Transserial Hardy fields*

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It is well known that Hardy fields can be extended with integrals, exponentials and solutions to Pfaffian first order differential equations $f' = P(f) / Q(f)$. From the formal point of view, the theory of transseries allows for the resolution of more general algebraic differential equations. However, until now, this theory did not admit a satisfactory analytic counterpart. In this paper, we will introduce the notion of a transserial Hardy field. Such fields combine the advantages of Hardy fields and transseries. In particular, we will prove that the field of differentially algebraic transseries over $\mathbb{R}\{\{x^{-1}\}\}$ carries a transserial Hardy field structure. Inversely, we will give a sufficient condition for the existence of a transserial Hardy field structure on a given Hardy field.

1. INTRODUCTION

A Hardy field is a field of infinitely differentiable germs of real functions near infinity. Since any non-zero element in a Hardy field $\mathcal{H}$ is invertible, it admits no zeros in a suitable neighbourhood of infinity, whence its sign remains constant. It follows that Hardy fields both carry a total ordering and a valuation. The ordering and valuation can be shown to satisfy several natural compatibility axioms with the differentiation, so that Hardy fields are models of the so called theory of H-fields [AD02, AD01, AD04].

Other natural models of the theory of H-fields are fields of transseries [Hoe97, Sch01, MMD97, MMD99, Kuh00, Hoe06]. Contrary to Hardy fields, these models are purely formal, which makes them particularly useful for the automation of asymptotic calculus [Hoe97]. Furthermore, the so called field of grid-based transseries $\mathbb{T}$ (for instance) satisfies several remarkable closure properties. Namely, $\mathbb{T}$ is differentially Henselian [Hoe06, theorem 8.21] and it satisfies the differential intermediate value theorem [Hoe06, theorem 9.33].

Now the purely formal nature of the theory of transseries is also a drawback, since it is not *a priori* clear how to associate a genuine real function to a transseries f, even in the case when f satisfies an algebraic differential equation over the set $\mathbb{R}\{\{x^{-1}\}\}$ of convergent power series in x^{-1} for $x \to \infty$. One approach to this problem is to develop Écalle's accelero-summation theory [Éca85, Éca87, Éca92, Éca93, Bra91, Bra92], which constitutes a more or less canonical way to associate analytic functions to formal transseries with a "natural origin". In this paper, we will introduce another approach, based on the concept of a *transserial Hardy field*.

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Roughly speaking, a transserial Hardy field is a truncation-closed differential subfield $\mathcal{T}$ of $\mathbb{T}$, which is also a Hardy field. The main objectives of this paper are to show the following two things:

1. The differentially algebraic closure in $\mathbb{T}$ of a transserial Hardy field can be given the structure of a transserial Hardy field.
2. Any differentially algebraic Hardy field extension of a transserial Hardy field, which is both differentially Henselian and closed under exponentiation, admits a transserial Hardy field structure.

We have chosen to limit ourselves to the context of grid-based transseries. More generally, an interesting question is which H-fields can be embedded in fields of well-based transseries and which differential fields of well-based transseries admit Hardy field representations. We hope that work in progress [ADH05, ADH] on the model theory of H-fields and asymptotic fields will enable us to answer these questions in the future.

The theory of Hardy fields admits a long history. Hardy himself proved that the field of so called L-functions is a Hardy field [Har10, Har11]. The definition of a Hardy field and the possibility to add integrals, exponentials and algebraic functions is due to Bourbaki [Bou61]. More generally, Hardy fields can be extended by the solutions to Pfaffian first order differential equations [Sin75, Bos81] and solutions to certain second order differential equations [Bos87]. Further results on Hardy fields can be found in [Ros83a, Ros83b, Ros87, Bos82, Bos86]. The theory of transserial Hardy fields can be thought of as a systematic way to deal with differentially algebraic extensions of any order.

The main idea behind the addition of solutions to higher order differential equations to a given transserial Hardy field $\mathcal{T}$ is to write such solutions in the form of "integral series" over $\mathcal{T}$ (see also [Hoe05]). For instance, consider a differential equations such as

$$f' = \mathrm{e}^{-2\mathrm{e}^x} + f^2,$$

for large $x \succ 1$. Such an equation may typically be written in integral form

$$f = \textstyle\int \mathrm{e}^{-2\mathrm{e}^x} + \int f^2.$$

The recursive replacement of the left-hand side by the right-hand side then yields a "convergent" expansion for f using iterated integrals

$$f = \textstyle\int \mathrm{e}^{-2\mathrm{e}^x} + \int (\int \mathrm{e}^{-2\mathrm{e}^x})^2 + 2\,(\int \mathrm{e}^{-2\mathrm{e}^x})\,(\int (\int \mathrm{e}^{-2\mathrm{e}^x})^2) + \cdots,$$

where we understand that each of the integrals in this expansion are taken from $+\infty$:

$$(\textstyle\int g)(x) = \displaystyle\int_\infty^x g(t)\,\mathrm{d}t.$$

In order to make this idea work, one has to make sure that the extension of $\mathcal{T}$ with a solution f of the above kind does not introduce any oscillatory behaviour. This is done using a combination of arguments from model theory and differential algebra.

More precisely, whenever a transseries solution f to an algebraic differential equation over $\mathcal{T}$ is not yet in $\mathcal{T}$, then we may assume the equation to be of minimal "complexity" (a notion which refines Ritt rank; see section 2.3). In section 2, we will show how to put the equation in normal form

$$L f = P(f), \tag{1.1}$$

where $P \in \mathcal{T}\{F\} = \mathcal{T}[F, F', F'', \ldots]$ is a "small" differential polynomial and $L \in \mathcal{T}[\partial]$ admits a factorization

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

over the complexification $\mathcal{T}[\mathrm{i}]$ of $\mathcal{T}$. In section 4, it will be show how to solve (1.1) using iterated integrals, using the fact that the equation $(\partial - \varphi)\, f = g$ admits $\mathrm{e}^{\int \varphi} \int \mathrm{e}^{-\int \varphi}\, g$ as a solution. Special care will be taken to ensure that the constructed solution is again real and that the solution admits the same asymptotic expansion over $\mathcal{T}$ as the formal solution.

Section 3 contains some general results about transserial Hardy fields. In particular, we prove the basic extension lemma: given a transseries f and a real germ $\hat{f}$ at infinity which behave similarly over $\mathcal{T}$ (both from the asymptotic and differentially algebraic points of view), there exists a transserial Hardy field extension of $\mathcal{T}$ in which f and $\hat{f}$ may be identified. The differential equivalence of f and $\hat{f}$ will be ensured by the fact that the equation (1.1) was chosen to be of minimal complexity. Using Zorn's lemma, it will finally be possible to close $\mathcal{T}$ under the resolution of real differentially algebraic equations. This will be the object of the last section 5. Throughout the paper, we will freely use notations from [Hoe06]. For the reader's convenience, some of the notations are recalled in section 2.1. We also included a glossary at the end.

It would be interesting to investigate whether the theory of transserial Hardy fields can be generalized so as to model some of the additional compositional structure on $\mathbb{T}$. A first step would be to replace all differential polynomials by restricted analytic functions [DMM94]. A second step would be to consider postcompositions with transseries $x+\delta$ such that $\delta = o(x)$, starting with "sufficiently flat" transseries f for which Taylor's formula holds:

$$f \circ (x+\delta) = f + f'\,\delta + \frac{1}{2} f''\,\delta^2 + \cdots.$$

This requires the existence of suitable analytic continuations of f in the complex domain. Typically, if $f \in \mathbb{T}_{\preccurlyeq g}$ with $g \in \mathbb{T}^{>,\succ}$ (see section 2.1 for the notations), then $f \circ g^{\mathrm{inv}}$ should be defined on some sector at infinity (notice that this can be forced for the constructions in this paper). Finally, more violent difference equations, such as

$$f(x) = \frac{1}{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{x}}}} + f(x+1),$$

generally give rise to quasi-analytic solutions. From the model theoretic point view, they can probably always be seen as convergent sums.

Finally, one may wonder about the respective merits of the theory of accelero-summation and the theory of transserial Hardy fields. Without doubt, the first theory is more canonical and therefore has a better behaviour with respect to composition. In particular, we expect it to be easier to prove o-minimality results [Dri98]. On the other hand, many technical details still have to be worked out in full detail. This will require a certain effort, even though the resulting theory can be expected to have many other interesting applications. The advantage of the theory of transserial Hardy fields is that it is more direct (given the current state of art) and that it allows for the association of Hardy field elements to transseries which are not necessarily accelero-summable.

2. PRELIMINARIES

2.1. Notations

Let $\mathbb{T} = \mathbb{R}[\![[x]\!]] = \mathbb{R}[\![\mathfrak{T}]\!]$ be the totally ordered field of grid-based transseries in $x \to \infty$ with real coefficients [Hoe06]. Any transseries is an infinite linear combination $f = \sum_{\mathfrak{m} \in \mathfrak{T}} f_{\mathfrak{m}}\, \mathfrak{m}$ of transmonomials, with grid-based support $\operatorname{supp} f \subseteq \mathfrak{T}$. If $f \neq 0$, then the largest element of $\operatorname{supp} f$ for $\preccurlyeq$ is written $\mathfrak{d}_f$ and called the *dominant monomial* of f. Transmonomials $\mathfrak{m}, \mathfrak{n}, \ldots$ are systematically written using the fraktur font. Each trans-monomial is an iterated logarithm $\log_l x$ of x or the exponential of a transseries g with $\mathfrak{n} \succ 1$ for each $\mathfrak{n} \in \operatorname{supp} g$. The asymptotic relations $\preccurlyeq, \prec, \asymp, \sim, \preccurlyeq\!\!\!\preccurlyeq, \prec\!\!\!\prec, \asymp\!\!\!\asymp$ and $\approx$ on $\mathbb{T}$ are defined by

$$\begin{array}{rcl} f \preccurlyeq g & \Longleftrightarrow & f = O(g) \\ f \prec g & \Longleftrightarrow & f = o(g) \\ f \asymp g & \Longleftrightarrow & f \preccurlyeq g \preccurlyeq f \\ f \sim g & \Longleftrightarrow & f - g \prec g \\ f \preccurlyeq\!\!\!\preccurlyeq g & \Longleftrightarrow & \log |f| \preccurlyeq \log |g| \\ f \prec\!\!\!\prec g & \Longleftrightarrow & \log |f| \prec \log |g| \\ f \asymp\!\!\!\asymp g & \Longleftrightarrow & \log |f| \asymp \log |g| \\ f \approx g & \Longleftrightarrow & \log |f| \sim \log |g|. \end{array}$$

Given $\mathfrak{v} \neq 1$, one also defines variants of $\preccurlyeq, \prec$, etc. modulo flatness:

$$\begin{array}{rcl} f \preccurlyeq_{\mathfrak{v}} g & \Longleftrightarrow & \exists \mathfrak{m} \prec\!\!\!\prec \mathfrak{v}, f \preccurlyeq g\, \mathfrak{m} \\ f \prec_{\mathfrak{v}} g & \Longleftrightarrow & \forall \mathfrak{m} \prec\!\!\!\prec \mathfrak{v}, f \prec g\, \mathfrak{m} \\ f \preccurlyeq_{\mathfrak{v}}^{*} g & \Longleftrightarrow & \exists \mathfrak{m} \preccurlyeq\!\!\!\preccurlyeq \mathfrak{v}, f \preccurlyeq g\, \mathfrak{m} \\ f \prec_{\mathfrak{v}}^{*} g & \Longleftrightarrow & \forall \mathfrak{m} \preccurlyeq\!\!\!\preccurlyeq \mathfrak{v}, f \prec g\, \mathfrak{m}. \end{array}$$

It is convenient to use relations as superscripts in order to filter elements, as in

$$\begin{array}{rcl} \mathbb{T}^{>} & = & \{f \in \mathbb{T} : f > 0\} \\ \mathbb{T}^{\neq} & = & \{f \in \mathbb{T} : f \neq 0\} \\ \mathbb{T}^{\succ} & = & \{f \in \mathbb{T} : f \succ 1\}. \end{array}$$

Similarly, we use subscripts for filtering on the support:

$$\begin{array}{rcl} f_{\succ} & = & \displaystyle\sum_{\mathfrak{m} \in \operatorname{supp} f, \mathfrak{m} \succ 1} f_{\mathfrak{m}}\, \mathfrak{m} \\ f_{\prec\!\!\!\prec \mathfrak{v}} & = & \displaystyle\sum_{\mathfrak{m} \in \operatorname{supp} f, \mathfrak{m} \preccurlyeq\!\!\!\preccurlyeq \mathfrak{v}} f_{\mathfrak{m}}\, \mathfrak{m} \\ \mathbb{T}_{\succ} & = & \{f_{\succ} : f \in \mathbb{T}\} \\ \mathbb{T}_{\prec\!\!\!\prec \mathfrak{v}} & = & \{f_{\prec\!\!\!\prec \mathfrak{v}} : f \in \mathbb{T}\}. \end{array}$$

We denote the derivation on $\mathbb{T}$ w.r.t. x by ∂ and the corresponding distinguished integration (with constant part zero) by $\int$. The logarithmic derivative of f is denoted by $f^{\dagger}$. The operations $\uparrow$ and $\downarrow$ of upward and downward shifting correspond to postcomposition with $\exp x$ resp. $\log x$. We finally write $f \trianglelefteq g$ if the transseries f is a truncation of g, i.e. $\mathfrak{m} \prec \operatorname{supp} f$ for all $\mathfrak{m} \in \operatorname{supp}(g - f)$.

2.2. Differential fields of transseries and cuts

Given $f \in \mathbb{T}$, we define the *canonical span* of f by

$$\operatorname{span} f = \max_{\preccurlyeq\!\!\!\preccurlyeq} \{\mathrm{e}^{-\partial(\log(\mathfrak{m}/\mathfrak{n}))} : \mathfrak{m}, \mathfrak{n} \in \operatorname{supp} f, \mathfrak{m} \neq \mathfrak{n}\}. \tag{2.1}$$

Here the maximum is taken over a finite set, since f is grid-based. By convention, $\operatorname{span} f = 1$ if $\operatorname{supp} f$ contains less than two elements. We also define the *ultimate canonical span* of f by

$$\operatorname{uspan} f = \min_{\preccurlyeq\!\!\!\preccurlyeq} \{\operatorname{span} f_{\prec \mathfrak{v}} : \mathfrak{v} \in \operatorname{supp} f\}. \tag{2.2}$$

Here the minimum is again taken over a finite set. We notice that $\operatorname{uspan} f \neq 1$ if and only if $\operatorname{supp} f$ admits no minimal element for $\preccurlyeq$.

Example 2.1. We have

$$\begin{aligned}\operatorname{span}\left(1+\frac{\mathrm{e}^{-x}}{1-x^{-1}}\right) &= \mathrm{e}^{-x}\\ \operatorname{uspan}\left(1+\frac{\mathrm{e}^{-x}}{1-x^{-1}}\right) &= x^{-1}\end{aligned}$$

Consider a differential subfield $\mathcal{T}$ of $\mathbb{T}$ and let $\mathfrak{v} \in \mathfrak{T}^{\prec}$. We say that $\mathcal{T}$ has span $\mathfrak{v}$, if $\operatorname{span} f \preceq\!\!\!\prec \mathfrak{v}$ for all $f \in \mathcal{T}$ and $\operatorname{span} f \asymp\!\!\!\asymp \mathfrak{v}$ for at least one $f \in \mathcal{T}$ (notice that we do not require $\mathfrak{v} \in \mathrm{e}^{-\mathfrak{T}}$). Since $\mathcal{T}$ is stable under differentiation, we have $\mathfrak{v} \succeq\!\!\!\succ x^{-1}$ as soon as $\mathcal{T} \nsubseteq \mathbb{R}$. Notice also that we must have $\mathcal{T} \subseteq \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$ if $\mathcal{T}$ has span $\mathfrak{v}$.

A transseries $f \in \mathbb{T} \setminus \mathcal{T}$ is said to be a *serial cut* over $\mathcal{T}$, if $\varphi \in \mathcal{T}$ for every $\varphi \lhd f$ and $\operatorname{supp} f$ admits no minimal element for $\preccurlyeq$. In that case, let $\mathfrak{m} \in \operatorname{supp} f$ be maximal for $\preccurlyeq$ such that $\mathfrak{m}^{-1} \operatorname{supp} f_{\preccurlyeq \mathfrak{m}} \preceq\!\!\!\prec \operatorname{uspan} f$. Then $\mathrm{H}_f = f_{\succ \mathfrak{m}}$ and $\mathrm{T}_f = f_{\preccurlyeq \mathfrak{m}}$ are called the *head* and the *tail* of f. We say that f is a *normal serial cut* if $f \in \mathbb{T}_{\preceq\!\!\!\prec \operatorname{uspan} f}$, which implies in particular that $\mathrm{H}_f = 0$.

