FILLING GAPS IN HARDY FIELDS

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

Abstract. We show how to fill "countable" gaps in Hardy fields. We use this to prove that any two maximal Hardy fields are back-and-forth equivalent.

Contents

Introduction

By a "Hardy field" we mean in this paper a Hardy field at $+\infty$: a subfield H of the ring of germs at $+\infty$ of real valued differentiable functions on intervals $(a, +\infty)$ ($a \in \mathbb{R}$) such that H is closed under differentiation. For basics on Hardy fields, see [26]. Each Hardy field is an ordered differential field, the (total) ordering given by $f \leqslant g$ iff $f(t) \leqslant g(t)$ eventually (that is, for all sufficiently large t). Among functions whose germs at $+\infty$ live in Hardy fields are all one-variable rational functions with real coefficients, the real exponential and logarithm functions (more generally, Hardy's logarithmico-exponential functions [18]), Euler's Γ-function and Riemann's ζ-function [27], and many other "regularly growing" functions arising in mathematical practice. As a case in point, by [15] every o-minimal expansion of the ordered field of real numbers gives rise to a Hardy field (of germs of definable functions). Our main result is as follows:

Theorem A. *Let H be a Hardy field, and let A, B be countable subsets of H such that $A < B$. Then $A < f < B$ for some f in a Hardy field extension of H.*

Some of the gaps $A < B$ in this theorem correspond to pseudo-cauchy sequences (*pc-sequences* in our abbreviated terminology). The relevant pc-sequences have

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length ω, and we can handle them using results from our book [ADH] and the manuscript [7] in an essential way, and various glueing techniques. This is done in Sections 3 and 4. This dependence on [ADH] and [7] makes this the deepest part of the present paper, but most of our work here deals with other gaps.

Sjödin [28] deals with the case $B = \emptyset$ for $\mathcal{C}^\infty$-Hardy fields (whose elements are germs of $\mathcal{C}^\infty$-functions). This provides an important clue for other kinds of gaps: Sjödin's construction of a suitable f can be varied in several ways, and that gives us a handle on the relevant remaining cases. In Section 5 we treat $B = \emptyset$, basically as in [28], and organized so that it helps in Section 6 where we deal with "wide" gaps. For the remaining gaps we use results about asymptotic couples from [6] and an elaboration of the "reverse engineering" in [28]; see Sections 8 and 9. (Sections 1 and 2 contain mainly analytic preliminaries, and Section 7 applies material in Sections 5 and 6 to show that there are $2^{\mathfrak{c}}$ many maximal Hardy fields where $\mathfrak{c} = 2^{\aleph_0}$ is the cardinality of the continuum. Here and below, "maximal" means "maximal under inclusion".)

Most of [7] concerns *differentially algebraic* extensions of Hardy fields. The present paper complements this with a "good enough" overview of *differentially transcendental* Hardy field extensions $H\langle y\rangle$ of a Liouville closed Hardy field $H \supseteq \mathbb{R}$.

An equivalent formulation of Theorem A is that every maximal Hardy field is η_1. The property η_1 (Hausdorff [20]) is defined at the end of the introduction. The main result of [7] is that all maximal Hardy fields, as ordered differential fields, are $\boldsymbol{\omega}$-free newtonian Liouville closed H-fields, and thus by [ADH, 15.0.2, 16.6.3] elementarily equivalent to $\mathbb{T}$, the ordered differential field of transseries. (On $\mathbb{T}$, see [ADH, Appendix A] or [4].) Combining this fact with Theorem A and a result from [5] we shall derive in Section 10:

Corollary B. *Assuming* CH (*the Continuum Hypothesis*), *every maximal Hardy field is isomorphic as an ordered differential field to the ordered field* $\mathbf{No}(\omega_1)$ *of surreal numbers of countable length equipped with the derivation* ∂_{BM} *of* [9].

Thus with CH, all maximal Hardy field are isomorphic as ordered differential fields. Without CH, the proof yields a nonempty back-and-forth system between any maximal Hardy field and the ordered differential field $\mathbf{No}(\omega_1)$. (See [ADH, B.5] for "back-and-forth system".) Then by Karp [21], cf. [8, Theorem 3], any maximal Hardy field and the ordered differential field $\mathbf{No}(\omega_1)$ are $\infty\omega$-equivalent. This is a strengthening of [7, Corollary 1].

Key ingredients for proving Theorem A include Lemma 3.4, the construction of a partition of unity in Section 4, the reduction to Case (b) stated in Lemma 8.11, and the elaborated reverse engineering in Section 9 that culminates in a diagonal argument. (The idea behind the original reverse engineering from [28] is sketched in the remarks that follow the statement of Theorem 5.12.)

Theorem A answers a question of Ehrlich [16] and establishes Conjecture B from [4]. (For Conjecture A, see [7, Theorem A].)

In this paper our Hardy fields are not assumed to be $\mathcal{C}^\infty$-Hardy fields, and we do not know if maximal $\mathcal{C}^\infty$-Hardy fields are necessarily maximal Hardy fields (even under CH). So the question arises if our main results go through for maximal $\mathcal{C}^\infty$-Hardy fields. This is indeed the case, and it is not hard to refine some of our proofs to that effect. The same question arises for the still more special $\mathcal{C}^\omega$-Hardy fields (analytic Hardy fields). Our main results also go through in that setting, but this is more delicate. We shall treat these refinements in a follow-up paper.

Notations and conventions. We let i, j, k, l, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. As in [ADH] the convention is that the ordering of an ordered set, ordered abelian group, or ordered field is a *total* ordering. Let S be an ordered set. For any element b in an ordered set extending S we set

$$S^{<b} := \{s \in S : s < b\}, \qquad S^{>b} := \{s \in S : s > b\}.$$

We have the usual notion of a set $P \subseteq S$ being cofinal in S (respectively, coinitial in S). In addition, sets $P, Q \subseteq S$ are said to be *cofinal* if for every $p \in P$ there exists $q \in Q$ with $p \leqslant q$ and for every $q \in Q$ there exists $p \in P$ with $q \leqslant p$; replacing here $\leqslant$ by $\geqslant$, we obtain the notion of P and Q being *coinitial.* Thus P and $P^{\downarrow} = \{s \in S : s \leqslant p \text{ for some } p \in P\}$ are cofinal, hence P, Q are cofinal iff $P^{\downarrow} = Q^{\downarrow}$. Likewise, P and $P^{\uparrow} = \{s \in S : s \geqslant p \text{ for some } p \in P\}$ are coinitial, and P, Q are coinitial iff $P^{\uparrow} = Q^{\uparrow}$. We let $\operatorname{cf}(S)$ and $\operatorname{ci}(S)$ denote the cofinality and coinitiality of S; see [ADH, 2.1]. We say that S is η_1 if for all countable $P, Q \subseteq S$ with $P < Q$ there exists an $s \in S$ with $P < s < Q$; in particular, such S is uncountable (cf. Lemma 7.2 below), has no least element, no largest element, and is dense in the sense that for all $p, q \in S$ with $p < q$ there exists $s \in S$ with $p < s < q$. We say that an ordered abelian group (ordered field) is η_1 if its underlying ordered set is η_1. For basic facts about various η_1-structures, see [25, Kapitel IV].

Let (a_ρ) be a well-indexed sequence. Its *length* is the (infinite limit) ordinal that is the order type of its well-ordered set of indices ρ (cf. [ADH, p. 73]). Note that if (a_ρ) has countable length, then its length has cofinality ω, and thus (a_ρ) has a cofinal subsequence (a_{ρ_n}) of length ω.

Let (Γ, ψ) be an asymptotic couple. As in [ADH, 6.5] we set $\Gamma_\infty := \Gamma \cup \{\infty\}$, and adopt the convention that $\psi(0) = \psi(\infty) = \infty > \Gamma$. For $\alpha \in \Gamma_\infty$ we use $\alpha^\dagger$ as an alternative notation for $\psi(\alpha)$ and define $\alpha^{\langle n \rangle} \in \Gamma_\infty$ by recursion on n by $\alpha^{\langle 0 \rangle} := \alpha$ and $\alpha^{\langle n+1 \rangle} := (\alpha^{\langle n \rangle})^\dagger$. We simplify terminology by calling an H-field *closed* ("H-closed" in [4, 7]) if it is $\boldsymbol{\omega}$-free, newtonian, and Liouville closed.

As in [7], $\mathcal{C}$ is the ring of germs at $+\infty$ of continuous functions $[a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, and $\mathcal{C}^r$ for $r \in \mathbb{N} \cup \{\infty\}$ its subring of germs of r times continuously differentiable functions $[a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$. Thus $\mathcal{C}^{<\infty} := \bigcap_n \mathcal{C}^n$ is a differential ring with the obvious derivation, and has $\mathcal{C}^\infty$ as a differential subring.

Given a Hardy field $H \supseteq \mathbb{R}$ we let $\operatorname{Li}(H)$ be the *Hardy-Liouville closure* of H, that is, the smallest real closed Hardy field extension of H that contains with any f also $\exp(f)$, and contains any $g \in \mathcal{C}^1$ whenever it contains g'; see [7, Section 5.3].

1. Preliminaries on Hausdorff Fields

This section contains basic facts about Hausdorff fields. After a subsection on pc-sequences of length ω in an ordered field we construct pseudolimits of such pc-sequences in the setting of Hausdorff fields, and show how to extend the value group of a Hausdorff field.

Ordered fields. Let K be an ordered field. We view $\mathbb{Q}$ as a subfield of K in the natural way, and consider K also as a valued field with respect to the standard valuation given by the valuation ring

$$\mathcal{O} = \{a \in K : |a| \leqslant n \text{ for some } n\},$$

the smallest convex subring of K; see [ADH, p. 175].

Lemma 1.1 (Alling [1, 2])**.** *The following two conditions on K are equivalent:*

(i) *K is η_1;*
(ii) *the residue field of K is isomorphic to $\mathbb{R}$, every pc-sequence of length ω in K has a pseudolimit in K, and the value group of K is η_1.*

This is well-known, see [24, 1.4] or [25, p. 160]. For a maximal Hardy field H we have $\mathbb{R} \subseteq H$, and so the residue field of H is indeed isomorphic to $\mathbb{R}$. Thus in order to show that H is η_1 it remains to show that all pc-sequences in H of length ω have a pseudolimit in H and that the value group of H is η_1. The former will be taken care of in Sections 3, 4, and the latter will be handled in Sections 5–9.

We continue with generalities on pc-sequences of length ω in our ordered field K.

Let (a_n) be a pc-sequence in K of length ω. When does (a_n) have a pseudolimit in K? We indicate below a reduction of this question to something that turns out to be more manageable. First, (a_n) and any infinite subsequence have the same pseudolimits in K, and so by passing to such a subsequence we can arrange that (a_n) is either strictly increasing or strictly decreasing. Replacing (a_n) by $(-a_n)$, the strictly decreasing case reduces to the strictly increasing case. Replacing (a_n) by $(a + a_n)$ for a suitable $a \in K$, the strictly increasing case reduces to the strictly increasing case where in addition all terms are positive. Next, assume (a_n) is strictly increasing and all terms are positive. Dropping some initial terms, if necessary, we arrange in addition that $a_n - a_{n-1} \succ a_{n+1} - a_n$ for all $n \geqslant 1$. Then we define b_n by $b_0 := a_0$ and $b_n := a_n - a_{n-1}$ for $n \geqslant 1$, so that $b_n > 0$, $b_n \succ b_{n+1}$, and $a_n = b_0 + \cdots + b_n$, for all n.

Reversing this last step, starting with a sequence (b_n) in K such that $b_n > 0$ and $b_n \succ b_{n+1}$ for all n, we obtain a strictly increasing pc-sequence (a_n) of positive terms a_n by $a_n = b_0 + \cdots + b_n$. This leads to:

Lemma 1.2. *The following are equivalent for K:*

(i) *all pc-sequences in K of length ω have a pseudolimit in K;*
(ii) *for every sequence (b_n) in K with $b_n > 0$ and $b_n \succ b_{n+1}$ for all n, the pc-sequence (a_n) with $a_n = b_0 + \cdots + b_n$ for all n has a pseudolimit in K.*

Hausdorff fields. As in [7] we define a *Hausdorff field* to be a subfield of $\mathcal{C}$, that is, a subring of $\mathcal{C}$ that happens to be a field. Let H be a Hausdorff field. Then

$$\{f \in H : f(t) > 0, \text{ eventually}\}$$

is the strictly positive cone for a (total) ordering on K that makes H an ordered field, and below we consider H as an ordered field in this way. This yields the convex subring

$$\mathcal{O} := \{f \in H : |f| \leqslant n \text{ for some } n\},$$

which is a valuation ring of H, and we consider H accordingly as a valued field as well. Restricting the relations $\preccurlyeq$, $\prec$, $\sim$ on $\mathcal{C}$ to H gives exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on H that it comes equipped with as a valued field.

Extending Hausdorff fields with pseudolimits. Let H be a Hausdorff field, and let a sequence

$$f_0 \succ f_1 \succ f_2 \succ \cdots$$

in $H^{>}$ be given. Then $(f_0 + \cdots + f_n)$ is a pc-sequence in H. We shall construct a pseudolimit of this pc-sequence in some Hausdorff field extension of H (possibly H itself). To conform with some later parts we let t range over real numbers $\geqslant 1$ in this subsection. We take for each n a continuous function $\mathbb{R}^{\geqslant 1} \to \mathbb{R}$ that represents the germ f_n, to be denoted also by f_n, such that $f_n(t) \geqslant 0$ and $f_{n+1}(t) \leqslant f_n(t)/2$ for all t. Now the sequence $(f_0 + \cdots + f_n)$ of partial sums converges pointwise to a function $f = \sum_{n=0}^{\infty} f_n \colon \mathbb{R}^{\geqslant 1} \to \mathbb{R}$, with the convergence being uniform on each compact subset of $\mathbb{R}^{\geqslant 1}$, so f is continuous. We claim that for all n,

$$f - (f_0 + \cdots + f_n) \ \prec\ f_n \ \text{ in } \mathcal{C}.$$

Let $\varepsilon > 0$, and take $t_n \in \mathbb{R}^{\geqslant 1}$ with $f_{n+1}(t) \leqslant \varepsilon f_n(t)$ for all $t \geqslant t_n$. Then for such t,

$$\begin{aligned} f(t) - \big(f_0(t) + \cdots + f_n(t)\big) \ &= \ f_{n+1}(t) + f_{n+2}(t) + f_{n+3}(t) + \cdots \\ &\leqslant \ f_{n+1}(t) + f_{n+1}(t)/2 + f_{n+1}(t)/4 + \cdots \\ &= \ 2f_{n+1}(t) \ \leqslant \ 2\varepsilon f_n(t), \end{aligned}$$

which proves the claim. As usual we denote the germ of f at $+\infty$ also by f, so that $f \in \mathcal{C}$. In [7, Section 5.1] we defined for $g, h \in \mathcal{C}$ that $g \leqslant h$ means $g(t) \leqslant h(t)$, eventually, and $g < h$ means $g \leqslant h$ and $g \neq h$. Here we define for $g, h \in \mathcal{C}$,

$$g <_{\mathrm{e}} h \quad :\Longleftrightarrow \quad g(t) < h(t), \text{ eventually},$$

so $g <_{\mathrm{e}} h \Rightarrow g < h$, and if $g, h \in H$, then $g <_{\mathrm{e}} h \Leftrightarrow g < h$.

Lemma 1.3. *Suppose $(f_0 + \cdots + f_n)$ has no pseudolimit in H. Let $g \in H$ be such that $g > f_0 + \cdots + f_n$ in $\mathcal{C}$, for all n. Then for all n we have*

$$f_0 + \cdots + f_n <_{\mathrm{e}} f <_{\mathrm{e}} g \quad \text{ in } \mathcal{C}.$$

Proof. As g is not a pseudolimit of $(f_0 + \cdots + f_n)$, we have $v\big(g - (f_0 + \cdots + f_n)\big) < v(f_{n+1})$ for some n. For such n we have, eventually, $g(t) - \big(f_0(t) + \cdots + f_n(t)\big) > 2f_{n+1}(t)$, and thus, eventually, $g(t) > f_0(t) + \cdots + f_n(t) + 2f_{n+1}(t) \geqslant f(t)$. □

In view of [7, Lemma 5.1.17] this yields:

Corollary 1.4. *If H is real closed and $(f_0 + \cdots + f_n)$ has no pseudolimit in H, then f generates over H an immediate Hausdorff field extension $H(f)$ of H such that $f_0 + \cdots + f_n \rightsquigarrow f$.*

Even if H is not real closed, $(f_0 + \cdots + f_n)$ pseudoconverges in some Hausdorff field extension of H, since we can pass to the real closure of H by [7, Proposition 5.1.4].

Extending the value group of a Hausdorff field. This is closely connected to filling *additive* gaps in Hausdorff fields: see Remark 1.7 and Lemma 1.11 below. For now, H is just an ordered field and $v \colon H^{\times} \to \Gamma$ is its standard valuation.

Lemma 1.5. *Let $A \subseteq H$. Then $A + A$, $2A$ are cofinal. Also, A, $2A$ are cofinal iff A, $\frac{1}{2}A$ are cofinal. Likewise with "coinitial" in place of "cofinal".*

Proof. From $2A \subseteq A + A$ and $a + b \leqslant 2\max(a, b)$ for $a, b \in A$ it follows that $A + A$ and $2A$ are cofinal. The rest is clear. □

Corollary 1.6. *Let $A, B \subseteq H^{>}$ be such that $A < B$ and there is no $h \in H$ with $A < h < B$. Then the following are equivalent:*

(i) *A, $A + A$ are cofinal;*
(ii) *A, $2A$ are cofinal;*
(iii) *B, $B + B$ are coinitial;*
(iv) *B, $\frac{1}{2}B$ are coinitial.*

Proof. The equivalence of (i) and (ii) follows from Lemma 1.5; likewise with (iii) and (iv), The equivalence of (ii) and (iv) is a consequence of $B^{\uparrow} = H^{>} \setminus A^{\downarrow}$. $\square$

An **additive gap** in H is a pair A, B of subsets of $H^{>}$ with $A < B$ such that there is no $h \in H$ with $A < h < B$, and one of the equivalent conditions (i)–(iv) in Corollary 1.6 holds.

Remark 1.7. As in [ADH], a *cut* in an ordered set S is a downward closed subset of S. Call a cut A in the ordered set $H^{>}$ *additive* if A, $B := H^{>} \setminus A$ is an additive gap in H. Then $A \mapsto A \cup (-A) \cup \{0\}$ defines an inclusion-preserving bijection

$$\big\{\text{additive cuts in } H^{>}\big\} \to \{\text{convex subgroups of } H\},$$

with inverse $D \mapsto D^{>}$. (In some places additive cuts in $H^{>}$ are therefore called "group cuts" in H; cf. [23].) Note: $D \mapsto v(D^{>})$ is an inclusion-preserving bijection

$$\{\text{convex subgroups of } H\} \to \{\text{upward closed subsets of } \Gamma\},$$

with inverse $P \mapsto v^{-1}(P) \cup \{0\}$.

In 1.8–1.10 below we assume that H is real closed. We have multiplicative versions of Lemma 1.5 and Corollary 1.6, obtained in the same way:

Lemma 1.8. *Let $A \subseteq H^{>}$. Then $A \cdot A$ and $\operatorname{sq}(A) := \{a^2 : a \in A\}$ are cofinal. Moreover, A and $\operatorname{sq}(A)$ are cofinal iff A and $\sqrt{A} := \{b \in H^{>} : b^2 \in A\}$ are cofinal. Likewise with "coinitial" in place of "cofinal".*

Corollary 1.9. *Let $A, B \subseteq H^{>}$ be such that $A < B$ and there is no $h \in H$ with $A < h < B$. Then the following are equivalent:*

(i) *A, $A \cdot A$ are cofinal;*
(ii) *A, $\operatorname{sq}(A)$ are cofinal;*
(iii) *B, $B \cdot B$ are coinitial;*
(iv) *B, $\sqrt{B}$ are coinitial.*

Lemma 1.10. *Let $A \subseteq H^{>\mathbb{Q}}$. If A, $\operatorname{sq}(A)$ are cofinal, then so are A, $2A$, and if A, $\sqrt{A}$ are coinitial, then so are A, $\frac{1}{2}A$.*

Proof. Let $a \in A$. For the first part, use $2a < a^2$; for the second, use $\sqrt{a} < \frac{a}{2}$. $\square$

Now suppose H is a Hausdorff field, turned into an ordered field as described earlier in this section. The following is Lemma 5.1.18 in [7]:

Lemma 1.11. *Suppose $\Gamma = v(H^{\times})$ is divisible. Let P be a nonempty upward closed subset of Γ, and let $f \in \mathcal{C}$ be such that $a < f$ for all $a \in H^{>}$ with $va \in P$, and $f < b$ for all $b \in H^{>}$ with $vb < P$. Then f generates a Hausdorff field $H(f)$ with $P > vf > \Gamma \setminus P$.*

We now specialize H even further: *in the rest of this subsection we assume that H is a Liouville closed Hardy field and $H \supseteq \mathbb{R}$.*

Lemma 1.12. *Let $A \subseteq H^{>\mathbb{R}}$. Then:*

(i) *if A and $\exp(A)$ are cofinal, then so are A and $\operatorname{sq}(A)$;*

Next, assume also that $\mathrm{e}^x \in A$, and that A and $\operatorname{sq}(A)$ are cofinal. Then:

(ii) *A and $A' := \{a' : a \in A\}$ are cofinal;*
(iii) *A and $\int A := \{b \in H : b' \in A\}$ are cofinal, and $\int A \subseteq H^{>\mathbb{R}}$.*

Proof. Item (i) follows from $a^2 \leqslant \exp a$ for $a \in A$.

Next, assume $\mathrm{e}^x \in A$, and A, $\operatorname{sq}(A)$ are cofinal. Then $\mathrm{e}^{nx} \in A^{\downarrow}$ if $n \geqslant 1$. Now for (ii), let $a \in A$. Then $1/a \prec 1$, so $-a'/a^2 = (1/a)' \prec 1$, and thus $0 < a' < a^2$. This yields $(A')^{\downarrow} \subseteq \operatorname{sq}(A)^{\downarrow} = A^{\downarrow}$. Suppose in addition $a \geqslant \mathrm{e}^x$, so $a^\dagger \succcurlyeq 1$. If $a^\dagger \succ 1$, then $a < a'$, and if $a^\dagger \asymp 1$, then [ADH, 9.1.11] yields $n \geqslant 1$ with $a \leqslant \mathrm{e}^{nx}$, and taking $b \in A$ with $b \geqslant \mathrm{e}^{(n+1)x} \succ \mathrm{e}^{nx}$ we get $b' > (\mathrm{e}^{nx})' \geqslant \mathrm{e}^{nx} \geqslant a$. Thus $A^{\downarrow} \subseteq (A')^{\downarrow}$.

As to (iii), let $a \in A$, $b \in H$, and $b' = a$. Then $b > \mathbb{R}$, even $b \succ x$. Moreover, $0 < b' = a < b^2$, so $\sqrt{a} < b$. Thus $A^{\downarrow} = (\sqrt{A})^{\downarrow} \subseteq (\int A)^{\downarrow}$. Next, assume also $a = b' \succ \mathrm{e}^x$. Then $b \succ \mathrm{e}^x$, since H is asymptotic, so $a/b = b^\dagger \succcurlyeq 1$, hence $b \preccurlyeq a \prec a^2$ and thus $b < a^2$. This yields $(\int A)^{\downarrow} \subseteq \operatorname{sq}(A)^{\downarrow} = A^{\downarrow}$. $\square$

Lemma 1.13. *Let $B \subseteq H$, $B > \mathrm{e}^x$, and assume B, $\sqrt{B}$ are coinitial. Then:*

(i) *B and $B' := \{b' : b \in B\}$ are coinitial;*
(ii) *B and $\int B := \{a \in H : a' \in B\}$ are coinitial;*
(iii) *B^{-1} and $-\int B^{-1} := \{-g : g \in H^{\prec 1},\, g' \in B^{-1}\}$ are cofinal.*

Proof. Since B, $\sqrt{B}$ are coinitial, so are B, $\frac{1}{2}B$, by Lemma 1.10. Thus B, $\mathbb{R}^{>}B$ are coinitial. Let $b \in B$. Then $b \succ \sqrt{b} > \mathrm{e}^x$, so $\beta := vb < 0$ gives $\beta^\dagger \leqslant 0$, hence $\beta' \leqslant \beta$, and thus $b' \geqslant b$. Also $\beta^\dagger = o(\beta)$ by [ADH, 9.2.10], so $\beta < \frac{1}{2}\beta + \beta^\dagger = (\frac{1}{2}\beta)'$ and thus $b \succ (\sqrt{b})' \succcurlyeq d'$ for some $d \in B$. This proves (i).

For (ii), let $a \in H$ and $a' = b \in B$. Then $a > \mathbb{R}$, and also $a \succ \mathrm{e}^x$, since $a \preccurlyeq \mathrm{e}^x$ gives $b = a' \preccurlyeq \mathrm{e}^x$, a contradiction. Hence $\alpha^\dagger \leqslant 0$ for $\alpha := va$, so $\alpha \geqslant \alpha' = \beta := vb$, which gives $a \preccurlyeq b$, and thus $a \leqslant b^2$. Since $b \in B$ was arbitrary and B and $\operatorname{sq}(B)$ are coinitial by Lemma 1.8, this shows that every element of B is $\geqslant a$ for some $a \in \int B$. With a and b as above we also have $\alpha < \beta/2$, so $a > \sqrt{b}$. This proves (ii).

As to (iii), let $b \in B$, $\beta := vb$, and $g \in H^{\prec 1}$ with $g' = b^{-1}$, so $g < 0$, and for $\gamma := vg$ we have $-\beta = \gamma + \gamma^\dagger$. We have $\sqrt{b} > \mathrm{e}^x$, so $b^{-1} < \mathrm{e}^{-2x}$. Claim: $\gamma^\dagger \leqslant 0$. If this claim does not hold, then $0 < \gamma^\dagger < v(x^{-2})$, so $g \succ \mathrm{e}^{-x}$, and $g^\dagger \succ x^{-2} \succ \mathrm{e}^{-x}$, and thus $b^{-1} = g' \succ \mathrm{e}^{-2x}$, a contradiction. Now $\gamma^\dagger \leqslant 0$ gives $-\beta \leqslant \gamma$, hence $b^{-1} = |b^{-1}| \succcurlyeq |g| = -g$, and thus $-g \leqslant d$ for some $d \in B^{-1}$. From $\gamma^\dagger = (-\gamma)^\dagger \leqslant 0$ we get $\gamma^\dagger = o(\gamma)$ by [ADH, 9.2.10], hence $-2\beta > \gamma$, and thus $b^{-2} < -g$. It remains to use that B^{-1}, $\operatorname{sq}(B^{-1})$ are cofinal. $\square$

2. Analytic Preliminaries

In this section a, b, c, s, t range over $\mathbb{R}$.

Constructing smooth functions. We prove here some facts about smooth functions needed later. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be the $\mathcal{C}^\infty$-function of [14, (8.12), Exercise 2(a)]. It is defined by

$$\rho(t) \ := \ \exp\!\left(-\frac{1}{(1+t)^2} - \frac{1}{(1-t)^2}\right) \text{ if } -1 < t < 1, \quad \rho(t) := 0 \text{ if } t \leqslant -1 \text{ or } t \geqslant 1.$$

Thus $\rho(t) > 0$ for $-1 < t < 1$, ρ is even, and $\rho(0) = \mathrm{e}^{-2}$. (See Figure 1.)
For any subset I of $\mathbb{R}$ and $r \in \mathbb{N} \cup \{\infty, \omega\}$ we define $\mathcal{C}^r(I)$ to be the set of $f\colon I \to \mathbb{R}$ for which $f = g|_I$ for some $\mathcal{C}^r$-function $g\colon U \to \mathbb{R}$ with U an open neighborhood of I in $\mathbb{R}$; instead of "$f \in \mathcal{C}^r(I)$" we also write "$f\colon I \to \mathbb{R}$ is a $\mathcal{C}^r$-function" or "$f\colon I \to \mathbb{R}$ is of class $\mathcal{C}^r$". We use this mainly for sets $I = [a, b]$ with $a < b$ and sets $I = [a, \infty)$. As in [7, Section 5.2] we denote $\mathcal{C}^r[a, +\infty)$ by $\mathcal{C}^r_a$, and $\mathcal{C}_a := \mathcal{C}^0_a$.

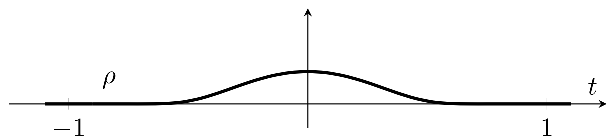

FIGURE 1. Sketch of ρ

Lemma 2.1. *There is a $\mathcal{C}^\infty$-function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha = 0$ on $[-\infty, 0]$, α is strictly increasing on $[0,1]$, and $\alpha = 1$ on $[1, +\infty)$.*

Proof. One can take $\alpha(t) := c^{-1}\int_{-\infty}^{t}\rho(2s-1)ds$ where $c := \int_{-\infty}^{\infty}\rho(2s-1)ds$. $\square$

Lemma 2.2. *Let $\theta\colon [a,\infty) \to \mathbb{R}^>$ be continuous. Then there exists a decreasing $\mathcal{C}^\infty$-function $\zeta\colon [a,\infty) \to \mathbb{R}^>$ such that $\theta(t) > \zeta(t)$ and $\zeta'(t) > -1$ for all $t \geqslant a$.*

Proof. Replacing θ by the function $t \mapsto \min_{a\leqslant s\leqslant t}\min\big(\theta(s),1\big)\colon [a,\infty) \to \mathbb{R}^>$ we arrange that θ is decreasing and $0 \leqslant \theta \leqslant 1$ on $[a,\infty)$. Next we follow Exercise 2 of [14, (8.12)], taking the convolution with ρ; in other words, we extend θ to all of $\mathbb{R}$ by setting $\theta(t) = 0$ for $t < a$, and then define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) := \int_{-\infty}^{\infty}\theta(s)\rho(t-s)\,ds = \int_{-\infty}^{\infty}\theta(t-s)\rho(s)\,ds.$$

Instead of $-\infty$, ∞ we can take in the left integral any real bounds $c \leqslant t-1$, $d \geqslant t+1$, and in the right integral any real bounds $c \leqslant -1$, $d \geqslant 1$. As in that exercise one shows that f is of class $\mathcal{C}^\infty$ (in fact, $f^{(p)}(t) = \int_{-\infty}^{\infty}\theta(s)\rho^{(p)}(t-s)ds$ for all $p \in \mathbb{N}$ and all t) and decreasing on $[a+1,\infty)$. For $t \geqslant a+1$ we have

$$0 < f(t) \;=\; \int_{-1}^{1}\theta(t-s)\rho(s)\,ds \;\leqslant\; 2\,\mathrm{e}^{-2}\,\theta(t-1) \;<\; \theta(t-1).$$

Using $\rho'(s) \geqslant 0$ for $-1 \leqslant s \leqslant 0$ and $\rho'(s) \leqslant 0$ for $0 \leqslant s \leqslant 1$, we obtain for all t,

$$f'(t) \;=\; \int_{-1}^{1}\theta(t-s)\rho'(s)\,ds \;\geqslant\; \int_{0}^{1}\theta(t-s)\rho'(s)\,ds \;\geqslant\; \int_{0}^{1}\rho'(s)\,ds \;=\; -\,\mathrm{e}^{-2} \;>\; -1.$$

Thus $\zeta\colon [a,\infty) \to \mathbb{R}^>$ defined by $\zeta(t) := f(t+1)$ has the desired property. $\square$

Lemma 2.3. *Let $a < b$ and $\phi, \zeta \in \mathcal{C}^\infty[a,b]$ be such that $\phi(a) = \zeta(a)$ and $\phi < \zeta$ on $(a,b]$, and let real numbers c_n be given with $\phi(b) < c_0 < \zeta(b)$. Then there exists a function $\theta \in \mathcal{C}^\infty[a,b]$ such that $\theta^{(n)}(a) = \phi^{(n)}(a)$ for all n, $\phi < \theta < \zeta$ on $(a,b]$, and $\theta^{(n)}(b) = c_n$ for all n.* (See Figure 2.)

Proof. By subtracting ϕ throughout we arrange $\phi = 0$. A result due to E. Borel [14, Exercise 4(a), p. 192] yields a function $\beta \in \mathcal{C}^\infty[a,b]$ such that $\beta^{(n)}(b) = c_n$ for all n. Take $\delta \in (0, b-a)$ with $\delta < \beta < \zeta - \delta$ on $[b-\delta, b]$, and then $\alpha \in \mathcal{C}^\infty[a,b]$ such that

- $\alpha = 0$ on $[a, b-\delta]$,
- α is strictly increasing on $[b-\delta, b-\frac{1}{2}\delta]$, and
- $\alpha = 1$ on $[b-\frac{1}{2}\delta, b]$.

Take $\varepsilon \in (0, \frac{1}{2})$ such that $\varepsilon\zeta < \delta$ on $[b-\delta, b]$, and take $\gamma \in \mathcal{C}^\infty[a,b]$ such that

- $\gamma^{(n)}(a) = 0$ for all n,
- $0 < \gamma < \varepsilon$ on $(a, b-\frac{1}{2}\delta)$, and

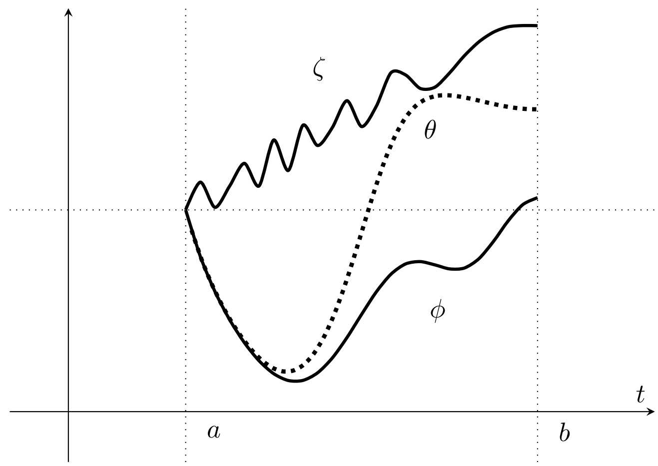

FIGURE 2. Sketch of ϕ, θ, ζ in Lemma 2.3

- $\gamma = 0$ on $[b - \frac{1}{2}\delta, b]$.

