\gamma^{\uparrow\sigma}\circ(\ell_\sigma+h)-\ell_\gamma)-\ell_\beta\\ &= (\ell_\beta^{\uparrow\gamma}\circ\ell_\gamma^{\uparrow\sigma})\circ(\ell_\sigma+h)-\ell_\beta=\ell_\beta^{\uparrow\sigma}\circ(\ell_\sigma+h)-\ell_\beta=\mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(\ell_\sigma,h)=G(h),\end{aligned}$$

so $(F-G)(h)=0$ for all $h\in\mathbb{L}_{<\alpha}^{\prec}$, and we conclude that $F=G$ by Lemma 2.7. □

We end this section with various extensions of the validity of Taylor series expansions and a proof that L_β is strictly increasing on $\mathbb{T}^{>,\succ}$.

Proposition 4.13. *Assume μ is a successor. For $s\in\mathbb{T}^{>,\succ}$, we have $L_\beta(L_{\omega^{\mu*}}(s))=L_\beta(s)-1$.*

Proof. By Corollary 4.10, there is some $n\in\mathbb{N}^{>}$ such that $\varepsilon:=L_{\omega^{\mu*}n}(s)-L_{\omega^{\mu*}n}(\mathfrak{d}_\beta(s))\prec 1$. We may write

$$L_{\omega^{\mu*}(n-1)}(L_{\omega^{\mu*}}(s))=L_{\omega^{\mu*}(n-1)}(L_{\omega^{\mu*}}(\mathfrak{d}_\beta(s)))+\varepsilon.$$

Note that $L_{\omega^{\mu*}}(\mathfrak{d}_\beta(s))$ is $L_{<\beta}$-atomic, so $\mathfrak{d}_\beta L_{\omega^{\mu*}}(s)=L_{\omega^{\mu*}}(\mathfrak{d}_\beta(s))$. For $k\in\mathbb{N}^{>}$ we have

$$(\ell_\beta^{\uparrow\omega^{\mu*}(n-1)})^{(k)}=(\ell_\beta+(n-1))^{(k)}=\ell_\beta^{(k)}=(\ell_\beta+n)^{(k)}=(\ell_\beta^{\uparrow\omega^{\mu*}n})^{(k)},$$

so $\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu*}(n-1)}}(\mathfrak{a},\varepsilon)=\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu*}n}}(\mathfrak{a},\varepsilon)$ for all $\mathfrak{a}\in\mathfrak{M}_\beta$. It follows that

$$\begin{aligned}L_\beta(L_{\omega^{\mu*}}(s)) &= L_\beta(L_{\omega^{\mu*}}(\mathfrak{d}_\beta(s)))+\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu*}(n-1)}}(L_{\omega^{\mu*}n}(\mathfrak{d}_\beta(s)),\varepsilon) && \text{(by Definition 4.11)}\\ &= L_\beta(\mathfrak{d}_\beta(s))-1+\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu*}n}}(L_{\omega^{\mu*}n}(\mathfrak{d}_\beta(s)),\varepsilon) && \text{(by }\mathbf{FE}_\mu)\\ &= L_\beta(s)-1 && \text{(by Definition 4.11)}\end{aligned}$$

This concludes the proof. □

Lemma 4.14. *For all $\gamma<\beta$, $\mathfrak{a}\in\mathfrak{M}_\beta$, and $\delta,\varepsilon\prec 1$, we have*

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),\delta)+\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a})+\delta,\varepsilon)=\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),\delta+\varepsilon).$$

Proof. By applying $\mathbf{C4}_\mu$ to $(\ell_\beta^{\uparrow\gamma})^{(k)}$ for $k>0$, we have

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a})+\delta,\varepsilon)=\sum_{k\in\mathbb{N}^{>}}\frac{1}{k!}\left((\ell_\beta^{\uparrow\gamma})^{(k)}\circ L_\gamma(\mathfrak{a})+\mathcal{T}_{(\ell_\beta^{\uparrow\gamma})^{(k)}}(L_\gamma(\mathfrak{a}),\delta)\right)\varepsilon^k.$$

Arguing as in the proof of Lemma 4.12, it is enough to show that

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,X)+\sum_{k\in\mathbb{N}^{>}}\frac{1}{k!}\left((\ell_\beta^{\uparrow\gamma})^{(k)}\circ\ell_\gamma+\mathcal{T}_{(\ell_\beta^{\uparrow\gamma})^{(k)}}(\ell_\gamma,X)\right)Y^k=\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,X+Y)$$

as power series in $\mathbb{L}_{<\beta}[[X,Y]]$. Let $f,g\in\mathbb{L}_{<\beta}^{\prec}$. We have

$$\begin{aligned}\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,f) &= \ell_\beta^{\uparrow\gamma}\circ(\ell_\gamma+f)-\ell_\beta,\\ \sum_{k\in\mathbb{N}^{>}}\frac{1}{k!}\left((\ell_\beta^{\uparrow\gamma})^{(k)}\circ\ell_\gamma+\mathcal{T}_{(\ell_\beta^{\uparrow\gamma})^{(k)}}(\ell_\gamma,f)\right)g^k &= \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma+f,g)=\ell_\beta^{\uparrow\gamma}\circ(\ell_\gamma+f+g)-\ell_\beta^{\uparrow\gamma}\circ(\ell_\gamma+f).\end{aligned}$$

Therefore,

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,f)+\sum_{k\in\mathbb{N}^{>}}\frac{1}{k!}\left((\ell_\beta^{\uparrow\gamma})^{(k)}\circ\ell_\gamma+\mathcal{T}_{(\ell_\beta^{\uparrow\gamma})^{(k)}}(\ell_\gamma,f)\right)g^k=\ell_\beta^{\uparrow\gamma}\circ(\ell_\gamma+f+g)-\ell_\beta.$$

Since $\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma,f+g)=\ell_\beta^{\uparrow\gamma}\circ(\ell_\gamma+f+g)-\ell_\beta$, we are done by Lemma 2.7. □

Proposition 4.15. *For $t \in \mathbb{T}^{>,\succ}$ and $\delta \in \mathbb{T}$ with $\delta \prec t$, we have*

$$L_\beta(t+\delta) = L_\beta(t) + \mathcal{T}_{\ell_\beta}(t,\delta).$$

Proof. Let $\gamma < \beta$ with $L_\gamma(t) - L_\gamma(\mathfrak{d}_\beta(t)) \prec 1$. Set $\varepsilon := L_\gamma(t+\delta) - L_\gamma(t)$. By Lemmas 2.8 and 2.9, the series $\mathcal{T}_{\ell_\beta}(t,\delta)$ exists in $\mathbb{T}$. We claim that $\mathcal{T}_{\ell_\beta}(t,\delta) = \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(t),\varepsilon)$. As $\varepsilon = \mathcal{T}_{\ell_\gamma}(\ell_0,\delta)$ by $\mathbf{C4}_\mu$, it suffices to show that

$$\mathcal{T}_{\ell_\beta}(\ell_0, X) = \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma, \mathcal{T}_{\ell_\gamma}(\ell_0, X))$$

as power series in $\mathbb{L}_{<\beta}[[X]]$. But this follows from Lemma 2.7, by noting that the equality holds when evaluated at any element of $\mathbb{L}_{<\beta}^{\prec}$.

Now set $\mathfrak{a} := \mathfrak{d}_\beta(t) = \mathfrak{d}_\beta(t+\delta)$ and let $h := L_\gamma(t) - L_\gamma(\mathfrak{a})$, so $L_\gamma(t+\delta) - L_\gamma(\mathfrak{a}) = h + \varepsilon$. By definition of $L_\beta(t)$ and the above claim, we have

$$L_\beta(t) + \mathcal{T}_{\ell_\beta}(t,\delta) = L_\beta(\mathfrak{a}) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),h) + \mathcal{T}_{\ell_\beta}(t,\delta) = L_\beta(\mathfrak{a}) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),h) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(t),\varepsilon).$$

Using $L_\gamma(t) = L_\gamma(\mathfrak{a}) + h$, it follows that

$$L_\beta(t) + \mathcal{T}_{\ell_\beta}(t,\delta) = L_\beta(\mathfrak{a}) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),h) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}) + h,\varepsilon).$$

By Lemma 4.14, we conclude that

$$L_\beta(t) + \mathcal{T}_{\ell_\beta}(t,\delta) = L_\beta(\mathfrak{a}) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),h+\varepsilon) = L_\beta(t+\delta),$$

where the last equality follows from the definition of $L_\beta(t+\delta)$. □

Lemma 4.16. *Let $\mathfrak{a} \in \mathfrak{M}_\beta$, let $s,t \in \mathcal{E}_\beta[\mathfrak{a}]$, and let $\gamma < \beta$ with*

$$L_\gamma(s) - L_\gamma(\mathfrak{a}) \prec 1, \qquad L_\gamma(t) - L_\gamma(\mathfrak{a}) \prec 1.$$

Then $L_\beta(t) = L_\beta(s) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(s),\varepsilon)$ where $\varepsilon := L_\gamma(t) - L_\gamma(s)$.

Proof. Set $\delta := L_\gamma(s) - L_\gamma(\mathfrak{a})$, so $\delta + \varepsilon = L_\gamma(t) - L_\gamma(\mathfrak{a})$. We have

$$\begin{aligned} L_\beta(t) &= L_\beta(\mathfrak{a}) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),\delta+\varepsilon) \\ &= L_\beta(\mathfrak{a}) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),\delta) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a})+\delta,\varepsilon) \\ &= L_\beta(s) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(s),\varepsilon) \end{aligned}$$

where the first and third equalities follow from the definition of L_β and where the second equality holds by Lemma 4.14. □

Lemma 4.17. *The function L_β is strictly increasing on $\mathbb{T}^{>,\succ}$.*

Proof. By induction on μ, we may assume that L_{ω^η} is strictly increasing on $\mathbb{T}^{>,\succ}$ for all $\eta < \mu$ (the $\eta = 0$ case follows from Proposition 4.3). As a composition of strictly increasing functions is strictly increasing, the function L_γ is strictly increasing on $\mathbb{T}^{>,\succ}$ for all $\gamma < \beta$. Given $s < t \in \mathbb{T}^{>,\succ}$, let us show that $L_\beta(s) < L_\beta(t)$. We start with the case when $\mathfrak{d}_\beta(s) = \mathfrak{d}_\beta(t) =: \mathfrak{a}$ and take $\gamma < \beta$ with $L_\gamma(s) - L_\gamma(\mathfrak{a}) \prec 1$ and $L_\gamma(t) - L_\gamma(\mathfrak{a}) \prec 1$. Then $\varepsilon := L_\gamma(t) - L_\gamma(s)$ is infinitesimal and positive by our induction hypothesis. By Lemma 4.16, we have

$$L_\beta(t) - L_\beta(s) = \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(s),\varepsilon) \sim ((\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(s))\,\varepsilon.$$

Since $\ell_\beta^{\uparrow\lambda} > \mathbb{R}$, we have $(\ell_\beta^{\uparrow\lambda})' > 0$, so $L_\beta(t) - L_\beta(s) > 0$.

Now we turn to the case when $\mathfrak{d}_\beta(s) \prec \mathfrak{d}_\beta(t)$. Set $\mathfrak{a} := \mathfrak{d}_\beta(s)$ and $\mathfrak{b} := \mathfrak{d}_\beta(t)$ and take an ordinal $\lambda := \omega^\eta n < \beta$ with

$$L_\lambda(s) - L_\lambda(\mathfrak{a}) \prec 1 \quad \text{and} \quad L_\lambda(t) - L_\lambda(\mathfrak{b}) \prec 1.$$

Set $\delta := L_\lambda(s) - L_\lambda(\mathfrak{a})$, so

$$L_\beta(s) - L_\beta(\mathfrak{a}) = \mathcal{T}_{\ell_\beta^{\uparrow\lambda}}(L_\lambda(\mathfrak{a}), \delta) \sim ((\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(\mathfrak{a}))\,\delta \prec (\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(\mathfrak{a})$$

Repeated applications of (2.4) with η in place of μ gives $\ell_\beta^{\uparrow\lambda} \sim \ell_\beta$, so $(\ell_\beta^{\uparrow\lambda})' \sim \ell_\beta'$ and

$$(\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(\mathfrak{a}) \sim \ell_\beta' \circ L_\lambda(\mathfrak{a}).$$

Since $\beta > 1$, we have $\ell_\beta \prec \ell_1$ so $\ell_\beta' \prec \ell_1' = \ell_0^{-1}$. Thus, $\ell_\beta' \circ L_\lambda(\mathfrak{a}) \prec L_\lambda(\mathfrak{a})^{-1}$. All together, this shows that $L_\beta(s) - L_\beta(\mathfrak{a}) \prec L_\lambda(\mathfrak{a})^{-1}$. Likewise, we have $L_\beta(t) - L_\beta(\mathfrak{b}) \prec L_\lambda(\mathfrak{b})^{-1}$. By the monotonicity axiom $\mathbf{M}_\mu$, we have $L_\beta(\mathfrak{a}) + L_\lambda(\mathfrak{a})^{-1} < L_\beta(\mathfrak{b}) - L_\lambda(\mathfrak{b})^{-1}$, so $L_\beta(s) < L_\beta(t)$. □

5 Compositions on confluent hyperserial skeletons

Throughout this section, ν stands for a fixed ordinal and $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ for a fixed confluent hyperserial skeleton of force ν. We let $\alpha := \omega^\nu$. Our aim is to construct a well-behaved external composition $\mathbb{L}_{<\alpha} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ that satisfies **C1**$_\nu$, **C2**$_\nu$, **C3**$_\nu$, and **C4**$_\nu$ from Theorem 4.1. We will also prove that the mapping $\mathfrak{L}_{<\alpha} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ s$ has relatively well-based support for all $s \in \mathbb{T}^{>,\succ}$. Throughout the section, we make the inductive assumption that Theorem 4.1 holds for all $\mu < \nu$ and that the mapping $\mathfrak{L}_{<\omega^\mu} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ s$ has relatively well-based support for all $\mu < \nu$ and $s \in \mathbb{T}^{>,\succ}$.

5.1 The case when $\nu = 0$

Here $\mathbb{T}$ is a 0-confluent hyperserial skeleton of force 0. The field $\mathbb{L}_{<1} \cong \mathbb{R}[[\ell_0^{\mathbb{R}}]]$ is the field of well-based series of real powers of the variable ℓ_0, with real coefficients.

Lemma 5.1. *If $I \subseteq \mathbb{R}$ is well-based, then $(s^r)_{r\in I}$ is well-based for all $s \in \mathbb{T}^{>,\succ}$.*

Proof. Let $s = c\,\mathfrak{m}\,(1+\varepsilon)$ with $c \in \mathbb{R}^>$, $\mathfrak{m} \in \mathfrak{M}^\succ$ and $\varepsilon \prec 1$. Note that $\operatorname{supp} s^r \subseteq \mathfrak{m}^r\,(\operatorname{supp} \varepsilon)^\infty$. Since $(\mathfrak{m}^r)_{r\in I}$ and $(\operatorname{supp} \varepsilon)^\infty$ are both well-based, we are done. □

Given $f = \sum_{r\in\mathbb{R}} f_r\,\ell_0^r \in \mathbb{L}_{<1}$ and $s \in \mathbb{T}^{>,\succ}$, the family $(f_r\,s^r)_{r\in\mathbb{R}}$ is well-based by the above lemma, so we may define

$$f \circ s := \sum_{r\in\mathbb{R}} f_r s^r.$$

One easily verifies that this composition satisfies **C1**$_0$ and **C2**$_0$. We next prove

Proposition 5.2. *Let $r \in \mathbb{R}$, $g \in \mathbb{L}_{<1}^>$, and $s \in \mathbb{T}^{>,\succ}$. We have $g^r \circ s = (g \circ s)^r$.*

Proof. Write $g = c\,\mathfrak{m}\,(1+\varepsilon)$ where $c \in \mathbb{R}^>$, $\mathfrak{m} := \mathfrak{d}_g$, and $\varepsilon \prec 1$. We have $g^r = c^r\,\mathfrak{m}^r \sum_{k\in\mathbb{N}} \binom{r}{k} \varepsilon^k$, so $g^r \circ s = c^r\,(\mathfrak{m}^r \circ s) \sum_{k\in\mathbb{N}} \binom{r}{k} \varepsilon^k \circ s$. We also have

$$(g \circ s)^r = c^r\,(\mathfrak{m} \circ s)^r\,(1 + \varepsilon \circ s)^r = c^r\,(\mathfrak{m} \circ s)^r \sum_{k\in\mathbb{N}} \binom{r}{k} (\varepsilon \circ s)^k.$$

Since $\varepsilon^k \circ s = (\varepsilon \circ s)^k$ by **C1**$_0$, we only need to show that $(\mathfrak{m} \circ s)^r = \mathfrak{m}^r \circ s$. Now $\mathfrak{m} = \ell_0^t$ for some $t \in \mathbb{R}$, so

$$(\mathfrak{m} \circ s)^r = (\ell_0^t \circ s)^r = s^{tr} = \ell_0^{tr} \circ s = \mathfrak{m}^r \circ s$$

by Proposition 2.6. □

Corollary 5.3. *Let $f \in \mathbb{L}_{<1}$, $g \in \mathbb{L}_{<1}^{>,>}$, and $s \in \mathbb{T}^{>,>}$. We have $f \circ (g \circ s) = (f \circ g) \circ s$.*

Proof. We have

$$f \circ (g \circ s) = \sum_{r \in \mathbb{R}} f_r\, (g \circ s)^r = \sum_{r \in \mathbb{R}} f_r\, (g^r \circ s) = \left(\sum_{r \in \mathbb{R}} f_r g^r \right) \circ s = (f \circ g) \circ s,$$

where the second equality follows from Proposition 5.2 and the third follows from strong linearity of composition with s. □

Proposition 5.4. *For $f \in \mathbb{L}_{<1}$, $t \in \mathbb{T}^{>,>}$ and $\delta \in \mathbb{T}$ with $\delta \prec t$, we have $f \circ (t + \delta) = f \circ t + \mathcal{T}_f(t, \delta)$.*

Proof. We first handle the case when $f = \ell_0^r$, where $r \in \mathbb{R}$. We have $\frac{\delta}{t} \prec 1$, so

$$\left(1 + \frac{\delta}{t}\right)^r = \sum_{k \in \mathbb{N}} \binom{r}{k} \left(\frac{\delta}{t}\right)^k.$$

For $k \in \mathbb{N}$, we also have $(\ell_0^r)^{(k)} = k! \binom{r}{k} \ell_0^{r-k}$, so $(\ell_0^r)^{(k)} \circ t = k! \binom{r}{k} t^{r-k}$. Therefore,

$$\begin{aligned} (t + \delta)^r &= t^r \left(1 + \frac{\delta}{t}\right)^r = t^r \sum_{k \in \mathbb{N}} \binom{r}{k} \left(\frac{\delta}{t}\right)^k = t^r \sum_{k \in \mathbb{N}} \frac{(\ell_0^r)^{(k)} \circ t}{k!\, t^{r-k}} \left(\frac{\delta}{t}\right)^k \\ &= \sum_{k \in \mathbb{N}} \frac{(\ell_0^r)^{(k)} \circ t}{k!} \delta^k = t^r + \mathcal{T}_{\ell_0^r}(t, \delta). \end{aligned}$$

Now, Lemma 2.8 gives that the the map $\mathbb{L}_{<1} \longrightarrow \mathbb{T}; f \longmapsto f \circ t + \mathcal{T}_f(t, \delta)$ is well-based and strongly linear, so for a general $f = \sum_{r \in \mathbb{R}} f_r\, \ell_0^r \in \mathbb{L}_{<1}$, we have

$$f \circ (t + \delta) = \sum_{r \in \mathbb{R}} f_r\, (t + \delta)^r = \sum_{r \in \mathbb{R}} f_r t^r + f_r\, \mathcal{T}_{\ell_0^r}(t, \delta) = f \circ t + \mathcal{T}_f(t, \delta). \qquad □$$

The above results show that $\circ$ satisfies **C3**$_0$, and **C4**$_0$. To complete the proof of Theorem 4.1 for $\nu = 0$, it remains to show uniqueness.

Proposition 5.5. *The function $\circ$ is unique to satisfy* **C1**$_0$, **C2**$_0$, **C3**$_0$, *and* **C4**$_0$.

Proof. Let $\bullet$ be a composition satisfying conditions **C1**$_0$, **C2**$_0$, **C3**$_0$, and **C4**$_0$. Write $s = c\, \mathfrak{m}\, (1 + \varepsilon) \in \mathbb{T}^{>,>}$, where $c \in \mathbb{R}^{\neq}$, $\mathfrak{m} := \mathfrak{d}_s$, and $\varepsilon \prec 1$. By strong linearity, it suffices to verify that $\ell_0^r \bullet s = s^r$ for any monomial in $\mathfrak{L}_{<1}$. Given $r \in \mathbb{R}$, the condition **C4**$_0$ implies

$$\ell_0^r \bullet s = \ell_0^r \bullet (c\, \mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_0^r)^{(k)} \bullet c\, \mathfrak{m}}{k!} (c\, \mathfrak{m}\, \varepsilon)^k.$$

We have $(\ell_0^r)^{(k)} = k! \binom{r}{k} \ell_0^{r-k}$, so

$$\ell_0^r \bullet s = \ell_0^r \bullet (c\, \mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \binom{r}{k} (\ell_0^{r-k} \bullet c\, \mathfrak{m}) (c\, \mathfrak{m}\, \varepsilon)^k.$$

We have $\ell_0^r \bullet (c\,\mathfrak{m}) = \ell_0^r \bullet (c\,\ell_0 \bullet \mathfrak{m}) = (\ell_0^r \circ (c\,\ell_0)) \bullet \mathfrak{m} = c^r(\ell_0^r \bullet \mathfrak{m})$ by $\mathbf{C3}_0$ and $\ell_0^r \bullet \mathfrak{m} = \mathfrak{m}^r$ by $\mathbf{C2}_0$, so $\ell_0^r \bullet (c\,\mathfrak{m}) = (c\,\mathfrak{m})^r$. Likewise, $\ell_0^{r-k} \bullet (c\,\mathfrak{m}) = (c\,\mathfrak{m})^{r-k}$, so

$$\ell_0^r \bullet s = (c\,\mathfrak{m})^r + \sum_{k \in \mathbb{N}^>} \binom{r}{k} (c\,\mathfrak{m})^{r-k} (c\,\mathfrak{m}\,\varepsilon)^k = (c\,\mathfrak{m})^r \left(1 + \sum_{k \in \mathbb{N}^>} \binom{r}{k} \varepsilon^k\right) = s^r. \qquad \square$$

5.2 $\mathbf{C1}_\nu$ and $\mathbf{C2}_\nu$ for $\nu > 0$

For the remainder of this section, we assume that $\nu > 0$. By the results in Section 4, we have a well-defined extension of L_γ to all of $\mathbb{T}^{>,\succ}$ for each $\gamma < \alpha$. Indeed, for $s \in \mathbb{T}^{>,\succ}$ and $\gamma < \alpha$, take n with $\gamma = \omega^{\nu_*} n + \sigma$ with $\sigma < \omega^{\nu_*}$ (so $n = 0$ if ν is a limit). Then we may set $L_\gamma(s) := L_\sigma(L_{\omega^{\nu_*}}^{\circ n}(s))$.

Given $\mathfrak{a} \in \mathfrak{M}_\alpha$ and $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{r_\gamma} \in \mathfrak{L}_{<\alpha}$, we have by $\mathbf{P}_\nu$ that $\sum_{\gamma < \alpha} r_\gamma L_{\gamma+1}(\mathfrak{a}) \in \log \mathfrak{M}$, so we set $\mathfrak{l} \circ \mathfrak{a} := \exp(\sum_{\gamma < \alpha} r_\gamma L_{\gamma+1}(\mathfrak{a})) \in \mathfrak{M}$. Clearly, the map $\mathfrak{L}_{<\alpha} \longrightarrow \mathfrak{M}; \mathfrak{l} \longmapsto \mathfrak{l} \circ \mathfrak{a}$ is an embedding of monomial groups which preserves real powers, and by $\mathbf{A}_\nu$, this embedding is order-preserving as well. For $f \in \mathbb{L}_{<\alpha}$, we set $f \circ \mathfrak{a} := \sum_{\mathfrak{l} \in \mathfrak{L}_{<\alpha}} f_{\mathfrak{l}} (\mathfrak{l} \circ \mathfrak{a})$. By Proposition 2.3, we have:

Lemma 5.6. *The map $\mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}; f \longmapsto f \circ \mathfrak{a}$ is a strongly linear ordered field embedding.*

By Lemma 2.8 with $\Phi(f) := f \circ \mathfrak{a}$, the series $\mathcal{T}_f(\mathfrak{a}, \varepsilon)$ exists in $\mathbb{T}$ for any $\varepsilon \prec \mathfrak{a}$.

Lemma 5.7. *Let $\mathfrak{a} \in \mathfrak{M}_\alpha$, $\varepsilon \in \mathbb{T}^\prec$, and $\rho < \alpha$. We have*

$$L_\rho(\mathfrak{a} + \varepsilon) = L_\rho(\mathfrak{a}) + \mathcal{T}_{\ell_\rho}(\mathfrak{a}, \varepsilon).$$

Proof. If ν is a limit ordinal, then the lemma follows from $\mathbf{C4}_\mu$ for any ordinal μ with $\rho < \omega^\mu$, so we may assume that ν is a successor. We prove the lemma by induction on ρ. The lemma is immediate when $\rho = 0$, so suppose $\rho > 0$ and take $n \in \mathbb{N}$ and $0 < \gamma \leqslant \omega^{\nu_*}$ with $\rho = \omega^{\nu_*} n + \gamma$. Our induction hypothesis yields

$$L_{\omega^{\nu_*} n}(\mathfrak{a} + \varepsilon) = L_{\omega^{\nu_*} n}(\mathfrak{a}) + \mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\mathfrak{a}, \varepsilon).$$

Note that $L_{\omega^{\nu_*} n}(\mathfrak{a}) \in \mathfrak{M}_\alpha$ and $\mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\mathfrak{a}, \varepsilon) \sim (\ell'_{\omega^{\nu_*} n} \circ \mathfrak{a})\,\varepsilon \prec 1$. We claim that

$$L_\gamma(L_{\omega^{\nu_*} n}(\mathfrak{a} + \varepsilon)) = L_\gamma(L_{\omega^{\nu_*} n}(\mathfrak{a})) + \mathcal{T}_{\ell_\gamma}(L_{\omega^{\nu_*} n}(\mathfrak{a}), \mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\mathfrak{a}, \varepsilon)).$$

When $\gamma < \omega^{\nu_*}$, this just follows from $\mathbf{C4}_{\nu_*}$. When $\gamma = \omega^{\nu_*} = 1$, this is Proposition 4.7 with $L_{\omega^{\nu_*} n}(\mathfrak{a})$ in place of t and $\mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\mathfrak{a}, \varepsilon)$ in place of δ. When $\gamma = \omega^{\nu_*} > 1$, this is Definition 4.11 with $L_{\omega^{\nu_*} n}(\mathfrak{a})$ in place of $\mathfrak{d}_\beta(s)$ and $\mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\mathfrak{a}, \varepsilon)$ in place of ε. Since $L_\gamma(L_{\omega^{\nu_*} n}(\mathfrak{a} + \varepsilon)) = L_\rho(\mathfrak{a} + \varepsilon)$ and $L_\gamma(L_{\omega^{\nu_*} n}(\mathfrak{a})) = L_\rho(\mathfrak{a})$, it remains to show that $\mathcal{T}_{\ell_\gamma}(L_{\omega^{\nu_*} n}(\mathfrak{a}), \mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\mathfrak{a}, \varepsilon)) = \mathcal{T}_{\ell_\rho}(\mathfrak{a}, \varepsilon)$. Now $\mathcal{T}_{\ell_\gamma}(\ell_{\omega^{\nu_*} n}, \mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\ell_0, X)) = \mathcal{T}_{\ell_\rho}(\ell_0, X)$ as series in $\mathbb{L}_{<\alpha}[[X]]$. Indeed, for $h \in \mathbb{L}_{<\alpha}^\prec$, we have

$$\begin{aligned}
\mathcal{T}_{\ell_\gamma}(\ell_{\omega^{\nu_*} n}, \mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\ell_0, h)) &= \ell_\gamma \circ (\ell_{\omega^{\nu_*} n} + \mathcal{T}_{\ell_{\omega^{\nu_*} n}}(\ell_0, h)) - \ell_\gamma \circ \ell_{\omega^{\nu_*} n} \\
&= \ell_\gamma \circ (\ell_{\omega^{\nu_*} n} + \ell_{\omega^{\nu_*} n} \circ (\ell_0 + h) - \ell_{\omega^{\nu_*} n}) - \ell_\gamma \circ \ell_{\omega^{\nu_*} n} \\
&= \ell_\gamma \circ (\ell_{\omega^{\nu_*} n} \circ (\ell_0 + h)) - \ell_\gamma \circ \ell_{\omega^{\nu_*} n} \\
&= \ell_\rho \circ (\ell_0 + h) - \ell_\rho = \mathcal{T}_{\ell_\rho}(\ell_0, h).
\end{aligned}$$

We conclude by Lemma 2.7. $\square$

Lemma 5.6 shows that **C1**$_\nu$ holds if $s=\mathfrak{a}$. In the general situation when $s\in\mathbb{T}^{>,\succ}$, our next goal is to show that the family $(L_{\gamma+1}(s))_{\gamma<\alpha}$ is well-based. For the remainder of this subsection, we fix $s\in\mathbb{T}^{>,\succ}$. By ν-confluence and Corollary 4.10, take $n\in\mathbb{N}$ and $\mu<\nu$ such that $L_\gamma(s)-L_\gamma(\mathfrak{d}_\alpha(s))\prec 1$ for all $\omega^\mu n\leqslant\gamma<\alpha$. If ν is a successor, we can arrange that $\mu=\nu_*$. Set $\mathfrak{a}:=\mathfrak{d}_\alpha(s)$, set $\varepsilon:=L_\gamma(s)-L_\gamma(\mathfrak{d}_\alpha(s))$, and set $\beta:=\omega^\mu$.

Lemma 5.8. *Let $f\in\mathbb{L}_{<\alpha}$ and let $m\in\mathbb{N}$. If ν is a successor or $f\in\bigcup_{\eta<\nu}\mathbb{L}_{<\omega^\eta}$, then the expression $f\circ L_{\beta m}(\mathfrak{a})$ is defined and equal to $(f\circ\ell_{\beta m})\circ\mathfrak{a}$.*

Proof. Suppose ν is a successor ordinal, so $\beta=\omega^{\nu_*}$. Then $L_{\beta m}(\mathfrak{a})\in\mathfrak{M}_\alpha$, so $f\circ L_{\beta m}(\mathfrak{a})$ is defined. As the maps $f\longmapsto(f\circ\ell_{\beta m})\circ\mathfrak{a}$ and $f\longmapsto f\circ L_{\beta m}(\mathfrak{a})$ are strongly linear, we may assume that f is a monomial $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}$. Since $\ell_\gamma\circ\ell_{\beta m}=\ell_{\beta m+\gamma}$ for $\gamma<\alpha$, we have

$$(\mathfrak{l}\circ\ell_{\beta m})\circ\mathfrak{a}=\left(\prod_{\gamma<\alpha}\ell_{\beta m+\gamma}^{\mathfrak{l}_\gamma}\right)\circ\mathfrak{a}=\exp\left(\sum_{\gamma<\alpha}\mathfrak{l}_\gamma L_{\beta m+\gamma}(\mathfrak{a})\right)=\exp\left(\sum_{\gamma<\alpha}\mathfrak{l}_\gamma L_\gamma(L_{\beta m}(\mathfrak{a}))\right)=\mathfrak{l}\circ L_{\beta m}(\mathfrak{a}).$$

Now suppose that ν is a limit and that $f\in\mathbb{L}_{<\omega^\eta}$ for some $\eta<\nu$. By increasing η, we may assume that $\beta m<\omega^\eta$, so $f,\ell_{\beta m}\in\mathbb{L}_{<\omega^\eta}$. Then **C2**$_\eta$ and **C3**$_\eta$ give

$$(f\circ\ell_{\beta m})\circ\mathfrak{a}=f\circ(\ell_{\beta m}\circ\mathfrak{a})=f\circ L_{\beta m}(\mathfrak{a}).\qquad\square$$

Lemma 5.9. *There is a well-based family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ from $\mathbb{L}^{\prec}_{[\beta n,\alpha)}$ such that*

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k$$

for each γ with $\beta n\leqslant\gamma<\alpha$.

Proof. Fix γ with $\beta n\leqslant\gamma<\alpha$. We first claim that

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon).$$

If ν is a limit, then take η with $\gamma<\omega^\eta<\alpha$. Then $\ell_{\gamma+1}^{\uparrow\beta n},\ell_{\beta n}\in\mathbb{L}_{<\omega^\eta}$, so **C4**$_\eta$ gives

$$L_{\gamma+1}(s)=\ell_{\gamma+1}^{\uparrow\beta n}\circ L_{\beta n}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon)$$

and **C3**$_\eta$ gives $\ell_{\gamma+1}^{\uparrow\beta n}\circ L_{\beta n}(\mathfrak{a})=L_{\gamma+1}(\mathfrak{a})$, thereby proving the claim. If ν is a successor, then take $\rho<\alpha$ with $\gamma+1=\beta n+\rho$. By Lemma 5.7 and the fact that $\ell_{\gamma+1}^{\uparrow\beta n}=\ell_\rho$, we have

$$\begin{aligned}L_{\gamma+1}(s) &= L_\rho(L_{\beta n}(s))=L_\rho(L_{\beta n}(\mathfrak{a}))+\mathcal{T}_{\ell_\rho}(L_{\beta n}(\mathfrak{a}),\varepsilon)\\ &= L_{\gamma+1}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon).\end{aligned}$$

Having proved our claim, let $k>0$ be given and set $f_{\gamma,k}:=\frac{1}{k!}\,(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\circ\ell_{\beta n}\in\mathbb{L}_{[\beta n,\alpha)}$. Lemma 2.9 yields $(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\prec 1$, whence $f_{\gamma,k}\prec 1$. If ν is a limit, then $\left(\ell_{\gamma+1}^{\uparrow\beta n}\right)^{(k)}\in\mathbb{L}_{<\omega^\eta}$, where η is as above. So in both the successor and limit cases, we may apply Lemma 5.8 with $\left(\ell_{\gamma+1}^{\uparrow\beta n}\right)^{(k)}$ in place of f to get

$$f_{\gamma,k}\circ\mathfrak{a}=\frac{1}{k!}(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\circ L_{\beta n}(\mathfrak{a}).$$

This gives

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon)=L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k.$$

It remains to show that the family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is well-based. Since $(\ell_{\gamma+1})_{\beta n\leqslant\gamma<\alpha}$ is a well-based family in $\mathbb{L}_{<\alpha}$ and $\mathbb{L}_{[\beta n,\alpha)}\longrightarrow\mathbb{L}_{<\alpha}; f\longmapsto f^{\uparrow\beta n}$ is strongly linear, the family $(\ell_{\gamma+1}^{\uparrow\beta n})_{\beta n\leqslant\gamma<\alpha}$ is well-based. Since $\operatorname{supp}_*\partial$ is well-based and infinitesimal, the family $((\ell_{\gamma+1}^{\uparrow\beta n})^{(k)})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is well-based. We conclude that the family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is well-based. □

Proposition 5.10. *Let $(r_\gamma)_{\gamma<\alpha}$ be a sequence of real numbers. Then the family $(L_{\gamma+1}(s))_{\gamma<\alpha}$ is well-based and the series $\sum_{\gamma<\alpha} r_\gamma L_{\gamma+1}(s)$ lies in $\log\mathbb{T}^>$.*

Proof. We will show the following:

a) For each $k<n$, the family $(L_{\gamma+1}(s))_{\beta k\leqslant\gamma<\beta(k+1)}$ is well-based and
$$\sum_{\beta k\leqslant\gamma<\beta(k+1)} r_\gamma L_{\gamma+1}(s)\ \in\ \log\mathbb{T}^>.$$

b) The family $(L_{\gamma+1}(s))_{\beta n\leqslant\gamma<\alpha}$ is well-based and
$$\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma L_{\gamma+1}(s)\ \in\ \log\mathbb{T}^>.$$

The proposition follows from (a) and (b), since the union of finitely many well-based families is well-based and $\log\mathbb{T}^>$ is closed under finite sums.

To see why (a) holds, let $k<n$ and note that
$$(L_{\gamma+1}(s))_{\beta k\leqslant\gamma<\beta(k+1)}=(L_{\rho+1}(L_{\beta k}(s)))_{\rho<\beta}.$$
Since $(\ell_{\rho+1})_{\rho<\beta}$ is well-based, $\mathbf{C1}_\mu$ gives that $(L_{\rho+1}(L_{\beta k}(s)))_{\rho<\beta}=(\ell_{\rho+1}\circ L_{\beta k}(s))_{\rho<\beta}$ is well-based. We have
$$\sum_{\beta k\leqslant\gamma<\beta(k+1)} r_\gamma L_{\gamma+1}(s)=\sum_{\rho<\beta} r_{\beta k+\rho}L_{\rho+1}(L_{\beta k}(s)).$$
Set $\mathfrak{l}:=\prod_{\rho<\beta}\ell_\rho^{r_{\beta k+\rho}}\in\mathfrak{L}_{<\beta}$. We claim that $\sum_{\rho<\beta} r_{\beta k+\rho}L_{\rho+1}(L_{\beta k}(s))=\log(\mathfrak{l}\circ L_{\beta k}(s))$. If $\mu=0$, then $\mathfrak{l}=\ell_0^{r_k}$ and
$$\sum_{\rho<\beta} r_{\beta k+\rho}L_{\rho+1}(L_{\beta k}(s))=r_k L_1(L_k(s))=\log(L_k(s)^{r_k})=\log(\mathfrak{l}\circ L_{\beta k}(s)).$$
If $\mu>0$, then $\mathbf{C3}_\mu$ gives
$$\sum_{\rho<\beta} r_{\beta k+\rho}L_{\rho+1}(L_{\beta k}(s))=\log(\mathfrak{l})\circ L_{\beta k}(s)=\log(\mathfrak{l}\circ L_{\beta k}(s)).$$

As for (b), let $\varepsilon:=L_{\beta n}(s)-L_{\beta n}(\mathfrak{a})$. By Lemma 5.9, there exists a well-based family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ from $\mathbb{L}^{\prec}_{[\beta n,\alpha)}$ such that
$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k.$$
The families $(L_{\gamma+1}(\mathfrak{a}))_{\beta n\leqslant\gamma<\alpha}$ and $(f_{\gamma,k}\circ\mathfrak{a})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ are well-based by Lemma 5.6 and the fact that $\mathfrak{a}\in\mathfrak{M}_\alpha$. Since the family $(\varepsilon^k)_{k\in\mathbb{N}}$ is also well-based, it follows that $((f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k)_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is again well-based. In particular,
$$(L_{\gamma+1}(s))_{\beta n\leqslant\gamma<\alpha}=\left(L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k\right)_{\beta n\leqslant\gamma<\alpha}$$

is well-based. Now

$$\sum_{\beta n \leqslant \gamma < \alpha} r_\gamma L_{\gamma+1}(s) = \sum_{\beta n \leqslant \gamma < \alpha} r_\gamma L_{\gamma+1}(\mathfrak{a}) + \sum_{\beta n \leqslant \gamma < \alpha} r_\gamma \sum_{k \in \mathbb{N}^>} (f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k.$$

Since $f_{\gamma,k}$ and ε^k are infinitesimal for all $k > 0$, we may write

$$\sum_{\beta n \leqslant \gamma < \alpha} r_\gamma L_{\gamma+1}(s) = \left(\sum_{\beta n \leqslant \gamma < \alpha} r_\gamma L_{\gamma+1}(\mathfrak{a}) \right) + \delta,$$

where $\delta \in \mathbb{T}^{\prec}$. By (4.1), we have $\delta = L(E(\delta)) \in \log \mathbb{T}^>$. Furthermore, P_ν implies

$$\sum_{\beta n \leqslant \gamma < \alpha} r_\gamma L_{\gamma+1}(\mathfrak{a}) \in \log \mathfrak{M} \subseteq \log \mathbb{T}^>.$$

We conclude that $\sum_{\beta n \leqslant \gamma < \alpha} r_\gamma L_{\gamma+1}(s) \in \log \mathbb{T}^>$. □

Let $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$. In light of Proposition 5.10, we define

$$\mathfrak{l} \circ s \;:=\; \exp\left(\sum_{\gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(s) \right).$$

We note that the map $\mathfrak{L}_{<\alpha} \longrightarrow \mathbb{T}^>; \mathfrak{l} \longmapsto \mathfrak{l} \circ s$ is an embedding of ordered multiplicative groups for each $s \in \mathbb{T}^{>,\succ}$.