Assuming that $\mathcal{T}$ has span $\mathfrak{v}$, any serial cut over $\mathcal{T}$ is necessarily in $\mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$. We will denote by $\hat{\mathcal{T}}$ the set of all $f \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$ which are either in $\mathcal{T}$ or serial cuts over $\mathcal{T}$ with $\operatorname{uspan} f \asymp\!\!\!\asymp \mathfrak{v}$. Notice that $\hat{\mathcal{T}}$ is again a differential subfield of $\mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}$.

The above definitions naturally adapt to the complexifications $\mathbb{T}[\mathrm{i}]$ and $\mathcal{T}[\mathrm{i}]$ of $\mathbb{T}$ and differential subfields $\mathcal{T}$ of $\mathbb{T}$. If $\mathcal{T}$ has span $\mathfrak{v}$, then the set $\hat{\mathcal{T}}[\mathrm{i}]$ coincides with the set of all $f \in \mathbb{T}_{\preceq\!\!\!\prec \mathfrak{v}}[\mathrm{i}] = \mathbb{T}[\mathrm{i}]_{\preceq\!\!\!\prec \mathfrak{v}}$ which are either in $\mathcal{T}[\mathrm{i}]$ or serial cuts over $\mathcal{T}[\mathrm{i}]$ with $\operatorname{uspan} f \asymp\!\!\!\asymp \mathfrak{v}$.

2.3. Complements on differential algebra

Let $\mathcal{T}$ be a differential field. Following [Kol73, Hoe06], we denote by $\mathcal{T}\{F\}$ the ring of differential polynomials in F over $\mathcal{T}$ and by $\mathcal{T}\langle F\rangle$ its quotient field. Given $P \in \mathcal{T}\{F\}$ and $i \in \mathbb{N}$, we recall that P_i denotes the homogeneous part of degree i of P. We will denote by L_P the linear operator in $\mathcal{T}[\partial]$ with $L_P F = P_1(F)$. Assuming that $P \in \mathcal{T}\{F\} \setminus \mathcal{T}$, we also denote the order of P by r_P, the degree of P in $F^{(r_P)}$ by s_P and the total degree of P by t_P. The *Ritt rank* of P is defined to be the pair (r_P, s_P). The triple $\chi_P = (r_P, s_P, t_P)$ will be called the *complexity* of P. Both Ritt ranks and complexities are ordered lexicographically. Sometimes, we will also write $\deg P = t_P$ for the total degree of P in $F, \ldots, F^{(r_P)}$ and $\operatorname{val} P \leqslant \deg P$ for the corresponding valuation of P.

As usual, we will denote the *initial* and *separant* of P by I_P resp. S_P and set $H_P = I_P S_P$. Given $P, Q \in \mathcal{T}\{F\}$ with $P \notin \mathcal{T}$, Ritt reduction of Q by P provides us with a relation

$$I_P^{\alpha} S_P^{\beta} Q = \boldsymbol{A} P + R, \tag{2.3}$$

where $\boldsymbol{A} \in \mathcal{T}\{F\}[\partial]$ is a linear differential operator, $\alpha, \beta \in \mathbb{N}$ and the remainder $R \in \mathcal{T}\{F\}$ satisfies $\chi_R < \chi_P$. If $R = 0$, then this simplifies into a relation

$$H_P^{\alpha} Q = \boldsymbol{A}\, P,$$

by replacing $\alpha \rightarrow \max(\alpha, \beta)$ and $\boldsymbol{A} \rightarrow I_P^{\max(\alpha,\beta)-\alpha}\, S_P^{\max(\alpha,\beta)-\beta} \boldsymbol{A}$.

Let $\mathcal{K}$ be a differential field extension of $\mathcal{T}$. An element $f \in \mathcal{K}$ is said to be *differentially algebraic* over $\mathcal{T}$ if there exists an annihilator $P \in \mathcal{T}\{F\} \setminus \mathcal{T}$ with $P(f) = 0$. An annihilator P of minimal complexity χ_P will then be called a *minimal annihilator* and $\chi_f = \chi_P$ is also called the *complexity* of f over $\mathcal{T}$. The order $r_f = r_P$ of such a minimal annihilator P is called the *order* of f over $\mathcal{T}$. We say that $\mathcal{K}$ is a *differentially algebraic extension* of $\mathcal{T}$ if each $f \in \mathcal{K}$ is differentially algebraic over $\mathcal{T}$.

We say that $\mathcal{T}$ is differentially closed in $\mathcal{K}$, if $\mathcal{K} \setminus \mathcal{T}$ contains no elements that are differentially algebraic over $\mathcal{T}$. Given $\chi \in \mathbb{N}^3$ (resp. $r \in \mathbb{N}$), we say that $\mathcal{T}$ is *χ-differentially closed* (resp. *r-differentially closed*) in $\mathcal{K}$ if $\chi_f > \chi$ (resp. $r_f > r$) for all $f \in \mathcal{K} \setminus \mathcal{T}$. We say that $\mathcal{T}$ is *weakly differentially closed* if every $P \in \mathcal{T}\{F\} \setminus \mathcal{T}$ admits a root in $\mathcal{T}$. We say that $\mathcal{T}$ is *weakly r-differentially closed* if every $P \in \mathcal{T}\{F\} \setminus \mathcal{T}$ of order $\leqslant r$ admits a root in $\mathcal{T}$.

Given a differential polynomial $P \in \mathcal{T}\{F\}$ and $\varphi \in \mathcal{T}$, we define the *additive* and *multiplicative conjugates* of P by φ:

$$\begin{aligned} P_{+\varphi}(F) &= P(F+\varphi) \\ P_{\times\varphi}(F) &= P(\varphi F). \end{aligned}$$

We have $P_{+\varphi}, P_{\times\varphi} \in \mathcal{T}\{F\}$ and

$$\begin{aligned} \chi_{P_{+\varphi}} &= \chi_P \\ \chi_{P_{\times\varphi}} &= \chi_P \\ I_{P_{+\varphi}} &= I_{P,+\varphi} \\ I_{P_{\times\varphi}} &= \varphi^{r_P} I_{P,\times\varphi} \\ S_{P_{+\varphi}} &= S_{P,+\varphi} \\ S_{P_{\times\varphi}} &= \varphi\, S_{P,\times\varphi} \end{aligned}$$

We also notice that additive and multiplicative conjugation are compatible with Ritt reduction: given $\varphi \in \mathcal{T}$ and assuming (2.3), we have

$$\begin{aligned} I_{P_{+\varphi}}^{\alpha} S_{P_{+\varphi}}^{\beta} Q_{+\varphi} &= \boldsymbol{A} P_{+\varphi} + R_{+\varphi} \\ I_{P_{\times\varphi}}^{\alpha} S_{P_{\times\varphi}}^{\beta} Q_{\times\varphi} &= \varphi^{\alpha r_P+\beta} \boldsymbol{A} P_{\times\varphi} + \varphi^{\alpha r_P+\beta} R_{\times\varphi}. \end{aligned}$$

Remark 2.2. The compatibility of Ritt's reduction theory with additive and multiplicative conjugation holds more generally for rings of differential polynomials in a finite number of commuting partial derivations (or with a finite dimensional Lie algebra of non-commuting derivations). Similar compatibility results hold for upward shiftings or changes of derivations (in the partial case, this requires the rankings to be order-preserving).

In the case when $\mathcal{T}$ is a differential subfield of $\mathbb{T} = \mathbb{R}[[\mathfrak{T}]]$, we recall that a differential polynomial $P \in \mathcal{T}\{F_1, \ldots, F_k\}$ may also be regarded as a series in $\mathbb{R}\{F_1, \ldots, F_k\}[[\mathfrak{T}]]$. In particular, we may write $P = D_P\, \mathfrak{d}_P + R_P$ for each $P \in \mathcal{T}\{F_1, \ldots, F_k\}$, where the *dominant part* $D_P \in \mathbb{R}\{F_1, \ldots, F_K\}$ is defined to be the coefficient of $\mathfrak{d}_P$ in P, so that $R_P \prec P$. Similarly, elements P/Q of the fraction field $\mathcal{T}\langle F_1, \ldots, F_k\rangle$ of $\mathcal{T}\{F_1, \ldots, F_k\}$ may be regarded as series with coefficients in $\mathbb{R}\langle F_1, \ldots, F_k\rangle$. Indeed, writing $P = D_P\, \mathfrak{d}_P + R_P$ and $Q = D_Q\, \mathfrak{d}_Q + R_Q$, where $D_P\, \mathfrak{d}_P$ denotes the dominant term of P, we may expand

$$\frac{P}{Q} = \frac{D_P}{D_Q} \cdot \frac{\mathfrak{d}_P}{\mathfrak{d}_Q} \cdot \frac{1 + \frac{R_P}{D_P\, \mathfrak{d}_P}}{1 + \frac{R_Q}{D_Q + \mathfrak{d}_Q}}$$

In the case when $P, Q \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]\{F_1, \ldots, F_k\}$ for some transbasis $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ in the sense of [Hoe06, section 4.4], then P and P/Q may also be expanded lexicographically with respect to $\mathfrak{b}_n, \ldots, \mathfrak{b}_1$.

2.4. Linear differential operators and factorization

Let $\mathcal{T}$ be a differential field and consider a linear differential operator $L \in \mathcal{T}[\partial]^{\neq}$. We will denote the order of L by r_L and recall from [Hoe06, section 7.2] that $(L f) / f$ is a differential polynomial of order $r_L - 1$ in $f^\dagger$, called the *differential Riccati polynomial* of L. Writing R_L for this polynomial, we thus have $L f = R_L(f^\dagger) f$. Given $\psi \in \mathcal{T}$, we define the *multiplicative conjugate* $L_{\times \psi}$ and the *twist* $L_{\ltimes \psi}$ by

$$\begin{aligned} L_{\times \psi} &= L \psi \\ L_{\ltimes \psi} &= \psi^{-1} L \psi \end{aligned}$$

We notice that $L_{\ltimes \psi}$ is also obtained by substituting $\partial + \psi^\dagger$ for ∂ in L. We say that L *splits* over $\mathcal{T}$, if it admits a complete factorization

$$L = c (\partial - \varphi_1) \cdots (\partial - \varphi_r) \tag{2.4}$$

with $c, \varphi_1, \ldots, \varphi_r \in \mathcal{T}$. In that case, each of the twists $L_{\ltimes \psi}$ of L also splits:

$$L_{\ltimes \psi} = c (\partial + \psi^\dagger - \varphi_1) \cdots (\partial + \psi^\dagger - \varphi_r).$$

We say that $\mathcal{T}$ is *r-linearly closed* if any linear differential operator of order $\leqslant r$ splits over $\mathcal{T}$.

PROPOSITION 2.3. *If $\mathcal{T}$ is weakly $(r-1)$-differentially closed, then $\mathcal{T}$ is r-linearly closed.*

Proof. The proof proceeds by induction on r. Let $L \in \mathcal{T}[\partial]$ be of order $r > 0$. Then the differential Riccati polynomial R_L has order $r - 1$, so it admits a root $\varphi_r \in \mathcal{T}$. Division of L by $\partial - \varphi_r$ in $\mathcal{T}[\partial]$ yields a factorization $L = \tilde{L} (\partial - \varphi_r)$ where $\tilde{L} \in \mathcal{T}[\partial]$ has order $r - 1$. By the induction hypothesis, $\tilde{L}$ splits over $\mathcal{T}$, whence so does L. □

PROPOSITION 2.4. *Let $L \in \mathcal{T}[\partial]^{\neq}$ be an operator which splits over $\mathcal{T}$ and let $A, B \in \mathcal{T}[\partial]$ be such that $L = A B$. Then A and B split over $\mathcal{T}$.*

Proof. This follows from the Jordan-Hölder theorem for submodules of $\mathcal{T}[\partial]$. □

Assume now that $\mathcal{T}$ is a totally ordered differential field. A monic operator $L \in \mathcal{T}[\partial]^{\neq}$ is said to be an *atomic real operator* if L has either one of the forms

$$\begin{aligned} L &= \partial - \varphi, & \varphi &\in \mathcal{T} \\ L &= (\partial - (\varphi - \psi \mathrm{i} + \psi^\dagger)) (\partial - (\varphi + \psi \mathrm{i})), & \varphi, \psi &\in \mathcal{T} \end{aligned}$$

A *real splitting* of an operator $L \in \mathcal{T}[\partial]^{\neq}$ over $\mathcal{T}$ is a factorization of the form

$$L = K_1 \cdots K_s, \tag{2.5}$$

where each K_i is an atomic real operator. A splitting (2.4) over $\mathcal{T}[\mathrm{i}]$ is said to *preserve realness*, if it gives rise to a real splitting (2.5) for $K_i = (\partial - \varphi_{i_j})$ or $K_i = (\partial - \varphi_{i_j}) (\partial - \varphi_{i_j + 1})$ and $i_1 < \cdots < i_s$.

PROPOSITION 2.5. *Let $L \in \mathcal{T}[\partial]^{\neq}$ be an operator which splits over $\mathcal{T}[\mathrm{i}]$. Then L admits a real splitting over $\mathcal{T}$.*

Proof. Assuming that $L \notin \mathcal{T}$, we claim that there exists an atomic real right factor $K \in \mathcal{T}[\partial]$ of L. Consider a splitting (2.4) over $\mathcal{T}[\mathrm{i}]$. If $\varphi_r \in \mathcal{T}$, then we may take $K = \partial - \varphi_r$. Otherwise, we write

$$L = \bar{c} (\partial - \bar{\varphi}_1) \cdots (\partial - \bar{\varphi}_r)$$

and take K to be the least common multiple of $\partial - \varphi_r$ and $\partial - \bar{\varphi}_r$ in $\mathcal{T}[\mathrm{i}]$. Since $K = \bar{K}$, we indeed have $K \in \mathcal{T}[\partial]$. Since $\partial - \varphi_r | L$ and $\partial - \bar{\varphi}_r | L$, we also have $K | L$. In particular, proposition 2.4 implies that K splits over $\mathcal{T}[\mathrm{i}]$. Such a splitting is necessarily of the form

$$K = (\partial - (\varphi - \psi\,\mathrm{i} + \psi^{\dagger}))\,(\partial - (\varphi + \psi\,\mathrm{i})), \quad \varphi, \psi \in \mathcal{T},$$

whence K is atomic. Having proved our claim, the proposition follows by induction on r. Indeed, let $\tilde{L} \in \mathcal{T}[\partial]$ be such that $\tilde{L}\,K = L$. By proposition 2.4, $\tilde{L}$ splits over $\mathcal{T}[\mathrm{i}]$. By the induction hypothesis, $\tilde{L}$ therefore admits a real splitting $\tilde{L} = K_1 \cdots K_s$ over $\mathcal{T}$. But then $L = K_1 \cdots K_s K$ is a real splitting of L. □

COROLLARY 2.6. *An operator $L \in \mathcal{T}[\partial]^{\neq}$ is atomic if and only if L is irreducible over $\mathcal{T}$ and L splits over $\mathcal{T}[\mathrm{i}]$.*

2.5. Factorization at cuts

Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Given $P \in \mathcal{T}[\mathrm{i}]\{F\}$ and $f \in \hat{\mathcal{T}}[\mathrm{i}]$, we say that P *splits* over $\hat{\mathcal{T}}[\mathrm{i}]$ at f, if $L_{P_{+f}}$ and P have the same order r and $L_{P_{+f}}$ splits over $\hat{\mathcal{T}}[\mathrm{i}]$.