Then the function $\theta := \alpha\beta + \gamma\zeta$ has the desired properties. $\square$

Lemma 2.4. *Let $a < b$, $f, g \in \mathcal{C}[a,b]$, and $f < g$ on $[a,b]$. Then there are $a_0 < a_1 < \cdots < a_n$ with $a_0 = a$, $a_n = b$, and a function $\phi\colon [a,b] \to \mathbb{R}$ such that*

(i) *$f < \phi < g$ on $[a,b]$,*
(ii) *$\phi(a) = \frac{1}{2}\big(f(a) + g(a)\big)$ and $\phi(b) = \frac{1}{2}\big(f(b) + g(b)\big)$, and*
(iii) *for $i = 0, \dots, n-1$, the restriction of ϕ to $[a_i, a_{i+1}]$ is the restriction of an affine function $\mathbb{R} \to \mathbb{R}$.*

Proof. Let $\varepsilon := \frac{1}{2}\min\big\{g(t) - f(t) : t \in [a,b]\big\}$, so $\varepsilon > 0$. Choose $n \geqslant 1$ such that for all $s, t \in [a,b]$ with $|s - t| \leqslant \delta := \frac{b-a}{n}$ we have $|f(s) - f(t)|, |g(s) - g(t)| < \varepsilon$. For $i = 0, \dots, n$ set $a_i := a + i\delta$, and for $i = 0, \dots, n-1$ take affine $\phi_i\colon \mathbb{R} \to \mathbb{R}$ with $\phi_i(a_i) = \frac{1}{2}\big(f(a_i) + g(a_i)\big)$ and $\phi_i(a_{i+1}) = \frac{1}{2}\big(f(a_{i+1}) + g(a_{i+1})\big)$. It suffices to show that $f < \phi_i < g$ on $[a_i, a_{i+1}]$ for $i = 0, \dots, n-1$. For such i and $s, t \in [a_i, a_{i+1}]$,

$$f(t) < \varepsilon + f(s) \leqslant \tfrac{1}{2}\big(f(s) + g(s)\big) \leqslant -\varepsilon + g(s) < g(t),$$

in particular, $f(t) < \phi_i(a_i), \phi_i(a_{i+1}) < g(t)$. Since $\phi_i(a_i) \leqslant \phi_i(t) \leqslant \phi_i(a_{i+1})$ or $\phi_i(a_{i+1}) \leqslant \phi_i(t) \leqslant \phi_i(a_i)$, we are done. $\square$

Lemma 2.5. *Let $f, g \in \mathcal{C}_a$ be such that $f < g$ on $[a, +\infty)$. Then there exists a function $y \in \mathcal{C}_a^\infty$ such that $f < y < g$ on $[a, +\infty)$.*

Proof. Lemma 2.4 yields a piecewise affine intermediary ϕ, more precisely, a strictly increasing sequence (a_n) in $\mathbb{R}$ with $a_0 = a$ and $a_n \to +\infty$ as $n \to \infty$ and a $\phi \in \mathcal{C}_a$ such that for each n the restriction of ϕ to $[a_n, a_{n+1}]$ is the restriction of an affine function $\mathbb{R} \to \mathbb{R}$, and such that $f < \phi < g$ on $[a, +\infty)$. This reduces the problem of constructing y to proving the next lemma. $\square$

Lemma 2.6. *Let $a < b < c$ and $\phi, \theta \in \mathcal{C}^\infty[a,c]$ be such that $\phi(b) = \theta(b)$, and let $0 < \varepsilon < b-a, c-b$. Then there exists $y \in \mathcal{C}^\infty[a,c]$ such that*

$$\begin{aligned} y(t) &= \phi(t) \text{ for } a \leqslant t \leqslant b-\varepsilon, & |y(t)-\phi(t)| &< \varepsilon \text{ for } b-\varepsilon \leqslant t \leqslant b, \\ y(t) &= \theta(t) \text{ for } b+\varepsilon \leqslant t \leqslant c, & |y(t)-\theta(t)| &< \varepsilon \text{ for } b \leqslant t \leqslant b+\varepsilon. \end{aligned}$$

Proof. Take $0 < \delta < \varepsilon$ such that $|\phi - \theta| \leqslant \varepsilon/2$ on $[b-\delta, b+\delta]$. Next, take $\beta \in \mathcal{C}^\infty[a,c]$ such that

- $\beta = 0$ on $[a, b-\delta]$,
- $0 \leqslant \beta \leqslant 1$ on $[b-\delta, b+\delta]$, and
- $\beta = 1$ on $[b+\delta, c]$.

Then $y := (1-\beta)\phi + \beta\theta$ has the desired property. □

Lemma 2.7. *For each n, let $f_n, g_n \in \mathcal{C}$ be such that $f_n \leqslant f_{n+1}$, $g_{n+1} \leqslant g_n$, and $f_n <_{\mathrm{e}} g_n$. Then there exists $\phi \in \mathcal{C}^\infty$ such that $f_n <_{\mathrm{e}} \phi <_{\mathrm{e}} g_n$ for each n.*

Proof. Take for each n representatives of f_n and g_n in $\mathcal{C}_0$, denoted also by f_n and g_n, such that $f_n < g_n$ on $[0,\infty)$. Next, take a strictly increasing sequence (a_n) of real numbers $\geqslant 0$ with $a_n \to \infty$ as $n \to \infty$, such that $f_n \leqslant f_{n+1}$ and $g_n \geqslant g_{n+1}$ on $[a_n, \infty)$, and take continuous functions $\alpha_n, \beta_n \colon [0,\infty) \to [0,1]$ with $\alpha_n(a_n) = 1$, $\alpha_n(a_{n+1}) = 0$ and $\alpha_n + \beta_n = 1$ on $[a_n, a_{n+1}]$. Let $f, g \colon [0,\infty) \to \mathbb{R}$ be given by

$$\begin{aligned} f &= f_0 \text{ on } [0, a_0], \quad & f &= \alpha_n f_n + \beta_n f_{n+1} \text{ on } [a_n, a_{n+1}], \\ g &= g_0 \text{ on } [0, a_0], \quad & g &= \alpha_n g_n + \beta_n g_{n+1} \text{ on } [a_n, a_{n+1}], \end{aligned}$$

so f, g are continuous, $f_n \leqslant f$ and $g \leqslant g_n$ on $[a_n, \infty)$, and $f < g$ on $[0,\infty)$. (See Figure 3.) Now Lemma 2.5 gives $\phi \in \mathcal{C}_0^\infty$ such that $f < \phi < g$ on $[0,\infty)$, and then its germ at $+\infty$, denoted also by ϕ, satisfies $f_n <_{\mathrm{e}} \phi <_{\mathrm{e}} g_n$ for all n. □

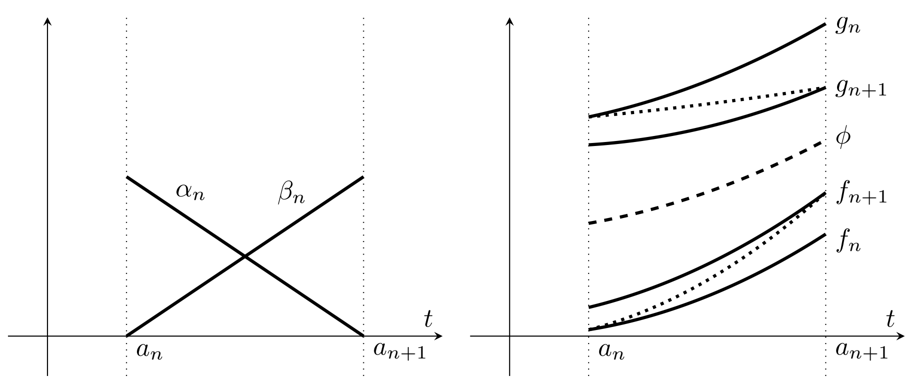

FIGURE 3. Constructing ϕ in the proof of Lemma 2.7

Corollary 2.8. *Let H be a Hausdorff field and A, B nonempty countable subsets of H with $A < B$. Then there exists $\phi \in \mathcal{C}^\infty$ such that $A <_{\mathrm{e}} \phi <_{\mathrm{e}} B$.*

Proof. Take an increasing and cofinal sequence (f_n) in A and a decreasing coinitial sequence (g_n) in B, and apply the previous lemma. □

We shall also use the following variant of Lemma 2.3:

Lemma 2.9. *Let $a < b$, $f, g \in \mathcal{C}[a,b]$, and $c_n, d_n \in \mathbb{R}$ for $n = 0, 1, 2, \dots$ be such that $f(a) < c_0 < g(a)$, $f < g$ on $[a,b]$, and $f(b) < d_0 < g(b)$. Then there exists $y \in \mathcal{C}^\infty[a,b]$ with $f < y < g$ on $[a,b]$ and $y^{(n)}(a) = c_n$, $y^{(n)}(b) = d_n$ for all n.*

Proof. Take $\varepsilon > 0$ such that $f(a) + \varepsilon < c_0$, $f + \varepsilon < g$ on $[a,b]$, and $f(b) + \varepsilon < d_0$. Lemma 2.5 gives $\phi \in \mathcal{C}^\infty[a,b]$ with $f < \phi < f + \varepsilon$ on $[a,b]$, and so replacing f by ϕ and then subtracting ϕ throughout (replacing g by $g - \phi$ and c_n, d_n by $c_n - \phi^{(n)}(a)$, $d_n - \phi^{(n)}(b)$, respectively) we arrange $f = 0$.

Borel's result gives $\alpha, \beta \in \mathcal{C}^\infty[a,b]$ with $\alpha^{(n)}(a) = c_n$ and $\beta^{(n)}(b) = d_n$ for all n. Take a real number $M > 0$ such that $|\alpha|, |\beta| \leqslant M$ on $[a,b]$. Take "small" real numbers $\eta_1, \eta_2 > 0$ such that $a + 2\eta_1 < b - 2\eta_1$, $2M\eta_1 < \eta_2$, and $2M\eta_1 + \eta_2 < g$ on $[a,b]$. Take $\gamma, \delta \in \mathcal{C}^\infty[a,b]$ such that

- $\gamma = 1$ on $[a, a + \eta_1]$,
- γ is decreasing on $[a + \eta_1, a + 2\eta_1]$,
- $\gamma = \eta_1$ on $[a + 2\eta_1, b - 2\eta_1]$,
- γ is decreasing on $[b - 2\eta_1, b - \eta_1]$, and
- $\gamma = 0$ on $[b - \eta_1, b]$,

and δ behaves similarly in the opposite direction:

- $\delta = 0$ on $[a, a + \eta_1]$,
- δ is increasing on $[a + \eta_1, a + 2\eta_1]$,
- $\delta = \eta_1$ on $[a + 2\eta_1, b - 2\eta_1]$,
- δ is increasing on $[b - 2\eta_1, b - \eta_1]$, and
- $\delta = 1$ on $[b - \eta_1, b]$.

Finally, take $\theta \in \mathcal{C}^\infty[a,b]$ such that

- $\theta = 0$ on $[a, a + \eta_1]$,
- θ is increasing on $[a + \eta_1, a + 2\eta_1]$,
- $\theta = \eta_2$ on $[a + 2\eta_1, b - 2\eta_1]$,
- θ is decreasing on $[b - 2\eta_1, b - \eta_1]$, and
- $\theta = 0$ on $[b - \eta_1, b]$.

(See Figure 4.) Then $y := \gamma\alpha + \delta\beta + \theta$ has the desired property, provided η_1, η_2 are sufficiently small. $\square$

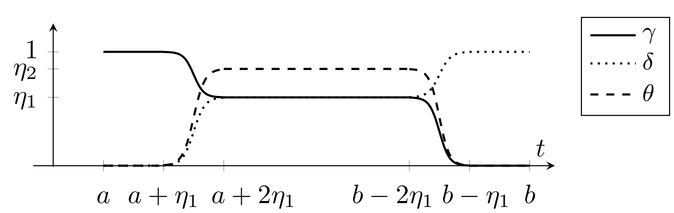

FIGURE 4. The functions γ, δ, θ

Constructing infinite sums. The next lemma follows from [14, (8.6.4)]:

Lemma 2.10. *Let (f_n) be a sequence of functions in $\mathcal{C}^1_a$ such that $f_n(a) \to c$ as $n \to \infty$, for some $c \in \mathbb{R}$. Suppose also that (f_n') converges to $g \in \mathcal{C}_a$, uniformly on $[a,b]$ for every $b > a$. Then (f_n) converges to a function $f \in \mathcal{C}^1_a$, uniformly on $[a,b]$ for every $b > a$, with $f' = g$.*

We use this for infinite series where the f_n are the partial sums, and with higher derivatives where the assumptions allow us to apply the lemma inductively. We shall also need a slight twist, where instead of the derivation $\partial\colon \mathcal{C}_a^1 \to \mathcal{C}_a$ we use $\derive\colon \mathcal{C}_a^1 \to \mathcal{C}_a$, with $\derive := \phi^{-1}\partial$, $\phi \in (\mathcal{C}_a)^\times$:

Lemma 2.11. *Let (f_n) be a sequence of functions in $\mathcal{C}_a^1$ such that $f_n(a) \to c$ as $n \to \infty$, for some c. Suppose also that $(\derive f_n)$ converges to $g \in \mathcal{C}_a$, uniformly on $[a,b]$ for every $b > a$. Then (f_n) converges to a function $f \in \mathcal{C}_a^1$, uniformly on $[a,b]$ for every $b > a$, with $\derive f = g$.*

This follows from the previous lemma in view of $\partial = \phi\derive$. Induction on m yields:

Corollary 2.12. *Let $m \geqslant 1$, $\phi \in (\mathcal{C}_a^{m-1})^\times$, and $\derive := \phi^{-1}\partial\colon \mathcal{C}_a^1 \to \mathcal{C}_a$. Then $\derive$ maps $\mathcal{C}_a^j$ into $\mathcal{C}_a^{j-1}$ for $j = 1,\dots,m$. Let (f_n) be a sequence of functions in $\mathcal{C}_a^m$ such that for $k = 0,\dots,m$ the series $\sum_{n=0}^\infty \derive^k f_n$ converges to $g_k \in \mathcal{C}_a$, uniformly on $[a,b]$ for every $b > a$. Then for $k = 0,\dots,m$ we have $g_k \in \mathcal{C}_a^{m-k}$ and $f := g_0 \in \mathcal{C}_a^m$ satisfies $\derive^k f = g_k$.*

This corollary and the following results on infinite sums will be used in the next two sections. Let $a \in \mathbb{R}$, and for $i = 0,1,2,\dots$, let a continuous function $f_i\colon [a,\infty) \to \mathbb{R}$ be given, and set $M_i^n := \max\limits_{a \leqslant t \leqslant a+n} |f_i(t)|$, so

$$0 \ \leqslant\ M_i^0 \ \leqslant\ M_i^1 \ \leqslant\ M_i^2 \ \leqslant\ \cdots.$$

Suppose the real numbers $\varepsilon_i > 0$ are such that $\sum_i \varepsilon_i M_i^n < \infty$ for every n. Then $\sum_i \varepsilon_i f_i$ converges uniformly on each set $[a, a+n]$, and so this sum defines a continuous function on $[a,\infty)$. We can certainly take real numbers $\varepsilon_i > 0$ such that $\sum_i \varepsilon_i M_i^i < \infty$, and then we do indeed have for every n that $\sum_i \varepsilon_i M_i^n < \infty$, since

$$\sum_i \varepsilon_i M_i^n \ =\ \sum_{i=0}^n \varepsilon_i M_i^n + \sum_{i>n} \varepsilon_i M_i^n \ \leqslant\ \sum_{i=0}^n \varepsilon_i M_i^n + \sum_{i>n} \varepsilon_i M_i^i.$$

Thus there exist ε_i as in the hypothesis of the next lemma. *In the rest of this subsection we assume that for every i we have $f_i \geqslant 0$ on $[a,\infty)$ and $f_i \prec f_{i+1}$ in $\mathcal{C}$.*

Lemma 2.13. *Let the reals $\varepsilon_i > 0$ be such that $\sum_i \varepsilon_i f_i$ converges to a function $f\colon [a,\infty) \to \mathbb{R}$, uniformly on each compact subset of $[a,\infty)$. Then $f \succ f_n$ (in $\mathcal{C}$) for all n. If all f_i are increasing, then so is $f = \sum_i \varepsilon_i f_i$.*

Proof. Note that $\sum_i \varepsilon_i f_i \geqslant \varepsilon_{n+1} f_{n+1} \succ f_n$. □

Lemma 2.14. *Let for each n a continuous function $g_n\colon [a,\infty) \to \mathbb{R}^>$ be given such that $f_i \prec g_n$, for all i and n. Then there exist reals $\varepsilon_i > 0$ for which $\sum_i \varepsilon_i f_i$ converges to a function $f\colon [a,\infty) \to \mathbb{R}$, uniformly on each compact subset of $[a,\infty)$, such that $f \leqslant g_n$ in $\mathcal{C}$ for all n.*

Proof. For the moment we just consider one continuous function $g\colon [a,\infty) \to \mathbb{R}^>$ with $f_i \prec g$ for all i. Then we pick the $\varepsilon_i > 0$ so small that $\sum_i \varepsilon_i M_i^i < \infty$ and $\varepsilon_i f_i \leqslant g/2^{i+1}$. This results in $f := \sum_i \varepsilon_i f_i \leqslant g$. Let a second continuous function $h\colon [a,\infty) \to \mathbb{R}^>$ be given with $f_i \prec h$ for all i. Take $b \geqslant a$ such that $\varepsilon_0 f_0 \leqslant h/2$ on $[b,\infty)$, and next decrease, if necessary, the ε_i with $i \geqslant 1$ so that $\varepsilon_i f_i \leqslant h/2^{i+1}$ on $[b,\infty)$ for the new values of ε_i. This results in $f \leqslant h$ on $[b,\infty)$ for the new f; note that we did not change ε_0. Starting with $g = g_0$ we apply this procedure successively to $g_1, g_2, \dots$ in the role of h: we recursively pick $b_1, b_2, \dots \geqslant a$, decreasing only the ε_i

for $i \geqslant n$ when dealing with g_n, $n \geqslant 1$. Then at the end we have not only $f \leqslant g_0$ on $[a, \infty)$, but also $f \leqslant g_n$ on $[b_n, \infty)$, for all $n \geqslant 1$ simultaneously. $\square$

Note that if in Lemma 2.14 we have $g_0 \succ g_1 \succ g_2 \succ \cdots$, then $f \prec g_n$ for all n. Lemmas 2.13 and 2.14 are more precise versions of results of du Bois-Reymond [10] and Hadamard [17, §19], respectively; cf. [19, Chapter II].

Assume next that the f_i are of class $\mathcal{C}^\infty$. Then we set

$$M_i^n \ := \max_{j \leqslant n,\ a \leqslant t \leqslant a+n} \bigl|f_i^{(j)}(t)\bigr|.$$

Again, $0 \leqslant M_i^0 \leqslant M_i^1 \leqslant M_i^2 \leqslant \cdots$. Taking the $\varepsilon_i > 0$ such that $\sum_i \varepsilon_i M_i^i < \infty$, we have $\sum_i \varepsilon_i M_i^n < \infty$ for every n, as before. Hence $\sum_i \varepsilon_i f_i$ converges, say to the continuous function $f \colon [a, \infty) \to \mathbb{R}^{\geqslant}$, uniformly on each $[a, a+n]$. Also $\sum_i \varepsilon_i f_i^{(j)}$ converges for every j to a continuous function $f^{(j)} \colon [a, \infty) \to \mathbb{R}$, uniformly on each $[a, a+n]$. An easy induction on j shows that f is in fact of class $\mathcal{C}^j$ with $f^{(j)}$ as its jth derivative, as suggested by the notation. Thus f is of class $\mathcal{C}^\infty$.

Useful inequalities in constructing Hardy fields. The lemmas in this subsection will be used in Sections 6 and 9.

Lemma 2.15. *Let $F, G \in \mathcal{C}_a^1$ satisfy $F'(t) \leqslant G'(t)$ for all $t \geqslant a$. Then there is a real constant c such that $F < G + c$ on $[a, \infty)$.*

Proof. The function $F - G$ is continuous and decreasing, hence on $[a, \infty)$ we have $F - G \leqslant F(a) - G(a) < c := F(a) - G(a) + 1$. $\square$

Here is a useful multiplicative version:

Lemma 2.16. *Let $F, G \colon [a, +\infty) \to \mathbb{R}^{>}$ of class $\mathcal{C}^1$ be such that $F^\dagger \leqslant G^\dagger$ on $[a, \infty)$. Then there is $c \in \mathbb{R}^{>}$ such that $F < cG$ on $[a, \infty)$.*

Proof. We have $F^\dagger = (\log F)'$ and $G^\dagger = (\log G)'$, so Lemma 2.15 yields $d \in \mathbb{R}$ with $\log F < d + \log G$ on $[a, \infty)$, and thus $F < c\,G$ on $[a, \infty)$ for $c := \mathrm{e}^d \in \mathbb{R}^{>}$. $\square$

Lemma 2.17. *Suppose $f \in \mathcal{C}$ lies in a Hardy field. Then the germ $f(x+1)$ satisfies:*

(i) $f(x) - x >_{\mathrm{e}} \mathbb{R} \implies f(x) + 1 <_{\mathrm{e}} f(x+1)$;
(ii) $0 < f(x) \succ \mathrm{e}^x \implies f(x) <_{\mathrm{e}} f(x+1)/2$;
(iii) $0 < f(x) \prec \mathrm{e}^{-x} \implies f(x) - f(x+1) >_{\mathrm{e}} f(x)/2$.

Proof. For (i), assume $f = x + g$ with $g \in \mathcal{C}$ and $g > \mathbb{R}$. Then g lies in a Hardy field, so g is eventually strictly increasing, hence $g = g(x) <_{\mathrm{e}} g(x+1)$, and thus

$$f(x) + 1 \ = \ x + 1 + g \ <_{\mathrm{e}} \ x + 1 + g(x+1) \ = \ f(x+1).$$

Next, assume $0 < f \succ \mathrm{e}^x$. Then $(\log f) - x >_{\mathrm{e}} \mathbb{R}$, so $\bigl(\log f(x)\bigr) + 1 <_{\mathrm{e}} \log f(x+1)$ by (i), and thus $\mathrm{e}\, f(x) <_{\mathrm{e}} f(x+1)$, hence $f(x) <_{\mathrm{e}} f(x+1)/\,\mathrm{e} <_{\mathrm{e}} f(x+1)/2$. As to (iii), assume $0 < f(x) \prec \mathrm{e}^{-x}$. Applying (ii) to f^{-1} gives $f(x) >_{\mathrm{e}} 2f(x+1)$, so $0 <_{\mathrm{e}} f(x+1) <_{\mathrm{e}} f(x)/2$, and thus $f(x) - f(x+1) >_{\mathrm{e}} f(x)/2$. $\square$

3. Pseudoconvergence in Hardy Fields

Let H be a Hardy field. Let a sequence

$$f_0 \succ f_1 \succ f_2 \succ \dots$$

in $H^>$ be given. Then we have the pc-sequence (F_i) in H, with $F_i := f_0 + \cdots + f_i$. Our aim in this section and the next is to show:

Theorem 3.1. *(F_i) pseudoconverges in some Hardy field extension of H.*

In view of Lemma 1.2 this has the following consequence:

Corollary 3.2. *Every pc-sequence of countable length in a maximal Hardy field has a pseudolimit in that Hardy field.*

Towards the proof of Theorem 3.1 we first recall from [7, Sections 5.1, 5.3] the following. Let $\ell \in \mathcal{C}^{<\infty}$ be such that $\ell > \mathbb{R}$ and $\ell' \in H$. Then ℓ lies in a Hardy field extension of H, $\phi := \ell' \in H^>$ is active in H, and the compositional inverse $\ell^{\text{inv}} > \mathbb{R}$ of ℓ yields an isomorphism $f \mapsto f^\circ := f \circ \ell^{\text{inv}} : (\mathcal{C}^{<\infty})^\phi \to \mathcal{C}^{<\infty}$ of differential rings that maps H onto the Hardy field $H^\circ := H \circ \ell^{\text{inv}}$; moreover, $f_1 \prec f_2 \Leftrightarrow f_1 \circ \ell \prec f_2 \circ \ell$, for all $f_1, f_2 \in \mathcal{C}^{<\infty}$. Thus (F_i°) is a pc-sequence in H°, and we have:

Lemma 3.3. *(F_i) pseudoconverges in some Hardy field extension of H if and only if (F_i°) pseudoconverges in some Hardy field extension of H°.*

We can also use [7, Theorem 6.7.22] to pass to an extension and arrange that $H \supseteq \mathbb{R}$ and H is closed. Then the following lemma is relevant.

Lemma 3.4. *Let $H \supseteq \mathbb{R}$ be closed. Suppose (F_i) has no pseudolimit in H, and let any element $F \in \mathcal{C}^{<\infty}$ be given. Then the following are equivalent:*

(i) *F generates a Hardy field $H\langle F\rangle$ over H with $F_i \rightsquigarrow F$;*
(ii) *for all k, m with $k < m$ and active $\phi \in H^>$ we have*

$$\delta^k\left(\frac{F - F_m}{f_m}\right) \preccurlyeq 1 \text{ in } \mathcal{C}^{<\infty}$$

where $\delta := \phi^{-1}\partial$ is construed as a derivation of $\mathcal{C}^{<\infty}$.

Proof. Assume (i). Then for all k, m and active $\phi \in H^>$ we have $(F - F_m)/f_m \prec 1$, and thus $\delta^k\big(\frac{F-F_m}{f_m}\big) \prec 1$. This proves (i) $\Rightarrow$ (ii). For (ii) $\Rightarrow$ (i), assume (ii). For $k = 0$ we get $F - F_m \preccurlyeq f_m$ for all $m \geqslant 1$. Let $P \in H\{Y\}^{\neq}$. Now (F_m) is of d-transcendental type over H by [ADH, 11.4, 14.0.2], so we have $m_0 \geqslant \operatorname{order}(P)$ in $\mathbb{N}^{\geqslant 1}$ such that $\operatorname{ndeg}_{\preccurlyeq f_{m+1}} P_{+F_m} = 0$ for all $m \geqslant m_0$, by [ADH, 11.4.11, 11.4.12]. Using $P_{+F_{m+1}} = (P_{+F_m})_{+f_{m+1}}$ and [ADH, 11.2.7] we obtain $\operatorname{ndeg}_{\preccurlyeq f_{m+1}} P_{+F_{m+1}} = 0$ for all $m \geqslant m_0$. Thus for $m_1 := m_0 + 1$ and $Q := P_{+F_{m_1}, \times f_{m_1}}$ we have an active $\phi_0 \in H^>$ with $\operatorname{ddeg} Q^\phi = 0$ for all active $\phi \preccurlyeq \phi_0$ in $H^>$. This gives $h \in H^\times$ such that, with $\boldsymbol{j}$ ranging over $\mathbb{N}^{m_0}$ and $Q_{\boldsymbol{j}}^{\phi_0} := (Q^{\phi_0})_{\boldsymbol{j}}$,

$$Q^{\phi_0}(Y) \;=\; h + \sum_{|\boldsymbol{j}|\neq 0} Q_{\boldsymbol{j}}^{\phi_0} Y^{\boldsymbol{j}}, \qquad Q_{\boldsymbol{j}}^{\phi_0} \prec h \text{ for } |\boldsymbol{j}| \neq 0.$$

Thus with $G := (F - F_{m_1})/f_{m_1}$ we have $G \preccurlyeq 1$ and $F = F_{m_1} + f_{m_1} G$, so

$$P(F) \;=\; Q^{\phi_0}(G) \;=\; h + \sum_{|\boldsymbol{j}|\neq 0} Q_{\boldsymbol{j}}^{\phi_0} G^{\boldsymbol{j}}$$

where the factors $G^{\boldsymbol{j}}$ are evaluated in $\mathcal{C}^{<\infty}$ using the derivation $\derivdelta = \phi_0^{-1}\partial$, and so $G^{\boldsymbol{j}} \preccurlyeq 1$ for $|\boldsymbol{j}| \neq 0$, by (ii). Hence $P(F) \sim h$. This yields (i). $\square$

Corollary 3.5. *In Lemma 3.4 we can replace* (ii) *by any of the two variants below:*

(ii)* *for all $m > k$ and active $\phi_0 \in H$ there is an active $\phi \preccurlyeq \phi_0$ in $H^>$ such that*

$$\delta^k\left(\frac{F - F_m}{f_m}\right) \preccurlyeq 1 \text{ in } \mathcal{C}^{<\infty}$$

where $\delta := \phi^{-1}\partial$ is construed as a derivation of $\mathcal{C}^{<\infty}$.

(ii)** *for all $m_0 \geqslant 1$ and active $\phi_0 \in H$ there is an active $\phi \preccurlyeq \phi_0$ in $H^>$ and an $m \geqslant m_0$ such that for $k = 1, \dots, m_0$,*

$$\delta^k\left(\frac{F - F_m}{f_m}\right) \preccurlyeq 1 \text{ in } \mathcal{C}^{<\infty}, \text{ with } \delta := \phi^{-1}\partial.$$

Proof. For (ii)** $\Rightarrow$ (i), assume (ii)** and take m_0, Q, ϕ_0 as in the proof of (ii) $\Rightarrow$ (i), and set $Q_m := P_{+F_{m+1}, \times f_{m+1}}$. For any $m \geqslant m_0$ and active $\phi \preccurlyeq \phi_0$ in $H^>$ we have $\operatorname{ddeg} Q^\phi = 0$, so $\operatorname{ddeg} Q_m^\phi = 0$ by [ADH, 6.6.12]. Now (ii) gives active $\phi \preccurlyeq \phi_0$ in $H^>$ and $m \geqslant m_0$ such that for $k = 1, \dots, m_0$ we have $\delta^k\big(\frac{F - F_m}{f_m}\big) \preccurlyeq 1$ in $\mathcal{C}^{<\infty}$, with $\delta := \phi^{-1}\partial$. In view of $\operatorname{ddeg} Q_m^\phi = 0$, the last part of the proof of the lemma with ϕ_0, Q replaced by ϕ, Q_m, and G replaced by $G_m := (F - F_{m+1})/f_{m+1}$, but $\boldsymbol{j}$ still ranging over $\mathbb{N}^{m_0}$, goes through, and yields the desired conclusion. $\square$

Rather than Lemma 3.4 we shall use in what follows the implications (ii)* $\Rightarrow$ (i) and (ii)** $\Rightarrow$ (i) that are implicit in the proof of that lemma, as we saw.

Expressing the powers δ^k in terms of ∂. To facilitate the use of Lemma 3.4 and its variants we shall express δ^k in terms of ∂. Let R be any differential ring with derivation ∂. Then $f \in R$ gives rise to a derivation $\delta := f\partial$ on the underlying ring of R. For $k \geqslant 1$, $0 \leqslant j \leqslant k$, we define $G_j^k(Y) \in \mathbb{Q}\{Y\} \subseteq R\{Y\}$ by recursion:

- $G_0^k = 0$,
- $G_k^k = Y^k$,
- $G_j^{k+1} = Y\big(\partial(G_j^k) + G_{j-1}^k\big)$ for $1 \leqslant j \leqslant k$.

(See also [ADH, 5.7].) For the additive operators ∂ and δ on the underlying ring R this recursion gives:

$$\delta^k \;=\; \sum_{j=1}^{k} G_j^k(f)\partial^j \qquad (k \geqslant 1).$$

For $1 \leqslant j \leqslant k$ the differential polynomial $G_j^k(Y)$ is homogeneous of degree k and of order $\leqslant k$, so we have a differential polynomial $R_j^k(Z) \in \mathbb{Q}\{Z\}$ of order $< k$ and depending only on j and k such that $G_j^k(f) = f^k R_j^k(f^\dagger)$ for all $f \in R^\times$. (See also [7, Section 5.3].) For $g \in R$, $\phi \in R^\times$, $\delta = \phi^{-1}\partial$, this gives

$$\delta^k(g) \;=\; \phi^{-k}\sum_{j=1}^{k} R_j^k(-\phi^\dagger)g^{(j)} \;\text{ with } g^{(j)} := \partial^j(g) \qquad (k \geqslant 1). \tag{3.1}$$

Given $a \in \mathbb{R}$, the identity (3.1) also holds for $g \in \mathcal{C}_a^k$ and $\phi \in (\mathcal{C}_a^k)^\times$, where δ^k and the ∂^j for $j \leqslant k$ are construed in the obvious way as maps $\mathcal{C}_a^k \to \mathcal{C}_a$.

For use in the next section we add the following observation:

Lemma 3.6. *Let $g \in H$ be active and $g \preccurlyeq h \in H$, and suppose $f \in \mathcal{C}^{<\infty}$ satisfies $(g^{-1}\partial)^k(f) \prec 1$ for $k = 0, \dots, m$. Then also $(h^{-1}\partial)^k(f) \prec 1$ for $k = 0, \dots, m$.*

Proof. Set $u := g/h \in H^{\preccurlyeq 1}$, $\delta_g := g^{-1}\partial$ and $\delta_h := h^{-1}\partial$. Then $\delta_h = u\delta_g$, as derivations on H and on $\mathcal{C}^{<\infty}$. For $k \geqslant 1$ we have by an earlier identity

$$\delta_h^k(f) \;=\; \sum_{j=1}^{k} G_j^k(u)\delta_g^j(f) \tag{3.2}$$

where each $G_j^k(u)$ is evaluated according to the small derivation δ_g on the asymptotic field H, and thus $G_j^k(u) \preccurlyeq 1$. This gives the desired result. □

Remark 3.7. For later use we note that the identity (3.2) also holds for $1 \leqslant k \leqslant m$, $f, g \in \mathcal{C}_a^m$, $h \in (\mathcal{C}_a^m)^\times$, with $a \in \mathbb{R}$ and $u := g/h$ (an element of $\mathcal{C}_a^m$), and where $\delta_g := g^{-1}\partial$, $\delta_h := h^{-1}\partial$ are taken as derivations $\mathcal{C}_a^j \to \mathcal{C}_a^{j-1}$, for $j = 1, \dots, m$, and each G_j^k is evaluated according to δ_g.