Our next objective is to show that the map $\mathfrak{L}_{<\alpha} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ s$ extends by strong linearity to a map $\mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}$ which satisfies $\mathbf{C1}_\nu$ and $\mathbf{C2}_\nu$. For this, we will show that $\mathfrak{l} \longmapsto \mathfrak{l} \circ s$ is a relatively well-based mapping, by using a similar "gluing" technique as for Proposition 5.10. Recall that our second induction hypothesis from the beginning of this section stipulated that the mapping $\mathfrak{L}_{<\omega^\mu} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ s$ is relatively well-based for all $\mu < \nu$ and $s \in \mathbb{T}^{>,\succ}$.

Proposition 5.11. *Let $\Phi: \mathfrak{L}_{<\alpha} \longrightarrow \mathbb{T}$ be the map $\Phi(\mathfrak{l}) := \mathfrak{l} \circ s$. Then Φ is relatively well-based.*

Proof. Let $\Phi_{\geqslant n}$ be the restriction of Φ to $\mathfrak{L}_{[\beta n, \alpha)}$ and for $k < n$, let Φ_k be the restriction of Φ to $\mathfrak{L}_{[\beta k, \beta(k+1))}$. Since

$$\operatorname{supp}_{\odot} \Phi \subseteq (\operatorname{supp}_{\odot} \Phi_0) \cdots (\operatorname{supp}_{\odot} \Phi_{n-1})\, (\operatorname{supp}_{\odot} \Phi_{\geqslant n}),$$

it suffices to show that each Φ_k and $\Phi_{\geqslant n}$ are relatively well-based. For the Φ_k, fix $k < n$. Our induction hypothesis implies that the map $\Psi_k: \mathfrak{L}_{[0,\beta)} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ L_{\beta k}(s)$ is relatively well-based. By Lemma 5.8 with $\mathfrak{l}$ in place of f, we have

$$\Phi_k(\mathfrak{l} \circ \ell_{\beta k}) = (\mathfrak{l} \circ \ell_{\beta k}) \circ s = \mathfrak{l} \circ L_{\beta k}(s) = \Psi_k(\mathfrak{l}).$$

It follows that Φ_k is also relatively well-based with $\operatorname{supp}_{\odot} \Phi_k = \operatorname{supp}_{\odot} \Psi_k$.

Now for $\Phi_{\geqslant n}$. Let $\mathfrak{l} = \prod_{\beta n < \gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{[\beta n, \alpha)}$. By Lemma 5.9, we have a well-based family $(f_{\gamma,k})_{\beta n \leqslant \gamma < \alpha, k \in \mathbb{N}^>}$ from $\mathbb{L}^{\prec}_{[\beta n, \alpha)}$ such that

$$\log(\Phi_{\geqslant n}(\mathfrak{l})) = \sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(s) = \sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(\mathfrak{a}) + \sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma \sum_{k \in \mathbb{N}^>} (f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k.$$

Exponentiating both sides, we obtain

$$\Phi_{\geqslant n}(\mathfrak{l}) = (\mathfrak{l} \circ \mathfrak{a})\, E\left(\sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma \sum_{k \in \mathbb{N}^>} (f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k \right)$$

so $\mathfrak{d}_{\Phi_{\geqslant n}(\mathfrak{l})} = \mathfrak{l} \circ \mathfrak{a}$. The set

$$\mathfrak{E} := \bigcup_{\beta n \leqslant \gamma < \alpha, k \in \mathbb{N}^{>}} \operatorname{supp}((f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k)$$

is well-based, infinitesimal and does not depend on $\mathfrak{l}$. Since

$$\frac{\operatorname{supp} \Phi_{\geqslant n}(\mathfrak{l})}{\mathfrak{d}_{\Phi_{\geqslant n}(\mathfrak{l})}} \subseteq \mathfrak{E}^{\infty}$$

for all $\mathfrak{l} \in \mathfrak{L}_{[\beta n, \alpha)}$, we conclude that $\operatorname{supp}_{\odot} \Phi_{\geqslant n} \subseteq \mathfrak{E}^{\infty}$ is well-based. □

We already noted that the map Φ from Proposition 5.11 is an order-preserving multiplicative embedding. By Proposition 2.5, it follows that Φ is well-based, so it extends uniquely into an order-preserving and strongly linear embedding $\hat{\Phi}\colon \mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}$. Taking $f \circ s := \hat{\Phi}(f)$ for all $f \in \mathbb{L}_{<\alpha}$, this proves **C1**$_\nu$. By construction, we also have **C2**$_\nu$. Note that $\circ$ extends the unique composition $\mathbb{L}_{<\omega^\eta} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ of Theorem 4.1 for $\eta < \nu$.

5.3 Properties C3$_\nu$ and C4$_\nu$ and uniqueness for $\nu > 0$

Proposition 5.12. *For $r \in \mathbb{R}, g \in \mathbb{L}_{<\alpha}^{>,\succ}$ and $s \in \mathbb{T}^{>,\succ}$, we have $g^r \circ s = (g \circ s)^r$.*

Proof. As in Proposition 5.2, it suffices to prove that $(\mathfrak{l} \circ s)^r = \mathfrak{l}^r \circ s$ holds for each $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$. For such $\mathfrak{l}$, we have

$$\log(\mathfrak{l}^r \circ s) = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma\, r\, L_{\gamma+1}(s).$$

By Proposition 4.6, we also have $\log((\mathfrak{l} \circ s)^r) = r \log\, (\mathfrak{l} \circ s) = r \sum_{\gamma < \omega^\eta} \mathfrak{l}_\gamma L_{\gamma+1}(s)$. By injectivity of the logarithm, we conclude that $(\mathfrak{l} \circ s)^r = \mathfrak{l}^r \circ s$. □

Lemma 5.13. *For all $h \in \mathbb{L}_{<\alpha}^{>}$ and all $s \in \mathbb{T}^{>,\succ}$, we have $\log(h \circ s) = (\log h) \circ s$.*

Proof. First, we note that for $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$, we have

$$(\log \mathfrak{l}) \circ s = \left(\sum_{\gamma < \alpha} \mathfrak{l}_\gamma\, \ell_{\gamma+1} \right) \circ s = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(s) = \log(\mathfrak{l} \circ s),$$

where the last equality uses the definition of $\mathfrak{l} \circ s$. Now, let $h \in \mathbb{L}_{<\alpha}^{>}$ and write $h = c\, \mathfrak{m}\, (1 + \varepsilon)$ with $c \in \mathbb{R}^{>}$, $\mathfrak{m} := \mathfrak{d}_h$, and $\varepsilon \prec 1$. Then $h \circ s = c\, (\mathfrak{m} \circ s)(1 + \varepsilon \circ s)$ and

$$\begin{aligned}(\log h) \circ s &= (\log \mathfrak{m}) \circ s + \log c + \sum_{k \in \mathbb{N}^{>}} \frac{(-1)^{k-1}}{k} \varepsilon^k \circ s \\ &= \log(\mathfrak{m} \circ s) + \log c + \sum_{k \in \mathbb{N}^{>}} \frac{(-1)^{k-1}}{k} (\varepsilon \circ s)^k, \\ &= \log(c\, (\mathfrak{m} \circ s)(1 + \varepsilon \circ s)) = \log(h \circ s).\end{aligned}$$

Here we used the facts that $(\log c) \circ s = \log c$ and that composition with s commutes with powers and infinite sums. □

Proposition 5.14. *The function $\circ$ satisfies **C3**$_\nu$, i.e. for all $f \in \mathbb{L}_{<\alpha}$, $g \in \mathbb{L}_{<\alpha}^{>,\succ}$, and $s \in \mathbb{T}^{>,\succ}$ we have $f \circ (g \circ s) = (f \circ g) \circ s$.*

Proof. We will show by induction on $\mu \leqslant \nu$ that $f\circ(g\circ s)=(f\circ g)\circ s$ for all $f\in\mathbb{L}_{<\omega^\mu}$, all $g\in\mathbb{L}_{<\alpha}^{>,\succ}$, and all $s\in\mathbb{T}^{>,\succ}$. If $\mu=0$, then this follows from Proposition 5.12 and strong linearity.

Let $\mu>0$, let g and s be fixed, and assume that the proposition holds whenever $f\in\mathbb{L}_{<\omega^\eta}$ for some $\eta<\mu$. By strong linearity, it suffices to prove that $\mathfrak{l}\circ(g\circ s)=(\mathfrak{l}\circ g)\circ s$ for all $\mathfrak{l}=\prod_{\gamma<\omega^\mu}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\omega^\mu}$. Lemma 5.13 gives

$$\begin{aligned}\log(\mathfrak{l}\circ(g\circ s)) &= (\log\mathfrak{l})\circ(g\circ s)=\sum_{\gamma<\omega^\mu}\mathfrak{l}_\gamma\,\ell_{\gamma+1}\circ(g\circ s),\\ \log((\mathfrak{l}\circ g)\circ s) &= (\log(\mathfrak{l}\circ g))\circ s=((\log\mathfrak{l})\circ g)\circ s=\left(\sum_{\gamma<\omega^\mu}\mathfrak{l}_\gamma\,\ell_{\gamma+1}\circ g\right)\circ s.\end{aligned}$$

Using the injectivity of log and strong linearity, we may thus reduce to the case when $\mathfrak{l}=\ell_\gamma$ for $\gamma<\omega^\mu$. Our induction hypothesis takes care of the case when μ is a limit ordinal or when $\gamma<\omega^{\mu_*}$, so we may assume that $\mathfrak{l}=\ell_\gamma$, where $\omega^{\mu_*}\leqslant\gamma<\omega^\mu$. By the inductive definitions of $L_\gamma(g\circ s)$ and $\ell_\gamma\circ g$, we may further reduce to the case when $\gamma=\omega^{\mu_*}$. Lemma 5.13 takes care of the case $\mu=1$, so we may assume that $\mu>1$. In summary, we thus need to show that $L_{\omega^{\mu_*}}(g\circ s)=(\ell_{\omega^{\mu_*}}\circ g)\circ s$, where $\mu>1$.

Set $\mathfrak{a}:=\mathfrak{d}_{\omega^{\mu_*}}(g)\in\mathfrak{L}_{<\alpha}$. We claim that $(\ell_{\omega^{\mu_*}}\circ\mathfrak{a})\circ s=L_{\omega^{\mu_*}}(\mathfrak{a}\circ s)$. We have $\mathfrak{a}=\ell_{\sigma+\omega^{\mu_{**}}k}$, where $\omega^{\mu_*}\leqslant\sigma<\alpha$, $k\in\mathbb{N}$, and $k=0$ if μ_* is a limit ordinal. As $\ell_{\sigma+\omega^{\mu_{**}}k}=\ell_{\omega^{\mu_{**}}k}\circ\ell_\sigma$, we have

$$\ell_{\omega^{\mu_*}}\circ\mathfrak{a}=\ell_{\omega^{\mu_*}}\circ(\ell_{\omega^{\mu_{**}}k}\circ\ell_\sigma)=(\ell_{\omega^{\mu_*}}\circ\ell_{\omega^{\mu_{**}}k})\circ\ell_\sigma=(\ell_{\omega^{\mu_*}}-k)\circ\ell_\sigma=\ell_{\sigma+\omega^{\mu_*}}-k.$$

This gives

$$\begin{aligned}(\ell_{\omega^{\mu_*}}\circ\mathfrak{a})\circ s &= (\ell_{\sigma+\omega^{\mu_*}}-k)\circ s=L_{\sigma+\omega^{\mu_*}}(s)-k=L_{\omega^{\mu_*}}(L_\sigma(s))-k\\ &= L_{\omega^{\mu_*}}(L_{\omega^{\mu_{**}}k}(L_\sigma(s)))=L_{\omega^{\mu_*}}(L_{\sigma+\omega^{\mu_{**}}k}(s))=L_{\omega^{\mu_*}}(\mathfrak{a}\circ s),\end{aligned}$$

where the first equality in the second line follows from Proposition 4.13.

Having proved our claim, let us now show that $(\ell_{\omega^{\mu_*}}\circ g)\circ s=L_{\omega^{\mu_*}}(g\circ s)$. Take $\gamma<\omega^{\mu_*}$ with $L_\gamma(g\circ s)-L_\gamma(\mathfrak{d}_{\omega^{\mu_*}}(g\circ s))\prec 1$, and $\varepsilon:=\ell_\gamma\circ g-\ell_\gamma\circ\mathfrak{a}\prec 1$. We have

$$\ell_{\omega^{\mu_*}}\circ g=\ell_{\omega^{\mu_*}}^{\uparrow\gamma}\circ(\ell_\gamma\circ g)=\ell_{\omega^{\mu_*}}^{\uparrow\gamma}\circ(\ell_\gamma\circ\mathfrak{a})+\mathcal{T}_{\ell_{\omega^{\mu_*}}^{\uparrow\gamma}}(\ell_\gamma\circ\mathfrak{a},\varepsilon)=\ell_{\omega^{\mu_*}}\circ\mathfrak{a}+\mathcal{T}_{\ell_{\omega^{\mu_*}}^{\uparrow\gamma}}(\ell_\gamma\circ\mathfrak{a},\varepsilon).$$

As $\ell_\gamma\in\mathbb{L}_{<\omega^{\mu_*}}$ and $(\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\in\mathbb{L}_{<\omega^{\mu_*}}$ for all $k>0$, by Lemma 2.9, our induction hypothesis applied to μ_* gives

$$((\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\circ(\ell_\gamma\circ\mathfrak{a}))\circ s=(\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\circ((\ell_\gamma\circ\mathfrak{a})\circ s)=(\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\circ L_\gamma(\mathfrak{a}\circ s)$$

for $k>0$. Along with **C1**$_\nu$, we thus have

$$\begin{aligned}\mathcal{T}_{\ell_{\omega^{\mu_*}}^{\uparrow\gamma}}(\ell_\gamma\circ\mathfrak{a},\varepsilon)\circ s &= \left(\sum_{k\in\mathbb{N}^>}\frac{(\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\circ(\ell_\gamma\circ\mathfrak{a})}{k!}\varepsilon^k\right)\circ s=\sum_{k\in\mathbb{N}^>}\frac{((\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\circ(\ell_\gamma\circ\mathfrak{a}))\circ s}{k!}\varepsilon^k\circ s\\ &= \sum_{k\in\mathbb{N}^>}\frac{(\ell_{\omega^{\mu_*}}^{\uparrow\gamma})^{(k)}\circ L_\gamma(\mathfrak{a}\circ s)}{k!}(\varepsilon\circ s)^k=\mathcal{T}_{\ell_{\omega^{\mu_*}}^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}\circ s),\varepsilon\circ s).\end{aligned}$$

Using also our claim that $(\ell_{\omega^{\mu_*}}\circ\mathfrak{a})\circ s=L_{\omega^{\mu_*}}(\mathfrak{a}\circ s)$, we obtain

$$(\ell_{\omega^{\mu_*}}\circ g)\circ s=(\ell_{\omega^{\mu_*}}\circ\mathfrak{a})\circ s+\mathcal{T}_{\ell_{\omega^{\mu_*}}^{\uparrow\gamma}}(\ell_\gamma\circ\mathfrak{a},\varepsilon)\circ s=L_{\omega^{\mu_*}}(\mathfrak{a}\circ s)+\mathcal{T}_{\ell_{\omega^{\mu_*}}^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}\circ s),\varepsilon\circ s).$$

It remains to show that $L_{\omega^{\mu_*}}(g\circ s)=L_{\omega^{\mu_*}}(\mathfrak{a}\circ s)+\mathcal{T}_{\ell^{\uparrow\gamma}_{\omega^{\mu_*}}}(L_\gamma(\mathfrak{a}\circ s),\varepsilon\circ s)$. Now

$$L_\gamma(g\circ s)-L_\gamma(\mathfrak{a}\circ s)=(\ell_\gamma\circ g)\circ s-(\ell_\gamma\circ\mathfrak{a})\circ s=\varepsilon\circ s\prec 1,$$

so $\mathfrak{d}_{\omega^{\mu_*}}(\mathfrak{a}\circ s)=\mathfrak{d}_{\omega^{\mu_*}}(g\circ s)$ and $L_\gamma(\mathfrak{a}\circ s)-L_\gamma(\mathfrak{d}_{\omega^{\mu_*}}(\mathfrak{a}\circ s))\prec 1$. We may thus apply Lemma 4.16 to $g\circ s$ and $\mathfrak{a}\circ s$, to conclude that $L_{\omega^{\mu_*}}(g\circ s)=L_{\omega^{\mu_*}}(\mathfrak{a}\circ s)+\mathcal{T}_{\ell^{\uparrow\gamma}_{\omega^{\mu_*}}}(L_\gamma(\mathfrak{a}\circ s),\varepsilon\circ s)$. □

Proposition 5.15. *For $\gamma<\alpha$, $t\in\mathbb{T}^{>,\succ}$ and $\delta\in\mathbb{T}$ with $\delta\prec t$, we have*

$$L_\gamma(t+\delta)=L_\gamma(t)+\mathcal{T}_{\ell_\gamma}(t,\delta).$$

Proof. Since $\mathbf{C4}_\eta$ holds for $\eta<\nu$, we need only consider the case when ν is a successor and $\omega^{\nu_*}\leqslant\gamma<\alpha$. We prove the result by induction on γ. By Proposition 4.7 (when $\nu=1$) or Proposition 4.15 (when $\nu>1$), we have $L_{\omega^{\nu_*}}(t+\delta)=L_{\omega^{\nu_*}}(t)+\mathcal{T}_{\ell_{\omega^{\nu_*}}}(t,\delta)$. Assume that $\gamma>\omega^{\nu_*}$ and write $\gamma=\omega^{\nu_*}+\sigma$ with $\sigma\leqslant\omega^{\nu_*}$. We have

$$L_\gamma(t+\delta)=L_\sigma(L_{\omega^{\nu_*}}(t+\delta))=L_\sigma(L_{\omega^{\nu_*}}(t)+\mathcal{T}_{\ell_{\omega^{\nu_*}}}(t,\delta)).$$

Since $\mathcal{T}_{\ell_{\omega^{\nu_*}}}(t,\delta)\prec L_{\omega^{\nu_*}}(t)$, the induction hypothesis yields

$$L_\sigma(L_{\omega^{\nu_*}}(t)+\mathcal{T}_{\ell_{\omega^{\nu_*}}}(t,\delta))=L_\sigma(L_{\omega^{\nu_*}}(t))+\mathcal{T}_{\ell_\sigma}(L_{\omega^{\nu_*}}(t),\mathcal{T}_{\ell_{\omega^{\nu_*}}}(t,\delta)).$$

Since $L_\sigma(L_{\omega^{\nu_*}}(t))=L_\gamma(t)$, it remains to show that $\mathcal{T}_{\ell_\sigma}(L_{\omega^{\nu_*}}(t),\mathcal{T}_{\ell_{\omega^{\nu_*}}}(t,\delta))=\mathcal{T}_{\ell_\gamma}(t,\delta)$. It is enough to show that $\mathcal{T}_{\ell_\sigma}(\ell_{\omega^{\nu_*}},\mathcal{T}_{\ell_{\omega^{\nu_*}}}(\ell_0,X))=\mathcal{T}_{\ell_\gamma}(\ell_0,X)$ as power series in $\mathbb{L}_{<\alpha}[[X]]$. This follows from Lemma 2.7, since

$$\begin{aligned}\mathcal{T}_{\ell_\sigma}(\ell_{\omega^{\nu_*}},\mathcal{T}_{\ell_{\omega^{\nu_*}}}(\ell_0,h)) &= \mathcal{T}_{\ell_\sigma}(\ell_{\omega^{\nu_*}},\ell_{\omega^{\nu_*}}\circ(\ell_0+h)-\ell_{\omega^{\nu_*}})=\ell_\sigma\circ(\ell_{\omega^{\nu_*}}\circ(\ell_0+h))-\ell_\sigma\circ\ell_{\omega^{\nu_*}}\\ &= \ell_\gamma\circ(\ell_0+h)-\ell_\gamma=\mathcal{T}_{\ell_\gamma}(\ell_0,h),\end{aligned}$$

for all $h\in\mathbb{L}^{\prec}_{<\alpha}$. □

Proposition 5.16. *The function $\circ$ satisfies $\mathbf{C4}_\nu$, i.e. for all $f\in\mathbb{L}_{<\alpha}$, all $t\in\mathbb{T}^{>,\succ}$ and all $\delta\in\mathbb{T}$ with $\delta\prec t$, we have*

$$f\circ(t+\delta)=f\circ t+\mathcal{T}_f(t,\delta).$$

Proof. Fix $t\in\mathbb{T}^{>,\succ}$ and $\delta\in\mathbb{T}$ with $\delta\prec t$. Let $T:\mathbb{L}_{<\alpha}\to\mathbb{T}$ be the map given by

$$T(f):=f\circ t+\mathcal{T}_f(t,\delta).$$

We need to show that $f\circ(t+\delta)=T(f)$ for all $f\in\mathbb{L}_{<\alpha}$. By Lemma 2.8, the map T is strongly linear, so it suffices to show that $\mathfrak{l}\circ(t+\delta)=T(\mathfrak{l})$ for all $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\alpha}$. Since log is injective, it is enough to show that $\log(\mathfrak{l}\circ(t+\delta))=\log T(\mathfrak{l})$. Now $\log(\mathfrak{l}\circ(t+\delta))=(\log\mathfrak{l})\circ(t+\delta)$ by Lemma 5.13 and $\log T(\mathfrak{l})=T(\log\mathfrak{l})$ by [12, Lemma 8.3]. By Proposition 5.15 and strong linearity, we have

$$T(\log\mathfrak{l})=T\left(\sum_{\gamma<\alpha}\mathfrak{l}_\gamma\,\ell_{\gamma+1}\right)=\sum_{\gamma<\alpha}\mathfrak{l}_\gamma\,T(\ell_{\gamma+1})=\sum_{\gamma<\alpha}\mathfrak{l}_\gamma L_{\gamma+1}(t+\delta)=(\log\mathfrak{l})\circ(t+\delta).$$

We conclude that $\log(\mathfrak{l}\circ(t+\delta))=(\log\mathfrak{l})\circ(t+\delta)=T(\log\mathfrak{l})=\log T(\mathfrak{l})$. □

To conclude our proof of Theorem 4.1, we prove the uniqueness of $\circ$.

Proposition 5.17. *The function $\circ$ is unique to satisfy $\mathbf{C1}_\nu$, $\mathbf{C2}_\nu$, $\mathbf{C3}_\nu$, and $\mathbf{C4}_\nu$.*

Proof. Let $\bullet$ be a composition satisfying conditions **C1**$_\nu$, **C2**$_\nu$, **C3**$_\nu$, and **C4**$_\nu$ and let $s \in \mathbb{T}^{>,>}$. We first show that $\ell_1 \bullet s = \ell_1 \circ s$. Write $s = c\,\mathfrak{m} + \delta$, with $c \in \mathbb{R}^>$, $\mathfrak{m} := \mathfrak{d}_s$, and $\delta \prec s$. By **C4**$_\nu$, we have

$$\ell_1 \bullet s = \ell_1 \bullet (c\,\mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \bullet (c\,\mathfrak{m})}{k!} \delta.$$

For $k > 0$, we have $\ell_1^{(k)} = (-1)^{k-1} (k-1)!\, \ell_0^{-k}$, so **C2**$_\nu$ gives

$$\ell_1^{(k)} \bullet (c\,\mathfrak{m}) = (-1)^{k-1} (k-1)!\, (c\,\mathfrak{m})^{-k} = \ell_1^{(k)} \circ (c\,\mathfrak{m}).$$

Thus, it remains to show that $\ell_1 \bullet (c\,\mathfrak{m}) = \ell_1 \circ (c\,\mathfrak{m})$. Using **C2**$_\nu$, **C3**$_\nu$, and the identity $c\,\mathfrak{m} = (c\,\ell_0) \bullet \mathfrak{m}$, we see that

$$\ell_1 \bullet (c\,\mathfrak{m}) = \ell_1 \bullet ((c\,\ell_0) \bullet \mathfrak{m}) = (\ell_1 \circ (c\,\ell_0)) \bullet \mathfrak{m} = (\ell_1 + \log c) \bullet \mathfrak{m} = L_1(\mathfrak{m}) + \log c.$$

Likewise $\ell_1 \circ (c\,\mathfrak{m}) = L_1(\mathfrak{m}) + \log c$.

Now we turn to the task of showing that $f \bullet s = f \circ s$ for $f \in \mathbb{L}_{<\alpha}$. We make the inductive assumption that for $\mu < \nu$ and $f \in \mathbb{L}_{<\omega^\mu}$, we have $f \bullet s = f \circ s$ (if $\mu = 0$, this is Proposition 5.5). By strong linearity, it suffices to verify that $\mathfrak{l} \bullet s = \mathfrak{l} \circ s$ for any monomial $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$. As $(\mathfrak{l} \bullet s)^{-1} = \mathfrak{l}^{-1} \bullet s$ and likewise for $\mathfrak{l} \circ s$, it suffices to show this only for $\mathfrak{l} \in \mathfrak{L}_{<\alpha}^{\succ}$. Given $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$, we have by **C3**$_\nu$ that

$$\begin{aligned} \ell_1 \bullet (\mathfrak{l} \bullet s) &= (\ell_1 \circ \mathfrak{l}) \bullet s = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma (\ell_{\gamma+1} \bullet s), \\ \ell_1 \circ (\mathfrak{l} \circ s) &= (\ell_1 \circ \mathfrak{l}) \circ s = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma (\ell_{\gamma+1} \circ s). \end{aligned}$$

Thus, it suffices to show that $\ell_\gamma \bullet s = \ell_\gamma \circ s$ for all $\gamma < \alpha$. By our induction hypothesis, we only need to handle the case that ν is a successor and $\gamma \geqslant \omega^{\nu_*}$. If $\gamma = \omega^{\nu_*}$, then by Proposition 4.9, there is an ordinal $\sigma < \omega^{\nu_*}$ with $\varepsilon := \ell_\sigma \circ s - L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s)) \prec 1$. Our inductive hypothesis and Lemma 2.9 yield

$$\begin{aligned} \ell_\sigma \bullet s &= \ell_\sigma \circ s = L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s)) + \varepsilon, \\ (\ell_{\omega^{\nu_*}}^{\uparrow \sigma})^{(k)} \bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s)) &= (\ell_{\omega^{\nu_*}}^{\uparrow \sigma})^{(k)} \circ L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s)) \qquad (\text{for } k \in \mathbb{N}^>) \end{aligned}$$

Thus,

$$\begin{aligned} \ell_{\omega^{\nu_*}} \bullet s &= \ell_{\omega^{\nu_*}}^{\uparrow \sigma} \bullet (\ell_\sigma \bullet s) = \ell_{\omega^{\nu_*}}^{\uparrow \sigma} \bullet (L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s)) + \varepsilon) && (\text{by } \mathbf{C3}_\nu) \\ &= \ell_{\omega^{\nu_*}}^{\uparrow \sigma} \bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s)) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_{\omega^{\nu_*}}^{\uparrow \sigma})^{(k)} \bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s))}{k!} \varepsilon^k && (\text{by } \mathbf{C4}_\nu) \\ &= (\ell_{\omega^{\nu_*}}^{\uparrow \sigma} \circ \ell_\sigma) \bullet \mathfrak{d}_{\omega^{\nu_*}}(s) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_{\omega^{\nu_*}}^{\uparrow \sigma})^{(k)} \bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s))}{k!} \varepsilon^k && (\text{by } \mathbf{C3}_\nu \text{ and } \mathbf{C2}_\nu) \\ &= L_{\omega^{\nu_*}}(\mathfrak{d}_{\omega^{\nu_*}}(s)) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_{\omega^{\nu_*}}^{\uparrow \sigma})^{(k)} \circ L_\sigma(\mathfrak{d}_{\omega^{\nu_*}}(s))}{k!} \varepsilon^k \\ &= \ell_{\omega^{\nu_*}} \circ s. \end{aligned}$$

Now suppose $\gamma > \omega^{\nu_*}$ and assume by induction that $\ell_\sigma \bullet s = \ell_\sigma \circ s$ for all $\sigma < \gamma$. Take $\sigma < \gamma$ with $\gamma = \omega^{\nu_*} + \sigma$. Then **C3**$_\nu$ and our inductive assumption gives

$$\ell_\gamma \bullet s = (\ell_\sigma \circ \ell_{\omega^{\nu_*}}) \bullet s = \ell_\sigma \bullet (\ell_{\omega^{\nu_*}} \bullet s) = \ell_\sigma \circ (\ell_{\omega^{\nu_*}} \circ s) = (\ell_\sigma \circ \ell_{\omega^{\nu_*}}) \circ s = \ell_\gamma \circ s.$$

This concludes the proof. $\square$

6 Hyperserial fields

We are now in a position to prove Theorems 1.1 and 1.2. Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series, let $\boldsymbol{\nu} \leqslant \mathbf{On}$, and let $\circ: \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ be a function. For $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}$, we define $\mathfrak{m}^r$ as follows: set $1^r := 1$, set $\mathfrak{m}^r := \ell_0^r \circ \mathfrak{m}$ if $\mathfrak{m} \succ 1$, and set $\mathfrak{m}^r := \ell_0^{-r} \circ \mathfrak{m}^{-1}$ if $\mathfrak{m} \prec 1$. For $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$, we define $\mathfrak{D}_\mu$ to be the class of series $s \in \mathbb{T}^{>,\succ}$ with $\ell_\gamma \circ s \in \mathfrak{M}^{\succ}$ for all $\gamma < \omega^\mu$. We say that $(\mathbb{T}, \circ)$ is a *hyperserial field of force* $\boldsymbol{\nu}$ if the following axioms are satisfied:

HF1. $\mathbb{L}_{<\omega^\nu} \longrightarrow \mathbb{T}; f \longmapsto f \circ s$ is a strongly $\mathbb{R}$-linear ordered field embedding for all $s \in \mathbb{T}^{>,\succ}$.

HF2. $f \circ (g \circ s) = (f \circ g) \circ s$ for all $f \in \mathbb{L}_{<\omega^\nu}$, $g \in \mathbb{L}_{<\omega^\nu}^{>,\succ}$, and $s \in \mathbb{T}^{>,\succ}$.

HF3. $f \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $f \in \mathbb{L}_{<\omega^\nu}$, $t \in \mathbb{T}^{>,\succ}$, and $\delta \in \mathbb{T}$ with $\delta \prec t$.

HF4. $\ell_{\omega^\mu}^{\uparrow\gamma} \circ s < \ell_{\omega^\mu}^{\uparrow\gamma} \circ t$ for all ordinals $\mu < \boldsymbol{\nu}$, $\gamma < \omega^\mu$, and all $s, t \in \mathbb{T}^{>,\succ}$ with $s < t$.

HF5. The map $\mathbb{R} \times \mathfrak{M} \to \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$ described above is a real power operation on $\mathfrak{M}$.

HF6. $\ell_1 \circ (s t) = \ell_1 \circ s + \ell_1 \circ t$ for all $s, t \in \mathbb{T}^{>,\succ}$.

HF7. $\operatorname{supp} \ell_1 \circ \mathfrak{m} \succ 1$ for all $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and
$\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ (\ell_\gamma \circ \mathfrak{a})^{-1}$ for all $1 \leqslant \mu < \boldsymbol{\nu}$, $\gamma < \omega^\mu$, and $\mathfrak{a} \in \mathfrak{D}_{\omega^\mu}$.

Axioms **HF6** and **HF7** only make sense when $\boldsymbol{\nu} > 0$, so they are assumed to hold trivially when $\boldsymbol{\nu} = 0$. We say that $(\mathbb{T}, \circ)$ is *confluent* if $\mathfrak{M} \neq 1$ and if for all $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$ and all $s \in \mathbb{T}^{>,\succ}$, there exist $\mathfrak{a} \in \mathfrak{D}_{\omega^\mu}$ and $\gamma < \omega^\mu$ with

$$\ell_\gamma \circ s \asymp \ell_\gamma \circ \mathfrak{a}.$$

For the remainder of this section, we fix a hyperserial field $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of force $\boldsymbol{\nu}$. For each $\mu < \boldsymbol{\nu}$, we define the function $L_{\omega^\mu}: \mathfrak{D}_{\omega^\mu} \longrightarrow \mathbb{T}; \mathfrak{a} \longmapsto \ell_{\omega^\mu} \circ \mathfrak{a}$. The *skeleton* of $(\mathbb{T}, \circ)$ is defined to be the structure $(\mathbb{T}, (L_{\omega^\mu})_{\mu < \boldsymbol{\nu}})$ equipped with the real power operation on $\mathfrak{M}$ given by **HF5**. The main purpose of this section is to prove the following refinement of Theorem 1.2.

Theorem 6.1. *The skeleton $(\mathbb{T}, (L_{\omega^\mu})_{\mu < \boldsymbol{\nu}})$ of $(\mathbb{T}, \circ)$ is a hyperserial skeleton. Moreover, if $(\mathbb{T}, \circ)$ is confluent, then so is its skeleton and $\circ$ coincides with the unique composition from Theorem 4.1.*

When $\boldsymbol{\nu} = 0$, then the skeleton of $\mathbb{T}$ is just the field $\mathbb{T}$ itself with the real power operation on $\mathfrak{M}$. Clearly, this is a hyperserial skeleton, as there are no axioms to verify. Moreover, it is 0-confluent so long as $(\mathbb{T}, \circ)$ is, so Theorem 6.1 follows from Proposition 5.5, since $\circ$ clearly satisfies $\mathbf{C1}_0$, $\mathbf{C2}_0$, $\mathbf{C3}_0$, and $\mathbf{C4}_0$. Therefore, we may assume that $\boldsymbol{\nu} > 0$. We will verify the various hyperserial skeleton axioms over the next few lemmas, beginning with the Domain of Definition axioms:

Lemma 6.2. *The skeleton $(\mathbb{T}, (L_{\omega^\mu})_{\mu < \boldsymbol{\nu}})$ satisfies the axioms $\mathbf{DD}_\mu$ for all $\mu < \boldsymbol{\nu}$.*

Proof. By definition, $\mathfrak{D}_0$ is the class of $s \in \mathbb{T}^{>,\succ}$ with $\ell_0 \circ s \in \mathfrak{M}^{\succ}$. Since $\ell_0 \circ s = s$ by **HF5**, the axiom $\mathbf{DD}_0$ holds. Let us fix $0 < \mu < \boldsymbol{\nu}$ and let us assume that $\mathbf{DD}_\eta$ holds for all $\eta < \mu$. If μ is a limit, then

$$\begin{aligned}\bigcap_{\eta < \mu} \operatorname{dom} L_{\omega^\eta} &= \bigcap_{\eta < \mu} \{s \in \mathbb{T}^{>,\succ} : \ell_\gamma \circ s \in \mathfrak{M}^{\succ} \text{ for all } \gamma < \omega^\eta\} \\ &= \{s \in \mathbb{T}^{>,\succ} : \ell_\gamma \circ s \in \mathfrak{M}^{\succ} \text{ for all } \gamma < \omega^\mu\} = \operatorname{dom} L_{\omega^\mu}.\end{aligned}$$

Suppose μ is a successor. The inclusion $\operatorname{dom} L_{\omega^\mu} \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\mu_*}}^{\circ n}$ holds by definition, so we show the other inclusion. Let $\gamma < \omega^\mu$ and let $s \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\mu_*}}^{\circ n}$. Take $n \in \mathbb{N}$ and $\sigma \ll \omega^{\mu_*}$ with $\gamma = \omega^{\mu_*} n + \sigma$. Then $L_{\omega^{\mu_*}}^{\circ n}(s) \in \operatorname{dom} L_{\omega^{\mu_*}}$, so $\ell_\sigma \circ L_{\omega^{\mu_*}}^{\circ n}(s) \in \mathfrak{M}^{\succ}$, by our inductive assumption. Repeated applications of **HF2** give $\ell_\sigma \circ L_{\omega^{\mu_*}}^{\circ n}(s) = \ell_\gamma \circ s$. Since $\gamma < \omega^\mu$ is arbitrary, this gives $s \in \operatorname{dom} L_{\omega^\mu}$. □

Now for the functional equations, asymptotics, regularity, and monotonicity axioms:

Lemma 6.3. *The skeleton $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\nu})$ satisfies the axioms* **FE**$_\mu$, **A**$_\mu$, *and* **R**$_\mu$ *for all $\mu < \nu$.*

Proof. Given $r \in \mathbb{R}^>$ and $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$, we have

$$\begin{aligned} L_1(\mathfrak{m}^r) &= \ell_1 \circ (\ell_0^r \circ \mathfrak{m}) = (\ell_1 \circ \ell_0^r) \circ \mathfrak{m} = (r\,\ell_1) \circ \mathfrak{m} = r\,(\ell_1 \circ \mathfrak{m}) = r\,L_1(\mathfrak{m}) && \text{(by HF2 and HF1)}\\ L_1(\mathfrak{m}\,\mathfrak{n}) &= \ell_1 \circ (\mathfrak{m}\,\mathfrak{n}) = \ell_1 \circ \mathfrak{m} + \ell_1 \circ \mathfrak{n} = L_1(\mathfrak{m}) + L_1(\mathfrak{n}), && \text{(by HF6)}\end{aligned}$$

so **FE**$_0$ holds. Let $0 < \mu < \nu$ be a successor ordinal and let $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$, so $L_{\omega^{\mu_*}}(\mathfrak{a})$ is defined and lies in $\mathfrak{M}_{\omega^\mu}$. The axiom **HF2** implies

$$L_{\omega^\mu}(L_{\omega^{\mu_*}}(\mathfrak{a})) = \ell_{\omega^\mu} \circ (\ell_{\omega^{\mu_*}} \circ \mathfrak{a}) = (\ell_{\omega^\mu} \circ \ell_{\omega^{\mu_*}}) \circ \mathfrak{a} = (\ell_{\omega^\mu} - 1) \circ \mathfrak{a} = L_{\omega^\mu}(\mathfrak{a}) - 1,$$

so **FE**$_\mu$ holds as well. The asymptotics axiom **A**$_0$ follows from the relation $\ell_1 \prec \ell_0$ in $\mathbb{L}_{<\omega^\nu}$ and **HF1**. Likewise, **A**$_\mu$ follows from the fact that $\ell_{\omega^\mu} \prec \ell_{\omega^\eta}$ for all $\eta < \mu$. By **HF1**, we note that the sets $(\ell_{<\omega^\mu} \circ s)^{-1}$ and $\{(\ell_{\omega^\eta n} \circ s)^{-1} : \eta < \mu \text{ and } n \in \mathbb{N}\}$ are mutually cofinal for each $s \in \mathbb{T}^{>,\succ}$. The regularity axioms **R**$_\mu$ for $\mu < \nu$ therefore follow from **HF7**. □

Lemma 6.4. *The skeleton $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\nu})$ satisfies the axioms* **M**$_\mu$ *for all $\mu < \nu$.*

Proof. The axiom **M**$_0$ follows from the fact that $\ell_1 > 0$. For $0 < \mu < \nu$, let $\gamma < \omega^\mu$ and take $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_{\omega^\mu}$ with $\mathfrak{a} < \mathfrak{b}$. We need to show

$$\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > (\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1}.$$

We first consider the case that $\mathfrak{a} \prec \ell_{\omega^\eta} \circ \mathfrak{b}$ for some $\eta < \mu$ with $\gamma < \omega^{\eta+1}$. Then **HF4** gives us that $\ell_{\omega^\mu}^{\uparrow\omega^\eta} \circ \mathfrak{a} < \ell_{\omega^\mu}^{\uparrow\omega^\eta} \circ (\ell_{\omega^\eta} \circ \mathfrak{b}) = \ell_{\omega^\mu} \circ \mathfrak{b}$. By (2.4), we have $\ell_{\omega^\mu}^{\uparrow\omega^\eta} = \ell_{\omega^\mu} + \mathfrak{l} + \varepsilon$, where $\mathfrak{l} = \frac{1}{\ell'_{\omega^{\eta+1}}} \ell'_{\omega^\mu} = \prod_{\omega^{\eta+1} \leqslant \sigma < \omega^\mu} \ell_\sigma^{-1}$ and $\varepsilon \prec \mathfrak{l}$. Since $\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu}^{\uparrow\omega^\eta} \circ \mathfrak{a} > 0$, we have

$$\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > \mathfrak{l} \circ \mathfrak{a} + \varepsilon \circ \mathfrak{a}.$$

Since $\gamma < \omega^{\eta+1}$, we have $\ell_\gamma^{-1} \prec \mathfrak{l}$, so $(\ell_\gamma \circ \mathfrak{a})^{-1} = \ell_\gamma^{-1} \circ \mathfrak{a} \prec \mathfrak{l} \circ \mathfrak{a}$. The axiom **HF4** gives $\ell_\gamma \circ \mathfrak{a} < \ell_\gamma \circ \mathfrak{b}$, so $(\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1} < 2(\ell_\gamma \circ \mathfrak{a})^{-1} \prec \mathfrak{l} \circ \mathfrak{a}$. Thus,

$$\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > (\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1}.$$

Now we handle the case that $\mathfrak{a} \succcurlyeq \ell_{\omega^\eta} \circ \mathfrak{b}$ for all $\eta < \mu$ with $\gamma < \omega^{\eta+1}$. We claim that the sets

$$\{(\ell_\sigma \circ \mathfrak{a})^{-1} : \sigma < \omega^\mu\} \text{ and } \{(\ell_\sigma \circ \mathfrak{b})^{-1} : \sigma < \omega^\mu\}$$

are mutually cofinal. Let $\sigma < \omega^\mu$ be given and take $\eta < \mu$ with $\gamma < \omega^{\eta+1}$ and $\sigma \lll \omega^\eta$. Then $\mathfrak{a} \succcurlyeq \ell_{\omega^\eta} \circ \mathfrak{b}$ by assumption, so $\mathfrak{a} > \ell_{\omega^\eta 2} \circ \mathfrak{b}$ and **HF4** gives $\ell_\sigma \circ \mathfrak{b} > \ell_\sigma \circ \mathfrak{a} > \ell_\sigma \circ (\ell_{\omega^\eta 2} \circ \mathfrak{b}) = \ell_{\omega^\eta 2 + \sigma} \circ \mathfrak{b}$. This proves the cofinality claim. Now, **HF7** gives $\operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{a}) \succ \{(\ell_\sigma \circ \mathfrak{a})^{-1} : \sigma < \omega^\mu\}$ and likewise, $\operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{b}) \succ \{(\ell_\sigma \circ \mathfrak{b})^{-1} : \sigma < \omega^\mu\}$. Thus,

$$\operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a}) \subseteq \operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{a}) \cup \operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{b}) \succ \{(\ell_\sigma \circ \mathfrak{a})^{-1}, (\ell_\sigma \circ \mathfrak{b})^{-1} : \sigma < \omega^\mu\}.$$

In particular, $\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > (\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1}$, as desired. □

Before proving the infinite powers axioms, we need a lemma:

Lemma 6.5. *Let $s = c\,\mathfrak{m}\,(1+\varepsilon) \in \mathbb{T}^{>,\succ}$ with $c \in \mathbb{R}^>$, $\mathfrak{m} := \mathfrak{d}_s$, and $\varepsilon \prec 1$. Then*

$$\ell_1 \circ s = \ell_1 \circ \mathfrak{m} + \log c + L(1+\varepsilon),$$

where L is as defined in Subsection 4.1.