LEMMA 2.7. *Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Let $P \in \mathcal{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathcal{T}}[\mathrm{i}]$ over $\mathcal{T}[\mathrm{i}]$, which splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at f. Then any minimal annihilator $Q \in \mathcal{T}[\mathrm{i}]\langle f\rangle\{\bar{F}\}$ of $\bar{f}$ over $\mathcal{T}[\mathrm{i}]\langle f\rangle$ splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\bar{f}$.*

Proof. Since $\bar{P}(\bar{f}) = 0$, Ritt division of $\bar{P}$ by Q yields

$$H_Q^{\alpha}\,\bar{P} = \boldsymbol{A}\,Q \tag{2.6}$$

for some $\alpha \in \mathbb{N}$ and $\boldsymbol{A} \in \mathcal{T}[\mathrm{i}]\langle f\rangle\{\bar{F}\}[\partial]$. Additive conjugation of (2.6) yields

$$H_{Q_{+\bar{f}}}^{\alpha}\,\bar{P}_{+\bar{f}} = \boldsymbol{A}\,Q_{+\bar{f}}. \tag{2.7}$$

By the minimality hypothesis for Q, we have $L_{Q_{+\bar{f}}, r_Q} = S_Q(\bar{f}) \neq 0$ and $H_Q(\bar{f}) \neq 0$, so that $\operatorname{val} Q_{+\bar{f}} = 1$ and $\operatorname{val} H_{Q_{+\bar{f}}} = 0$. Similarly, we have $\operatorname{val} \bar{P}_{+\bar{f}} = 1$. Consequently, when considering the linear part of the equation (2.7), we obtain

$$H_{Q_{+\bar{f}},0}^{\alpha}\,L_{\bar{P}_{+\bar{f}}} = \boldsymbol{A}_0\,L_{Q_{+\bar{f}}},$$

whence $L_{Q_{+\bar{f}}}$ divides $L_{\bar{P}_{+\bar{f}}}$ in $\mathcal{T}[\mathrm{i}]\langle f\rangle[\partial]$. Now $L_{P_{+f}}$ splits over $\hat{\mathcal{T}}[\mathrm{i}][\partial]$, whence so does $L_{\bar{P}_{+\bar{f}}}$. By proposition 2.4, we infer that $L_{Q_{+\bar{f}}}$ splits over $\hat{\mathcal{T}}[\mathrm{i}][\partial]$. Since $S_Q(\bar{f}) \neq 0$, we also have $r_{L_{Q_{+\bar{f}}}} = r_Q$ and we conclude that Q splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\bar{f}$. □

COROLLARY 2.8. *Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Let $P \in \mathcal{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathcal{T}}[\mathrm{i}]$ over $\mathcal{T}[\mathrm{i}]$, which splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at f. Then any minimal annihilator $R \in \mathcal{T}[\mathrm{i}]\langle f\rangle\{G\}$ of $\operatorname{Re} f$ over $\mathcal{T}[\mathrm{i}]\langle f\rangle$ splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\operatorname{Re} f$.*

Proof. Applying the lemma to $Q = R_{/2,-f}$, we see that $L_{Q_{+\bar{f}}}$ splits over $\hat{\mathcal{T}}[\mathrm{i}]$. Now $Q_{+\bar{f}} = R_{+\operatorname{Re} f,/2}$, whence $L_{R_{+\operatorname{Re} f,/2}}$ and $L_{R_{\operatorname{Re} f}} = L_{R_{+\operatorname{Re} f,/2},\times 2}$ also split over $\hat{\mathcal{T}}[\mathrm{i}]$. □

LEMMA 2.9. *Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$, such that $\hat{\mathcal{T}}[\mathrm{i}]$ is r-linearly closed. Let $P \in \mathcal{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathcal{T}}[\mathrm{i}]$ over $\mathcal{T}[\mathrm{i}]$, such that P has order r. Assume that $\operatorname{Re} f \notin \mathcal{T}$ and let $S \in \mathcal{T}\{G\}$ be a minimal annihilator of $\operatorname{Re} f$ over $\mathcal{T}$. Then S splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\operatorname{Re} f$.*

Proof. Let R be as in the above corollary, so that R splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\operatorname{Re} f$. Since R has minimal complexity and $S(\operatorname{Re} f)=0$, Ritt division of S by R yields

$$H_R^{\alpha} S = \boldsymbol{A} R$$

for some $\alpha \in \mathbb{N}$ and $\boldsymbol{A} \in \mathbb{T}[\mathrm{i}]\langle f\rangle\{G\}[\partial]$. Additive conjugation and extraction of the linear part yields

$$H_{S_{+\operatorname{Re} f},0}^{\alpha} L_{S_{+\operatorname{Re} f}} = \boldsymbol{A}_0 L_{R_{+\operatorname{Re} f}},$$

so $L_{R_{+\operatorname{Re} f}}$ divides $L_{S_{+\operatorname{Re} f}}$ in $\hat{\mathcal{T}}[\mathrm{i}][\partial]$. Since the separants of R and S do not vanish at $\operatorname{Re} f$, we have

$$\begin{array}{rcrcl} r_{L_{R_{+\operatorname{Re} f}}} & = & r_R & = & \operatorname{tr\,deg}(\mathcal{T}[\mathrm{i}]\langle f, \operatorname{Re} f\rangle : \mathcal{T}[\mathrm{i}]\langle f\rangle) \\ & & & = & \operatorname{tr\,deg}(\mathcal{T}[\mathrm{i}]\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathcal{T}[\mathrm{i}]) - \operatorname{tr\,deg}(\mathcal{T}[\mathrm{i}]\langle f\rangle : \mathcal{T}[\mathrm{i}]) \\ & & & = & \operatorname{tr\,deg}(\mathcal{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathcal{T}) - \operatorname{tr\,deg}(\mathcal{T}[\mathrm{i}]\langle f\rangle : \mathcal{T}[\mathrm{i}]) \\ r_{L_{S_{+\operatorname{Re} f}}} & = & r_S & = & \operatorname{tr\,deg}(\mathcal{T}\langle \operatorname{Re} f\rangle : \mathcal{T}) \\ & & & = & \operatorname{trdeg}(\mathcal{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathcal{T}) - \operatorname{tr\,deg}(\mathcal{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathcal{T}\langle \operatorname{Re} f\rangle) \end{array}$$

and

$$r_S - r_R = \operatorname{tr\,deg}(\mathcal{T}[\mathrm{i}]\langle f\rangle : \mathcal{T}[\mathrm{i}]) - \operatorname{tr\,deg}(\mathcal{T}\langle \operatorname{Re} f, \operatorname{Im} f\rangle : \mathcal{T}\langle \operatorname{Re} f\rangle) \leqslant r.$$

Consequently, the quotient of $L_{S_{+\operatorname{Re} f}}$ and $L_{R_{+\operatorname{Re} f}}$ has order at most r, whence it splits over $\hat{\mathcal{T}}[\mathrm{i}]$. It follows that $L_{S_{+\operatorname{Re} f}}$ splits over $\hat{\mathcal{T}}[\mathrm{i}]$ and S splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\operatorname{Re} f$. □

2.6. Normalization of linear operators

Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v} \succ\!\!\succ x$. Recall from [Hoe06, Section 7.7] that $L h = 0$ with $L \in \mathcal{T}[\mathrm{i}][\partial]$ admits a canonical fundamental system of oscillatory transseries solutions $\Sigma_L = \{h_1, \ldots, h_r\} \subseteq \mathbb{O}$ with $\log h_1, \ldots, \log h_r \in \mathbb{T}_{\preccurlyeq\!\!\preccurlyeq \mathfrak{v}}[\mathrm{i}]$. We will denote by $\mathfrak{H}_L$ the set of dominant monomials of $h_1, \ldots, h_r$. The neglection relation on $\mathbb{T}$ is extended to $\mathbb{O}$ by $f \prec 1$ if and only if $f = f_{;\psi_1} \mathrm{e}^{\mathrm{i}\psi_1} + \cdots + f_{;\psi_p} \mathrm{e}^{\mathrm{i}\psi_p}$ with $f_{;\psi_1}, \ldots, f_{;\psi_p} \in \mathbb{T}[\mathrm{i}]^{\prec}$ and $\psi_1, \ldots, \psi_p \in \mathbb{T}$.

We say that L is *normal*, if we have $h_i \succ_{\mathfrak{v}} 1$ or $\operatorname{Re} \log h_i \succ \log \mathfrak{v}$ for each i. In that case, any quasi-linear equation of the form

$$L f = g, \qquad f \preccurlyeq_{\mathfrak{v}} 1$$

with $g \in \mathbb{T}_{\preccurlyeq\!\!\preccurlyeq \mathfrak{v}}[\mathrm{i}]$ admits $L^{-1} g$ as its only solution in $\mathbb{T}_{\preccurlyeq\!\!\preccurlyeq \mathfrak{v}}[\mathrm{i}]$. If L is a first order operator of the form $L = \partial - \varphi$, then L is normal if and only if $\operatorname{Re} \varphi \geqslant c\, \mathfrak{v}^{\dagger}$ for some $c > 0$ or $\operatorname{Re} \varphi \succ \mathfrak{v}^{\dagger}$. In particular, we must have $\varphi \succcurlyeq_{\mathfrak{v}} 1$ and $\operatorname{Re} \varphi \succcurlyeq \mathfrak{v}^{\dagger}$.

PROPOSITION 2.10. *Let $L \in \mathcal{T}[\mathrm{i}][\partial] \setminus \mathcal{T}[\mathrm{i}]$.*

a) There exists a $\lambda \in \mathbb{R}$ such that $L_{\ltimes \mathfrak{v}^{\lambda}}$ is normal.

b) If L is normal and $\lambda \geqslant 0$, then $L_{\ltimes \mathfrak{v}^{\lambda}}$ is normal.

Proof. Let $\Sigma_L = \{h_1, \ldots, h_r\}$. For each $\lambda \in \mathbb{R}$, the operator $L_{\ltimes \mathfrak{v}^{\lambda}}$ admits $h_1 / \mathfrak{v}^{\lambda}, \ldots, h_r / \mathfrak{v}^{\lambda}$ as solutions, which implies in particular that $\mathfrak{H}_{L_{\ltimes \mathfrak{v}^{\lambda}}} = \mathfrak{v}^{-\lambda} \mathfrak{H}_L$. Now $\operatorname{Re} \log (h_i / \mathfrak{v}^{\lambda}) \preccurlyeq \log \mathfrak{v} \Leftrightarrow \operatorname{Re} \log h_i \preccurlyeq \log \mathfrak{v}$ for all i. Choosing λ sufficiently large, it follows that $h_i / \mathfrak{v}^{\lambda} \succ_{\mathfrak{v}} 1$ for all i with $\operatorname{Re} \log (h_i / \mathfrak{v}^{\lambda}) \preccurlyeq \log \mathfrak{v}$, so that $L_{\ltimes \mathfrak{v}^{\lambda}}$ is normal. Similarly, if $h_i \succ_{\mathfrak{v}} 1$ for some i with $\operatorname{Re} \log (h_i / \mathfrak{v}^{\lambda}) \preccurlyeq \log \mathfrak{v}$, then $h_i \succ_{\mathfrak{v}} \mathfrak{v}^{\lambda}$ for all $\lambda \geqslant 0$. □

PROPOSITION 2.11. *Consider a normal operator* $L \in \mathcal{T}[\mathrm{i}][\partial]$*, which admits a splitting*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

with $\varphi_1, \ldots, \varphi_r \in \mathcal{T}[\mathrm{i}]$*. Then each* $\partial - \varphi_i$ *is a normal operator.*

Proof. We will call $h \in \mathbb{T}_{\preccurlyeq\mathfrak{v}}[\mathrm{i}]\, \mathrm{e}^{\mathrm{i}\mathbb{T}_{\preccurlyeq\mathfrak{v}}}$ normal, if $\partial - h^\dagger$ is normal. Let us first prove the following auxiliary result: given $\varphi \in \mathcal{T}[\mathrm{i}]$ and $h \in \mathbb{T}_{\preccurlyeq\mathfrak{v}}[\mathrm{i}]\, \mathrm{e}^{\mathrm{i}\mathbb{T}_{\preccurlyeq\mathfrak{v}}}$ such that $\partial - \varphi$ and h are normal and $\mathfrak{h} = \mathfrak{d}_h \notin \mathfrak{H}_{\partial - \varphi}$, then $(\partial - \varphi)\, h$ is also normal. If $\operatorname{Re} \log h \succ \log \mathfrak{v}$, then $0 \neq (\partial - \varphi)\, h \asymp_{\mathfrak{v}}^* h$, whence $\operatorname{Re} \log\, (\partial - \varphi)\, h = \operatorname{Re} \log h + O(\log \mathfrak{v}) \succ \log \mathfrak{v}$. In the other case, we have $h \succ_{\mathfrak{v}} 1$. Now if $\mathfrak{h}^\dagger \not\sim \varphi$, then $(\partial - \varphi)\, h \succ_{\mathfrak{v}} 1$, since $\varphi \succcurlyeq_{\mathfrak{v}} 1$. If $\mathfrak{h}^\dagger \sim \varphi$, then $\mathfrak{h} \notin \mathfrak{H}_{\partial - \varphi}$ implies $1 \notin \mathfrak{H}_{(\partial - \varphi)_{\ltimes h}}$, whence $\varphi - h^\dagger \succ 1 / (x \log x \cdots)$. It again follows that $(\partial - \varphi)\, h \succcurlyeq_{\mathfrak{v}} h / (x \log x \cdots) \succ_{\mathfrak{v}} 1$.

Let us now prove the proposition by induction on r. For $r = 1$, we have nothing to do, so assume that $r > 1$. Since $\tilde{L} = (\partial - \varphi_2) \cdots (\partial - \varphi_r)$ is normal, the induction hypothesis implies that $\partial - \varphi_i$ is normal for all $i \geqslant 2$. Now let h be the unique element in $\Sigma_L \setminus \Sigma_{\tilde{L}}$. Since h is normal, $(\partial - \varphi_i) \cdots (\partial - \varphi_r)\, h$ is also normal for $i = r, \ldots, 2$, by the auxiliary result. We conclude that $\partial - \varphi_1$ is normal, since $\varphi_1 = (\tilde{L}\, h)^\dagger$. □

Let L and $\Sigma_L = \{h_1, \ldots, h_r\}$ be as above. The smallest real number $\nu \geqslant 0$ with $\log h_i \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-\nu}$ for all i will be called the *growth rate* of L, and we denote $\sigma_L = \nu$. For all $\alpha \in \mathbb{R}$, we notice that $\sigma_{L_{\ltimes \mathfrak{v}^\alpha}} = \sigma_L$.

PROPOSITION 2.12. *Let* $K, L \in \mathcal{T}[\mathrm{i}][\partial]$ *be operators of the same order with*

$$K = L + o_{\mathfrak{v}}(\mathfrak{v}^{r_L \sigma_L} L).$$

Then $\mathfrak{H}_K = \mathfrak{H}_L$*.*

Proof. Given $h \in \Sigma_L$, we have

$$K_{\ltimes h} = L_{\ltimes h} + o_{\mathfrak{v}}(L_{\ltimes h}),$$

since $h^\dagger \asymp_{\mathfrak{v}} \log h \preccurlyeq \mathfrak{v}^{-\sigma_L}$. In particular, $K_{\ltimes h, 0} \prec_{\mathfrak{v}} K$, whence $1 \in \mathfrak{H}_{K_{\ltimes h}}$ and $\mathfrak{d}_h \in \mathfrak{H}_K$. □

PROPOSITION 2.13. *Given a splitting*

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r)$$

with $\varphi_1, \ldots, \varphi_r \in \mathbb{T}_{\preccurlyeq\mathfrak{v}}[\mathrm{i}]$*, we have* $\varphi_i \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$ *for all* i*.*

Proof. Assume for contradiction that $\varphi_i \succ_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$ for some i and choose i maximal with this property. Setting

$$K = (\partial - \varphi_{i+1}) \cdots (\partial - \varphi_r),$$

the transseries

$$h = K^{-1}(\mathrm{e}^{\int \varphi_i}) \in \mathbb{T}_{\preccurlyeq\mathfrak{v}}[\mathrm{i}]\, \mathrm{e}^{\int \varphi_i}$$

satisfies $L h = 0$, as well as $\log h \asymp_{\mathfrak{v}} \varphi_i \succ_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$. But such an h cannot be a linear combination of the h_i with $\log h_i \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-\sigma_L}$. □

Remark 2.14. It can be shown (although this will not be needed in what follows) that an operator $L \in \mathcal{T}[\mathrm{i}][\partial]$ splits over $\hat{\mathcal{T}}[\mathrm{i}]$ if and only if there exists an approximation $\tilde{L} \in \mathcal{T}[\mathrm{i}][\partial]$ with $\tilde{L} - L \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^\lambda$ which splits over $\mathcal{T}[\mathrm{i}]$ for every $\lambda \in \mathbb{R}$. In particular, $\hat{\mathcal{T}}[\mathrm{i}]$ is r-linearly closed if and only if $\mathcal{T}[\mathrm{i}]$ is r-linearly closed over $\hat{\mathcal{T}}[\mathrm{i}]$.

2.7. Normalization of quasi-linear equations

Assume now that $\mathcal{T}$ is a differential subfield of $\mathbb{T}$ of span $\mathfrak{v} \succ\!\!\succ x$. We say that P is *normal* if L_P is normal of order r_P and $P_{\neq 1} \prec_{\mathfrak{v}} \mathfrak{v}^{r_P \sigma_{L_P}} L_P$. In that case, the equation

$$P(f) = 0, \qquad f \preccurlyeq_{\mathfrak{v}} 1 \tag{2.8}$$

is quasi-linear and it admits a unique solution in $\mathbb{T}_{\preccurlyeq \mathfrak{v}}$. Indeed, let $f \in \mathbb{T}_{\preccurlyeq \mathfrak{v}}$ be the distinguished solution to (2.8). By proposition 2.12, the operator $L_{P_{+f}}$ is normal. If $\tilde{f} \in \mathbb{T}_{\preccurlyeq \mathfrak{v}}$ were another solution to (2.8), then $\mathfrak{d}_{\tilde{f}-f}$ would be in $\mathfrak{H}_{L_{+f}}$, whence $\tilde{f} \succ 1$, which is impossible.