Bump functions. In this subsection t ranges over $\mathbb{R}$. From Lemma 2.1 we obtain an increasing $\mathcal{C}^\infty$-function $\alpha\colon \mathbb{R} \to \mathbb{R}$ with $\alpha(t) = 0$ for $t \leqslant 0$ and $\alpha(t) = 1$ for $t \geqslant 1$, and below we fix such an α. (Cf. Figure 5.)

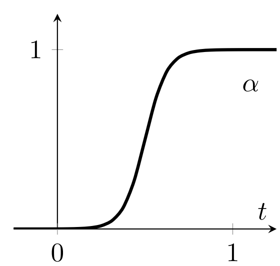

FIGURE 5. The bump function α

For each n we take a real constant C_n such that $1 \leqslant C_0 \leqslant C_1 \leqslant C_2 \leqslant \cdots$ and

$$|\alpha^{(n)}(t)| \;\leqslant\; C_n \quad \text{for all } n \text{ and } t. \tag{3.3}$$

For reals $a < b$ we define the increasing $\mathcal{C}^\infty$-function $\alpha_{a,b}\colon \mathbb{R} \to \mathbb{R}$ by

$$\alpha_{a,b}(t) \;:=\; \alpha\left(\frac{t-a}{b-a}\right), \tag{3.4}$$

so $\alpha_{a,b}(t) = 0$ for $t \leqslant a$ and $\alpha_{a,b}(t) = 1$ for $t \geqslant b$. Also,

$$\left|\alpha_{a,b}^{(m)}(t)\right| \;\leqslant\; \frac{C_m}{(b-a)^m} \quad \text{for all } m \text{ and } t. \tag{3.5}$$

Constructing F^*. We go back to our Hardy field H (not necessarily ω-free or newtonian) and its elements f_n and $F_n := f_0 + \cdots + f_n$, and in the rest of this section t ranges over $\mathbb{R}^{\geqslant 1}$. First, we take for each n a continuous function $\mathbb{R}^{\geqslant 1} \to \mathbb{R}$ that represents the germ f_n, to be denoted also by f_n, such that $f_n(t) > 0$ and $f_{n+1}(t) \leqslant f_n(t)/2$ for all t: first choose the function f_0, then f_1, next f_2, and so on.

For each n we fix an $a_n \in \mathbb{R}^{\geqslant 1}$ such that $f_0, \dots, f_n$ are of class $\mathcal{C}^n$ on $[a_n, +\infty)$. Next, let $c_0 < c_1 < c_2 < \cdots$ be real numbers $\geqslant 1$ with $c_n \to \infty$ as $n \to \infty$. We define $\alpha_n\colon \mathbb{R}^{\geqslant 1} \to \mathbb{R}$ by $\alpha_n(t) := \alpha_{c_n,c_{n+1}}(t)$, so α_n is an increasing $\mathcal{C}^\infty$-function with $\alpha_n(t) = 0$ for $t \leqslant c_n$ and $\alpha_n(t) = 1$ for $t \geqslant c_{n+1}$, and we set

$$f_n^* \ := \ \alpha_n f_n \ : \mathbb{R}^{\geqslant 1} \to \mathbb{R}^{\geqslant 0},$$

so $f_n^*(t) = 0$ for $t \leqslant c_n$ and $f_n^*(t) = f_n(t)$ for $t \geqslant c_{n+1}$. Thus f_n and f_n^* have the same germ at $+\infty$, and we still have $f_{n+1}^*(t) \leqslant f_n^*(t)/2$ for all n and t. As we saw in the subsection on Hausdorff fields in Section 1, this yields a continuous function

$$F^* \ := \ \sum_{n=0}^{\infty} f_n^* \ : \mathbb{R}^{\geqslant 1} \to \mathbb{R}$$

such that $F^* - F_n \prec f_n$ (in $\mathcal{C}$) for all n.

Lemma 3.8. *Assume $c_n > a_0, \dots, a_n$ for all n. Then for all n, f_n^* is of class $\mathcal{C}^n$, and F^* is of class $\mathcal{C}^n$ on $[c_n, +\infty)$. So the germ of F^* at $+\infty$ belongs to $\mathcal{C}^{<\infty}$.*

Proof. We have $f_n^* = 0$ on $[1, c_n]$, and f_n is of class $\mathcal{C}^n$ on $[a_n, +\infty)$, so f_n^* is of class $\mathcal{C}^n$. For $t \leqslant c_{n+1}$ we have $F^*(t) = f_0^*(t) + \cdots + f_n^*(t)$, so F^* is of class $\mathcal{C}^n$ on $[a_n, c_{n+1}]$. Likewise, F^* is of class $\mathcal{C}^{n+1}$ on $[a_{n+1}, c_{n+2}]$. Continuing this way we obtain that F^* is of class $\mathcal{C}^n$ on $[c_n, +\infty)$. □

We consider the a_n as fixed, with the $c_n > a_0, \dots, a_n$ to be chosen as needed later. We set $\varepsilon_m := f_{m+1}/f_m$, so $0 < \varepsilon_m(t) \leqslant 1/2$ for all t and $\varepsilon_m \prec 1$ in H. For any $n > m$ we also set $\varepsilon_{n,m} := f_n/f_m$, so $\varepsilon_{n,m}$ is of class $\mathcal{C}^n$ on $[a_n, +\infty)$ and $0 < \varepsilon_{n,m}(t) \leqslant 2\varepsilon_m(t)/2^{n-m}$ for all t. Then for $n > m$ we have $\varepsilon_{n,m} \preccurlyeq \varepsilon_m \prec 1$ in H, so $\varepsilon_{n,m}^{(k)} \prec x^{-1}$ for all $k \geqslant 1$: use [ADH, 9.1.9(iv), 9.1.10], first passing from H to a Hardy field extension containing x if necessary.

Proof in the fluent case. This case of Theorem 3.1 is as follows:

Proposition 3.9. *Suppose $\varepsilon \in H^{\prec 1}$ is such that $f_{i+1}/f_i \prec \varepsilon$ for all i. Then (F_i) pseudoconverges in some Hardy field extension of H.*

Proof. By passing to a suitable extension we first arrange that $H \supseteq \mathbb{R}$ is closed. Then $\ell := -\log|\varepsilon| \in H^{>\mathbb{R}}$, $|\varepsilon| = \mathrm{e}^{-\ell}$, so $|\varepsilon| \circ \ell^{\mathrm{inv}} = \mathrm{e}^{-x}$, and thus $(f_{i+1}/f_i) \circ \ell^{\mathrm{inv}} \prec \mathrm{e}^{-x}$ for all i. Replacing H by $H \circ \ell^{\mathrm{inv}}$ and renaming we can arrange in view of Lemma 3.3 that $f_{i+1}/f_i \prec \mathrm{e}^{-x}$ for all i, and this is what we assume below. Note that then $(f_n/f_m)^{(k)} \prec \mathrm{e}^{-x}$ for all $n > m$ and all k. We also assume that (F_i) does not pseudoconverge in H.

As in the subsection on constructing F^* we choose for each germ f_n a continuous representative $\mathbb{R}^{\geqslant 1} \to \mathbb{R}$, also to be denoted by f_n, and real numbers $a_0, a_1, a_2, \dots$, $c_0, c_1, c_2, \dots$ with the properties listed there, and with $c_n > a_0, \dots, a_n$ for all n: the a_n are fixed and the c_n are adjustable. As in that subsection this yields an $F^* = \sum_{n=0}^{\infty} f_n^* \in \mathcal{C}^{<\infty}$ with $f_n^* = \alpha_n f_n$ for all n, and we introduce the functions $\varepsilon_m = f_{m+1}/f_m$ and $\varepsilon_{n,m} = f_n/f_m$ for $n > m$. For each n we take $b_n \geqslant a_0, \dots, a_n$ such that for all k, m with $0 \leqslant k \leqslant m < n$,

$$t \geqslant b_n \implies \left|\varepsilon_{n,m}^{(k)}(t)\right| \leqslant \frac{\mathrm{e}^{-t}}{2^{n-m}}.$$

Next, with the C_n from (3.3), take $c_n > b_0, \dots, b_n$ such that $c_{n+1} - c_n \geqslant C_n$ (so $c_n \to \infty$). For $m \leqslant n$ we have $C_m \leqslant C_n$, so $|\alpha_n^{(m)}(t)| \leqslant 1$ for all t, by (3.5).

Let $\phi \in H^>$ be active and $\phi \prec 1$, so $\phi \succ x^{-2}$ and $\phi^\dagger \preccurlyeq x^{-1}$. This gives a derivation $\mathfrak{d} := \phi^{-1}\partial$ on $\mathcal{C}^{<\infty}$. Now we use (ii)* $\Rightarrow$ (i) from Corollary 3.5. It tells us that for F^* to generate a Hardy field over H with $F_i \rightsquigarrow F^*$, it is enough to establish that the present assumptions on ϕ imply:

Claim: *for all* $m > k$ *we have* $\delta^k\big(\frac{F^* - F_m}{f_m}\big) \prec 1$ *in* $\mathcal{C}^{<\infty}$.

Let $m \geqslant 1$ be given and represent the germ ϕ by a $\mathcal{C}^m$-function $\mathbb{R}^{\geqslant 1} \to \mathbb{R}^{>}$, to be denoted also by ϕ. For $1 \leqslant j < k$, the coefficient of Y^k in the homogeneous differential polynomial G^k_j of degree k is 0, so $G^k_j(1) = R^k_j(0) = 0$. Also $R^k_k = 1$ for $k \geqslant 1$. Hence we can take a real number $c^*_m \geqslant c_m$ such that for all $t \geqslant c^*_m$,

$$\phi(t) \geqslant t^{-2}, \quad \left|R^k_j(-\phi^\dagger)(t)\right| \leqslant 1 \text{ whenever } 1 \leqslant j \leqslant k \leqslant m.$$

Then (3.1) yields

$$\left|\delta^k\left(\frac{f^*_n}{f_m}\right)(t)\right| \leqslant t^{2k}\sum_{j=1}^{k}\left|\left(\frac{f^*_n}{f_m}\right)^{(j)}(t)\right| \qquad (1 \leqslant k \leqslant m < n,\ t \geqslant c^*_m).$$

Here it is relevant that the f^*_n/f_m are of class $\mathcal{C}^m$ on $[c_m, +\infty)$ for the derivatives to exist. Next, for $1 \leqslant j \leqslant m < n$ and $t \geqslant c^*_m$,

$$\begin{aligned}\left|\left(\frac{f^*_n}{f_m}\right)^{(j)}(t)\right| &\leqslant \sum_{i=0}^{j}\binom{j}{i}\left|\alpha_n^{(j-i)}(t)\cdot\varepsilon^{(i)}_{n,m}(t)\right| \\ &\leqslant \sum_{i=0}^{j}\binom{j}{i}\frac{\mathrm{e}^{-t}}{2^{n-m}} = 2^j\frac{\mathrm{e}^{-t}}{2^{n-m}}.\end{aligned}$$

Combining this with the previous inequality we get

$$\left|\delta^k\left(\frac{f^*_n}{f_m}\right)(t)\right| \leqslant 2^{k+1}t^{2k}\frac{\mathrm{e}^{-t}}{2^{n-m}} \qquad (1 \leqslant k \leqslant m < n,\ t \geqslant c^*_m).$$

Now $F^*_m := f^*_0 + \cdots + f^*_m$ is of class $\mathcal{C}^m$ on $[c^*_m, \infty)$, so by Lemma 3.8 the function

$$\frac{F^* - F^*_m}{f_m} = \sum_{n=m+1}^{\infty}\frac{f^*_n}{f_m}$$

is of class $\mathcal{C}^m$ on $[c^*_m, \infty)$. Using also Corollary 2.12 we have for $t \geqslant c^*_m$,

$$\left|\delta^k\left(\frac{F^* - F^*_m}{f_m}\right)(t)\right| \leqslant 2^{k+1}t^{2k}\,\mathrm{e}^{-t} \qquad (1 \leqslant k \leqslant m).$$

Hence $\delta^k\left(\frac{F^* - F^*_m}{f_m}\right) \prec 1$ in $\mathcal{C}^{<\infty}$ for $1 \leqslant k \leqslant m$. As F^*_m and F_m are equal as germs in $\mathcal{C}^{<\infty}$, this proves the claim when $k \geqslant 1$. For $k = 0$, use that $F^* - F_n \prec f_n$ for all n. □

Corollary 3.10. *If* $H^{>\mathbb{R}}$ *has uncountable coinitiality, then* (F_i) *pseudoconverges in some Hardy field extension of* H.

Thus to prove Theorem 3.1 it would be enough to show that in every maximal Hardy field its set of positive infinite elements has uncountable coinitiality. However, we were not able to prove the latter directly, and so couldn't exploit this remark. Instead we refine in the next section the previous constructions in the remaining case where $H^{>\mathbb{R}}$ has countable coinitiality.

Remarks on $H^{>\mathbb{R}}$ having countable coinitiality. We show that the property of $H^{>\mathbb{R}}$ having countable coinitiality is fairly robust; this is not used later but has independent interest. More generally, in this subsection K is a pre-H-field with $\Gamma := v(K^\times) \neq \{0\}$. Note: $K^{>\mathcal{O}} = K^{>C}$ if K is an H-field. If K is ungrounded, then $\operatorname{ci}(K^{>\mathcal{O}}) = \operatorname{cf}(\Gamma^<) \geqslant \omega$, and $\operatorname{cf}(\Gamma^<) = \omega$ iff K has a logarithmic sequence (as defined in [ADH, 11.5]) of countable length. First we refine [7, Lemma 1.4.20]:

Lemma 3.11. *Suppose K is not $\uplambda$-free, and L is a Liouville closed* d*-algebraic H-field extension of K. Then L is $\upomega$-free with a logarithmic sequence of length ω, and $\Gamma^<$ is not cofinal in $\Gamma_L^<$.*

Proof. Suppose first that K is grounded. Let $K_\upomega$ be the $\upomega$-free pre-H-field extension of K introduced before [ADH, 11.7.16] (with K in place of F there), identified with a pre-H-subfield of L containing K as in the proof of [7, Lemma 1.4.18]. The sequence (f_n) constructed before [ADH, 11.7.16] is a logarithmic sequence in $K_\upomega$ with $\Gamma^< < v(f_n) < 0$ for all $n \geqslant 1$. By [7, Theorem 1.4.1], L is $\upomega$-free and $\Gamma^<_{K_\upomega}$ is cofinal in $\Gamma_L^<$, so (f_n) remains a logarithmic sequence in L, and $\Gamma^<$ is not cofinal in $\Gamma_L^<$. If K is not grounded we reduce to the grounded case by following the proofs of [7, Lemmas 1.4.18–1.4.20]. □

Next, let $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ be a pre-$\Lambda\Omega$-field with Newton-Liouville closure $\boldsymbol{K}^{\mathrm{nl}} = (K^{\mathrm{nl}}, \dots)$; see [ADH, 16.4]. Recall that K^{nl} is differentially algebraic over K. The following proposition is analogous to the characterizations of rational asymptotic integration and of $\uplambda$-freeness in [7, Propositions 1.4.8, 1.4.12]:

Proposition 3.12. *The following are equivalent:*

- (i) *K is $\upomega$-free;*
- (ii) *$\Gamma^<$ is cofinal in $\Gamma_L^<$ for every* d*-algebraic H-field extension L of K;*
- (iii) *$\Gamma^<$ is cofinal in $\Gamma^<_{K^{\mathrm{nl}}}$.*

Moreover, if K is not $\upomega$-free, then K^{nl} has a logarithmic sequence of length ω.

Proof. The implication (i) $\Rightarrow$ (ii) holds by [7, Theorem 1.4.1], and (ii) $\Rightarrow$ (iii) is clear. For the rest, note that if K is not $\uplambda$-free, then K^{nl} has a logarithmic sequence of length ω and $\Gamma^<$ is not cofinal in $\Gamma^<_{K^{\mathrm{nl}}}$, by Lemma 3.11. Suppose now that K is $\uplambda$-free but not $\upomega$-free. Then [ADH, 11.8.30] gives $\upomega \in K$ with $\omega\big(\Lambda(K)\big) < \upomega < \sigma\big(\Gamma(K)\big)$. By the proof of [ADH, 16.4.6], either $\Omega = \omega(K)^{\downarrow}$ or $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^{\uparrow}$. If $\Omega = \omega(K)^{\downarrow}$, then the proof of [ADH, 16.4.6] yields a $\upgamma \in K^{\mathrm{nl}}$ such that $\upgamma > 0$, $\sigma(\upgamma) = \upomega$, and the pre-$H$-subfield $K_\upgamma := K\langle\upgamma\rangle$ of K^{nl} has a gap. Replacing $\boldsymbol{K}$ by the pre-$\Lambda\Omega$-subfield $(K_\upgamma, \dots)$ of $\boldsymbol{K}^{\mathrm{nl}}$ we reduce to the case that K is not $\uplambda$-free. If $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^{\uparrow}$, then the proof of [ADH, 16.4.6] yields $\uplambda \in K^{\mathrm{nl}}$ such that $\omega(\uplambda) = \upomega$ and the pre-H-subfield $K_\uplambda := K\langle\uplambda\rangle$ of K^{nl} is not $\uplambda$-free, so we can argue as before, with $K_\uplambda$ in place of $K_\upgamma$. □

Now assume $H \supseteq \mathbb{R}$. Let M be a maximal Hardy field extension of H and H^{da} the d-closure of H in M, with canonical $\Lambda\Omega$-expansions $\boldsymbol{H}$, $\boldsymbol{H}^{\mathrm{da}}$, respectively; see [7, Sections 7.1, 7.2]. Then $\boldsymbol{H}^{\mathrm{da}}$ is a Newton-Liouville closure of $\boldsymbol{H}$, by [7, Corollary 7.2.17], and so the H-field H^{da} is closed. If M^* is also a maximal Hardy field extending H, then there is an H-field embedding $H^{\mathrm{da}} \to M^*$ over H whose image is the d-closure of H in M^*, by [ADH, 16.4.9]. By Proposition 3.12:

Corollary 3.13. *If H is $\upomega$-free, then $H^{>\mathbb{R}}$ is coinitial in $(H^{\mathrm{da}})^{>\mathbb{R}}$. If H is not $\upomega$-free, then H^{da} has a logarithmic sequence of length ω. In particular, if $H^{>\mathbb{R}}$ has countable coinitiality, then so does $(H^{\mathrm{da}})^{>\mathbb{R}}$.*

4. The Remaining Case

We keep the assumptions on H and (f_i) from the beginning of Section 3, and let t range over $\mathbb{R}^{\geqslant 1}$. For use in the "remaining case" we first derive bounds like those of clause (ii) in Lemma 3.4 for $\phi = 1$.

Useful bounds. We adopt the conventions and notations in the subsection on constructing F^* from the previous section; in particular, the a_n are fixed, and the c_n will be adjusted so as to get the desired bounds on certain derivatives of the functions f_n^*/f_m with $n > m$. For each n we take $b_n \geqslant a_0, \dots, a_n$ such that for all k, m with $1 \leqslant k \leqslant m < n$,

$$t \geqslant b_n \quad \Longrightarrow \quad \left|\varepsilon_{n,m}^{(k)}(t)\right| \leqslant \frac{t^{-1}}{2^{n-m}}.$$

Next, with the C_n from (3.3), we take for each n a $c_n > b_0, \dots, b_n$ with $c_{n+1} - c_n \geqslant C_n$ (so $c_n \to \infty$ as $n \to \infty$). Then $|\alpha_n^{(m)}(t)| \leqslant 1$ for all t whenever $m \leqslant n$, using that $C_m \leqslant C_n$ for such m, n. Note also that for $m \leqslant n$ the function f_n^*/f_m is of class $\mathcal{C}^n$ on its entire domain $[1, \infty)$ in view of $f_n^*(t) = 0$ for $t \leqslant c_n$.

Lemma 4.1. *For all k, m with $k \leqslant m$ we have*

$$\partial^k\left(\frac{F^* - F_m}{f_m}\right) \prec 1 \quad \text{in } \mathcal{C}^{<\infty}.$$

Proof. Let $1 \leqslant k \leqslant m < n$. From $f_n^*/f_m = \alpha_n \varepsilon_{n,m}$ we get for $t \geqslant c_n$,

$$\begin{aligned}\left|\left(\frac{f_n^*}{f_m}\right)^{(k)}(t)\right| &\leqslant \sum_{j=0}^{k}\binom{k}{j}\left|\alpha_n^{(k-j)}(t)\cdot\varepsilon_{n,m}^{(j)}(t)\right| \\ &\leqslant |\alpha_n^{(k)}(t)|\frac{2\varepsilon_m(t)}{2^{n-m}} + \sum_{j=1}^{k}\binom{k}{j}\frac{t^{-1}}{2^{n-m}} \leqslant \frac{2\varepsilon_m(t)}{2^{n-m}} + \frac{2^k t^{-1}}{2^{n-m}}.\end{aligned}$$

This also holds for $t < c_n$, since $(f_n^*/f_m)(t) = 0$ for such t. Now fix $m \geqslant 1$ and set $F_m^* := f_0^* + \dots + f_m^*$. By Corollary 2.12 the function

$$\frac{F^* - F_m^*}{f_m} = \sum_{n=m+1}^{\infty}\frac{f_n^*}{f_m}$$

is of class $\mathcal{C}^{m+1}$ on its entire domain $[1, \infty)$, and for all t,

$$\left|\partial^k\left(\frac{F^* - F_m^*}{f_m}\right)(t)\right| \leqslant 2\varepsilon_m(t) + 2^k t^{-1} \qquad (k = 1, \dots, m).$$

Hence $\partial^k\big(\frac{F^* - F_m^*}{f_m}\big) \prec 1$ in $\mathcal{C}^{<\infty}$ for $k = 1, \dots, m$. As F_m^* and F_m are equal as germs in $\mathcal{C}^{<\infty}$, this gives the desired result when $k \geqslant 1$. For $k = 0$, use that $F^* - F_n \prec f_n$ for all n. ∎

For later use we record the following consequence:

Corollary 4.2. *Let $\phi \in H^{>}$ be active, and $\mathfrak{d} := \phi^{-1}\partial$, as a derivation on $\mathcal{C}^{<\infty}$. Then there exists $F_\phi \in \mathcal{C}^{<\infty}$ such that for all k, m with $k \leqslant m$,*

$$\mathfrak{d}^k\left(\frac{F_\phi - F_m}{f_m}\right) \prec 1 \quad \text{in } \mathcal{C}^{<\infty}.$$

Proof. Take an ℓ in a Hardy field extension of H with $\ell' = \phi$; note that $\ell > \mathbb{R}$. The lemma above applied to the sequence $(f_i \circ \ell^{\text{inv}})$ in $H \circ \ell^{\text{inv}}$ yields $F^* \in \mathcal{C}^{<\infty}$ with $\partial^k\left(\frac{F^* - F_m \circ \ell^{\text{inv}}}{f_m \circ \ell^{\text{inv}}}\right) \prec 1$ in $\mathcal{C}^{<\infty}$ for all $m \geqslant k$. For $F_\phi := F^* \circ \ell \in \mathcal{C}^{<\infty}$ this gives the desired result. □

In view of Lemma 3.4, the problem is that F_ϕ depends on ϕ. The idea, to be carried out in the next subsections, is to show that for suitable ϕ_n and a kind of partition of unity (β_n) the infinite sum $\sum_n \beta_n F_{\phi_n}$ has the desired properties. In the previous section we proved Theorem 3.1 in the so-called fluent case, which includes the case that $H^{>\mathbb{R}}$ has uncountable coinitiality. The remaining case where $H^{>\mathbb{R}}$ has countable coinitiality will lead to the suitable ϕ_n and the partition of unity (β_n) that we alluded to. The a_n and b_n below are still real numbers but have little to do with the earlier a_n and b_n; reusing these symbols with another meaning simply reflects the limitations of the alphabet.

Towards constructing a good partition of unity. Until further notice the Hardy field $H \supseteq \mathbb{R}$ is Liouville closed and $H^{>\mathbb{R}}$ has countable coinitiality. It follows that there is a sequence (ϕ_n) of active elements in $H^{>}$ such that $\big(v(\phi_n)\big)$ is strictly increasing and cofinal in Ψ_H. Below we fix such a sequence (ϕ_n), and set $\mathfrak{d}_n := \phi_n^{-1}\partial$, a derivation on $\mathcal{C}^{<\infty}$. Then Corollary 4.2 provides for each n a $\Phi_n \in \mathcal{C}^{<\infty}$ such that for all k, m with $k \leqslant m$,

$$\mathfrak{d}_n^k\left(\frac{\Phi_n - F_m}{f_m}\right) \ \prec\ 1 \quad \text{in } \mathcal{C}^{<\infty},$$

and thus by Lemma 3.6, for all $k \leqslant m$ and all $i \leqslant n$,

$$\mathfrak{d}_i^k\left(\frac{\Phi_n - F_m}{f_m}\right) \ \prec\ 1 \quad \text{in } \mathcal{C}^{<\infty}.$$

We represent the germs ϕ_n, f_n, and Φ_n by $\mathcal{C}^n$-functions $\mathbb{R}^{\geqslant 1} \to \mathbb{R}^{>}$, denoted also by ϕ_n, f_n, and Φ_n. These functions ϕ_n, f_n and Φ_n are fixed in the rest of this section, and the notion of "admissible sequence" defined below is relative to these given sequences (ϕ_n), (f_n), (Φ_n). Suppose the real numbers $a_n \geqslant 1$ are such that:

(I) for each n, $f_0, \dots, f_n$ and $\phi_0, \dots, \phi_n$ are of class $\mathcal{C}^n$ on $[a_n, +\infty)$;

(II) for all i, k, m, n with $k \leqslant m \leqslant n$, $i \leqslant n$, and all $t \geqslant a_n$ we have

$$\left|\mathfrak{d}_i^k\left(\frac{\Phi_n - F_m}{f_m}\right)(t)\right| \ \leqslant\ 1.$$

Note that (II) makes sense in view of (I), and that (I) and (II) remain valid upon increasing all a_n. We have $\phi_n/\phi_i \prec 1$ in H for $i < n$ and $\phi_n/\phi_i = 1$ for $i = n$, and thus $\mathfrak{d}_n^k(\phi_n/\phi_i) \prec 1$ in H for $i \leqslant n$ and $k \geqslant 1$. Note also that $\mathfrak{d}_n^k(\phi_n/\phi_i)(t)$ is defined for $i, k \leqslant n$ and $t \geqslant a_n$, since ϕ_n/ϕ_i is of class $\mathcal{C}^n$ on $[a_n, +\infty)$ for $i \leqslant n$. Thus by taking the a_n large enough we can arrange in addition to (I) and (II):

(III) for all n and $i, k \leqslant n$ and all $t \geqslant a_n$ we have

$$|\mathfrak{d}_n^k(\phi_n/\phi_i)(t)| \ \leqslant\ 1.$$

An **admissible sequence** is a sequence $\big((a_n, b_n, \beta_n)\big)_{n\geqslant 0}$ of triples (a_n, b_n, β_n) such that:

(i) (a_n) is a strictly increasing sequence of real numbers $\geqslant 1$ with $a_n \to \infty$ as $n \to \infty$ for which (I), (II), (III) hold;

and such that for all n:

(ii) b_n is a real number with $a_n < b_n < a_{n+1}$;
(iii) β_n is a function $\mathbb{R}^{\geqslant 1} \to \mathbb{R}$ of class $\mathcal{C}^n$;
(iv) $\beta_n(t) = 0$ if $t \leqslant a_n$, β_n is increasing on $[a_n, b_n]$, $\beta_n(t) = 1$ if $b_n \leqslant t \leqslant a_{n+1}$, β_n is decreasing on $[a_{n+1}, b_{n+1}]$, and $\beta_n(t) = 0$ for $t \geqslant b_{n+1}$;
(v) $\beta_n + \beta_{n+1} = 1$ on $[a_{n+1}, b_{n+1}]$.

(See Figure 6.)

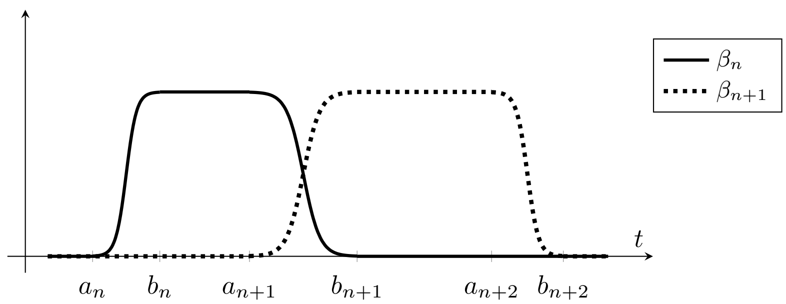

FIGURE 6. The functions β_n, β_{n+1}

In the rest of this subsection $\big((a_n, b_n, \beta_n)\big)$ *denotes an admissible sequence.* Note that $\operatorname{supp}\beta_n \subseteq [a_n, b_{n+1}]$ by (iv), and that (v) expresses the "partition of unity" requirement. The series $\sum_n \beta_n\Phi_n$ converges pointwise on $\mathbb{R}^{\geqslant 1}$ to a continuous function Φ such that on each segment $[b_n, a_{n+2}]$ we have

$$\beta_n + \beta_{n+1} \ = \ 1 \quad\text{and}\quad \Phi \ = \ \beta_n\Phi_n + \beta_{n+1}\Phi_{n+1},$$

so Φ is of class $\mathcal{C}^n$ on $[b_n, a_{n+2})$. Likewise, Φ is of class $\mathcal{C}^{n+1}$ on the set $(b_{n+1}, a_{n+3}]$, which overlaps the previous set. Continuing this way we see that Φ is of class $\mathcal{C}^n$ on $[b_n, +\infty)$, and thus the germ of Φ lies in $\mathcal{C}^{<\infty}$.

Lemma 4.3. *Suppose for all m and $i, k \leqslant m$ there is a positive constant $C = C(i, k, m)$ such that for all $n \geqslant m$,*

$$\left|\delta_i^k\left(\frac{\beta_n\Phi_n + \beta_{n+1}\Phi_{n+1} - F_m}{f_m}\right)\right| \ \leqslant\ C \quad \text{on} \quad [a_{n+1}, b_{n+1}].$$

Then for all m and $i, k \leqslant m$ we have

$$\delta_i^k\left(\frac{\Phi - F_m}{f_m}\right) \ \preccurlyeq\ 1 \qquad \text{in } \mathcal{C}^{<\infty}.$$

Proof. Let $i, k \leqslant m$, and take a $C \geqslant 1$ as in the hypothesis. Then for all $n \geqslant m$ we have $|\delta_i^k\big(\frac{\Phi - F_m}{f_m}\big)| \leqslant C$ on $[a_{n+1}, b_{n+1}]$, and also by (II) above,

$$\left|\delta_i^k\left(\frac{\Phi - F_m}{f_m}\right)\right| \leqslant 1 \text{ on } [b_n, a_{n+1}], \qquad \left|\delta_i^k\left(\frac{\Phi - F_m}{f_m}\right)\right| \leqslant 1 \text{ on } [b_{n+1}, a_{n+2}],$$

and thus $|\mathfrak{d}_i^k\big(\frac{\Phi-F_m}{f_m}\big)| \leqslant C$ on $[b_n, a_{n+2}]$. Taking the union over all $n \geqslant m$ we obtain

$$\left|\mathfrak{d}_i^k\left(\frac{\Phi - F_m}{f_m}\right)\right| \leqslant C \text{ on } [b_m, +\infty),$$

which gives the desired result. □

We didn't use (III) yet, but we need it for a further reduction:

Lemma 4.4. *Suppose for all m and $k \leqslant m$ there is a positive constant $C(k,m)$ such that for all $n \geqslant m$,*

$$\left|\mathfrak{d}_n^k\left(\frac{\beta_n\Phi_n + \beta_{n+1}\Phi_{n+1} - F_m}{f_m}\right)\right| \;\leqslant\; C(k,m) \quad \textit{on} \quad [a_{n+1}, b_{n+1}].$$

Then for all m and $i,k \leqslant m$ we have

$$\mathfrak{d}_i^k\left(\frac{\Phi - F_m}{f_m}\right) \;\preccurlyeq\; 1 \qquad \textit{in } \mathcal{C}^{<\infty}.$$

Proof. Let $C(m) := \max_{k\leqslant m} C(k,m)$ where the $C(k,m)$ are as in the hypothesis. Let $i,k \leqslant m \leqslant n$, and let F,g,h be the restrictions of $\frac{\beta_n\Phi_n+\beta_{n+1}\Phi_{n+1}-F_m}{f_m}$, ϕ_n, ϕ_i to $[a_{n+1}, +\infty)$, respectively; these functions are of class $\mathcal{C}^n$. For $j = 1,\dots,m$ we denote the derivations

$$f \mapsto g^{-1}f' \;:\; \mathcal{C}^j_{a_{n+1}} \to \mathcal{C}^{j-1}_{a_{n+1}}, \quad f \mapsto h^{-1}f' \;:\; \mathcal{C}^j_{a_{n+1}} \to \mathcal{C}^{j-1}_{a_{n+1}}$$

by $\mathfrak{d}_g$ and $\mathfrak{d}_h$, suppressing for convenience the dependence on j. Let $u := g/h \in \mathcal{C}^m_{a_{n+1}}$. Then Remark 3.7 gives for $f \in \mathcal{C}^m_{a_{n+1}}$:

$$\mathfrak{d}_h^k(f) \;=\; \sum_{j=0}^{k} G_j^k(u)\mathfrak{d}_g^j(f) \qquad (\text{with } G_0^0 := 1 \text{ to handle the case } k=0),$$

where G_j^k is evaluated according to the derivation $\mathfrak{d}_g$. By our hypothesis,

$$|\mathfrak{d}_g^j(F)| \leqslant C(m) \;\text{ on } [a_{n+1}, b_{n+1}],\; j = 0,\dots,k.$$

Now (III) provides a positive constant $B(m)$ depending on m but not on n, such that $|G_j^k(u)| \leqslant B(m)$ on $[a_{n+1}, b_{n+1}]$ for $j = 0,\dots,k$. Hence

$$|\mathfrak{d}_h^k(F)| \;\leqslant\; (k+1)B(m)C(m) \;\text{ on } [a_{n+1}, b_{n+1}],$$

and so the hypothesis of the previous lemma is satisfied. □

Corollary 4.5. *If H is closed, (F_i) has no pseudolimit in H, and the hypothesis of Lemma 3.4 is satisfied, then Φ generates a Hardy field over H and $F_i \rightsquigarrow \Phi$.*

Proof. Use the conclusion of Lemma 4.4, and the implication (ii)** ⇒ (i) (with Φ in the role of F) from Corollary 3.5. □

In order to make a further reduction, note that on $[a_{n+1}, b_{n+1}]$ we have $\beta_n = 1 - \beta_{n+1}$, and so, on $[a_{n+1}, b_{n+1}]$,

$$(*) \quad \frac{\beta_n\Phi_n + \beta_{n+1}\Phi_{n+1} - F_m}{f_m} \;=\; \beta_{n+1}\cdot\left(\frac{\Phi_{n+1}-\Phi_n}{f_m}\right) + \frac{\Phi_n - F_m}{f_m}$$
$$= \beta_{n+1}\cdot\left(\frac{\Phi_{n+1}-F_m}{f_m} - \frac{\Phi_n - F_m}{f_m}\right) + \frac{\Phi_n - F_m}{f_m}.$$

This leads to a further simplification:

Lemma 4.6. *If for all k there is a constant $B(k) > 0$ such that for all $n \geqslant k$, $|\delta_n^k(\beta_{n+1})| \leqslant B(k)$ on $[a_{n+1}, b_{n+1}]$, then the hypothesis of Lemma 4.4 is satisfied.*

Proof. To simplify notation, set $G_{n,m} := \frac{\Phi_n - F_m}{f_m}$, and let $k \leqslant m \leqslant n$. Then by $(*)$,

$$\delta_n^k\left(\frac{\beta_n\Phi_n + \beta_{n+1}\Phi_{n+1} - F_m}{f_m}\right) \;=\; \sum_{j=0}^{k}\binom{k}{j}\delta_n^j(\beta_{n+1})\delta_n^{k-j}(G_{n+1,m} - G_{n,m}) + \delta_n^k(G_{n,m})$$

on $[a_{n+1}, b_{n+1}]$. Suppose $B \in \mathbb{R}^{>}$ and $|\delta_n^j(\beta_{n+1})| \leqslant B$ on $[a_{n+1}, b_{n+1}]$ for $j = 0, \dots, k$. Then the above identity and (II) gives that on $[a_{n+1}, b_{n+1}]$,

$$\left|\delta_n^k\left(\frac{\beta_n\Phi_n + \beta_{n+1}\Phi_{n+1} - F_m}{f_m}\right)\right| \;\leqslant\; \left[\sum_{j=0}^{k}\binom{k}{j}\cdot B\cdot 2\right] + 1 \;=\; 2^{k+1}B + 1,$$

which gives the desired result. $\square$

Using composition. In this subsection we explore how we might arrange that our admissible sequence $\big((a_n, b_n, \beta_n)\big)$ satisfies the hypothesis of Lemma 4.6, and thus of Lemma 4.4. In the next subsection we then construct such a sequence.