Proof. Set $\delta := c\,\mathfrak{m}\,\varepsilon$, so $\delta \prec c\,\mathfrak{m}$ and $s = c\,\mathfrak{m} + \delta$. The axiom **HF3** gives

$$\ell_1 \circ s = \ell_1 \circ (c\,\mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta.$$

We have $\ell_1 \circ (c\,\mathfrak{m}) = \ell_1 \circ ((c\,\ell_0) \circ \mathfrak{m}) = (\ell_1 \circ (c\,\ell_0)) \circ \mathfrak{d}_{\mathfrak{l} \circ \mathfrak{a}}$ by **HF2**, and $\ell_1 \circ (c\,\ell_0) = \ell_1 + \log c$. Hence

$$\ell_1 \circ s = (\ell_1 + \log c) \circ \mathfrak{m} + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta = \ell_1 \circ \mathfrak{m} + \log c + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta.$$

Given $k > 0$, we have $\ell_1^{(k)} \circ t = (-1)^{k-1} (k-1)!\, t^{-k}$, so for $\delta \prec t$, we have

$$\frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta = \frac{(-1)^{k-1}}{k} \left(\frac{\delta}{c\,\mathfrak{m}}\right)^k = \frac{(-1)^{k-1}}{k} \varepsilon.$$

Thus, $\sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta^k = L(1+\varepsilon)$. □

Lemma 6.6. *The skeleton $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\nu})$ satisfies the axioms $\mathbf{P}_\mu$ for all $\mu \in \mathbf{On}$ with $\mu \leqslant \nu$.*

Proof. let $\mu \in \mathbf{On}$ with $\mu \leqslant \nu$, let $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and let $(r_\gamma)_{\gamma<\omega^\mu}$ be a sequence of real numbers. We need to show that $\sum_{\gamma<\omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{a}) \in \log \mathfrak{M}$, where $\log \mathfrak{m} := -\ell_1 \circ \mathfrak{m}^{-1}$ for $\mathfrak{m} \in \mathfrak{M}^\prec$ and where $\log 1 := 0$. Set $\mathfrak{l} := \prod_{\gamma<\omega^\mu} \ell_\gamma^{r_\gamma}$. We may assume that $\mathfrak{l} \neq 1$ and, by negating each r_γ if need be, we further assume that $\mathfrak{l} \succ 1$. Hence $\ell_1 \circ \mathfrak{l}$ is defined. The axioms **HF1** and **HF2** give

$$\sum_{\gamma<\omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{a}) = (\ell_1 \circ \mathfrak{l}) \circ \mathfrak{a} = \ell_1 \circ (\mathfrak{l} \circ \mathfrak{a}),$$

so it remains to show that $\mathfrak{l} \circ \mathfrak{a} \in \mathfrak{M}^\succ$. For each $\gamma < \omega^\mu$, we have $L_{\gamma+1}(\mathfrak{a}) \in \mathfrak{M}^\succ$. This gives $\operatorname{supp} \ell_1 \circ (\mathfrak{l} \circ \mathfrak{a}) \subseteq \mathfrak{M}^\succ$. Take $c \in \mathbb{R}^>$ and $\varepsilon \prec 1$ with $\mathfrak{l} \circ \mathfrak{a} = c\,\mathfrak{d}_{\mathfrak{l} \circ \mathfrak{a}}(1+\varepsilon)$. Lemma 6.5 gives

$$\ell_1 \circ (\mathfrak{l} \circ \mathfrak{a}) = \ell_1 \circ \mathfrak{d}_{\mathfrak{l} \circ \mathfrak{a}} + \log c + L(1+\varepsilon).$$

The axiom **HF7** gives $\operatorname{supp}(\ell_1 \circ \mathfrak{d}_{\mathfrak{l} \circ \mathfrak{a}}) \succ 1$. If $\varepsilon \neq 0$, then $L(1+\varepsilon) \sim \varepsilon$, so $\mathfrak{d}_\varepsilon \in \operatorname{supp} L(1+\varepsilon)$. If $c \neq 1$, we have $\operatorname{supp} \log c = \{1\}$. As we have established that $\operatorname{supp} \ell_1 \circ (\mathfrak{l} \circ \mathfrak{a}) \subseteq \mathfrak{M}^\succ$, it follows that $c = 1$ and $\varepsilon = 0$. Thus $\mathfrak{l} \circ \mathfrak{a} = \mathfrak{d}_{\mathfrak{l} \circ \mathfrak{a}} \in \mathfrak{M}$, as desired. □

This shows that $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\nu})$ is a hyperserial skeleton of force ν. Now we turn to confluence. First, we need a lemma:

Lemma 6.7. *Let $s, t \in \mathbb{T}^{>,\succ}$ and let $\gamma < \omega^\nu$. If $\ell_\gamma \circ s \asymp \ell_\gamma \circ t$, then $\ell_{\gamma+1} \circ s - \ell_{\gamma+1} \circ t \preccurlyeq 1$ and $\ell_\sigma \circ s - \ell_\sigma \circ t \prec 1$ for all σ with $\gamma + 2 \leqslant \sigma < \omega^\nu$. In particular, $\ell_\sigma \circ s \asymp \ell_\sigma \circ t$ for all σ with $\gamma \leqslant \sigma < \omega^\nu$.*

Proof. The proof is essentially the same as the proof of Lemma 4.8. Take $c \in \mathbb{R}^>$ and $\varepsilon \prec 1$ with $\ell_\gamma \circ s = c (\ell_\gamma \circ t)(1+\varepsilon)$. By Lemma 6.5, we have

$$\ell_{\gamma+1} \circ s = \ell_1 \circ (\ell_\gamma \circ s) = \ell_1 \circ (c\,(\ell_\gamma \circ t)\,(1+\varepsilon)) = \ell_{\gamma+1} \circ t + \log c + L(1+\varepsilon),$$

so $\ell_{\gamma+1} \circ s \sim \ell_{\gamma+1} \circ t$. Set $\delta := (\ell_{\gamma+1} \circ t)^{-1} (\log c + L(1+\varepsilon)) \prec 1$, so $\ell_{\gamma+1} \circ s = (\ell_{\gamma+1} \circ t)\,(1+\delta)$. Again, Lemma 6.5 gives

$$\ell_{\gamma+2} \circ s = \ell_1 \circ (\ell_{\gamma+1} \circ s) = \ell_1 \circ ((\ell_{\gamma+1} \circ t)\,(1+\delta)) = \ell_{\gamma+2} \circ t + L(1+\delta),$$

so $\ell_{\gamma+2} \circ s - \ell_{\gamma+2} \circ t = L(1+\delta) \sim \delta \prec 1$. Now set $h := (\ell_{\gamma+2} \circ s - \ell_{\gamma+2} \circ t) \prec 1$ and fix σ with $\gamma + 2 \leqslant \sigma < \omega^{\nu}$. We have

$$\begin{aligned} \ell_{\sigma} \circ s - \ell_{\sigma} \circ t &= \ell_{\sigma}^{\uparrow\gamma+2} \circ (\ell_{\gamma+2} \circ s) - \ell_{\sigma}^{\uparrow\gamma+2} \circ (\ell_{\gamma+2} \circ t) \\ &= \ell_{\sigma}^{\uparrow\gamma+2} \circ ((\ell_{\gamma+2} \circ t) + h) - \ell_{\sigma}^{\uparrow\gamma+2} \circ (\ell_{\gamma+2} \circ t) \\ &= \mathcal{T}_{\ell_{\sigma}^{\uparrow\gamma+2}}(\ell_{\gamma+2} \circ t, h) \sim (\ell_{\sigma}^{\uparrow\gamma+2})' \circ (\ell_{\gamma+2} \circ t)\, h. \end{aligned}$$

Since $(\ell_{\sigma}^{\uparrow\gamma+2})', h \prec 1$, we have $\ell_{\sigma} \circ s - \ell_{\sigma} \circ t \prec 1$. □

Lemma 6.8. *Suppose $(\mathbb{T}, \circ)$ is confluent. Then $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ is confluent as well.*

Proof. The skeleton $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ is 0-confluent since $\mathfrak{M}$ is non-trivial. Let $\mu \in \mathbf{On}$ with $0 < \mu \leqslant \nu$ and assume that $(\mathbb{T}, (L_{\omega^{\eta}})_{\eta<\nu})$ is η-confluent for all $\eta < \mu$. We also make the inductive assumption that for $s \in \mathbb{T}^{>,\succ}$ and $\eta < \mu$, we have $\ell_{\gamma} \circ s \asymp \ell_{\gamma} \circ \mathfrak{d}_{\omega^{\eta}}(s)$ for some $\gamma < \omega^{\eta}$. Let $s \in \mathbb{T}^{>,\succ}$ and take $\gamma < \omega^{\mu}$ and $\mathfrak{a} \in \mathfrak{M}_{\omega^{\mu}}$ with $\ell_{\gamma} \circ s \asymp \ell_{\gamma} \circ \mathfrak{a}$. We will show that $\mathfrak{d}_{\omega^{\mu}}(s) = \mathfrak{a}$. We first handle the case that μ is a successor. Take $n \in \mathbb{N}^{>}$ with $\gamma < \omega^{\mu_*} n$. Lemma 6.7 gives $\ell_{\omega^{\mu_*} n} \circ s \asymp \ell_{\omega^{\mu_*} n} \circ \mathfrak{a}$. By assumption, we have $\ell_{\rho} \circ \mathfrak{d}_{\omega^{\mu_*}}(s) \asymp \ell_{\rho} \circ s$ for some $\rho < \omega^{\mu_*}$, so $\ell_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}}(s) \asymp \ell_{\omega^{\mu_*}} \circ s$, again by Lemma 6.7. Induction on m gives $(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ m}(s) \asymp \ell_{\omega^{\mu_*} m} \circ s$ for all $m \in \mathbb{N}^{>}$, so

$$(L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(s) \asymp \ell_{\omega^{\mu_*} n} \circ s \asymp \ell_{\omega^{\mu_*} n} \circ \mathfrak{a} = (L_{\omega^{\mu_*}} \circ \mathfrak{d}_{\omega^{\mu_*}})^{\circ n}(\mathfrak{a}),$$

and $\mathfrak{d}_{\omega^{\mu}}(s) = \mathfrak{a}$. The case that μ is a limit is similar, though this time we take $\eta < \mu$ with $\gamma < \omega^{\eta}$ and use that

$$L_{\omega^{\eta}}(\mathfrak{d}_{\omega^{\eta}}(s)) \asymp \ell_{\omega^{\eta}} \circ s \asymp \ell_{\omega^{\eta}} \circ \mathfrak{a} \asymp L_{\omega^{\eta}}(\mathfrak{d}_{\omega^{\eta}}(\mathfrak{a}))$$

to see that $\mathfrak{d}_{\omega^{\mu}}(s) = \mathfrak{a}$. Since s was arbitrary, this gives that $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ is μ-confluent. □

Proof of Theorem 6.1. Lemmas 6.2, 6.3, 6.4, and 6.6 show that $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ of $(\mathbb{T}, \circ)$ is a hyperserial skeleton. The composition $\circ$ clearly satisfies $\mathbf{C1}_{\nu}$, $\mathbf{C2}_{\nu}$, $\mathbf{C3}_{\nu}$, and $\mathbf{C4}_{\nu}$. If $(\mathbb{T}, \circ)$ is confluent, then $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ is confluent by Lemma 6.8 and Proposition 5.17 implies that $\circ$ coincides with the unique composition from Theorem 4.1. □

Given a confluent hyperserial skeleton $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ of force ν, it is clear that the unique composition $\circ\colon \mathbb{L}_{<\omega^{\nu}} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ in Theorem 4.1 satisfies all of the hyperserial field axioms except for possibly **HF4**. In the course of our inductive proof in Sections 7 and 8, we will prove the following lemma (Lemma 7.5):

Lemma 6.9. *Let $(\mathbb{T}, (L_{\omega^{\eta}})_{\eta<\nu})$ be a confluent hyperserial skeleton of force ν and let $\circ$ be the composition law established in Theorem 4.1. Then the function $\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}; s \longmapsto \ell_{\omega^{\mu}}^{\uparrow\gamma} \circ s$ is strictly increasing for all ordinals $\gamma < \omega^{\mu} < \omega^{\nu}$.*

Thus, the unique composition in Theorem 4.1 satisfies **HF4** as well. The proof of Lemma 7.5 will not rely on any of the results from this section. This gives us the following refinement of Theorem 1.1:

Theorem 6.10. *If $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$ is a confluent hyperserial skeleton of force ν, then there is a unique function $\circ$ such that $(\mathbb{T}, \circ)$ is a confluent hyperserial field of force ν with skeleton $(\mathbb{T}, (L_{\omega^{\mu}})_{\mu<\nu})$.*

7 Hyperexponentiation

Our goal for Sections 7 and 8 is to prove Theorem 1.3. We will actually prove a "relative" version of the theorem, for which we first need a few more definitions. Given a confluent hyperserial skeleton $\mathbb{T}$ of force $\boldsymbol{\nu} \leqslant \mathbf{On}$, we let $\circ : \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ be the composition from Theorem 4.1. For each $\gamma < \omega^\nu$, we let $L_\gamma : \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ be the map given by $L_\gamma(s) := \ell_\gamma \circ s$.

Definition 7.1. *Let $\mathbb{T}$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. We say that $\mathbb{T}$ has* **force** $(\boldsymbol{\nu}, \boldsymbol{\mu})$ *if for each $\eta < \boldsymbol{\mu}$, the function $L_{\omega^\eta} : \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ is bijective.*

Note that if $\mathbb{T}$ has force $(\boldsymbol{\nu}, \boldsymbol{\mu})$, then $L_\gamma : \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ is bijective for all $\gamma < \omega^{\boldsymbol{\mu}}$.

Remark 7.2. Every confluent hyperserial skeleton of force $\boldsymbol{\nu}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, 0)$. Given a set-sized field of transseries $\mathbb{T}$, we recall that the exponential function cannot be total [23, Proposition 2.2]. Thus, any confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ with $\boldsymbol{\mu} > 0$ is necessarily a proper class.

Remark 7.3. Let $\mathbb{T}$ be a hyperserial skeleton of force $\mathbf{On}$. Then $\mathbb{T}$ is hyperserial of force $(\mathbf{On}, \mu)$ if and only if $(\mathbb{T}, (L_{\omega^\eta})_{\eta < \nu})$ is hyperserial of force (ν, μ) for all $\nu \geqslant \mu$. Similarly, $\mathbb{T}$ is hyperserial of force $(\mathbf{On}, \mathbf{On})$ if and only if $\mathbb{T}$ is hyperserial of force $(\mathbf{On}, \mu)$ for all μ.

We can now state the relative version of Theorem 1.3 that we are after:

Theorem 7.4. *Let $\mathbb{T}$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. Then $\mathbb{T}$ has a confluent extension $\mathbb{T}_{(<\boldsymbol{\mu})}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ with the following property: if $\mathbb{U}$ is another confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ and if $\Phi : \mathbb{T} \longrightarrow \mathbb{U}$ is an embedding of force $\boldsymbol{\nu}$, then there is a unique embedding $\Psi : \mathbb{T}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{U}$ of force $\boldsymbol{\nu}$ that extends Φ.*

Theorem 1.3 follows from Theorem 7.4 by taking $\boldsymbol{\nu} = \boldsymbol{\mu} = \mathbf{On}$. Throughout Section 7 and Subsections 8.1, 8.2, 8.3, and 8.4, we fix a confluent hyperserial skeleton $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of force $\boldsymbol{\nu}$ and an ordinal $\mu < \boldsymbol{\nu}$, and we make the induction hypothesis that Theorem 7.4 holds for $(\boldsymbol{\nu}, \mu)$. Note that this holds trivially if $\mu = 0$. Our main objective is to show that Theorem 7.4 still holds for $(\boldsymbol{\nu}, \mu + 1)$ instead of $(\boldsymbol{\nu}, \mu)$.

For this, we have to show how to define missing hyperexponentials of the form $E_{\omega^\mu}(s)$ with $s \in \mathbb{T}^{>,\succ}$. In this section, we start by giving a formula for hyperexponentials $E_{\omega^\mu}(s)$ that are already defined in $\mathbb{T}^{>,\succ}$. In the next section, we show how to adjoin the missing hyperexponentials to $\mathbb{T}$.

Before we continue, let us fix some notation. Let

$$\begin{aligned} \boldsymbol{\alpha} &:= \omega^{\boldsymbol{\nu}} \\ \beta &:= \omega^\mu \end{aligned}$$

Given $\gamma < \beta$, we set

$$\ell_{[\gamma,\beta)} := \prod_{\gamma \leqslant \sigma < \beta} \ell_\sigma \in \mathfrak{L}_{[\gamma,\beta)}, \qquad \ell_{(\gamma,\beta)} := \prod_{\gamma < \sigma < \beta} \ell_\sigma, \qquad \ell_{<\beta} := \ell_{[0,\beta)}.$$

Note that $\ell'_\beta = \ell_{<\beta}^{-1}$ and that $\ell_{[\gamma,\beta)}^{\uparrow\gamma} = \prod_{\gamma \leqslant \sigma < \beta} \ell_\sigma^{\uparrow\gamma}$. Given $s \in \mathbb{T}^{>,\succ}$, we set

$$L_{[\gamma,\beta)}(s) := \ell_{[\gamma,\beta)} \circ s, \qquad L_\beta^{\uparrow\gamma}(s) := \ell_\beta^{\uparrow\gamma} \circ s, \qquad L_{[\gamma,\beta)}^{\uparrow\gamma}(s) := \ell_{[\gamma,\beta)}^{\uparrow\gamma} \circ s,$$

and we view $L_{[\gamma,\beta)}$, $L_{\beta}^{\uparrow\gamma}$, and $L_{[\gamma,\beta)}^{\uparrow\gamma}$ as functions from $\mathbb{T}^{>,\succ}$ to $\mathbb{T}^{>,\succ}$. We define $L_{(\gamma,\beta)}$ and $L_{(\gamma,\beta)}^{\uparrow\gamma}$ analogously.

Given $\gamma<\boldsymbol{\alpha}$, we say that $E_{\gamma}(s)$ is defined if $s\in L_{\gamma}(\mathbb{T}^{>,\succ})$. If $\mathbb{T}$ is of force $(\boldsymbol{\nu},\mu)$, then $E_{\gamma}(s)$ is defined for all $\gamma<\omega^{\mu}$ and $s\in\mathbb{T}^{>,\succ}$. Lemma 4.17 tells us that L_{γ} is strictly increasing; in particular, it is injective. We let $E_{\gamma}\colon L_{\gamma}(\mathbb{T}^{>,\succ})\longrightarrow\mathbb{T}^{>,\succ}$ be its functional inverse, which is again strictly increasing. We may also consider E_{γ} as a partially defined function on $\mathbb{T}^{>,\succ}$.

Our induction hypothesis, that $\mathbb{T}_{(<\mu)}$ exists, has the following consequence:

Lemma 7.5. *For $\gamma<\beta$, the function $L_{\beta}^{\uparrow\gamma}$ is strictly increasing on $\mathbb{T}^{>,\succ}$.*

Proof. Let $s,t\in\mathbb{T}^{>,\succ}$ with $s<t$. By our inductive assumption, $E_{\gamma}(s)$ and $E_{\gamma}(t)$ both exist in $\mathbb{T}_{(<\mu)}$. As E_{γ} and L_{β} are strictly increasing on $\mathbb{T}_{(<\mu)}^{>,\succ}$ and $s<t$, we have $E_{\gamma}(s)<E_{\gamma}(t)$ and

$$L_{\beta}^{\uparrow\gamma}(s)=L_{\beta}(E_{\gamma}(s))<L_{\beta}(E_{\gamma}(t))=L_{\beta}^{\uparrow\gamma}(t). \qquad \square$$

7.1 Local inversion of the hyperlogarithms

In this subsection, we study the range of the functions $L_{\beta}^{\uparrow\gamma}$ for $\gamma<\beta$ and give a formula for their partial functional inverses. We fix $a\in\mathbb{T}^{>,\succ}$ and set $\varphi:=L_{\beta}(a)\in\mathbb{T}^{>,\succ}$. We also fix $\lambda<\beta$. For $k\in\mathbb{N}$, we define series $t_k\in\mathbb{L}_{<\beta}$ inductively by

$$\begin{aligned} t_0 &:= \ell_{\lambda} \\ t_{k+1} &:= \ell_{<\beta}\, t_k' \end{aligned}$$

Intuitively speaking, $t_k\circ a$ behaves like $(\ell_{\lambda}^{\uparrow\beta})^{(k)}\circ\varphi$, whereas the sum $\sum_{k\in\mathbb{N}}\frac{t_k\circ a}{k!}\varepsilon^k$ behaves like $L_{\lambda}(E_{\beta}(\varphi+\varepsilon))$ for $\varepsilon\prec L_{(\lambda,\beta)}(a)^{-1}$, and thereby provides a functional inverse of $L_{\beta}^{\uparrow\lambda}$ on a neighborhood of φ.

Proposition 7.6. *Let $\varepsilon\in\mathbb{T}$ with $\varepsilon\prec L_{(\lambda,\beta)}(a)^{-1}$. Then the family $((t_k\circ a)\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based and $t_0\circ a\succ(t_k\circ a)\,\varepsilon^k$ for $k>0$.*

Proof. Consider the derivative $\partial_{[\lambda,\beta)}:=\ell_{[\lambda,\beta)}^{\uparrow\lambda}\,\partial$ on $\mathbb{L}_{<\beta}$. We claim that $t_k=\partial_{[\lambda,\beta)}^k(\ell_0)\circ\ell_{\lambda}$ for all $k\in\mathbb{N}$. This is clear for $k=0$. Assuming that the claim holds for a given k, we have

$$\begin{aligned} t_{k+1} &= \ell_{<\beta}\,t_k'=\ell_{<\beta}\,(\partial_{[\lambda,\beta)}^k(\ell_0)\circ\ell_{\lambda})'=\ell_{<\beta}\,(\partial_{[\lambda,\beta)}^k(\ell_0)'\circ\ell_{\lambda})\,\ell_{\lambda}' \\ &= \ell_{[\lambda,\beta)}\,(\partial_{[\lambda,\beta)}^k(\ell_0)'\circ\ell_{\lambda})=(\ell_{[\lambda,\beta)}^{\uparrow\lambda}\,\partial_{[\lambda,\beta)}^k(\ell_0)')\circ\ell_{\lambda}=\partial_{[\lambda,\beta)}^{k+1}(\ell_0)\circ\ell_{\lambda}. \end{aligned}$$

In light of this claim, we have $t_k\circ a=\partial_{[\lambda,\beta)}^k(\ell_0)\circ L_{\lambda}(a)$. Recall that ∂ has well-based operator support $\operatorname{supp}_*\partial=\{\ell_{\gamma+1}':\gamma<\beta\}\preccurlyeq\ell_0^{-1}$ as an operator on $\mathbb{L}_{<\beta}$, so

$$\operatorname{supp}_*\partial_{[\lambda,\beta)}\preccurlyeq\ell_0^{-1}\,\ell_{[\lambda,\beta)}^{\uparrow\lambda}=\ell_0^{-1}\prod_{\lambda\leqslant\gamma<\beta}\ell_{\gamma}^{\uparrow\lambda}=\prod_{\lambda<\gamma<\beta}\ell_{\gamma}^{\uparrow\lambda}=\ell_{(\lambda,\beta)}^{\uparrow\lambda}.$$

Consider the strongly linear map

$$\begin{aligned} \Phi\colon\mathbb{L}_{<\beta} &\longrightarrow \mathbb{T} \\ f &\longmapsto f\circ L_{\lambda}(a) \end{aligned}$$

and set

$$\mathfrak{A}:=\bigcup_{\mathfrak{m}\in\operatorname{supp}_*\partial_{[\lambda,\beta)}}\operatorname{supp}\Phi(\mathfrak{m}),$$

so $\mathfrak{A}$ is well-based and $\mathfrak{A} \preccurlyeq L_{(\lambda,\beta)}^{\uparrow\lambda}(L_\lambda(a)) = L_{(\lambda,\beta)}(a)$. For $k \in \mathbb{N}$, we have $t_k \circ a = \Phi(\partial_{[\lambda,\beta)}^k(\ell_0))$, so for $\mathfrak{m} \in \operatorname{supp}(t_k \circ a)$, there exist $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \operatorname{supp}_* \partial_{[\lambda,\beta)}$ with

$$\mathfrak{m} \in (\operatorname{supp} \Phi(\mathfrak{m}_1) \cdots \operatorname{supp} \Phi(\mathfrak{m}_k)) \cdot \operatorname{supp} \Phi(\ell_0).$$

This gives us

$$\operatorname{supp}(t_k \circ a) \subseteq \mathfrak{A}^k \cdot \operatorname{supp} \Phi(\ell_0)$$

and it follows that

$$\operatorname{supp}((t_k \circ a)\, \varepsilon^k) \subseteq (\mathfrak{A} \cdot \operatorname{supp} \varepsilon)^k \cdot \operatorname{supp} \Phi(\ell_0).$$

As $\varepsilon \prec L_{(\lambda,\beta)}(a)^{-1}$, we have $\mathfrak{A} \cdot \operatorname{supp} \varepsilon \prec 1$, so we deduce that $((t_k \circ a)\, \varepsilon^k)_{k \in \mathbb{N}}$ is well-based and that $t_0 \circ a \succ (t_k \circ a)\, \varepsilon^k$ for $k > 0$. □

For our next result, we need a combinatorial lemma for power series over a differential field. Let (K, ∂) be a differential field. Then the ring $K[[X]]$ is naturally equipped with two derivations:

$$\begin{aligned} \left(\sum_{n=0}^{\infty} a_n X^n\right)' &:= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} X^n, \\ \partial\left(\sum_{n=0}^{\infty} a_n X^n\right) &:= \sum_{n=0}^{\infty} \partial(a_n)\, X^n. \end{aligned}$$

We also have a composition $\circ: K[[X]] \times X K[[X]] \longrightarrow K[[X]]$ given by

$$R \circ (XS) \longmapsto R(XS)$$

for $R, S \in K[[X]]$. This composition cooperates with our derivations as follows:

$$\partial(R \circ (XS)) = (\partial R) \circ (XS) + (R' \circ (XS))\, X \partial S, \qquad (R \circ (XS))' = (R' \circ (XS))\, (XS)'$$

Lemma 7.7. *Let $S = \sum_{n \in \mathbb{N}} a_n X^n \in K[[X]]$ and $R = \sum_{m \in \mathbb{N}} b_m X^m \in K[[X]]$. Write $F := R \circ (XS)$ and assume that we have*

$$u\, a_0\, \partial b_0 = 1, \qquad (n+2)\, a_{n+1} = u\, a_0\, \partial a_n, \qquad (m+1)\, b_{m+1} = u\, \partial b_m$$

for each n and m, where $u \in K$. Then $F = b_0 + X$.

Proof. The last two assumptions give us the following identities

$$R' = u \partial R, \quad \text{and} \tag{7.1}$$
$$(XS)' = a_0 (1 + u X \partial S). \tag{7.2}$$

We claim that $(\partial b_0) F' = \partial F$. Indeed, we have

$$\begin{aligned} \partial F &= \partial(R \circ (XS)) \\ &= (\partial R) \circ (XS) + (R' \circ (XS))\, X \partial S \\ &= (u^{-1} R') \circ (XS) + (R' \circ (XS))\, X \partial S && \text{(by (7.1))} \\ &= (u^{-1} + X \partial S)\, (R' \circ (XS)) \\ &= u^{-1} (1 + u X \partial S)\, (R' \circ (XS)) \\ &= u^{-1} a_0^{-1} (XS)' (R' \circ (XS)) && \text{(by (7.2))} \\ &= (\partial b_0)\, (XS)' (R' \circ (XS)) && (\text{since } u\, a_0\, \partial b_0 = 1) \\ &= (\partial b_0)\, (R \circ (XS))' \\ &= (\partial b_0)\, F'. \end{aligned}$$

Write $F=\sum_{k=0}^{\infty} F_k X^k$. The identity $(\partial b_0)\, F'=\partial F$ yields $F_{k+1}=\frac{1}{(k+1)\,\partial b_0}\partial F_k$ for each k. Since $F_0=b_0$, we conclude that $F_1=1$ and $F_k=0$ for $k>1$. □

Lemma 7.8. *Let $\varepsilon\in\mathbb{T}$ with $\varepsilon\prec L_{(\lambda,\beta)}(a)^{-1}$. Then*

$$L_\beta^{\uparrow\lambda}\left(\sum_{n\in\mathbb{N}}\frac{t_n\circ a}{n!}\varepsilon^n\right)=\varphi+\varepsilon. \tag{7.3}$$

Proof. We have

$$L_\beta^{\uparrow\lambda}\left(\sum_{n\in\mathbb{N}}\frac{t_n\circ a}{n!}\varepsilon^n\right)=L_\beta^{\uparrow\lambda}\left(t_0\circ a+\sum_{n\geqslant 1}\frac{t_n\circ a}{n!}\varepsilon^n\right)=\sum_{m\in\mathbb{N}}\frac{(\ell_\beta^{\uparrow\lambda})^{(m)}\circ(t_0\circ a)}{m!}\left(\sum_{n\geqslant 1}\frac{t_n\circ a}{n!}\varepsilon^n\right)^m.$$

Consider the formal power series

$$F(X)=\sum_{m\in\mathbb{N}}\frac{(\ell_\beta^{\uparrow\lambda})^{(m)}\circ t_0}{m!}\left(\sum_{n\geqslant 1}\frac{t_n}{n!}X^n\right)^m\in\mathbb{L}_{<\alpha}[[X]].$$

Writing $F(X)=\sum_{k\in\mathbb{N}} F_k X^k$, we have

$$\sum_{k\in\mathbb{N}}(F_k\circ a)\,\varepsilon^k=L_\beta^{\uparrow\lambda}\left(\sum_{n\in\mathbb{N}}\frac{t_n\circ a}{n!}\varepsilon^n\right).$$

Thus, it suffices to show that $F(X)=\ell_\beta+X$.

Let $a_n:=\frac{1}{(n+1)!}t_{n+1}$ and $b_m:=\frac{1}{m!}(\ell_\beta^{\uparrow\lambda})^{(m)}\circ t_0$. Then by factoring out X from the inner sum and reindexing, we have

$$F(X)=\sum_{m\in\mathbb{N}} b_m\left(X\sum_{n\in\mathbb{N}} a_n X^n\right)^m.$$

Note that the sequence (a_n) satisfies the identities:

$$a_0=t_1=\ell_{<\beta}\,\ell_\lambda'=\ell_{[\lambda,\beta)},\qquad a_{n+1}=\frac{t_{n+2}}{(n+2)!}=\frac{\ell_{<\beta}\,t_{n+1}'}{(n+2)!}=\frac{\ell_{<\beta}\,a_n'}{n+2}.$$

Since $((\ell_\beta^{\uparrow\lambda})^{(m)}\circ t_0)'=((\ell_\beta^{\uparrow\lambda})^{(m+1)}\circ t_0)\,t_0'=((\ell_\beta^{\uparrow\lambda})^{(m+1)}\circ t_0)\,\ell_\lambda'$, the sequence (b_m) satisfies the identities

$$b_0=\ell_\beta^{\uparrow\lambda}\circ t_0=\ell_\beta,\qquad b_{m+1}=\frac{1}{(m+1)!}(\ell_\beta^{\uparrow\lambda})^{(m+1)}\circ t_0=\frac{b_m'}{(m+1)\,\ell_\lambda'}.$$

Setting $u:=\ell_{<\lambda}$, we have

$$u\,a_0 b_0'=\ell_{<\beta}\,b_0'=1,\qquad (n+2)\,a_{n+1}=\ell_{<\beta}a_n'=u\,a_0 a_n',\qquad (m+1)\,b_{m+1}=\frac{b_m'}{\ell_\lambda'}=u\,b_m'.$$

Using Lemma 7.7, we conclude that $F(X)=b_0+X=\ell_\beta+X$. □

Proposition 7.9. *The map $s\longmapsto L_\beta^{\uparrow\lambda}(s)$ is a bijection from $L_\lambda(a)+\mathbb{T}^{\prec L_\lambda(a)}$ to $L_\beta(a)+\mathbb{T}^{\prec L_{(\lambda,\beta)}(a)^{-1}}$.*

Proof. Let $\delta\prec L_\lambda(a)$ and let $s:=L_\lambda(a)+\delta$. We have $L_\beta^{\uparrow\lambda}(s)=L_\beta(a)+\mathcal{T}_{\ell_\beta^{\uparrow\lambda}}(L_\lambda(a),\delta)$, so

$$L_\beta^{\uparrow\lambda}(s)-L_\beta(a)\sim((\ell_\beta^{\uparrow\lambda})'\circ L_\lambda(a))\,\delta\prec((\ell_\beta^{\uparrow\lambda})'\circ L_\lambda(a))\,L_\lambda(a).$$

Since $\ell_\beta'=(\ell_\beta^{\uparrow\lambda}\circ\ell_\lambda)'=((\ell_\beta^{\uparrow\lambda})'\circ\ell_\lambda)\,\ell_\lambda'$, we have

$$((\ell_\beta^{\uparrow\lambda})'\circ\ell_\lambda)\,\ell_\lambda=\frac{\ell_\beta'}{\ell_\lambda'}\,\ell_\lambda=\ell_{(\lambda,\beta)}^{-1},$$

so $L_\beta^{\uparrow\lambda}(s) - L_\beta(a) \prec L_{(\lambda,\beta)}(a)^{-1}$. This gives $L_\beta^{\uparrow\lambda}(s) \in L_\beta(a) + \mathbb{T}^{\prec L_{(\lambda,\beta)}(a)^{-1}}$.