PROPOSITION 2.15. *Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$. Let $P \in \mathcal{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathcal{T}}[\mathrm{i}]$ over $\mathcal{T}[\mathrm{i}]$. Then there exists a truncation $\varphi \lhd f$ and $\lambda \in \mathbb{R}$ such that $P_{+\varphi, \times \mathfrak{v}^\lambda}$ is normal.*

Proof. Let $\tilde{P} = P_{+f}$ and $\nu = r_{L_{\tilde{P}}} \sigma_{L_{\tilde{P}}}$. Modulo a multiplicative conjugation by $\mathfrak{v}^\alpha$ for some $\alpha \geqslant 0$, we may assume without loss of generality that $\tilde{P} \asymp L_{\tilde{P}}$. Modulo an additive conjugation by $f_{\succcurlyeq_{\mathfrak{v}} 1}$, we may also assume that $f \prec_{\mathfrak{v}} 1$. For any $\lambda, \mu \geqslant 0$ and $\varphi = f_{\succcurlyeq_{\mathfrak{v}} \mathfrak{v}^\mu} \lhd f$, we have

$$P_{+\varphi} = \tilde{P}_{+\varphi - f} = \tilde{P} + o_{\mathfrak{v}}(\mathfrak{v}^\mu \tilde{P}),$$

whence

$$P_{+\varphi, \times \mathfrak{v}^\lambda} = \tilde{P}_{1, \times \mathfrak{v}^\lambda} + O_{\mathfrak{v}}(\mathfrak{v}^{2\lambda} \tilde{P}) + o_{\mathfrak{v}}(\mathfrak{v}^\mu \tilde{P}). \tag{2.9}$$

Since $S_P(f) \neq 0$, we have $\tilde{P}_1 \neq 0$. By proposition 2.10, there exists a $\lambda > \nu$ for which $L_{\tilde{P}, \times \mathfrak{v}^\lambda}$ is normal. Now take $\mu = \lambda + \nu$. Denoting $N = P_{+\varphi, \times \mathfrak{v}^\lambda}$, proposition 2.12 and (2.9) imply that L_N is normal with $\sigma_{L_N} = \nu$ and $N_{\neq 1} \prec_{\mathfrak{v}} \mathfrak{v}^\nu \tilde{P}_{1, \times \mathfrak{v}^\lambda} \asymp \mathfrak{v}^\nu L_N$. □

We say that $P \in \mathcal{T}[\mathrm{i}]\{F\}$ is *split-normal*, if P is normal and L_P can be decomposed $L_P = L + K$ such that L splits over $\mathcal{T}[\mathrm{i}]$ and $K \prec_{\mathfrak{v}} \mathfrak{v}^{r_L \sigma_L} L$. In that case, we may also decompose $P(F) = LF + R(F)$ for $R(F) = P_{\neq 1}(F) + KF$ with $R \prec_{\mathfrak{v}} \mathfrak{v}^{r_L \sigma_L} L$. If L is monic, then we say that P is *monic split-normal*. Any split-normal equation (2.8) is clearly equivalent to a monic split-normal equation of the same form.

PROPOSITION 2.16. *Let $\mathcal{T}$ be a differential subfield of $\mathbb{T}$ of span $\mathfrak{v}$ such that $\hat{\mathcal{T}}[\mathrm{i}]$ is r-linearly closed. Let $P \in \mathcal{T}[\mathrm{i}]\{F\}$ be a minimal annihilator of a differentially algebraic cut $f \in \hat{\mathcal{T}}[\mathrm{i}]$ of order r over $\mathcal{T}[\mathrm{i}]$. Let $S \in \mathcal{T}\{F\}$ be a minimal annihilator of $\operatorname{Re} f$ and assume that $r_S \geqslant r_P$. Then there exists a truncation $\varphi \lhd \operatorname{Re} f$ and $\lambda \in \mathbb{R}$ such that $S_{+\varphi, \times \mathfrak{v}^\lambda}$ is split-normal.*

Proof. By proposition 2.15 and modulo a replacement of f by $\mathfrak{v}^{-\lambda} (f - \varphi)$, we may assume without loss of generality that S is normal. By lemma 2.9, S splits over $\hat{\mathcal{T}}[\mathrm{i}]$ at $\operatorname{Re} f$. Let $c, \varphi_1, \ldots, \varphi_s \in \hat{\mathcal{T}}[\mathrm{i}]$ be such that

$$L_{S_{+f}} = c\, (\partial - \varphi_1) \cdots (\partial - \varphi_s).$$

Setting $\nu = s\, \sigma_{L_S}$, we notice that $L_S = L_{S_{+f}} + o_{\mathfrak{v}}(\mathfrak{v}^\nu L_S)$. Now take

$$L = c_{\succcurlyeq_{\mathfrak{v}} \mathfrak{v}^\nu \mathfrak{d}_c} (\partial - \varphi_{1, \succcurlyeq_{\mathfrak{v}} \mathfrak{v}^\nu}) \cdots (\partial - \varphi_{s, \succcurlyeq_{\mathfrak{v}} \mathfrak{v}^\nu}) \in \mathcal{T}[\mathrm{i}][\partial].$$

Then $L = L_S + o_{\mathfrak{v}}(\mathfrak{v}^\nu L_S)$ and proposition 2.12 implies that L is normal, with $\sigma_L = \sigma_{L_S} = \sigma_{L_{S_{+f}}}$. Denoting $R(F) = S(F) - LF$, we finally have $R \prec_{\mathfrak{v}} \mathfrak{v}^{s \sigma_L} L$. □

3. TRANSSERIAL HARDY FIELDS

3.1. Transserial Hardy fields

Let $\mathbb{T} = \mathbb{R}[\![\![x]\!]\!] = \mathbb{R}[\![\mathfrak{T}]\!]$ be the field of grid-based transseries [Hoe06] and $\mathcal{G}$ the set of infinitely differentiable germs at infinity. A *transserial Hardy* field is a differential subfield $\mathcal{T} \supseteq \mathbb{R}$ of $\mathbb{T}$, together with a monomorphism ρ: $\mathcal{T} \rightarrow \mathcal{G}$ of ordered differential $\mathbb{R}$-algebras, such that

TH1. For every $f \in \mathcal{T}$, we have $\operatorname{supp} f \subseteq \mathcal{T}$.

TH2. For every $f \in \mathcal{T}$, we have $f_{\prec} \in \mathcal{T}$.

TH3. There exists an integer $d \in \mathbb{Z}$ such that $\log \mathfrak{m} \in \mathcal{T} + \mathbb{R} \log_d x$ for all $\mathfrak{m} \in \mathfrak{T} \cap \mathcal{T}$.

TH4. The set $\mathfrak{T} \cap \mathcal{T}$ is stable under taking real powers.

TH5. We have $\rho(\log f) = \log \rho(f)$ for all $f \in \mathcal{T}^{>}$ with $\log f \in \mathcal{T}$.

In what follows, we will always identify $\mathcal{T}$ with its image under ρ, which is necessarily a Hardy field in the classical sense.

We always have $d \geqslant 0$, since $\mathcal{T}$ is stable under differentiation. The incomplete transbasis theorem (see [Hoe06, section 4.4] and below) implies the following properties. If there exist an $\mathfrak{m} \in \mathfrak{T} \cap \mathcal{T}$ with $\log \mathfrak{m} \notin \mathcal{T}$, then the integer d in **TH3** is unique and called the *depth* of $\mathcal{T}$. In that case, $f{\uparrow}_d$ is exponential for all $f \in \mathcal{T}$ and $\mathcal{T}$ contains $\log_{d-1} x$. If $\mathcal{T} \neq \mathbb{R}$ and $\log \mathfrak{m} \in \mathcal{T}$ for all $\mathfrak{m} \in \mathfrak{T} \cap \mathcal{T}$, then **TH3** is satisfied for all sufficiently large d and the depth of $\mathcal{T}$ is defined to be $+\infty$. Notice that $\mathcal{T}$ contains $\log_k x$ for all sufficiently large k in that case.

Example 3.1. The field $\mathcal{T} = \mathbb{R}$ is clearly a transserial Hardy field. As will follow from theorem 3.12 below, other examples are

$$\begin{aligned}
\mathbb{R}(x^{\mathbb{R}}) &= \bigcup_{\alpha_1, \ldots, \alpha_k \in \mathbb{R}} \mathbb{R}(x^{\alpha_1}, \ldots, x^{\alpha_k}) \\
\mathbb{R}(\mathrm{e}^{\mathbb{R} x}) &= \bigcup_{\alpha_1, \ldots, \alpha_k \in \mathbb{R}} \mathbb{R}(\mathrm{e}^{\alpha_1 x}, \ldots, \mathrm{e}^{\alpha_k x}).
\end{aligned}$$

Remark 3.2. Although the axioms **TH4** and **TH5** are not really necessary, **TH4** allows for the simplification of several proofs, whereas it is natural to enforce **TH5**. Notice that **TH5** automatically holds for $f \in \mathcal{T}^{>}$ with $f \asymp 1$ since

$$\rho(\log f)' = \rho((\log f)') = \rho(f'/f) = \rho(f)'/\rho(f) = (\log \rho(f))',$$

whence $\rho(\log f) = \log \rho(f) + c$ for some $c \in \mathbb{R}$. Since both $\rho(\log f) - \log f_{\asymp}$ and $\log \rho(f) - \log f_{\asymp}$ are infinitesimal in $\mathcal{G}$, we have $c = 0$. Consequently, it suffices to check **TH5** for monomials $f \in \mathcal{T} \cap \mathfrak{T}$ with $\log f \in \mathcal{T}$.

PROPOSITION 3.3. *Let $\mathcal{T}$ be a transserial Hardy field with $x \in \mathcal{T}$. Then the upward shift $\mathcal{T}{\uparrow}$ of $\mathcal{T}$ carries a natural transserial Hardy field structure with $\rho(f{\uparrow}) = \rho(f) \circ \mathrm{e}^x$.*

Proof. The field $\mathcal{T}{\uparrow}$ is stable under differentiation, since $f{\uparrow}' = (x f'){\uparrow}$ for all $f \in \mathcal{T}$. □

COROLLARY 3.4. *If $\mathcal{T}$ has depth $d < \infty$, then $\mathcal{T}{\uparrow}_d$ is a transserial Hardy field of depth* 0.

We recall that a *transbasis* $\mathfrak{B}$ is a finite set of transmonomials $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ with

TB1. $\mathfrak{b}_1, \ldots, \mathfrak{b}_n \succ 1$ and $\mathfrak{b}_1 \prec\!\!\prec \cdots \prec\!\!\prec \mathfrak{b}_n$.

TB2. $\mathfrak{b}_1 = \log_{d-1} x$ for some $d \in \mathbb{Z}$.

TB3. $\log \mathfrak{b}_i \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]]$ for all $1 < i \leqslant n$.

If $d = 0$, then $\mathfrak{B}$ is called a *plane* transbasis and $\mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$ is stable under differentiation. The incomplete transbasis theorem for $\mathbb{T}$ also holds for transserial Hardy fields:

PROPOSITION 3.5. *Let $\mathfrak{B} \subseteq \mathcal{T}$ be a transbasis and $f \in \mathcal{T}$. Then there exists a transbasis $\hat{\mathfrak{B}} \subseteq \mathcal{T}$ with $\hat{\mathfrak{B}} \supseteq \mathfrak{B}$ and $f \in \mathbb{R}[[\mathfrak{B}^{\mathbb{R}}]]$. Moreover, if $\mathfrak{B}$ is plane and f is exponential, then $\hat{\mathfrak{B}}$ may be taken to be plane.*

Proof. The same proof as for [Hoe06, Theorem 4.15] may be used, since all field operations, logarithms and truncations used in the proof can be carried out in $\mathcal{T}$. $\square$

Given a set $\mathcal{F}$ of exponential transseries in $\mathcal{T}$, the *transrank* of $\mathcal{F}$ is the minimal cardinal n of a plane transbasis $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ with $\mathcal{F} \subseteq \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$. This notion may be extended to allow for differential polynomials P in $\mathcal{F}$ (modulo the replacement of P by its set of coefficients).

Remark 3.6. The span and ultimate span of $f \in \mathcal{T}$ are not necessarily in $\mathcal{T}$. Nevertheless, if $\operatorname{span} f \neq 1$ and $\mathfrak{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} \subseteq \mathcal{T}$ is a transbasis for f, then we do have $\operatorname{span} f \asymp\!\!\!\!\asymp \mathfrak{b}_i$ for some i (and similarly for the ultimate span of f).

3.2. Cuts in transserial Hardy fields

Let $\mathcal{T}$ be a transserial Hardy field. Given $f \in \mathbb{T}$ and $\hat{f} \in \mathcal{G}$, we write $f \sim \hat{f}$ if there exists a $\varphi \in \mathcal{T}$ with

$$f \sim_{\mathbb{T}} \varphi \sim_{\mathcal{G}} \hat{f}.$$

We say that f and $\hat{f}$ are *asymptotically equivalent* over $\mathcal{T}$ if for each $\varphi \in \mathcal{T}$ (or, equivalently, for each $\varphi \lhd f$), we have

$$f - \varphi \sim \hat{f} - \varphi.$$

We say that f and $\hat{f}$ are *differentially equivalent* over $\mathcal{T}$ if

$$P(f) = 0 \Leftrightarrow P(\hat{f}) = 0$$

for all $P \in \mathcal{T}\{F\}$.

LEMMA 3.7. *Let $\mathcal{T}$ be a transserial Hardy field and let $f \in \mathbb{T} \setminus \mathcal{T}$ be differentially algebraic over $\mathcal{T}$. Let $\mathfrak{m} \in \operatorname{supp} f$ be maximal for $\succcurlyeq$, such that $\varphi = f_{\succ \mathfrak{m}} \notin \mathcal{T}$. If φ is a serial cut over $\mathcal{T}$, then φ is differentially algebraic over $\mathcal{T}$ and $\chi_\varphi \leqslant \chi_f$.*

Proof. Let $P \in \mathcal{T}\{F\}$ be a minimal annihilator of f. Modulo upward shifting, we may assume without loss of generality that P and f are exponential. Since $\varphi \in \hat{\mathcal{T}}$, all monomials in $\operatorname{supp} \varphi$ are in $\mathcal{T}$, whence there exists a plane transbasis $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\} \subseteq \mathcal{T}$ for P and φ. Modulo subtraction of H_φ from f and φ, we may assume without loss of generality that $\mathrm{H}_\varphi = 0$. Let k be such that $\operatorname{uspan} \varphi \asymp\!\!\!\!\asymp \mathfrak{b}_k$ and let $\mathfrak{b}_1^{\alpha_1} \cdots \mathfrak{b}_n^{\alpha_n}$ be the dominant monomial of φ. Modulo division of f and φ by $\mathfrak{b}_{k+1}^{\alpha_{k+1}} \cdots \mathfrak{b}_n^{\alpha_n}$, we may also assume that φ is a normal serial cut. But then the equation $P(f) = 0$ gives rise to the equation $P_{\preccurlyeq\!\!\!\!\prec \mathfrak{b}_k}(\varphi) = 0$ for $\varphi = f_{\preccurlyeq\!\!\!\!\prec \mathfrak{b}_k}$. The complexity of $P_{\preccurlyeq\!\!\!\!\prec \mathfrak{b}_k}$ is clearly bounded by $\chi_P = \chi_f$. $\square$

LEMMA 3.8. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathfrak{v} \in \mathcal{T} \cap \mathfrak{T}^{\prec}$. Let $f \in \widehat{\mathcal{T}_{\preccurlyeq\!\!\!\!\prec \mathfrak{v}}}$ and $\hat{f} \in \mathcal{G}$ be such that f and $\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}_{\preccurlyeq\!\!\!\!\prec \mathfrak{v}}$. Then f and $\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}$.*

Proof. Given $\varphi \in \mathcal{T}$, we either have $\varphi \succ_{\mathfrak{v}}^* 1$ and

$$f - \varphi \sim_{\mathbb{T}} -\varphi \sim_{\mathcal{G}} \hat{f} - \varphi$$

or $\varphi \preccurlyeq_{\mathfrak{v}}^* 1$, in which case

$$f - \varphi \sim_{\mathbb{T}} f - \varphi_{\asymp_{\mathfrak{v}}^* 1} \sim \hat{f} - \varphi_{\asymp_{\mathfrak{v}}^* 1} \sim_{\mathcal{G}} \hat{f} - \varphi.$$

This proves that f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}$.

As to their differential equivalence, let us first assume that f is differentially transcendent over $\mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}$. Given $R \in \mathcal{T}\{F\}^{\neq}$, let us denote

$$D_R = \mathfrak{d}_R^{-1} R_{\asymp_{\mathfrak{v}}^* \mathfrak{d}_R} \in \mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}.$$

We have $D_R(f) \neq 0$, $D_R(\hat{f}) \neq 0$ and

$$R(f) \sim_{\mathfrak{v}}^* D_R(f)\, \mathfrak{d}_R \tag{3.1}$$

$$R(\hat{f}) \sim_{\mathfrak{v}}^* D_R(\hat{f})\, \mathfrak{d}_R, \tag{3.2}$$

whence $R(f) \neq 0$ and $R(\hat{f}) \neq 0$.