By (iv) there is no problem for $k = 0$, since $0 \leqslant \beta_{n+1} \leqslant 1$. Assume $1 \leqslant k \leqslant n$ and set $\phi := \phi_n$, so $\delta := \phi^{-1}\partial = \delta_n$, and set $a := a_{n+1}$, $b := b_{n+1}$, $\beta := \beta_{n+1}$. We wish to bound $|\delta^k(\beta)|$ on $[a, b]$ by a positive constant that may depend on k but not on $n \geqslant k$. To achieve this goal we introduce the strictly increasing bijection $g\colon \mathbb{R}^{\geqslant 1} \to \mathbb{R}^{\geqslant 0}$ given by $g(r) = \int_1^r \phi(t)\,dt$, so $g \in \mathcal{C}_1^{n+1}$, $g' = \phi$, and g has as compositional inverse the strictly increasing bijection $g^{\text{inv}}\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 1}$ of class $\mathcal{C}^{n+1}$. Induction on $j \leqslant n+1$ gives $\delta^j\beta = (\beta \circ g^{\text{inv}})^{(j)} \circ g$ on $\mathbb{R}^{\geqslant 1}$. For $j = k$ this identity gives for any $B \in \mathbb{R}^{>}$ the equivalence

$$|\delta^k(\beta)| \leqslant B \text{ on } [a, b] \quad\Longleftrightarrow\quad |(\beta \circ g^{\text{inv}})^{(k)}| \;\leqslant\; B \text{ on } \big[g(a), g(b)\big]. \tag{4.1}$$

We shall arrange below that b is given in terms of a by $g(b) = g(a) + 1$ and that on $\big[g(a), g(b)\big]$ the function $\beta \circ g^{\text{inv}}$ equals $\alpha_{g(a),g(b)}$ with α the bump function from Section 3. In the next subsection we show that then for sufficiently fast growing (a_n) all our constraints are satisfied.

The construction. In view of the dependence of the function g on n in the story above we restore here indices, defining the strictly increasing bijection

$$g_n \;:\; \mathbb{R}^{\geqslant 1} \to \mathbb{R}^{\geqslant 0}, \qquad g_n(r) \;:=\; \int_1^r \phi_n(t)\,dt,$$

so $g_n \in \mathcal{C}_1^{n+1}$, $g_n' = \phi_n$, and g_n has as compositional inverse the strictly increasing bijection $g_n^{\text{inv}}\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 1}$ of class $\mathcal{C}^{n+1}$.

Next we take a strictly increasing sequence (a_n) of real numbers $\geqslant 1$ such that $a_n \to \infty$ and (I), (II), (III) hold, and such that for every n we have $b_{n+1} < a_{n+2}$ where the real number b_{n+1} is defined by $g_n(b_{n+1}) = g_n(a_{n+1}) + 1$. (It will be clear that there are such sequences.) With b_0 any real number satisfying $a_0 < b_0 < a_1$, we now have $a_n < b_n < a_{n+1}$ for all n. We define for each n the $\mathcal{C}^\infty$-function

$$\alpha_n := \alpha_{g_n(a_{n+1}),\, g_n(b_{n+1})}\colon \mathbb{R} \to \mathbb{R}.$$

The bump function α came with constants $C_k > 0$ such that $|\alpha^{(k)}| \leqslant C_k$ on $\mathbb{R}$, thus $|\alpha_n^{(k)}| \leqslant C_k$ on $\mathbb{R}$ for all k, n in view of $g_n(b_{n+1}) - g_n(a_{n+1}) = 1$ and (3.5). Since $\alpha_n\big(g_n(b_{n+1})\big) = 1 = 1 - \alpha_{n+1}\big(g_{n+1}(b_{n+1})\big)$ we can define $\beta_{n+1} \colon \mathbb{R}^{\geqslant 1} \to \mathbb{R}$ by

$$\beta_{n+1}(t) \;=\; \begin{cases} \alpha_n\big(g_n(t)\big) & \text{for } t \leqslant b_{n+1}, \\ 1 - \alpha_{n+1}\big(g_{n+1}(t)\big) & \text{for } t \geqslant b_{n+1}. \end{cases}$$

Then β_{n+1} is continuous. We also take a continuous function $\beta_0 \colon \mathbb{R}^{\geqslant 1} \to \mathbb{R}$ such that (iii), (iv), (v) hold for $n = 0$. We now have constructed a sequence $\big((a_n, b_n, \beta_n)\big)$ that satisfies conditions (i), (ii), (iv), and (v) (and (iii) for $n = 0$). In fact, it fulfills all our wishes:

Proposition 4.7. *The sequence $\big((a_n, b_n, \beta_n)\big)$ is admissible, and for all k and $n \geqslant k$ we have $|\boldsymbol{\delta}_n^k(\beta_{n+1})| \leqslant C_k$ on $[a_{n+1}, b_{n+1}]$.*

Proof. Clearly β_{n+1} is of class $\mathcal{C}^{n+1}$ on $\mathbb{R}^{\geqslant 1} \setminus \{b_{n+1}\}$. Now $\alpha_n \circ g_n = 1$ on $[b_{n+1}, \infty)$, so $(\alpha_n \circ g_n)^{(j)}(b_{n+1}) = 0$ for $j = 1, \dots, n+1$. Moreover, $\alpha_{n+1} \circ g_{n+1} = 0$ on $[1, a_{n+2}]$, so β_{n+1} is $\mathcal{C}^{n+1}$ on all of $\mathbb{R}^{\geqslant 1}$. Therefore condition (iii) is satisfied, and so $\big((a_n, b_n, \beta_n)\big)$ is admissible. The bound $|\boldsymbol{\delta}_n^k(\beta_{n+1}| \leqslant C_k$ on $[a_{n+1}, b_{n+1}]$ for $n \geqslant k$ is clear from $\beta_{n+1} \circ g_n^{\text{inv}} = \alpha_n$ on $\big[g_n(a_{n+1}), g_n(b_{n+1})\big]$ and the equivalence (4.1). $\square$

Finishing the proof of Theorem 3.1. As already mentioned we can use [7, Theorem 6.7.22] to pass to an extension and arrange that $H \supseteq \mathbb{R}$ is closed. If $H^{>\mathbb{R}}$ has uncountable coinitiality, we are done by Corollary 3.10. Suppose $H^{>\mathbb{R}}$ has countable coinitiality. Then we have an admissible sequence as in Proposition 4.7, and so by Lemma 4.6 and Corollary 4.5, if (F_i) has no pseudolimit in H, then Φ is a pseudolimit of (F_i) in a Hardy field extension of H. This concludes the proof.

Corollary 4.8. *Suppose H is a maximal Hardy field. Then $\operatorname{ci}(H^{>\mathbb{R}}) > \omega$.*

Proof. If $\operatorname{ci}(H^{>\mathbb{R}}) = \omega$, then H being $\boldsymbol{\lambda}$-free yields a divergent pc-sequence $(\boldsymbol{\lambda}_\rho)$ in H whose well-ordered index set has cofinality ω, contradicting Corollary 3.2. $\square$

5. Constructing Overhardian Germs

Our goal in this section is the following:

Theorem 5.1. *If $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and $\phi \in \mathcal{C}$, $\phi >_{\mathrm{e}} H$, then some $y \in \mathcal{C}^\infty$ with $y >_{\mathrm{e}} \phi$ generates a Hardy field $H\langle y \rangle$ over H.*

This is Sjödin's main result in [28], except that he considers only $\mathcal{C}^\infty$-Hardy fields. Our construction of y follows that of Sjödin, with the material organized so that much of it will also be useful in the next section where we fill more general gaps.

Boshernitzan [12, Theorems 1.1, 1.2] (see also [7, proof of Corollary 5.4.23]) showed that y in Theorem 5.1 can be taken in $\mathcal{C}^\omega$, using a result of Kneser [22] on solutions $E \in \mathcal{C}^\omega$ to the functional equation $\exp \circ E = E \circ (x+1)$. (Our follow-up paper will have a different argument that yields a $y \in \mathcal{C}^\omega$ in Theorem 5.1.)

We state here an easy consequence of Theorem 5.1:

Corollary 5.2. *If H is a maximal Hardy field, then $\operatorname{cf}(H) > \omega$, and thus*

$$\operatorname{ci}(H) \;=\; \operatorname{cf}(H^{<a}) \;=\; \operatorname{ci}(H^{>a}) \;>\; \omega \quad \textit{for all } a \in H.$$

Proof. If H is a maximal Hardy field with a strictly increasing cofinal sequence (h_n) in H, then Lemma 2.13 yields a $\phi \in \mathcal{C}$ such that $h_n <_{\mathrm{e}} \phi$ for all n, contradicting Theorem 5.1. For any Hausdorff field F and $a \in F$ we have $\mathrm{cf}(F) = \mathrm{ci}(F) = \mathrm{cf}(F^{<a}) = \mathrm{ci}(F^{>a})$ (use fractional linear transformations). □

This corollary yields Theorem A in the case where A or B is finite.

Lemmas on logarithmic derivatives. Let $f \in \mathcal{C}^1_a$. Note that if $f(t) > 0$ and $f'(t) > 0$ for all $t \geqslant a$, then f is strictly increasing, and thus $f(t) \geqslant f(a) > 0$ for all $t \geqslant a$. It is convenient to replace here f' by $f^\dagger$, noting that if $f(t) > 0$ for all $t \geqslant a$, then $f^\dagger(t)$ is defined for all $t \geqslant a$. Thus if $f(t) > 0$ and $f^\dagger(t) > 0$ for all $t \geqslant a$, then f is strictly increasing, and thus $f(t) \geqslant f(a) > 0$ for all $t \geqslant a$.

Lemma 5.3. *Let $f \in \mathcal{C}^2_a$. Assume that $f(t), f^\dagger(t), f^{\dagger\dagger}(t) > 0$ for all $t \geqslant a$. Then $f(t) \to +\infty$ as $t \to +\infty$.*

Proof. Applying the remark preceding the lemma to $f^\dagger$ in the role of f gives $f^\dagger(t) \geqslant f^\dagger(a)$ for $t \geqslant a$, so $f'(t) = f(t)f^\dagger(t) \geqslant f(a)f^\dagger(a)$ for $t \geqslant a$, where we apply that same remark also to f. Hence for $t \geqslant a$,

$$f(t) \;=\; f(a) + \int_a^t f'(s)\,ds \;\geqslant\; f(a) + (t-a)f(a)f^\dagger(a),$$

which gives the desired conclusion. □

Lemma 5.4. *Let $f \in \mathcal{C}^3_a$, and suppose that for all $t \geqslant a$,*

$$f(t) > 0, \quad f^\dagger(t) > 0, \quad f^{\dagger\dagger}(t) > 0, \quad \textit{and } f^\dagger(t) \;>\; f^{\dagger\dagger}(t) \;>\; f^{\dagger\dagger\dagger}(t).$$

Then $f(t)/f^\dagger(t) \to +\infty$ as $t \to +\infty$.

Proof. We have $(f/f^\dagger)^\dagger = f^\dagger - f^{\dagger\dagger}$, so $(f/f^\dagger)^\dagger(t) > 0$ for $t \geqslant a$. Also

$$\begin{aligned}(f/f^\dagger)^{\dagger\prime} \;&=\; (f^\dagger - f^{\dagger\dagger})' \;=\; f^{\dagger\prime} - f^{\dagger\dagger\prime}\\ &=\; f^{\dagger\dagger}f^\dagger - f^{\dagger\dagger\dagger}f^{\dagger\dagger} \;=\; f^{\dagger\dagger}(f^\dagger - f^{\dagger\dagger\dagger})\end{aligned}$$

and thus $(f/f^\dagger)^{\dagger\dagger}(t) > 0$ for all $t \geqslant a$. Applying Lemma 5.3 to $f/f^\dagger$ in the role of f now gives the desired result. □

Lemma 5.5. *Let $f \in \mathcal{C}^4_a$, and suppose that for all $t \geqslant a$,*

$$\begin{aligned}&f(t) > 0, \quad f^\dagger(t) > 0, \quad f^{\dagger\dagger}(t) > 0, \quad f^{\dagger\dagger\dagger}(t) > 0, \ \textit{and}\\ &f^\dagger(t) \;>\; f^{\dagger\dagger}(t) \;>\; f^{\dagger\dagger\dagger}(t) \;>\; f^{\dagger\dagger\dagger\dagger}(t).\end{aligned}$$

Then $f^\dagger(t) \to +\infty$ as $t \to +\infty$, and for every n, $f(t) > f^\dagger(t)^n$, eventually.

Proof. Applying Lemma 5.3 to $f^\dagger$ in the role of f gives $f^\dagger(t) \to +\infty$ as $t \to +\infty$. Applying Lemma 5.4 to $f^\dagger$ in the role of f gives $f^\dagger(t)/f^{\dagger\dagger}(t) \to +\infty$ as $t \to +\infty$. Let $n \geqslant 1$ and take $a_n \geqslant a$ such that $f^\dagger(t)/n > f^{\dagger\dagger}(t)$ for all $t \geqslant a_n$. So the assumptions of Lemma 5.4 are satisfied for a_n and the restriction of $f^{1/n}$ to $[a_n, +\infty)$ in the role of a and f, hence $f(t)^{1/n}/\big(f^\dagger(t)/n\big) \to +\infty$ as $t \to +\infty$, and thus $f(t)/f^\dagger(t)^n \to +\infty$ as $t \to +\infty$. □

Hardian and overhardian germs. Let $y \in \mathcal{C}^{<\infty}$. Following the terminology of [28] we say that y is **hardian** if y generates a Hardy field $\mathbb{Q}\langle y\rangle$.

Lemma 5.6. *If y is hardian and $y >_{\mathrm{e}} 0$, $y^\dagger >_{\mathrm{e}} 0$, then $y >_{\mathrm{e}} (y^\dagger)^n$ for all n.*

Proof. Suppose y is hardian and $y >_{\mathrm{e}} 0$, $y^\dagger >_{\mathrm{e}} 0$. The case $y \prec 1$ is impossible, since it would give $y^\dagger <_{\mathrm{e}} 0$. If $y \asymp 1$, then $y^\dagger \prec 1$, and we are done. If $y \succ 1$ and $y^\dagger \prec 1$, we are done. If $y \succ 1$ and $y^\dagger \succcurlyeq 1$, then $v(y^\dagger) = o(vy)$ by [ADH, 9.2.10], which gives the desired conclusion. $\square$

We set $y^{\langle 0\rangle} := y$, and inductively, if $y^{\langle i\rangle} \in \mathcal{C}^{<\infty}$ is defined and $y^{\langle i\rangle} \in (\mathcal{C}^{<\infty})^\times$ (so either $y^{\langle i\rangle} <_{\mathrm{e}} 0$ or $y^{\langle i\rangle} >_{\mathrm{e}} 0$), then $y^{\langle i+1\rangle} := (y^{\langle i\rangle})^\dagger$, and otherwise $y^{\langle i+1\rangle}$ is not defined. As in [28] we call y **overhardian** if for all i,

$$y^{\langle i\rangle} \text{ is defined}, \quad y^{\langle i\rangle} >_{\mathrm{e}} 0, \text{ and } y^{\langle i\rangle} >_{\mathrm{e}} y^{\langle i+1\rangle}.$$

If y is overhardian, then so is $y^\dagger$. By Lemma 5.5:

Corollary 5.7. *If y is overhardian, then for all i, n we have*

$$y^{\langle i\rangle} >_{\mathrm{e}} \mathbb{R}, \quad y^{\langle i\rangle} >_{\mathrm{e}} (y^{\langle i+1\rangle})^n.$$

Next we recall from [ADH, 4.3] that a differential polynomial $P(Y) \in K\{Y\}$ over a differential field K has a unique logarithmic decomposition

$$P(Y) \;=\; \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} Y^{\langle \boldsymbol{i}\rangle} \qquad (P_{\langle \boldsymbol{i}\rangle} \in K).$$

If K is a Hardy field and $y^{\langle i\rangle}$ is defined for all i, then we can substitute y for the indeterminate Y to get $P(y) = \sum_{\boldsymbol{i}} P_{\langle i\rangle} y^{\langle \boldsymbol{i}\rangle}$ in $\mathcal{C}^{<\infty}$, where of course

$$y^{\langle \boldsymbol{i}\rangle} := (y^{\langle 0\rangle})^{i_0} \cdots (y^{\langle r\rangle})^{i_r} \quad \text{for } \boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}.$$

Such a substitution is in particular possible if y is overhardian. Thus for overhardian y and $P \in \mathbb{R}\{Y\}^{\neq}$ we obtain $P(y) \in (\mathcal{C}^{<\infty})^\times$ from Corollary 5.7. Therefore:

Corollary 5.8. *If y is overhardian, then y is hardian.*

Lemma 5.9. *If y is overhardian, then $\log y \prec y^\dagger$.*

Proof. More generally, let y be hardian, $y >_{\mathrm{e}} \mathbb{R}$, $y^\dagger >_{\mathrm{e}} \mathbb{R}$, and $y^{\dagger\dagger} >_{\mathrm{e}} \mathbb{R}$; we claim that then $\log y \prec y^\dagger$. To prove this, take a Liouville closed Hardy field $H \supseteq \mathbb{R}$ with $y \in H$. Applying [ADH, 9.2.18] to $\alpha = vy$ in the asymptotic couple (Γ, ψ) of H gives $\log y \asymp y^\dagger / y^{\dagger\dagger} \prec y^\dagger$. $\square$

Given a Hardy field H, we say that a germ $y \in \mathcal{C}$ is **H-hardian** if y is contained in a Hardy field extension of H; see also [7, Section 5.3].

Corollary 5.10. *Suppose $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and $y >_{\mathrm{e}} H$. Then the following are equivalent:*

(i) *y is overhardian;*
(ii) *y is H-hardian;*
(iii) *y is hardian.*

Proof. From $\exp(H) \subseteq H$ and $y >_{\mathrm{e}} H$ we obtain $\log y >_{\mathrm{e}} H$. If y is overhardian, this gives by induction on n and Lemma 5.9 that $y^{\langle n\rangle} >_{\mathrm{e}} H$ for all n, and so $P(y) <_{\mathrm{e}} H$ or $P(y) >_{\mathrm{e}} H$ for all $P(Y) \in H\{Y\} \setminus H$, hence y is H-hardian. This proves (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is trivial. To show (iii) $\Rightarrow$ (i), assume (iii).

From $\log y >_{\mathrm{e}} H$ and $\exp(x^2) \in H$ we obtain $\log y \succ \exp(x^2)$. Working in a Hardy field containing y, $\log y$, x, and $\exp(x^2)$, we have $(\log y)' \succ \exp(x^2)'$, so

$$v(y^\dagger) \;<\; v\big(x\exp(x^2)\big) \;<\; v\big(\exp(x^2)\big) \;<\; 0,$$

hence $v(y^{\dagger\dagger}) \leqslant v\big(\exp(x^2)^\dagger\big) = v(x) < 0$, and thus $y^{\dagger\dagger} >_{\mathrm{e}} \mathbb{R}$. Hence $y^\dagger >_{\mathrm{e}} \log y >_{\mathrm{e}} H$ by the proof of Lemma 5.9. Since $y^\dagger$ is hardian, we can iterate this argument, which by induction shows that all $y^{\langle n\rangle}$ are defined and $>_{\mathrm{e}} H$. This yields (i). □

Corollary 5.10 combines [28, Theorems 3 and 4]; the implication (iii) ⇒ (ii) also follows from [11, Theorem 12.23].

Corollary 5.11. *If y is overhardian, then so is $\log y$. Moreover,*

$$y \text{ is overhardian} \iff y \text{ is hardian and } y >_{\mathrm{e}} \exp_n(x) \text{ for all } n.$$

Proof. Suppose y is overhardian. Then y is hardian and hence so is $\log y$. Moreover, $\log y >_{\mathrm{e}} \mathbb{R}$, and $\log y \asymp y^\dagger/y^{\dagger\dagger}$ by the proof of Lemma 5.9. Hence

$$(\log y)^\dagger \;\sim\; \big(y^\dagger/y^{\dagger\dagger}\big)^\dagger \;=\; y^{\dagger\dagger} - y^{\dagger\dagger\dagger} \;\sim\; y^{\dagger\dagger} \;=\; y^{\langle 2\rangle}.$$

Now an easy induction shows that all $(\log y)^{\langle n\rangle}$ are defined, and that for $n \geqslant 1$ we have $(\log y)^{\langle n\rangle} \sim y^{\langle n+1\rangle}$. This proves the first claim of the corollary. Also $x \prec y$, since $1 \prec y \preccurlyeq x$ would give $y^\dagger \preccurlyeq x^\dagger = 1/x$, contradicting $y^\dagger >_{\mathrm{e}} \mathbb{R}$. Applying this to $\log_n y$ (which we now know to be overhardian), gives $\log_n y >_{\mathrm{e}} x$, and thus $y >_{\mathrm{e}} \exp_n(x)$, proving the direction ⇒ of the equivalence.

For the converse, assume y is hardian and $y >_{\mathrm{e}} \exp_n(x)$ for all n. Then $H := \operatorname{Li}\big(\mathbb{R}(x)\big)$, the Liouville closure of $\mathbb{R}(x)$ as a Hardy field, embeds as an H-field over $\mathbb{R}(x)$ into the Liouville closed H-field extension $\mathbb{T}$ of $\mathbb{R}(x)$. Since the sequence $\big(\exp_n(x)\big)$ is cofinal in $\mathbb{T}$, this is also the case in H, so $y >_{\mathrm{e}} H$, and hence y is overhardian by Corollary 5.10. □

Constructing overhardian germs. Our goal is the following:

Theorem 5.12. *For any $\phi \in \mathcal{C}$ there is an overhardian $y \in \mathcal{C}^\infty$ such that $y^{\langle m\rangle} >_{\mathrm{e}} \phi$ for all m.*

Note that Theorem 5.1 follows from Corollary 5.10 and Theorem 5.12. To get an idea of how to construct a y as in Theorem 5.12, consider an overhardian y represented by a function in $\mathcal{C}^\infty_a$, to be denoted also by y. Then we have a strictly increasing sequence (a_m) of real numbers $\geqslant a$ tending to $+\infty$ such that $y^{\langle m\rangle}(t)$ is defined for $t \geqslant a_m$, for every m, and thus

$$y^{\langle m-1\rangle}(t) \;=\; y^{\langle m-1\rangle}(a_m)\cdot \exp\int_{a_m}^t y^{\langle m\rangle}(s)\,ds \quad \text{for } m \geqslant 1,\ t \geqslant a_m.$$

It follows that y is determined as a function on $[a_0,+\infty)$ by the family of restrictions $\big(y^{\langle m\rangle}|_{[a_m,a_{m+1}]}\big)$: y on $[a_0,a_1]$ and $y^{\langle 1\rangle}$ on $[a_1,a_2]$ determine y on $[a_0,a_2]$; likewise, $y^{\langle 1\rangle}$ on $[a_1,a_2]$ and $y^{\langle 2\rangle}$ on $[a_2,a_3]$ determine $y^{\langle 1\rangle}$ on $[a_1,a_3]$, and thus y on $[a_0,a_3]$, and so on. We use this as a clue to reverse engineer overhardian elements.

We start with $a \in \mathbb{R}$ and a strictly increasing sequence (a_m) in $\mathbb{R}^{\geqslant a}$ tending to $+\infty$ and for each $m \geqslant 1$ a continuous function $y_{m-1,m}\colon [a_{m-1},a_m] \to \mathbb{R}$.

Let $m \geqslant 1$. We define the continuous function $y_{k,m} : [a_k, a_m] \to \mathbb{R}$ for $0 \leqslant k < m$ by downward recursion on k: $y_{m-1,m}$ is already given to us, and for $1 \leqslant k < m$,

$$y_{k-1,m}(t) \ := \ \begin{cases} y_{k-1,k}(t) & \text{for } a_{k-1} \leqslant t \leqslant a_k, \\ y_{k-1,k}(a_k) \cdot \exp \displaystyle\int_{a_k}^{t} y_{k,m}(s)\, ds & \text{for } a_k \leqslant t \leqslant a_m. \end{cases}$$

(See Figure 7.)

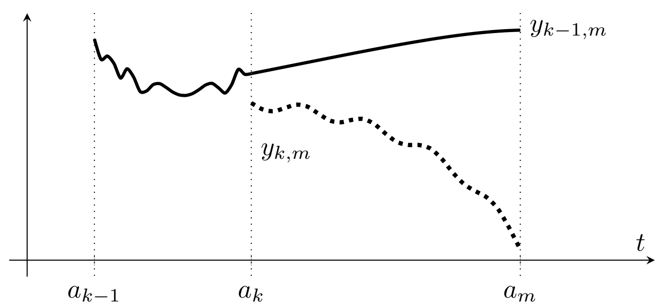

FIGURE 7. Passing from $y_{k,m}$ to $y_{k-1,m}$

Downward induction on k gives $y_{k,m} = y_{k,m+1}$ on $[a_k, a_m]$ for $k < m$. This fact gives for each k a continuous function $y_k : [a_k, +\infty) \to \mathbb{R}$ such that $y_k = y_{k,m}$ on $[a_k, a_m]$, for all $m > k$. Thus for $k \geqslant 1$ we have

$$y_{k-1}(t) \ = \ y_{k-1}(a_k) \cdot \exp \int_{a_k}^{t} y_k(s)\, ds \quad \text{for } t \geqslant a_k.$$

In the next lemma we use the notation $E(t)^{(r)}_{t=a}$ where the expression $E(t)$ defines a function $t \mapsto E(t)$ in $\mathcal{C}^r(I)$, where $I = [b, c]$ ($b < c$ in $\mathbb{R}$) and $a \in I$. With f this function, $E(t)^{(r)}_{t=a} := f^{(r)}(a)$.

Lemma 5.13. *Assume the following holds for all $k \geqslant 1$:*

(i) $y_{k-1,k} \in \mathcal{C}^\infty[a_{k-1}, a_k]$ *and* $y_{k-1,k}(t) > 0$ *for* $a_{k-1} \leqslant t \leqslant a_k$*;*
(ii) $y^{(r)}_{k-1,k}(a_k) = y_{k-1,k}(a_k) \cdot \big(\exp \int_{a_k}^{t} y_{k,k+1}(s)\, ds\big)^{(r)}_{t=a_k}$ *for all* $r \in \mathbb{N}^{\geqslant 1}$.

Then for all k we have $y_k \in \mathcal{C}^\infty_{a_k}$, $y_k(t) > 0$ for $t \geqslant a_k$, and $y_k^\dagger = y_{k+1}$ on $[a_{k+1}, +\infty)$. Thus y_0 is overhardian if $y_k >_{\mathrm{e}} y_{k+1}$ for all k.

Proof. Downward induction on k shows that $y_{k,m}$ for $k < m$ has the corresponding properties. Note: $y_k^{\langle m \rangle}$ is defined in $\mathcal{C}^{<\infty}$ and equals y_{k+m} in $\mathcal{C}^{<\infty}$ for all k, m. □

Towards proving Theorem 5.12 we may assume $\phi \in \mathcal{C}$ to be represented by a continuous function $\phi : [a, +\infty) \to \mathbb{R}^>$, so ϕ denotes the function and its germ.

Lemma 5.14. *There exists an increasing $\mathcal{C}^\infty$-function $f : [a, +\infty) \to \mathbb{R}$ such that $\phi(t) < f(t)$ and $f(t) > f^\dagger(t)$ for all $t \geqslant a$.*

Proof. Lemma 2.2 yields a decreasing $\mathcal{C}^\infty$-function $\zeta : [a, +\infty) \to \mathbb{R}^>$ with $1/\phi(t) > \zeta(t)$ and $\zeta'(t) > -1$ for all $t \geqslant a$. Then $f := 1/\zeta$ works. □

Replacing ϕ by f and renaming, we arrange that $\phi\colon [a,+\infty) \to \mathbb{R}^{>}$ is increasing of class $\mathcal{C}^\infty$ and $\phi(t) > \phi^\dagger(t)$ for all $t \geqslant a$. With these assumptions:

Lemma 5.15. *Suppose for all $k \geqslant 1$ we have $y_{k-1,k}(t) > \phi(t)$ for $a_{k-1} < t \leqslant a_k$. Then for all k we have $y_k(t) > \phi(t)$ for $t > a_k$.*

Proof. Let $1 \leqslant k < m$, and assume as an inductive assumption that $y_{k,m}(t) > \phi(t)$ for $a_k < t \leqslant a_m$. Our job is to show that then $y_{k-1,m}(t) > \phi(t)$ for $a_{k-1} < t \leqslant a_m$, and this amounts to showing for $a_k \leqslant t \leqslant a_m$ that

$$y_{k-1,k}(a_k) \cdot \exp \int_{a_k}^t y_{k,m}(s)\,ds \ > \ \phi(t).$$

This holds for $t = a_k$, and for $a_k < t \leqslant a_m$ we have

$$\begin{aligned} y_{k-1,k}(a_k) \cdot \exp \int_{a_k}^t y_{k,m}(s)\,ds \ &> \ \phi(a_k) \cdot \exp \int_{a_k}^t \phi(s)\,ds \\ &> \ \phi(a_k) \cdot \exp \int_{a_k}^t \phi^\dagger(s)\,ds \\ &= \ \phi(a_k) \cdot \exp\big(\log\phi(t) - \log\phi(a_k)\big) \ = \ \phi(t), \end{aligned}$$

which gives the desired result. □

For $b \geqslant a$ we define the $\mathcal{C}^\infty$-function $\phi_b\colon [a,+\infty) \to \mathbb{R}^{>}$ by

$$\phi_b(t) \ = \ \phi(b) \cdot \exp \int_b^t \phi(s)\,ds, \tag{5.1}$$

so $\phi(t) < \phi_b(t)$ for $t > b$, using again that $\phi(s) > \phi^\dagger(s)$ for $s > b$.