Conversely, given $\varepsilon \prec L_{(\lambda,\beta)}(a)^{-1}$, Lemma 7.8 yields $L_\beta^{\uparrow\lambda}\left(\sum_{k\in\mathbb{N}} \frac{t_k\circ a}{k!}\varepsilon^k\right) = L_\lambda(a) + \varepsilon$. Let us show by induction on $k \geqslant 1$ that $t_k \preccurlyeq \ell_{(\lambda,\beta)}^k \ell_\lambda$. We have $t_1 = \ell_{<\beta}\,\ell_\lambda' = \ell_{[\lambda,\beta)} = \ell_{(\lambda,\beta)}\,\ell_\lambda$. Assuming that $t_k \preccurlyeq \ell_{(\lambda,\beta)}^k \ell_\lambda$, we have

$$t_{k+1} = \ell_{<\beta}\,t_k' \preccurlyeq \ell_{<\beta}\,(\ell_{(\lambda,\beta)}^k\,\ell_\lambda)' = \ell_{<\beta}\,(k\,\ell_{(\lambda,\beta)}^{k-1}\,\ell_{(\lambda,\beta)}'\,\ell_\lambda + \ell_{(\lambda,\beta)}^k\,\ell_\lambda').$$

We have

$$\ell_{(\lambda,\beta)}'\,\ell_\lambda = \ell_\lambda \sum_{\lambda<\sigma<\beta} \ell_\sigma^{-1}\,\ell_{<\sigma}^{-1}\,\ell_{(\lambda,\beta)} \sim \ell_\lambda\,\ell_{\lambda+1}^{-1}\,\ell_{<(\lambda+1)}^{-1}\,\ell_{(\lambda,\beta)} \prec \ell_\lambda\,\ell_{<(\lambda+1)}^{-1}\,\ell_{(\lambda,\beta)} = \ell_{<\lambda}^{-1}\,\ell_{(\lambda,\beta)} = \ell_\lambda'\,\ell_{(\lambda,\beta)},$$

so $k\,\ell_{(\lambda,\beta)}^{k-1}\,\ell_{(\lambda,\beta)}'\,\ell_\lambda + \ell_{(\lambda,\beta)}^k\,\ell_\lambda' \sim \ell_{(\lambda,\beta)}^k\,\ell_\lambda'$. This gives

$$t_{k+1} \preccurlyeq \ell_{<\beta}\,\ell_{(\lambda,\beta)}^k\,\ell_\lambda' = \frac{\ell_{<\beta}\,\ell_{(\lambda,\beta)}^k}{\ell_{<\lambda}} = \ell_{[\lambda,\beta)}\,\ell_{(\lambda,\beta)}^k = \ell_{(\lambda,\beta)}^{k+1}\,\ell_\lambda.$$

It follows that $(t_k\circ a)\,\varepsilon^k \prec (t_k\circ a)\,L_{(\lambda,\beta)}(a)^{-k} \preccurlyeq L_\lambda(a)$ for each $k>0$, so $\sum_{k\in\mathbb{N}^>} \frac{t_k\circ a}{k!}\varepsilon^k \prec L_\lambda(a)$. Since $t_0\circ a = L_\lambda(a)$, we conclude that $\sum_{k\in\mathbb{N}} \frac{t_k\circ a}{k!} \in L_\lambda(a) + \mathbb{T}^{\prec L_\lambda(a)}$. □

7.2 Truncated series

Definition 7.10. *We say that a series $\varphi \in \mathbb{T}$ is* **1-truncated** *if it is purely infinite, i.e. if* $\operatorname{supp}\varphi \subseteq \mathfrak{M}^{\succ}$. *For $0<\eta\leqslant\mu$, we say that $\varphi\in\mathbb{T}^{>,\succ}$ is* $\boldsymbol{\omega^\eta}$**-truncated** *if $\varphi > L_{\omega^\eta}^{\uparrow\gamma}(\mathfrak{m}^{-1})$ for all $\mathfrak{m}\in(\operatorname{supp}\varphi)^{\prec}$ and all $\gamma<\omega^\eta$. Let $\mathbb{T}_{>,\omega^\eta}$ denote the class of ω^η-truncated series in $\mathbb{T}$.*

In Subsection 3.3, we showed for $\eta<\nu$ that the class $\mathbb{T}^{>,\succ}$ can be partitioned into convex subclasses $\mathcal{E}_{\omega^\eta}[s], s\in\mathbb{T}^{>,\succ}$, each of which contains a unique $L_{<\omega^\eta}$-atomic element $\mathfrak{d}_{\omega^\eta}(s)$. In this section, we describe a different partition of $\mathbb{T}$ into convex subclasses, each of which will contain a unique ω^η-truncated series $\sharp_{\omega^\eta}(s)$. We will then show that L_β is bijective provided that $\mathbb{T}_{>,\beta}\subseteq L_\beta(\mathbb{T}^{>,\succ})$. This was done in [29] for $\beta=1$, but we provide a short proof below. First, for $s\in\mathbb{T}$, set $\sharp_1(s) := s_{\succ 1}$, so $\sharp_1(s)$ is 1-truncated and $s-\sharp_1(s)\preccurlyeq 1$. We also set $\mathcal{L}_1[s] := \{t\in\mathbb{T} : \sharp_1(s)=\sharp_1(t)\}$.

Proposition 7.11. [29, Proposition 2.3.8] *For $s\in\mathbb{T}^>$, we have $s\in\log\mathbb{T}^>$ if and only if $\sharp_1(s)\in\log\mathbb{T}^>$. Thus, the function* $\log\colon\mathbb{T}^>\longrightarrow\mathbb{T}$ *is bijective if and only if $\mathbb{T}_{>,1}\subseteq\log\mathbb{T}^>$.*

Proof. Let $s\in\mathbb{T}^>$ and let $r\in\mathbb{R}$, and $\varepsilon\in\mathbb{T}^{\prec}$ with $s=\sharp_1(s)+r+\varepsilon$. We have $r+\varepsilon\in\log\mathbb{T}^>$, since $\exp(r+\varepsilon)=\exp(r)\,E(\varepsilon)\in\mathbb{T}^>$. Thus, we have $s\in\log\mathbb{T}^>$ if and only if $\sharp_1(s)\in\log\mathbb{T}^>$, since $\log\mathbb{T}^>$ is an additive subgroup of $\mathbb{T}$. □

As a related fact for later use, we also note the following:

Lemma 7.12. *For $s\in\mathbb{T}^>$, we have $\sharp_1(\log s)=\log\mathfrak{d}_s$. Thus, $s\in\mathfrak{M}$ if and only if $\log s\in\mathbb{T}_{>,1}$. Moreover, L_1 is a bijection between $\mathcal{E}_1[\mathfrak{m}]$ and $\mathcal{L}_1[L_1(\mathfrak{m})]$ for each $\mathfrak{m}\in\mathfrak{M}^{\succ}$.*

Proof. Given $s\in\mathbb{T}^>$, write $s=c\,\mathfrak{m}\,(1+\varepsilon)$, where $c\in\mathbb{R}^>$, $\mathfrak{m}:=\mathfrak{d}_s$, and $\varepsilon\prec 1$. We have

$$\log s = \log\mathfrak{m} + \log c + L(1+\varepsilon)$$

$\mathbf{R}_0$ and $\mathbf{M}_0$ give that $L_1(\mathfrak{M}^{\succ}) \subseteq \mathbb{T}_{>,1}^{\geqslant}$ and, as $\mathbb{T}_{>,1}$ is a subgroup of $\mathbb{T}$, it follows that $\log \mathfrak{M} \subseteq \mathbb{T}_{>,1}$. Thus, $\log \mathfrak{m}$ is 1-truncated. If $c \neq 1$, then $\operatorname{supp} \log c = \{1\}$ and if $\varepsilon \neq 0$, then $L(1+\varepsilon) \sim \varepsilon$, so $\operatorname{supp} L(1+\varepsilon) \preccurlyeq \mathfrak{d}_{\varepsilon}$. Thus, $\sharp_1(\log s) = \log \mathfrak{m}$, as desired. The fact that $s \in \mathfrak{M}$ if and only if $\log s \in \mathbb{T}_{>,1}$ follows from this and the fact that log is injective. Now assume that $s \succ 1$ and let $\mathfrak{n} \in \mathfrak{M}^{\succ}$. Then

$$s \in \mathcal{E}_1[\mathfrak{n}] \iff \mathfrak{m} = \mathfrak{n} \iff \sharp_1(L_1(s)) = L_1(\mathfrak{m}) = L_1(\mathfrak{n}) \iff L_1(s) \in \mathcal{L}_1[L_1(\mathfrak{n})],$$

so $L_1(\mathcal{E}_1[\mathfrak{n}]) = \mathcal{L}_1[L_1(\mathfrak{n})] \cap L_1(\mathbb{T}^{>,\succ})$. By Proposition 7.11, $\mathcal{L}_1[L_1(\mathfrak{n})] \cap L_1(\mathbb{T}^{>,\succ}) = \mathcal{L}_1[L_1(\mathfrak{n})]$. □

For the remainder of this subsection, we assume that $\mu > 0$.

Lemma 7.13. *We have $\mathbb{T}_{>,\beta} + \mathbb{R}^{\geqslant} \subseteq \mathbb{T}_{>,\beta}$. If μ is a successor, then $\mathbb{T}_{>,\beta} + \mathbb{R} = \mathbb{T}_{>,\beta}$.*

Proof. For $\varphi \in \mathbb{T}_{>,\beta}$ and $r \in \mathbb{R}^{\geqslant}$, we have $(\operatorname{supp} \varphi + r)^{\prec} = (\operatorname{supp} \varphi)^{\prec}$ and $\varphi + r \geqslant \varphi$ so $\varphi + r \in \mathbb{T}_{>,\beta}$. Assume now that μ is a successor and let $\varphi \in \mathbb{T}_{>,\beta}$ and $r \in \mathbb{R}$. Again, $(\operatorname{supp} \varphi + r)^{\prec} = (\operatorname{supp} \varphi)^{\prec}$. Take $n \in \mathbb{N}$ with $n > -r$. Then for all $\gamma < \beta$ and $\mathfrak{m} \in (\operatorname{supp} \varphi)^{\prec}$, we have

$$\varphi > L_{\beta}^{\uparrow \gamma + \omega^{\mu_*} n}(\mathfrak{m}^{-1}) = L_{\beta}^{\uparrow \gamma}(\mathfrak{m}^{-1}) + n > L_{\beta}^{\uparrow \gamma}(\mathfrak{m}^{-1}) - r,$$

so $\varphi + r > L_{\beta}^{\uparrow \gamma}(\mathfrak{m}^{-1})$. □

Lemma 7.14. *Let $a \in \mathbb{T}^{>,\succ}$ and let $\varphi := L_{\beta}(a) \in \mathbb{T}^{>,\succ}$. Then φ is β-truncated if and only if $\operatorname{supp} \varphi \succ L_{\gamma}(a)^{-1}$ for all $\gamma < \beta$.*

Proof. We have $(\operatorname{supp} \varphi)^{\succcurlyeq} \succ L_{\gamma}(a)^{-1}$ for all $\gamma < \beta$ since the series $L_{\gamma}(a)$ is infinite. Let $\mathfrak{m} \in (\operatorname{supp} \varphi)^{\prec}$ and let $\gamma < \beta$. By Lemma 7.5, the function $L_{\beta}^{\uparrow \gamma}$ is strictly increasing, so we have $\varphi = L_{\beta}^{\uparrow \gamma}(L_{\gamma}(a)) > L_{\beta}^{\uparrow \gamma}(\mathfrak{m}^{-1})$ if and only if $L_{\gamma}(a) > \mathfrak{m}^{-1}$, hence the result. □

By Lemma 7.14 and $\mathbf{R}_{\mu}$, the series $L_{\beta}(\mathfrak{a})$ is β-truncated for all $\mathfrak{a} \in \mathfrak{M}_{\beta}$. The axiom $\mathbf{R}_0$ also gives that $L_1(\mathfrak{m})$ is 1-truncated for $\mathfrak{m} \in \mathfrak{M}_1$.

Lemma 7.15. *Let $s, t \in \mathbb{T}^{>,\succ}$ with $s \succcurlyeq t$ and let $\gamma < \beta$. Then $L_{\beta}^{\uparrow \gamma + 1}(s) > L_{\beta}^{\uparrow \gamma}(t)$.*

Proof. Take $r \in \mathbb{R}^{>}$ with $rs > t$. Then Lemma 7.5 gives $L_{\beta}^{\uparrow \gamma}(rs) > L_{\beta}^{\uparrow \gamma}(t)$, so it is enough to prove that $L_{\beta}^{\uparrow \gamma + 1}(s) > L_{\beta}^{\uparrow \gamma}(rs)$. For this, we may show that $\ell_{\beta}^{\uparrow \gamma + 1} > \ell_{\beta}^{\uparrow \gamma} \circ (r\,\ell_0)$ in $\mathbb{L}$. As the map $\mathbb{L} \longrightarrow \mathbb{L}; f \longmapsto f \circ \ell_1$ is order-preserving, it is enough to show that

$$\ell_{\beta}^{\uparrow \gamma} = \ell_{\beta}^{\uparrow \gamma + 1} \circ \ell_1 > (\ell_{\beta}^{\uparrow \gamma} \circ (r\,\ell_0)) \circ \ell_1 = \ell_{\beta}^{\uparrow \gamma} \circ (r\,\ell_1).$$

This follows from Lemma 7.5 and the fact that $r\,\ell_1 < \ell_0$. □

Definition 7.16. *For $t \in \mathbb{T}^{>,\succ}$, we define*

$$\mathcal{L}_{\beta}[t] := \{s \in t + \mathbb{T}^{\prec} : s = t \text{ or } (s \neq t \text{ and } t < L_{\beta}^{\uparrow \gamma}(|s-t|^{-1}) \text{ for some } \gamma < \beta)\}.$$

Proposition 7.17. *The classes $\mathcal{L}_{\beta}[t]$ form a partition of $\mathbb{T}^{>,\succ}$ into convex subclasses.*

Proof. Let $t \in \mathbb{T}^{>,\succ}$. The convexity of $\mathcal{L}_{\beta}[t]$ follows immediately from the definition of $\mathcal{L}_{\beta}[t]$ and Lemma 7.5. Let $s \in \mathcal{L}_{\beta}[t]$. We claim that $\mathcal{L}_{\beta}[t] \subseteq \mathcal{L}_{\beta}[s]$, from which it follows by symmetry that $\mathcal{L}_{\beta}[t] = \mathcal{L}_{\beta}[s]$. This clearly holds if $s = t$, so assume that $s \neq t$.

We first show that $t\in\mathcal{L}_\beta[s]$. Let $\varepsilon:=s-t\prec 1$ and let $\gamma<\beta$ with $t<L_\beta^{\uparrow\gamma}(|\varepsilon|^{-1})$ for some $\gamma<\beta$. Given σ with $\beta>\sigma>\gamma$, we have $\ell_\sigma^{\uparrow\gamma}\circ\ell_\gamma=\ell_\sigma<\ell_\gamma$, whence $\ell_\sigma^{\uparrow\gamma}<\ell_0$. Therefore,

$$L_\beta^{\uparrow\gamma}(|\varepsilon|^{-1})=L_\beta^{\uparrow\sigma}(L_\sigma^{\uparrow\gamma}(|\varepsilon|^{-1}))<L_\beta^{\uparrow\sigma}(|\varepsilon|^{-1})$$

by Lemma 7.5, so $t<L_\beta^{\uparrow\sigma}(|\varepsilon|^{-1})$ for all such σ.

If μ is a successor, take $n<\omega$ with $\gamma<\omega^{\mu_*}n$. Then $t<L_\beta^{\uparrow\omega^{\mu_*}n}(|\varepsilon|^{-1})$ and since $s-t=\varepsilon\prec 1$, we have

$$s=t+\varepsilon<L_\beta^{\uparrow\omega^{\mu_*}n}(|\varepsilon|^{-1})+\varepsilon<L_\beta(|\varepsilon|^{-1})+n+1=L_\beta^{\uparrow\omega^{\mu_*}(n+1)}(|\varepsilon|^{-1}).$$

If μ is a limit, take $\eta<\mu$ with $\gamma<\omega^\eta$, so that $t<L_\beta^{\uparrow\omega^\eta}(|\varepsilon|^{-1})$. Let us show that $s<L_\beta^{\uparrow\omega^{\eta+1}}(|\varepsilon|^{-1})$. Suppose for contradiction that $s\geqslant L_\beta^{\uparrow\omega^{\eta+1}}(|\varepsilon|^{-1})$. By (2.4), we have

$$\ell_\beta^{\uparrow\omega^\eta}-\ell_\beta\sim\frac{1}{\ell'_{\omega^{\eta+1}}}\ell'_\beta,\qquad \ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\sim\frac{1}{\ell'_{\omega^{\eta+2}}}\ell'_\beta.$$

Since $\ell'_{\omega^{\eta+2}}\prec\ell'_{\omega^{\eta+1}}$, we have $\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta>\ell_\beta^{\uparrow\omega^\eta}-\ell_\beta$, so

$$\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta^{\uparrow\omega^\eta}=\left(\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\right)-\left(\ell_\beta^{\uparrow\omega^\eta}-\ell_\beta\right)\sim\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\sim\frac{1}{\ell'_{\omega^{\eta+2}}}\ell'_\beta=\ell^{-1}_{[\omega^{\eta+2},\beta)}.$$

Therefore,

$$\varepsilon=s-t\geqslant L_\beta^{\uparrow\omega^{\eta+1}}(|\varepsilon|^{-1})-L_\beta^{\uparrow\omega^\eta}(|\varepsilon|^{-1})\sim L_{[\omega^{\eta+2},\beta)}(|\varepsilon|^{-1})^{-1}.$$

This means that $|\varepsilon|^{-1}\preccurlyeq L_{[\omega^{\eta+2},\beta)}(|\varepsilon|^{-1})$: a contradiction since $\ell_{[\omega^{\eta+2},\beta)}\prec\ell_0$.

Now let $u\in\mathcal{L}_\beta[t]$ and let us show $u\in\mathcal{L}_\beta[s]$. This is clear if $u=s$ or if $u=t$, so we assume that u, s, and t are pairwise distinct. By our claim, we have $t\in\mathcal{L}_\beta[s]$ and $t\in\mathcal{L}_\beta[u]$, so take $\gamma<\beta$ with $s<L_\beta^{\uparrow\gamma}(|t-s|^{-1})$ and $u<L_\beta^{\uparrow\gamma}(|t-u|^{-1})$. Note that

$$|s-u|\leqslant|t-s|+|t-u|\leqslant 2\max(|t-s|,|t-u|),$$

thus, $|s-u|^{-1}\geqslant\frac{1}{2}\min(|t-s|^{-1},|t-u|^{-1})$. Lemmas 7.5 and 7.15 yield

$$L_\beta^{\uparrow\gamma+1}(|s-u|^{-1})>L_\beta^{\uparrow\gamma+1}(2|s-u|^{-1})>\min(L_\beta^{\uparrow\gamma}(|t-s|^{-1}),L_\beta^{\uparrow\gamma}(|t-u|^{-1})).$$

If $L_\beta^{\uparrow\gamma+1}(|s-u|^{-1})>L_\beta^{\uparrow\gamma}(|t-s|^{-1})>s$, then $u\in\mathcal{L}_\beta[s]$ by definition. If $L_\beta^{\uparrow\gamma+1}(|s-u|^{-1})>L_\beta^{\uparrow\gamma}(|t-u|^{-1})>u$, then $s\in\mathcal{L}_\beta[u]$, so $u\in\mathcal{L}_\beta[s]$ by our claim. □

Proposition 7.18. *Let $t\in\mathbb{T}^{>,\succ}$. Then the class $\mathcal{L}_\beta[t]$ contains exactly one β-truncated element.*

Proof. Let us first show that $\mathcal{L}_\beta[t]$ contains a β-truncated element. Suppose that t itself is not β-truncated, let $\mathfrak{m}\in(\operatorname{supp}t)^\prec$ be greatest such that $t\leqslant L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1})$ for some $\gamma<\beta$. Setting $\varphi:=t_{\succ\mathfrak{m}}$, we have $\varphi-t\asymp\mathfrak{m}$, so $L_\beta^{\uparrow\gamma+1}(|\varphi-t|^{-1})>L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1})$ by Lemma 7.15. Our assumption on $\mathfrak{m}$ therefore yields $L_\beta^{\uparrow\gamma+1}(|\varphi-t|^{-1})>t$, whence $\varphi\in\mathcal{L}_\beta[t]$.

We claim that φ is β-truncated. Fix $\mathfrak{n}\in(\operatorname{supp}\varphi)^\prec$. By definition of φ, we have $t>L_\beta^{\uparrow\gamma+1}(\mathfrak{n}^{-1})$ for all $\gamma<\beta$. Since $t-\varphi_{\succ\mathfrak{n}}\asymp\mathfrak{n}$, Lemma 7.15 gives $L_\beta^{\uparrow\gamma+1}(\mathfrak{n}^{-1})>L_\beta^{\uparrow\gamma}(|t-\varphi_{\succ\mathfrak{n}}|^{-1})$ for all $\gamma<\beta$, so $\varphi_{\succ\mathfrak{n}}\notin\mathcal{L}_\beta[t]=\mathcal{L}_\beta[\varphi]$. By definition, this means that $\varphi\geqslant L_\beta^{\uparrow\gamma+1}(|\varphi-\varphi_{\succ\mathfrak{n}}|^{-1})$ for all $\gamma<\beta$. Since $\varphi-\varphi_{\succ\mathfrak{n}}\asymp\mathfrak{n}$, we have $L_\beta^{\uparrow\gamma+1}(|\varphi-\varphi_{\succ\mathfrak{n}}|^{-1})>L_\beta^{\uparrow\gamma}(\mathfrak{n}^{-1})$, by Lemma 7.15. Thus, $\varphi>L_\beta^{\uparrow\gamma}(\mathfrak{n}^{-1})$, as claimed.

Now let $\varphi, \psi \in \mathbb{T}^{>,\succ}$ be β-truncated series with $\varphi \in \mathcal{L}_\beta[\psi]$. We need to show that $\varphi = \psi$. Take $\gamma < \beta$ with $\varphi < L_\beta^{\uparrow\gamma}(|\varphi - \psi|^{-1})$. For $\mathfrak{m} \in (\operatorname{supp} \varphi)^{\prec}$, we have $\varphi > L_\beta^{\uparrow\gamma+1}(\mathfrak{m}^{-1})$ since φ is β-truncated. Therefore,

$$L_\beta^{\uparrow\gamma}(|\varphi - \psi|^{-1}) > \varphi > L_\beta^{\uparrow\gamma+1}(\mathfrak{m}^{-1}),$$

so $|\varphi - \psi|^{-1} \succ \mathfrak{m}^{-1}$ by Lemma 7.15. Thus $(\operatorname{supp} \varphi)^{\prec} \succ |\varphi - \psi|$. Since $|\varphi - \psi| \prec 1$, we deduce $\operatorname{supp} \varphi \succ |\varphi - \psi|$, so $\varphi \trianglelefteq \psi$. We also have $\psi \in \mathcal{L}_\beta[\varphi]$, so the same argument gives $\psi \trianglelefteq \varphi$ and we conclude that $\varphi = \psi$. □

For $t \in \mathbb{T}^{>,\succ}$, we define $\sharp_\beta(t)$ to be the unique β-truncated series in $\mathcal{L}_\beta[t]$. Note that this definition extends the previous definition of $\sharp_1$. It follows from the proof of Proposition 7.18 that $\sharp_\beta(t) \trianglelefteq s$ for all $s \in \mathcal{L}_\beta[t]$ and that

$$\mathcal{L}_\beta[t] \; = \; \{s \in \mathbb{T}^{>,\succ} : \sharp_\beta(s) = \sharp_\beta(t)\}.$$

Proposition 7.19. *For $a \in \mathbb{T}^{>,\succ}$ we have*

$$\mathcal{L}_\beta[L_\beta(a)] = \{s \in \mathbb{T}^{>,\succ} : s - L_\beta(a) \prec L_{[\gamma,\beta)}(a)^{-1} \text{ for some } \gamma < \beta\}.$$

Proof. We have $s \in \mathcal{L}_\beta[L_\beta(a)] \setminus \{L_\beta(a)\}$ if and only if $L_\beta^{\uparrow\rho}(|s - L_\beta(a)|^{-1}) > L_\beta(a)$ for some $\rho < \beta$. Since $L_\beta(a) = L_\beta^{\uparrow\rho}(L_\rho(a))$ for each $\rho < \beta$, this is in turn equivalent to $|s - L_\beta(a)|^{-1} > L_\rho(a)$ by Lemma 7.5 . Thus, $s \in \mathcal{L}_\beta[L_\beta(a)]$ if and only if $|s - L_\beta(a)| < L_\rho(a)^{-1}$ for some $\rho < \beta$, and it remains to show that $|s - L_\beta(a)| < L_\rho(a)^{-1}$ for some $\rho < \beta$ if and only if $|s - L_\beta(a)| \prec L_{[\gamma,\beta)}(a)^{-1}$ for some $\gamma < \beta$. This follows from the fact that if $\rho < \gamma < \beta$, then $\ell_\rho \succ \ell_{[\gamma,\beta)} \succ \ell_\gamma$, so $L_\rho(a)^{-1} \prec L_{[\gamma,\beta)}(a)^{-1} \prec L_\gamma(a)^{-1}$. □

Proposition 7.20. *For each $a \in \mathbb{T}^{>,\succ}$ we have $L_\beta(\mathcal{E}_\beta[a]) \subseteq \mathcal{L}_\beta[L_\beta(a)]$.*

Proof. Let $u \in \mathcal{E}_\beta[a]$. Then there is $\lambda = \omega^\eta n < \beta$ with $L_\lambda(u) - L_\lambda(a) \prec 1$. Thus, $L_\lambda(u) \in L_\lambda(a) + \mathbb{T}^{\prec}$ and so $L_\beta(u) = L_\beta^{\uparrow\lambda}(L_\lambda(u)) \in L_\beta(a) + \mathbb{T}^{\prec L_{[\lambda,\beta)}(a)^{-1}}$ by Proposition 7.9. Therefore, $L_\beta(u) \in \mathcal{L}_\beta[L_\beta(a)]$ by Proposition 7.19. □

Corollary 7.21. *We have $\sharp_\beta \circ L_\beta = L_\beta \circ \mathfrak{d}_\beta$ on $\mathbb{T}^{>,\succ}$. Thus, for $s \in \mathbb{T}^{>,\succ}$, we have $s \in \mathfrak{M}_\beta$ if and only if $L_\beta(s) \in \mathbb{T}_{\succ,\beta}$.*

Proof. Let $s \in \mathbb{T}^{>,\succ}$. Then $L_\beta(\mathfrak{d}_\beta(s)) \in \mathcal{L}_\beta[L_\beta(s)]$ by Proposition 7.20 and $L_\beta(\mathfrak{d}_\beta(s))$ is β-truncated by $\mathbf{R}_\mu$ and Lemma 7.14. Thus $L_\beta(\mathfrak{d}_\beta(s)) = \sharp_\beta(L_\beta(s))$. The fact that $s \in \mathfrak{M}_\beta$ if and only if $L_\beta(s) \in \mathbb{T}_{\succ,\beta}$ follows from this and the fact that L_β is injective. □

Proposition 7.22. *Assume that $\mathbb{T}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \mu)$. Then $L_\beta(\mathcal{E}_\beta[s]) = \mathcal{L}_\beta[L_\beta(s)]$ for all $s \in \mathbb{T}^{>,\succ}$. In particular, if $E_\beta(t)$ is defined for $t \in \mathbb{T}^{>,\succ}$, then E_β is defined on $\mathcal{L}_\beta[t]$.*

Proof. We prove this by induction on μ. Let $s \in \mathbb{T}^{>,\succ}$. By Proposition 7.20, we need only prove that $L_\beta(\mathcal{E}_\beta[s]) \supseteq \mathcal{L}_\beta[L_\beta(s)]$. Let $t \in \mathcal{L}_\beta[L_\beta(s)]$. By Proposition 7.19, there is a $\lambda = \omega^\eta n < \beta$ with $t \in L_\beta(s) + \mathbb{T}^{\prec L_{[\lambda,\beta)}^{-1}(s)}$. By Proposition 7.9, there is a $v \in L_\lambda(s) + \mathbb{T}^{\prec L_\lambda(s)}$ with $t = L_\beta^{\uparrow\lambda}(v)$. Since $\mathbb{T}$ is hyperserial of force $(\boldsymbol{\nu}, \mu)$, the hyperexponential $E_\lambda(v)$ is defined and

$$E_\beta(t) = E_\lambda(v).$$

Finally, since $v \sim L_\lambda(s)$, Lemma 4.8 and Proposition 4.9 imply $E_\lambda(v) \in \mathcal{E}_\beta[s]$. □

Corollary 7.23. *Assume that $\mathbb{T}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \mu)$. Then we have $E_\beta \circ \sharp_\beta = \mathfrak{d}_\beta \circ E_\beta$ whenever one of the sides is defined.*

Corollary 7.24. *The following are equivalent:*

a) *$\mathbb{T}$ has force $(\boldsymbol{\nu}, \mu + 1)$.*

b) *For all $\eta \leqslant \mu$, the function E_{ω^η} is defined on $\mathbb{T}_{>,\omega^\eta}$.*

c) *For all $\eta \leqslant \mu$ and $s \in \mathbb{T}^{>,\succ}$, the hyperexponential $E_{\omega^\eta}(t)$ is defined for some $t \in \mathcal{L}_{\omega^\eta}[s]$.*

d) *For all $\eta \leqslant \mu$, we have $L_{\omega^\eta}(\mathfrak{M}_{\omega^\eta}) = \mathbb{T}_{>,\omega^\eta}$.*

Proof. The equivalence between *a*) and *b*) follows from Proposition 7.22 and the fact that we have

$$\mathbb{T}^{>,\succ} = \bigsqcup_{\varphi \in \mathbb{T}_{>,\omega^\eta}} \mathcal{L}_{\omega^\eta}[\varphi]$$

for all $\eta \leqslant \mu$. The equivalence between *b*) and *c*) follows directly from Proposition 7.22. The equivalence between *b*) and *d*) follows from Corollary 7.21. □

7.3 Useful properties of truncation

Throughout this subsection, we let $0 < \mu < \boldsymbol{\nu}$ and we set $\beta := \omega^\mu$ and $\theta := \omega^{\mu_*}$. Given $s, t \in \mathbb{T}^{>,\succ}$, it will be convenient to introduce the following notations:

$$\begin{aligned} s <_\beta t &\iff \mathcal{L}_\beta[s] < \mathcal{L}_\beta[t] \iff \sharp_\beta(s) < \sharp_\beta(t) \\ s =_\beta t &\iff \mathcal{L}_\beta[s] = \mathcal{L}_\beta[t] \iff \sharp_\beta(s) = \sharp_\beta(t) \end{aligned}$$

Lemma 7.25. *Let $s \in \mathbb{T}^{>,\succ}$, $\gamma < \beta$, and $r \in \mathbb{R}^>$. We have*

$$L_\beta^{\uparrow\gamma}(r L_\gamma(s)) =_\beta L_\beta(s)$$

Proof. We claim that if $\ell_\beta \neq \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma)$, then $\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma) \prec 1$ and

$$\ell_\beta < \ell_\beta^{\uparrow\gamma+1} \circ |\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma)|^{-1}.$$

Assuming that $\ell_\beta \neq \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma)$, we have

$$\ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma) = \ell_\beta^{\uparrow\gamma+1} \circ \log(r\,\ell_\gamma) = \ell_\beta^{\uparrow\gamma+1} \circ (\ell_{\gamma+1} + \log r) = \sum_{k \in \mathbb{N}} \frac{(\ell_\beta^{\uparrow\gamma+1})^{(k)} \circ \ell_{\gamma+1}}{k!} (\log r)^k,$$

whence $\ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma) - \ell_\beta \sim ((\ell_\beta^{\uparrow\gamma+1})' \circ \ell_{\gamma+1}) \log r$. Now

$$(\ell_\beta^{\uparrow\gamma+1})' \circ \ell_{\gamma+1} = \frac{\ell_\beta'}{\ell_{\gamma+1}'} = \ell_{[\gamma+1,\beta)}^{-1},$$

so $\ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma) - \ell_\beta \asymp \ell_{[\gamma+1,\beta)}^{-1} \prec 1$. Since $\ell_{[\gamma+1,\beta)} \succ \ell_{\gamma+1}$, we have $|\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma)|^{-1} > \ell_{\gamma+1}$, so Lemma 7.5 gives

$$\ell_\beta^{\uparrow\gamma+1} \circ |\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_\gamma)|^{-1} > \ell_\beta^{\uparrow\gamma+1} \circ \ell_{\gamma+1} = \ell_\beta,$$

as desired. Composing with s gives that if $L_\beta(s) \neq L_\beta^{\uparrow\gamma}(rL_\gamma(s))$, then $L_\beta(s) - L_\beta^{\uparrow\gamma}(rL_\gamma(s)) \prec 1$ and

$$L_\beta(s) < L_\beta^{\uparrow\gamma+1}(|L_\beta(s) - L_\beta^{\uparrow\gamma}(rL_\gamma(s))|^{-1}),$$

From which it follows that $L_\beta^{\uparrow\gamma}(rL_\gamma(s)) \in \mathcal{L}_\beta[L_\beta(s)]$. □

Corollary 7.26. *Let $s,t \in \mathbb{T}^{>,\succ}$ with $t \preccurlyeq s$. Then $L_\beta(st) =_\beta L_\beta(s)$.*

Proof. We have $L_\beta(st) = L_\beta^{\uparrow 1}(L_1(st)) = L_\beta^{\uparrow 1}(L_1(s) + L_1(t))$. Let $n > 0$ with $t < ns$. We have $0 < L_1(t) < L_1(s) + \log n < 2L_1(s)$, so

$$L_\beta^{\uparrow 1}(L_1(s)) < L_\beta^{\uparrow 1}(L_1(s) + L_1(t)) < L_\beta^{\uparrow 1}(3L_1(s))$$

by Lemma 7.5. Since $L_\beta(s) = L_\beta^{\uparrow 1}(L_1(s)) =_\beta L_\beta^{\uparrow 1}(3L_1(s))$ by Lemma 7.25 and $\mathcal{L}_\beta[s]$ is convex, we are done. □

Lemma 7.27. *For each $s \in \mathbb{T}^{>,\succ}$ and each $\gamma < \theta$, we have*

$$L_\beta^{\uparrow\gamma}(s) =_\beta L_\beta(L_\gamma(s)) =_\beta L_\beta(s).$$

Proof. Take $\lambda = \omega^\eta n$ with $\gamma \leqslant \lambda < \theta$. Since $\ell_\lambda^{\uparrow\gamma} \leqslant \ell_0$, we have $\ell_\beta^{\uparrow\gamma} = \ell_\beta^{\uparrow\lambda} \circ \ell_\lambda^{\uparrow\gamma} \leqslant \ell_\beta^{\uparrow\lambda} \circ \ell_0$ by Lemma 7.5. This gives

$$\ell_\beta^{\uparrow\gamma} \leqslant \ell_\beta^{\uparrow\lambda} = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ \ell_{\omega^{\eta+1}}^{\uparrow\lambda} = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (\ell_{\omega^{\eta+1}} + n) < \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (2\,\ell_{\omega^{\eta+1}}).$$

Thus, $L_\beta^{\uparrow\gamma}(s) < L_\beta^{\uparrow\omega^{\eta+1}}(2L_{\omega^{\eta+1}}(s))$. Likewise, since $\ell_\gamma \geqslant \ell_\lambda$, we have

$$\ell_\beta \circ \ell_\gamma \geqslant \ell_\beta \circ \ell_\lambda = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (\ell_{\omega^{\eta+1}} \circ \ell_\lambda) = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (\ell_{\omega^{\eta+1}} - n) > \ell_\beta^{\uparrow\omega^{\eta+1}} \circ \left(\frac{1}{2}\,\ell_{\omega^{\eta+1}}\right),$$

so $L_\beta(L_\gamma(s)) > L_\beta^{\uparrow\omega^{\eta+1}}\left(\frac{1}{2}L_{\omega^{\eta+1}}(s)\right)$. Lemma 7.5 gives $\ell_\beta^{\uparrow\gamma} = \ell_\beta^{\uparrow\gamma} \circ \ell_0 \geqslant \ell_\beta^{\uparrow\gamma} \circ \ell_\gamma = \ell_\beta \geqslant \ell_\beta \circ \ell_\gamma$, so we have

$$L_\beta^{\uparrow\omega^{\eta+1}}(2L_{\omega^{\eta+1}}(s)) > L_\beta^{\uparrow\gamma}(s) \geqslant L_\beta(L_\gamma(s)) > L_\beta^{\uparrow\omega^{\eta+1}}\left(\frac{1}{2}L_{\omega^{\eta+1}}(s)\right).$$

By Lemma 7.25, both $L_\beta^{\uparrow\omega^{\eta+1}}(2L_{\omega^{\eta+1}}(s))$ and $L_\beta^{\uparrow\omega^{\eta+1}}\left(\frac{1}{2}L_{\omega^{\eta+1}}(s)\right)$ are elements of $\mathcal{L}_\beta[L_\beta(s)]$. Since $\mathcal{L}_\beta[L_\beta(s)]$ is convex, this means that it also contains $L_\beta^{\uparrow\gamma}(s)$ and $L_\beta(L_\gamma(s))$. □

We have the following useful consequence:

Corollary 7.28. *Let $s,t \in \mathbb{T}^{>,\succ}$ be such that $L_\gamma(s) \asymp L_\sigma(t)$ for some $\gamma, \sigma < \theta$. Then*

$$L_\beta(s) =_\beta L_\beta(t).$$

Proof. Take $n \in \mathbb{N}^>$ with $\frac{1}{n}L_\gamma(s) < L_\sigma(t) < nL_\gamma(s)$. Then

$$L_\beta\left(\frac{1}{n}L_\gamma(s)\right) < L_\beta(L_\sigma(t)) < L_\beta(nL_\gamma(s)).$$

We have $L_\beta(nL_\gamma(s)) =_\beta L_\beta^{\uparrow\gamma}(nL_\gamma(s))$ by Lemma 7.27 and we have $L_\beta^{\uparrow\gamma}(nL_\gamma(s)) =_\beta L_\beta(s)$ by Lemma 7.25, so $L_\beta(nL_\gamma(s)) =_\beta L_\beta(s)$. Likewise, $L_\beta\left(\frac{1}{n}L_\gamma(s)\right) =_\beta L_\beta(s)$. Since $\mathcal{L}_\beta[L_\beta(s)]$ is convex, this yields $L_\beta(L_\sigma(t)) =_\beta L_\beta(s)$. Since $L_\beta(L_\sigma(t)) =_\beta L_\beta(t)$ by Lemma 7.27, we conclude that $L_\beta(t) =_\beta L_\beta(s)$. □

Corollary 7.29. *Let $s, t \in \mathbb{T}^{>,\succ}$ with $L_\beta(s) <_\beta L_\beta(t)$. Then $s^{-1} t \in \mathbb{T}^{>,\succ}$ and $L_\beta(s^{-1} t) =_\beta L_\beta(t)$.*

Proof. As L_β is strictly increasing, we have $s \leqslant t$, which gives $L_1(s) \leqslant L_1(t)$. We first claim that $L_1(s) \nsim L_1(t)$. If $\mu > 1$, then Corollary 7.28 gives that $L_1(s) \not\asymp L_1(t)$, so we may focus on the case when $\mu = 1$. Suppose towards contradiction that $L_\omega(s) <_\omega L_\omega(t)$ and that $L_1(t) = L_1(s) + \varepsilon$ for some $\varepsilon \prec L_1(s)$. Then

$$L_\omega(t) - L_\omega(s) = L_\omega^{\uparrow 1}(L_1(s) + \varepsilon) - L_\omega^{\uparrow 1}(L_1(s)) = \mathcal{T}_{\ell_\omega^{\uparrow 1}}(L_1(s), \varepsilon) \sim ((\ell_\omega^{\uparrow 1})' \circ L_1(s))\, \varepsilon.$$

Since $(\ell_\omega^{\uparrow 1}) = \ell_\omega + 1$, we have $(\ell_\omega^{\uparrow 1})' = \ell_\omega' = \ell_{[0,\omega)}^{-1}$, so $(\ell_\omega^{\uparrow 1})' \circ L_1(s) = \ell_{[0,\omega)}^{-1} \circ L_1(s) = L_{[1,\omega)}(s)^{-1}$. Since $\varepsilon \prec L_1(s)$, we have

$$L_\omega(t) - L_\omega(s) \sim ((\ell_\omega^{\uparrow 1})' \circ L_1(s))\, \varepsilon \prec L_{[2,\omega)}(s)^{-1},$$

so $L_\omega(s) =_\omega L_\omega(t)$ by Proposition 7.19, a contradiction.

From our claim, we get $0 < L_1(s^{-1} t) = L_1(t) - L_1(s) \asymp L_1(t)$. This yields $s^{-1} t \in \mathbb{T}^{>,\succ}$, as $L_1(s^{-1} t) \in \mathbb{T}^{>,\succ}$. Take $r \in \mathbb{R}^{>1}$ with $r^{-1} L_1(t) < L_1(s^{-1} t) < r L_1(t)$. Lemma 7.25 gives

$$L_\beta(t) = L_\beta^{\uparrow 1}(L_1(t)) =_\beta L_\beta^{\uparrow 1}(r^{-1} L_1(t)) =_\beta L_\beta^{\uparrow 1}(r L_1(t)),$$

so $L_\beta(t) =_\beta L_1(s^{-1} t)$ since $\mathcal{L}_\beta[L_\beta(t)]$ is convex and $L_\beta^{\uparrow 1}$ is strictly increasing. □

8 Hyperexponential extensions

In this section, $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ is a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and $\boldsymbol{\alpha} = \omega^{\boldsymbol{\nu}}$. Given $\mu < \boldsymbol{\nu}$ we consider a class $\mathbf{T}$ of ω^μ-truncated series without ω^μ-hyperexponentials in $\mathbb{T}$ and show how to extend $\mathbb{T}$ into a minimal confluent hyperserial skeleton $\mathbb{T}_{\mathbf{T}} = \mathbb{R}[[\mathfrak{M}_{\mathbf{T}}]]$ of force $\boldsymbol{\nu}$ that contains $E_{\omega^\mu}(\varphi)$ for all series $\varphi \in \mathbf{T}$. Most of the work in the case $\mu = 0$ has already been done in [29], but Subsection 8.1 contains a self-contained treatment for our setting. For the construction of $\mathbb{T}_{\mathbf{T}}$ in Subsections 8.1, 8.2, 8.3, and 8.4, we recall that we made the induction hypothesis that Theorem 7.4 holds for $(\boldsymbol{\nu}, \mu)$. After the construction of $\mathbb{T}_{\mathbf{T}}$, we conclude this section with the proofs of Theorems 7.4 and 1.3.