Assume now that f is differentially algebraic over $\mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}$ and let $P \in \mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}\{F\}$ be a minimal annihilator. Given $Q \in \mathcal{T}\{F\}$, Ritt reduction of Q w.r.t. P gives

$$I_P^{\alpha} S_P^{\beta} Q = \boldsymbol{A} P + R,$$

where $\boldsymbol{A} \in \mathcal{T}\{F\}[\partial]$ and $R \in \mathcal{T}\{F\}$ is such that $\chi_R < \chi_P$. Since $\chi_{H_P} < \chi_P$ and $H_P \in \mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}$, we both have $H_P(f) \neq 0$ and $H_P(\hat{f}) \neq 0$, whence

$$Q(f) = \frac{R(f)}{I_P(f)^{\alpha}\, S_P(f)^{\beta}}$$

$$Q(\hat{f}) = \frac{R(\hat{f})}{I_P(\hat{f})^{\alpha}\, S_P(f)^{\beta}}.$$

If $R = 0$, this clearly implies $R(f) = R(\hat{f}) = 0$. Otherwise, D_R vanishes neither at f nor at $\hat{f}$ and the relations (3.1) and (3.2) again yield $R(f) \neq 0$ and $R(\hat{f}) \neq 0$. We conclude that either $Q(f) = Q(\hat{f}) = 0$ or $Q(f)\, Q(\hat{f}) \neq 0$. □

LEMMA 3.9. *Let $\mathcal{T}$ be a transserial Hardy field and let $f \in \hat{\mathcal{T}} \setminus \mathcal{T}$ be a differentially algebraic cut over $\mathcal{T}$ with minimal annihilator P. Let $\hat{f} \in \mathcal{G}$ be a root of P such that f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}$. Then f and $\hat{f}$ are differentially equivalent over $\mathcal{T}$.*

Proof. Let $\mathfrak{v} \in \mathcal{T}$ be such that uspan $f \asymp \mathfrak{v}$. Modulo some upward shiftings, we may assume without loss of generality that f and P are exponential. Modulo an additive conjugation by H_f and a multiplicative conjugation by $\mathfrak{d}_f$, we may also assume that f is a normal cut. Modulo a division of P by $\mathfrak{d}_P$ and replacing P by $P_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}$, we may finally assume that $P \in \mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}\{F\}$.

Now consider $Q \in \mathcal{T}_{\preccurlyeq\!\!\!\!\preccurlyeq \mathfrak{v}}\{F\}^{\neq}$ with $\chi_Q < \chi_P$. Since $Q(f) \neq 0$, there exists a $\varphi \lhd f$ with $f - \varphi \prec_{\mathfrak{v}} 1$ and $Q_{+\varphi, \neq 0} \prec_{\mathfrak{v}} Q(\varphi)$. But then

$$Q(\hat{f}) = Q(\varphi) + Q_{+\varphi, \neq 0}(\hat{f} - \varphi) \sim Q(\varphi) \neq 0.$$

For general $Q \in \mathcal{T}\{F\}$, we use Ritt reduction of Q w.r.t. P and conclude in a similar way as in the proof of lemma 3.8. □

3.3. Elementary extensions

Lemma 3.10. *Let $f \in \mathbb{T} \setminus \mathcal{T}$ and $\hat{f} \in \mathcal{G} \setminus \mathcal{T}$ be such that*

i. f is a serial cut over $\mathcal{T}$.

ii. f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}$.

iii. f and $\hat{f}$ are differentially equivalent over $\mathcal{T}$.

Then $\mathcal{T}\langle f\rangle$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}\langle f\rangle \rightarrow \mathcal{G}$ over $\mathcal{T}$ with $\rho(f) = \hat{f}$.

Proof. Modulo upward shifting, an additive conjugation by H_f and a multiplicative conjugation by $\mathfrak{d}_f$, we may assume without loss of generality that f is an exponential normal serial cut. Let $\mathfrak{v} \in \mathcal{T}$ be such that uspan $f \asymp\!\!\!\!\asymp \mathfrak{v}$. We have to show that $\mathcal{T}\langle f\rangle$ is closed under truncation and that $P(f) \sim P(\hat{f})$ for all $P \in \mathcal{T}\{F\}$ with $P(f) \neq 0$ (this implies in particular that ρ is increasing). Notice that supp $f \subseteq \mathcal{T}$ implies $\mathcal{T}\langle f\rangle \cap \mathfrak{T} = \mathcal{T} \cap \mathfrak{T}$.

Truncation closedness. Given $P \in \mathcal{T}\langle F\rangle$, let us prove by induction on the transrank n of $\{P, f\}$ that $P(f)_{\succ} \in \mathcal{T}\langle f\rangle$. So let $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a plane transbasis for P and f. Assume first that $\mathfrak{b}_n \succ\!\!\succ \mathfrak{v}$. Writing

$$P = \sum_{\alpha \in \mathbb{R}} P_\alpha \, \mathfrak{b}_n^\alpha \in \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{n-1}]] \langle F\rangle [[\mathfrak{b}_n]],$$

the sum

$$P_{\succ \mathfrak{b}_n} = \sum_{\alpha > 0} P_\alpha \, \mathfrak{b}_n^\alpha$$

is finite, whence

$$P(f)_{\succ \mathfrak{b}_n} = P_{\succ \mathfrak{b}_n}(f) = \sum_{\alpha > 0} P_\alpha(f) \, \mathfrak{b}_n^\alpha \in \mathcal{T}\langle f\rangle.$$

By the induction hypothesis, we also have $P_0(f)_{\succ} \in \mathcal{T}\langle f\rangle$ and $P(f)_{\succ} \in \mathcal{T}\langle f\rangle$. If $\mathfrak{b}_n \asymp\!\!\!\!\asymp \mathfrak{v}$, then

$$P(f)_{\succ} = P(\varphi)_{\succ}$$

for a sufficiently large truncation $\varphi \lhd f$, whence $P(f)_{\succ} \in \mathcal{T}$.

Preservation of dominant terms. Given $P \in \mathcal{T}\{F\}$ with $P(f) \neq 0$, let us prove by induction on the transrank n of $\{P, f\}$ that $P(f) \sim P(\hat{f})$. Let $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a plane transbasis for P and f and assume first that $\mathfrak{v} \prec\!\!\prec \mathfrak{b}_n$. Since $P(f) \neq 0$, there exists a maximal α with $P_\alpha(f) \neq 0$, when considering $P = \sum_{\alpha \in \mathbb{R}} P_\alpha \, \mathfrak{b}_n^\alpha$ as a series in $\mathfrak{b}_n$. But then

$$P(f) \sim P_\alpha(f) \, \mathfrak{b}_n^\alpha \sim P_\alpha(\hat{f}) \, \mathfrak{b}_n^\alpha \sim P(\hat{f}),$$

by the induction hypothesis. If $\mathfrak{b}_n \asymp\!\!\!\!\asymp \mathfrak{v}$, then there exists an $\alpha \in \mathbb{R}$ such that, for all sufficiently large truncations $\varphi \lhd f$, the Taylor series expansion of $P(\varphi + (f - \varphi))$ yields

$$\begin{aligned} P(f) &= P(\varphi) + O_{\mathfrak{v}}((f - \varphi) \, \mathfrak{v}^\alpha) \\ P(\hat{f}) &= P(\varphi) + O_{\mathfrak{v}}((\hat{f} - \varphi) \, \mathfrak{v}^\alpha). \end{aligned}$$

Taking $\varphi \lhd f$ such that $(f - \varphi) \, \mathfrak{v}^\alpha \prec_{\mathfrak{v}} P(f)$, we obtain

$$P(f) \sim P(\varphi) \sim P(\hat{f}).$$

This completes the proof. □

Theorem 3.11. *Let $\mathcal{T}$ be a transserial Hardy field. Then its real closure $\mathcal{T}^{\mathrm{rcl}}$ admits a unique transserial Hardy field structure which extends the one of $\mathcal{T}$.*

Proof. Assume that $\mathcal{T}^{\mathrm{rcl}} \neq \mathcal{T}$ and choose $f \in \mathcal{T}^{\mathrm{rcl}} \setminus \mathcal{T}$ of minimal complexity. By lemma 3.7, we may assume without loss of generality that f is a serial cut. Consider the monic minimal polynomial $P \in \mathcal{T}[F]$ of f. Since $P'(f) \neq 0$, we have

$$\deg_{\preccurlyeq f-\varphi} P_{+\varphi} = 1$$

for a sufficiently large truncation $\varphi \lhd f$ of f (we refer to [Hoe06, Section 8.3] for a definition of the Newton degrees $\deg_{\preccurlyeq \psi} P$). But then

$$P_{+\varphi}(g) = 0, \quad g \preccurlyeq f - \varphi \tag{3.3}$$

admits unique solutions g and $\hat{g}$ in $\mathbb{T}$ resp. $\mathcal{G}$, by the implicit function theorem. It follows in particular that $f = \varphi + g$. Let $\hat{f} = \varphi + \hat{g}$ and consider ψ with $\varphi \trianglelefteq \psi \lhd f$. Then

$$\begin{aligned} P(f) - P(\psi) &\sim P_{+\psi,1}(f - \psi) \\ P(\hat{f}) - P(\psi) &\sim P_{+\psi,1}(\hat{f} - \psi) \end{aligned}$$

Since $P(f) = P(\hat{f}) = 0$, we obtain $f - \psi \sim \hat{f} - \psi$, whence f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}$. By lemmas 3.9 and 3.10, it follows that $\mathcal{T}\langle f \rangle$ carries a transserial Hardy field structure which extends the one on $\mathcal{T}$. Since (3.3) has a unique solution $\hat{g}$ in $\mathcal{G}$, this structure is unique. We conclude by Zorn's lemma. □

3.4. Exponential and logarithmic extensions

THEOREM 3.12. *Let $\mathcal{T}$ be a transserial Hardy field and let $\varphi \in \mathcal{T}_{\succ}$ be such that $\mathrm{e}^{\varphi} \notin \mathcal{T}$. Then the set $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho(\mathrm{e}^{\lambda\varphi}) = \mathrm{e}^{\lambda\rho(\varphi)}$ for all $\lambda \in \mathbb{R}$.*

Proof. Each element in $f = \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ is of the form $f = R(\mathrm{e}^{\lambda_1\varphi}, \ldots, \mathrm{e}^{\lambda_k\varphi})$ for $R \in \mathcal{T}(F_1, \ldots, F_k)$ and $\mathbb{Q}$-linearly independent $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Given $R \in \mathcal{T}(F_1, \ldots, F_k)$, let $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be a transbasis for R. We may write

$$\mathrm{e}^{\varphi} = \mathrm{e}^{\tilde{\varphi}} \mathfrak{b}_i^{\alpha_i} \cdots \mathfrak{b}_n^{\alpha_n}$$

with $\mathfrak{b}_{i-1} \prec\!\!\prec \mathrm{e}^{\tilde{\varphi}} \prec\!\!\prec \mathfrak{b}_i$ (or the obvious adaptations if $i = 1$ or $i = n + 1$). Modulo the substitution of φ by $\alpha_i \log \mathfrak{b}_i + \cdots + \alpha_n \log \mathfrak{b}_n + \tilde{\varphi}$, we may assume without loss of generality that $\alpha_i = \cdots = \alpha_n = 0$.

If $\mathfrak{b}_n \prec\!\!\prec \mathrm{e}^{\varphi}$, then we may regard $f = \sum_{\mu \in \mathbb{R}} f_\mu \mathrm{e}^{\mu\varphi}$ as a convergent grid-based series in e^{φ} with coefficients in $\mathcal{T} \cap \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_n]]$. In particular,

$$f_{\succ} = \left[\sum_{\mu \operatorname{sign} \varphi > 0} f_\mu \mathrm{e}^{\mu\varphi}\right] + f_{0,\succ} \in \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}).$$

Furthermore, if f admits ν as its dominant exponent in e^{φ}, then $f \sim f_\nu \mathrm{e}^{\nu\varphi}$ holds both in $\mathbb{T}$ and in $\mathcal{G}$.

If $\mathrm{e}^{\varphi} \prec\!\!\prec \mathfrak{b}_n$, then we may consider R as a series

$$R \in \mathcal{S} := (\mathcal{T} \cap \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]])(F_1, \ldots, F_k)[[\mathfrak{b}_i; \ldots; \mathfrak{b}_n]]$$

in $\mathfrak{b}_i, \ldots, \mathfrak{b}_n$. Since $\mathcal{T}$ is closed under truncation, both $R_{\succ \mathfrak{b}_i}$ and $R_{\preceq \mathfrak{b}_i}$ lie in $\mathcal{S}$, whence

$$f_{\succ} = R_{\succ \mathfrak{b}_i}(\mathrm{e}^{\lambda_1\varphi}, \ldots, \mathrm{e}^{\lambda_k\varphi}) + R_{\preceq \mathfrak{b}_i}(\mathrm{e}^{\lambda_1\varphi}, \ldots, \mathrm{e}^{\lambda_k\varphi})_{\succ} \in \mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}),$$

by what precedes. Similarly, if $R_{\nu_i, \ldots, \nu_n} \mathfrak{b}_i^{\nu_i} \cdots \mathfrak{b}_n^{\nu_n}$ is the dominant term of R as a series in $\mathfrak{b}_i, \ldots, \mathfrak{b}_n$ and $c\,\mathrm{e}^{\nu\varphi}$ is the dominant term of $R_{\nu_i, \ldots, \nu_n}(\mathrm{e}^{\lambda_1\varphi}, \ldots, \mathrm{e}^{\lambda_k\varphi})$ as a series in e^{φ} (with $c \in \mathcal{T} \cap \mathbb{R}[[\mathfrak{b}_1; \ldots; \mathfrak{b}_{i-1}]]$), then $f \sim c\,\mathrm{e}^{\nu\varphi} \mathfrak{b}_i^{\nu_i} \cdots \mathfrak{b}_n^{\nu_n}$ holds both in $\mathbb{T}$ and in $\mathcal{G}$.

This shows that $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ is truncation closed and that the extension of ρ to $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ is increasing. We also have $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi}) \cap \mathfrak{T} = (\mathcal{T} \cap \mathfrak{T})\,\mathrm{e}^{\mathbb{R}\varphi}$. In other words, $\mathcal{T}(\mathrm{e}^{\mathbb{R}\varphi})$ is a transserial Hardy field. □

THEOREM 3.13. *Let $\mathcal{T}$ be a transserial Hardy field of depth $d < \infty$. Then $\mathcal{T}((\log_d x)^{\mathbb{R}})$ carries the structure of a transserial Hardy field for the unique differential morphism $\rho: \mathcal{T}((\log_d x)^{\mathbb{R}}) \to \mathcal{G}$ over $\mathcal{T}$ with $\rho((\log_d x)^{\lambda}) = (\log_d x)^{\lambda}$ for all $\lambda \in \mathbb{R}$.*

Proof. The proof is similar to the proof of theorem 3.12, when replacing e^{φ} by $\log_l x$. □

3.5. Complex transserial Hardy fields

Let $\mathcal{T}$ be a transserial Hardy field. Asymptotic and differential equivalence over $\mathcal{T}[\mathrm{i}]$ are defined in a similar way as over $\mathcal{T}$.

PROPOSITION 3.14. *Let $\mathcal{T}$ be a transserial Hardy field. Let $f \in \mathbb{T}[\mathrm{i}]$ be a serial cut over $\mathcal{T}[\mathrm{i}]$ and $\hat{f} \in \mathcal{G}[\mathrm{i}]$. Then f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$ if and only if* $\operatorname{Re} f$ *and* $\operatorname{Re} \hat{f}$ *as well as* $\operatorname{Im} f$ *and* $\operatorname{Im} \hat{f}$ *are asymptotically equivalent over $\mathcal{T}$.*

Proof. Assume that f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$ and let $\varphi \lhd \operatorname{Re} f$. Consider $\psi = (\operatorname{Im} f)_{\succ \operatorname{Re} f - \varphi} \trianglelefteq \operatorname{Im} f$. We have $\varphi + \psi\,\mathrm{i} \lhd f$, so that $f - \varphi - \psi\,\mathrm{i} \sim \hat{f} - \varphi - \psi\,\mathrm{i}$. Moreover, $f - \varphi - \psi\,\mathrm{i} \asymp \operatorname{Re} f - \varphi$, whence $\operatorname{Re} f - \varphi \sim \operatorname{Re} \hat{f} - \varphi$ and $\operatorname{Re} f \sim \operatorname{Re} \hat{f}$. The relation $\operatorname{Im} f \sim \operatorname{Im} \hat{f}$ is proved similarly. Inversely, assume that $\operatorname{Re} f$ and $\operatorname{Re} \hat{f}$ as well as $\operatorname{Im} f$ and $\operatorname{Im} \hat{f}$ are asymptotically equivalent over $\mathcal{T}$. Given $\varphi \lhd f$, we have $\operatorname{Re} \varphi, \operatorname{Im} \varphi \in \mathcal{T}$, whence there exist $g, h \in \mathcal{T}$ with $\operatorname{Re} f - \operatorname{Re} \varphi \sim g \sim \operatorname{Re} \hat{f} - \operatorname{Re} \varphi$ and $\operatorname{Im} f - \operatorname{Im} \varphi \sim h \sim \operatorname{Im} \hat{f} - \operatorname{Im} \varphi$. It follows that $f - \varphi \sim g + h\,\mathrm{i} \sim \hat{f} - \varphi$, whence $f \sim \hat{f}$. □

PROPOSITION 3.15. *Let $\mathcal{T}$ be a transserial Hardy field, $f \in \mathbb{T}$ and $\hat{f} \in \mathcal{G}$. Then f and $\hat{f}$ are differentially equivalent over $\mathcal{T}[\mathrm{i}]$ if and only if they are differentially equivalent over $\mathcal{T}$.*

Proof. Differential equivalence over $\mathcal{T}[\mathrm{i}]$ clearly implies differential equivalence over $\mathcal{T}$. Assuming that f and $\hat{f}$ are differentially equivalent over $\mathcal{T}$, we also have

$$P(f) = 0 \Leftrightarrow (\operatorname{Re} P)(f) = 0 \wedge (\operatorname{Im} P)(f) = 0 \Leftrightarrow (\operatorname{Re} P)(\hat{f}) = 0 \wedge (\operatorname{Im} P)(\hat{f}) = 0 \Leftrightarrow P(\hat{f}) = 0$$

for every $P \in \mathcal{T}[\mathrm{i}]\{F\}$. □

Remark 3.16. Given $f \in \mathbb{T}$ and $\hat{f} \in \mathcal{G}$, it can happen that f and $\hat{f}$ are differentially equivalent over $\mathcal{T}[\mathrm{i}]$, without $\operatorname{Re} f$ and $\operatorname{Re} \hat{f}$ being differentially equivalent over $\mathcal{T}$. This is for instance the case for $\mathcal{T} = \mathbb{R}(x^{\mathbb{R}})$, $f = \mathrm{e}^x$ and $\hat{f} = \mathrm{i}\,\mathrm{e}^x$. Indeed, the differential ideals which annihilate f resp. $\hat{f}$ are both $F' - F$.