Lemma 5.16. *Suppose that for all $k \geqslant 1$ we have $\phi < y_{k-1,k} \leqslant \phi_{a_{k-1}}$ on $(a_{k-1}, a_k]$. Then for $k+1 < m$ we have $y_{k,m} > y_{k+1,m}$ on $[a_{k+1}, a_m]$.*

Proof. For $m = k+2$ and $a_{k+1} \leqslant t \leqslant a_m$ we have

$$\begin{aligned} y_{k,m}(t) \ &= \ y_{k,k+1}(a_{k+1}) \cdot \exp \int_{a_{k+1}}^t y_{k+1,m}(s)\,ds \\ &> \ \phi(a_{k+1}) \exp \int_{a_{k+1}}^t \phi(s)\,ds \ = \ \phi_{a_{k+1}}(t) \ \geqslant \ y_{k+1,m}(t). \end{aligned}$$

Let $1 \leqslant k < k+1 < m$ and assume inductively that $y_{k,m}(t) > y_{k+1,m}(t)$ whenever $a_{k+1} \leqslant t \leqslant a_m$. Then for $a_k \leqslant t \leqslant a_{k+1}$ the special case above yields

$$y_{k-1,m}(t) \ = \ y_{k-1,k+1}(t) \ > \ y_{k,k+1}(t) \ = \ y_{k,m}(t),$$

and for $a_{k+1} \leqslant t \leqslant a_m$ the inductive assumption gives

$$\begin{aligned} y_{k-1,m}(t) \ &= \ y_{k-1,k}(a_k) \cdot \exp \int_{a_k}^t y_{k,m}(s)\,ds \\ &= \ y_{k-1,k}(a_k) \cdot \exp \int_{a_k}^{a_{k+1}} y_{k,m}(s)\,ds \cdot \exp \int_{a_{k+1}}^t y_{k,m}(s)\,ds \\ &= \ y_{k-1,m}(a_{k+1}) \cdot \exp \int_{a_{k+1}}^t y_{k,m}(s)\,ds \\ &> \ y_{k,m}(a_{k+1}) \exp \int_{a_{k+1}}^t y_{k+1,m}(s)\,ds \ = \ y_{k,m}(t), \end{aligned}$$

which concludes the induction. ☐

Corollary 5.17. *Suppose that for all $k \geqslant 1$ we have*

(i) $y_{k-1,k} \in \mathcal{C}^\infty[a_{k-1}, a_k]$;
(ii) $\phi < y_{k-1,k} \leqslant \phi_{a_{k-1}}$ *on* $(a_{k-1}, a_k]$;
(iii) $y^{(r)}_{k-1,k}(a_{k-1}) \ = \ \phi^{(r)}(a_{k-1})$ *for all* $r \in \mathbb{N}$;
(iv) $y^{(r)}_{k-1,k}(a_k) = y_{k-1,k}(a_k) \cdot \big(\exp \int_{a_k}^t \phi(s)\, ds\big)^{(r)}_{t=a_k}$ *for all* $r \in \mathbb{N}^{\geqslant 1}$.

Then $y := y_0 \in \mathcal{C}^\infty_{a_0}$, *$y$ is overhardian, and* $y^{\langle k\rangle} >_{\mathrm{e}} \phi$ *for all* k.

Proof. This follows from Lemmas 5.13, 5.15, and 5.16. ☐

The phrase "y is overhardian" in the corollary above is short for "the germ of y at $+\infty$ is overhardian". Given the strictly increasing sequence (a_k) of real numbers $\geqslant a$ tending to $+\infty$ and the increasing $\mathcal{C}^\infty$-function $\phi\colon [a, +\infty) \to \mathbb{R}^>$ such that $\phi(t) > \phi^\dagger(t)$ for all $t \geqslant a$, it follows from Lemma 2.3 that there exist functions $y_{k-1,k}$ for $k \geqslant 1$ satisfying conditions (i)–(iv) of Corollary 5.17, where each value $y_{k-1,k}(a_k)$ can be chosen arbitrarily in the interval $\big(\phi(a_k), \phi_{a_{k-1}}(a_k)\big)$. Now the conclusion of that corollary yields Theorem 5.12, which in turn gives us Theorem 5.1.

6. Filling Wide Gaps

We now adapt the material from the previous section to filling a *wide gap*. To describe this situation, let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field. By a **wide gap** in H we mean a pair A, B of nonempty subsets of $H^{>\mathbb{R}}$ such that $A < B$, there is no $h \in H$ with $A < h < B$, and A and $\exp A$ are cofinal; note that then A and $\log A$ are cofinal, that B, $\exp B$, $\log B$ are coinitial, and that for any $\phi \in \mathcal{C}$ with $A <_{\mathrm{e}} \phi <_{\mathrm{e}} B$ we have $A <_{\mathrm{e}} \log\phi, \exp\phi <_{\mathrm{e}} B$. Moreover, if A, B is a wide gap in H, then it is an additive gap in H, and A, $\mathrm{sq}(A)$ are cofinal, and B, $2B$, $\sqrt{B}$ are coinitial, by Corollary 1.9 and Lemmas 1.5, 1.10, and 1.12(i). Let us also record the following, although we shall not explicitly use it:

Lemma 6.1. *Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field and A, B nonempty subsets of $H^{>\mathbb{R}}$ such that $A < B$ and there is no $h \in H$ with $A < h < B$. Suppose there exists $\phi \in \mathcal{C}$ such that $A <_{\mathrm{e}} \phi$ and $\mathrm{e}^\phi <_{\mathrm{e}} B$. Then A, B is a wide gap in H.*

Proof. For ϕ as above and $h \in A$ we have $h <_{\mathrm{e}} \phi$, so $\mathrm{e}^h <_{\mathrm{e}} \mathrm{e}^\phi <_{\mathrm{e}} B$, and thus $\mathrm{e}^h \leqslant f$ for some $f \in A$. ☐

Here is the main result of this section:

Theorem 6.2. *If $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and A, B is a wide gap in H with $\mathrm{cf}(A) = \mathrm{ci}(B) = \omega$, then some $y \in \mathcal{C}^{<\infty}$ with $A <_{\mathrm{e}} y <_{\mathrm{e}} B$ is H-hardian.*

Wide gaps as in Theorem 6.2 do actually occur, as we show in the next subsection. Towards proving Theorem 6.2 and some variants we begin with a result that is mainly an exercise in valuation theory:

Lemma 6.3. *Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field, let A, B be a wide gap in H, and let $y \in \mathcal{C}^{<\infty}$ be overhardian with $A <_{\mathrm{e}} y <_{\mathrm{e}} B$. Then y is H-hardian and* d*-transcendental over H.*

Proof. It will be convenient to work with the $y^{\langle n\rangle}$. Note that Lemma 5.9 and the cofinality of A and $\exp(A)$ give $A <_{\mathrm{e}} \log y <_{\mathrm{e}} y^{\dagger} <_{\mathrm{e}} y <_{\mathrm{e}} B$. Using this inductively we obtain $A <_{\mathrm{e}} y^{\langle i\rangle} <_{\mathrm{e}} B$ for all i. We prove by induction on n the claim that $y, y', \dots, y^{(n)}$ generate a Hausdorff field extension $H_n := H(y, y', \dots, y^{(n)})$ of H. For $n = 0$ this claim follows by applying Lemma 1.11 to

$$P := \big\{vh : h \in H^{>},\ h \preccurlyeq g \text{ for some } g \in A\big\}.$$

Assume the claim holds for a certain n. It is easy to check that then $y^{\langle 0\rangle}, \dots, y^{\langle n\rangle}$ lie in H_n, that $H_n = H\big(y^{\langle 0\rangle}, \dots, y^{\langle n\rangle}\big)$, and that H_n has value group

$$v(H_n^{\times}) = v(H^{\times}) \oplus \mathbb{Z}vy^{\langle 0\rangle} \oplus \cdots \oplus \mathbb{Z}vy^{\langle n\rangle}$$

with $vB < vy^{\langle i\rangle} < vA$ for all $i \leqslant n$ and $vy^{\langle i+1\rangle} = o(vy^{\langle i\rangle})$ for all $i < n$. Note that $vA < 0$. Let Δ be the smallest convex subgroup of $v(H^{\times})$ that includes vA. Then vA is coinitial in Δ, and $\Delta + \mathbb{Z}vy^{\langle 0\rangle} + \cdots + \mathbb{Z}vy^{\langle n\rangle}$ is a convex subgroup of $v(H_n^{\times})$ with $vB < \Delta + \mathbb{Z}vy^{\langle 0\rangle} + \cdots + \mathbb{Z}vy^{\langle n\rangle}$. Hence the real closure H_n^{rc} of H_n, taken as a Hausdorff field extension of H_n, has value group

$$v\big(H_n^{\mathrm{rc},\times}\big) = v(H^{\times}) \oplus \mathbb{Q}vy^{\langle 0\rangle} \oplus \cdots \oplus \mathbb{Q}vy^{\langle n\rangle},$$

and Δ as well as $\Delta + \mathbb{Q}vy^{\langle 0\rangle} + \cdots + \mathbb{Q}vy^{\langle n\rangle}$ are convex subgroups of $v(H_n^{\mathrm{rc},\times})$ with $vB < \Delta + \mathbb{Q}vy^{\langle 0\rangle} + \cdots + \mathbb{Q}vy^{\langle n\rangle}$. (See Figure 8.) In view of $A <_{\mathrm{e}} (y^{\langle n+1\rangle})^i <_{\mathrm{e}} y^{\langle n\rangle}$ for all $i \geqslant 1$ it now follows from Lemma 1.11 (with H_n^{rc} in the role of H) that $y^{(n+1)}$ generates a Hausdorff field over H_n^{rc}. $\square$

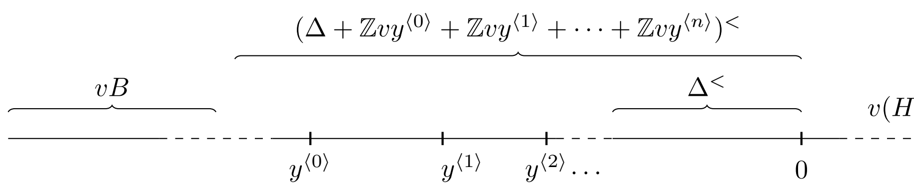

FIGURE 8. Value group of H_n

Let us now consider the slightly different situation where $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and $y \in \mathcal{C}^{<\infty}$ is overhardian with $y >_{\mathrm{e}} H$. Then the proof of the lemma above goes through for $A := H^{>\mathbb{R}}$ and $B = \emptyset$, although this pair A, B is not a wide gap. The proof not only shows in this situation that y generates a Hardy field $H\langle y\rangle$, but also that $v(H^{\times})$ is a convex subgroup of $v\big(H\langle y\rangle^{\times}\big)$, and that $v\big(H\langle y\rangle^{\times}\big) = v(H^{\times}) \oplus \bigoplus_i \mathbb{Z}vy^{\langle i\rangle}$, and so y is d-transcendental over H. (See also [ADH, 16.6.10].)

Constructing "countable" wide gaps. The Liouville closed Hardy field $\mathrm{Li}(\mathbb{R}) = \mathrm{Li}\big(\mathbb{R}(x)\big)$ is d-algebraic over $\mathbb{R}$. Hence by [3, Theorem 3.4] the sequence $\big(\exp_n(x)\big)$ is cofinal in $\mathrm{Li}(\mathbb{R})$, so $\mathrm{cf}\big(\mathrm{Li}(\mathbb{R})\big) = \omega$. More generally, let $H \supseteq \mathbb{R}$ be any Liouville closed Hardy field with $\mathrm{cf}(H) = \omega$. Then [7, remarks after Lemma 5.4.17] yields a $\phi \in \mathcal{C}$ with $\phi >_{\mathrm{e}} H$ and so Theorem 5.1 gives an H-hardian $y \in \mathcal{C}^{<\infty}$ such that $y >_{\mathrm{e}} H$. We now consider the Hardy-Liouville closure $\operatorname{Li}(H\langle y\rangle)$ of $H\langle y\rangle$. We have a wide gap A, B in $\operatorname{Li}(H\langle y\rangle)$ given by

$$\begin{aligned} A &:= \{f \in \operatorname{Li}(H\langle y\rangle) : \mathbb{R} < f < h \text{ for some } h \in H\}, \\ B &:= \{g \in \operatorname{Li}(H\langle y\rangle) : g > H\}. \end{aligned}$$

Note that $\operatorname{cf}(H) = \omega$ gives $\operatorname{cf}(A) = \omega$. Moreover:

Lemma 6.4. $B = \{g \in \operatorname{Li}(H\langle y\rangle) : g > \log_n y$ *for some* $n\}$.

Proof. By the remarks preceding this subsection y is d-transcendental over H and $\{y^n : n = 0, 1, 2, \dots\}$ is cofinal in $H\langle y\rangle$. Now $\operatorname{Li}(H\langle y\rangle)$ is d-algebraic over $H\langle y\rangle$, so $\{\exp_n(y) : n = 0, 1, 2, \dots\}$ is cofinal in $\operatorname{Li}(H\langle y\rangle)$ by [3, Theorem 3.4] applied to $K = H\langle y\rangle$. In particular, $\operatorname{Li}(H\langle y\rangle)$ has d-transcendence degree 1 over H. Towards a contradiction, suppose $g \in B$ and $g < \log_n y$ for all n. With g instead of y we conclude that $\{\exp_n(g) : n = 0, 1, 2, \dots\}$ is cofinal in $\operatorname{Li}(H\langle g\rangle)$ and $\operatorname{Li}(H\langle g\rangle)$ has d-transcendence degree 1 over H. Hence $y > \operatorname{Li}(H\langle g\rangle)$, and so with $\operatorname{Li}(H\langle g\rangle)$ in the role of H we conclude that $\operatorname{Li}(H\langle y\rangle) = \operatorname{Li}(\operatorname{Li}(H\langle g\rangle)\langle y\rangle)$ has d-transcendence degree 1 over $\operatorname{Li}(H\langle g\rangle)$ and thus d-transcendence degree 2 over H, a contradiction. □

Thus A, B is a wide gap in $\operatorname{Li}(H\langle y\rangle)$ with $\operatorname{cf}(A) = \operatorname{ci}(B) = \omega$. See also Figure 9.

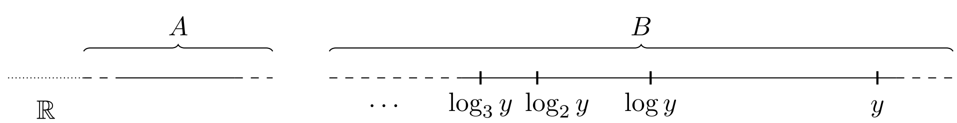

FIGURE 9. The countable wide gap A, B

Upper bounds. Assume that $a \geqslant 1$, $\phi\colon [a, +\infty) \to \mathbb{R}^{>}$ is $\mathcal{C}^{\infty}$ and increasing, and $\phi > \phi^{\dagger}$ on $[a, +\infty)$. Let (a_m) be a strictly increasing sequence of real numbers $\geqslant a$ tending to $+\infty$, and let for each $m \geqslant 1$ a continuous function

$$y_{m-1,m}\colon\ [a_{m-1}, a_m] \to \mathbb{R}$$

be given. As in the previous section this gives rise to functions $y_{k,m}$ for $k < m$ and functions y_m and $y := y_0$. Finally, assume that $y_{k-1,k} \leqslant \phi_{a_{k-1}}$ on $(a_{k-1}, a_k]$, for all $k \geqslant 1$. (See (5.1) for the definition of ϕ_b for $b \geqslant a$.) Our goal is to find an upper bound for y on $[a_0, a_n]$ for $n \geqslant 1$ that depends only on ϕ and n, not on the sequence (a_m) or the functions $y_{m-1,m}$.

For $n \geqslant 1$ and $a_{n-1} \leqslant t \leqslant a_n$, we have

$$\begin{aligned} y_{n-1,n}(t) \ &\leqslant\ \phi_{a_{n-1}}(t) \ =\ \phi(a_{n-1}) \exp \int_{a_{n-1}}^{t} \phi(s)\, ds \\ &\leqslant\ \phi(a_{n-1}) \exp\big((t - a_{n-1})\phi(t)\big) \ =\ \frac{\phi(a_{n-1})}{\exp\big(a_{n-1}\phi(t)\big)} \exp\big(t\phi(t)\big) \\ &\leqslant\ \exp\big(t\phi(t)\big). \end{aligned}$$

Let $1 \leqslant k < n$. Then $y_{k-1,n}(t) \leqslant \exp\big(t\phi(t)\big)$ for $a_{k-1} \leqslant t \leqslant a_k$. We assume inductively that for $a_k \leqslant t \leqslant a_n$ we have $y_{k,n}(t) \leqslant \exp_{n-k}\big(t\phi(t) + (n-k)t\big)$. Then for $a_k \leqslant t \leqslant a_n$,

$$\begin{aligned} y_{k-1,n}(t) &= y_{k-1,k}(a_k) \exp \int_{a_k}^t y_{k,n}(s)\, ds \\ &\leqslant y_{k-1,k}(a_k) \exp\big[(t-a_k)\exp_{n-k}\big(t\phi(t)+(n-k)t\big)\big] \\ &\leqslant \frac{y_{k-1,k}(a_k)}{\exp\big[a_k \exp_{n-k}\big(t\phi(t)+(n-k)t\big)\big]} \exp\big[t\exp_{n-k}\big(t\phi(t)+(n-k)t\big)\big] \\ &\leqslant \exp_{n-(k-1)}\big(t\phi(t)+(n-k+1)t\big), \end{aligned}$$

where we use that for $t \geqslant a_k$ we have the inequalities

$$\begin{aligned} y_{k-1,k}(a_k) \ &\leqslant\ \exp\big(a_k\phi(a_k)\big) \ \leqslant\ \exp\big[a_k \exp_{n-k}\big(t\phi(t)+(n-k)t\big)\big], \\ t\exp_{n-k}\big(t\phi(t)+(n-k)t\big) \ &\leqslant\ \exp_{n-k}\big(t\phi(t)+(n-k+1)t\big), \end{aligned}$$

the latter being a special case of the easily verified fact that

$$t\exp_n\big(t\phi(t)+nt\big) \ \leqslant\ \exp_n\big(t\phi(t)+(n+1)t\big) \qquad (n \geqslant 1,\ t \geqslant 1).$$

We have now proved by downward induction on k that for all $k < n$,

$$y_{k,n}(t) \ \leqslant\ \exp_{n-k}\big(t\phi(t)+(n-k)t\big) \qquad \text{for } a_k \leqslant t \leqslant a_n.$$

For $y := y_0$ this yields

$$y(t) \ \leqslant\ \exp_n\big(t\phi(t)+nt\big) \ \text{ for } n \geqslant 1 \text{ and } a_0 \leqslant t \leqslant a_n.$$

To simplify notation, let $\phi_n \colon [a,+\infty) \to \mathbb{R}$ be the function given by

$$\phi_n(t) \ :=\ \exp_n\big(t\phi(t)+nt\big),$$

so that that $\phi < \phi_1 < \phi_2 < \phi_3 < \cdots$ on $[a,+\infty)$, and the bound above takes the form that for all $n \geqslant 1$ we have $y \leqslant \phi_n$ on $[a_0, a_n]$.

Back to wide gaps. *In the rest of this section H is a Liouville closed Hardy field with $\mathbb{R} \subseteq H$, and A, B is a wide gap in H.* We say that $\phi \in \mathcal{C}$ **lies between** A and B if $A <_{\mathrm{e}} \phi <_{\mathrm{e}} B$. By an **intermediary** for A, B we mean a $\phi \in \mathcal{C}^\infty$ lying between A and B such that $0 <_{\mathrm{e}} \phi^\dagger <_{\mathrm{e}} \phi$; note that the condition $0 <_{\mathrm{e}} \phi^\dagger$ implies that ϕ is eventually strictly increasing.

Proposition 6.5. *Suppose $x \in A$, $\operatorname{ci}(B) = \omega$, and ϕ is an intermediary for A, B. Then there exists an overhardian $y \in \mathcal{C}^\infty$ such that $\phi <_{\mathrm{e}} y^{\langle n \rangle} <_{\mathrm{e}} B$ for all n, in particular $A <_{\mathrm{e}} y <_{\mathrm{e}} B$, and so y is H-hardian, by Lemma 6.3.*

Proof. Take a strictly increasing $\mathcal{C}^\infty$-function $\phi \colon [a,+\infty) \to \mathbb{R}^>$ representing the germ ϕ such that $a \geqslant 1$ and $0 < \phi^\dagger < \phi$ on $[a,+\infty)$. For $n \geqslant 1$, let $\phi_n \colon [a,+\infty) \to \mathbb{R}$ be the function from the previous subsection given by

$$\phi_n(t) \ =\ \exp_n\big(t\phi(t)+nt\big).$$

From $x \in A$ and B and $\log B$ being cofinal we obtain $\phi_n <_{\mathrm{e}} \phi_{n+1} <_{\mathrm{e}} B$.

Take a strictly decreasing sequence $g_1 > g_2 > g_3 > \cdots$ in B, coinitial in B. Let g_n also denote a continuous function $[a,+\infty) \to \mathbb{R}$ representing the germ g_n. Choose a strictly increasing sequence $b_1 < b_2 < b_3 < \cdots$ of real numbers $\geqslant a$ tending to $+\infty$ such that $\phi_n < g_n$ and $g_{n+1} < g_n$, on $[b_n,+\infty)$. Next, set $a_n := b_{n+1}$, and choose functions $y_{k-1,k} \in \mathcal{C}^\infty[a_{k-1}, a_k]$ for $k \geqslant 1$ such that conditions (i)–(iv) of

Corollary 5.17 are satisfied. (The discussion following that corollary indicates how to construct such functions, using Lemma 2.3.) This yields an overhardian $y := y_0 \in \mathcal{C}^{\infty}_{a_0}$ as in that corollary, with $y^{\langle k\rangle} >_{\mathrm{e}} \phi$ for all k.

Let $n \geqslant 1$. The upper bound from the previous subsection gives $y \leqslant \phi_n$ on $[a_0, a_n]$, so $y < g_n$ on $[b_n, b_{n+1}]$. With $n+1$ instead of n this gives $y < g_{n+1}$ on $[b_{n+1}, b_{n+2}]$, and as $g_{n+1} < g_n$ on $[b_n, +\infty)$, we get $y < g_n$ on $[b_n, b_{n+2}]$. Continuing this way we get $y < g_n$ on $[b_n, b_{n+3}]$, and so on, and thus $y < g_n$ on $[b_n, +\infty)$. Since this holds for all $n \geqslant 1$, this yields $y <_{\mathrm{e}} B$. □

Lemma 6.6. *Suppose $x \in A$, and some element of $\mathcal{C}$ lies between A and B. Then there exists an intermediary for A, B.*

Proof. Let $f\colon [a, +\infty) \to \mathbb{R}^{>}$ be a continuous function whose germ at $+\infty$ lies between A and B. Lemma 2.5 gives a $\mathcal{C}^{\infty}$-function $f^*\colon [a, +\infty) \to \mathbb{R}^{>}$ such that $f < f^* < f + 1$ on $[a, +\infty)$. Then $f^* <_{\mathrm{e}} B$, and so replacing f by f^* we have arranged that $f \in \mathcal{C}^{\infty}_a$. Defining $F(t) := 1 + \int_a^t f(s)\,ds$ we obtain a strictly increasing $F \in \mathcal{C}^{\infty}_a$ with $F' = f$. By Lemma 2.15 we have $\int A <_{\mathrm{e}} F <_{\mathrm{e}} \int B$, and so $A <_{\mathrm{e}} F <_{\mathrm{e}} B$ by Lemma 1.12(iii) and Lemma 1.13(ii). Thus we can replace f by F and arrange in this way that f is also strictly increasing and $f \geqslant 1$. Next, consider the strictly decreasing $\mathcal{C}^{\infty}$-function $\theta\colon [a, +\infty) \to (0, 1]$ given by

$$\theta(t) \ := \ \int_t^{t+1} f(s)^{-1}\,ds \ = \ \int_0^1 f^{-1}(s+t)\,ds, \qquad f^{-1}(s) := f(s)^{-1} \text{ for } s \geqslant a.$$

Claim: $\theta' > -1$ *on* $[a, +\infty)$, *and* $B^{-1} <_{\mathrm{e}} \theta <_{\mathrm{e}} A^{-1}$.

That $\theta' > -1$ on $[a, +\infty)$ is clear from

$$\theta'(t) \ = \ \int_0^1 (f^{-1})'(s+t)\,ds \ = \ f(t+1)^{-1} - f(t)^{-1}.$$

Also $\theta(t) < f(t)^{-1}$ for $t \geqslant a$, so $\theta <_{\mathrm{e}} f^{-1} <_{\mathrm{e}} A^{-1}$.

To establish the claim it remains to show that $B^{-1} <_{\mathrm{e}} \theta$, and this is where we shall need Lemma 1.13(iii). Let $g \in \int B^{-1}$, so $g \in H^{\prec 1}$ and $g' = h^{-1}$ with $h \in B$. We have $h \succ \mathrm{e}^x$, so after increasing a if necessary we can assume that the germ h is represented by a continuous function $h\colon [a, +\infty) \to \mathbb{R}$ with $h(t) > \mathrm{e}^t$ and thus $0 < h(t)^{-1} < \mathrm{e}^{-t}$, for all $t \geqslant a$. This yields a $\mathcal{C}^1$-function

$$t \mapsto \int_{+\infty}^t h(s)^{-1}\,ds \ := \ -\int_t^{+\infty} h(s)^{-1}\,ds \ : \ [a, +\infty) \to \mathbb{R}$$

with derivative h^{-1} and tending to 0 as $t \to +\infty$, so this function represents the germ g, and will be denoted below by g. Thus for $t \geqslant a$,

$$g(t+1) - g(t) \ = \ \int_t^{t+1} h(s)^{-1}\,ds.$$

Moreover, $h(s)^{-1} < f(s)^{-1}$ for all sufficiently large $s \geqslant a$, and thus

$$g(t+1) - g(t) \ < \ \theta(t) \qquad \text{for all sufficiently large } t \geqslant a. \tag{6.1}$$

From $h^{-1} \prec \mathrm{e}^{-x}$ we get $0 < -g \prec \mathrm{e}^{-x}$, and so Lemma 2.17(iii) applied to $-g$ and combined with (6.1) gives $-g(t)/2 < \theta(t)$ for all sufficiently large $t \geqslant a$. In view of Lemma 1.13(iii) and coinitiality of B, $2B$ this yields $B^{-1} <_{\mathrm{e}} \theta$, as claimed.

From the claim it follows that the germ of $\phi := \theta^{-1}\colon [a,+\infty) \to \mathbb{R}$ is an intermediary for A, B. ☐

Corollary 6.7. *Suppose $x \in A$ and $\operatorname{cf}(A) = \operatorname{ci}(B) = \omega$. Then there exists an overhardian $y \in \mathcal{C}^{\infty}$ with $A <_{\mathrm{e}} y <_{\mathrm{e}} B$, thus generating a Hardy field over H.*

Proof. Using $\operatorname{cf}(A) = \operatorname{ci}(B) = \omega$, Lemmas 2.13 and 2.14 give an element of $\mathcal{C}$ that lies between A and B. Then Lemma 6.6 provides an intermediary for A, B, which in view of Proposition 6.5 gives the desired result. ☐

Proof of Theorem 6.2. We assume $\operatorname{cf}(A) = \operatorname{ci}(B) = \omega$. Our job is to obtain a $y \in \mathcal{C}^{<\infty}$ such that $A <_{\mathrm{e}} y <_{\mathrm{e}} B$ and y generates a Hardy field over H. Take any $g \in A$. Then $g > \mathbb{R}$, so g' is active in H, and we pass to the compositional conjugate $H \circ g^{\mathrm{inv}}$, which is again a Liouville closed Hardy field containing $\mathbb{R}$ as a subfield, and having $A \circ g^{\mathrm{inv}}, B \circ g^{\mathrm{inv}}$ as a wide gap with $x = g \circ g^{\mathrm{inv}} \in A \circ g^{\mathrm{inv}}$. Now Corollary 6.7 yields a $y \in \mathcal{C}^{\infty}$ such that $A \circ g^{\mathrm{inv}} <_{\mathrm{e}} y <_{\mathrm{e}} B \circ g^{\mathrm{inv}}$ and y generates a Hardy field over $H \circ g^{\mathrm{inv}}$. It follows that $y \circ g \in \mathcal{C}^{<\infty}$, $A <_{\mathrm{e}} y \circ g <_{\mathrm{e}} B$, and $y \circ g$ generates a Hardy field over H. This concludes the proof. ☐

If A in Theorem 6.2 contains an element of $\mathcal{C}^{\infty}$, then we can take y in the conclusion of that theorem to be in $\mathcal{C}^{\infty}$ as well: in the proof, take $g \in \mathcal{C}^{\infty}$.

7. The Number of Maximal Hardy Fields

Since $\mathcal{C}$ has cardinality $\mathfrak{c} = 2^{\aleph_0}$, the number of Hardy fields (and thus of maximal Hardy fields) is at most $2^{\mathfrak{c}}$. By Proposition 3.7 in [13] there are $\geqslant \mathfrak{c}$ many maximal Hardy fields. In this short section we show:

Theorem 7.1. *The number of maximal Hardy fields is equal to $2^{\mathfrak{c}}$.*

This is mainly an application of the previous two sections. Let S be an ordered set. Define a **countable gap** in S to be a pair P, Q of countable subsets of S such that $P < Q$ and there is no $s \in S$ with $P < s < Q$; for example, if P is a countable cofinal subset of S, then P, $\emptyset$ is a countable gap in S. Also, S is η_1 iff it has no countable gap. We thank Ilijas Farah for pointing out that the following well-known lemma might be useful in proving statements like Theorem 7.1 via a suitable binary tree construction:

Lemma 7.2. *If S has cardinality $< \mathfrak{c}$, then S has a countable gap.*

Proof. Suppose S has no countable gap. Then S is in particular dense: for any $p < q$ in S there is an $s \in S$ with $p < s < q$. Thus we can embed the ordered set $(\mathbb{Q}; <)$ of rational numbers into S. Identifying $\mathbb{Q}$ with its image under such an embedding, there is for every $r \in \mathbb{R} \setminus \mathbb{Q}$ an $s \in S$ such that for all $t \in \mathbb{Q}$: $s > t$ in S iff $r > t$ in $\mathbb{R}$. Thus the cardinality of S is at least that of $\mathbb{R}$, which is $\mathfrak{c}$. ☐

Below $H \supseteq \mathbb{R}$ is a Hardy field. We set

$$H^{\mathrm{te}} := \{f \in H : f \text{ is overhardian}\},$$

the **transexponential** (or **overhardian**) **part** of H. By Corollary 5.11 we have

$$H^{\mathrm{te}} = \big\{f \in H : f > \exp_n(x) \text{ for all } n\big\},$$

so H^{te} is closed upward in $H^{>\mathbb{R}}$. On H^{te} we define the equivalence relation $\sim_{\exp}$ of **exponential equivalence** by

$$\begin{aligned} f \sim_{\exp} g \quad &:\Longleftrightarrow \quad f \leqslant \exp_n(g) \text{ and } g \leqslant \exp_n(f) \text{ for some } n \\ &\Longleftrightarrow \quad f \leqslant \exp_m(g) \text{ and } g \leqslant \exp_n(f) \text{ for some } m,\, n. \end{aligned}$$

Let $*f$ be the exponential equivalence class of $f \in H^{\mathrm{te}}$, a convex subset of H^{te}. We linearly order the set $*H^{\mathrm{te}}$ of exponential equivalence classes by:

$$*f < *g \quad :\Longleftrightarrow \quad \exp_n(f) < g \text{ for all } n \qquad (f, g \in H^{\mathrm{te}}).$$

For a Hardy field extension H_1 of H we have $H_1^{\mathrm{te}} \cap H = H^{\mathrm{te}}$, and we identify $*H^{\mathrm{te}}$ with a subset of $*H_1^{\mathrm{te}}$ via the order-preserving embedding

$$*f \,(\text{in } *H^{\mathrm{te}}) \;\mapsto\; *f \,(\text{in } *H_1^{\mathrm{te}}) \quad \text{ for } f \in H^{\mathrm{te}}.$$

Note that $\mathbb{R}(x)^{\mathrm{te}} = \emptyset$. If H is Liouville closed, then $\exp(*f) = \log(*f) = *f$ for $f \in H^{\mathrm{te}}$. We record a few other properties of $\sim_{\exp}$ used later:

Lemma 7.3. *Let $f \in H^{\mathrm{te}}$. Then*

(i) $(*f) \cdot (*f) = *f$;
(ii) *if $g \in H^{>\mathbb{R}}$ and $[vf] = [vg]$, then $g \in H^{\mathrm{te}}$ and $*f = *g$;*
(iii) $(*f)^{\dagger} = *f$ *and* $\partial(*f) = *f$.

Proof. Parts (i) and (ii) follow easily from the definitions. For (iii), note first that $f^{\dagger}$ is overhardian by a remark before Corollary 5.7 and $\log f < f^{\dagger}$ by Lemma 5.9, so $f < \exp(f^{\dagger})$, and $f^{\dagger} < f < \exp(f)$, and thus $f^{\dagger} \sim_{\exp} f$. This yields $(*f)^{\dagger} = f*$, hence $\partial(*f) = *f$ by (i). $\square$

Using results of the previous section we shall prove:

Proposition 7.4. *Suppose P, Q is a countable gap in $*H^{\mathrm{te}}$. Then H has Hardy field extensions $H_0 = H\langle f_0\rangle$, $H_1 = H\langle f_1\rangle$ with $f_0 \in H_0^{\mathrm{te}}$, $f_1 \in H_1^{\mathrm{te}}$, such that*

$$P < *f_0 < Q, \qquad P < *f_1 < Q,$$

H_0 and H_1 have no common Hardy field extension, and

$$*H_0^{\mathrm{te}} = *H^{\mathrm{te}} \cup \{*f_0\}, \qquad *H_1^{\mathrm{te}} = *H^{\mathrm{te}} \cup \{*f_1\}.$$

We accept this for the moment, and indicate how it enables a binary tree construction leading to Theorem 7.1. Let $|X|$ denote the cardinality of the set X, and identify as usual a cardinal with the least ordinal of that cardinality, where an ordinal λ is considered as the set of ordinals $< \lambda$. Let $\mathcal{H}$ be the set of all Hardy fields $H \supseteq \mathbb{R}$ such that $|*H^{\mathrm{te}}| < \mathfrak{c}$. We build by transfinite recursion a binary tree in $\mathcal{H}$ by assigning to each ordinal $\lambda < \mathfrak{c}$ and function $s\colon \lambda \to \{0,1\}$ a Hardy field $H_s \in \mathcal{H}$ with $|*H_s^{\mathrm{te}}| \leqslant |\lambda|$. For $\lambda = 0$ the function s has empty domain and we take $H_s = \mathbb{R}$. Suppose $s\colon \lambda \to \{0,1\}$ as above and $H_s \in \mathcal{H}$ are given with $|*H_s^{\mathrm{te}}| \leqslant |\lambda|$. Then Lemma 7.2 provides a countable gap P, Q in $*H_s^{\mathrm{te}}$. Let $s0, s1\colon \lambda+1 \to \{0,1\}$ be the obvious extensions of s, and let $H_{s0}, H_{s1} \in \mathcal{H}$ be obtained from H_s as H_0, H_1 are obtained from H in Proposition 7.4. Let $\lambda < \mathfrak{c}$ be an infinite limit ordinal and $s\colon \lambda \to \{0,1\}$; assume that for every $\alpha < \lambda$ there is given $H_{s|\alpha} \in \mathcal{H}$ with $H_{s|\alpha} \subseteq H_{s|\beta}$ whenever $\alpha \leqslant \beta < \lambda$. Then we set $H_s := \bigcup_{\alpha<\lambda} H_{s|\alpha}$. Assuming also inductively that $|*H_{s|\alpha}^{\mathrm{te}}| \leqslant |\alpha|$ for all $\alpha < \lambda$, we obtain $|*H_s^{\mathrm{te}}| \leqslant |\lambda| \cdot |\lambda| = |\lambda|$, as desired. This finishes the construction of our tree. It yields for any function $s\colon \mathfrak{c} \to \{0,1\}$ a Hardy field $H_s := \bigcup_{\lambda<\mathfrak{c}} H_{s|\lambda}$, and the way we constructed the tree guarantees that if $s, s' : \mathfrak{c} \to \{0,1\}$ are different, then H_s and $H_{s'}$ have no common Hardy field extension. Thus there are $2^{\mathfrak{c}}$ many maximal Hardy fields.