8.1 Adjoining exponentials

Let $\mathbf{T}_0(\mathbb{T})$ be the class of all 1-truncated series $\varphi \in \mathbb{T}_{>,1}$ for which $\exp \varphi$ is not defined. Let $\mathbf{T}$ be a non-empty subclass of $\mathbf{T}_0(\mathbb{T})$ and let $\langle \mathbf{T} \rangle$ be the $\mathbb{R}$-subspace of $\mathbb{T}_{>,1}$ generated by $\mathbf{T}$ and $\log \mathfrak{M}$. By Lemma 7.12, $\langle \mathbf{T} \rangle$ consists only of 1-truncated series.

Group of monomials. We associate to each $\varphi \in \langle \mathbf{T} \rangle$ a formal symbol e^φ and we let $\mathfrak{M}_{\mathbf{T}}$ denote the multiplicative $\mathbb{R}$-vector space of all such symbols, where $\mathrm{e}^\varphi \mathrm{e}^\psi = \mathrm{e}^{\varphi + \psi}$ and $(\mathrm{e}^\varphi)^r = \mathrm{e}^{r\varphi}$. We use 1 in place of e^0. We order this space by setting $\mathrm{e}^\varphi \succ \mathrm{e}^\psi \Longleftrightarrow \varphi > \psi$. It is easy to see that $(\mathfrak{M}_{\mathbf{T}}, \times, \prec, \mathbb{R})$ is an ordered $\mathbb{R}$-vector space which is isomorphic to $(\langle \mathbf{T} \rangle, +, <, \mathbb{R})$. We identify $\mathfrak{M}$ with the $\mathbb{R}$-subspace $\mathrm{e}^{\log \mathfrak{M}}$ of $\mathfrak{M}_{\mathbf{T}}$ via the embedding $\mathfrak{m} \longmapsto \mathrm{e}^{\log \mathfrak{m}}$. Let $\mathbb{T}_{\mathbf{T}} := \mathbb{R}[[\mathfrak{M}_{\mathbf{T}}]]$, so the identification $\mathfrak{M} \subseteq \mathfrak{M}_{\mathbf{T}}$ induces an identification $\mathbb{T} \subseteq \mathbb{T}_{\mathbf{T}}$. In the special case when $\mathbf{T} = \mathbf{T}_0(\mathbb{T})$, we write $\mathfrak{M}_{(0)} := \mathfrak{M}_{\mathbf{T}}$ and $\mathbb{T}_{(0)} := \mathbb{T}_{\mathbf{T}}$.

Extending the logarithm. For $\mathrm{e}^\varphi \in \mathfrak{M}_{\mathbf{T}}$, we set

$$\log \mathrm{e}^\varphi := \varphi.$$

We let L_1 be the restriction of log to $\mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$. Note the following:

1. By construction, $(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies **DD₀** and **FE₀**. Moreover, $L_1(\mathfrak{m}) = L_1(\mathrm{e}^{L_1(\mathfrak{m})})$ for $\mathfrak{m} \in \mathfrak{M}^{\succ}$, so $(\mathbb{T}, L_1) \subseteq (\mathbb{T}_{\mathbf{T}}, L_1)$
2. We claim that $(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{A}_0$. Suppose for contradiction that $\varphi = L_1(\mathrm{e}^{\varphi}) \succcurlyeq \mathrm{e}^{\varphi}$, where $\mathrm{e}^{\varphi} \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$. Then $\mathfrak{d}_{\varphi} \succcurlyeq \mathrm{e}^{\varphi}$, so $L_1(\mathfrak{d}_{\varphi}) \succcurlyeq \varphi$ by definition. This gives $L_1(\mathfrak{d}_{\varphi}) \succcurlyeq \mathfrak{d}_{\varphi}$, which contradicts the fact that $(\mathbb{T}, L_1)$ satisfies $\mathbf{A}_0$.
3. By definition, we have $\mathrm{e}^{\varphi} \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$ if and only if $L_1(\mathrm{e}^{\varphi}) > 0$, so $(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{M}_0$.
4. Since $L_1(\mathrm{e}^{\varphi}) = \varphi \in \mathbb{T}_{>,1}$ for $\mathrm{e}^{\varphi} \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$, the axiom $\mathbf{R}_0$ is satisfied.
5. As remarked in Remark 3.2, $\mathbf{P}_0$ follows from **FE₀**.

Extending L_ω. For $\varphi \in \langle \mathbf{T} \rangle$ with $\mathrm{e}^{\varphi} \succ 1$, we have $L_1(\mathrm{e}^{\varphi}) \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$ if and only if $\varphi \in \mathfrak{M}^{\succ}$, so $\mathrm{e}^{\varphi} \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\varphi \in \mathfrak{M}_\omega$. Accordingly, we set

$$\operatorname{dom} L_\omega := \{\mathrm{e}^{\varphi} : \varphi \in \langle \mathbf{T} \rangle \cap \mathfrak{M}_\omega\}, \qquad L_\omega(\mathrm{e}^{\varphi}) := L_\omega(\varphi) + 1.$$

This ensures that $\mathbf{DD}_1$ holds. Note that if $\mathfrak{a} \in \mathfrak{M}_\omega$, then $L_1(\mathfrak{a}) \in \langle \mathbf{T} \rangle \cap \mathfrak{M}_\omega$, so $\mathfrak{a} = \mathrm{e}^{L_1(\mathfrak{a})} \in \operatorname{dom} L_\omega$ and $L_\omega(\mathrm{e}^{L_1(\mathfrak{a})}) = L_\omega(L_1(\mathfrak{a})) + 1 = L_\omega(\mathfrak{a})$. Thus, $\mathfrak{M}_\omega \subseteq \operatorname{dom} L_\omega$ and $(\mathbb{T}, L_1, L_\omega) \subseteq (\mathbb{T}_{\mathbf{T}}, L_1, L_\omega)$. We also have the following:

1. For $\mathrm{e}^{\varphi} \in \operatorname{dom} L_\omega$, we have

$$L_\omega(L_1(\mathrm{e}^{\varphi})) = L_\omega(\varphi) = L_\omega(\mathrm{e}^{\varphi}) - 1$$

so $(\mathbb{T}_{\mathbf{T}}, L_1, L_\omega)$ satisfies $\mathbf{FE}_1$.

2. For $\mathrm{e}^{\varphi} \in \operatorname{dom} L_\omega$, we have $L_\omega(\varphi) + 1 = (\ell_\omega + 1) \circ \varphi \prec \ell_0 \circ \varphi = \varphi$, since $\ell_\omega + 1 \prec \ell_0$. Thus

$$L_\omega(\mathrm{e}^{\varphi}) = L_\omega(\varphi) + 1 \prec \varphi = L_1(\mathrm{e}^{\varphi}),$$

which proves $\mathbf{A}_1$.

3. $(\mathbb{T}_{\mathbf{T}}, L_1, L_\omega)$ satisfies $\mathbf{M}_1$. To see this, let $\mathrm{e}^{\varphi}, \mathrm{e}^{\psi} \in \operatorname{dom} L_\omega$ with $\mathrm{e}^{\varphi} \prec \mathrm{e}^{\psi}$ and let $n \in \mathbb{N}$. We want to show that $L_\omega(\mathrm{e}^{\varphi}) + L_n(\mathrm{e}^{\varphi})^{-1} < L_\omega(\mathrm{e}^{\psi}) - L_n(\mathrm{e}^{\psi})^{-1}$. Since $L_{n+1}(\mathrm{e}^{\varphi}) \prec L_n(\mathrm{e}^{\varphi})$ and $L_{n+1}(\mathrm{e}^{\psi}) \prec L_n(\mathrm{e}^{\psi})$ by $\mathbf{A}_0$, we may assume without loss of generality that $n > 0$. Now

$$\begin{aligned} L_\omega(\mathrm{e}^{\varphi}) + L_n(\mathrm{e}^{\varphi})^{-1} &= L_\omega(\varphi) + 1 + L_{n-1}(\varphi)^{-1} \\ L_\omega(\mathrm{e}^{\psi}) - L_n(\mathrm{e}^{\psi})^{-1} &= L_\omega(\psi) + 1 - L_{n-1}(\psi)^{-1}. \end{aligned}$$

Since $\varphi, \psi \in \mathfrak{M}_\omega$ and since $(\mathbb{T}, L_1, L_\omega)$ satisfies $\mathbf{M}_1$, we have

$$L_\omega(\varphi) + L_{n-1}^{-1}(\varphi) < L_\omega(\psi) - L_{n-1}^{-1}(\psi).$$

4. Let $\mathrm{e}^{\varphi} \in \operatorname{dom} L_\omega$. Since $\varphi \in \mathfrak{M}_\omega$ and $(\mathbb{T}, L_1, L_\omega)$ satisfies $\mathbf{R}_1$, the hyperlogarithm $L_\omega(\varphi)$ is ω-truncated by Lemma 7.14. It follows from Lemma 7.13 that $L_\omega(\mathrm{e}^{\varphi}) = L_\omega(\varphi) + 1$ is also ω-truncated, so $(\mathbb{T}_{\mathbf{T}}, L_1, L_\omega)$ satisfies $\mathbf{R}_1$.
5. Let $\mathrm{e}^{\varphi} \in \operatorname{dom} L_\omega$ and let $(r_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. To show that $(\mathbb{T}_{\mathbf{T}}, L_1, L_\omega)$ satisfies $\mathbf{P}_1$, we need to show that the sum $s = \sum_{n \in \mathbb{N}} r_n L_{n+1}(\mathrm{e}^{\varphi})$ is in $\log \mathfrak{M}_{\mathbf{T}}$. We have

$$s = \sum_{n \in \mathbb{N}} r_n L_{n+1}(\mathrm{e}^{\varphi}) = r_0 \varphi + \sum_{n \in \mathbb{N}^{>}} r_n L_n(\varphi).$$

Since $\varphi\in\mathfrak{M}_\omega$ and since $(\mathbb{T}_{\mathbf{T}},L_1,L_\omega)$ satisfies $\mathbf{P}_1$, we have $\sum_{n\in\mathbb{N}^>} r_n L_n(\varphi)\in\log\mathfrak{M}$. It remains to note that $r_0\varphi=r_0L_1(\mathrm{e}^\varphi)=\log\mathrm{e}^{r_0\varphi}\in\log\mathfrak{M}_{\mathbf{T}}$ and that $\log\mathfrak{M}_{\mathbf{T}}$ is closed under finite sums.

Extending L_{ω^η} for $1<\eta<\nu$. Let $1<\eta<\nu$ and set $\operatorname{dom}L_{\omega^\eta}:=\mathfrak{M}_{\omega^\eta}$. We need to show that $\mathbf{DD}_\eta$ holds for each η, and for this, it suffices to show that $\mathbf{DD}_2$ holds. Let $\mathrm{e}^\varphi\in\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_\omega^{\circ n}$ and take n with $L_{\omega(n+1)}(\varphi)\asymp L_{\omega(n+1)}(\mathfrak{d}_{\omega^2}(\varphi))$. Since $L_\omega(\varphi)+1\asymp L_\omega(\varphi)$, Lemma 4.8 gives that

$$L_{\omega(n+1)}(\mathrm{e}^\varphi)=L_{\omega n}(L_\omega(\varphi)+1)\asymp L_{\omega n}(L_\omega(\varphi))=L_{\omega(n+1)}(\varphi)\asymp L_{\omega(n+1)}(\mathfrak{d}_{\omega^2}(\varphi)).$$

Since $L_{\omega(n+1)}(\mathrm{e}^\varphi)$ and $L_{\omega(n+1)}(\mathfrak{d}_{\omega^2}(\varphi))$ are both monomials, they must be equal. The axiom $\mathbf{M}_1$ gives $\mathrm{e}^\varphi=\mathfrak{d}_{\omega^2}(\varphi)\in\mathfrak{M}_{\omega^2}=\operatorname{dom}L_{\omega^2}$.

Now $\mathbf{FE}_\eta$, $\mathbf{A}_\eta$, $\mathbf{M}_\eta$, $\mathbf{R}_\eta$, and $\mathbf{P}_\eta$ hold for each $1<\eta<\nu$, since they hold in $\mathbb{T}$. Furthermore, $\mathbf{P}_\nu$ holds if $\nu\in\mathbf{On}$; this is clear if $\nu>1$ and the above proof of $\mathbf{P}_1$ still goes through when $\nu=1$. Thus, $(\mathbb{T}_{\mathbf{T}},(L_{\omega^\mu})_{\eta<\nu})$ is a hyperserial skeleton of force ν which extends $(\mathbb{T},(L_{\omega^\mu})_{\eta<\nu})$.

Proposition 8.1. *Assume that $\mathfrak{M}_\omega\subseteq\langle\mathbf{T}\rangle$. Then $\mathbb{T}_{\mathbf{T}}$ is ν-confluent.*

Proof. Clearly, $\mathbb{T}_{\mathbf{T}}$ is 0-confluent. Let $s\in\mathbb{T}_{\mathbf{T}}^{>,\succ}$ and take $\varphi\in\langle\mathbf{T}\rangle$ with $\mathfrak{d}_s=\mathrm{e}^\varphi\in\mathfrak{M}_{\mathbf{T}}^{\succ}$. We have $L_1(\mathfrak{d}_1(s))=L_1(\mathrm{e}^\varphi)=\varphi\in\mathbb{T}$. Let $\mathfrak{a}:=\mathfrak{d}_\omega(\varphi)$ and take n with $(L_1\circ\mathfrak{d}_1)^{\circ n}(\varphi)\asymp(L_1\circ\mathfrak{d}_1)^{\circ n}(\mathfrak{a})$. We have $L_1(\mathfrak{d}_1(s))=\varphi$ and, by assumption, $\mathfrak{a}\in\langle\mathbf{T}\rangle$, so

$$(L_1\circ\mathfrak{d}_1)^{\circ(n+1)}(s)\asymp(L_1\circ\mathfrak{d}_1)^{\circ n}(\mathfrak{a})=(L_1\circ\mathfrak{d}_1)^{\circ(n+1)}(\mathrm{e}^\mathfrak{a}).$$

By definition, $\mathfrak{a}\in\mathfrak{M}_\omega$ implies $\mathrm{e}^\mathfrak{a}\in\operatorname{dom}L_\omega$, so $\mathfrak{d}_\omega(s)=\mathrm{e}^\mathfrak{a}$. We have

$$L_\omega(\mathfrak{d}_\omega(s))=L_\omega(\mathrm{e}^\mathfrak{a})=L_\omega(\mathfrak{a})+1\asymp L_\omega(\mathfrak{a})=L_\omega(\mathfrak{d}_\omega(\mathfrak{a})),$$

so $\mathfrak{d}_{\omega^2}(s)=\mathfrak{d}_{\omega^2}(\mathfrak{a})$ and, more generally, $\mathfrak{d}_{\omega^\eta}(s)=\mathfrak{d}_{\omega^\eta}(\mathfrak{a})$ for $2\leqslant\eta\leqslant\nu$. Thus, the skeleton $\mathbb{T}_{\mathbf{T}}$ is ν-confluent. □

Let us summarize:

Proposition 8.2. *The field $\mathbb{T}_{\mathbf{T}}$ is a confluent hyperserial skeleton of force ν. It is an extension of $\mathbb{T}$ of force ν with $\langle\mathbf{T}\rangle\subseteq L_1(\mathfrak{M}_{\mathbf{T}})$.*

Using the composition from Theorem 4.1, we can check whether an embedding Φ of confluent hyperserial skeletons is of force ν without having to verify that $\Phi(\mathfrak{M}_{\omega^\eta})\subseteq\mathfrak{N}_{\omega^\eta}$ for all η.

Lemma 8.3. *Let $\mathbb{U}=\mathbb{R}[[\mathfrak{N}]]$ be a confluent hyperserial skeleton of force ν with the external composition $\circ\colon\mathbb{L}_{<\alpha}\times\mathbb{U}^{>,\succ}\longrightarrow\mathbb{U}$ from Theorem 4.1 and let $\Phi\colon\mathbb{T}\longrightarrow\mathbb{U}$ be a strongly linear field embedding. Suppose that $\Phi(\mathfrak{M})\subseteq\mathfrak{N}$, that $\Phi(\mathfrak{m}^r)=\Phi(\mathfrak{m})^r$ for all $\mathfrak{m}\in\mathfrak{M}$ and all $r\in\mathbb{R}$, and that $\Phi(L_{\omega^\eta}(\mathfrak{a}))=L_{\omega^\eta}(\Phi(\mathfrak{a}))$ for all $\eta<\nu$ and all $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$. Then Φ is an embedding of force ν.*

Proof. We will show by induction on $\eta<\nu$ that $\Phi(\mathfrak{M}_{\omega^\eta})\subseteq\mathfrak{N}_{\omega^\eta}$. For $\eta=0$, this holds as Φ is order-preserving. Let $\eta>0$ and assume that $\Phi(\mathfrak{M}_{\omega^\iota})\subseteq\mathfrak{N}_{\omega^\iota}$ for all $\iota<\eta$. If η is a limit, then by $\mathbf{DD}_\eta$, we have

$$\Phi(\mathfrak{M}_{\omega^\eta})=\Phi\left(\bigcap_{\iota<\eta}\mathfrak{M}_{\omega^\iota}\right)=\bigcap_{\iota<\eta}\Phi(\mathfrak{M}_{\omega^\iota})\subseteq\bigcap_{\iota<\eta}\mathfrak{N}_{\omega^\iota}=\mathfrak{N}_{\omega^\eta}.$$

Suppose η is a successor and let $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$. We have $L_{\omega^{\eta_*}n}(\mathfrak{a})\in\mathfrak{M}_{\omega^{\eta_*}}$ for all $n\in\mathbb{N}$ by $\mathbf{DD}_\eta$. Our induction hypothesis gives $L_{\omega^{\eta_*}n}(\Phi(\mathfrak{a}))=\Phi(L_{\omega^{\eta_*}n}(\mathfrak{a}))\in\mathfrak{N}_{\omega^{\eta_*}}$ for all $n\in\mathbb{N}$. Applying $\mathbf{DD}_\eta$ again gives $\Phi(\mathfrak{a})\in\mathfrak{N}_{\omega^\eta}$, so $\Phi(\mathfrak{M}_{\omega^\iota})\subseteq\mathfrak{N}_{\omega^\iota}$. □

Proposition 8.4. *Let $\mathbb{U}=\mathbb{R}[[\mathfrak{N}]]$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu}$ and let $\Phi:\mathbb{T}\longrightarrow\mathbb{U}$ be an embedding of force $\boldsymbol{\nu}$. If $\Phi(\mathbf{T})\subseteq\log(\mathbb{U}^>)$, then there is a unique embedding*

$$\Psi:\mathbb{T}_{\mathbf{T}}\longrightarrow\mathbb{U}$$

of force $\boldsymbol{\nu}$ that extends Φ.

Proof. As $\mathbb{U}$ is hyperserial of force $\boldsymbol{\nu}$, we have an external composition $\circ:\mathbb{L}_{<\boldsymbol{\alpha}}\times\mathbb{U}^{>,\succ}\longrightarrow\mathbb{U}$. Since $\Phi(\mathbf{T})\subseteq\log\mathbb{U}^>$, Φ is $\mathbb{R}$-linear, and $\log\mathbb{U}^>$ is an $\mathbb{R}$-subspace of $\mathbb{U}$ containing $\Phi(\log\mathfrak{M})$, we have $\Phi(\langle\mathbf{T}\rangle)\subseteq\log\mathbb{U}^>$.

Since $\Phi(\mathfrak{M}^\succ)\subseteq\mathfrak{N}^\succ$, we have $\Phi(\mathbb{T}_{\succ,1})\subseteq\mathbb{U}_{\succ,1}$ so $\Phi(\langle\mathbf{T}\rangle)\subseteq\log\mathbb{U}^>\cap\mathbb{U}_{\succ,1}$. Thus, $\exp(\Phi(\varphi))$ is a monomial for $\varphi\in\langle\mathbf{T}\rangle$ by Lemma 7.12. We define a map $\Psi:\mathfrak{M}_{\mathbf{T}}\longrightarrow\mathfrak{N}$ by setting

$$\Psi(\mathrm{e}^\varphi):=\exp(\Phi(\varphi)).$$

It is routine to check that $\Psi:\mathfrak{M}_{\mathbf{T}}\longrightarrow\mathfrak{N}$ is an embedding of ordered monomial groups with $\mathbb{R}$-powers. By Proposition 2.3, this embedding Ψ uniquely extends into a strongly linear field embedding of $\mathbb{T}_{\mathbf{T}}$ into $\mathbb{U}$, which we will still denote by Ψ. Note that if $\mathfrak{m}\in\mathfrak{M}$, then $\Psi(\mathrm{e}^{\log(\mathfrak{m})})=\exp(\Phi(\log\mathfrak{m}))=\exp(\log(\Phi(\mathfrak{m})))=\Phi(\mathfrak{m})$, so Ψ extends Φ.

We now prove that Ψ is a force $\boldsymbol{\nu}$ embedding. By Lemma 8.3, we need only show that Ψ commutes with logarithms and hyperlogarithms. Given $\mathrm{e}^\varphi\in\mathfrak{M}_{\mathbf{T}}$, we have

$$\Psi(\log(\mathrm{e}^\varphi))=\Psi(\varphi)=\Phi(\varphi)=\log(\exp(\Phi(\varphi)))=\log(\Psi(\mathrm{e}^\varphi)).$$

Now let $\mu<\boldsymbol{\nu}$ with $\mu>0$ and let $\mathrm{e}^\varphi\in(\mathfrak{M}_{\mathbf{T}})_{\omega^\mu}$. If $\mu>1$, then $\mathrm{e}^\varphi\in\mathfrak{M}_{\omega^\mu}$, so we automatically have $L_{\omega^\mu}(\Psi(\mathrm{e}^\varphi))=\Psi(L_{\omega^\mu}(\mathrm{e}^\varphi))$, since Ψ extends Φ. If $\mu=1$, then $\varphi\in\langle\mathbf{T}\rangle\cap\mathfrak{M}_\omega$, so

$$L_\omega(\Psi(\mathrm{e}^\varphi))=L_\omega(\exp(\Phi(\varphi)))=L_\omega(\Phi(\varphi))+1=\Phi(L_\omega(\varphi)+1)=\Phi(L_\omega(\mathrm{e}^\varphi))=\Psi(L_\omega(\mathrm{e}^\varphi)).$$

Let us finally assume that $\Lambda:\mathbb{T}_{\mathbf{T}}\longrightarrow\mathbb{U}$ is another embedding of force $\boldsymbol{\nu}$ that extends Φ. To see that $\Lambda=\Psi$, it suffices to show that $\Lambda(\mathrm{e}^\varphi)=\Psi(\mathrm{e}^\varphi)$ for $\varphi\in\langle\mathbf{T}\rangle$. Now

$$\log(\Lambda(\mathrm{e}^\varphi))=\Lambda(\log(\mathrm{e}^\varphi))=\Lambda(\varphi)=\Phi(\varphi),$$

so $\Lambda(\mathrm{e}^\varphi)=\exp(\Phi(\varphi))=\Psi(\mathrm{e}^\varphi)$. □

8.2 Adjoining hyperexponentials

From this point until Subsection 8.5, we let $0<\mu<\boldsymbol{\nu}$ and set

$$\begin{aligned}\beta &:= \omega^\mu\\ \theta &:= \omega^{\mu_*}.\end{aligned}$$

Note that $\beta=\theta\,\omega$ if μ is a successor and $\beta=\theta$ if μ is a limit. Let $\mathbf{T}_\mu(\mathbb{T})$ be the class of all β-truncated series $\varphi\in\mathbb{T}_{\succ,\beta}$ for which $E_\beta(\varphi)$ is not defined. Let $\mathbf{T}$ be a non-empty subclass of $\mathbf{T}_\mu(\mathbb{T})$. Consider $\varphi\in\mathbf{T}$ and $s\in\mathbb{T}^{>,\succ}$. We have $\sharp_\beta(L_\beta(s))=L_\beta(\mathfrak{d}_\beta(s))\in L_\beta(\mathbb{T}^{>,\succ})$ by Corollary 7.21. Since $\mathcal{L}_\beta[L_\beta(s)]$ contains a unique β-truncated element, φ is β-truncated and $\varphi\notin L_\beta(\mathbb{T}^{>,\succ})$, it follows that $\varphi\notin\mathcal{L}_\beta[L_\beta(s)]$. Thus, we have

$$\begin{aligned}\varphi\leqslant L_\beta(s) &\iff \varphi<_\beta L_\beta(s)\\ \varphi\geqslant L_\beta(s) &\iff \varphi>_\beta L_\beta(s).\end{aligned}$$

If μ is a successor, then let $\langle \mathbf{T} \rangle$ be the smallest class containing $\mathbf{T}$ such that $\varphi - 1 \in \langle \mathbf{T} \rangle$ whenever $\varphi \in \langle \mathbf{T} \rangle$ and $\varphi - 1 \notin L_\beta(\mathbb{T}^{>,\succ})$. By Lemma 7.13, the class $\langle \mathbf{T} \rangle$ also consists only of β-truncated series. If μ is a limit, then set $\langle \mathbf{T} \rangle := \mathbf{T}$. Note that $\mathbf{T}_\mu(\mathbb{T}) = \langle \mathbf{T}_\mu(\mathbb{T}) \rangle$. For the remainder of this subsection (with the exception of Proposition 8.30) we assume that $\mathbf{T} = \langle \mathbf{T} \rangle$.

Remark 8.5. Let $\varphi \in \mathbf{T}$ and suppose μ is a successor. If $\varphi - n \notin \mathbf{T}$ for some $n \in \mathbb{N}$, so $E_\beta(\varphi - n)$ is defined in $\mathbb{T}$, then for each $m \in \mathbb{N}$, we have

$$\varphi - (n+m) = L_\beta(E_\beta(\varphi - n)) - m = L_\beta(L_{\theta m}(E_\beta(\varphi - n))),$$

so $L_{\theta m}(E_\beta(\varphi - n)) = E_\beta(\varphi - (n+m))$. In particular, $\varphi - (n+m) \notin \mathbf{T}$.

Group of monomials. We associate to each $\mathfrak{l} \in \mathfrak{L}_{<\theta}$ and each $\varphi \in \mathbf{T}$ a formal symbol $\mathfrak{l}[\mathrm{e}_\beta^\varphi]$. This should be thought of as $\mathfrak{l} \circ \mathrm{e}_\beta^\varphi$ if e_β^φ is an element in a hyperserial extension of $\mathbb{T}$. Accordingly, we write e_β^φ in place of $\ell_0[\mathrm{e}_\beta^\varphi]$ and 1 in place of $1[\mathrm{e}_\beta^\varphi]$.

We define the group $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ as follows. If μ is a limit, then $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ is the group generated by the elements $\mathfrak{l}[\mathrm{e}_\beta^\varphi]$ with $\mathfrak{l} \in \mathfrak{L}_{<\theta}$ and satisfying the relations $\mathfrak{l}_1[\mathrm{e}_\beta^\varphi]\, \mathfrak{l}_2[\mathrm{e}_\beta^\varphi] = (\mathfrak{l}_1 \mathfrak{l}_2)[\mathrm{e}_\beta^\varphi]$. Hence $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ is the group of products

$$\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi], \qquad \mathfrak{t}_\varphi \in \mathfrak{L}_{<\theta},$$

for which the *hypersupport*

$$\operatorname{hsupp} \mathfrak{t} := \{\varphi \in \mathbf{T} : \mathfrak{t}_\varphi \neq 1\}$$

of $\mathfrak{t}$ is finite. If μ is a successor, then let $\frown$ be the equivalence relation on $\mathbb{T}$ defined by

$$s \frown t \iff t - s \in \mathbb{Z}.$$

We let $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ be the group of formal products

$$\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi], \qquad \mathfrak{t}_\varphi \in \mathfrak{L}_{<\theta},$$

for which the hypersupport $\operatorname{hsupp} \mathfrak{t}$ is well-based and $\operatorname{hsupp} \mathfrak{t} / \frown$ is finite. Given $\mathfrak{s}, \mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$, we note that $\operatorname{hsupp} \mathfrak{s}^{-1} \mathfrak{t} \subseteq \operatorname{hsupp} \mathfrak{s} \cup \operatorname{hsupp} \mathfrak{t}$, whence $\mathfrak{s}^{-1} \mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$. Hence $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ is indeed a group.

For $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]^{\neq 1}$, we define $\varphi_\mathfrak{t} := \max \operatorname{hsupp} \mathfrak{t}$ and $\gamma_\mathfrak{t} := \min \{\gamma < \theta : (\mathfrak{t}_{\varphi_\mathfrak{t}})_\gamma \neq 0\}$. We set $\mathfrak{t} \succ 1$ if $\mathfrak{t}_{\varphi_\mathfrak{t}} \succ 1$, which happens if and only if $(\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} > 0$. The following facts will be used frequently, where $\mathfrak{t}, \mathfrak{u}$ range over $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$:

- $\varphi_{\mathfrak{t}^{-1}} = \varphi_\mathfrak{t}$ for $\mathfrak{t} \neq 1$,
- $\varphi_{\mathfrak{t}\mathfrak{u}} \leqslant \max(\varphi_\mathfrak{t}, \varphi_\mathfrak{u})$, and if $\varphi_\mathfrak{t} \neq \varphi_\mathfrak{u}$ then $\varphi_{\mathfrak{t}\mathfrak{u}} = \max(\varphi_\mathfrak{t}, \varphi_\mathfrak{u})$
- If $1 \prec \mathfrak{t} \preccurlyeq \mathfrak{u}$ or $\mathfrak{u} \preccurlyeq \mathfrak{t} \prec 1$, then $\varphi_\mathfrak{t} \leqslant \varphi_\mathfrak{u}$,
- If $\mathfrak{t} \succ 1$ and $\mathfrak{u} \succcurlyeq 1$ or if $\mathfrak{t} \prec 1$ and $\mathfrak{u} \preccurlyeq 1$ then $\varphi_{\mathfrak{t}\mathfrak{u}} = \max(\varphi_\mathfrak{t}, \varphi_\mathfrak{u})$.

Let $\mathfrak{M}_{\mathbf{T}}$ denote the direct product $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}] \times \mathfrak{M}$. We denote by $\mathfrak{t}\,\mathfrak{m}$ a general element $(\mathfrak{t}, \mathfrak{m})$ of this group, where we implicitly understand that $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ and $\mathfrak{m} \in \mathfrak{M}$; we also identify $(\mathfrak{t}, 1)$ and $(1, \mathfrak{m})$ with $\mathfrak{t}$ and $\mathfrak{m}$, respectively. In the special case when $\mathbf{T} = \mathbf{T}_\mu(\mathbb{T})$, we write $\mathfrak{M}_{(\mu)} := \mathfrak{M}_{\mathbf{T}}$.

Remark 8.6. Assume that μ is a successor and consider $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ as above. The advantage of the representation of $\mathfrak{t}$ as an infinite product of terms of the form $\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi]$ with $\mathfrak{t}_\varphi \in \mathfrak{L}_{<\theta}$ is that such a representation is unique. Alternatively, it is possible to represent $\mathfrak{t}$ as a finite product of terms of the form $\mathfrak{l}[\mathrm{e}_\beta^\varphi]$ with $\mathfrak{l} \in \mathfrak{L}_{<\beta}$, but uniqueness is lost, since $\ell_0[\mathrm{e}_\beta^\varphi] = \ell_\theta[\mathrm{e}_\beta^{\varphi+1}]$.

Nevertheless, we may construct a privileged representation as a finite product as follows. Since $\operatorname{hsupp} \mathfrak{t} / \frown$ is finite, there exist $\varphi_1 > \cdots > \varphi_n \in \mathbf{T}$ with $\varphi_i \not\frown \varphi_j$ for $i \neq j$ and $\operatorname{hsupp} \mathfrak{t} / \frown = \{\varphi_1, \ldots, \varphi_n\} / \frown$. Since $\operatorname{hsupp} \mathfrak{t}$ is well-based, we may also take $\varphi_i = \max \{\varphi \in \operatorname{hsupp} \mathfrak{t} : \varphi \frown \varphi_i\}$ for all i. Then

$$\mathfrak{t} = \prod_{1 \leqslant i \leqslant n} \prod_{\substack{m \in \mathbb{N} \\ \varphi_i - m \in \mathbf{T}}} \mathfrak{t}_{\varphi_i - m}[\mathrm{e}_\beta^{\varphi_i - m}].$$

Fix $i \in \{1, \ldots, n\}$ and set $A_i := \{m \in \mathbb{N} : \varphi_i - m \in \mathbf{T}\}$. For each $m \in A_i$, we have $\log \mathfrak{t}_{\varphi_i - m} = \sum_{\gamma < \theta} (\mathfrak{t}_{\varphi_i - m})_\gamma \, \ell_{\gamma+1}$, so

$$\begin{aligned} \sum_{m \in A_i} \log(\mathfrak{t}_{\varphi_i - m} \circ \ell_{\theta m}) &= \sum_{m \in A_i} (\log \mathfrak{t}_{\varphi_i - m}) \circ \ell_{\theta m} = \sum_{m \in A_i} \left(\sum_{\gamma < \theta} (\mathfrak{t}_{\varphi_i - m})_\gamma \, \ell_{\gamma+1} \right) \circ \ell_{\theta m} \\ &= \sum_{m \in A_i} \sum_{\gamma < \theta} (\mathfrak{t}_{\varphi_i - m})_\gamma \, \ell_{\theta m + \gamma + 1} = \log\left(\prod_{m \in A_i} \prod_{\gamma < \theta} \ell_{\theta m + \gamma}^{(\mathfrak{t}_{\varphi_i - m})_\gamma} \right). \end{aligned}$$

Set

$$\mathfrak{t}_{\varphi_i}^* := \prod_{m \in A_i} \prod_{\gamma < \theta} \ell_{\theta m + \gamma}^{(\mathfrak{t}_{\varphi_i - m})_\gamma} \in \mathfrak{L}_{<\beta}.$$

This gives us the finite representation

$$\mathfrak{t} = \prod_{1 \leqslant i \leqslant n} \mathfrak{t}_{\varphi_i}^*[\mathrm{e}_\beta^{\varphi_i}].$$

Note that $\mathfrak{t} \succ 1 \Longleftrightarrow \mathfrak{t}_{\varphi_1} \succ 1 \Longleftrightarrow \mathfrak{t}_{\varphi_1}^* \succ 1$.

Ordering. Let $\mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$ be the set of all elements $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{\mathbf{T}}$ that satisfy one of the following conditions:

$$\mathfrak{t} \succ 1,\ \mathfrak{m} \prec 1, \text{ and } \varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1}) \tag{I}$$

$$\mathfrak{t} \prec 1,\ \mathfrak{m} \succ 1, \text{ and } \varphi_{\mathfrak{t}} < L_\beta(\mathfrak{m}) \tag{II}$$

$$\mathfrak{t} \succcurlyeq 1 \text{ and } \mathfrak{m} \succ 1 \tag{III}$$

$$\mathfrak{t} \succ 1 \text{ and } \mathfrak{m} \succcurlyeq 1 \tag{IV}$$

We define the relation $\prec$ on $\mathfrak{M}_{\mathbf{T}}$ by $\mathfrak{t}\,\mathfrak{m} \prec \mathfrak{u}\,\mathfrak{n}$ if and only if $(\mathfrak{u}\,\mathfrak{t}^{-1})\,(\mathfrak{n}\,\mathfrak{m}^{-1}) \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$.

Proposition 8.7. *The relation $\prec$ is an order on $\mathfrak{M}_{\mathbf{T}}$ that extends the orderings on both $\mathfrak{M}$ and $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$.*

Proof. By definition, the relation $\prec$ extends the orderings on $\mathfrak{M}$ and $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$. In order to show that $\prec$ is an order, it suffices to check that $\mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$ is a total positive cone on $\mathfrak{M}_{\mathbf{T}}$.

Let $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{\mathbf{T}} \setminus \{1\}$. By the definition of $\mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$ and the fact that $\varphi_{\mathfrak{t}^{-1}} = \varphi_{\mathfrak{t}}$, it is clear that $\mathfrak{t}\,\mathfrak{m}$ and $(\mathfrak{t}\,\mathfrak{m})^{-1}$ cannot both be in $\mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$. Let us show that either $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$ or $(\mathfrak{t}\,\mathfrak{m})^{-1} \in \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$. Assume that $\mathfrak{t}\,\mathfrak{m} \notin \mathfrak{M}_{\mathbf{T}}^{\succcurlyeq}$. If $\mathfrak{t} \prec 1$ and $\mathfrak{m} \preccurlyeq 1$ or $\mathfrak{t} \preccurlyeq 1$ and $\mathfrak{m} \prec 1$, then $(\mathfrak{t}\,\mathfrak{m})^{-1}$ satisfies (III) or (IV). Suppose that $\mathfrak{t} \succ 1$, $\mathfrak{m} \prec 1$, and $\varphi_{\mathfrak{t}} \leqslant L_\beta(\mathfrak{m}^{-1})$. Then $\varphi_{\mathfrak{t}} < L_\beta(\mathfrak{m}^{-1})$ since $\varphi_{\mathfrak{t}} \notin L_\beta(\mathbb{T}^{>,\succ})$, so $\varphi_{\mathfrak{t}^{-1}} = \varphi_{\mathfrak{t}} < L_\beta(\mathfrak{m}^{-1})$. Since $\mathfrak{t}^{-1} \prec 1$ and $\mathfrak{m}^{-1} \succ 1$, we conclude that $(\mathfrak{t}\,\mathfrak{m})^{-1}$ satisfies (II). If $\mathfrak{t} \prec 1$, $\mathfrak{m} \succ 1$, and $\varphi_{\mathfrak{t}} \geqslant L_\beta(\mathfrak{m})$ then $(\mathfrak{t}\,\mathfrak{m})^{-1}$ satisfies (I), for similar reasons.

Now let $\mathfrak{t}\,\mathfrak{m}, \mathfrak{u}\,\mathfrak{n} \in \mathfrak{M}_{\mathbb{T}}^{\succcurlyeq}$. We will show that $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n}) \in \mathfrak{M}_{\mathbb{T}}^{\succcurlyeq}$. If both $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{s}\,\mathfrak{n}$ satisfy one of the last two rules, then this is clear. Thus, we may assume without loss of generality that $\mathfrak{t}\,\mathfrak{m}$ satisfies either rule (I) or rule (II). We consider the following cases:

Case 1: $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{u}\,\mathfrak{n}$ both satisfy (I) or they both satisfy (II). Suppose that they both satisfy (I). Then $\mathfrak{t}\,\mathfrak{u} \succ 1$ and $\mathfrak{m}\,\mathfrak{n} \prec 1$, so we need to verify that $\varphi_{\mathfrak{tu}} > L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1})$. By Corollary 7.26, we have $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}) =_\beta \max(L_\beta(\mathfrak{m}^{-1}), L_\beta(\mathfrak{n}^{-1}))$. Since $\mathfrak{t}, \mathfrak{u} \succ 1$, we also have $\varphi_{\mathfrak{tu}} = \max(\varphi_{\mathfrak{t}}, \varphi_{\mathfrak{u}})$, whence $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}) <_\beta \varphi_{\mathfrak{tu}}$. The case when $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{u}\,\mathfrak{n}$ both satisfy (II) is similar.