Most results from the previous sections generalize to the complex setting in a straightforward way. In particular, lemmas 3.7, 3.8 and 3.9 also hold over $\mathcal{T}[\mathrm{i}]$. However, the fundamental extension lemma 3.10 admits no direct analogue: when taking $f \in \mathbb{T}[\mathrm{i}] \setminus \mathcal{T}[\mathrm{i}]$ and $\hat{f} \in \mathcal{G}[\mathrm{i}] \setminus \mathcal{T}[\mathrm{i}]$ such that the complexified conditions *i*, *ii* and *iii* hold, we cannot necessarily give $\mathcal{T}\langle \operatorname{Re} f \rangle$ the structure of a transserial Hardy field. This explains why some results such as lemmas 2.9 and 2.16 have to be proved over $\mathcal{T}$ instead of $\mathcal{T}[\mathrm{i}]$. Of course, theorem 3.11 does imply the following:

THEOREM 3.17. *Let $\mathcal{T}$ be a transserial Hardy field. Then there exists a unique algebraic transserial Hardy field extension $\mathcal{T}^{\mathrm{rcl}}$ of $\mathcal{T}$ such that $\mathcal{T}^{\mathrm{rcl}}[\mathrm{i}]$ is algebraically closed.*

4. ANALYTIC RESOLUTION OF DIFFERENTIAL EQUATIONS

Recall that $\mathcal{G}$ stands for the differential algebra of infinitely differentiable germs of real functions at $+\infty$. Given $x_0 \in \mathbb{R}$, we will denote by $\mathcal{G}_{x_0}$ the differential subalgebra of infinitely differentiable functions on $[x_0, \infty)$. We define a norm on $\mathcal{G}_{x_0}^{\preccurlyeq} = \{f \in \mathcal{G}_{x_0} : f \preccurlyeq 1\}$ by

$$\|f\|_{x_0} = \sup_{x \geqslant x_0} |f(x)|$$

Given $r \in \mathbb{N}$, we also denote $\mathcal{G}_{x_0;r}^{\preccurlyeq} = \{f \in \mathcal{G}_{x_0} : f, \ldots, f^{(r)} \preccurlyeq 1\}$ and define a norm on $\mathcal{G}_{x_0;r}^{\preccurlyeq}$ by

$$\|f\|_{x_0;r} = \max\{\|f\|_{x_0}, \ldots, \|f^{(r)}\|_{x_0}\}.$$

Notice that

$$\|f g\|_{x_0;r} \leqslant 2^r \, \|f\|_{x_0;r} \, \|g\|_{x_0;r}.$$

An operator $K: \mathcal{G}_{x_0} \rightarrow \mathcal{G}_{x_0}$ (resp. $K: \mathcal{G}_{x_0} \rightarrow \mathcal{G}_{x_0;r}$) is said to be *continuous* if there exists an $M \in \mathbb{R}$ with $\|K f\|_{x_0} \leqslant M \, \|f\|_{x_0}$ (resp. $\|K f\|_{x_0;r} \leqslant M \, \|f\|_{x_0}$) for all $f \in \mathcal{G}_{x_0}$. The smallest such M is called the *norm* of K and denoted by $|||K|||_{x_0}$ (resp. $|||K|||_{x_0;r}$). The above definitions generalize in an obvious way to the complexifications $\mathcal{G}_{x_0}^{\preccurlyeq}[\mathrm{i}]$ and $\mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$.

4.1. Continuous right-inverses of first order operators

Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq\!\!\!\succ \mathrm{e}^x$. Consider a normal operator $\partial - \varphi$ with $\varphi \in \mathcal{T}[\mathrm{i}]$ and let x_0 be sufficiently large such that $\operatorname{Re} \varphi$ does not change sign on $[x_0, \infty)$. We define a primitive $\Phi \in \mathcal{G}$ of φ by

$$\Phi(x) = \begin{cases} \int_\infty^x \varphi(t) \, \mathrm{d}t & \text{if } \varphi \text{ is integrable at } \infty \\ \int_{x_0}^x \varphi(t) \, \mathrm{d}t & \text{otherwise} \end{cases}$$

Decomposing $\Phi = \Re + \Im \, \mathrm{i}$, we are either in one of the following two cases:

1. The repulsive case when $\mathrm{e}^{\Re} \succ_{\mathfrak{v}} 1$.
2. The attractive case when both $\mathrm{e}^{\Re} \prec_{\mathfrak{v}} 1$ and $\mathrm{e}^{\Re} \not\prec\!\!\!\prec \mathfrak{v}$.

Notice that the hypothesis $\mathfrak{v} \succcurlyeq\!\!\!\succ \mathrm{e}^x$ implies $\Re' = \operatorname{Re} \varphi \succcurlyeq \mathfrak{v}^\dagger \succcurlyeq 1$.

PROPOSITION 4.1. *The operator $J = (\partial - \varphi)_{x_0}^{-1}$, defined by*

$$(J f)(x) = \begin{cases} \mathrm{e}^{\Phi(x)} \int_\infty^x \mathrm{e}^{-\Phi(t)} \, f(t) \, \mathrm{d}t & \textit{(repulsive case)} \\ \mathrm{e}^{\Phi(x)} \int_{x_0}^x \mathrm{e}^{-\Phi(t)} \, f(t) \, \mathrm{d}t & \textit{(attractive case)} \end{cases} \tag{4.1}$$

is a continuous right-inverse of $L = \partial - \varphi$ on $\mathcal{G}^{\preccurlyeq}[\mathrm{i}]$, with

$$|||J|||_{x_0} \leqslant \left\| \frac{1}{\operatorname{Re} \varphi} \right\|_{x_0}. \tag{4.2}$$

Proof. In the repulsive case, the change of variables $\Re(t) = u$ yields

$$(J f)(x) = \mathrm{e}^{\Phi(x)} \int_\infty^{\Re(x)} \mathrm{e}^{-u - \Im(\Re^{\mathrm{inv}}(u)) \mathrm{i}} \, \frac{f(\Re^{\mathrm{inv}}(u))}{\Re'(\Re^{\mathrm{inv}}(u))} \, \mathrm{d}u \, .$$

It follows that

$$|(J f)(x)| \leqslant \mathrm{e}^{\Re(x)} \int_\infty^{\Re(x)} \mathrm{e}^{-u} \, \|f\|_x \left\| \frac{1}{\Re'} \right\|_x \mathrm{d}u = \|f\|_x \left\| \frac{1}{\Re'} \right\|_x$$

for all $x \geqslant x_0$, whence (4.2). In the attractive case, the change of variables $-\Re(t) = u$ leads in a similar way to the bound

$$\begin{aligned} |(Jf)(x)| &\leqslant \mathrm{e}^{\Re(x)} \int_{-\Re(x_0)}^{-\Re(x)} \mathrm{e}^u \, \|f\|_{x_0} \left\| \frac{1}{\Re'} \right\|_{x_0} \mathrm{d}u \\ &= [1 - \mathrm{e}^{\Re(x) - \Re(x_0)}] \, \|f\|_{x_0} \left\| \frac{1}{\Re'} \right\|_{x_0} \\ &\leqslant \|f\|_{x_0} \left\| \frac{1}{\Re'} \right\|_{x_0}, \end{aligned}$$

for all $x \geqslant x_0$, using the monotonicity of $\Re$. Again, we have (4.2). □

COROLLARY 4.2. *In the attractive case, the operator*

$$J_\lambda\colon f \longmapsto (Jf)(x) + \lambda \, \mathrm{e}^{\Phi(x)} \, \|f\|_{x_0}$$

is a continuous right-inverse of L *on* $\mathcal{G}^{\preccurlyeq}[\mathrm{i}]$, *for any* $\lambda \in \mathbb{C}$.

4.2. Continuous right-inverses of higher order operators

Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \not\succeq \mathrm{e}^x$. A monic operator $L \in \mathcal{T}[\mathrm{i}][\partial]$ is said to be *split-normal*, if it is normal and if it admits a splitting

$$L = (\partial - \varphi_1) \cdots (\partial - \varphi_r) \tag{4.3}$$

with $\varphi_1, \ldots, \varphi_r \in \mathcal{T}[\mathrm{i}]$. In that case, proposition 2.11 implies that each $\partial - \varphi_i$ is a normal first order operator. For a sufficiently large x_0, it follows that L admits a continuous “factorwise” right-inverse $J_r \cdots J_1$ on $\mathcal{G}[\mathrm{i}]^{\preccurlyeq}$, where $J_i = (\partial - \varphi_i)_{x_0}^{-1}$. We have

$$|||J_r \cdots J_1|||_{x_0} \leqslant |||J_r|||_{x_0} \cdots |||J_1|||_{x_0}.$$

PROPOSITION 4.3. $\mathfrak{v}^\nu J_r \cdots J_1\colon \mathcal{G}_{x_0}^{\preccurlyeq}[\mathrm{i}] \to \mathcal{G}_{x_0;r}^{\preccurlyeq}[\mathrm{i}]$ *is a continuous operator for every* $\nu > r\,\sigma_L$.

Proof. Given $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$, the the first r derivatives of $(\mathfrak{v}^\nu J_r \cdots J_1)\, f$ satisfy

$$[(\mathfrak{v}^\nu J_r \cdots J_1)\, f]^{(i)} = \sum_{j=r-i}^{r} c_{i,j} \, (\mathfrak{v}^\nu J_j \cdots J_1)\, f,$$

with

$$\begin{aligned} c_{0,r} &= 1 \\ c_{i+1,j} &= c_{i,j}' + \nu\, \mathfrak{v}^\dagger c_{i,j} + \varphi_j\, c_{i,j} + \tfrac{1}{\psi_{j+1}}\, c_{i,j+1}. \end{aligned}$$

By proposition 2.13 and induction on i, we have $c_{i,j} \preccurlyeq_{\mathfrak{v}} \mathfrak{v}^{-i\sigma_L}$ for all i, j. Since $\nu > r\,\sigma_L$, it follows that

$$\|[(\mathfrak{v}^\nu J_r \cdots J_1)\, f]^{(i)}\|_{x_0} \leqslant C_i \, \|f\|_{x_0}, \tag{4.4}$$

for all $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ and i, where

$$C_i = \sum_{j=r-1}^{r} \|\mathfrak{v}^\nu c_{i,j}\|_{x_0} \, |||J_j|||_{x_0} \cdots |||J_1|||_{x_0}.$$

We conclude that

$$|||\mathfrak{v}^\nu J_r \cdots J_1|||_{x_0;r} \leqslant \max\{C_0, \ldots, C_r\}.$$ □

PROPOSITION 4.4. *If* $L \in \mathcal{T}[\partial]$ *and the splitting* (4.3) *preserves realness, then* $J_r \cdots J_1$ *preserves realness in the sense that it maps* $\mathcal{G}_{x_0}^{\preccurlyeq}$ *into itself.*

Proof. It clearly suffices to prove the proposition for an atomic real operator L. If L has order 1, then the result is clear. Otherwise, we have

$$L = (\partial - (a - b\,\mathrm{i} + b^{\dagger}))\,(\partial - (a + b\,\mathrm{i}))$$

for certain $a, b \in \mathcal{T}$. In particular, we are in the same case (attractive or repulsive) for both factors of L. Setting $\varphi = a + b\,\mathrm{i}$, let $\Phi = \Re + \Im\,\mathrm{i}$ be as in the previous section. Consider $f \in \mathcal{G}_{x_0}^{\preccurlyeq}$ and $g = J_2\,J_1\,f$. In the repulsive case, we have

$$g(x) = b(x)\,\mathrm{e}^{\bar{\Phi}(x)} \int_{x_0}^{x} \frac{\mathrm{e}^{2\mathrm{i}\Im(t)}}{b(t)} \int_{x_0}^{t} \mathrm{e}^{-\Phi(u)}\,f(u)\,\mathrm{d}u\,\mathrm{d}t.$$

In particular, we have $g(x_0) = g'(x_0) = 0$, whence $g \in \mathcal{G}_{x_0}^{\preccurlyeq}$, since g satisfies the differential equation $L g = f$ of order 2 with real coefficients. In the attractive case, we have

$$g(x) = b(x)\,\mathrm{e}^{\bar{\Phi}(x)} \int_{\infty}^{x} \frac{\mathrm{e}^{2\mathrm{i}\Im(t)}}{b(t)} \int_{\infty}^{t} \mathrm{e}^{-\Phi(u)}\,f(u)\,\mathrm{d}u\,\mathrm{d}t,$$

so that $g, g' \preccurlyeq_{\mathfrak{v}} 1$. Since $L g = L\,\bar{g} = f$, the difference $\bar{g} - g$ satisfies $L(\bar{g} - g) = 0$. Now 0 is the only solution with $h \preccurlyeq_{\mathfrak{v}} 1$ to the equation $L h = 0$. This proves that $\bar{g} = g$. □

4.3. The fixed point theorem

Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq\!\!\!\!/\; \mathrm{e}^x$ and consider a monic split-normal quasi-linear equation

$$L f = P(f), \quad f \prec 1, \tag{4.5}$$

where $L \in \mathcal{T}[\mathrm{i}][\partial]$ has order r and $P \in \mathcal{T}[\mathrm{i}]\{F\}$ has degree d. Of course, we understand that L is a monic split-normal operator with $P \prec_{\mathfrak{v}} \mathfrak{v}^{r\sigma_L}$. We will denote by $v_P > r\,\sigma_L$ the valuation of P in $\mathfrak{v}$ (i.e. $P \asymp_{\mathfrak{v}} \mathfrak{v}^{v_P}$ for $P \neq 0$ and $v_0 = \infty$). We will show how to construct a solution to (4.5) using the fixed-point technique.

Proposition 4.5. *Given ν with $r\ \sigma_L < \nu < v_P$, let $J_{r,\ltimes\mathfrak{v}^{\nu}} \cdots J_{1,\ltimes\mathfrak{v}^{\nu}}$ be a continuous factorwise right-inverse of $L_{\ltimes\mathfrak{v}^{\nu}}$ beyond x_0 and consider the operator*

$$\Xi \colon f \longmapsto (J_r \cdots J_1)(P(f)) \tag{4.6}$$

on $\mathcal{G}_{x_0;r}^{\preccurlyeq}$. Then there exists a constant C_{x_0} with

$$\|\Xi(f + \delta) - \Xi(f)\|_{x_0;r} \leqslant C_{x_0}\,(1 + \cdots + \|f\|_{x_0;r}^d)\,(\|\delta\|_{x_0;r} + \cdots + \|\delta\|_{x_0;r}^d), \tag{4.7}$$

for all $f, \delta \in \mathcal{G}_{x_0;r}^{\preccurlyeq}$.