It remains to prove Proposition 7.4. This goes via some lemmas.

Lemma 7.5. *Let K be an asymptotic field with value group Γ and $\Psi := \psi(\Gamma^{\neq})$ its Ψ-set. Let L be an asymptotic field extension of K of finite transcendence degree over K. Then $\Psi_L \setminus \Psi$ is finite, where Ψ_L is the Ψ-set of L.*

Proof. By [ADH, 3.1.11] (Zariski-Abhyankar), Γ_L/Γ has finite rational rank. Now use that if $\gamma_1, \dots, \gamma_n \in \Gamma_L^{\neq}$ and $\gamma_1^\dagger, \dots, \gamma_n^\dagger$ are pairwise distinct and not in Ψ, then $\gamma_1, \dots, \gamma_n$ are $\mathbb{Z}$-linearly independent modulo Γ. □

Lemma 7.6. *Let $L \supseteq H \supseteq \mathbb{R}$ be Hardy fields where L is* d*-algebraic over H. Then $*H^{\mathrm{te}} = *L^{\mathrm{te}}$. (In particular, $*H^{\mathrm{te}} = *\operatorname{Li}(H)^{\mathrm{te}}$.)*

Proof. Let $y \in L^{\mathrm{te}}$; we claim that $y \sim_{\exp} h$ for some $h \in H^{\mathrm{te}}$. To prove this, set $\gamma := vy$. Then $\gamma < \psi(\gamma) < \cdots < \psi^n(\gamma) < \psi^{n+1}(\gamma) < \cdots < 0$, so Lemma 7.5 gives $n \geqslant 1$ with $\psi^n(\gamma) \in \Gamma := v(H^\times)$, say $\psi^n(\gamma) = vh$ with $h \in H^>$. Then h is overhardian and $y \sim_{\exp} h$ by Lemma 7.3. □

Lemma 7.7. *Let $H \supseteq \mathbb{R}$ be a Hardy field and let P, Q be a countable gap in $*H^{\mathrm{te}}$. Then H has a Hardy field extension $H\langle y\rangle$ with overhardian $y \in \mathcal{C}^{<\infty}$ such that $P < *y < Q$. For any such y we have $*H\langle y\rangle^{\mathrm{te}} = *H^{\mathrm{te}} \cup \{*y\}$.*

Proof. Using Lemma 7.6 we arrange that H is Liouville closed, in particular, $\exp_n(x) \in H$ for all n. Assume for now that $Q \neq \emptyset$. Then P, Q gives rise to a wide gap A, B in H by

$$A := \{\exp_n(x) : n = 0,1,2,\dots\} \cup \{\exp_n(a) : n = 0,1,2,\dots,\ a \in H^{\mathrm{te}},\ *a \in P\},$$
$$B := \{\log_n(b) : n = 0,1,2,\dots,\ b \in H^{\mathrm{te}},\ *b \in Q\},$$

with $\mathrm{cf}(A) = \omega$ and $\mathrm{ci}(B) = \omega$. Then Corollary 6.7 yields an overhardian $y \in \mathcal{C}^{<\infty}$ with $A <_{\mathrm{e}} y <_e B$. Given any such y it generates a Hardy field $H\langle y\rangle$ over H by Lemma 6.3, with $P < *y < Q$. Moreover, by the proof of that lemma,

$$v\big(H\langle y\rangle^\times\big) \;=\; v(H^\times) \oplus \bigoplus_n \mathbb{Z}v\big(y^{\langle n\rangle}\big),$$

with convex subgroups Δ of $v(H^\times)$ (as defined in that proof) and $\Delta + D$ of $v\big(H\langle y\rangle^\times\big)$ with $D := \bigoplus_n \mathbb{Z}v\big(y^{\langle n\rangle}\big)$. Let $f \in H\langle y\rangle^{\mathrm{te}}$. There are three possibilities:

(1) $vf \in \Delta$. Then $*f \in *H^{\mathrm{te}}$.
(2) $vf \in \Delta + D$, $vf < \Delta$. Then $vf = mvy^{\langle i\rangle} + o(vy^{\langle i\rangle})$ for some i and some $m \geqslant 1$, hence $*f = *y$ by Lemma 7.3.
(3) $vf < \Delta + D$. Then an easy argument gives $b > A$ in H with $vf = vb + o(vb)$, hence $*f = *b \in *H^{\mathrm{te}}$ by Lemma 7.3.

If $Q = \emptyset$, then we set $A := H^{>\mathbb{R}}$, $B := \emptyset$, and proceed as before, using results from Section 5 instead of Corollary 6.7 to obtain the existence of an overhardian $y \in \mathcal{C}^{<\infty}$ with $A <_{\mathrm{e}} y$, and using instead of Lemma 6.3 the remark following the proof of that lemma. □

The following consequence of Lemmas 7.7 and 7.2 is worth recording. (It also uses the fact that any Hardy field, as a subset of $\mathcal{C}$, has cardinality $\leqslant \mathfrak{c}$.)

Corollary 7.8. *If H is a maximal Hardy field, then the ordered set $*H^{\mathrm{te}}$ is η_1, and $|*H^{\mathrm{te}}| = \mathfrak{c}$.*

As to the H-field $\mathbb{T}$ of transseries that was studied extensively in [ADH], we usually think of $\mathbb{T}$ as rather large, but $H^{\mathrm{te}} = \emptyset$ for any Hardy field $H \supseteq \mathbb{R}$ which embeds into $\mathbb{T}$ (as H-fields); those H are dwarfed by any maximal Hardy field.

Next, given $\phi \in \mathcal{C}^{<\infty}$, call ϕ **hardy-small** if $\phi^{(n)} \prec 1$ for all n, and **hardy-bounded** if $\phi^{(n)} \preccurlyeq 1$ for all n. For example, $\sin x$ is hardy-bounded. Here are some simple observations about these notions: If $\phi, \theta \in \mathcal{C}^{<\infty}$ are hardy-small, then so is $\phi + \theta$. If $\phi \in \mathcal{C}^{<\infty}$ is hardian and $\phi \prec 1$, then ϕ is hardy-small. If $\phi \in \mathcal{C}^{<\infty}$ is hardian and $\phi \preccurlyeq 1$, then ϕ is hardy-bounded. If $\phi \in \mathcal{C}^{<\infty}$ is hardy-bounded and $\theta \in \mathcal{C}^{<\infty}$ is hardy-small, then $\phi\theta$ is hardy-small. If $\phi, \theta \in \mathcal{C}^{<\infty}$ are hardy-bounded, then so are $\phi + \theta$ and $\phi\theta$. A routine computation gives:

Lemma 7.9. *If $\phi \in \mathcal{C}^{<\infty}$ is hardy-small, then $(1+\phi)^{-1} = 1 + \theta$ for some hardy-small $\theta \in \mathcal{C}^{<\infty}$, and and so $(1+\phi)^{-1}$ is hardy-bounded.*

For the proof of Proposition 7.4 we shall use (see also [7, Corollary 5.4.14]):

Lemma 7.10 (Boshernitzan [11, Theorem 13.6])**.** *Suppose $\phi \in \mathcal{C}^{<\infty}$ is overhardian and $\theta \in \mathcal{C}^{<\infty}$ is hardy-bounded. Then $\phi + \theta$ is overhardian.*

Proof. Note that $\phi + \theta \in (\mathcal{C}^{<\infty})^{\times}$ by Corollary 5.7. Moreover,

$$(\phi+\theta)^{\dagger} \;=\; \left[\phi \cdot \left(1 + \frac{\theta}{\phi}\right)\right]^{\dagger} \;=\; \phi^{\dagger} + \left(1 + \frac{\theta}{\phi}\right)^{\dagger} \;=\; \phi^{\dagger} + \frac{(\theta/\phi)'}{1 + (\theta/\phi)}.$$

Now ϕ^{-1} is hardy-small, so θ/ϕ and $(\theta/\phi)'$ are hardy-small. Hence by Lemma 7.9, $\big(1 + (\theta/\phi)\big)^{-1}$ is hardy-bounded, and so $\frac{(\theta/\phi)'}{1+(\theta/\phi)}$ is hardy-small. Therefore, as $\phi^{\dagger}$ is still overhardian, we can iterate the above to obtain

$$(\phi+\theta)^{\langle n\rangle} \;=\; \phi^{\langle n\rangle} + \theta_n, \qquad \text{with hardy-small } \theta_n \text{ for } n \geqslant 1.$$

Thus $\phi + \theta$ is overhardian in view of Corollary 5.7. □

In particular, if $\phi \in \mathcal{C}^{<\infty}$ is overhardian, then so is $\phi + \sin x$. Thus for H, P, Q as in the hypothesis of Proposition 7.4 and taking y as in Lemma 7.7, the conclusion of that proposition holds for $f_0 := y$ and $f_1 := y + \sin x$ by the proof of Lemma 7.7.

8. The H-couple of a Maximal Hardy Field

Taking into account Lemma 1.1, proving Theorem A has now been reduced to showing that the value group of every maximal Hardy field is η_1. Let H be a maximal Hardy field. Then H is an asymptotic field in the sense of [ADH, 9.1], so it has an H-asymptotic couple (Γ, ψ) where Γ is the value group of H.

Let us consider more generally any asymptotic field K with its asymptotic couple (Γ, ψ). Recall that $\psi\colon \Gamma^{\neq} \to \Gamma$ is given by $\psi(\gamma) = v(g^{\dagger})$, with $g \in K^{\times}$ such that $v(g) = \gamma$, and that $\psi(\gamma)$ is also written as $\gamma^{\dagger}$. Recall also that ψ is a valuation on Γ. Let (γ_ρ) be a pc-sequence in Γ with respect to the valuation ψ on Γ. Take $g_\rho \in K^{\times}$ with $v(g_\rho) = \gamma_\rho$. Then $(g_\rho^{\dagger})$ is a pc-sequence in K, since $v(g_\sigma^{\dagger} - g_\rho^{\dagger}) = (\gamma_\sigma - \gamma_\rho)^{\dagger}$ for $\sigma > \rho$, provided ρ is sufficiently large. Suppose $g_\rho^{\dagger} \rightsquigarrow g^{\dagger}$ with nonzero g in some asymptotic field extension L of K (possibly $L = K$). We claim that then for $\gamma = vg$ we have $\gamma_\rho \rightsquigarrow \gamma$. This is because eventually $(\gamma - \gamma_\rho)^{\dagger} = v(g^{\dagger} - g_\rho^{\dagger})$, and the latter is eventually strictly increasing, using also that eventually $\gamma_\rho \neq \gamma$. In particular, if $K = K^\dagger$ and every pc-sequence in K of length ω has a pseudolimit in K, then every pc-sequence in Γ of length ω has a pseudolimit in Γ. Thus by Corollaries 3.2, 4.8, and 5.2:

Corollary 8.1. *If H is a maximal Hardy field with asymptotic couple (Γ, ψ), then every pc-sequence in (Γ, ψ) of length ω has a pseudolimit in (Γ, ψ), and*

$$\operatorname{cf}(\Gamma^<) \ = \ \operatorname{ci}(\Gamma^>) \ > \ \omega, \qquad \operatorname{ci}(\Gamma) \ = \ \operatorname{cf}(\Gamma) \ > \ \omega.$$

Corollary 8.1 includes [7, Proposition 5.6.6]: every maximal Hardy field contains a germ ℓ which is *translogarithmic,* that is, $\mathbb{R} < \ell \leqslant \ell_n$ for all n, where ℓ_n is inductively defined by $\ell_0 := x$ and $\ell_{n+1} := \log \ell_n$.

Ordered vector spaces and H-couples over an ordered field. In the rest of this section we fix an ordered field $\boldsymbol{k}$ (only the case $\boldsymbol{k} = \mathbb{R}$ is really needed) and use notation, terminology, and results from [6]. Let Γ be an ordered vector space over $\boldsymbol{k}$ (as defined there). For $\alpha \in \Gamma$ we defined its *$\boldsymbol{k}$-archimedean class*

$$[\alpha]_{\boldsymbol{k}} \ := \ \big\{\gamma \in \Gamma : \, |\gamma| \leqslant c|\alpha| \text{ and } |\alpha| \leqslant c|\gamma| \text{ for some } c \in \boldsymbol{k}^>\big\},$$

and we linearly ordered the set $[\Gamma]_{\boldsymbol{k}}$ of $\boldsymbol{k}$-archimedean classes. We defined Γ to be a *Hahn space* if for all $\alpha, \gamma \in \Gamma^{\neq}$ with $[\alpha]_{\boldsymbol{k}} = [\gamma]_{\boldsymbol{k}}$ there is a scalar $c \in \boldsymbol{k}^\times$ such that $[\alpha - c\gamma]_{\boldsymbol{k}} < [\alpha]_{\boldsymbol{k}}$. (If $\boldsymbol{k} = \mathbb{R}$, then the $\boldsymbol{k}$-archimedean class $[\alpha]_{\boldsymbol{k}}$ of an element α in an ordered vector space over $\boldsymbol{k}$ equals its archimedean class $[\alpha]$, and every ordered vector space over $\boldsymbol{k}$ is a Hahn space.) For an ordered vector space Δ over $\boldsymbol{k}$ extending Γ we identify $[\Gamma]_{\boldsymbol{k}}$ with a subset of $[\Delta]_{\boldsymbol{k}}$ via the order-preserving embedding $[\gamma]_{\boldsymbol{k}} \mapsto [\gamma]_{\boldsymbol{k}} \colon [\Gamma]_{\boldsymbol{k}} \to [\Delta]_{\boldsymbol{k}}$.

Let now (Γ, ψ) be an H-couple over $\boldsymbol{k}$, as defined in [6], so for all $\alpha, \beta \in \Gamma^{\neq}$,

$$[\alpha]_{\boldsymbol{k}} \leqslant [\beta]_{\boldsymbol{k}} \implies \psi(\alpha) \geqslant \psi(\beta).$$

We defined (Γ, ψ) to be of *Hahn type* if for all $\alpha, \beta \in \Gamma^{\neq}$ with $\psi(\alpha) = \psi(\beta)$ there exists a scalar $c \in \boldsymbol{k}^\times$ such that $\psi(\alpha - c\beta) > \psi(\alpha)$; a consequence of "Hahn type" is that for all $\alpha, \beta \in \Gamma^{\neq}$,

$$[\alpha]_{\boldsymbol{k}} \leqslant [\beta]_{\boldsymbol{k}} \iff \psi(\alpha) \geqslant \psi(\beta),$$

and so the underlying ordered vector space Γ over $\boldsymbol{k}$ is a Hahn space. We defined (Γ, ψ) to be *closed* if $\Psi := \psi(\Gamma^{\neq})$ is downward closed in the ordered set Γ, and (Γ, ψ) has asymptotic integration.

Let now K be a Liouville closed H-field. Recall from [6] that its value group Γ is then an ordered vector space over its (ordered) constant field C, with scalar multiplication given by $c\, vf = vg$ whenever $f, g \in K^\times$ and $cf^\dagger = g^\dagger$. Its asymptotic couple (Γ, ψ) with this scalar multiplication is a closed H-couple over C of Hahn type. For a Liouville closed Hardy field $H \supseteq \mathbb{R}$ its constant field is $\mathbb{R}$, and we construe its asymptotic couple as an H-couple over $\mathbb{R}$ as indicated.

Elements of countable type. Let Γ be an ordered vector space over $\boldsymbol{k}$. Let β be an element in an ordered vector space over $\boldsymbol{k}$ that extends Γ. Then we say that β **has countable type over** Γ if $\beta \notin \Gamma$ and $\operatorname{cf}(\Gamma^{<\beta}), \operatorname{ci}(\Gamma^{>\beta}) \leqslant \omega$; in that case every element in $(\Gamma + \boldsymbol{k}\beta) \setminus \Gamma$ has countable type over Γ. See [ADH, 2.2] for immediate extensions of valued abelian groups and [ADH, 2.4] for the $\boldsymbol{k}$-valuation of an ordered vector space over $\boldsymbol{k}$.

Lemma 8.2. *Suppose β has countable type over Γ and the ordered vector space $\Gamma + \boldsymbol{k}\beta$ over $\boldsymbol{k}$ is an immediate extension of Γ with respect to the $\boldsymbol{k}$-valuation. Then β is a pseudolimit of a divergent pc-sequence in Γ of length ω.*

Proof. The assumptions yield a countable (necessarily infinite) set $A \subseteq \Gamma$ such that for every $\gamma \in \Gamma$ there exists an $\alpha \in A$ with $[\alpha - \beta]_{\boldsymbol{k}} < [\gamma - \beta]_{\boldsymbol{k}}$. This easily yields a divergent pc-sequence (α_n) in Γ with all $\alpha_n \in A$ such that $\alpha_n \rightsquigarrow \beta$. ☐

Lemma 8.3. *Suppose* $\operatorname{cf}(\Gamma), \operatorname{cf}(\Gamma^{<}) > \omega$, *and β has countable type over Γ. Then*

$$\operatorname{cf}(\Gamma^{<\beta}) \ = \ \operatorname{ci}(\Gamma^{>\beta}) \ = \ \omega.$$

Proof. If $\beta < \Gamma$, then $\operatorname{ci}(\Gamma^{>\beta}) = \operatorname{ci}(\Gamma) = \operatorname{cf}(\Gamma) > \omega$, contradicting $\operatorname{ci}(\Gamma^{>\beta}) \leqslant \omega$. Thus $\Gamma^{<\beta} \neq \emptyset$. If $\operatorname{cf}(\Gamma^{<\beta}) \neq \omega$, then $\Gamma^{<\beta}$ has a largest element γ, so $\Gamma^{>\beta} = \Gamma^{>\gamma}$, contradicting $\operatorname{ci}(\Gamma^{>\gamma}) = \operatorname{cf}(\Gamma^{<}) > \omega$. Thus $\operatorname{cf}(\Gamma^{<\beta}) = \omega$; likewise, $\operatorname{ci}(\Gamma^{<\beta}) = \omega$. ☐

For us the relevant fact relating "countable type" to the η_1-property is as follows: given an H-couple (Γ, ψ) over $\boldsymbol{k}$,

$$\Gamma \text{ is } \eta_1 \quad \Longleftrightarrow \quad \begin{cases} \text{there is no } H\text{-couple over } \boldsymbol{k} \text{ extending } (\Gamma, \psi) \\ \text{with an element of countable type over } \Gamma. \end{cases}$$

(For "$\Leftarrow$" use model-theoretic compactness.)

Lemma 8.4. *Let (Γ, ψ) be a closed H-couple over $\boldsymbol{k}$, and suppose β in an H-couple over $\boldsymbol{k}$ extending (Γ, ψ) has countable type over Γ and $\beta^\dagger \notin \Gamma$. Then $\beta^\dagger$ has countable type over Γ.*

Proof. Without loss of generality we assume $\beta > 0$. Consider first the case where we have a strictly increasing sequence (α_m) in $\Gamma^{>}$ and a strictly decreasing sequence (γ_n) in $\Gamma^{>}$, such that $\alpha_m < \beta < \gamma_n$ for all m, n, and (α_m) is cofinal in $\Gamma^{<\beta}$, and (γ_n) is coinitial in $\Gamma^{>\beta}$. Then $(\alpha_m^\dagger)$ is decreasing, $(\gamma_n^\dagger)$ is increasing, $\alpha_m^\dagger > \beta^\dagger > \gamma_n^\dagger$ for all m, n. Using that the H-couple (Γ, ψ) is closed we also obtain that $(\alpha_m^\dagger)$ is coinitial in $\Gamma^{>\beta^\dagger}$ and that $(\gamma_n^\dagger)$ is cofinal in $\Gamma^{<\beta^\dagger}$. Thus $\beta^\dagger$ has countable type over Γ. Next consider the case $\beta > \Gamma$. Then the cofinality of Γ is ω, $\beta^\dagger < \Gamma$, and so $\beta^\dagger$ has countable type over Γ, since the coinitiality of Γ is also ω.

The case that there are $\alpha, \gamma \in \Gamma^{>}$ with $\alpha < \beta < \gamma$ and there is a largest $\alpha \in \Gamma^{>}$ with $\alpha < \beta$ or a least $\gamma \in \Gamma^{>}$ with $\beta < \gamma$ cannot occur, since for such a largest α we would have $\alpha < \beta < 2\alpha$, so $\alpha^\dagger = \beta^\dagger$, contradicting $\beta^\dagger \notin \Gamma$ (and a least such γ yields the same contradiction).

It remains to consider the case $0 < \beta < \Gamma^{>}$. Then β being of countable type over Γ yields a strictly decreasing sequence (γ_n) in $\Gamma^{>}$ that is coinitial in $\Gamma^{>}$. Then $(\gamma_m^\dagger)$ is increasing, (γ_n') is decreasing, $\gamma_m^\dagger < \beta^\dagger < \gamma_n'$ for all m, n, and $(\gamma_m^\dagger)$ is cofinal in $\Gamma^{<\beta^\dagger}$ and (γ_n') is coinitial in $\Gamma^{>\beta^\dagger}$. So here $\beta^\dagger$ is also of countable type over Γ. ☐

Good approximations. Let Γ be an ordered vector space over $\boldsymbol{k}$, and let α, γ range over Γ. An *extension* of Γ is an ordered vector space over $\boldsymbol{k}$ extending Γ.

Lemma 8.5. *Let $\beta \notin \Gamma$ be an element in an extension of Γ. Then for any α,*

$$[\beta - \alpha]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}} \implies [\beta - \alpha]_{\boldsymbol{k}} = \min_{\gamma} \, [\beta - \gamma]_{\boldsymbol{k}}.$$

If $\Gamma + \boldsymbol{k}\beta$ is a Hahn space, then this implication turns into an equivalence.

Proof. If $[\beta-\gamma]_{\boldsymbol{k}} < [\beta-\alpha]_{\boldsymbol{k}}$, then $(\beta-\alpha)-(\beta-\gamma)=\gamma-\alpha$ yields $[\beta-\alpha]_{\boldsymbol{k}} = [\gamma-\alpha]_{\boldsymbol{k}} \in [\Gamma]_{\boldsymbol{k}}$; this gives (the contrapositive of) "$\Rightarrow$". Suppose $\Gamma+\boldsymbol{k}\beta$ is a Hahn space. If $[\beta-\alpha]\in[\Gamma]_{\boldsymbol{k}}$, say $[\beta-\alpha]_{\boldsymbol{k}}=[\gamma]_{\boldsymbol{k}}$, then $\big[\beta-(\alpha+c\gamma)\big]_{\boldsymbol{k}} < [\beta-\alpha]_{\boldsymbol{k}}$ for some $c\in\boldsymbol{k}^\times$; this proves (the contrapositive of) "$\Leftarrow$". □

Suppose $\beta\notin\Gamma$ lies in an extension of Γ. Then a **good approximation** of β in Γ is by definition an α such that $[\beta-\alpha]_{\boldsymbol{k}}\notin[\Gamma]_{\boldsymbol{k}}$. Note that a good approximation of β in Γ exists iff $[\Gamma+\boldsymbol{k}\beta]_{\boldsymbol{k}}\neq[\Gamma]_{\boldsymbol{k}}$. Together with Lemma 8.2 this yields:

Corollary 8.6. *Suppose β lies in an extension Γ^* of Γ and Γ^* is a Hahn space. Assume also that there is no divergent pc-sequence of length ω in Γ and that β has countable type over Γ. Then β has a good approximation in Γ.*

Lemma 8.7. *Suppose $\beta\notin\Gamma$ in an extension of Γ has a good approximation α in Γ. Then the following holds:*

(i) *if $[\beta]_{\boldsymbol{k}}\in[\Gamma]_{\boldsymbol{k}}$, then $\alpha\neq 0$, $[\beta-\alpha]_{\boldsymbol{k}}<[\beta]_{\boldsymbol{k}}=[\alpha]_{\boldsymbol{k}}$; and*
(ii) *for all γ, if $\operatorname{sign}(\beta-\gamma)\neq\operatorname{sign}(\beta-\alpha)$, then $[\alpha-\gamma]_{\boldsymbol{k}}=[\beta-\gamma]_{\boldsymbol{k}}>[\beta-\alpha]_{\boldsymbol{k}}$.*

Proof. Part (i) is clear. For (ii), assume $\alpha<\beta<\gamma$; the case $\gamma<\beta<\alpha$ reduces to this case by taking negatives. Then $\gamma-\alpha>\beta-\alpha>0$, so $[\gamma-\alpha]_{\boldsymbol{k}}>[\beta-\alpha]_{\boldsymbol{k}}$, since $[\beta-\alpha]_{\boldsymbol{k}}\notin[\Gamma]_{\boldsymbol{k}}$. Thus $[\beta-\gamma]_{\boldsymbol{k}}=\big[(\beta-\alpha)+(\alpha-\gamma)\big]_{\boldsymbol{k}}=[\alpha-\gamma]_{\boldsymbol{k}}$. □

In the rest of this section (Γ,ψ) is an H-couple over $\boldsymbol{k}$, and α, γ range over Γ. By an extension *of (Γ,ψ) we mean an H-couple over $\boldsymbol{k}$ that extends (Γ,ψ). We consider (Γ,ψ) as a valued ordered vector space over $\boldsymbol{k}$ with the valuation on Γ given by ψ, so $\alpha\sim\gamma$ means $(\alpha-\gamma)^\dagger>\alpha^\dagger$.* For $\alpha\neq0$ we set $\alpha^\sim:=\{\gamma:\alpha\sim\gamma\}$.

Lemma 8.8. *Suppose (Γ,ψ) is closed and $\alpha\neq0$. Then*

$$\operatorname{cf}(\alpha^\sim)\ =\ \operatorname{ci}(\alpha^\sim)\ =\ \operatorname{cf}(\Gamma^<)\ =\ \operatorname{ci}(\Gamma^>).$$

Proof. We have $\alpha^\sim=\big\{\alpha+\gamma:\gamma^\dagger>\alpha^\dagger\big\}$. The map $\alpha+\gamma\mapsto\alpha-\gamma$ is a decreasing permutation of $\alpha^\sim$, so $\operatorname{cf}(\alpha^\sim)=\operatorname{ci}(\alpha^\sim)$. We also have the decreasing map

$$\alpha+\gamma\mapsto\gamma^\dagger\colon\alpha^\sim\cap\Gamma^{>\alpha}\to\Gamma^{>\alpha^\dagger}$$

whose image is coinitial in $\Gamma^{>\alpha^\dagger}$, since (Γ,ψ) is closed. Hence $\operatorname{cf}(\alpha^\sim)=\operatorname{ci}(\Gamma^{>\alpha^\dagger})=\operatorname{ci}(\Gamma^>)$ by [ADH, 2.1.4]. □

Lemma 8.9. *Suppose (Γ,ψ) is of Hahn type, closed, and $\operatorname{cf}(\Gamma),\operatorname{cf}(\Gamma^<)>\omega$. Let β in an extension of (Γ,ψ) have countable type over Γ with $[\beta]_{\boldsymbol{k}}\notin[\Gamma]_{\boldsymbol{k}}$. Then $\beta^\dagger\notin\Gamma$, and so $\beta^\dagger$ has countable type over Γ by Lemma 8.4.*

Proof. We may replace β by $-\beta$, and so we arrange $\beta>0$. For $0<\alpha<\beta<\gamma$ we have $[\alpha]_{\boldsymbol{k}}<[\beta]_{\boldsymbol{k}}<[\gamma]_{\boldsymbol{k}}$, and so $\alpha^\dagger\geqslant\beta^\dagger\geqslant\gamma^\dagger$, but $\alpha^\dagger>\gamma^\dagger$ by the Hahn type assumption. Suppose towards a contradiction that $\beta^\dagger\in\Gamma$. We distinguish two cases. First case: $\alpha^\dagger=\beta^\dagger$ for some α with $0<\alpha<\beta$. Then $\beta^\dagger>\gamma^\dagger$ for all $\gamma>\beta$, but then $\operatorname{ci}(\Gamma^{>\beta})=\omega$ (by Lemma 8.3) and (Γ,ψ) being closed gives for such α that $\operatorname{cf}(\Gamma^{<\alpha^\dagger})=\operatorname{cf}(\Gamma^{<\beta^\dagger})\leqslant\omega$, contradicting $\operatorname{cf}(\Gamma^{<\alpha^\dagger})>\omega$. Second case: $\beta^\dagger=\gamma^\dagger$ for some $\gamma>\beta$. This leads to a contradiction in a similar way. □

We say that (Γ,ψ) is **countably spherically complete** if every pc-sequence in it of length ω pseudoconverges in it. In particular, if (Γ,ψ) is the H-couple of a maximal Hardy field (with $\boldsymbol{k}=\mathbb{R}$), then (Γ,ψ) is of Hahn type, closed, countably spherically complete, and $\operatorname{cf}(\Gamma),\operatorname{cf}(\Gamma^<)>\omega$. (See Corollary 8.1.)

If (Γ,ψ) is of Hahn type, then the valuation ψ on Γ is equivalent to the $\boldsymbol{k}$-valuation of Γ [ADH, p. 82]. If in addition (Γ,ψ) is countably spherically complete, then by Corollary 8.6, any β in an extension of (Γ,ψ) and of countable type over Γ and such that $\Gamma+\boldsymbol{k}\beta$ is a Hahn space has a good approximation in Γ.

In the next lemma only part (i) of the conclusion is needed later. The other parts are included for their independent interest.

Lemma 8.10. *Suppose (Γ,ψ) is of Hahn type, closed, and $\operatorname{cf}(\Gamma),\operatorname{cf}(\Gamma^{<})>\omega$. Let β in an extension of (Γ,ψ) have countable type over Γ, with $[\beta]_{\boldsymbol{k}}\in[\Gamma]_{\boldsymbol{k}}$, and let α_0 be a good approximation of β in Γ. Then*

(i) *$\beta_*:=(\beta-\alpha_0)^\dagger\notin\Gamma$, and β_* has countable type over Γ;*
(ii) *if $\alpha_0<\beta$, then there is a sequence (γ_n) in $\Gamma^{>\beta}$ such that*
$$[\beta-\alpha_0]_{\boldsymbol{k}}<[\gamma_n-\beta]_{\boldsymbol{k}}<[\beta]_{\boldsymbol{k}},\quad \textit{for all } n,$$
and $\big([\gamma_n-\beta]_{\boldsymbol{k}}\big)$ is strictly decreasing and coinitial in $[\Gamma]_{\boldsymbol{k}}^{>[\beta-\alpha_0]_{\boldsymbol{k}}}$;
(iii) *if $\beta<\alpha_0$, then there is a sequence (γ_n) in $\Gamma^{<\beta}$ such that*
$$[\beta-\alpha_0]_{\boldsymbol{k}}<[\beta-\gamma_n]_{\boldsymbol{k}}<[\beta]_{\boldsymbol{k}},\quad \textit{for all } n,$$
and $\big([\beta-\gamma_n]_{\boldsymbol{k}}\big)$ is strictly decreasing and coinitial in $[\Gamma]_{\boldsymbol{k}}^{>[\beta-\alpha_0]_{\boldsymbol{k}}}$; and
(iv) *$\alpha_0\sim\beta$, that is, $\beta_*>\alpha_0^\dagger=\beta^\dagger$.*

Proof. Applying Lemma 8.9 to $\beta-\alpha_0$ in the role of β gives (i). As to (ii), let $\alpha_0<\beta$ and suppose $[\alpha]_{\boldsymbol{k}}>[\beta-\alpha_0]_{\boldsymbol{k}}$; then $[\alpha]_{\boldsymbol{k}}=[\gamma-\beta]_{\boldsymbol{k}}$ for some $\gamma>\beta$: taking $\alpha>0$, this holds with $\gamma:=\alpha_0+\alpha$. Hence
$$\big\{[\gamma-\beta]_{\boldsymbol{k}}:\gamma>\beta\big\}\ =\ [\Gamma]_{\boldsymbol{k}}^{>[\beta-\alpha_0]_{\boldsymbol{k}}}$$
by Lemma 8.7(ii). Using also (i) we have a decreasing bijection
$$[\gamma-\beta]_{\boldsymbol{k}}\mapsto(\gamma-\beta)^\dagger\ :\ [\Gamma]_{\boldsymbol{k}}^{>[\beta-\alpha_0]_{\boldsymbol{k}}}\to\Gamma^{<\beta_*}\qquad(\gamma>\beta).$$
Thus $\operatorname{ci}\big([\Gamma]_{\boldsymbol{k}}^{>[\beta-\alpha_0]_{\boldsymbol{k}}}\big)=\operatorname{cf}(\Gamma^{<\beta_*})=\omega$ by (i) and Lemma 8.3 applied to β_* in the role of β, and $[\beta]_{\boldsymbol{k}}>[\beta-\alpha_0]_{\boldsymbol{k}}$ by Lemma 8.7(i). This proves (ii), and taking negatives we obtain (iii). For (iv) first note that $\operatorname{cf}(\alpha_0^\sim)=\operatorname{cf}(\Gamma^<)>\omega$ by Lemma 8.8. If $\alpha_0<\gamma<\beta$, then $\alpha_0\sim\gamma$: otherwise $\alpha_0<\gamma<\beta$ and $[\gamma-\alpha_0]_{\boldsymbol{k}}\geqslant[\alpha_0]_{\boldsymbol{k}}>[\beta-\alpha_0]_{\boldsymbol{k}}$, which is impossible. The set $\alpha_0^\sim$ must contain elements $>\beta$, since otherwise $\alpha_0^\sim$ would be a cofinal subset of $\Gamma^{<\beta}$, contradicting $\operatorname{cf}(\Gamma^{<\beta})=\omega$. Thus $\alpha_0\sim\beta$. □

Case (b) extensions. *In this subsection (Γ,ψ) is an H-couple over $\boldsymbol{k}$ with asymptotic integration, and $\beta\notin\Gamma$ is in an H-couple (Γ^*,ψ^*) over $\boldsymbol{k}$ that extends (Γ,ψ).* Let $(\Gamma\langle\beta\rangle,\psi_\beta)$ be the H-couple over $\boldsymbol{k}$ generated by β over (Γ,ψ) in (Γ^*,ψ^*). The structure of the extension $\big(\Gamma\langle\beta\rangle,\psi_\beta\big)$ of (Γ,ψ) is described in detail in [6, Section 4]: the possibilities are listed in [6, Proposition 4.1] as (a), (b), $(\mathrm{c})_n$, and $(\mathrm{d})_n$. Case (b) is as follows:

(b) We have a sequence (α_i) in Γ and a sequence (β_i) in Γ^* that is $\boldsymbol{k}$-linearly independent over Γ, such that $\beta_0=\beta-\alpha_0$ and $\beta_{i+1}=\beta_i^\dagger-\alpha_{i+1}$ for all i, and such that $\Gamma\langle\beta\rangle=\Gamma\oplus\bigoplus_{i=0}^\infty\boldsymbol{k}\beta_i$.