Case 2: $\mathfrak{t}\,\mathfrak{m}$ satisfies (I) and $\mathfrak{u}\,\mathfrak{n}$ satisfies (III) or (IV). We have $\mathfrak{t}\,\mathfrak{u} \succ 1$, so if $\mathfrak{m}\,\mathfrak{n} \succcurlyeq 1$, then $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (IV). Suppose that $\mathfrak{m}\,\mathfrak{n} \prec 1$. If $\mathfrak{n} = 1$, then $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}) = L_\beta(\mathfrak{m}^{-1})$ and if $\mathfrak{n} \succ 1$, then $(\mathfrak{m}\,\mathfrak{n})^{-1} \prec \mathfrak{m}^{-1}$, so $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}) < L_\beta(\mathfrak{m}^{-1})$ as L_β is strictly increasing. Since $\mathfrak{t}\,\mathfrak{m}$ satisfies rule (I) and $\mathfrak{u} \succcurlyeq 1$, we have

$$\varphi_{\mathfrak{tu}} = \max(\varphi_{\mathfrak{t}}, \varphi_{\mathfrak{u}}) \geqslant \varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1}) > L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}),$$

so $(\mathfrak{t}\,\mathfrak{u})(\mathfrak{m}\,\mathfrak{n})$ satisfies (I).

Case 3: $\mathfrak{t}\,\mathfrak{m}$ satisfies (II) and $\mathfrak{u}\,\mathfrak{n}$ satisfies (III) or (IV). We have $\mathfrak{m}\,\mathfrak{n} \succcurlyeq \mathfrak{m} \succ 1$, so if $\mathfrak{t}\,\mathfrak{u} \succcurlyeq 1$, then $(\mathfrak{t}\,\mathfrak{u})(\mathfrak{m}\,\mathfrak{n})$ satisfies (IV). Suppose that $\mathfrak{t}\,\mathfrak{u} \prec 1$. If $\mathfrak{u} \succ 1$, then $1 \prec \mathfrak{u} \prec \mathfrak{t}^{-1}$, so $\varphi_{\mathfrak{u}} \leqslant \varphi_{\mathfrak{t}^{-1}} = \varphi_{\mathfrak{t}}$ and $\varphi_{\mathfrak{tu}} \leqslant \max(\varphi_{\mathfrak{t}}, \varphi_{\mathfrak{u}}) = \varphi_{\mathfrak{t}}$. Since $\mathfrak{t}\,\mathfrak{m}$ satisfies rule (II), we have $\varphi_{\mathfrak{t}} <_\beta L_\beta(\mathfrak{m})$, so

$$\varphi_{\mathfrak{tu}} \leqslant \varphi_{\mathfrak{t}} <_\beta L_\beta(\mathfrak{m}) \leqslant L_\beta(\mathfrak{m}\,\mathfrak{n}).$$

Hence $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (II).

Case 4: One of the monomials $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{u}\,\mathfrak{n}$ satisfies (I) and the other one satisfies (II). Without loss of generality, we may assume that $\mathfrak{t}\,\mathfrak{m}$ satisfies (I) and $\mathfrak{u}\,\mathfrak{n}$ satisfies (II). Let us first consider the case when $\mathfrak{t}\,\mathfrak{u} \prec 1$. Then $1 \prec \mathfrak{t} \prec \mathfrak{u}^{-1}$, so $\varphi_{\mathfrak{t}} \leqslant \varphi_{\mathfrak{u}^{-1}} = \varphi_{\mathfrak{u}}$ and $\varphi_{\mathfrak{tu}} \leqslant \varphi_{\mathfrak{u}}$. Since $\varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1})$ and $\varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n})$, we deduce that $L_\beta(\mathfrak{m}^{-1}) <_\beta L_\beta(\mathfrak{n})$, so $L_\beta(\mathfrak{m}\,\mathfrak{n}) =_\beta L_\beta(\mathfrak{n})$ by Corollary 7.29. Since $\mathfrak{u}\,\mathfrak{n}$ satisfies (II), we have $\varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n})$, so

$$\varphi_{\mathfrak{tu}} \leqslant \varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n}) =_\beta L_\beta(\mathfrak{m}\,\mathfrak{n})$$

and $(\mathfrak{t}\,\mathfrak{s})(\mathfrak{m}\,\mathfrak{n})$ satisfies (II).

Let us now consider the case when $\mathfrak{t}\,\mathfrak{u} \succcurlyeq 1$. If $\mathfrak{m}\,\mathfrak{n} \succ 1$, then $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (III). If $\mathfrak{m}\,\mathfrak{n} = 1$ and $\mathfrak{t}\,\mathfrak{u} \succ 1$, then $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (IV). If $\mathfrak{m}\,\mathfrak{n} = \mathfrak{t}\,\mathfrak{u} = 1$, then $\mathfrak{m}\,\mathfrak{n} = (\mathfrak{t}\,\mathfrak{u})^{-1}$, so $\mathfrak{t}\,\mathfrak{m} = (\mathfrak{u}\,\mathfrak{n})^{-1}$, contradicting that $\mathfrak{t}\,\mathfrak{m}, \mathfrak{u}\,\mathfrak{n} \in \mathfrak{M}_{\mathbb{T}}^{\succcurlyeq}$. It remains to consider the case that $\mathfrak{m}\,\mathfrak{n} \prec 1$. Then $\mathfrak{m}^{-1} \succ \mathfrak{n} \succ 1$, so $L_\beta(\mathfrak{m}^{-1}) > L_\beta(\mathfrak{n})$ as L_β is strictly increasing. Since $\varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1})$ and $\varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n})$, we deduce that $\varphi_{\mathfrak{t}} > \varphi_{\mathfrak{u}}$, so $\varphi_{\mathfrak{tu}} = \varphi_{\mathfrak{t}}$. Since $\mathfrak{n}^{-1} \prec 1$, we have $(\mathfrak{m}\,\mathfrak{n})^{-1} \prec \mathfrak{m}^{-1}$, so $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}) < L_\beta(\mathfrak{m}^{-1})$. This gives

$$\varphi_{\mathfrak{tu}} = \varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1}) > L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}),$$

so $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (I). □

Remark 8.8. Given $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbb{T}}]^{\succ}$, we have

$$\mathfrak{m} \prec \mathfrak{t} \Longleftrightarrow \mathfrak{m}^{-1}\mathfrak{t} \succ 1 \Longleftrightarrow L_\beta(\mathfrak{m}) < \varphi_{\mathfrak{t}}.$$

Since $\mathfrak{m} \neq \mathfrak{t}$, we also have $\mathfrak{m} \succ \mathfrak{t} \Longleftrightarrow L_\beta(\mathfrak{m}) > \varphi_{\mathfrak{t}}$. More generally, for $s \in \mathbb{T}^{>,\succ}$, we have

$$s \prec \mathfrak{t} \Longleftrightarrow L_\beta(s) < \varphi_{\mathfrak{t}}, \qquad s \succ \mathfrak{t} \Longleftrightarrow L_\beta(s) > \varphi_{\mathfrak{t}}.$$

This is because $L_\beta(s) =_\beta L_\beta(\mathfrak{d}_s)$ by Corollary 7.28 with $\sigma = \gamma = 0$.

Extending the real power operation. For $r \in \mathbb{R}$ and $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{\mathbf{T}}$, define $(\mathfrak{t}\,\mathfrak{m})^r := \mathfrak{t}^r \mathfrak{m}^r$ where $\mathfrak{m}^r$ is as defined in $\mathfrak{M}$, and

$$\mathfrak{t}^r := \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi^r[\mathrm{e}_\beta^\varphi] \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}].$$

It is easy to check that this defines a real power operation on $\mathfrak{M}_{\mathbf{T}}$. Note that $\varphi_{\mathfrak{t}^r} = \varphi_{\mathfrak{t}}$ for each non-zero $r \in \mathbb{R}$.

Now that we have defined an ordering and a real power operation on $\mathfrak{M}_{\mathbf{T}}$, we let $\mathbb{T}_{\mathbf{T}} := \mathbb{R}[[\mathfrak{M}_{\mathbf{T}}]]$. Then $\mathbb{T}_{\mathbf{T}}$ is a field of well-based series extending $\mathbb{T}$. When $\mathbf{T} = \mathbf{T}_\mu(\mathbb{T})$, we write $\mathbb{T}_{(\mu)} := \mathbb{T}_{\mathbf{T}}$.

8.3 Extending the hyperlogarithmic structure

In this subsection, we extend the hyperlogarithms L_{ω^η} from $\mathbb{T}$ to $\mathbb{T}_{\mathbf{T}}$, while verifying that they satisfy the axioms for hyperserial skeletons. We separate various cases as a function of η, including the case of the ordinary logarithm when $\eta = 0$.

In each case, we start with the definition of the domain $\operatorname{dom} L_{\omega^\eta}$ of the extended hyperlogarithm L_{ω^η} on $\mathbb{T}_{\mathbf{T}}$ and then define L_{ω^η} on the elements of $\operatorname{dom} L_{\omega^\eta}$ which do not already lie in $\mathfrak{M}_{\omega^\eta}$. We next check that $(\mathbb{T}, (L_{\omega^\eta})_{\iota \leqslant \eta})$ satisfies the domain definition axioms $\mathbf{DD}_\eta$, as well as the other axioms for hyperserial skeletons.

Extending the logarithm when $\mu = 1$. Suppose that $\mu = 1$, so $\beta = \omega$ and $\theta = 1$. For $\ell_0^r \in \mathfrak{L}_{<1}$ and $\varphi \in \mathbf{T}$, we define

$$\log(\ell_0^r[\mathrm{e}_\omega^\varphi]) := \begin{cases} r\,\mathrm{e}_\omega^{\varphi-1} & \text{if } \varphi - 1 \in \mathbf{T} \\ r\,E_\omega(\varphi - 1) & \text{otherwise.} \end{cases}$$

We extend log to $\mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$ by setting

$$\log \mathfrak{t} := \sum_{\varphi \in \mathbf{T}} \log(\ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi])$$

for $\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi] \in \mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$. Note that $\log(\ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi]) \neq 0$ if and only if $\varphi \in \operatorname{hsupp} \mathfrak{t}$. We claim that $\log \mathfrak{t}$ is well-defined. Let $\varphi_1 > \cdots > \varphi_n \in \mathbf{T}$ and $A_1, \ldots, A_n \subseteq \mathbb{N}$ be as in Remark 8.6, so $\mathfrak{t} = \prod_{i=1}^n \prod_{m \in A_i} \ell_0^{r_{\varphi_i - m}}[\mathrm{e}_\omega^{\varphi_i - m}]$ and

$$\log \mathfrak{t} = \sum_{i=1}^n \sum_{m \in A_i} \log(\ell_0^{r_{\varphi_i - m}}[\mathrm{e}_\omega^{\varphi_i - m}]) = \sum_{i=1}^n \sum_{m \in A_i} r_{\varphi_i - m}\,\mathrm{e}_\omega^{\varphi_i - m - 1}.$$

Each sum $\sum_{m \in A_i} r_{\varphi_i - m}\,\mathrm{e}_\omega^{\varphi_i - m - 1}$ exists in $\mathbb{T}_{\mathbf{T}}$, since the support $(\mathrm{e}_\omega^{\varphi_i - m - 1})_{m \in A_i}$ is a strictly decreasing sequence in $\mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$. Thus, $\log \mathfrak{t}$ is well-defined. If $\mathfrak{t} \neq 1$, then we note that

$$\log \mathfrak{t} \sim \log(\ell_0^{r_{\varphi_{\mathfrak{t}}}}[\mathrm{e}_\omega^{\varphi_{\mathfrak{t}}}]) \sim \begin{cases} r_{\varphi_{\mathfrak{t}}}\,\mathrm{e}_\omega^{\varphi_{\mathfrak{t}} - 1} & \text{if } \varphi_{\mathfrak{t}} - 1 \in \mathbf{T} \\ r_{\varphi_{\mathfrak{t}}}\,E_\omega(\varphi_{\mathfrak{t}} - 1) & \text{otherwise.} \end{cases}$$

Finally, we extend log to all of $\mathfrak{M}_{\mathbf{T}}$ by setting $\log(\mathfrak{t}\,\mathfrak{m}) := \log \mathfrak{t} + \log \mathfrak{m}$. We let L_1 be the restriction of log to the class $\mathfrak{M}_{\mathbf{T}}^{\succ}$, so $(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{DD}_0$.

Using the definition of real powers, it is straightforward to check that $(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{FE}_0$. Let $\varphi \in \mathbf{T}$. If $\varphi - 1 \in \mathbf{T}$, then $\mathrm{e}_\omega^{\varphi-1} \in \mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]^{\succ}$. If $\varphi - 1 \notin \mathbf{T}$, then $E_\omega(\varphi - 1) \in \mathfrak{M}_\omega \subseteq \mathfrak{M}^{\succ}$ by Corollary 7.21. In either case, $\log(\ell_0^r[\mathrm{e}_\omega^\varphi]) \succ 1$ for all $r \in \mathbb{R}$. Hence,

$$\operatorname{supp} L_1(\mathfrak{t}\,\mathfrak{m}) \subseteq \operatorname{supp} \log \mathfrak{t} \cup \operatorname{supp} \log \mathfrak{m} \succ 1$$

for $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{\mathbf{T}}^{\succ}$ and $\mathbf{R}_0$ is satisfied. The axiom $\mathbf{P}_0$ follows from $\mathbf{FE}_0$, so it remains to be shown that $(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{A}_0$ and $\mathbf{M}_0$.

Lemma 8.9. *$(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{A}_0$.*

Proof. Given $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{\mathbf{T}}^{\succ}$, we must show that $L_1(\mathfrak{t}\,\mathfrak{m})\prec\mathfrak{t}\,\mathfrak{m}$. We proceed by case distinction:

1. If $\mathfrak{t}=1$, then $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{m})\prec\mathfrak{m}=\mathfrak{t}\,\mathfrak{m}$ since $(\mathbb{T},L_1)$ satisfies $\mathbf{A}_0$.
2. If $\mathfrak{m}=1$, then $\mathfrak{t}\succ 1$ and
$$\mathfrak{d}_{L_1(\mathfrak{t})}=\begin{cases} \mathrm{e}_{\omega}^{\varphi_{\mathfrak{t}}-1} & \text{if } \varphi_{\mathfrak{t}}-1\in\mathbf{T}\\ E_{\omega}(\varphi_{\mathfrak{t}}-1) & \text{otherwise.}\end{cases}$$
If $\varphi_{\mathfrak{t}}-1\in\mathbf{T}$, then $\mathfrak{d}_{L_1(\mathfrak{t})}\in\mathfrak{L}_{<1}[\mathrm{e}_{\omega}^{\mathbf{T}}]$ and $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}}=\varphi_{\mathfrak{t}}-1$. Thus $\mathfrak{d}_{L_1(\mathfrak{t})}\prec\mathfrak{t}$ since $\varphi_{\mathfrak{t}}-1<\varphi_{\mathfrak{t}}$. If $\varphi_{\mathfrak{t}}-1\notin\mathbf{T}$, then $L_{\omega}(\mathfrak{d}_{L_1(\mathfrak{t})})=\varphi_{\mathfrak{t}}-1<\varphi_{\mathfrak{t}}$, so Remark 8.8 again yields $\mathfrak{d}_{L_1(\mathfrak{t})}\prec\mathfrak{t}$. In either case, $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})\prec\mathfrak{t}=\mathfrak{t}\,\mathfrak{m}$.
3. Suppose $\mathfrak{t}\succ 1$, $\mathfrak{m}\prec 1$, and $\varphi_{\mathfrak{t}}>L_{\omega}(\mathfrak{m}^{-1})$. We have $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})-L_1(\mathfrak{m}^{-1})$, so it is enough to show that $L_1(\mathfrak{t})\prec\mathfrak{t}\,\mathfrak{m}$ and $L_1(\mathfrak{m}^{-1})\prec\mathfrak{t}\,\mathfrak{m}$. We have
$$L_{\omega}(\mathfrak{m}^{-2})=L_{\omega}^{\uparrow 1}(2L_1(\mathfrak{m}^{-1}))=_{\omega}L_{\omega}(\mathfrak{m}^{-1})$$
by Lemma 7.25, so $\varphi_{\mathfrak{t}}>L_{\omega}(\mathfrak{m}^{-2})$, whence $\mathfrak{t}\,\mathfrak{m}^2\succ 1$ and $\mathfrak{t}\,\mathfrak{m}\succ\mathfrak{m}^{-1}\succ L_1(\mathfrak{m}^{-1})$. Since $\varphi_{\mathfrak{t}^{1/2}}=\varphi_{\mathfrak{t}}>L_{\omega}(\mathfrak{m}^{-1})$, we also have $\mathfrak{t}^{1/2}\mathfrak{m}\succ 1$, so
$$\mathfrak{t}\,\mathfrak{m}\succ\mathfrak{t}^{1/2}\succ L_1(\mathfrak{t}^{1/2})\asymp L_1(\mathfrak{t}).$$
4. Suppose $\mathfrak{t}\prec 1$, $\mathfrak{m}\succ 1$, and $\varphi_{\mathfrak{t}}<L_{\omega}(\mathfrak{m})$. This time, we need to show that $L_1(\mathfrak{t}^{-1})\prec\mathfrak{t}\,\mathfrak{m}$ and $L_1(\mathfrak{m})\prec\mathfrak{t}\,\mathfrak{m}$. Using that $\varphi_{\mathfrak{t}^2}=\varphi_{\mathfrak{t}}$ and that $L_{\omega}(\mathfrak{m}^{1/2})=_{\omega}L_{\omega}(\mathfrak{m})$, we have $\mathfrak{t}^2\mathfrak{m}$, $\mathfrak{t}\,\mathfrak{m}^{1/2}\succ 1$, so
$$\mathfrak{t}\,\mathfrak{m}\succ\mathfrak{t}^{-1}\succ L_1(\mathfrak{t}^{-1}),\qquad \mathfrak{t}\,\mathfrak{m}\succ\mathfrak{m}^{1/2}\succ L_1(\mathfrak{m}^{1/2})\asymp L_1(\mathfrak{m}).$$
5. If $\mathfrak{t}\succ 1$ and $\mathfrak{m}\succ 1$, then $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})+L_1(\mathfrak{m})$. So the result follows from the fact that $L_1(\mathfrak{t})\prec\mathfrak{t}\prec\mathfrak{t}\,\mathfrak{m}$ and $L_1(\mathfrak{m})\prec\mathfrak{m}\prec\mathfrak{t}\,\mathfrak{m}$. □

Lemma 8.10. *$(\mathbb{T}_{\mathbf{T}}, L_1)$ satisfies $\mathbf{M}_0$.*

Proof. Given $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{\mathbf{T}}^{\succ}$, we need to show that $L_1(\mathfrak{t}\,\mathfrak{m})>0$. If $\mathfrak{t}=1$, then $\mathfrak{m}\succ 1$ so $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{m})>0$ since $(\mathbb{T},L_1)$ satisfies $\mathbf{M}_0$. If $\mathfrak{m}=1$, then $\mathfrak{t}\succ 1$, so $r_{\varphi_{\mathfrak{t}}}>0$. Since
$$L_1(\mathfrak{t})\sim\begin{cases} r_{\varphi_{\mathfrak{t}}}\mathrm{e}_{\omega}^{\varphi_{\mathfrak{t}}-1} & \text{if } \varphi_{\mathfrak{t}}-1\in\mathbf{T}\\ r_{\varphi_{\mathfrak{t}}}E_{\omega}(\varphi_{\mathfrak{t}}-1) & \text{otherwise,}\end{cases}$$
we have $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})>0$. If $\mathfrak{t},\mathfrak{m}\succ 1$, then $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})+L_1(\mathfrak{m})>0$.

Consider now the case that $\mathfrak{t}\prec 1$, $\mathfrak{m}\succ 1$, and $\varphi_{\mathfrak{t}}<L_{\omega}(\mathfrak{m})$. Since $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{m})-L_1(\mathfrak{t}^{-1})$, we need to show that $L_1(\mathfrak{t}^{-1})<L_1(\mathfrak{m})$. For each $r\in\mathbb{R}^{>}$, we have
$$L_{\omega}(\mathfrak{m})=_{\omega}L_{\omega}^{\uparrow 1}(rL_1(\mathfrak{m}))=L_{\omega}(rL_1(\mathfrak{m}))+1$$
by Lemma 7.25. As $\varphi_{\mathfrak{t}}<_{\omega}L_{\omega}(\mathfrak{m})$, this gives $\varphi_{\mathfrak{t}}-1<L_{\omega}(rL_1(\mathfrak{m}))$. If $\varphi_{\mathfrak{t}}-1\in\mathbf{T}$, then we have
$$L_1(\mathfrak{t}^{-1})\asymp\mathrm{e}_{\omega}^{\varphi_{\mathfrak{t}}-1}\prec rL_1(\mathfrak{m})\asymp L_1(\mathfrak{m})$$
by Remark 8.8. If $\varphi_{\mathfrak{t}}-1\notin\mathbf{T}$, so $E_{\omega}(\varphi_t-1)$ is defined in $\mathbb{T}$, then for each $r\in\mathbb{R}^{>}$, we have
$$E_{\omega}(\varphi_{\mathfrak{t}}-1)<rL_1(\mathfrak{m})$$

since E_ω is strictly increasing. As $r\in\mathbb{R}^>$ is arbitrary, this gives $L_1(\mathfrak{t}^{-1})\asymp E_\omega(\varphi_\mathfrak{t}-1)\prec L_1(\mathfrak{m})$.

Finally, suppose $\mathfrak{t}\succ 1$, $\mathfrak{m}\prec 1$, and $\varphi_\mathfrak{t}>L_\omega(\mathfrak{m}^{-1})$. The same argument as above gives $\varphi_\mathfrak{t}-1>L_\omega(rL_1(\mathfrak{m}^{-1}))$ for $r\in\mathbb{R}^>$, so $L_1(\mathfrak{t})\succ L_1(\mathfrak{m}^{-1})$ and $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})-L_1(\mathfrak{m}^{-1})>0$. □

Extending the logarithm when $\mu>1$. For $\mathfrak{l}=\prod_{\gamma<\theta}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\theta}$ and $\varphi\in\mathbf{T}$, we define

$$\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi]):=\sum_{\gamma<\theta}\mathfrak{l}_\gamma\,\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi].$$

This sum is well-defined, as $\ell_{\sigma+1}[\mathrm{e}_\beta^\varphi]\prec\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]$ for $\gamma<\sigma<\theta$. For $\mathfrak{t}\in\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^\mathbf{T}]$, we set

$$\log\mathfrak{t}:=\sum_{\varphi\in\operatorname{hsupp}\mathfrak{t}}\log(\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi])=\sum_{\varphi\in\operatorname{hsupp}\mathfrak{t}}\sum_{\gamma<\theta}(\mathfrak{t}_\varphi)_\gamma\,\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi].$$

This sum is also well-defined, as $\operatorname{hsupp}\mathfrak{t}$ is well-based and $\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]\prec\ell_{\sigma+1}[\mathrm{e}_\beta^\psi]$ for all $\gamma,\sigma<\theta$, and $\varphi,\psi\in\mathbf{T}$ with $\varphi<\psi$. If $\mathfrak{t}\neq 1$, then note that $\log\mathfrak{t}\sim(\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}}\,\ell_{\gamma_\mathfrak{t}+1}[\mathrm{e}_\beta^\varphi]$, so

$$\mathfrak{d}_{\log\mathfrak{t}}=\ell_{\gamma_\mathfrak{t}+1}[\mathrm{e}_\beta^{\varphi_\mathfrak{t}}]=\mathfrak{d}_{\log\mathfrak{t}_{\varphi_\mathfrak{t}}}[\mathrm{e}_\beta^{\varphi_\mathfrak{t}}]$$

and $\log\mathfrak{t}>0$ whenever $\mathfrak{t}\succ 1$. Finally, we extend log to all of $\mathfrak{M}_\mathbf{T}$ by setting

$$\log(\mathfrak{t}\,\mathfrak{m}):=\log\mathfrak{t}+\log\mathfrak{m}.$$

for $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_\mathbf{T}$. As before, we let L_1 be the restriction of log to $\mathfrak{M}_\mathbf{T}^\succ$, so $(\mathbb{T}_\mathbf{T},L_1)$ satisfies $\mathbf{DD}_0$.

The axiom $\mathbf{FE}_0$ (and thus $\mathbf{P}_0$) follow easily from the definition of L_1 and the axiom $\mathbf{R}_0$ holds since $\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]\succ 1$ for each γ. Let us prove that $\mathbf{A}_0$ holds for $\mathfrak{t}\in\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^\mathbf{T}]^\succ$. Given $\mathfrak{t}\succ 1$, we need to show that $\mathfrak{t}\,\mathfrak{d}_{L_1(\mathfrak{t})}^{-1}\succ 1$. Since $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}}=\varphi_\mathfrak{t}$, it suffices to show that $(\mathfrak{t}\,\mathfrak{d}_{L_1(\mathfrak{t})}^{-1})_{\varphi_\mathfrak{t}}=\mathfrak{t}_{\varphi_\mathfrak{t}}(\mathfrak{d}_{L_1(\mathfrak{t})}^{-1})_{\varphi_\mathfrak{t}}\succ 1$. Since $(\mathfrak{d}_{L_1(\mathfrak{t})}^{-1})_{\varphi_\mathfrak{t}}=\mathfrak{d}_{L_1(\mathfrak{t}_{\varphi_\mathfrak{t}})}^{-1}$, this further reduces to showing that $\mathfrak{t}_{\varphi_\mathfrak{t}}\succ L_1(\mathfrak{t}_{\varphi_\mathfrak{t}})$. But this follows from the fact that $\mathbf{A}_0$ holds for $\mathbb{L}_{<\theta}$. The proof that $\mathbf{A}_0$ holds for a general element $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_\mathbf{T}^\succ$ is identical to cases 3–5 of Lemma 8.9. Let us now show that $(\mathbb{T}_\mathbf{T},L_1)$ also satisfies $\mathbf{M}_0$.

Lemma 8.11. $(\mathbb{T}_\mathbf{T},L_1)$ *satisfies* $\mathbf{M}_0$.

Proof. We have $L_1(\mathfrak{t})>0$ for $\mathfrak{t}\in\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^\mathbf{T}]^\succ$ and $L_1(\mathfrak{m})>0$ for $\mathfrak{m}\in\mathfrak{M}^\succ$. It follows that $L_1(\mathfrak{t}\,\mathfrak{m})>0$ for $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_\mathbf{T}^\succ$ so long as $\mathfrak{t},\mathfrak{m}\succcurlyeq 1$. Suppose that $\mathfrak{t}\succ 1$, $\mathfrak{m}\prec 1$, and $\varphi_\mathfrak{t}>L_\beta(\mathfrak{m}^{-1})$. Then $L_1(\mathfrak{t}\,\mathfrak{m})=L_1(\mathfrak{t})-L_1(\mathfrak{m}^{-1})$, so it is enough to show that $L_1(\mathfrak{t})\succ L_1(\mathfrak{m}^{-1})$. As shown in the argument that $\mathbf{A}_0$ holds, we have $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}}=\varphi_\mathfrak{t}$. By Lemma 7.27, we also have $L_\beta(\mathfrak{m}^{-1})=_\beta L_\beta(L_1(\mathfrak{m}^{-1}))$. Thus, $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}}>L_\beta(L_1(\mathfrak{m}^{-1}))$, so $L_1(\mathfrak{t})\asymp\mathfrak{d}_{L_1(\mathfrak{t})}\succ L_1(\mathfrak{m}^{-1})$; see Remark 8.8. The case that $\mathfrak{t}\prec 1$, $\mathfrak{m}\succ 1$, and $\varphi_\mathfrak{t}<L_\beta(\mathfrak{m})$ is similar. □

Extending L_{ω^η} when $0<\eta<\mu_*$. Given $0<\eta<\mu_*$, we set

$$\operatorname{dom}L_{\omega^\eta}:=\mathfrak{M}_{\omega^\eta}\cup\{\ell_\gamma[\mathrm{e}_\beta^\varphi]:\varphi\in\mathbf{T}\text{ and }\omega^{\eta_*}\leqslant\gamma<\theta\}.$$

Given γ with $\omega^{\eta_*}\leqslant\gamma<\theta$, we decompose $\gamma=\gamma_{\geqslant\omega^\eta}+\omega^{\eta_*}n$, and define

$$L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]):=\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]-n.$$

Note that $n=0$ and $L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi])=\ell_{\gamma+\omega^\eta}[\mathrm{e}_\beta^\varphi]$ whenever η is a limit ordinal. More generally, we have

$$L_{\omega^\iota}(\ell_\gamma[\mathrm{e}_\beta^\varphi])=\ell_{\gamma+\omega^\iota}[\mathrm{e}_\beta^\varphi]$$

whenever $\iota\leqslant\eta_*$ (including the case when $\iota=0$).

Lemma 8.12. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu_*})$ satisfies* $\mathbf{DD}_\eta$ *for each* $\eta<\mu_*$.

Proof. We prove this by induction on $\eta<\mu_*$, beginning with $\eta=1$. Let $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{\mathbf{T}}^{\succ}$, so

$$L_1(\mathfrak{t}\,\mathfrak{m})=\log\mathfrak{m}+\sum_{\varphi\in\operatorname{hsupp}\mathfrak{t}}\sum_{\gamma<\theta}(\mathfrak{t}_\varphi)_\gamma\,\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi].$$

If $L_1(\mathfrak{t}\,\mathfrak{m})\in\mathfrak{M}_{\mathbf{T}}^{\succ}$, then either $\mathfrak{t}=1$ or $\mathfrak{m}=1$. If $\mathfrak{t}=1$, then $\mathfrak{m}\in\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_1^{\circ n}$ if and only if $\mathfrak{m}\in\mathfrak{M}_\omega$. If $\mathfrak{m}=1$, then $L_1(\mathfrak{t})\in\mathfrak{M}_{\mathbf{T}}^{\succ}$ if and only if $\mathfrak{t}=\ell_\gamma[\mathrm{e}_\beta^\varphi]\in\operatorname{dom}L_\omega$. It remains to note that $L_n(\ell_\gamma[\mathrm{e}_\beta^\varphi])=\ell_{\gamma+n}[\mathrm{e}_\beta^\varphi]\in\mathfrak{M}_{\mathbf{T}}^{\succ}$ for all n.

Now assume that $\eta>1$ and that $\mathbf{DD}_\iota$ holds for all $\iota<\eta$. Since $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{DD}_\eta$ for each $\eta<\mu_*$, we may focus on elements of the form $\ell_\gamma[\mathrm{e}_\beta^\varphi]$ where $\gamma<\theta$ and $\varphi\in\mathbf{T}$. For the remainder of the proof, we fix such an element. If η is a successor, then we need to show that $\ell_\gamma[\mathrm{e}_\beta^\varphi]\in\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_{\omega^{\eta_*}}^{\circ n}$ if and only if $\gamma\gtrsim\omega^{\eta_*}$. One direction is clear: if $\gamma\gtrsim\omega^{\eta_*}$ then $L_{\omega^{\eta_*}n}(\ell_\gamma[\mathrm{e}_\beta^\varphi])=\ell_{\gamma+\omega^{\eta_*}n}[\mathrm{e}_\beta^\varphi]\in\operatorname{dom}L_{\omega^{\eta_*}}$ for each n. For the other direction, if $\ell_\gamma[\mathrm{e}_\beta^\varphi]\in\operatorname{dom}L_{\omega^{\eta_*}}$, then $\gamma\gtrsim\omega^{\eta_{**}}$, so write $\gamma=\gamma_{\gtrsim\omega^{\eta_*}}+\omega^{\eta_{**}}m$ and note that $L_{\omega^{\eta_*}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])=\ell_{\gamma_{\gtrsim\omega^{\eta_*}}+\omega^{\eta_*}}[\mathrm{e}_\beta^\varphi]-m$ is a monomial if and only if $m=0$. If η is a limit, then $\gamma\gtrsim\omega^{\iota_*}$ for all $\iota<\eta$ if and only if $\gamma\gtrsim\omega^{\eta_*}=\omega^\eta$, so we have $\ell_\gamma[\mathrm{e}_\beta^\varphi]\in\operatorname{dom}L_{\omega^\eta}$ if and only if $\ell_\gamma[\mathrm{e}_\beta^\varphi]\in\operatorname{dom}L_{\omega^\iota}$ for all $\iota<\eta$. □

Lemma 8.13. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu_*})$ satisfies* $\mathbf{A}_\eta$ *for each* $\eta<\mu_*$.

Proof. Let $\varphi\in\mathbf{T}$ and $\eta,\iota,\gamma\in\mathbf{On}$ with $0\leqslant\iota<\eta<\mu_*$ and $\omega^{\eta_*}\lesssim\gamma<\theta$. Since $(\mathbb{T},(L_{\omega^\lambda})_{\lambda<\mu_*})$ satisfies $\mathbf{A}_\lambda$ for each $\lambda<\mu_*$, it suffices to show that $L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi])<L_{\omega^\iota}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$. Decomposing $\gamma=\gamma_{\gtrsim\omega^\eta}+\omega^{\eta_*}n$, we have $\gamma_{\gtrsim\omega^\eta}+\omega^\eta>\gamma+\omega^\iota$, so

$$L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi])=\ell_{\gamma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]-n\leqslant\ell_{\gamma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]<\ell_{\gamma+\omega^\iota}[\mathrm{e}_\beta^\varphi]=L_{\omega^\iota}(\ell_\gamma[\mathrm{e}_\beta^\varphi]).\qquad\square$$

Let $0<\eta<\mu_*$, let $\omega^{\eta_*}\lesssim\gamma<\theta$, and let $\varphi\in\mathbf{T}$. We note that $L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$ has no infinitesimal terms in its support, so $\mathbf{R}_\eta$ is satisfied since it holds in $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu_*})$. To see that $(\mathbb{T}_{\mathbf{T}},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{FE}_\eta$, suppose that η is a successor and write $\gamma=\gamma_{\gtrsim\omega^\eta}+\omega^{\eta_*}n$. Then

$$L_{\omega^\eta}(L_{\omega^{\eta_*}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]))=L_{\omega^\eta}(\ell_{\gamma_{\gtrsim\omega^\eta}+\omega^{\eta_*}(n+1)}[\mathrm{e}_\beta^\varphi])=\ell_{\gamma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]-(n+1)=L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi])-1.$$

Lemma 8.14. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu_*})$ satisfies* $\mathbf{M}_\eta$ *for each* $\eta<\mu_*$ *with* $\eta>0$.

Proof. Let $\eta<\mu_*$ with $\eta>0$, let $\mathfrak{a},\mathfrak{b}\in(\mathfrak{M}_{\mathbf{T}})_{\omega^\eta}$ with $\mathfrak{a}\prec\mathfrak{b}$, and let $\omega^\iota n<\omega^\eta$. We want to show that

$$L_{\omega^\eta}(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1}<L_{\omega^\eta}(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

If $\mathfrak{a},\mathfrak{b}\in\mathfrak{M}_{\omega^\eta}$, then this holds because $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{M}_\eta$. Consider the following cases:

1. If $\mathfrak{a}=\ell_\gamma[\mathrm{e}_\beta^\varphi]$ and $\mathfrak{b}=\ell_\sigma[\mathrm{e}_\beta^\psi]$, then write $\gamma=\gamma_{\gtrsim\omega^\eta}+\omega^{\eta_*}m$ and $\sigma=\sigma_{\gtrsim\omega^\eta}+\omega^{\eta_*}k$. We have

$$\begin{aligned}L_{\omega^\eta}(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1}&=\ell_{\gamma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]-m+\ell_{\gamma+\omega^\eta n}^{-1}[\mathrm{e}_\beta^\varphi]\\L_{\omega^\eta}(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1}&=\ell_{\sigma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\psi]-k-\ell_{\sigma+\omega^\eta n}^{-1}[\mathrm{e}_\beta^\psi].\end{aligned}$$

   Since $\mathfrak{a}\prec\mathfrak{b}$, we have $\varphi\leqslant\psi$. If $\varphi<\psi$, then $\ell_{\gamma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]\prec\ell_{\sigma_{\gtrsim\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\psi]$. If $\varphi=\psi$, then $\gamma>\sigma$, so either $\gamma_{\gtrsim\omega^\eta}>\sigma_{\gtrsim\omega^\eta}$ or $\gamma_{\gtrsim\omega^\eta}=\sigma_{\gtrsim\omega^\eta}$ and $m>k$. In both cases, we have $L_{\omega^\eta}(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1}<L_{\omega^\eta}(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1}$.

2. If $\mathfrak{a}=\ell_\gamma[\mathrm{e}_\beta^\varphi]$ and $\mathfrak{b}\in\mathfrak{M}_{\omega^\eta}$, then we must have $\varphi<L_\beta(\mathfrak{b})$ by Remark 8.8. Writing $\gamma=\gamma_{\geqslant\omega^\eta}+\omega^{\eta_*}m$, we have $L_{\omega^\eta}(\mathfrak{a})=\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]-m$, so $\mathfrak{d}_{L_{\omega^\eta}(\mathfrak{a})}=\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]$. By Corollary 7.28, we have $L_\beta(L_{\omega^\eta}(\mathfrak{b}))=_\beta L_\beta(\mathfrak{b})>\varphi$, so
$$L_{\omega^\eta}(\mathfrak{b})\succ\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]\asymp L_{\omega^\eta}(\mathfrak{a}),$$
again by Remark 8.8. In particular, $L_{\omega^\eta}(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1}<L_{\omega^\eta}(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1}$.