Proof. Consider the Taylor series expansion

$$\begin{aligned} P(f + \delta) &= \sum_{\boldsymbol{i}} P^{(\boldsymbol{i})}(f)\,\delta^{(\boldsymbol{i})} \\ &= \sum_{\boldsymbol{i}} \left[\sum_{\boldsymbol{j}} P_{\boldsymbol{j}}^{(\boldsymbol{i})}\,f^{(\boldsymbol{j})} \right] \delta^{(\boldsymbol{i})} \end{aligned}$$

Since $P_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec_{\mathfrak{v}} \mathfrak{v}^{\nu}$ for all $\boldsymbol{i}$ and $\boldsymbol{j}$, we may define A_{x_0} by

$$A_{x_0} = \sum_{\boldsymbol{i},\boldsymbol{j}} \left\|\mathfrak{v}^{-\nu}\,P_{\boldsymbol{j}}^{(\boldsymbol{i})}\right\|_{x_0} \tag{4.8}$$

and obtain

$$\|\mathfrak{v}^{-\nu}\,(P(f + \delta) - P(f))\|_{x_0} \leqslant A_{x_0}\,(1 + \cdots + \|f\|_{x_0;r}^d)\,(\|\delta\|_{x_0;r} + \cdots + \|\delta\|_{x_0;r}^d).$$

On the other hand, for each $g \in \mathcal{G}_{x_0}$ with $g \preccurlyeq \mathfrak{v}^\nu$, we have

$$\|(J_r \cdots J_1)(g)\|_{x_0;r} = \|(\mathfrak{v}^\nu J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu})(\mathfrak{v}^{-\nu} g)\|_{x_0;r} \leqslant B_{x_0} \|\mathfrak{v}^{-\nu} g\|_{x_0},$$

where

$$B_{x_0} = |||\mathfrak{v}^\nu J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu}|||_{x_0;r} \tag{4.9}$$

Consequently, the proposition holds for $C_{x_0} = A_{x_0} B_{x_0}$. □

THEOREM 4.6. *Let (4.5) be a monic split-normal equation and let ν be such that $r\,\sigma_L < \nu < v_P$. Then for any sufficiently large x_0, there exists a continuous factorwise right-inverse $J_{r,\ltimes \mathfrak{v}^\nu} \cdots J_{1,\ltimes \mathfrak{v}^\nu}$ of $L_{\ltimes \mathfrak{v}^\nu}$, such that the operator (4.6) satisfies*

$$\|\Xi(f+\delta) - \Xi(f)\|_{x_0;r} \leqslant \frac{1}{2} \|\delta\|_{x_0;r} \tag{4.10}$$

for all

$$f, \delta \in \mathcal{B}\Big(\mathcal{G}_{x_0;r}^{\preccurlyeq}, \tfrac{1}{2}\Big) = \Big\{f \in \mathcal{G}_{x_0;r}^{\preccurlyeq} : \|f\|_{x_0;r} \leqslant \tfrac{1}{2}\Big\}.$$

Moreover, taking x_0 such that $\|P_0\|_{x_0;r} \leqslant \frac{1}{4}$, the sequence $\Xi^{(n)}(0)$ tends to a unique fixed point $f \in \mathcal{B}(\mathcal{G}_{x_0;r}^{\preccurlyeq}, \frac{1}{2})$ for the operator Ξ.

Proof. Since $\mathfrak{v}^{-\nu} P_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec 1$ for all i, j, the number A_{x_0} from (4.8) tends to 0 for $x_0 \to \infty$. When constructing $J_{1,\ltimes \mathfrak{v}^\nu}$, ..., $J_{r,\ltimes \mathfrak{v}^\nu}$ using proposition 4.1, the number B_{x_0} from (4.9) decreases as a function of x_0. Taking x_0 sufficiently large so that $C_{x_0} = A_{x_0} B_{x_0} \leqslant \frac{1}{4}$, we obtain (4.10). By induction over n, it follows that

$$\begin{aligned} \|\Xi^n(0) - \Xi^{n-1}(0)\|_{x_0;r} &\leqslant \frac{1}{2^{n+1}} \\ \|\Xi^n(0)\|_{x_0;r} &\leqslant \frac{1}{2} - \frac{1}{2^{n+1}}. \end{aligned}$$

Now let $\hat{\mathcal{G}}_{x_0;r}^{\preccurlyeq}$ be the space of r times continuously differentiable functions f on $[x_0, \infty)$, such that $f, ..., f^{(r)}$ are bounded. This space is complete, whence $\Xi^n(0)$ converges to a limit $f \in \mathcal{B}(\hat{\mathcal{G}}_{x_0;r}^{\preccurlyeq}, \frac{1}{2})$. Since this limit satisfies the equation (4.5), the function f is actually infinitely differentiable, i.e. $f \in \mathcal{B}(\mathcal{G}_{x_0;r}^{\preccurlyeq}, \frac{1}{2})$. □

4.4. Asymptotic analysis

With the notations from the previous section, assume now that $\mathcal{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]_{\preccurlyeq \mathfrak{v}}$, i.e. any solution $f \in \mathbb{T}[\mathrm{i}]_{\preccurlyeq \mathfrak{v}}$ to an equation $(\partial - \varphi) f = g$ with $\varphi, g \in \mathcal{T}[\mathrm{i}]$ is already in $\mathcal{T}[\mathrm{i}]$. Each J_i is the right-inverse of an operator $\partial - \varphi_i$ with $\varphi_i \in \mathcal{T}[\mathrm{i}]$. Now $\partial - \varphi_i$ also admits a formal distinguished right-inverse $\tilde{J}_i$. Consequently, the operator Ξ also admits a formal counterpart

$$\tilde{\Xi}: f \longmapsto (\tilde{J}_r \cdots \tilde{J}_1)(P(f)).$$

For each $n \in \mathbb{N}$, we have

$$\tilde{\Xi}^{n+1}(0) - \tilde{\Xi}^n(0) \prec_{\mathfrak{v}} \tilde{\Xi}^n(0)$$

so the sequence $\tilde{\Xi}^n(0)$ also admits a formal limit $\tilde{f}$ in $\hat{\mathcal{T}}[\mathrm{i}]$. In order to show that the fixed point f from proposition 4.6 and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$, we need some further notations. Given $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ and $\tilde{f} \in \mathcal{T}[\mathrm{i}]$, let us write $f \approx \tilde{f}$ if $f - \tilde{f} \prec \mathfrak{v}^{\mathbb{R}}$, i.e. $f - \tilde{f} \prec \mathfrak{v}^\alpha$ for all $\alpha \in \mathbb{R}$. We also write $f \approx_r \tilde{f}$ if $f \approx \tilde{f}, ..., f^{(r)} \approx \tilde{f}^{(r)}$.

PROPOSITION 4.7. *For $f, g \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$, $\tilde{f}, \tilde{g} \in \mathcal{T}[\mathrm{i}]$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned} f \approx_r \tilde{f} \wedge g \approx_r \tilde{g} &\Rightarrow f + g \approx_r \tilde{f} + \tilde{g} \\ f \approx_r \tilde{f} \wedge g \approx_r \tilde{g} &\Rightarrow f g \approx_r \tilde{f} \tilde{g} \\ f \approx_{r+1} \tilde{f} &\Rightarrow f' \approx_r \tilde{f}' \end{aligned}$$

Proof. Trivial. □

PROPOSITION 4.8. *For $f \in \mathcal{G}^{\preccurlyeq}[\mathrm{i}]$, $\tilde{f} \in \mathcal{T}[\mathrm{i}]$ and $r \in \mathbb{N}$ with $f, \tilde{f} \prec_{\mathfrak{v}} \mathfrak{v}^{\nu}$, we have*

$$f \approx_r \tilde{f} \Rightarrow J_i f \approx_{r+1} \tilde{J}_i \tilde{f}.$$

Proof. Let us first show that

$$f \approx 0 \Rightarrow J_i f \approx_1 0. \tag{4.11}$$

Given $\alpha \geqslant \nu$ with $f \preccurlyeq \mathfrak{v}^{\alpha}$, we have $J_{i, \ltimes \mathfrak{v}^{\alpha}} (\mathfrak{v}^{-\alpha} f) \preccurlyeq 1$, whence $J_i f \preccurlyeq \mathfrak{v}^{\alpha}$. Moreover,

$$(J_i f)' = \psi_i^{-1} f + \varphi (J_i f), \tag{4.12}$$

whence $f \preccurlyeq \mathfrak{v}^{\alpha} \Rightarrow (J_i f)' \preccurlyeq \mathfrak{v}^{\alpha + \beta}$ for some fixed β. This proves (4.11). More generally, r additional applications of (4.12) yield

$$f \approx_r 0 \Rightarrow J_i f \approx_{r+1} 0.$$

Now assume that $f \approx_r \tilde{f}$ and write

$$J_i f - \tilde{J}_i \tilde{f} = J_i (f - \tilde{f}) + (J_i - \tilde{J}_i) (\tilde{f}).$$

By what precedes, we have $J_i (f - \tilde{f}) \approx_{r+1} 0$. On the other hand,

$$(J_i - \tilde{J}_i)(\tilde{f}) = c \, \mathrm{e}^{\int \varphi_i}$$

for some $c \in \mathbb{C}$. Since $\partial - \varphi_i$ is normal, we either have $\mathrm{e}^{\int \varphi_i} \prec \mathfrak{v}^{\mathbb{R}}$ (in which case $(\mathrm{e}^{\int \varphi_i})^{(i)} \prec \mathfrak{v}^{\mathbb{R}}$ for all $i \in \mathbb{N}$) or $c = 0$. In both cases, we get $(J_i - \tilde{J}_i)(\tilde{f}) \approx_{r+1} 0$, so that $J_i f \approx_{r+1} \tilde{J}_i \tilde{f}$. □

THEOREM 4.9. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq \mathrm{e}^x$ such that $\mathcal{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}_{\preccurlyeq\!\!\!\prec \mathfrak{v}}$. Consider a monic split-normal quasi-linear equation (4.5) without solutions in $\mathcal{T}$. Then there exist solutions $f \in \mathcal{G}[\mathrm{i}]$ and $\tilde{f} \in \hat{\mathcal{T}}[\mathrm{i}]$ to (4.5), such that f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$.*

Proof. With the above notations, let f and $\tilde{f}$ be the limits in $\mathcal{G}[\mathrm{i}]$ resp. $\hat{\mathcal{T}}[\mathrm{i}]$ of the sequences $\Xi^n(0)$ resp. $\tilde{\Xi}^n(0)$. Given $g \in \mathcal{T}[\mathrm{i}]$, there exists an n with

$$\Xi^{n+1}(0) - \Xi^n(0) \prec_{\mathfrak{v}} g.$$

At that point, we have

$$f - g \sim \Xi^n(0) - g \approx \tilde{\Xi}^n(0) - g \sim \tilde{f} - g$$

In other words, f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}[\mathrm{i}]$. □

THEOREM 4.10. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \succcurlyeq \mathrm{e}^x$. Consider a monic split-normal quasi-linear equation (4.5) without solutions in $\mathcal{T}$ such that L and P have coefficients in $\mathcal{T}$. Assume that one of the following conditions holds:*

a) *$\mathcal{T}$ is $(1,1,1)$-differentially closed in $\mathbb{T}_{\preccurlyeq\!\!\!\prec \mathfrak{v}}$ and $r_L = r_P = 1$.*

b) *$\mathcal{T}[\mathrm{i}]$ is $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]_{\preccurlyeq\!\!\!\prec \mathfrak{v}}$.*

Then there exist solutions $f \in \mathcal{G}$ and $\tilde{f} \in \hat{\mathcal{T}}$ to (4.5), such that f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$.

Proof. In view of propositions 2.5 and 4.4, we may assume that $J_r \cdots J_1$ and Ξ preserve realness in all results from sections 4.3 and 4.4. In particular, the solutions f and $\tilde{f}$ in the conclusion of theorem 4.9 are both real. □

5. DIFFERENTIALLY ALGEBRAIC HARDY FIELDS

5.1. First order extensions

LEMMA 5.1. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v} \not\succcurlyeq \mathrm{e}^x$. Let $L = \partial - \varphi \in \mathcal{T}[\partial]$ be a normal operator. Let $\tilde{f} \in \hat{\mathcal{T}}^{\preccurlyeq}$ and $g \in \mathcal{T}^{\preccurlyeq}$ be such that $\tilde{f}$ is transcendental over $\mathcal{T}$ and $L\tilde{f} = g$. Then there exists an $f \in \mathcal{G}^{\preccurlyeq}$ with $Lf = g$, such that f and $\tilde{f}$ are both differentially and asymptotically equivalent over $\mathcal{T}$.*

Proof. With the notations of section 4.1, let $f = Jg$. Given a truncation $\psi \lhd \tilde{f}$, we claim that

$$f - \psi \approx J(g - (\psi' - \varphi\,\psi)).$$

Indeed, consider

$$\delta = \psi - J(\psi' - \varphi\,\psi) \in \mathbb{R}\,\mathrm{e}^{\Phi}.$$

In the attractive case, $\psi \prec_{\mathfrak{v}} \mathrm{e}^{\Phi}$ implies $\delta = 0$. In the repulsive case, we have $\mathrm{e}^{\Phi} \prec_{\mathfrak{v}}^{*} 1$ and again $\delta \approx 0$. By proposition 4.8, we also have

$$\tilde{f} - \psi = \tilde{J}(g - \psi' + \varphi\,\psi) \approx J(g - \psi' + \varphi\,\psi).$$

Since $\psi' - \varphi\,\psi \neq g$, it follows that $\tilde{f} - \psi \sim f - \psi$, whence f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$. Furthermore, $LF - g$ is a minimal annihilator of $\tilde{f}$ over $\mathcal{T}$, since $\tilde{f}$ is transcendental over $\mathcal{T}$. Lemma 3.9 therefore implies that f and $\tilde{f}$ are differentially equivalent over $\mathcal{T}$. □

THEOREM 5.2. *Let $\mathcal{T}$ be a transserial Hardy field. Let $\mathcal{T}^{\mathrm{fo}} \supseteq \mathcal{T}$ be the smallest differential subfield of $\mathbb{T}$, such that for any $P \in \mathcal{T}^{\mathrm{fo}}\{F\}^{\neq}$ with $r_P \leqslant 1$ and $f \in \mathbb{T}$ we have $P(f) = 0 \Rightarrow f \in \mathcal{T}^{\mathrm{fo}}$. Then the transserial Hardy field structure of $\mathcal{T}$ can be extended to $\mathcal{T}^{\mathrm{fo}}$.*

Proof. By theorems 3.11, 3.12 and 3.13, we may assume that $\mathcal{T}$ is closed under the resolution of real algebraic equations, exponentiation and logarithm. Assume that $\mathcal{T}^{\mathrm{fo}} \neq \mathcal{T}$ and let $P \in \mathcal{T}\{F\}^{\neq}$ be of minimal complexity $\chi_P = (1, s, t)$, such that $P(f) = 0$ for some $f \in \mathcal{T}^{\mathrm{fo}}$. Without loss of generality, we may make the following assumptions:

- f and P are exponential (modulo upward shifting).
- f is a serial cut (by lemma 3.7).
- f is a normal cut (modulo additive and multiplicative conjugations by H_f resp. $\mathfrak{d}_f$).
- $P \in \mathcal{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}\{F\}$, where $\mathfrak{v} \in \mathcal{T} \cap \mathfrak{T}$ satisfies uspan $f \asymp \mathfrak{v}$ (modulo replacing P by $P_{\preccurlyeq\mathfrak{v}}$).
- P is monic split-normal (modulo proposition 2.16, additive and multiplicative conjugations, and division by $\mathfrak{d}_P$).

By Zorn's lemma, it suffices to show that $\mathcal{T}\langle f\rangle$ carries the structure of a transserial Hardy field, which extends the structure of $\mathcal{T}$.

If $s=t=1$, then lemma 5.1 implies the existence of an $\hat{f}\in\mathcal{G}^{\preccurlyeq}$ such that f and $\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$. Hence, the result follows from lemmas 3.8 and 3.10.