Lemma 8.11. *Suppose (Γ,ψ) is of Hahn type, closed, countably spherically complete, and $\operatorname{cf}(\Gamma),\operatorname{cf}(\Gamma^{<})>\omega$. Assume also that Γ^* is a Hahn space, and β has countable type over Γ. Then β falls under Case* (b).

Proof. Suppose β falls under Case (a). This means $(\Gamma + \boldsymbol{k}\beta)^\dagger = \Gamma^\dagger$. In particular, $\beta^\dagger \in \Gamma$, hence $[\beta]_{\boldsymbol{k}} \in [\Gamma]_{\boldsymbol{k}}$ by Lemma 8.9, so $(\beta - \alpha)^\dagger \notin \Gamma$ for some α, by Lemma 8.10(i) and the remark preceding that lemma, contradicting $(\Gamma + \boldsymbol{k}\beta)^\dagger = \Gamma^\dagger$.

Next, assume β falls under Case $(\mathrm{c})_n$. Then we have $\alpha_0, \dots, \alpha_n \in \Gamma$, and nonzero $\beta_0, \dots, \beta_n \in \Gamma^*$ such that $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \beta_i^\dagger - \alpha_{i+1}$ for $0 \leqslant i < n$, the vectors $\beta_0, \dots, \beta_n, \beta_n^\dagger$ are $\boldsymbol{k}$-linearly independent over Γ, and $(\Gamma + \boldsymbol{k}\beta_n^\dagger)^\dagger = \Gamma^\dagger$. As β has countable type over Γ, an induction using Lemma 8.4 gives that $\beta_0, \dots, \beta_n, \beta_n^\dagger$ have countable type over Γ. But then Case (a) would apply to $\beta_n^\dagger$ in the role of β, and we already excluded that possibility.

The cases $(\mathrm{d})_n$ are excluded because (Γ, ψ) is closed, as noted after the proof of Proposition 4.1 in [6]. □

Here is more information about Case (b):

Lemma 8.12. *Let (α_i) and (β_i) be as in* (b). *Then:*

(i) *$\beta_i^\dagger \notin \Gamma$ for all i, and thus $[\beta_i]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$ for all i;*
(ii) *α_0 is a good approximation of β in Γ;*
(iii) *α_{i+1} is a good approximation of $\beta_i^\dagger$ in Γ, for all i;*
(iv) *$\beta_i^{\dagger\dagger} \leqslant \beta_{i+1}^\dagger$ for all i;*
(v) *$(\beta_i^\dagger)$ is strictly increasing, and thus $\big([\beta_i]_{\boldsymbol{k}}\big)$ is strictly decreasing;*
(vi) *$\big[\Gamma\langle\beta\rangle\big]_{\boldsymbol{k}} = [\Gamma]_{\boldsymbol{k}} \cup \big\{[\beta_i]_{\boldsymbol{k}} : i \in \mathbb{N}\big\}$, and thus $\Psi_\beta = \Psi \cup \big\{\beta_i^\dagger : i \in \mathbb{N}\big\}$;*
(vii) *there is no $\delta \in \Gamma\langle\beta\rangle$ with $\Psi < \delta < (\Gamma^>)'$;*
(viii) *$\Gamma^<$ is cofinal in $\Gamma\langle\beta\rangle^<$.*

If (Γ, ψ) is closed and η in an extension of (Γ, ψ) realizes the same cut in Γ as β, then there is an isomorphism $\big(\Gamma\langle\beta\rangle, \psi_\beta\big) \to \big(\Gamma\langle\eta\rangle, \psi_\eta\big)$ of H-couples over $\boldsymbol{k}$ that is the identity on Γ and sends β to η. If (Γ, ψ) is of Hahn type, then so is $(\Gamma\langle\beta\rangle, \psi_\beta)$.

Proof. Except for (ii), (iii), (iv), and the isomorphism claim this is in [6, Lemma 4.2]. Now (ii) holds by $[\beta - \alpha_0]_{\boldsymbol{k}} = [\beta_0]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$, and (iii) by $[\beta_i^\dagger - \alpha_{i+1}]_{\boldsymbol{k}} = [\beta_{i+1}]_{\boldsymbol{k}} \notin [\Gamma]_{\boldsymbol{k}}$. As to (iv), this is because $[\beta_i^\dagger]_{\boldsymbol{k}} \geqslant [\beta_i^\dagger - \alpha_{i+1}]_{\boldsymbol{k}} = [\beta_{i+1}]_{\boldsymbol{k}}$ by (iii) and Lemma 8.5.

Now assume (Γ, ψ) is closed and η in an extension (Γ_1, ψ_1) of (Γ, ψ) realizes the same cut in Γ as β, in particular, $\eta \notin \Gamma$. The case $(\Gamma_1, \psi_1) = (\Gamma^*, \psi^*)$ is actually part of [6, Lemma 4.2], and one can reduce to that case: the theory of closed H-couples over $\boldsymbol{k}$ has QE in the language specified in [6, Section 3], and so there is an H-couple (Γ_1^*, ψ_1^*) extending (Γ, ψ) with embeddings $(\Gamma^*, \psi^*) \to (\Gamma_1^*, \psi_1^*)$ and $(\Gamma_1, \psi_1) \to (\Gamma_1^*, \psi_1^*)$ over Γ. □

We add the following observations:

Corollary 8.13. *Suppose $(\alpha_i) = (\alpha_0, \alpha_1, \alpha_2, \dots)$ and $(\beta_i) = (\beta_0, \beta_1, \beta_2, \dots)$ are as in* (b). *Then $-\beta$ falls under Case* (b) *with associated sequences $(-\alpha_0, \alpha_1, \alpha_2, \dots)$ and $(-\beta_0, \beta_1, \beta_2, \dots)$. Also, for any i, $\beta_i^\dagger$ falls under Case* (b) *with associated sequences $(\alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \dots)$ and $(\beta_{i+1}, \beta_{i+2}, \beta_{i+3}, \dots)$.*

Corollary 8.14. *Suppose (Γ, ψ) is closed, β has countable type over Γ, $\alpha < \beta < \gamma$ for some α, γ, and (α_i), (β_i) are as in* (b). *Then $\operatorname{cf}(\Gamma^{<\beta_i}) = \operatorname{ci}(\Gamma^{>\beta_i}) = \omega$ for all i.*

Proof. Induction using Lemma 8.4 shows that every β_i has countable type over Γ and for every i there are α, γ with $\alpha < \beta_i < \gamma$. It follows from Lemma 8.12(viii) that for any $\eta \in \Gamma\langle\beta\rangle \setminus \Gamma$ the ordered set $\Gamma^{<\eta}$ has no largest element and the ordered set $\Gamma^{>\eta}$ has no least element. Applying this to the β_i gives the desired result. □

In the next corollary we let $\boldsymbol{k}_0$ be an ordered subfield of $\boldsymbol{k}$. Then (Γ, ψ), (Γ^*, ψ^*) are also H-couples over $\boldsymbol{k}_0$.

Corollary 8.15. *Let (α_i) be a sequence in Γ and (β_i) be a sequence in Γ^*. Then β falls under Case* (b) *with respect to (α_i), (β_i) iff β falls under Case* (b) *with respect to (α_i), (β_i) when (Γ, ψ) and (Γ^*, ψ^*) are viewed as H-couples over $\boldsymbol{k}_0$.*

Proof. Use Lemma 8.12(i),(v) and $\beta_i^\dagger = \beta_{i+1} + \alpha_{i+1}$. □

Although the element β of (Γ^*, ψ^*) does not determine uniquely the sequence (β_i) in Case (b), it follows from Lemma 8.12(i),(v),(vi) that β does determine uniquely the sequences $(\beta_i^\dagger)$ and $\big([\beta_i]_{\boldsymbol{k}}\big)$. Without changing β we still have considerable flexibility in choosing the α_i and β_i:

Lemma 8.16. *Let (α_i), (β_i) be as in* (b). *Let α_0^* be a good approximation of β in Γ, and α_{i+1}^* a good approximation of $\beta_i^\dagger$ in Γ, for all i. Set $\beta_0^* := \beta - \alpha_0^*$ and $\beta_{i+1}^* := \beta_i^\dagger - \alpha_{i+1}^*$. Then (α_i^*) and (β_i^*) are also as in* (b), *with $[\beta_i^*]_{\boldsymbol{k}} = [\beta_i]_{\boldsymbol{k}}$ and $\beta_i^* - \beta_i \in \Gamma$ for all i.*

Proof. We have $\beta_i^* - \beta_i = \alpha_i - \alpha_i^* \in \Gamma$ for each i and so $\Gamma\langle\beta\rangle = \Gamma \oplus \bigoplus_{i=0}^\infty \boldsymbol{k}\beta_i^*$. From Lemma 8.5 and Lemma 8.12(ii),(iii) we get $[\beta_i^*]_{\boldsymbol{k}} = [\beta_i]_{\boldsymbol{k}}$ for all i, and so $\beta_{i+1}^* = \beta_i^\dagger - \alpha_{i+1}^* = (\beta_i^*)^\dagger - \alpha_{i+1}^*$ as required. □

Next we consider a shift $(\Gamma, \psi - \gamma)$ of (Γ, ψ) and replace β by $\beta - \gamma$, viewed as an element of the extension $(\Gamma^*, \psi^* - \gamma)$ of $(\Gamma, \psi - \gamma)$:

Lemma 8.17. *Let (α_i), (β_i) be as in* (b). *Then $\beta - \gamma$ falls under* (b) *with respect to the indicated shifts, as witnessed by the sequences $(\alpha_i - \gamma)$, (β_i).*

At the end of the introduction we defined $\alpha^{\langle n\rangle}$. This comes into play now.

Lemma 8.18. *Let (α_i) and (β_i) be as in* (b), *and suppose that $\beta_i^\dagger < 0$ for all i. Then $\beta_i^{\langle n+1\rangle} \leqslant \beta_{i+n}^\dagger < 0$ for all i and all n.*

Proof. This is trivial for $n = 0$. Suppose $\beta_i^{\langle n+1\rangle} \leqslant \beta_{i+n}^\dagger$. Then by Lemma 8.12(iv),

$$\beta_i^{\langle n+2\rangle} \ \leqslant\ \beta_{i+n}^{\dagger\dagger} \ \leqslant\ \beta_{i+n+1}^\dagger.$$ □

We next discuss a situation where we can arrange that $\beta_i^\dagger < 0$ for all i.

Remark 8.19. Suppose $\operatorname{cf}(\Gamma^<) > \omega$ and (α_i), (β_i) are as in (b). Then $\operatorname{cf}(\Psi) = \operatorname{cf}(\Gamma^<) > \omega$, so we have $\gamma \in \Psi$ with $\beta_i^\dagger < \gamma$ for all i, hence $\beta_i^\dagger - \gamma < 0$ for all i. Thus $\beta - \gamma$ falls under Case (b) with respect to the shifts $(\Gamma, \psi - \gamma)$ and $(\Gamma^*, \psi^* - \gamma)$ and for the associated sequences $(\alpha_i - \gamma)$, (β_i) we have $(\psi^* - \gamma)(\beta_i) < 0$ for all i, so that the hypothesis of Lemma 8.18 is satisfied for this shifted situation.

Constructing a case (b)-extension. Let K be a Liouville closed H-field; below we view its asymptotic couple (Γ, ψ) as an H-couple over $\boldsymbol{k} := \mathbb{Q}$. Assume $\beta \notin \Gamma$ in an extension (Γ^*, ψ^*) of (Γ, ψ) falls under Case (b). We show:

Proposition 8.20. *There exists an H-field extension $K\langle y\rangle$ of K such that:*

(i) *$y > 0$ and $vy \notin \Gamma$ realizes the same cut in Γ as β;*
(ii) *for any H-field extension M of K and any $z \in M^>$ such that $vz \notin \Gamma$ and vz realizes the same cut in Γ as β, there is an H-field embedding $K\langle y\rangle \to M$ over K sending y to z.*

Proof. Model-theoretic compactness gives a Liouville closed H-field extension L of K with $y \in L^{>}$ such that $vy \notin \Gamma$ realizes the same cut in Γ as β. Lemma 8.12 then yields an isomorphism $\big(\Gamma\langle\beta\rangle, \psi_\beta\big) \to \big(\Gamma\langle vy\rangle, \psi_y\big)$ of H-couples over $\mathbb{Q}$ that is the identity on Γ and sends β to vy. (Here $\big(\Gamma\langle vy\rangle, \psi_y\big)$ is the H-couple over $\mathbb{Q}$ generated by $\Gamma \cup \{vy\}$ in the H-couple of L over $\mathbb{Q}$.) It follows that $\Gamma\langle vy\rangle/\Gamma$ has infinite dimension as a vector space over $\mathbb{Q}$, so y is differentially transcendental over K in view of $\Gamma\langle vy\rangle \subseteq v(K_y^\times)$ where K_y is the real closure of $K\langle y\rangle$ in L. We claim that $K\langle y\rangle$ has the properties stated in the proposition; in particular, we show that $K\langle y\rangle$ is an H-subfield of L, not just an asymptotic (ordered) subfield of L.

Let (α_i) and (β_i) be as in (b); for each i, take $f_i \in K^{>}$ such that $vf_i = \alpha_i$. We define $y_i \in K\langle y\rangle$ by recursion: $y_0 := y/f_0$, and $y_{i+1} = y_i^\dagger/f_{i+1}$; to make this recursion possible we simultaneously show by induction on i that $y_i \neq 0$ and $vy_i \notin \Gamma$ realizes the same cut in Γ as β_i, and $v(y_i^\dagger) \notin \Gamma$ realizes the same cut in Γ as $\beta_i^\dagger$. This is all straightforward using the above isomorphism

$$\big(\Gamma\langle\beta\rangle, \psi_\beta\big) \to \big(\Gamma\langle vy\rangle, \psi_y\big), \tag{8.1}$$

which sends β_i to vy_i for all i. Likewise we obtain that for all n,

$$K_n \ := \ K\big(y, y', \dots, y^{(n)}\big) \ = \ K(y_0, \dots, y_n) \ = \ K\big(y, \dots, y^{\langle n\rangle}\big), \text{ and}$$
$$v(K_n^\times) \ = \ \Gamma \oplus \mathbb{Z}vy_0 \oplus \cdots \oplus \mathbb{Z}vy_n \ \subseteq \ \Gamma\langle vy\rangle,$$

with the above isomorphism (8.1) restricting to an isomorphism

$$\Gamma \oplus \mathbb{Z}\beta_0 \oplus \cdots \oplus \mathbb{Z}\beta_n \ \to \ \Gamma \oplus \mathbb{Z}vy_0 \oplus \cdots \oplus \mathbb{Z}vy_n, \qquad \beta_i \mapsto vy_i \quad (i = 0, \dots, n)$$

of ordered abelian groups. Hence the residue field $\operatorname{res}(K_n)$ of the valued subfield K_n of L is algebraic over $\operatorname{res}(K)$ by [ADH, 3.1.11] (Zariski-Abhyankar), and so $\operatorname{res}(K)$ being real closed gives $\operatorname{res}(K_n) = \operatorname{res}(K)$. Then from $K\langle y\rangle = \bigcup_n K_n$ we obtain $\operatorname{res}\big(K\langle y\rangle\big) = \operatorname{res}(K)$, so $K\langle y\rangle$ is an H-subfield of L with the same constant field as K, by [ADH, 9.1.2]. So far we only used $y \neq 0$ rather than $y > 0$.

Next, let M be any H-field extension of K and $z \in M^\times$ such that $vz \notin \Gamma$ realizes the same cut as β in Γ. By increasing M we can assume M is Liouville closed, and then all the above goes through with z instead of y. In particular, setting $z_0 := z/f_0$ and $z_{i+1} := z_i^\dagger/f_{i+1}$, we obtain for each n an isomorphism of the valued subfield K_n of L onto the valued subfield $K(z_0, \dots, z_n)$ of M over K, sending y_i to z_i for $i = 0, \dots, n$. These have a common extension to a valued differential field isomorphism $K\langle y\rangle \to K\langle z\rangle$ over K sending y to z. For this isomorphism to preserve the ordering, we now assume besides $y > 0$ that also $z > 0$. Induction on i then shows that y_i and z_i are both positive, or both negative, for each i: use that all $f_i > 0$ and that for any g in any H-field we have:

$$g \succ 1 \ \Rightarrow \ g^\dagger > 0, \qquad g \prec 1 \ \Rightarrow \ g^\dagger < 0.$$

The valuation determines for every polynomial $P(Y_0, \dots, Y_n) \in K[Y_0, \dots, Y_n]^{\neq}$ the unique dominant term in $P(y_0, \dots, y_n)$ and in $P(z_0, \dots, z_n)$ in the same way, so this isomorphism $K\langle y\rangle \to K\langle z\rangle$ is also order-preserving. $\square$

Remark 8.21. Let $K\langle y\rangle$ be an H-field extension of K with $y > 0$ such that vy realizes the same cut in Γ as β, with real closure $F := K\langle y\rangle^{\mathrm{rc}}$. By the proof above F has the same constant field as K, and the H-couple of F over $\mathbb{Q}$ is generated over (Γ, ψ) by vy, as witnessed by an isomorphism $\big(\Gamma\langle\beta\rangle, \psi_\beta\big) \to (\Gamma_F, \psi_F)$ over Γ sending β to vy.

With f_i, y_i as in the proof above (so $y_i^\dagger = f_{i+1}y_{i+1}$ for all i), we think informally of the element y in Proposition 8.20 as given in terms of the f_i by

$$y \;=\; f_0 y_0 \;=\; f_0\,\mathrm{e}^{\int f_1 y_1} \;=\; f_0\,\mathrm{e}^{\int f_1\,\mathrm{e}^{\int f_2 y_2}} \;=\; \cdots \;=\; f_0\,\mathrm{e}^{\int f_1\,\mathrm{e}^{\int f_2\,\mathrm{e}^{\int \cdots}}}$$

In the next section we show how to construct such a y analytically when K is a Liouville closed Hardy field containing $\mathbb{R}$, under additional hypotheses on β.

9. Filling Gaps of Type (b)

In Section 8—see in particular the remark at the beginning of that section and the remark preceding Lemma 8.4—we showed that Theorem A reduces to:

Lemma 9.1. *Let H be a maximal Hardy field with H-couple (Γ,ψ) over $\mathbb{R}$. Then no element in any extension of (Γ,ψ) has countable type over Γ.*

Proof. Suppose towards a contradiction that β in some extension of (Γ,ψ) has countable type over Γ. Then β falls under Case (b) by the remarks that precede Lemma 8.10 and by Lemma 8.11. Let (α_i), (β_i) be as in (b). Then $(\beta_i^\dagger)$ is strictly increasing by Lemma 8.12(v). Since $\operatorname{cf}(\Gamma^<) = \operatorname{cf}(\Psi) > \omega$, we can take $\gamma\in\Psi$ such that $\beta_i^\dagger < \gamma$ for all i. Take $g\in H^>$ with $vg=\gamma$, and $\ell\in H$ with $\ell' = g$, so $\ell > \mathbb{R}$. Composing with ℓ^{inv} yields a maximal Hardy field $H\circ\ell^{\mathrm{inv}}$ whose H-couple over $\mathbb{R}$ we identify with the shift $(\Gamma,\psi-\gamma)$ of (Γ,ψ). As indicated in Remark 8.19 this allows us to replace H by $H\circ\ell^{\mathrm{inv}}$ and β by $\beta-\gamma$. By renaming we thus arrange that $\beta_i^\dagger < 0$ for all i. This situation is impossible by Theorem 9.2 below. □

Theorem 9.2 is of interest independent of Theorem A and Lemma 9.1, since it involves a new way of constructing certain Hardy field extensions.

Theorem 9.2. *Let $H\supseteq\mathbb{R}$ be a Liouville closed Hardy field with H-couple (Γ,ψ) over $\mathbb{R}$. Suppose β in an extension of (Γ,ψ) and of countable type over Γ falls under Case* (b)*, and $\beta_i^\dagger < 0$ for all i, where (α_i), (β_i) are as in* (b)*. Then there exists $y\neq 0$ in a Hardy field extension of H such that vy realizes the same cut in Γ as β.*

The special cases $\beta<\Gamma$ and $\beta>\Gamma$ of Theorem 9.2 are taken care of by Section 5: say $\beta<\Gamma$; then $\operatorname{ci}(\Gamma)=\operatorname{cf}(H)=\omega$, and so there are overhardian $y >_{\mathrm{e}} H$, and any such y has the desired property by Corollary 5.10.

The rest of this section proves Theorem 9.2 in the case where $\alpha<\beta<\gamma$ for some $\alpha,\gamma\in\Gamma$. As we saw, this is also the final step in proving Theorem A.

Some useful inclusions. Let $H\supseteq\mathbb{R}$ be a Liouville closed Hardy field with H-couple (Γ,ψ) over $\mathbb{R}$, and let α, γ range over Γ. Let β in an extension of (Γ,ψ) of Hahn type be such that $\alpha<\beta<\gamma$ for some α, γ, $[\beta]\notin[\Gamma]$ (so $\beta^\dagger\notin\Gamma$), and $\beta^{\langle n\rangle}<0$ for all $n\geqslant 1$. (We do allow $\beta>0$, but use $-|\beta|$ below to arrange a value <0, with $(-|\beta|)^{\langle n\rangle}=\beta^{\langle n\rangle}$ for $n\geqslant 1$.) Set

$$A \;:=\; \big\{h\in H^{>\mathbb{R}} :\, -|\beta| < vh\big\},\qquad B:=\;\big\{h\in H^{>\mathbb{R}} : vh < -|\beta|\big\}.$$

Then $A\cup B = H^{>\mathbb{R}}$, $A<B$, and so there is no $h\in H$ with $A<h<B$. Also

$$vA\cup vB \;=\; \Gamma^<,\quad vB \;<\; -|\beta| \;<\; vA \;<\; 0,$$

and so there is no α with $vB < \alpha < vA$.

Lemma 9.3. *The sets A and B have the following properties:*

(i) $e_n := \exp_n(x) \in A$ *for all* n, *and* $B \neq \emptyset$;
(ii) $A = \operatorname{sq}(A)$ *and* $B = \sqrt{B}$.

Proof. As to (i), an easy induction shows that $e_n^\dagger = e_1 \cdots e_{n-1}$ for $n \geqslant 1$ and $e_n^{\langle m \rangle} \sim e_{n-m+1}^\dagger$ for $n \geqslant m \geqslant 1$. In particular, $e_n^{\langle n \rangle} \sim 1$ for $n \geqslant 1$. Since $\beta^{\langle n \rangle} < 0$ for all $n \geqslant 1$, this gives $v(e_n) > \beta$ for all n. Item (ii) follows from $[\beta] \notin [\Gamma]$. □

We now set

$$A^\dagger \ := \ \{a^\dagger : a \in A,\ a^\dagger \succ 1\}, \quad B^\dagger \ := \ \{b^\dagger : b \in B\}, \text{ so in view of } \beta^\dagger \notin \Gamma:$$
$$A^\dagger \ = \ \{h \in H^{>\mathbb{R}} : \beta^\dagger < vh\}, \quad B^\dagger \ = \ \{h \in H^{>\mathbb{R}} : vh < \beta^\dagger\}.$$

Thus $A^\dagger \cup B^\dagger = H^{>\mathbb{R}}$, $A^\dagger < B^\dagger$, and there is no $h \in H$ with $A^\dagger < h < B^\dagger$. Also

$$v(A^\dagger) \cup v(B^\dagger) \ = \ \Gamma^<, \qquad v(B^\dagger) \ < \ \beta^\dagger \ < \ v(A^\dagger) \ < \ 0,$$

and there is no α with $v(B^\dagger) < \alpha < v(A^\dagger)$. Note also that $e_n^\dagger \in A^\dagger$ for all $n \geqslant 2$.

Corollary 9.4. $\log A \subseteq A^\dagger \subseteq A$ *and* $\log B \supseteq B^\dagger \supseteq B$.

Proof. If $h \in H$, $h \geqslant e_2$, then $\log h \preccurlyeq (\log h)' = h^\dagger$. Then by Lemma 9.3(i) we have $\log A \subseteq A^\dagger$. Now use $\log A < \log B$ and $\log A \cup \log B = A^\dagger \cup B^\dagger = H^{>\mathbb{R}}$. As to $A^\dagger \subseteq A$: if $h \in H^{>\mathbb{R}}$ and $h^\dagger \succ 1$, then $vh^\dagger = o(vh)$ by [ADH, 9.2.10(iv)]. □

To indicate the dependence of A, B, $A^\dagger$, $B^\dagger$ on β we may denote these sets by

$$A(\beta), \quad B(\beta), \quad A^\dagger(\beta), \quad B^\dagger(\beta).$$

In fact, these four sets depend only on $[\beta]$ rather than β, in view of $[\beta] \notin [\Gamma]$.

Recall that $\beta^\dagger \notin \Gamma$ and $\beta^\dagger < 0$, so if $\beta^\dagger$ has a good approximation in Γ, it has a good approximation $\leqslant 0$ in Γ. Note: if $[\beta^\dagger] \notin [\Gamma]$, then 0 is a good approximation of β in Γ and any good approximation $\alpha \leqslant 0$ to $\beta^\dagger$ in Γ satisfies $\beta^\dagger < \alpha$.

Suppose now that $\alpha \leqslant 0$ is a good approximation of $\beta^\dagger$ in Γ, so $[\beta^\dagger - \alpha] \notin [\Gamma]$. Set $\beta_{\text{next}} := \beta^\dagger - \alpha$, and assume also that $\beta_{\text{next}}^{\langle n \rangle} < 0$ for all $n \geqslant 1$. This means that the conditions we imposed earlier on β are now also satisfied by β_{next}. Since $[\beta_{\text{next}}]$ does not depend on the particular good approximation $\alpha \leqslant 0$ of $\beta^\dagger$ in Γ,

$$A(\beta_{\text{next}}) \ = \ \{h \in H^{>\mathbb{R}} : vh > -|\beta_{\text{next}}|\}$$

doesn't either, and the assumption that $\beta_{\text{next}}^{\langle n \rangle} < 0$ for all $n \geqslant 1$ will still be satisfied for any such α.

Lemma 9.5. $A(\beta_{\text{next}}) \subseteq \log A(\beta)$.

Proof. Let $h \in A(\beta_{\text{next}})$; it suffices to show that then $e^h \in A(\beta)$. Suppose towards a contradiction that $e^h \in B$. Then $v\, e^h < -|\beta|$, so $v(e^h)^\dagger = vh' < \beta^\dagger < 0$. If $[\beta^\dagger] \in \Gamma$, then $[\beta^\dagger] > [\beta_{\text{next}}]$, so $h' \in B(\beta_{\text{next}})$, and thus $h \in B(\beta_{\text{next}})$ by Lemma 1.13(ii) applied to $B(\beta_{\text{next}})$ in the role of B. If $[\beta^\dagger] \notin \Gamma$, then $[\beta^\dagger] = [\beta_{\text{next}}]$, and again $h' \in B(\beta_{\text{next}})$, so $h \in B(\beta_{\text{next}})$. In both cases we contradict $h \in A(\beta_{\text{next}})$. □

The diagram in Figure 10 depicts the gaps

$$(A, B) = \big(A(\beta), B(\beta)\big), \quad (A^\dagger, B^\dagger), \quad (\log A, \log B), \quad \big(A(\beta_{\text{next}}), B(\beta_{\text{next}})\big)$$

in H and hypothetical H-hardian germs y, y_{next} with $A < y < B$ and $A(\beta_{\text{next}}) < y_{\text{next}} < B(\beta_{\text{next}})$, as well as $y^\dagger$ and $\log y$.

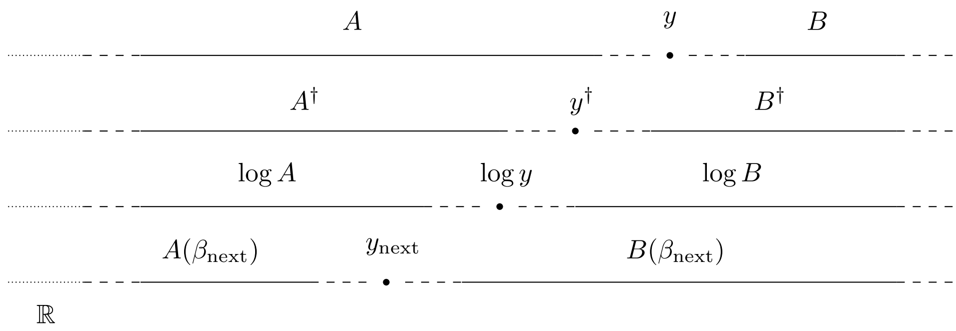

FIGURE 10. Various gaps in H associated to (A, B)

Lemma 9.6. *We have $[\beta] > [\beta_{\text{next}}]$. If there is no γ such that $[\beta] > [\gamma] > [\beta_{\text{next}}]$, then $A(\beta) = A(\beta_{\text{next}})$, $B(\beta) = B(\beta_{\text{next}})$, and A, B is a wide gap.*

Proof. From [ADH, 9.2.10(iv)] and $\beta^\dagger < 0$ we get $\beta^\dagger = o(\beta)$, so $[\beta] > [\beta^\dagger] \geqslant [\beta_{\text{next}}]$. Suppose there is no γ with $[\beta] > [\gamma] > [\beta_{\text{next}}]$. Then clearly $A(\beta) = A(\beta_{\text{next}})$ and $B(\beta) = B(\beta_{\text{next}})$, so $A \subseteq \log A$ by Lemma 9.5. Thus A, B is a wide gap. □

By Lemma 9.3 we have $\mathrm{e}_n \in A(\beta_{\text{next}})$ for all n. In combination with the next result this gives further information about the behavior of A and B and of the gap between them. For $p, \phi \in \mathcal{C}$ with $\phi >_{\mathrm{e}} 0$ we have the germ $\phi^p \in \mathcal{C}$. Let $p \in H$; then $h \in H^>$ gives $h^p = \exp(p \log h) \in H^>$, and for $S \subseteq H^>$ we set $S^p := \{h^p : h \in S\} \subseteq H^>$.

Proposition 9.7. *The sets A, $A^\dagger$, $A(\beta_{\text{next}})$, B have the following properties:*

(i) *$A(\beta_{\text{next}}) \cdot A^\dagger \subseteq A^\dagger$;*
(ii) *if $p \in A(\beta_{\text{next}})$, then $A^p \subseteq A$ and $B^{1/p} \subseteq B$;*
(iii) *if $p \in A(\beta_{\text{next}})$, $\phi \in \mathcal{C}$, and $A <_{\mathrm{e}} \phi <_{\mathrm{e}} B$, then $A <_{\mathrm{e}} \phi^{1/p} <_{\mathrm{e}} \phi <_{\mathrm{e}} \phi^p <_{\mathrm{e}} B$.*

Proof. For (i) we distinguish two cases. Suppose first that $\beta^\dagger < \alpha$. Let $p \in A(\beta_{\text{next}})$, $h \in A^\dagger$; we need to show $ph \in A^\dagger$, that is, $v(ph) > \beta^\dagger$, equivalently, $vp > \beta^\dagger - vh$. Since $vp > \beta_{\text{next}} = \beta^\dagger - \alpha$, we do have $vp > \beta^\dagger - vh$ if $vh \geqslant \alpha$. If $vh < \alpha$, then $\beta^\dagger < vh < \alpha$, so vh is also a good approximation of $\beta^\dagger$ in Γ, and then replacing α by vh yields $vp > \beta^\dagger - vh$ in view of remarks made earlier about $A(\beta_{\text{next}})$.

Next, suppose $\alpha < \beta^\dagger$, so $[\beta^\dagger] \in [\Gamma]$ by an earlier remark. Let $p \in A(\beta_{\text{next}})$, $h \in A^\dagger$; as before we need to show $vp > \beta^\dagger - vh$. Now $\alpha < \beta^\dagger < vh$ gives $[\alpha - \beta^\dagger] < [\beta^\dagger - vh]$ by Lemma 8.7(ii). Since $\alpha - \beta^\dagger$ and $\beta^\dagger - vh$ are both negative, this yields $\alpha - \beta^\dagger > \beta^\dagger - vh$, which together with $vp > \alpha - \beta^\dagger$ gives $vp > \beta^\dagger - vh$.

As to (ii), let $p \in A(\beta_{\text{next}})$ and $h \in A$. We have $h^p > \mathbb{R}$ and

$$(h^p)^\dagger \;=\; (p \log h)' \;=\; p' \log h + ph^\dagger,$$

and $ph^\dagger \preccurlyeq 1$ or $ph^\dagger \in A^\dagger$ by (i). Also $p' \in A(\beta_{\text{next}})$ or $0 < p' \preccurlyeq 1$, by Lemma 1.12, and $\log p \in A^\dagger$ by Corollary 9.3, so $p' \log h \preccurlyeq 1$ or $p' \log h \in A^\dagger$ by (i). Hence $(h^p)^\dagger \in A^\dagger$, and thus $h^p \in A$. Next, let $p \in A(\beta_{\text{next}})$ and $h \in B$. Then $h^{1/p} \notin B$ would mean $h^{1/p} \in A$ or $0 < h^{1/p} \preccurlyeq 1$, and in either case $h = (h^{1/p})^p$ would give $h \in A$ or $h \preccurlyeq 1$, contradicting $h \in B$. This concludes the proof of (ii).