3. If $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$ and $\mathfrak{b}=\ell_\gamma[\mathrm{e}_\beta^\varphi]$, then $\varphi>L_\beta(\mathfrak{a})$. Arguing as in the previous case, we have $L_{\omega^\eta}(\mathfrak{b})\asymp\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]\succ L_{\omega^\eta}(\mathfrak{a})$. □

Lemma 8.15. *$(\mathbb{T}_{\mathbf{T}},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{P}_\eta$ for each $0<\eta<\mu_*$.*

Proof. Let $\mathfrak{a}\in(\mathfrak{M}_{\mathbf{T}})_{\omega^\eta}$ and let $(r_\gamma)_{\gamma<\omega^\eta}$ be a sequence of real numbers. Consider the sum $s:=\sum_{\gamma<\omega^\eta}r_\gamma L_{\gamma+1}(\mathfrak{a})$. We need to show that $s\in\log\mathfrak{M}_{\mathbf{T}}$. If $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$, then $s\in\log\mathfrak{M}$. Suppose $\mathfrak{a}=\ell_\sigma[\mathrm{e}_\beta^\varphi]$ with $\omega^{\eta_*}\leqslant\sigma<\theta$. Then $L_\gamma(\mathfrak{a})=\ell_{\sigma+\gamma}[\mathrm{e}_\beta^\varphi]$ for all $\gamma<\omega^\eta$, so
$$s=\sum_{\gamma<\omega^\eta}r_\gamma L_{\gamma+1}(\ell_\sigma[\mathrm{e}_\beta^\varphi])=\sum_{\gamma<\omega^\eta}r_\gamma\,\ell_{\sigma+\gamma+1}[\mathrm{e}_\beta^\varphi]=\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi])$$
where $\mathfrak{l}:=\prod_{\gamma<\omega^\eta}\ell_{\sigma+\gamma}^{r_\gamma}\in\mathfrak{L}_{<\theta}$. □

Extending L_θ if $\mu>1$ is a successor. Assume that $\mu>1$ is a successor and let $\xi:=\omega^{\mu_{**}}$. We take
$$\operatorname{dom}L_\theta:=\mathfrak{M}_\theta\cup\{\ell_\gamma[\mathrm{e}_\beta^\varphi]:\varphi\in\mathbf{T}\text{ and }\xi\leqslant\gamma<\theta\}.$$
Note that $\xi\leqslant\gamma<\theta$ implies $\gamma=\xi n$ for some $n\in\mathbb{N}$. Moreover, if μ_* is a limit, then $n=0$. In other words,
$$\operatorname{dom}L_\theta\;=\;\begin{cases}\mathfrak{M}_\theta\cup\{\ell_{\xi n}[\mathrm{e}_\beta^\varphi]:\varphi\in\mathbf{T}\text{ and }n\in\mathbb{N}\} & \text{if }\mu_*\text{ is a successor.}\\ \mathfrak{M}_\theta\cup\{\mathrm{e}_\beta^\varphi:\varphi\in\mathbf{T}\} & \text{if }\mu_*\text{ is a limit.}\end{cases}$$
We define
$$L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]):=\begin{cases}\mathrm{e}_\beta^{\varphi-1}-n & \text{if }\varphi-1\in\mathbf{T}\\ E_\beta(\varphi-1)-n & \text{otherwise.}\end{cases}$$
The proofs of Lemmas 8.12 and 8.15 can be amended to show that $(\mathbb{T}_{\mathbf{T}},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{DD}_{\mu_*}$ and $\mathbf{P}_{\mu_*}$; just replace η with μ_*. Since $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{R}_{\mu_*}$, $\mathbf{FE}_{\mu_*}$, and $\mathbf{A}_{\mu_*}$, it suffices to check these axioms for elements of the form $\ell_{\xi n}[\mathrm{e}_\beta^\varphi]$, where $\varphi\in\mathbf{T}$ and $\xi n<\theta$. If $\varphi-1\in\mathbf{T}$, then $\mathrm{e}_\beta^{\varphi-1}\in\mathfrak{M}_{\mathbf{T}}^{\succ}$ and if $\varphi-1\notin\mathbf{T}$, then $\varphi-1$ is β-truncated by Lemma 7.13 (recall that μ is a successor), whence $E_\beta(\varphi-1)\in\mathfrak{M}_\beta\subseteq\mathfrak{M}^{\succ}$ by Corollary 7.21. In either case, we have $\operatorname{supp}L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi])\succcurlyeq1$, so $(\mathbb{T}_{\mathbf{T}},(L_{\omega^\eta})_{\eta<\mu_*})$ satisfies $\mathbf{R}_{\mu_*}$. As for $\mathbf{FE}_{\mu_*}$, suppose that μ_* is a successor. We have
$$L_\theta(L_\xi(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]))=L_\theta(\ell_{\xi(n+1)}[\mathrm{e}_\beta^\varphi])=\mathrm{e}_\beta^{\varphi-1}-(n+1)=L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi])-1$$
if $\varphi-1\in\mathbf{T}$ and we have
$$L_\theta(L_\xi(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]))=L_\theta(\ell_{\xi(n+1)}[\mathrm{e}_\beta^\varphi])=E_\beta(\varphi-1)-(n+1)=L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi])-1$$
otherwise.

Lemma 8.16. *$(\mathbb{T}_{\mathbf{T}},(L_{\omega^\eta})_{\eta<\mu})$ satisfies $\mathbf{A}_{\mu_*}$.*

Proof. Let $\varphi \in \mathbf{T}$, $\xi n < \theta$, and $\iota < \mu_*$. If $\varphi - 1 \in \mathbf{T}$, then we have

$$L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^{\varphi-1} - n \prec \ell_{\xi n + \omega^\iota}[\mathrm{e}_\beta^\varphi] = L_{\omega^\iota}(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]).$$

If $\varphi - 1 \notin \mathbf{T}$, then $L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]) = E_\beta(\varphi - 1) - n$. Since $L_\beta(E_\beta(\varphi - 1)) = \varphi - 1 < \varphi$, we have $L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]) \asymp E_\beta(\varphi - 1) \prec \ell_{\xi n + \omega^\iota}[\mathrm{e}_\beta^\varphi]$ by Remark 8.8. □

Lemma 8.17. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu})$ satisfies $\mathbf{M}_{\mu_*}$.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in (\mathfrak{M}_{\mathbf{T}})_\theta$ with $\mathfrak{a} \prec \mathfrak{b}$ and let $\omega^\iota n < \theta$. We need to show that

$$L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

We proceed by case distinction.

1. If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\theta$, then this holds because $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu})$ satisfies $\mathbf{M}_{\mu_*}$.
2. Suppose $\mathfrak{a} = \ell_{\xi m}[\mathrm{e}_\beta^\varphi]$ and $\mathfrak{b} = \ell_{\xi k}[\mathrm{e}_\beta^\psi]$ for some $\xi m, \xi k < \theta$ and some $\varphi, \psi \in \mathbf{T}$. Then

$$L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} = \begin{cases} \mathrm{e}_\beta^{\varphi-1} - m + \ell_{\xi m + \omega^\iota n}[\mathrm{e}_\beta^\varphi]^{-1} & \text{if } \varphi - 1 \in \mathbf{T} \\ E_\beta(\varphi - 1) - m + \ell_{\xi m + \omega^\iota n}[\mathrm{e}_\beta^\varphi]^{-1} & \text{otherwise,} \end{cases}$$

$$L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1} = \begin{cases} \mathrm{e}_\beta^{\psi-1} - k - \ell_{\xi k + \omega^\iota n}[\mathrm{e}_\beta^\psi]^{-1} & \text{if } \psi - 1 \in \mathbf{T} \\ E_\beta(\psi - 1) - k - \ell_{\xi k + \omega^\iota n}[\mathrm{e}_\beta^\psi]^{-1} & \text{otherwise.} \end{cases}$$

   If $\varphi - 1, \psi - 1 \in \mathbf{T}$, then either $\varphi < \psi$ or $\varphi = \psi$ and $m > k$. In either case, we have $L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$. If $\varphi - 1 \in \mathbf{T}$ and $\psi - 1 \notin \mathbf{T}$, then $\varphi < \psi$ so

$$L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} \asymp \mathrm{e}_\beta^{\varphi-1} \prec E_\beta(\psi - 1) \asymp L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$$

   by Remark 8.8. A similar argument handles the case that $\varphi - 1 \notin \mathbf{T}$ and $\psi - 1 \in \mathbf{T}$. Finally, if $\varphi - 1 \notin \mathbf{T}$ and $\psi - 1 \notin \mathbf{T}$, then again, either $\varphi < \psi$ or $\varphi = \psi$ and $m > k$. In the first case, we have $E_\beta(\varphi - 1) < E_\beta(\psi - 1)$, so $E_\beta(\varphi - 1) \prec E_\beta(\psi - 1)$ since both $E_\beta(\varphi - 1)$ and $E_\beta(\psi - 1)$ are monomials by Lemma 7.13 and Corollary 7.21. In the second case, we have $L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$ since $L_\theta(\mathfrak{b}) - L_\theta(\mathfrak{a}) = m - k \asymp 1$ and $L_{\omega^\iota n}(\mathfrak{a})^{-1}, L_{\omega^\iota n}(\mathfrak{b})^{-1} \prec 1$.
3. Suppose $\mathfrak{a} = \ell_{\xi m}[\mathrm{e}_\beta^\varphi]$ for some $\xi m < \theta$ and some $\varphi \in \mathbf{T}$ and $\mathfrak{b} \in \mathfrak{M}_\theta$. Then $\varphi < L_\beta(\mathfrak{b})$ since $\mathfrak{a} \prec \mathfrak{b}$. For each $r \in \mathbb{R}^>$, we have $L_\beta(\mathfrak{b}) =_\beta L_\beta^{\uparrow\theta}(r L_\theta(\mathfrak{b})) = L_\beta(r L_\theta(\mathfrak{b})) + 1$ by Lemma 7.25, so $\varphi - 1 < L_\beta(r L_\theta(\mathfrak{b}))$. If $\varphi - 1 \in \mathbf{T}$, then $L_\theta(\mathfrak{a}) \asymp \mathrm{e}_\beta^{\varphi-1} \prec r L_\theta(\mathfrak{b}) \asymp L_\theta(\mathfrak{b})$ by Remark 8.8. Hence,

$$L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} \asymp \mathrm{e}_\beta^{\varphi-1} \prec L_\theta(\mathfrak{b}) \asymp L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

   If $\varphi - 1 \notin \mathbf{T}$, then $E_\beta(\varphi - 1) < r L_\theta(\mathfrak{b})$ since E_β is strictly increasing. Since $r \in \mathbb{R}^>$ is arbitrary, this gives $E_\beta(\varphi - 1) \prec L_\theta(\mathfrak{b})$, so

$$L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} \asymp E_\beta(\varphi - 1) \prec L_\theta(\mathfrak{b}) \asymp L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

4. Suppose $\mathfrak{a} \in \mathfrak{M}_\theta$ and $\mathfrak{b} = \ell_{\xi m}[\mathrm{e}_\beta^\varphi]$ for some $\xi m < \theta$ and some $\varphi \in \mathbf{T}$. Then $\varphi > L_\beta(\mathfrak{a})$, so similar arguments as above give $\varphi - 1 > L_\beta(r L_\theta(\mathfrak{a}))$ for each $r \in \mathbb{R}^>$. Again, we conclude that $L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} \prec L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$. □

Extending L_β. We define

$$\begin{aligned} \operatorname{dom} L_\beta &:= \mathfrak{M}_\beta \cup \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T}\} \\ L_\beta(\mathrm{e}_\beta^\varphi) &:= \varphi. \end{aligned}$$

Lemma 8.18. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{DD}_\mu$.*

Proof. If $\mu = 1$, let $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{\mathbf{T}}^{\succ}$, with $\mathfrak{t} = \prod_{\varphi\in\mathbf{T}} \ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi]$. We have

$$L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{m}) + \sum_{\varphi\in\mathbf{T}} L_1(\ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi]) = L_1(\mathfrak{m}) + \sum_{\varphi-1\in\mathbf{T}} r_\varphi\, \mathrm{e}_\omega^{\varphi-1} + \sum_{\varphi-1\notin\mathbf{T}} r_\varphi\, E_\omega(\varphi-1).$$

If $L_1(\mathfrak{t}\,\mathfrak{m}) \in \mathfrak{M}_{\mathbf{T}}^{\succ}$, then either $\mathfrak{t} = 1$ or $\mathfrak{m} = 1$. If $\mathfrak{t} = 1$, then $\mathfrak{m} \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\mathfrak{m} \in \mathfrak{M}_\omega$. If $\mathfrak{m} = 1$, then $L_1(\mathfrak{t}) \in \mathfrak{M}_{\mathbf{T}}^{\succ}$ if and only if $\mathfrak{t} = \mathrm{e}_\omega^\varphi \in \operatorname{dom} L_\omega$. By Remark 8.5, we have

$$L_n(\mathrm{e}_\omega^\varphi) = \begin{cases} \mathrm{e}_\omega^{\varphi-n} & \text{if } \varphi - n \in \mathbf{T} \\ E_\omega(\varphi-n) & \text{otherwise} \end{cases}$$

for all n, where $E_\beta(\varphi-n) \in \mathfrak{M}_\omega$ by Lemma 7.13 and Corollary 7.21.

If $\mu > 1$ is a successor, then let $\varphi \in \mathbf{T}$ and $\xi m < \theta$. We need to show that $\ell_{\xi m}[\mathrm{e}_\beta^\varphi] \in \bigcap \operatorname{dom} L_\theta^{\circ n}$ if and only if $m = 0$. This holds since

$$L_\theta(\ell_{\xi m}[\mathrm{e}_\beta^\varphi]) = \begin{cases} \mathrm{e}_\beta^{\varphi-1} - m & \text{if } \varphi - 1 \in \mathbf{T} \\ E_\beta(\varphi-1) - m & \text{otherwise} \end{cases}$$

and since $E_\beta(\varphi-1) \in \mathfrak{M}_\beta$ whenever $\varphi - 1 \notin \mathbf{T}$, by Corollary 7.21.

Finally, if μ is a non-zero limit, then we use that

$$\bigcap_{\eta<\mu} \{\ell_\gamma[\mathrm{e}_\beta^\varphi] : \varphi \in \mathbf{T} \text{ and } \omega^{\eta_*} \leqslant \gamma < \theta\} = \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T}\}. \qquad \square$$

To see that $\mathbf{R}_\mu$ is satisfied, let $\varphi \in \mathbf{T}$ and let $\omega^\eta n < \beta$. By Remark 8.5, we have

$$L_{\omega^\eta n}(\mathrm{e}_\beta^\varphi)^{-1} := \begin{cases} \ell_{\omega^\eta n}[\mathrm{e}_\beta^\varphi]^{-1} & \text{if } \eta < \mu_* \\ (\mathrm{e}_\beta^{\varphi-n})^{-1} & \text{if } \eta = \mu_* \text{ and } \varphi - n \in \mathbf{T} \\ E_\beta(\varphi-n)^{-1} & \text{if } \eta = \mu_* \text{ and } \varphi - n \notin \mathbf{T} \end{cases}$$

Let $\mathfrak{m} \in (\operatorname{supp} \varphi)^{\prec}$. Since φ is β-truncated, we have $\varphi > L_\beta(\mathfrak{m}^{-1})$. This gives $\ell_{\omega^\eta n}[\mathrm{e}_\beta^\varphi]^{-1} \prec \mathfrak{m}$ for $\eta < \mu_*$. If $\eta = \mu_*$, then $\varphi - n$ is also β-truncated by Lemma 7.13, so $\varphi - n > L_\beta(\mathfrak{m}^{-1})$ since $(\operatorname{supp} \varphi)^{\prec} = (\operatorname{supp} (\varphi - n))^{\prec}$. This gives $(\mathrm{e}_\beta^{\varphi-n}) > \mathfrak{m}^{-1}$ if $\varphi - n \in \mathbf{T}$. If $\varphi - n \notin \mathbf{T}$, then $E_\beta(\varphi - n) > \mathfrak{m}^{-1}$ since E_β is strictly increasing. Since $E_\beta(\varphi - n)$ is a monomial by Corollary 7.21, this gives $E_\beta(\varphi - n) \succ \mathfrak{m}^{-1}$. In all three cases, we have $L_{\omega^\eta n}(\mathrm{e}_\beta^\varphi)^{-1} \prec \mathfrak{m}$, so

$$\operatorname{supp} L_\beta(\mathrm{e}_\beta^\varphi) = \operatorname{supp} \varphi \succ L_{\omega^\eta n}(\mathrm{e}_\beta^\varphi)^{-1},$$

as desired.

If μ is a successor, then either $L_\theta(\mathrm{e}_\beta^\varphi) = \mathrm{e}_\beta^{\varphi-1}$ or $L_\theta(\mathrm{e}_\beta^\varphi) = E_\beta(\varphi-1)$. In both cases,

$$L_\beta(L_\theta(\mathrm{e}_\beta^\varphi)) = \varphi - 1 = L_\beta(\mathrm{e}_\beta^\varphi) - 1,$$

so $\mathbf{FE}_\mu$ is satisfied. As for $\mathbf{A}_\mu$, let $\iota < \mu$. Since $\ell_0 > \ell_\beta$, we have $\varphi > L_\beta(\varphi)$, so Remark 8.8 with $\mathfrak{t} = \ell_{\omega^\iota}[\mathrm{e}_\beta^\varphi]$ and $s = \varphi$ gives

$$L_{\omega^\iota}(\mathrm{e}_\beta^\varphi) = \ell_{\omega^\iota}[\mathrm{e}_\beta^\varphi] \succ \varphi = L_\beta(\mathrm{e}_\beta^\varphi).$$

Lemma 8.19. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{M}_\mu$.*

Proof. Let $\mathfrak{a} \prec \mathfrak{b} \in \operatorname{dom} L_\beta$ and let $\omega^\iota n < \beta$. We want to show that

$$L_{\omega^\iota n}(\mathfrak{a})^{-1} + L_{\omega^\iota n}(\mathfrak{b})^{-1} < L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a}).$$

Note that $L_\beta(\mathfrak{a}), L_\beta(\mathfrak{b}) \in \mathbb{T}_{>,\beta}$. We claim that $L_\beta(\mathfrak{a}) < L_\beta(\mathfrak{b})$. If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$, then this follows from the fact that $(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta < \mu+1})$ satisfies $\mathbf{M}_\mu$. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ and $\mathfrak{b} = \mathrm{e}_\beta^\psi$, then we have $L_\beta(\mathfrak{a}) = \varphi < \psi = L_\beta(\mathfrak{b})$. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ and $\mathfrak{b} \in \mathfrak{M}_\beta$, then $L_\beta(\mathfrak{a}) = \varphi < L_\beta(\mathfrak{b})$ by Remark 8.8 and likewise, if $\mathfrak{a} \in \mathfrak{M}_\beta$ and $\mathfrak{b} = \mathrm{e}_\beta^\psi$, then $L_\beta(\mathfrak{a}) < \psi = L_\beta(\mathfrak{b})$.

Now suppose towards contradiction that $L_{\omega^\iota n}(\mathfrak{b})^{-1} + L_{\omega^\iota n}(\mathfrak{a})^{-1} \geqslant L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})$. We will show that $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$. As $L_\beta(\mathfrak{a})$ is the unique β-truncated element in $\mathcal{L}_\beta[L_\beta(\mathfrak{a})]$ and $L_\beta(\mathfrak{b})$ is β-truncated, this is a contradiction.

Since $L_{\omega^\iota n}(\mathfrak{a})^{-1} > L_{\omega^\iota n}(\mathfrak{b})^{-1}$ by $\mathbf{M}_\iota$, we have $2 L_{\omega^\iota n}(\mathfrak{a})^{-1} > L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})$, so

$$\frac{1}{2} L_{\omega^\iota n}(\mathfrak{a}) < |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}.$$

By $\mathbf{A}_0$, we have $L_1(L_{\omega^\iota n}(\mathfrak{a})) \prec L_{\omega^\iota n}(\mathfrak{a}) \asymp \frac{1}{2} L_{\omega^\iota n}(\mathfrak{a})$, so

$$L_{\omega^\iota n+1}(\mathfrak{a}) < |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}.$$

If $L_{\omega^\iota n+1}(\mathfrak{a}) \in \mathbb{T}^{>,\succ}$, then Lemma 7.5 gives

$$L_\beta(\mathfrak{a}) = L_\beta^{\uparrow \omega^\iota n+1}(L_{\omega^\iota n+1}(\mathfrak{a})) < L_\beta^{\uparrow \omega^\iota n+1}(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}),$$

so $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$. Suppose $L_{\omega^\iota n+1}(\mathfrak{a}) \notin \mathbb{T}^{>,\succ}$ and let $\varphi \in \mathbf{T}$ with $\mathfrak{a} = \mathrm{e}_\beta^\varphi$. If $\iota < \mu_*$, then

$$L_{\omega^\iota n+1}(\mathfrak{a}) = \ell_{\omega^\iota n+1}[\mathrm{e}_\beta^\varphi] < |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1},$$

so $\varphi < L_\beta(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1})$ by Remark 8.8. As $\varphi = L_\beta(\mathfrak{a})$, this too gives $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$. Finally, if $\iota = \mu_* < \mu$, then

$$L_{\omega^\iota n+1}(\mathfrak{a}) = \ell_1[\mathrm{e}_\beta^{\varphi - n}] < |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1},$$

so $\varphi - n < L_\beta(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1})$ by Remark 8.8. As $\ell_\beta + n = \ell_\beta^{\uparrow \theta n}$, we have

$$\varphi < L_\beta(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}) + n = L_\beta^{\uparrow \theta n}(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}),$$

so $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$ once again. □

Lemma 8.20. $(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta < \mu+1})$ *satisfies* $\mathbf{P}_\mu$.

Proof. Let $\mathfrak{a} \in \operatorname{dom} L_\beta$ and let $(r_\gamma)_{\gamma < \beta}$ be a sequence of real numbers. Consider the sum $s := \sum_{\gamma < \beta} r_\gamma L_{\gamma+1}(\mathfrak{a})$. If $\mathfrak{a} \in \mathfrak{M}_\beta$, then $s \in \log \mathfrak{M}$ since $(\mathbb{T}, (L_{\omega^\eta})_{\eta < \mu+1})$ satisfies $\mathbf{P}_\mu$. Assume therefore that $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ for some $\varphi \in \mathbf{T}$. If μ is a limit, then $\beta = \theta$ and

$$s = \sum_{\gamma < \theta} r_\gamma L_{\gamma+1}(\mathrm{e}_\beta^\varphi) = \sum_{\gamma < \theta} r_\gamma \, \ell_{\gamma+1}[\mathrm{e}_\beta^\varphi] = \log(\mathfrak{l}[\mathrm{e}_\beta^\varphi])$$

where $\mathfrak{l} := \prod_{\gamma < \theta} \ell_\gamma^{r_\gamma} \in \mathfrak{L}_{<\theta}^{\succ}$. If μ is a successor, then we may write

$$s = \sum_{n \in \mathbb{N}} \sum_{\gamma < \theta} r_{\theta n+\gamma} L_{\theta n+\gamma+1}(\mathrm{e}_\beta^\varphi) = \sum_{n \in \mathbb{N}} \sum_{\gamma < \theta} r_{\theta n+\gamma} L_{\gamma+1}(L_{\theta n}(\mathrm{e}_\beta^\varphi)).$$

If $\varphi - n \in \mathbf{T}$ for all n, then

$$\sum_{n \in \mathbb{N}} \sum_{\gamma < \theta} r_{\theta n+\gamma} L_{\gamma+1}(L_{\theta n}(\mathrm{e}_\beta^\varphi)) = \sum_{n \in \mathbb{N}} \sum_{\gamma < \theta} r_{\theta n+\gamma} L_{\gamma+1}(\mathrm{e}_\beta^{\varphi - n}) = \log\left(\prod_{n \in \mathbb{N}} \mathfrak{l}_n[\mathrm{e}_\beta^{\varphi - n}]\right)$$

where $\mathfrak{l}_n := \prod_{\gamma<\theta} \ell_\gamma^{r_{\theta n+\gamma}} \in \mathfrak{L}_{<\theta}^{\succ}$. If $\varphi - n \notin \mathbf{T}$ for some n, then let $n_0 > 0$ be minimal with $\varphi - n_0 \notin \mathbf{T}$. We have $s = s_1 + s_2$, where

$$s_1 = \sum_{n<n_0} \sum_{\gamma<\theta} r_{\theta n+\gamma} L_{\gamma+1}(\mathrm{e}_\beta^{\varphi-n}), \qquad s_2 = \sum_{n\geqslant n_0} \sum_{\gamma<\theta} r_{\theta n+\gamma} (L_{\theta(n-n_0)+\gamma+1}(E_\beta(\varphi-n_0))).$$

Note that

$$s_1 = \log\left(\prod_{n<n_0} \prod_{\gamma<\theta} \ell_\gamma^{r_{\theta n+\gamma}}[\mathrm{e}_\beta^{\varphi-n}]\right) \in \log(\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]),$$

so it remains to show that $s_2 \in \log \mathfrak{M}$. Since $E_\beta(\varphi - n_0) \in \mathfrak{M}_\beta$ by Lemma 7.13 and Corollary 7.21, this follows from the fact that $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{P}_\mu$. □

Extending $L_{\omega^{\mu+1}}$. Suppose $\nu > \mu + 1$. We define

$$\begin{aligned} \operatorname{dom} L_{\omega^{\mu+1}} &:= \mathfrak{M}_{\omega^{\mu+1}} \cup \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}\} \\ L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) &:= L_{\omega^{\mu+1}}(\varphi) + 1. \end{aligned}$$

For $\varphi \in \mathbf{T}$, we have $\mathrm{e}_\beta^\varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_\beta^{\circ n}$ if and only if $\varphi \in \mathfrak{M}_{\omega^{\mu+1}}$ since $\varphi = L_\beta(\mathrm{e}_\beta^\varphi)$. This proves that $(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{DD}_{\mu+1}$. Let $\varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}$. We have

$$L_{\omega^{\mu+1}}(L_\beta(\mathrm{e}_\beta^\varphi)) = L_{\omega^{\mu+1}}(\varphi) = L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) - 1,$$

so $\mathbf{FE}_{\mu+1}$ is satisfied. As for $\mathbf{A}_{\mu+1}$, it suffices to show that $L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) < L_\beta(\mathrm{e}_\beta^\varphi)$ since $L_\beta(\mathrm{e}_\beta^\varphi) < L_{\omega^\iota}(\mathrm{e}_\beta^\varphi)$ for all $\iota < \mu$ by $\mathbf{A}_\mu$. Since $\ell_{\omega^{\mu+1}} + 1 \prec \ell_0$, we have

$$L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) = L_{\omega^{\mu+1}}(\varphi) + 1 = (\ell_{\omega^{\mu+1}} + 1) \circ \varphi \prec \varphi = L_\beta(\mathrm{e}_\beta^\varphi).$$

Now for $\mathbf{R}_{\mu+1}$, let $\omega^\iota n < \omega^{\mu+1}$. Since $L_{\beta(n+1)}(\mathrm{e}_\beta^\varphi) \leqslant L_{\omega^\iota n}(\mathrm{e}_\beta^\varphi)$ by $\mathbf{A}_\mu$, it suffices to show that $\operatorname{supp} L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) \succ L_{\beta(n+1)}(\mathrm{e}_\beta^\varphi)^{-1}$. Since

$$\operatorname{supp} L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) = \operatorname{supp} L_{\omega^{\mu+1}}(\varphi) \cup \{1\}, \qquad L_{\beta(n+1)}(\mathrm{e}_\beta^\varphi)^{-1} = L_{\beta n}(\varphi)^{-1},$$

it is enough to show that $\operatorname{supp} L_{\omega^{\mu+1}}(\varphi) \succ L_{\beta n}(\varphi)^{-1}$. This holds because $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{R}_{\mu+1}$ and $\varphi \in \mathfrak{M}_{\omega^{\mu+1}}$.

Lemma 8.21. *$(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies* $\mathbf{M}_{\mu+1}$.

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \operatorname{dom} L_{\omega^{\mu+1}}$ with $\mathfrak{a} \prec \mathfrak{b}$ and let $\omega^\iota n < \omega^{\mu+1}$. We want to show that $L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_{\omega^{\mu+1}}(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$. Since $L_{\beta(n+1)}(\mathfrak{a}) \leqslant L_{\omega^\iota n}(\mathfrak{a})$ and likewise for $\mathfrak{b}$, it is enough to show that

$$L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\beta(n+1)}(\mathfrak{a})^{-1} < L_{\omega^{\mu+1}}(\mathfrak{b}) - L_{\beta(n+1)}(\mathfrak{b})^{-1}.$$

We proceed by case distinction:

1. If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$, then the result follows from $\mathbf{M}_{\mu+1}$ for $\mathbb{T}$.
2. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ and $\mathfrak{b} = \mathrm{e}_\beta^\psi$, then

$$\begin{aligned} L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\beta(n+1)}(\mathfrak{a})^{-1} &= L_{\omega^{\mu+1}}(\varphi) + 1 + L_{\beta n}(\varphi)^{-1} \\ L_{\omega^{\mu+1}}(\mathfrak{b}) - L_{\beta(n+1)}(\mathfrak{b})^{-1} &= L_{\omega^{\mu+1}}(\psi) + 1 - L_{\beta n}(\psi)^{-1}. \end{aligned}$$

Since $\varphi, \psi \in \mathfrak{M}_{\omega^{\mu+1}}$ and $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{M}_{\mu+1}$, we have

$$L_{\omega^{\mu+1}}(\varphi) + L_{\beta n}(\varphi)^{-1} < L_{\omega^{\mu+1}}(\psi) - L_{\beta n}(\psi)^{-1}.$$

3. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ and $\mathfrak{b} \in \mathfrak{M}_{\omega^{\mu+1}}$, then $\varphi < L_\beta(\mathfrak{b})$. Since $\varphi, L_\beta(\mathfrak{b}) \in \mathfrak{M}_{\omega^{\mu+1}}$ and $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{M}_{\mu+1}$, we have

$$L_{\omega^{\mu+1}}(\varphi) + L_{\beta n}(\varphi)^{-1} < L_{\omega^{\mu+1}}(L_\beta(\mathfrak{b})) - L_{\beta n}(L_\beta(\mathfrak{b}))^{-1} = L_{\omega^{\mu+1}}(\mathfrak{b}) - 1 + L_{\beta(n+1)}(\mathfrak{b})^{-1}.$$

Thus,

$$L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\beta(n+1)}(\mathfrak{a})^{-1} = L_{\omega^{\mu+1}}(\varphi) + 1 + L_{\beta n}(\varphi)^{-1} < L_{\omega^{\mu+1}}(\mathfrak{b}) + L_{\beta(n+1)}(\mathfrak{b})^{-1}.$$

4. If $\mathfrak{a} \in \mathfrak{M}_\beta$ and $\mathfrak{b} = \mathrm{e}_\beta^\psi$, then the argument is similar to the previous case. □

Lemma 8.22. *$(\mathbb{T}_\mathbf{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{P}_{\mu+1}$.*

Proof. Let $\mathfrak{a} \in (\mathfrak{M}_\mathbf{T})_{\omega^{\mu+1}}$ and let $(r_\gamma)_{\gamma<\omega^{\mu+1}}$ be a sequence of real numbers. We need to show that the sum $s = \sum_{\gamma<\omega^{\mu+1}} r_\gamma L_{\gamma+1}(\mathfrak{a})$ is in $\log \mathfrak{M}_\mathbf{T}$. If $\mathfrak{a} \in \mathfrak{M}_{\omega^\eta}$, then $s \in \log \mathfrak{M}$. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ for some $\varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}$, then

$$s = \sum_{n\in\mathbb{N}} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\beta n+\gamma+1}(\mathrm{e}_\beta^\varphi) = \sum_{\gamma<\beta} r_\gamma L_{\gamma+1}(\mathrm{e}_\beta^\varphi) + \sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\gamma+1}(L_{\beta n}(\mathrm{e}_\beta^\varphi)).$$

We have $\sum_{\gamma<\beta} r_\gamma L_{\gamma+1}(\mathrm{e}_\beta^\varphi) \in \log \mathfrak{M}_\mathbf{T}$, since $(\mathbb{T}_\mathbf{T}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{P}_\mu$. We also have

$$\begin{aligned}\sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\gamma+1}(L_{\beta n}(\mathrm{e}_\beta^\varphi)) &= \sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\gamma+1}(L_{\beta(n-1)}(\varphi)) \\ &= \sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\beta(n-1)+\gamma+1}(\varphi) \in \log \mathfrak{M},\end{aligned}$$

since $\varphi \in \mathfrak{M}_{\omega^{\mu+1}}$ and $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{P}_{\mu+1}$. We conclude by noting that $\log \mathfrak{M}_\mathbf{T}$ is closed under addition. □

Remark 8.23. In the case that $\nu = \mu + 1$, the argument that $\mathbf{DD}_{\mu+1}$ is satisfied gives

$$(\mathfrak{M}_\mathbf{T})_{\omega^\nu} = \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_\beta^{\circ n} = \mathfrak{M}_{\omega^\nu} \cup \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^\nu}\}$$

and the proof of Lemma 8.22 also tells us that $(\mathbb{T}_\mathbf{T}, (L_{\omega^\eta})_{\eta<\nu})$ satisfies $\mathbf{P}_\nu$.

Extending L_{ω^η} when $\mu+1<\eta<\nu$. If $\nu > \mu + 1$, then we will not extend the hyperlogarithms L_{ω^η} with $\eta > \mu + 1$. So for $\eta < \nu$ with $\eta > \mu + 1$, we simply set

$$\operatorname{dom} L_{\omega^\eta} := \mathfrak{M}_{\omega^\eta}.$$

Lemma 8.24. *$(\mathbb{T}_\mathbf{T}, (L_{\omega^\eta})_{\eta<\nu})$ satisfies $\mathbf{DD}_\eta$ for all $\eta < \nu$.*

Proof. It suffices [JORIS: is this really clear at this point?] to show that $(\mathbb{T}_\mathbf{T}, (L_{\omega^\eta})_{\eta<\nu})$ satisfies $\mathbf{DD}_{\mu+2}$. Suppose towards contradiction that there is some $\varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}$ with $\mathrm{e}_\beta^\varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\mu+1}}^{\circ n}$. Take $n > 0$ with $L_{\omega^{\mu+1}n}(\varphi) \asymp L_{\omega^{\mu+1}n}(\mathfrak{d}_{\omega^{\mu+2}}(\varphi))$. Since $L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) = L_{\omega^{\mu+1}}(\varphi) + 1 \asymp L_{\omega^{\mu+1}}(\varphi)$, Lemma 4.8 gives

$$L_{\omega^{\mu+1}n}(\mathrm{e}_\beta^\varphi) = L_{\omega^{\mu+1}(n-1)}(L_{\omega^{\mu+1}}(\varphi) + 1) \asymp L_{\omega^{\mu+1}(n-1)}(L_{\omega^{\mu+1}}(\varphi)) \asymp L_{\omega^{\mu+1}n}(\mathfrak{d}_{\omega^{\mu+2}}(\varphi)).$$

Since $L_{\omega^{\mu+1}n}(\mathrm{e}_\beta^\varphi)$ and $L_{\omega^{\mu+1}n}(\mathfrak{d}_{\omega^{\mu+2}}(\varphi))$ are both monomials, they must be equal. The axiom $\mathbf{M}_{\mu+1}$ gives $\mathrm{e}_\beta^\varphi = \mathfrak{d}_{\omega^{\mu+2}}(\varphi) \in \mathbb{T}$, a contradiction. □

For all $\eta < \nu$ with $\eta > \mu + 1$, the axioms $\mathbf{FE}_\eta$, $\mathbf{A}_\eta$, $\mathbf{M}_\eta$, $\mathbf{R}_\eta$ and $\mathbf{P}_\eta$ automatically hold in $\mathbb{T}_\mathbf{T}$ since they hold in $\mathbb{T}$, as does the axiom $\mathbf{P}_\nu$ if $\nu > \mu + 1$ is an ordinal.

8.4 The extended hyperserial skeleton

We have completed the proof of the following:

Proposition 8.25. *$(\mathbb{T}_\mathbf{T}, (L_{\omega^\eta})_{\eta<\nu})$ is a hyperserial skeleton of force ν.*

Let us finally examine the confluence and universality of $\mathbb{T}_\mathbf{T}$.

Proposition 8.26. *Assume that $\mathfrak{M}_{\omega^{\mu+1}} \subseteq \mathbf{T} \cup L_\beta(\mathbb{T}^{>,\succ})$. Then $\mathbb{T}_\mathbf{T}$ is ν-confluent. In particular, $\mathbb{T}_{(\mu)}$ is ν-confluent.*

Proof. Clearly, $\mathbb{T}_\mathbf{T}$ is 0-confluent. Consider $s \in \mathbb{T}_\mathbf{T}^{>,\succ}$ and write $\mathfrak{d}_1(s) = \mathfrak{d}_s = \mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_\mathbf{T}^\succ$. By our definition of L_1, we either have have $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \mathfrak{d}_1(L_1(\mathfrak{m}))$ or $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \ell_{\gamma_\mathfrak{t}+1}[\mathrm{e}_\beta^{\varphi_\mathfrak{t}}]$. If $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \mathfrak{d}_1(L_1(\mathfrak{m}))$, then $\mathfrak{d}_\omega(s) = \mathfrak{d}_\omega(\mathfrak{m})$ and, more generally, $\mathfrak{d}_{\omega^\eta}(s) = \mathfrak{d}_{\omega^\eta}(\mathfrak{m}) \in \mathfrak{M}_{\omega^\eta}$ for all $\eta \in \mathbf{On}$ with $1 \leqslant \eta \leqslant \nu$, since $\mathcal{E}_\omega[\mathfrak{d}_\omega(\mathfrak{m})] = \mathcal{E}_\omega[\mathfrak{m}] \subseteq \mathcal{E}_{\omega^\eta}[\mathfrak{m}]$ by Lemma 3.7. Assume from now on that $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \ell_{\gamma_\mathfrak{t}+1}[\mathrm{e}_\beta^{\varphi_\mathfrak{t}}]$.

We set $\gamma := \gamma_\mathfrak{t}$ and $\varphi := \varphi_\mathfrak{t}$. For $1 \leqslant \eta \leqslant \mu$, let us first show by induction that

$$\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\succcurlyeq \omega^{\eta *}}}[\mathrm{e}_\beta^\varphi] \in \mathfrak{M}_{\omega^\eta}.$$

If $\eta = 1$, then $\gamma = \gamma_{\succcurlyeq 1}$ and

$$\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \ell_{\gamma+1}[\mathrm{e}_\beta^\varphi] = L_1(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_1(L_1(\ell_\gamma[\mathrm{e}_\beta^\varphi])),$$

so we indeed have $\mathfrak{d}_\omega(s) = \ell_{\gamma_{\succcurlyeq 1}}[\mathrm{e}_\beta^\varphi]$. Let $1 < \eta \leqslant \mu$ and suppose that $\mathfrak{d}_{\omega^\sigma}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\succcurlyeq \omega^{\sigma *}}}[\mathrm{e}_\beta^\varphi]$ for $1 \leqslant \sigma < \eta$. If $1 < \eta < \mu$ and η is a successor, then our induction hypothesis yields

$$L_{\omega^{\eta *}}(\mathfrak{d}_{\omega^{\eta *}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^{\eta *}}(\ell_{\gamma_{\succcurlyeq \omega^{\eta **}}}[\mathrm{e}_\beta^\varphi]).$$

Writing $\gamma_{\succcurlyeq \omega^{\eta **}} = \gamma_{\succcurlyeq \omega^{\eta *}} + \omega^{\eta_{**}} n$, we have

$$L_{\omega^{\eta *}}(\ell_{\gamma_{\succcurlyeq \omega^{\eta **}}}[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\succcurlyeq \omega^{\eta *}} + \omega^{\eta *}}[\mathrm{e}_\beta^\varphi] - n \asymp \ell_{\gamma_{\succcurlyeq \omega^{\eta *}} + \omega^{\eta *}}[\mathrm{e}_\beta^\varphi] = L_{\omega^{\eta *}}(\ell_{\gamma_{\succcurlyeq \omega^{\eta *}}}[\mathrm{e}_\beta^\varphi]),$$

so

$$\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^\eta}(\mathfrak{d}_{\omega^{\eta *}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = \ell_{\gamma_{\succcurlyeq \omega^{\eta *}}}[\mathrm{e}_\beta^\varphi].$$

If $1 < \eta \leqslant \mu$ and $\eta = \eta_*$ is a limit, then there is $\sigma < \eta$ such that $\gamma_{\succcurlyeq \omega^{\eta *}} = \gamma_{\succcurlyeq \omega^{\sigma *}}$. For this σ, we have

$$L_{\omega^\sigma}(\mathfrak{d}_{\omega^\sigma}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^\sigma}(\ell_{\gamma_{\succcurlyeq \omega^{\sigma *}}}[\mathrm{e}_\beta^\varphi]) = L_{\omega^\sigma}(\ell_{\gamma_{\succcurlyeq \omega^{\eta *}}}[\mathrm{e}_\beta^\varphi]),$$

so $\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\succcurlyeq \omega^{\eta *}}}[\mathrm{e}_\beta^\varphi] \in \mathfrak{M}_{\omega^\eta}$. Finally, if $\eta = \mu$ and μ is a successor, then $\gamma_{\succcurlyeq \omega^{\mu **}} = \omega^{\mu_{**}} n$, where $n = 0$ if μ_* is a limit. This gives

$$L_\theta(\mathfrak{d}_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_\theta(\ell_{\omega^{\mu **} n}[\mathrm{e}_\beta^\varphi]) = \begin{cases} \mathrm{e}_\beta^{\varphi - 1} - n & \text{if } \varphi - 1 \in \mathbf{T} \\ E_\beta(\varphi - 1) - n & \text{otherwise.} \end{cases}$$

In both cases, we have $L_\theta(\mathfrak{d}_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) \asymp L_\theta(\mathrm{e}_\beta^\varphi)$, so $\mathfrak{d}_\beta(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^\varphi = \ell_{\gamma_{\succcurlyeq \theta}}[\mathrm{e}_\beta^\varphi]$, since $\gamma_{\succcurlyeq \theta} = 0$.