Property (iii) is a routine consequence of (ii). □

Part (iii) of Proposition 9.7 is only relevant if there is any $\phi \in \mathcal{C}$ with $A <_{\mathrm{e}} \phi <_{\mathrm{e}} B$. There are indeed such ϕ if $\mathrm{cf}(A) = \mathrm{ci}(B) = \omega$, by Corollary 2.8.

To describe $A(\beta_{\text{next}})$ directly in terms of $A^\dagger$, take $f \in H^{>}$ with $vf = \alpha$. Then:

Lemma 9.8. *If $\beta^\dagger < \alpha$, then $f \in A^\dagger$ or $f \asymp 1$, and*

$$A(\beta_{\text{next}}) \ = \ H^{>\mathbb{R}} \cap f^{-1}A^\dagger, \qquad B(\beta_{\text{next}}) \ = \ f^{-1}B^\dagger.$$

If $\alpha < \beta^\dagger$, then $f \in B^\dagger$, $[\beta^\dagger - \alpha] < [\beta^\dagger] = [\alpha] \in [\Gamma]$, and

$$A(\beta_{\text{next}}) \ = \ H^{>\mathbb{R}} \cap f(B^\dagger)^{-1}, \quad f(A^\dagger)^{-1} \textit{ is a coinitial subset of } B(\beta_{\text{next}}).$$

Proof. If $\beta^\dagger < \alpha$, then the inclusion $A(\beta_{\text{next}}) \supseteq H^{>\mathbb{R}} \cap f^{-1}A^\dagger$ and the equality $B(\beta_{\text{next}}) = f^{-1}B^\dagger$ are almost obvious, and one can use $\alpha \leqslant 0$ to prove the inclusion $A(\beta_{\text{next}}) \subseteq H^{>\mathbb{R}} \cap f^{-1}A^\dagger$.

Next, suppose $\alpha < \beta^\dagger$. Then for the inclusion $A(\beta_{\text{next}}) \supseteq H^{>\mathbb{R}} \cap f(B^\dagger)^{-1}$, use $\beta^\dagger < \alpha - \beta^\dagger$, and for the statement about $B(\beta_{\text{next}})$, note that

$$B(\beta_{\text{next}}) \ = \ \big\{h \in H^{>\mathbb{R}} : vh < \alpha - \beta^\dagger\big\}, \text{ and}$$
$$f(A^\dagger)^{-1} \ = \ \big\{h \in H^{>\mathbb{R}} : \alpha < vh < \alpha - \beta^\dagger\big\}. \qquad \square$$

Using the first part of Lemma 9.8 we obtain:

Corollary 9.9. *Suppose $\beta_{\text{next}} < 0$ and $z_0, z_1 \in \mathcal{C}^{<\infty}$ are such that*

$$z_0 >_{\mathrm{e}} 0, \qquad z_0^\dagger \ = \ fz_1, \qquad A(\beta_{\text{next}}) <_{\mathrm{e}} z_1 <_{\mathrm{e}} B(\beta_{\text{next}}).$$

Then $A <_{\mathrm{e}} z_0 <_{\mathrm{e}} B$.

Proof. Let $h \in A$. Then $f^{-1}h^\dagger \in A(\beta_{\text{next}})$ or $f^{-1}h^\dagger \preccurlyeq 1$, so $f^{-1}h^\dagger <_{\mathrm{e}} z_1 = f^{-1}z_0^\dagger$, and thus $h^\dagger <_{\mathrm{e}} z_0^\dagger$. Then Lemma 2.16 gives $c \in \mathbb{R}^{>}$ with $ch <_{\mathrm{e}} z_0$. Applying this argument to h^2 instead of h gives $d \in \mathbb{R}^{>}$ with $dh^2 <_{\mathrm{e}} z_0$, which in view of $h <_{\mathrm{e}} dh^2$ gives $h <_{\mathrm{e}} z_0$. In the same way one shows that if $h \in B$, then $z_0 <_{\mathrm{e}} h$. $\square$

Likewise, using the second part of Lemma 9.8:

Corollary 9.10. *If $\beta_{\text{next}} > 0$ and $z_0, z_1 \in \mathcal{C}^{<\infty}$ are such that*

$$z_0, z_1 >_{\mathrm{e}} 0, \qquad z_0^\dagger \ = \ f/z_1, \qquad A(\beta_{\text{next}}) <_{\mathrm{e}} z_1 <_{\mathrm{e}} B(\beta_{\text{next}}),$$

then $A <_{\mathrm{e}} z_0 <_{\mathrm{e}} B$.

What remains to be done. Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field with H-couple (Γ, ψ) over $\mathbb{R}$, and let α, γ range over Γ. Suppose β in an extension of (Γ, ψ) is of countable type over Γ and falls under Case (b), with (α_i), (β_i) as in (b), and $\beta_i^\dagger < 0$, $\alpha_{i+1} \leqslant 0$ for all i. Assume also that $\alpha < \beta < \gamma$ for some α, γ. Then $\big(\Gamma\langle\beta\rangle, \psi_\beta\big)$ is of Hahn type, by the last claim in Lemma 8.12, and so all β_i lie in this extension of (Γ, ψ) of Hahn type; this is significant because of the initial assumption on β in the previous subsection. Note also that for all i there are α, γ with $\alpha < \beta_i < \gamma$, and that by Lemma 8.18 we have $\beta_i^{\langle n\rangle} < 0$ for all i and all $n \geqslant 1$. Thus we can apply the previous subsection to each β_i in the role of β there. Set

$$A_i \ := \ \big\{h \in H^{>\mathbb{R}} : vh > -|\beta_i|\big\}, \qquad B_i \ := \ \big\{h \in H : vh < -|\beta_i|\big\},$$

so $A_i = A(\beta_i)$, $B_i = B(\beta_i)$, and $\beta_{i+1} = (\beta_i)_{\text{next}}$ in the notation of the previous subsection. Thus by lemmas in that subsection:

(i) $\mathrm{e}_n \in A_i$ for all i, n;
(ii) A_i and $\mathrm{sq}(A_i)$ are cofinal, and B_i and $\sqrt{B_i}$ are coinitial;

(iii) $h \in A_i \Rightarrow h^{\mathrm{e}_n} \in A_i$, and $h \in B_i \Rightarrow h^{1/\mathrm{e}_n} \in B_i$;

By Corollary 8.14 we have $\mathrm{cf}(\Gamma^{<\beta_i}) = \mathrm{ci}(\Gamma^{>\beta_i}) = \omega$ for all i. Hence $\mathrm{cf}(A_i) = \mathrm{ci}(B_i) = \omega$ for all i. What remains to be done is to show the existence of a $y > 0$ in a Hardy field extension of H such that vy realizes the same cut in Γ as β.

For each i, take $f_i \in H^>$ with $vf_i = \alpha_i$, and $f_i \succcurlyeq 1$ for $i \geqslant 1$. To get the right idea for our reverse engineering, suppose $y > 0$ is H-hardian and vy realizes the same cut in Γ as β. As in the proof of Proposition 8.20, let $y_i \in H\langle y\rangle$ be given by $y_0 := y/f_0$, and $y_{i+1} = y_i^\dagger/f_{i+1}$. Then vy_i realizes the same cut in Γ as β_i, so $y_i \succ 1$ if $\beta_i < 0$ and $y_i \prec 1$ if $\beta_i > 0$. To have only positive infinite germs, set

$$z_i \ :=\ |y_i| \ \text{ if } \ \beta_i < 0, \qquad z_i \ :=\ |y_i|^{-1} \ \text{ if } \ \beta_i > 0.$$

One verifies easily that then $A_i <_{\mathrm{e}} z_i <_{\mathrm{e}} B_i$, and

$$\beta_{i+1} < 0 \implies z_i^\dagger \ =\ f_{i+1}z_{i+1}, \qquad \beta_{i+1} > 0 \implies z_i^\dagger \ =\ f_{i+1}/z_{i+1}.$$

We first deal with a "wide gap" case:

Lemma 9.11. *Suppose for some n there is no γ with $[\beta_n] > [\gamma] > [\beta_{n+1}]$. Then there exists H-hardian $y > 0$ such that vy realizes the same cut in Γ as β.*

Proof. Let n be as in the hypothesis. Then A_n, B_n is a wide gap by Lemma 9.6, hence Section 6 gives an H-hardian z_n such that $A_n <_{\mathrm{e}} z_n <_{\mathrm{e}} B_n$. Let $L := \mathrm{Li}\big(H\langle z_n\rangle\big)$. Then we have $z_{n-1}, \dots, z_0 \in L^>$ such that for all $i < n$:

$$\beta_{i+1} < 0 \implies z_i^\dagger \ =\ f_{i+1}z_{i+1}, \qquad \beta_{i+1} > 0 \implies z_i^\dagger \ =\ f_{i+1}/z_{i+1}.$$

Downward induction on i using Corollaries 9.9 and 9.10 then gives $A_i <_{\mathrm{e}} z_i <_{\mathrm{e}} B_i$ for all $i \leqslant n$. Thus if $\beta_0 < 0$, then $v(z_0)$ realizes the same gap in Γ as β_0, and so $y := f_0 z_0$ has the desired property. If $\beta_0 > 0$, then $v(z_0)$ realizes the same gap in Γ as $-\beta_0$, and so $y := f_0/z_0$ has the desired property. $\square$

It remains to consider the case that for all i there exists γ with $[\beta_i] > [\gamma] > [\beta_{i+1}]$. We assume this for the rest of this section. The goal of our reverse engineering will be to construct germs z_i as in the next lemma:

Lemma 9.12. *Let the germs $z_i \in \mathcal{C}^{<\infty}$ be such that for all i, $A_i <_{\mathrm{e}} z_i <_{\mathrm{e}} B_i$ and*

$$\beta_{i+1} < 0 \implies z_i^\dagger \ =\ f_{i+1}z_{i+1}, \qquad \beta_{i+1} > 0 \implies z_i^\dagger \ =\ f_{i+1}/z_{i+1}.$$

Then there exists H-hardian $y > 0$ such that vy realizes the same cut in Γ as β.

Proof. Note that for each n we have the ordered subgroup $\Gamma \oplus \mathbb{Z}\beta_0 \oplus \cdots \oplus \mathbb{Z}\beta_n$ of $\Gamma\langle\beta\rangle$, and likewise with $\mathbb{Q}$ instead of $\mathbb{Z}$. We prove by induction on n that $z_0, \dots, z_n$ generate a Hausdorff field $H_n := H(z_0, \dots, z_n)$ over H, with

$$v(H_n^\times) \ =\ \Gamma \oplus \mathbb{Z}vz_0 \oplus \cdots \oplus \mathbb{Z}vz_n,$$

and with an ordered abelian group isomorphism that is the identity on Γ:

$$\Gamma \oplus \mathbb{Z}\beta_0 \oplus \cdots \oplus \mathbb{Z}\beta_n \ \to\ \Gamma \oplus \mathbb{Z}vz_0 \oplus \cdots \oplus \mathbb{Z}vz_n, \qquad -|\beta_i| \mapsto vz_i \quad (i = 0, \dots, n).$$

For $n = 0$ this follows from Lemma 1.11. Assume that the above holds for a certain n. Then for the real closure H_n^{rc} of H_n as a Hausdorff field extension of H_n,

$$v\big(H_n^{\mathrm{rc},\times}\big) \ =\ \Gamma \oplus \mathbb{Q}vz_0 \oplus \cdots \oplus \mathbb{Q}vz_n,$$

with an ordered abelian group isomorphism that is the identity on Γ:

$$\Gamma \oplus \mathbb{Q}\beta_0 \oplus \cdots \oplus \mathbb{Q}\beta_n \ \to\ \Gamma \oplus \mathbb{Q}vz_0 \oplus \cdots \oplus \mathbb{Q}vz_n, \qquad -|\beta_i| \mapsto vz_i \quad (i = 0, \dots, n).$$

Thus $\big[v\big(H_n^{\mathrm{rc},\times}\big)\big] = [\Gamma] \cup \big\{[vz_0],\dots,[vz_n]\big\}$ by Lemma 8.12.

Claim: *For each $f \in H_n^{\mathrm{rc},>}$, either $f \preccurlyeq h$ for some $h \in A_{n+1}$, or $f \succcurlyeq h$ for some $h \in B_{n+1}$.*

Otherwise we have $f \in H_n^{\mathrm{rc}}$ with $A_{n+1} < f < B_{n+1}$, so $[vf] \notin [\Gamma]$ and vf realizes the same cut in Γ as $-|\beta_{n+1}|$. Taking γ with $[\beta_n] > [\gamma] > [\beta_{n+1}]$ we obtain $[vz_0] > \cdots > [vz_n] > [\gamma] > [vf]$, contradicting $vf \in v\big(H_n^{\mathrm{rc},\times}\big)$.

The claim and Lemma 1.11 give a Hausdorff field extension $H_n^{\mathrm{rc}}(z_{n+1})$ of H_n^{rc}, and the resulting Hausdorff field extension $H_{n+1} = H_n(z_{n+1})$ of H has the properties that the inductive step requires. This concludes the proof by induction.

An easy induction on n now shows that for $z := z_0$ the elements $z, z', \dots, z^{(n)}$ of $\mathcal{C}^{<\infty}$ generate the Hausdorff field $H(z, z', \dots, z^{(n)}) = H_n$ over H, and so we have a Hardy field $H\langle z\rangle$ over H. If $\beta_0 < 0$, then $y := f_0 z_0$ has the desired property, and if $\beta_0 > 0$, then $y := f_0/z_0$ has the desired property. $\square$

First step in reverse engineering. To construct germs z_i as in Lemma 9.12 we first take for each i a continuous function $[0,+\infty) \to \mathbb{R}^>$ that represents the germ $f_i \in H$ and to be denoted also by f_i, and with $f_i \geqslant 1$ on $[0,+\infty)$ for $i \geqslant 1$. Next, let (a_i) be a strictly increasing sequence of real numbers $\geqslant 0$ tending to $+\infty$ such that $f_0,\dots,f_m$ are of class $\mathcal{C}^m$ on $[a_m,+\infty)$. Let there also be given for each $m \geqslant 1$ a continuous function $z_{m-1,m}\colon [a_{m-1},a_m] \to \mathbb{R}^>$. Then we define the continuous function $z_{k,m}\colon [a_k,a_m] \to \mathbb{R}^>$ for $0 \leqslant k < m$ by downward recursion: $z_{m-1,m}$ for $m \geqslant 1$ is already given to us, and for $1 \leqslant k < m$,

$$z_{k-1,m}(t) \ := \ \begin{cases} z_{k-1,k}(t) & \text{for } a_{k-1} \leqslant t \leqslant a_k, \\ z_{k-1,k}(a_k)\cdot \exp\displaystyle\int_{a_k}^t f_k(s) z_{k,m}(s)\,ds & \text{for } a_k \leqslant t \leqslant a_m, \text{ if } \beta_k < 0, \\ z_{k-1,k}(a_k)\cdot \exp\displaystyle\int_{a_k}^t \frac{f_k(s)}{z_{k,m}(s)}\,ds & \text{for } a_k \leqslant t \leqslant a_m, \text{ if } \beta_k > 0. \end{cases}$$

Downward induction on k gives $z_{k,m} = z_{k,m+1}$ on $[a_k,a_m]$ for $k < m$. This fact gives for each $k \in \mathbb{N}$ a continuous function $z_k\colon [a_k,+\infty) \to \mathbb{R}^>$ such that $z_k = z_{k,m}$ on $[a_k,a_m]$, for all $m > k$. Thus for $k \geqslant 1$ and $t \geqslant a_k$ we have

$$\begin{aligned} \beta_k < 0 \implies z_{k-1}(t) \ &= \ z_{k-1}(a_k)\cdot \exp\int_{a_k}^t f_k(s) z_k(s)\,ds, \\ \beta_k > 0 \implies z_{k-1}(t) \ &= \ z_{k-1}(a_k)\cdot \exp\int_{a_k}^t \frac{f_k(s)}{z_k(s)}\,ds, \end{aligned}$$

so z_{k-1} is of class $\mathcal{C}^1$ on $[a_k,+\infty)$, and:

$$\begin{aligned} \beta_k < 0 &\implies z_{k-1}^\dagger = f_k\, z_k \text{ on } [a_k,+\infty), \\ \beta_k > 0 &\implies z_{k-1}^\dagger = f_k/z_k \text{ on } [a_k,+\infty). \end{aligned}$$

Hence induction on m gives that z_k is of class $\mathcal{C}^m$ on $[a_{k+m},+\infty)$ (for all k, m), and thus (the germ of) each z_k lies in $\mathcal{C}^{<\infty}$.

The above is a general construction of functions whose germs satisfy the *equalities* in Lemma 9.12. More work is needed to satisfy also the *inequalities* $A_i <_{\mathrm{e}} z_i <_{\mathrm{e}} B_i$ in that lemma. We now turn to this task.

Second step in reverse engineering. Assume in this subsection that $z_{k-1,k} \geqslant 1$ on $[a_{k-1}, a_k]$, for all $k \geqslant 1$. Then $z_k \geqslant 1$ on $[a_k, +\infty)$ for all k. For $k \geqslant 1$ we have $z_{k-1}(t), z_{k-1}^{\dagger}(t) > 0$ for all $t \geqslant a_k$, so z_{k-1} is strictly increasing on $[a_k, +\infty)$. For each k, let $p_k, q_k \colon [a_k, +\infty) \to \mathbb{R}^{>}$ be continuous functions such that

$$p_{m-1} \ \leqslant\ z_{m-1,m} \ \leqslant\ q_{m-1} \quad \text{on } [a_{m-1}, a_m], \text{ for all } m \geqslant 1.$$

We try to find conditions on the families (p_k) and (q_k) so that these inequalities extend to $p_k \leqslant z_{k,m} \leqslant q_k$ on $[a_k, a_m]$ for all k, m with $k < m$ (and thus $p_k \leqslant z_k \leqslant q_k$ on $[a_k, +\infty)$ for all k). Let $1 \leqslant k < m$ and assume inductively that $p_k \leqslant z_{k,m} \leqslant q_k$ on $[a_k, a_m]$. On $[a_{k-1}, a_k]$ we have $p_{k-1} \leqslant z_{k-1,k} \leqslant q_{k-1}$, so $p_{k-1} \leqslant z_{k-1,m} \leqslant q_{k-1}$, as desired.

First suppose $\beta_k < 0$. Then for $a_k \leqslant t \leqslant a_m$ we have

$$z_{k-1,m}(t) \ =\ z_{k-1,k}(a_k) \exp \int_{a_k}^{t} f_k(s) z_{k,m}(s)\,ds,$$

hence

$$p_{k-1}(a_k) \exp \int_{a_k}^{t} f_k(s) p_k(s)\,ds \ \leqslant\ z_{k-1,m}(t) \ \leqslant\ q_{k-1}(a_k) \exp \int_{a_k}^{t} f_k(s) q_k(s)\,ds,$$

and so the desired $p_{k-1} \leqslant z_{k-1,m} \leqslant q_{k-1}$ on $[a_{k-1}, a_m]$ would follow if

$$(\mathrm{I}_k) \qquad p_{k-1}(t) \ \leqslant\ p_{k-1}(a_k) \exp \int_{a_k}^{t} f_k(s) p_k(s)\,ds \qquad \text{for all } t \geqslant a_k,$$

$$(\mathrm{II}_k) \qquad q_{k-1}(t) \ \geqslant\ q_{k-1}(a_k) \exp \int_{a_k}^{t} f_k(s) q_k(s)\,ds \qquad \text{for all } t \geqslant a_k.$$

Now assume $\beta_k > 0$. Then for $a_k \leqslant t \leqslant a_m$ we have

$$z_{k-1,m}(t) \ =\ z_{k-1,k}(a_k) \exp \int_{a_k}^{t} \frac{f_k(s)}{z_{k,m}(s)}\,ds,$$

and so

$$p_{k-1}(a_k) \exp \int_{a_k}^{t} \frac{f_k(s)}{q_k(s)}\,ds \ \leqslant\ z_{k-1,m}(t) \ \leqslant\ q_{k-1}(a_k) \exp \int_{a_k}^{t} \frac{f_k(s)}{p_k(s)}\,ds,$$

and so the desired $p_{k-1} \leqslant z_{k-1,m} \leqslant q_{k-1}$ on $[a_{k-1}, a_m]$ would follow if

$$(\mathrm{III}_k) \qquad p_{k-1}(t) \ \leqslant\ p_{k-1}(a_k) \exp \int_{a_k}^{t} \frac{f_k(s)}{q_k(s)}\,ds \qquad \text{for all } t \geqslant a_k,$$

$$(\mathrm{IV}_k) \qquad q_{k-1}(t) \ \geqslant\ q_{k-1}(a_k) \exp \int_{a_k}^{t} \frac{f_k(s)}{p_k(s)}\,ds \qquad \text{for all } t \geqslant a_k.$$

The above leads to the following:

Lemma 9.13. *Let $p_i \in A_i$ and $q_i \in B_i$ for $i = 0, 1, 2, \dots$ be given such that*

$$\beta_{i+1} \ <\ 0 \implies p_i^{\dagger} \ \leqslant\ f_{i+1} p_{i+1},\ \ q_i^{\dagger} \ \geqslant\ f_{i+1} q_{i+1} \qquad (\text{in } H),$$
$$\beta_{i+1} \ >\ 0 \implies p_i^{\dagger} \ \leqslant\ f_{i+1}/q_{i+1},\ q_i^{\dagger} \ \geqslant\ f_{i+1}/p_{i+1} \qquad (\text{in } H).$$

Then there are germs $z_i \in \mathcal{C}^{<\infty}$ $(i = 0, 1, 2, \dots)$ such that for all i,

$$p_i \leqslant z_i \leqslant q_i \ \text{ in } \mathcal{C}, \quad \beta_{i+1} < 0 \implies z_i^{\dagger} = f_{i+1} z_{i+1}, \quad \beta_{i+1} > 0 \implies z_i^{\dagger} = f_{i+1}/z_{i+1}.$$

Proof. Take a strictly increasing sequence (a_i) of real numbers $\geqslant 0$ tending to $+\infty$ and represent p_i, q_i for each i by $\mathcal{C}^1$-functions $[a_i,+\infty)\to\mathbb{R}^>$, also to be denoted by p_i, q_i, such that for all m,

- $f_0,\dots,f_m$ are of class $\mathcal{C}^m$ on $[a_m,+\infty)$;
- $1\leqslant p_m\leqslant q_m$ on $[a_m,+\infty)$;
- $\beta_{m+1}<0 \implies p_m^\dagger\leqslant f_{m+1}p_{m+1},\quad q_m^\dagger\geqslant f_{m+1}q_{m+1}$ on $[a_{m+1},+\infty)$;
- $\beta_{m+1}>0 \implies p_m^\dagger\leqslant f_{m+1}/q_{m+1},\ q_m^\dagger\geqslant f_{m+1}/p_{m+1}$ on $[a_{m+1},+\infty)$.

Upon replacing (a_i) by a strictly increasing sequence (b_i) of reals with $a_i\leqslant b_i$ for all i and the p_i, q_i by their restrictions to $[b_i,+\infty)$, for each i, the conditions above are obviously still satisfied. For $t\geqslant a_m$ we have

$$p_m(t)\ =\ p_m(a_{m+1})\exp\int_{a_{m+1}}^t p_m^\dagger(s)\,ds,\qquad q_m(t)\ =\ q_m(a_{m+1})\exp\int_{a_{m+1}}^t q_m^\dagger(s)\,ds,$$

and so for all $k\geqslant 1$ conditions (I_k) and (II_k) are satisfied if $\beta_k<0$, and conditions (III_k) and (IV_k) are satisfied if $\beta_k>0$. Thus by the above we can take any continuous function $z_{m-1,m}\colon[a_{m-1},a_m]\to\mathbb{R}$ with $p_{m-1}\leqslant z_{m-1,m}\leqslant q_{m-1}$ on $[a_{m-1},a_m]$ for $m=1,2,\dots$ to give germs z_i for $i=0,1,\dots$ as required. □

Final step in reverse engineering. This step involves a diagonalization. We take $p_{i,n}\in A_i$ and $q_{i,n}\in B_i$ (for $i=0,1,2,\dots,\ n=0,1,2,\dots$) such that for all i, n:

- $p_{i,n}\prec p_{i,n+1}$, and $\{p_{i,0},p_{i,1},p_{i,2},\dots\}$ is cofinal in A_i;
- $q_{i,n}\succ q_{i,n+1}$, and $\{q_{i,0},q_{i,1},q_{i,2},\dots\}$ is coinitial in B_i;
- if $\beta_{i+1}<0$, then $p_{i+1,n}=p_{i,N}^\dagger/f_{i+1}$ and $q_{i+1,n}=q_{i,N}^\dagger/f_{i+1}$ for some $N=N(i,n)>n$;
- if $\beta_{i+1}>0$, then $p_{i+1,n}=f_{i+1}/q_{i,N}^\dagger$ and $q_{i+1,n}=f_{i+1}/p_{i,N}^\dagger$ for some $N=N(i,n)>n$.

It follows from Lemma 9.8 that there is such a family $\big((p_{i,n},q_{i,n})\big)$. Setting $p_i:=p_{i,i}$ and $q_i:=q_{i,i}$ we note that the hypotheses of Lemma 9.13 are satisfied, and this gives us germs $z_i\in\mathcal{C}^{<\infty}$ for $i=0,1,2,\dots$ such that for all i,

(1) $p_i\leqslant z_i\leqslant q_i$ in $\mathcal{C}$;
(2) $\beta_{i+1}<0\Longrightarrow z_i^\dagger=f_{i+1}z_{i+1},\qquad \beta_{i+1}>0\Longrightarrow z_i^\dagger=f_{i+1}/z_{i+1}$.

We claim that then $A_i<_{\mathrm{e}}z_i<_{\mathrm{e}}B_i$ for all i. (Establishing this claim achieves our goal by Lemma 9.12.) To prove this claim, suppose for a certain pair i,n with $i<n$ we have $p_{i+1,n}\leqslant z_{i+1}\leqslant q_{i+1,n}$. (See Figure 11.) As a subclaim we show that then $p_{i,n}\leqslant z_i\leqslant q_{i,n}$. Consider first the case $\beta_{i+1}<0$. Then for $N:=N(i,n)>n$,

$$f_{i+1}^{-1}p_{i,N}^\dagger\ \leqslant\ z_{i+1}\ =\ f_{i+1}^{-1}z_i^\dagger\ \leqslant\ f_{i+1}^{-1}q_{i,N}^\dagger,\quad\text{so}\quad p_{i,N}^\dagger\ \leqslant z_i^\dagger\ \leqslant\ q_{i,N}^\dagger,$$

which by Lemma 2.16 gives constants $c_1,c_2>0$ with $c_1p_{i,N}\leqslant z_i\leqslant c_2q_{i,N}$, and so $p_{i,N-1}\leqslant z_i\leqslant q_{i,N-1}$. Now $N-1\geqslant n$, and thus $p_{i,n}\leqslant z_i\leqslant q_{i,n}$ as promised. The case $\beta_{i+1}>0$ is handled in the same way. Given $i<n$ we have $p_{n,n}\leqslant z_n\leqslant q_{n,n}$, and so $p_{i,n}\leqslant z_i\leqslant q_{i,n}$ by iterated application of the subclaim. For any fixed i this yields $A_i<_{\mathrm{e}}z_i<_{\mathrm{e}}B_i$ by the cofinality and coinitiality requirements we imposed on the $p_{i,n}$ and $q_{i,n}$. This proves the claim, and concludes the proof of Theorem 9.2, and thus of Theorem A. □

$$
\begin{array}{ccccccccccccc}
\boxed{p_{0,0}} & \prec & p_{0,1} & \prec \cdots \prec & p_{0,n} & \prec \cdots & z_0 & \cdots \prec & q_{0,n} & \prec \cdots \prec & q_{0,1} & \prec & \boxed{q_{0,0}} \\
p_{1,0} & \prec & \boxed{p_{1,1}} & \prec \cdots \prec & p_{1,n} & \prec \cdots & z_1 & \cdots \prec & q_{1,n} & \prec \cdots \prec & \boxed{q_{1,1}} & \prec & q_{1,0} \\
\vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \iddots & \vdots & & \vdots \\
\vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \iddots & \vdots & & \vdots \\
p_{n,0} & \prec & p_{n,1} & \prec \cdots \prec & \boxed{p_{n,n}} & \prec \cdots & z_n & \cdots \prec & \boxed{q_{n,n}} & \prec \cdots \prec & q_{n,1} & \prec & q_{n,0} \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & \iddots & \vdots & & \vdots & & \vdots
\end{array}
$$

FIGURE 11

10. ISOMORPHISM OF MAXIMAL HARDY FIELDS

The cardinality of any Hardy field extending $\mathbb{R}$ is $2^{\aleph_0}$. By Theorem A, all maximal Hardy fields are η_1 and thus $\aleph_1$-saturated as real closed ordered fields; in particular, under CH they are all isomorphic as ordered fields. However, they are not $\aleph_1$-saturated as ordered *differential* fields, since their constant field $\mathbb{R}$ isn't. Thus to show they are isomorphic (under CH), we need to argue in a different way, and this is what we do in this section.

Lemma 10.1. *Let K be a countable closed H-field with archimedean constant field C, let L be a closed H-field with constant field $\mathbb{R}$ and small derivation, and assume L is η_1. Let E be an ω-free H-subfield of K, and let $i\colon E \to L$ be an H-field embedding. Then i extends to an H-field embedding $K \to L$.*

$$
\begin{tikzcd}
 & L \\
K \arrow[ur, dashed] & \\
E \arrow[u, "\subseteq"] \arrow[uur, "i"'] &
\end{tikzcd}
$$

Proof. We identify C in the usual (and only possible) way with a subfield of $\mathbb{R}$, and note that then i is the identity on $C_E \subseteq \mathbb{R}$. Then [ADH, 10.5.15, 10.5.16] yield an extension of i to an H-field embedding $E(C) \to L$ that is the identity on $C \subseteq \mathbb{R}$. The H-subfield $E(C)$ of K is d-algebraic over E, so is ω-free. Replacing E by $E(C)$ we reduce to the case that $C_E = C$. Recall that η_1-ordered sets are $\aleph_1$-saturated. Hence [ADH, 16.2.3] applies and gives the desired conclusion. $\square$

Lemma 10.2. *Let L_1, L_2 be closed H-fields with small derivation and common constant field $\mathbb{R}$, and assume that L_1 and L_2 are η_1. Then the collection of H-field isomorphisms $K_1 \to K_2$ between countable closed H-subfields K_1 of L_1 and K_2 of L_2 is nonempty and is a back-and-forth system between L_1 and L_2. In particular, L_1 and L_2 are back-and-forth equivalent.*

Proof. The theory of closed H-fields with small derivation has a (countable) prime model, by [ADH, p. 705], and so there is an H-field isomorphism between copies of that prime model in L_1 and in L_2. Also, any countable subset of a closed H-field L is contained in a countable closed H-subfield of L, by downward Löwenheim-Skolem [ADH, B.5.10]. It remains to use Lemma 10.1. $\square$

A standard argument (cf. proof of [ADH, B.5.3]) using Lemma 10.2 now yields:

Corollary 10.3. *Let L_1, L_2 as in Lemma 10.2 have cardinality $2^{\aleph_0}$. Assume* CH. *Then L_1 and L_2 are isomorphic as H-fields.*

Next we recall that Berarducci and Mantova [9] defined a derivation ∂_{BM} on the real closed field **No** of surreal numbers and proved that **No** with ∂_{BM} is a Liouville closed H-field with $\mathbb{R} \subseteq \mathbf{No}$ as its field of constants. Below we consider **No** as an H-field in this way, and recall also that its derivation ∂_{BM} is small. We proved in [5, Theorems 1 and 2] that **No** is even a closed H-field, that its real closed subfield $\mathbf{No}(\omega_1)$ is closed under ∂_{BM}, and that $\mathbf{No}(\omega_1)$ as a differential subfield of **No** is a closed H-field as well. Moreover, $\mathbf{No}(\omega_1)$ has cardinality $2^{\aleph_0}$, and is η_1 as an ordered set. In combination with Theorem A and Corollary 10.3, with L_1 any maximal Hardy field and $L_2 = \mathbf{No}(\omega_1)$, this yields Corollary B in the introduction; more precisely, also using [8, Theorem 3], we have:

Corollary 10.4. *Let M be a maximal Hardy field. Then the ordered differential fields M and $\mathbf{No}(\omega_1)$ are back-and-forth equivalent. Hence M and $\mathbf{No}(\omega_1)$ are $\infty\omega$-equivalent, and assuming* CH, *M and $\mathbf{No}(\omega_1)$ are isomorphic.*

We finish with a lemma on $\infty\omega$-elementary embeddings:

Lemma 10.5. *Let L_1 and L_2 be as in Lemma 10.2, with L_1 an H-subfield of L_2. Then $L_1 \preccurlyeq_{\infty\omega} L_2$.*

Proof. Let Φ be the back-and-forth system from Lemma 10.2. By downward Löwenheim-Skolem there is for all $a_1, \dots, a_n \in L_1$ a countable closed H-subfield K_1 of L_1 containing $a_1, \dots, a_n$, and then the identity map $K_1 \to K_1$ belongs to Φ. This yields $L_1 \preccurlyeq_{\infty\omega} L_2$ by [8, Theorem 4]. □

This will be used in the follow-up paper on maximal analytic Hardy fields.

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The citation [ADH] refers to our book

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Kurt Gödel Research Center for Mathematical Logic, Universität Wien, 1090 Wien, Austria
Email address: matthias.aschenbrenner@univie.ac.at

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.
Email address: vddries@illinois.edu

CNRS, LIX (UMR 7161), Campus de l'École Polytechnique, 91120 Palaiseau, France
Email address: vdhoeven@lix.polytechnique.fr