Let us now show that $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$ exists. Let $\mathfrak{a} := \mathfrak{d}_{\omega^{\mu+1}}(\varphi)$, so $\mathfrak{a} \in \mathbf{T} \cup L_\beta(\mathbb{T}^{>,\succ})$ by our assumption that $\mathfrak{M}_{\omega^{\mu+1}} \subseteq \mathbf{T} \cup L_\beta(\mathbb{T}^{>,\succ})$. Take n with $(L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\varphi) \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{a})$. We have $L_\beta(\mathfrak{d}_\beta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_\beta(\mathrm{e}_\beta^\varphi) = \varphi$, so

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{a}) = \begin{cases} (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathrm{e}_\beta^{\mathfrak{a}}) & \text{if } \mathfrak{a} \in \mathbf{T} \\ (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(E_\beta(\mathfrak{a})) & \text{otherwise.} \end{cases}$$

Since $\mathfrak{a}$ is an infinite monomial, it is $\omega^{\mu+1}$-truncated, so $E_\beta(\mathfrak{a}) \in \mathfrak{M}_{\omega^{\mu+1}}$ so long as it is defined. Thus, $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$ is either equal to $\mathrm{e}_\beta^{\mathfrak{a}}$ or $E_\beta(\mathfrak{a})$.

If $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = E_\beta(\mathfrak{a})$, then $\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^\eta}(E_\beta(\mathfrak{a}))$ for $\eta \in \mathbf{On}$ with $\mu+1 \leqslant \eta \leqslant \nu$. On the other hand, if $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^{\mathfrak{a}}$, then

$$L_{\omega^{\mu+1}}(\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^{\mu+1}}(\mathrm{e}_\beta^{\mathfrak{a}}) = L_{\omega^{\mu+1}}(\mathfrak{a}) + 1 \asymp L_{\omega^{\mu+1}}(\mathfrak{a}).$$

Take $n \in \mathbb{N}$ with $(L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ n}(\mathfrak{a}) \asymp (L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ n}(\mathfrak{d}_{\omega^{\mu+2}}(\mathfrak{a}))$. Then

$$(L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ(n+1)}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) \asymp (L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ(n+1)}(\mathfrak{a}) \asymp (L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ(n+1)}(\mathfrak{d}_{\omega^{\mu+2}}(\mathfrak{a})),$$

so $\mathfrak{d}_{\omega^{\mu+2}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^{\mu+2}}(\mathfrak{a})$ and, more generally, $\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^\eta}(\mathfrak{a})$ when $\eta \in \mathbf{On}$ and $\mu+2 \leqslant \eta \leqslant \nu$. □

Propositions 8.25 and 8.26 yield:

Corollary 8.27. *If $\mathfrak{M}_{\omega^{\mu+1}} \subseteq \mathbf{T} \cup L_\beta(\mathbb{T}^{>,\succ})$, then $(\mathbb{T}_{\mathbf{T}}, (L_{\omega^\eta})_{\eta<\nu})$ is a confluent hyperserial skeleton of force ν.*

Remark 8.28. Let $0 < \eta \leqslant \mu_*$. Then

$$(\mathfrak{M}_{\mathbf{T}})_{\omega^\eta} = \mathfrak{M}_{\omega^\eta} \cup \{\mathfrak{l}[\mathrm{e}_\beta^\varphi] : \mathfrak{l} \in (\mathfrak{L}_{<\theta})_{\omega^\eta} \text{ and } \varphi \in \mathbf{T}\}.$$

Given $\gamma < \omega^\eta$ and $\mathfrak{l}[\mathrm{e}_\beta^\varphi] \in (\mathfrak{M}_{\mathbf{T}})_{\omega^\eta} \setminus \mathfrak{M}_{\omega^\eta}$, we have $L_\gamma(\mathfrak{l}[\mathrm{e}_\beta^\varphi]) = L_\gamma(\mathfrak{l})[\mathrm{e}_\beta^\varphi]$. Given $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$, we have $\mathfrak{d}_{\omega^\eta}(\mathfrak{t}) = \mathfrak{d}_{\omega^\eta}(\mathfrak{t}_{\varphi_{\mathfrak{t}}})[\mathrm{e}_\beta^{\varphi_{\mathfrak{t}}}]$.

Let us now show that $\mathbb{T}_{\mathbf{T}}$ satisfies a universal property. We start with a lemma.

Lemma 8.29. *For any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$ with $\mathfrak{a} \prec \mathfrak{b}$ and any $\gamma, \sigma < \theta$, we have $L_\sigma(\mathfrak{a}) \prec L_\gamma(\mathfrak{b})$.*

Proof. Choose $\eta < \mu_*$ and $n \in \mathbb{N}$ such that $\gamma, \sigma < \omega^\eta n$. Then $L_\sigma(\mathfrak{a}) \prec \mathfrak{a}$ and $L_{\omega^\eta n}(\mathfrak{b}) \prec L_\gamma(\mathfrak{b})$ so it suffices to show that $\mathfrak{a} \prec L_{\omega^\eta n}(\mathfrak{b})$. Since $L_{\omega^{\eta+1}}(\mathfrak{a}), L_{\omega^{\eta+1}}(\mathfrak{b})$ are monomials and $L_{\omega^{\eta+1}}(\mathfrak{a}) < L_{\omega^{\eta+1}}(\mathfrak{b})$, we have

$$L_{\omega^{\eta+1}}(\mathfrak{a}) \prec L_{\omega^{\eta+1}}(\mathfrak{b}) \asymp L_{\omega^{\eta+1}}(\mathfrak{b}) - n = L_{\omega^{\eta+1}}(L_{\omega^\eta n}(\mathfrak{b})).$$

The monotonicity of $L_{\omega^{\eta+1}}$ gives $\mathfrak{a} < L_{\omega^\eta n}(\mathfrak{b})$. We conclude that $\mathfrak{a} \prec L_{\omega^\eta n}(\mathfrak{b})$, since $\mathfrak{a}$ and $L_{\omega^\eta n}(\mathfrak{b})$ are monomials. □

Proposition 8.30. *Let $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ be a confluent hyperserial skeleton of force $\nu \leqslant \mathbf{On}$ and let $\Phi \colon \mathbb{T} \longrightarrow \mathbb{U}$ be an embedding of force ν. Let $\mathbf{T} \subseteq \mathbf{T}_\mu(\mathbb{T})$ be a subclass (we no longer require that $\mathbf{T} = \langle \mathbf{T} \rangle$). If $\Phi(\mathbf{T}) \subseteq L_\beta(\mathbb{U}^{>,\succ})$, then there is a unique embedding*

$$\Psi \colon \mathbb{T}_{\langle \mathbf{T} \rangle} \longrightarrow \mathbb{U}$$

of force ν that extends Φ.

Proof. Since $\mathbb{U}$ is confluent, we have an external composition $\circ: \mathbb{L}_{<\alpha} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$. We first claim that $\Phi(\langle \mathbf{T} \rangle) \subseteq L_\beta(\mathbb{U}^{>,\succ})$. If μ is a limit, then $\mathbf{T} = \langle \mathbf{T} \rangle$ so there is nothing to show. If μ is a successor, then for $\varphi \in \langle \mathbf{T} \rangle$, there is $\varphi_0 \in \mathbf{T}$ with $\varphi = \varphi_0 - n$. We have

$$L_\beta(L_{\theta n}(E_\beta(\Phi(\varphi_0)))) = L_\beta(E_\beta(\Phi(\varphi_0)) - n = \Phi(\varphi_0) - n = \Phi(\varphi),$$

so $E_\beta(\Phi(\varphi)) = L_{\theta n}(E_\beta(\Phi(\varphi_0))) \in \mathbb{U}^{>,\succ}$. Having shown our claim, we may assume without loss of generality that $\mathbf{T} = \langle \mathbf{T} \rangle$.

Given $\varphi \in \mathbf{T}$, the series $\Phi(\varphi)$ is β-truncated, so $E_\beta(\Phi(\varphi))$ is β-atomic, by Remark 7.21. We set $\mathfrak{a}_\varphi := E_\beta(\Phi(\varphi)) \in \mathfrak{N}_\beta$. Note that for $\varphi \in \mathbf{T}$ and $\mathfrak{l} = \prod_{\gamma<\beta} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\beta}$, the series

$$\mathfrak{l} \circ \mathfrak{a}_\varphi = \exp\left(\sum_{\gamma<\beta} \mathfrak{l}_\gamma L_{\gamma+1}(\mathfrak{a}_\varphi) \right)$$

exists in $\mathfrak{N}$ by $\mathbf{P}_\mu$. Let us define a map $\Psi: \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}] \longrightarrow \mathfrak{N}$. Let $\mathfrak{t} \in \mathfrak{L}_\theta[\mathrm{e}_\beta^{\mathbf{T}}]$. If μ is a limit, then $\operatorname{hsupp} \mathfrak{t}$ is finite and we define

$$\Psi(\mathfrak{t}) := \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi \circ \mathfrak{a}_\varphi \in \mathfrak{N}.$$

If μ is a successor, let $\varphi_1 > \cdots > \varphi_n \in \mathbf{T}$ and $\mathfrak{t}^*_{\varphi_i}$ be as in Remark 8.6. We define

$$\Psi(\mathfrak{t}) := \prod_{i=1}^n \mathfrak{t}^*_{\varphi_i} \circ \mathfrak{a}_{\varphi_i}.$$

Note that in both the limit and successor case, we have

$$\log \Psi(\mathfrak{t}) = \sum_{\varphi \in \mathbf{T}} \log(\mathfrak{t}_\varphi \circ \mathfrak{a}_\varphi) = \sum_{\varphi \in \mathbf{T}} \sum_{\gamma<\theta} (\mathfrak{t}_\varphi)_\gamma L_{\gamma+1}(\mathfrak{a}_\varphi).$$

Given $\varphi < \psi \in \mathbf{T}$ and $\gamma, \sigma < \theta$, we have $L_\sigma(\mathfrak{a}_\varphi) \prec L_\gamma(\mathfrak{a}_\psi)$ by Lemma 8.29 and, if $\gamma < \sigma$, then $L_\sigma(\mathfrak{a}_\varphi) \prec L_\gamma(\mathfrak{a}_\varphi)$. Thus, $\log \Psi(\mathfrak{t}) \sim (\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}})$ for $\mathfrak{t} \neq 1$. In particular, Ψ is order preserving, since

$$\mathfrak{t} \succ 1 \Longleftrightarrow (\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} > 0 \Longleftrightarrow \log \Psi(\mathfrak{t}) > 0 \Longleftrightarrow \Psi(\mathfrak{t}) \succ 1.$$

Next, we extend Ψ to all of $\mathfrak{M}_\mathbf{T}$ by setting $\Psi(\mathfrak{t}\,\mathfrak{m}) = \Psi(\mathfrak{t})\,\Phi(\mathfrak{m})$ for $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_\mathbf{T}$. Note that Ψ extends Φ. It is straightforward to check that $\Psi: \mathfrak{M}_\mathbf{T} \longrightarrow \mathfrak{N}$ is an embedding of monomial groups which respects real powers. We need to show that Ψ preserves the order. Let $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_\mathbf{T}^\succ$. If both $\mathfrak{t}, \mathfrak{m} \succcurlyeq 1$, then $\Psi(\mathfrak{t}\,\mathfrak{m}) = \Psi(\mathfrak{t})\,\Phi(\mathfrak{m}) \succ 1$. This leaves us two cases to consider:

1. Suppose $\mathfrak{t} \succ 1$, $\mathfrak{m} \prec 1$, and $\varphi_\mathfrak{t} > L_\beta(\mathfrak{m}^{-1})$. Set $r := (\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} > 0$. We claim that $L_\beta(\mathfrak{m}^{-1}) =_\beta L_\beta^{\uparrow \gamma_\mathfrak{t}+1}(2\,r^{-1}\,L_1(\mathfrak{m}^{-1}))$. If $\mu = 1$, then $\gamma_\mathfrak{t} = 0$, so this follows from Lemma 7.25. If $\mu > 1$, then $1, \gamma_\mathfrak{t}+1 < \theta$, so this follows from Lemmas 7.25 and 7.27. In either case, we have $\varphi_\mathfrak{t} > L_\beta^{\uparrow \gamma_\mathfrak{t}+1}(2r^{-1}L_1(\mathfrak{m}^{-1}))$, so $\Phi(\varphi_\mathfrak{t}) > L_\beta^{\uparrow \gamma_\mathfrak{t}+1}(2r^{-1}L_1(\Phi(\mathfrak{m}^{-1})))$. From this, we see that

   $$L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}}) = L_{\gamma_\mathfrak{t}+1}(E_\beta(\Phi(\varphi_\mathfrak{t}))) > 2\,r^{-1}\,L_1(\Phi(\mathfrak{m}^{-1})),$$

   so $\frac{1}{2}\,r\,L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}}) > L_1(\Phi(\mathfrak{m}^{-1}))$. Since $L_1(\Psi(\mathfrak{t})) \sim r\,L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}})$, this gives $L_1(\Psi(\mathfrak{t})) > L_1(\Phi(\mathfrak{m}^{-1}))$. Thus

   $$\log(\Psi(\mathfrak{t}\,\mathfrak{m})) = L_1(\Psi(\mathfrak{t})) - L_1(\Phi(\mathfrak{m}^{-1})) > 0,$$

   so $\Psi(\mathfrak{t}\,\mathfrak{m}) \succ 1$.

2. Suppose $\mathfrak{t} \prec 1$, $\mathfrak{m} \succ 1$, and $\varphi_{\mathfrak{t}} < L_\beta(\mathfrak{m})$. Set $r := (\mathfrak{t}_{\varphi_{\mathfrak{t}}})_{\gamma_{\mathfrak{t}}} < 0$. As before, Lemmas 7.25 and 7.27 give $\Phi(\varphi_{\mathfrak{t}}) < L_\beta^{\uparrow\gamma_{\mathfrak{t}}+1}\left(-\frac{1}{2} r^{-1} L_1(\Phi(\mathfrak{m}))\right)$, so

$$-2\, r\, L_{\gamma_{\mathfrak{t}}+1}(\mathfrak{a}_{\varphi_{\mathfrak{t}}}) < L_1(\Phi(\mathfrak{m})).$$

Since $L_1(\Psi(\mathfrak{t})^{-1}) = -\log(\Psi(\mathfrak{t})) \sim -r\, L_{\gamma_{\mathfrak{t}}+1}(\mathfrak{a}_{\varphi_{\mathfrak{t}}})$, this gives $L_1(\Psi(\mathfrak{t})^{-1}) < L_1(\Phi(\mathfrak{m}))$, so

$$\log(\Psi(\mathfrak{t}\,\mathfrak{m})) = L_1(\Phi(\mathfrak{m})) - L_1(\Psi(\mathfrak{t})^{-1}) > 0$$

and $\Psi(\mathfrak{t}\,\mathfrak{m}) \succ 1$.

By Proposition 2.3, the function $\Psi\colon \mathfrak{M}_{\mathbf{T}} \longrightarrow \mathfrak{N}$ extends uniquely into a strongly linear strictly increasing embedding $\mathbb{T}_{\mathbf{T}} \longrightarrow \mathbb{U}$, which we still denote by Ψ.

We claim that Ψ is an embedding of force ν. By Lemma 8.3, we need only show that Ψ commutes with logarithms and hyperlogarithms. We begin with logarithms. Let $\mathfrak{l} \in \mathfrak{L}_{<\theta}$ and $\varphi \in \mathbf{T}$. If $\mu = 1$, then $\mathfrak{l} = \ell_0^r$ for some $r \in \mathbb{R}$ and

$$\log(\Psi(\ell_0^r[\mathrm{e}_\omega^\varphi])) = r\, L_1(\mathfrak{a}_\varphi) = r\, L_1(E_\omega(\Phi(\varphi))) = r\, E_\omega(\Phi(\varphi-1)).$$

If $\varphi - 1 \in \mathbf{T}$, then

$$r\, E_\omega(\Phi(\varphi-1)) = r\, \mathfrak{a}_{\varphi-1} = \Psi(\log(\ell_0^r[\mathrm{e}_\omega^\varphi])).$$

If $\varphi - 1 \notin \mathbf{T}$, then $\log(\ell_0^r[\mathrm{e}_\omega^\varphi]) \in \mathbb{T}$ and

$$r\, E_\omega(\Phi(\varphi-1)) = \Phi(r\, E_\omega(\varphi-1)) = \Phi(\log(\ell_0^r[\mathrm{e}_\omega^\varphi])) = \Psi(\log(\ell_0^r[\mathrm{e}_\omega^\varphi])).$$

If $\mu > 0$, then

$$\log(\Psi(\mathfrak{l}[\mathrm{e}_\beta^\varphi])) = \sum_{\gamma<\theta} \mathfrak{l}_\gamma L_{\gamma+1}(\mathfrak{a}_\varphi) = \Psi\left(\sum_{\gamma<\theta} \mathfrak{l}_\gamma\, \ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]\right) = \Psi(\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi])).$$

In all cases, we have, $\log(\Psi(\mathfrak{l}[\mathrm{e}_\beta^\varphi])) = \Psi(\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi]))$. For $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{\mathbf{T}}$, we have

$$\begin{aligned}\log \Psi(\mathfrak{t}\,\mathfrak{m}) &= \log \Psi(\mathfrak{t}) + \log \Psi(\mathfrak{m}) = \sum_{\varphi\in\mathbf{T}} \log(\Psi(\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi])) + \log \Phi(\mathfrak{m}) \\ &= \sum_{\varphi\in\mathbf{T}} \Psi(\log(\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi])) + \Phi(\log \mathfrak{m}) = \Psi(\log \mathfrak{t}) + \Psi(\log \mathfrak{m}) = \Psi(\log(\mathfrak{t}\,\mathfrak{m})).\end{aligned}$$

Now, let $0 < \eta \leqslant \mu + 1$ and let $\mathfrak{t} = \ell_\gamma[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_{\omega^\eta} \setminus \mathfrak{M}_{\omega^\eta}$. Note that $\Psi(\mathfrak{t}) = L_\gamma(\mathfrak{a}_\varphi)$, so we need to show that $\Psi(L_{\omega^\eta}(\mathfrak{t})) = L_{\omega^\eta}(L_\gamma(\mathfrak{a}_\varphi))$. Write $\gamma = \gamma_{\geqslant\omega^\eta} + \omega^{\eta_*} n$. If $\eta < \mu_*$, then

$$\Psi(L_{\omega^\eta}(\mathfrak{t})) = \Psi(\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] - n) = L_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}(\mathfrak{a}_\varphi) - n = L_{\omega^\eta}(L_\gamma(\mathfrak{a}_\varphi)).$$

If $\eta = \mu_* < \mu$, then $\gamma = \omega^{\mu_{**}} n$. If $\varphi - 1 \in \mathbf{T}$, then

$$\Psi(L_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = \Psi\left(\mathrm{e}_\beta^{\varphi-1}\right) - n = \mathfrak{a}_{\varphi-1} - n = L_\theta(\mathfrak{a}_\varphi) - n = L_\theta(L_\gamma(\mathfrak{a}_\varphi)).$$

If $\varphi - 1 \notin \mathbf{T}$, then

$$\Psi(L_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = \Psi(E_\beta(\varphi-1)) - n = \Phi(E_\beta(\varphi-1)) - n = L_\theta(\mathfrak{a}_\varphi) - n = L_\theta(L_\gamma(\mathfrak{a}_\varphi)).$$

If $\eta = \mu$, then $\gamma = 0$ and

$$\Psi(L_\beta(\mathfrak{t})) = \Psi(\varphi) = \Phi(\varphi) = L_\beta(\mathfrak{a}_\varphi).$$

If $\eta = \mu + 1$, then $\gamma = 0$ and

$$\Psi(L_{\omega^{\mu+1}}(\mathfrak{t})) = \Psi(L_{\omega^{\mu+1}}(\varphi) + 1) = \Phi(L_{\omega^{\mu+1}}(\varphi) + 1) = L_{\omega^{\mu+1}}(\Phi(\varphi)) + 1 = L_{\omega^{\mu+1}}(\mathfrak{a}_{\varphi}).$$

Since $\Psi(L_{\omega^{\eta}}(\mathfrak{m})) = \Phi(L_{\omega^{\eta}}(\mathfrak{m})) = L_{\omega^{\eta}}(\Phi(\mathfrak{m})) = L_{\omega^{\eta}}(\Psi(\mathfrak{m}))$ for $\mathfrak{m} \in \mathfrak{M}_{\omega^{\eta}}$ and since $\operatorname{dom} L_{\omega^{\eta}} = \mathfrak{M}_{\omega^{\eta}}$ for $\eta > \mu + 1$, this completes the proof of our claim that Ψ is an embedding of force $\boldsymbol{\nu}$.

We finish with the uniqueness of Ψ. Let $\Lambda: \mathbb{T}_{\mathbf{T}} \longrightarrow \mathbb{U}$ be another embedding of force $\boldsymbol{\nu}$ that extends Φ. To see that $\Lambda = \Psi$, we only need to show that $\Lambda(\mathfrak{t}) = \Psi(\mathfrak{t})$ for all $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_{\beta}^{\mathbf{T}}]$. For $\varphi \in \mathbf{T}$, we have

$$L_{\beta}(\Lambda(\mathrm{e}_{\beta}^{\varphi})) = \Lambda(L_{\beta}(E_{\beta}(\varphi))) = \Lambda(\varphi) = \Phi(\varphi),$$

so $\Lambda(\mathrm{e}_{\beta}^{\varphi}) = \mathfrak{a}_{\varphi}$. For $\gamma < \theta$, we deduce that

$$\Lambda(\ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}]) = \Lambda(L_{\gamma+1}(\mathrm{e}_{\beta}^{\varphi})) = L_{\gamma+1}(\Lambda(\mathrm{e}_{\beta}^{\varphi})) = L_{\gamma+1}(\mathfrak{a}_{\varphi}) = \Psi(\ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}]).$$

Since Λ is strongly linear, this gives $\log \Lambda(\mathfrak{t}) = \Lambda(\log \mathfrak{t}) = \Psi(\log \mathfrak{t}) = \log \Psi(\mathfrak{t})$ for $\mathfrak{t} \in \mathfrak{L}_{<\beta}[\mathrm{e}_{\beta}^{\mathbf{T}}]$, so $\Lambda(\mathfrak{t}) = \Psi(\mathfrak{t})$ by the injectivity of log. □

8.5 Fields with bijective hyperlogarithms

In this subsection, we prove Theorem 7.4. Recall that $\mathbb{T}$ is a confluent hyperserial skeleton of force $\boldsymbol{\nu}$.

Definition 8.31. *Let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. For $\gamma \in \mathbf{On}$ and $\boldsymbol{\eta} \leqslant \boldsymbol{\mu}$, we define $\mathfrak{M}_{(\gamma,\boldsymbol{\eta})}$ as follows:*

- $\mathfrak{M}_{(0,0)} := \mathfrak{M}$.
- $\mathfrak{M}_{(\gamma,\boldsymbol{\eta})} := (\mathfrak{M}_{(\gamma,\boldsymbol{\eta}_*)})_{(\boldsymbol{\eta}_*)}$ *if $\boldsymbol{\eta}$ is a successor.*
- $\mathfrak{M}_{(\gamma,\boldsymbol{\eta})} := \bigcup_{\sigma<\boldsymbol{\eta}} \mathfrak{M}_{(\gamma,\sigma)}$ *if $\boldsymbol{\eta}$ is a non-zero limit.*
- $\mathfrak{M}_{(\gamma,0)} := \bigcup_{\lambda<\gamma} \mathfrak{M}_{(\lambda,\boldsymbol{\mu})}$ *if $\gamma > 0$.*

We set $\mathbb{T}_{(\gamma,\boldsymbol{\eta})} := \mathbb{R}[[\mathfrak{M}_{(\gamma,\boldsymbol{\eta})}]]$, so $\mathbb{T}_{(0,0)} = \mathbb{T}$ and we have the force $\boldsymbol{\nu}$ inclusion $\mathbb{T}_{(\lambda,\boldsymbol{\sigma})} \subseteq \mathbb{T}_{(\gamma,\boldsymbol{\eta})}$ whenever $\lambda < \gamma$ or $\lambda = \gamma$ and $\boldsymbol{\sigma} \leqslant \boldsymbol{\eta}$. We set

$$\mathfrak{M}_{(<\boldsymbol{\mu})} := \bigcup_{\gamma \in \mathbf{On}} \mathfrak{M}_{(\gamma,0)}, \qquad \mathbb{T}_{(<\boldsymbol{\mu})} := \bigcup_{\gamma \in \mathbf{On}} \mathbb{T}_{(\gamma,0)}$$

Note that $\mathbb{T}_{(<\boldsymbol{\mu})} = \mathbb{R}[[\mathfrak{M}_{(<\boldsymbol{\mu})}]]$ by Lemma 2.1. Note also that $\mathfrak{M}_{(<0)} = \mathfrak{M}$ and $\mathbb{T}_{(<0)} = \mathbb{T}$. Theorem 7.4 is a consequence of the next two propositions:

Proposition 8.32. *$\mathbb{T}_{(<\boldsymbol{\mu})}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$.*

Proof. By Corollary 7.24, it suffices to show that

$$(\mathbb{T}_{(<\boldsymbol{\mu})})_{\succ,\omega^{\eta}} \subseteq L_{\omega^{\eta}}(\mathbb{T}_{(<\boldsymbol{\mu})}^{>,\succ})$$

for $\eta < \boldsymbol{\mu}$. Fix $\eta < \boldsymbol{\mu}$ and fix $s \in (\mathbb{T}_{<\boldsymbol{\mu}})_{\succ,\omega^{\eta}}$. Fix also a limit ordinal γ with $s \in \mathbb{T}_{(\gamma,0)}^{>,\succ}$. Then $s \in \mathbb{T}_{(\gamma,\eta)}^{>,\succ}$ so either $E_{\omega^{\eta}}(s)$ exists in $\mathbb{T}_{(\gamma,\eta)}$ or $E_{\omega^{\eta}}(s)$ exists in $(\mathbb{T}_{(\gamma,\eta)})_{(\eta)} = \mathbb{T}_{(\gamma,\eta+1)}$. In either case, $E_{\omega^{\eta}}(s) \in \mathbb{T}_{(\gamma+1,0)}^{>,\succ}$. □

Proposition 8.33. *Let $\mathbb{U}$ be a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ and let $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ be a force $\boldsymbol{\nu}$ embedding. Then there is a unique force $\boldsymbol{\nu}$ embedding $\Psi: \mathbb{T}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{U}$ extending Φ.*

Proof. We will show for each $\gamma \in \mathbf{On}$ and each $\boldsymbol{\eta} \leqslant \boldsymbol{\mu}$ that there is a unique force $\boldsymbol{\nu}$ embedding $\Psi_{(\gamma,\boldsymbol{\eta})}\colon \mathbb{T}_{(\gamma,\boldsymbol{\eta})} \longrightarrow \mathbb{U}$ extending Φ. We have $\Psi_{(0,0)} = \Phi$, so suppose that we have defined this unique embedding $\Psi_{(\lambda,\sigma)}$ when $\lambda < \gamma$, $\boldsymbol{\sigma} \leqslant \boldsymbol{\mu}$ and when $\lambda = \gamma$, $\boldsymbol{\sigma} < \boldsymbol{\eta}$. If $\boldsymbol{\eta}$ is a successor, then $\mathbb{T}_{(\gamma,\boldsymbol{\eta})} = (\mathbb{T}_{(\gamma,\boldsymbol{\eta}_*)})_{(\boldsymbol{\eta}_*)}$, so by Proposition 8.4 (if $\boldsymbol{\eta}_* = 0$) or Proposition 8.30 (if $\boldsymbol{\eta}_* > 0$), the embedding $\Psi_{(\gamma,\boldsymbol{\eta}_*)}$ extends uniquely to an embedding

$$\Psi_{(\gamma,\boldsymbol{\eta})}\colon \mathbb{T}_{(\gamma,\boldsymbol{\eta})} \longrightarrow \mathbb{U}.$$

Since $\Psi_{(\gamma,\boldsymbol{\eta})}$ uniquely extends $\Psi_{(\gamma,\boldsymbol{\eta}_*)}$ and since $\Psi_{(\gamma,\boldsymbol{\eta}_*)}$ uniquely extends Φ, we see that $\Psi_{(\gamma,\boldsymbol{\eta})}$ uniquely extends Φ. If $\boldsymbol{\eta}$ is a limit, then we set $\Psi_{(\gamma,\boldsymbol{\eta})} := \bigcup_{\sigma<\boldsymbol{\eta}} \Psi_{(\gamma,\sigma)}$. The map $\Psi_{(\gamma,\boldsymbol{\eta})}$ is only defined on $\bigcup_{\sigma<\boldsymbol{\eta}} \mathbb{T}_{(\gamma,\sigma)}$, which may not equal $\mathbb{T}_{(\gamma,\boldsymbol{\eta})}$, but $\Psi_{(\gamma,\boldsymbol{\eta})}$ *is* defined on all of $\mathfrak{M}_{(\gamma,\boldsymbol{\eta})}$ and so $\Psi_{(\gamma,\boldsymbol{\eta})}$ extends uniquely to a force $\boldsymbol{\nu}$ embedding $\mathbb{T}_{(\gamma,\boldsymbol{\eta})} \longrightarrow \mathbb{U}$, which we also denote by $\Psi_{(\gamma,\boldsymbol{\eta})}$. As each $\Psi_{(\gamma,\sigma)}$ uniquely extends Φ, we see that $\Psi_{(\gamma,\boldsymbol{\eta})}$ uniquely extends Φ as well. Likewise, we define $\Psi_{(\gamma,0)}$ to be the unique force $\boldsymbol{\nu}$ embedding extending $\bigcup_{\lambda<\gamma} \Psi_{(\lambda,\boldsymbol{\mu})}$. □

Theorem 7.4 follows from Propositions 8.32 and 8.33.

Bibliography

[1] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. On numbers, germs, and transseries. In *Proc. Int. Cong. of Math. 2018*, volume 1, pages 1–24. Rio de Janeiro, 2018.

[2] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. The surreal numbers as a universal *H*-field. *J. Eur. Math. Soc.*, 21(4):1179–1199, 2019.

[3] V. Bagayoko, J. van der Hoeven, and V. Mantova. Defining a surreal hyperexponential. Technical Report, HAL, 2020. http://hal.archives-ouvertes.fr/hal-02861485.

[4] A. Berarducci and V. Mantova. Surreal numbers, derivations and transseries. Technical Report http://arxiv.org/abs/1503.00315, Arxiv, 2015.

[5] É. Borel. *Leçons sur les séries divergentes*. Gauthier-Villars, 2-nd edition, 1928. Reprinted by Jacques Gabay.

[6] M. Boshernitzan. Hardy fields, existence of transexponential functions. *Æquationes Math.*, 30:258–280, 1986.

[7] N. Bourbaki. *Fonctions d'une variable réelle*. Éléments de Mathématiques (Chap. 5). Hermann, 2-nd edition, 1961.

[8] J. H. Conway. *On numbers and games*. Academic Press, 1976.

[9] B. I. Dahn and P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae*, 127:45–50, 1986.

[10] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Math. Soc. Lect. Note*. Cambridge University Press, 1998.

[11] L. van den Dries and Ph. Ehrlich. Fields of surreal numbers and exponentiation. *Fundamenta Mathematicae*, 167(2):173–188, 2001.

[12] L. van den Dries, J. van der Hoeven, and E. Kaplan. Logarithmic hyperseries. *Trans. of the AMS*, 372(7):5199–5241, 2019.

[13] L. van den Dries, A. Macintyre, and D. Marker. Logarithmic-exponential power series. *Journal of the London Math. Soc.*, 56(2):417–434, 1997.

[14] P. du Bois-Reymond. Sur la grandeur relative des infinis des fonctions. *Annali di Matematica Pura ed Applicata (1867-1897)*, 4(1):338–353, 1870.

[15] P. du Bois-Reymond. Über asymptotische Werte, infinitäre Approximationen und infinitäre Auflösung von Gleichungen. *Math. Ann.*, 8:363–414, 1875.

[16] P. du Bois-Reymond. Über die Paradoxen des Infinitärscalcüls. *Math. Ann.*, 11:149–167, 1877.

[17] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Hermann, collection: Actualités mathématiques, 1992.

[18] H. Gonshor. *An Introduction to the Theory of Surreal Numbers*. Cambridge Univ. Press, 1986.

[19] H. Hahn. Über die nichtarchimedischen Größensysteme. *Sitz. Akad. Wiss. Wien*, 116:601–655, 1907.

[20] G. H. Hardy. *Orders of infinity*. Cambridge Univ. Press, 1910.

[21] G. H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London Mathematical Society*, 10(2):54–90, 1911.

[22] D. Harvey and J. van der Hoeven. Faster polynomial multiplication over finite fields using cyclotomic coefficient rings. Accepted for publication in J. of Complexity, 2019.

[23] J. van der Hoeven. *Automatic asymptotics*. PhD thesis, École polytechnique, Palaiseau, France, 1997.

[24] J. van der Hoeven. Transséries fortement monotones. Chapter 1 of unpublished CNRS activity report, https://www.texmacs.org/joris/schmeling/rap1-2000.pdf, 2000.

[25] J. van der Hoeven. Operators on generalized power series. *Journal of the Univ. of Illinois*, 45(4):1161–1190, 2001.

[26] J. van der Hoeven. *Transseries and real differential algebra*, volume 1888 of *Lecture Notes in Mathematics*. Springer-Verlag, 2006.

[27] J. van der Hoeven. Transserial Hardy fields. In Felipe Cano, Frank Loray, Juan José Moralez-Ruiz, Paulo Sad, and Mark Spivakovsky, editors, *Differential Equations and Singularities. 60 years of J. M. Aroca*, volume 323 of *Astérisque*, pages 453–487. SMF, 2009.

[28] H. Kneser. Reelle analytische Lösungen der Gleichung $\phi(\phi(x)) = e^x$ und verwandter Funktionalgleichungen. *Jour. f. d. reine und angewandte Math.*, 187(1/2):56–67, 1950.

[29] M. C. Schmeling. *Corps de transséries*. PhD thesis, Université Paris-VII, 2001.

Glossary

On	class of ordinal numbers	7
$\boldsymbol{\nu} \leqslant \mathbf{On}$	$\boldsymbol{\nu} \in \mathbf{On}$ or $\boldsymbol{\nu} = \mathbf{On}$	7
μ_*	$\mu = \mu_* + 1$ if μ is a successor, $\mu = \mu_*$ if μ is a limit	7
$\rho \ll \sigma$	$\rho < \omega^\eta$ for each exponent η of σ	7
$\rho \leqq \sigma$	$\rho \leqslant \omega^\eta$ for each exponent η of σ	7
$\gamma_{\geqslant \omega^\eta}$	unique ordinal with $\gamma_{\geqslant \omega^\eta} \geqq \omega^\eta$ and $\exists \iota < \omega^\eta, \gamma = \gamma_{\geqslant \omega^\eta} + \iota$	7
$\mathbb{R}[[\mathfrak{M}]]$	field of well-based series over $\mathfrak{M}$	8
$\operatorname{supp} f$	set of monomials $\mathfrak{m}$ with $f_\mathfrak{m} \neq 0$	8
$\mathfrak{d}_f$	dominant monomial $\mathfrak{d}_f = \max \operatorname{supp} f$ of f	8
$f \prec g$	$\mathbb{R}^> \lvert f \rvert < \lvert g \rvert$	8
$f \preccurlyeq g$	$\exists r \in \mathbb{R}^>, \lvert f \rvert \geqslant r \lvert g \rvert$	8
$f \asymp g$	$f \preccurlyeq g \preccurlyeq f$	8
$f \sim g$	$f - g \prec f$	8
$g \trianglelefteq f$	g is a truncation of f	9
$\mathbb{T}_\succ$	$\{f \in \mathbb{T} : \operatorname{supp} f \subseteq \mathfrak{M}^\succ\}$	9
$\mathbb{T}^\prec$	$\{f \in \mathbb{T} : f \prec 1\}$	9
$\mathbb{T}^{>,\succ}$	$\{f \in \mathbb{T} : f > \mathbb{R}\}$	9
$\operatorname{supp}_* \Psi$	operator support $\operatorname{supp}_* \Psi = \bigcup_{\mathfrak{m} \in \mathfrak{M}} \operatorname{supp} \Psi(\mathfrak{m}) / \mathfrak{m}$ of Ψ	10
$\operatorname{supp}_\odot \Phi$	relative support of Φ	10
s^r	$s > 0$ to the power $r \in \mathbb{R}$	10
$\mathbb{L}$	field of logarithmic hyperseries	11
ℓ_γ	formal hyperlogarithm $\ell_\gamma \in \mathbb{L}$ of strength γ	11
$\mathfrak{L}_{<\alpha}$	group of logarithmic monomials of strength α	11
$\mathbb{L}_{<\alpha}$	field of logarithmic series of strength α	12
$\mathfrak{L}$	group of logarithmic (hyper)monomials	12
∂	derivation on $\mathbb{L}$	12
$f^{(k)}$	k-th derivative $\partial^k(f)$ of $f \in \mathbb{L}$	12
$g^{\uparrow\gamma}$	for $g \in \mathbb{L}_{[\gamma,\alpha)}$, unique series with $g = (g^{\uparrow\gamma}) \circ \ell_\gamma$	13
L_{ω^μ}	hyperlogarithm of force μ	14
$\mathbf{DD}_\mu$	axiom of domain definition	14
$\mathfrak{M}_{\omega^\mu}$	class of $L_{<\omega^\mu}$-atomic elements	14
$\mathbf{FE}_\mu$	axiom of functional equations	15
$\mathbf{A}_\mu$	axiom of asymptotics	15
$\mathbf{M}_\mu$	axiom of monotonicity	15
$\mathbf{R}_\mu$	axiom of regularity	15
$\mathbf{P}_\mu$	axiom of transfinite products	15
$\mathfrak{d}_\alpha$	projection $\mathbb{T}^{>,\succ} \longrightarrow \mathfrak{M}_\alpha$	16
$\mathcal{E}_\alpha[s]$	class of series t with $\exists \gamma < \alpha, L_\gamma(t) \asymp L_\gamma(s)$	16
$\mathcal{E}_\alpha[s]$	class of series t with $\exists \gamma < \alpha, L_\gamma(t) \asymp L_\gamma(s)$	16
$L(1+\varepsilon)$	Taylor expansion of log at $1+\varepsilon$	20
$E(\varepsilon)$	Taylor expansion of exp at ε	20
HF1 – HF7	axioms for hyperserial fields	36
$\mathbb{T}_{(<\boldsymbol{\mu})}$	minimal extension of $\mathbb{T}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$	40
$L_{[\gamma,\beta)}$	$s \longmapsto \prod_{\gamma \leqslant \sigma < \beta} \ell_\sigma \circ s : \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$	40
$L_\beta^{\uparrow\gamma}$	$\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}; s \longmapsto \ell_\beta^{\uparrow\gamma} \circ s$	40
$L_{[\gamma,\beta)}^{\uparrow\gamma}$	$s \longmapsto \prod_{\gamma \leqslant \sigma < \beta} \ell_\sigma^{\uparrow\gamma} \circ s : \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$	40
E_γ	(partially defined) functional inverse of $L_\gamma : \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$	41
$\mathbb{T}_{\succ,\omega^\eta}$	class of ω^η-truncated series	44
$\mathcal{L}_\alpha[s]$	$\{t \in s + \mathbb{T}^\prec : t = s$ or $(t \neq s$ and $s < L_\alpha^{\uparrow\gamma}(\lvert t - s \rvert^{-1})$ for some $\gamma < \alpha)\}$	45
$\sharp_{\omega^\eta}(s)$	unique ω^η-truncated series in $\mathcal{L}_{\omega^\eta}[s]$	47
$s =_\beta t$	$\sharp_\beta(s) = \sharp_\beta(t)$	48
$s <_\beta t$	$\sharp_\beta(s) < \sharp_\beta(t)$	48
$\mathbb{T}_\mathbf{T}$	minimal extension of $\mathbb{T}$ with $E_\beta(\mathbf{T}) \subseteq \mathbb{T}_\mathbf{T}$	50

Index