If $t>1$, then $\mathcal{T}$ and $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$ are $(1,1,1)$-differentially closed in $\mathbb{T}$ resp. $\mathbb{T}_{\preccurlyeq\mathfrak{v}}$. Now $\mathfrak{v}\succcurlyeq\mathrm{e}^x$, since f is exponential. Therefore, theorem 4.10 provides us with an $\hat{f}\in\mathcal{G}^{\preccurlyeq}$ with $P(\hat{f})=0$, such that f and $\hat{f}$ are asymptotically equivalent over $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$. We conclude by lemmas 3.9, 3.8 and 3.10. □

5.2. Higher order extensions

LEMMA 5.3. *Let $\mathcal{T}$ be a transserial Hardy field of span $\mathfrak{v}\succcurlyeq\mathrm{e}^x$. Let $L=\partial-\varphi\in\mathcal{T}[\mathrm{i}][\partial]$ be a normal operator. Let $\tilde{f}\in\hat{\mathcal{T}}[\mathrm{i}]^{\preccurlyeq}$ and $g\in\mathcal{T}[\mathrm{i}]^{\preccurlyeq}$ be such that $\operatorname{Re}\tilde{f}$ has order 2 over $\mathcal{T}$ and $L\,\tilde{f}=g$. Then there exists an $f\in\mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ with $L f=g$, such that $\operatorname{Re}f$ and $\operatorname{Re}\tilde{f}$ are both differentially and asymptotically equivalent over $\mathcal{T}$.*

Proof. The fact that f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$ is proved in a similar way as for lemma 5.1. It follows in particular that $\operatorname{Re}f$ and $\operatorname{Re}\tilde{f}$ are asymptotically equivalent. Since $\operatorname{lcm}(L,\bar{L})$ annihilates f, $\bar{f}$, $\tilde{f}$ and $\bar{\tilde{f}}$, it also annihilates both $\operatorname{Re}f$ and $\operatorname{Re}\tilde{f}$. The fact that $\operatorname{Re}\tilde{f}$ has complexity $(2,1,1)$ over $\mathcal{T}$ now guarantees that $\operatorname{lcm}(L,\bar{L})$ is a minimal annihilator of $\operatorname{Re}\tilde{f}$. We conclude by lemma 3.9. □

THEOREM 5.4. *Let $\mathcal{T}$ be a transserial Hardy field. Let $\mathcal{T}^{\mathrm{dalg}}\supseteq\mathcal{T}$ be the smallest differential subfield of $\mathbb{T}$, such that for any $P\in\mathcal{T}^{\mathrm{dalg}}\{F\}^{\neq}$ and $f\in\mathbb{T}$ we have $P(f)=0\Rightarrow f\in\mathcal{T}^{\mathrm{dalg}}$. Then the transserial Hardy field structure of $\mathcal{T}$ can be extended to $\mathcal{T}^{\mathrm{dalg}}$.*

Proof. By theorems 3.12, 3.13 and 5.2, we may assume that $\mathcal{T}$ is closed under exponentiation, logarithm and the resolution of first order differential equations. Assume that $\mathcal{T}^{\mathrm{dalg}}\neq\mathcal{T}$ and let $P\in\mathcal{T}[\mathrm{i}]\{F\}^{\neq}$ be of minimal complexity $\chi_P=(r,s,t)$, such that $P(f)=0$ for some $f\in\mathcal{T}^{\mathrm{dalg}}[\mathrm{i}]$ with $\operatorname{Re}f\notin\mathcal{T}$. Let $Q\in\mathcal{T}\{F\}$ be a minimal annihilator of $\operatorname{Re}f$ and notice that $r_Q\geqslant r_P$, since $\operatorname{Re}f\notin\mathcal{T}$. Without loss of generality, we may make the following assumptions:

- f, P and Q are exponential (modulo upward shifting).
- f is a serial cut (by the complexified version of lemma 3.7).
- f is a normal cut (modulo additive and multiplicative conjugations by H_f resp. $\mathfrak{d}_f$).
- $P\in\mathcal{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}\{F\}$ and $Q\in\mathcal{T}_{\preccurlyeq\mathfrak{v}}\{F\}$, where $\mathfrak{v}\in\mathcal{T}\cap\mathfrak{T}$ satisfies $\operatorname{uspan}f\asymp\mathfrak{v}$ (modulo the replacement of P and Q by $P_{\preccurlyeq\mathfrak{v}}$ resp. $Q_{\preccurlyeq\mathfrak{v}}$).
- Q is monic split-normal (modulo proposition 2.16, additive and multiplicative conjugations, and division by $\mathfrak{d}_Q$).

By Zorn's lemma, it now suffices to show that $\mathcal{T}\langle\operatorname{Re}f\rangle$ carries the structure of a transserial Hardy field, which extends the structure of $\mathcal{T}$.

If $r=s=t=1$, then lemma 5.3 and the fact that $\mathcal{T}$ is 1-differentially closed imply the existence of an $\hat{f}\in\mathcal{G}^{\preccurlyeq}[\mathrm{i}]$ such that $\operatorname{Re}f$ and $\operatorname{Re}\hat{f}$ are both asymptotically and differentially equivalent over $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$. The result follows by lemmas 3.8 and 3.10.

If $\chi_P\neq(1,1,1)$, then $\mathcal{T}[\mathrm{i}]$ and $\mathcal{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}$ are $(1,1,1)$-differentially closed in $\mathbb{T}[\mathrm{i}]$ resp. $\mathbb{T}[\mathrm{i}]_{\preccurlyeq\mathfrak{v}}$. Now $\mathfrak{v}\succcurlyeq\mathrm{e}^x$, since f is exponential. Therefore, theorem 4.10 provides us with a $g\in\mathcal{G}^{\preccurlyeq}$ with $Q(g)=0$, such that $\operatorname{Re}f$ and g are asymptotically equivalent over $\mathcal{T}_{\preccurlyeq\mathfrak{v}}$. We conclude by lemmas 3.9, 3.8 and 3.10. □

COROLLARY 5.5. *There exists a transserial Hardy field $\mathcal{T}$, such that for any $P\in\mathcal{T}\{F\}$ and $f,g\in\mathcal{T}$ with $f<g$ and $P(f)\,P(g)<0$, there exists a $h\in\mathcal{T}$ with $f<h<g$ and $P(h)=0$.*

Proof. Take $\mathcal{T} = \mathbb{R}(x^{\mathbb{R}})^{\mathrm{dalg}}$ and endow it with a transserial Hardy field structure. Let $P \in \mathcal{T}\{F\}$ and $f, g \in \mathcal{T}$ with $f < g$ be such that $P(f)\, P(g) < 0$. By [Hoe06, Theorem 9.33], there exists a $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$. But $P(h) = 0$ implies $h \in \mathcal{T}$. □

Corollary 5.6. *There exists a transserial Hardy field $\mathcal{T}$, such that $\mathcal{T}[\mathrm{i}]$ is weakly differentially closed.*

Proof. Take $\mathcal{T} = \mathbb{R}^{\mathrm{dalg}}$. By a straightforward adaptation of [Hoe06, Chapter 8] (see also [Hoe01, theorem 9.3]), it can be shown that any differential equation $P(f) = 0$ of degree d with $P \in \mathcal{T}[\mathrm{i}]\{F\}$ admits d distinguished solutions in $\mathbb{T}[\mathrm{i}]$ when counting with multiplicities. Let f be such a solution. Since $P(f) = \bar{P}(\bar{f}) = 0$, both $\operatorname{Re} f$ and $\operatorname{Im} f$ are differentially algebraic over $\mathcal{T}$, whence $f \in \mathcal{T}[\mathrm{i}]$. □

Corollary 5.7. *There exists a differentially Henselian transserial Hardy field $\mathcal{T}$, i.e., such that any quasi-linear differential equation over $\mathcal{T}$ admits a solution in $\mathcal{T}$.*

5.3. Differential Newton polynomials for Hardy fields

Let $\mathcal{H}$ be a differentially algebraic Hardy field extension of a transserial Hardy field $\mathcal{T}$.

Proposition 5.8. *Given $\varepsilon \in \mathcal{H}^{\prec}$, there exists an $l \in \mathbb{N}$ with $\varepsilon \prec (\log_l x)^{-1}$.*

Proof. The functional inverse $|\varepsilon^{-1}|^{\mathrm{inv}}$ of $|\varepsilon^{-1}|$ satisfies an algebraic differential equation $P(|\varepsilon^{-1}|^{\mathrm{inv}}) = 0$ over $\mathcal{T}$. Let $P_{\langle \boldsymbol{i} \rangle}\, f^{\langle \boldsymbol{i} \rangle}$ be the leading term of P for its logarithmic decomposition. As in [Hoe06, Section 8.1.4]. there exists an $l \in \mathbb{N}$ with $P(f) \sim P_{\langle \boldsymbol{i} \rangle}\, f^{\langle \boldsymbol{i} \rangle}$ for all $f \succcurlyeq \exp_l x$. It follows that $|\varepsilon^{-1}|^{\mathrm{inv}} \prec \exp_l x$ and $\varepsilon \prec (\log_l x)^{-1}$. □

Given a differential polynomial $P \in \mathcal{H}\{F\}^{\neq}$, we define its *dominant part* to be the unique monic $D_P \in \mathbb{R}\{F\}$ such that $P = \ell_P\, (D_P + E_P)$ for some $\ell_P \in \mathcal{H}$ and $E_P \in \mathcal{H}\{F\}^{\prec}$. Here D_P is said to be monic if its leading coefficient w.r.t. $F^{(r_P)}, \ldots, F$ equals 1.

Theorem 5.9. *Given $P \in \mathcal{H}\{F\}^{\neq}$, there exists a polynomial $N_P \in \mathbb{R}[F]\,(F')^{\mathbb{N}}$ with*

$$\begin{aligned} D_{P\uparrow_l} &= N_P \\ E_{P\uparrow_l} &= o_{\mathrm{e}^x}(1) \end{aligned}$$

for all sufficiently large $l \in \mathbb{N}$.

Proof. As in the proof of [Hoe06, Theorem 8.6], we have

$$\operatorname{wt} D_P \geqslant \operatorname{wv} D_P \geqslant \operatorname{wt} D_{P\uparrow} \geqslant \operatorname{wv} D_{P\uparrow} \geqslant \cdots,$$

so we may assume without loss of generality that $\operatorname{wt} D_{P\uparrow_i} = \operatorname{wv} D_{P\uparrow_i} = w$ is constant for all $i \in \mathbb{N}$. Now

$$\begin{aligned} P\!\uparrow &= \ell_{P\uparrow}\, (D_{P\uparrow} + E_{P\uparrow}) \\ &= \ell_P\!\uparrow (D_P\!\uparrow + E_P\!\uparrow) \\ &= \ell_P\!\uparrow (\mathrm{e}^{-wx}\, D_{P\uparrow} + E_P\!\uparrow), \end{aligned}$$

whence

$$\ell_{P\uparrow} = \ell_P\!\uparrow \mathrm{e}^{-wx} \tag{5.1}$$

$$D_{P\uparrow} = D_{D_P\uparrow} \tag{5.2}$$

$$E_{P\uparrow} = E_P\!\uparrow \mathrm{e}^{wx}. \tag{5.3}$$

Indeed, we must have

$$E_{P}{\uparrow}\,\mathrm{e}^{wx}=(E_{P_{[<w]}}{\uparrow}+E_{P_{[\geqslant w]}}{\uparrow})\,\mathrm{e}^{wx}\prec 1,$$

because $E_{P_{[<w]}}{\uparrow}\,\mathrm{e}^{wx}\succcurlyeq 1$ would imply $\operatorname{wt} D_{P\uparrow}<w$. Applying [Hoe06, Lemma 8.5] to (5.2), and similarly for $P{\uparrow}, P{\uparrow}{\uparrow}, \ldots$, we get

$$D_{P\uparrow_l}=D_P\in\mathbb{R}[F]\,(F')^w$$

for all $l\in\mathbb{N}$.

By proposition 5.8 and (5.3), we have $E_{P,[\geqslant v]}\prec_{\log_l x}1$ and $E_{P\uparrow_{l+1},[\geqslant v]}\prec_{\mathrm{e}^x}1$ for some $l\in\mathbb{N}$. Modulo upward shiftings, we may thus assume without loss of generality that $E_{P,[\geqslant v]}\prec_{\mathrm{e}^x}1$. More generally, assume that $E_{P,[>v]}\prec_{\mathrm{e}^x}1$ for some $v<w$. By (5.3), this implies $E_{P\uparrow_l,[>v]}\prec_{\mathrm{e}^x}1$ for all $l\in\mathbb{N}$ and

$$\begin{aligned}E_{P\uparrow,[\boldsymbol{\omega}]} &= (E_{P,[v]}{\uparrow}_{[\boldsymbol{\omega}]}+E_{P,[>v]}{\uparrow}_{[\boldsymbol{\omega}]})\,\mathrm{e}^{wx}\\ &= \mathrm{e}^{(w-v)x}\,(E_{P,[\boldsymbol{\omega}]}{\uparrow}+o_{\mathrm{e}^x}(1)),\end{aligned}\tag{5.4}$$

for all $\boldsymbol{\omega}$ of weight v. We claim that there exists an $l\in\mathbb{N}$ with

$$E_{P,[v]}\prec[(\log_l^{-1}x)']^{w-v}.\tag{5.5}$$

Assume the contrary and consider a coefficient $E_{P,[\boldsymbol{\omega}]}$ of weight v with

$$\psi=\sqrt[w-v]{E_{P,[\boldsymbol{\omega}]}}\succcurlyeq(\log_l^{-1}x)'$$

for all $l\in\mathbb{N}$. Without loss of generality, we may assume that ψ and $\int\psi$ are in $\mathcal{H}$. Then proposition 5.8 implies $\int\psi\succcurlyeq 1$ and even $\int\psi\succ 1$ (by integrating from $+\infty$ when possible). Again by proposition 5.8, it follows that $\int\psi\succ\log_l x$ and $\psi\succ(\log_l x)'$ for some $l\in\mathbb{N}$. But then (5.4) yields

$$E_{P\uparrow_l,[\boldsymbol{\omega}]}=[(\log_l x)']^{v-w}{\uparrow}_l\,(E_{P,[\boldsymbol{\omega}]}{\uparrow}_l+o_{\mathrm{e}^x}(1))\succ 1,$$

which contradicts the fact that $E_{P\uparrow_l}\prec 1$. The relations (5.5) and (5.4) imply the existence of an $l\in\mathbb{N}$ with $E_{P\uparrow_{l+1},[v]}\prec_{\mathrm{e}^x}1$. By induction on $v=w,w-1,\ldots,0$ and modulo upward shiftings, we may thus ensure that $E_{P,[\geqslant v]}\prec_{\mathrm{e}^x}1$ for all $v\leqslant w$. □

The polynomial N_P in theorem 5.9 is called the *differential Newton polynomial* of P. The generalization of this concept to $\mathcal{H}$ allows us to mimic a lot of the theory from [Hoe06, chapter 8] in $\mathcal{H}$. In what follows, we will mainly need a few more definitions. The *Newton degree* of an equation

$$P(f)=0,\qquad f\prec\varphi\tag{5.6}$$

with $P\in\mathcal{H}\{F\}$ and $\varphi\in\mathcal{H}^{\neq}$ is defined by $\deg_{\prec\varphi}P=\deg N_{P_{\times\varphi}}$. Setting

$$\hat{\gamma}=\frac{1}{x\log x\log_2 x\cdots}$$

we also define

$$\deg_{\prec\hat{\gamma}}P=\min_{\varphi\succ\hat{\gamma}}\deg_{\prec\varphi}P.$$

We say that $f\prec\varphi$ is a solution to (5.6) modulo $o(\psi)$, $\psi\in\mathcal{T}\cup\{\hat{\gamma}\}$ if $\deg_{\prec\psi}P_{+f}>0$. We say that $\mathcal{H}$ is *differentially Henselian*, if every quasi-linear equation over $\mathcal{H}$ admits a solution. Given a solution f to (5.6), we say that f has *algebraic type* if $N_{P_{\times f}}$ is not homogeneous and *differential type* in the other case. The following proposition is proved along the same lines as [Hoe06, proposition 8.16]:

PROPOSITION 5.10. *Let f be a solution to (5.6) of differential type and let i be the degree of $N_{P_{\times f}}$. Then $f^{\dagger}$ is a solution modulo $o(\hat{\gamma})$ of R_{P_i}.*

Remark 5.11. In this section, we assumed that $\mathcal{H}$ is a differentially algebraic Hardy field extension of a transserial Hardy field $\mathcal{T}$. We expect that the theory can be adapted to even more general H-field. This is one of the objectives of a current collaboration with Lou van den Dries and Matthias Aschenbrenner [ADH].

5.4. Transserial models of differentially algebraic Hardy fields

THEOREM 5.12. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian and stable under exponentiation. Then there exists a transserial Hardy field structure on $\mathcal{H}$ which extends the structure on $\mathcal{T}$.*

Proof. By theorems 3.11, 3.12 and 5.2, we may assume that $\mathcal{T}$ is closed under the resolution of real algebraic equations, exponentiation and integration. Assume that $\mathcal{H} \neq \mathcal{T}$ and choose $P \in \mathcal{T}\{F\}$ of minimal complexity $\chi_P = (r, s, t)$, such that either

C1. $P(f) = 0$ for some $f \in \mathcal{H}$.

C2. $P(f) = 0$ modulo $o(\mathfrak{m}\,\hat{\gamma})$ for some $f \in \mathcal{H}$, $\mathfrak{m} \in \mathcal{T} \cap \mathfrak{T}$ and P admits no roots in $\mathcal{T}$ modulo $o(\mathfrak{m}\,\hat{\gamma})$. Moreover, $\mathcal{T}$ is χ_P-differentially closed in $\mathcal{H}$.

Modulo upward shifting, we may assume without loss of generality that P is exponential. In view of Zorn's lemma, it suffices to show that there exists a transserial Hardy field structure on $\mathcal{T}\langle f\rangle$ which extends the structure on $\mathcal{T}$.

Let Φ be the set of $\tilde{f} \in \mathcal{T}$ such that $f - \tilde{f} \prec \operatorname{supp} \tilde{f}$. The set Φ is totally ordered for $\trianglelefteq$, so there exists a minimal well-based transseries $\tilde{f}$ with $\varphi \trianglelefteq \tilde{f}$ for all $\varphi \in \Phi$. We call $\tilde{f}$ the *initializer* of f over $\mathcal{T}$. Assume first that $\tilde{f} \in \mathcal{T}$. Then we may assume without loss of generality that $\varphi = 0$, modulo an additive conjugation by φ. Now f is of differential type, since $f \asymp \mathfrak{m}$ for no $\mathfrak{m} \in \mathcal{T} \cap \mathfrak{T}$. Let $i \in \mathbb{N}$ be such that $R_{P_i}(f^{\dagger}) = 0$ modulo $o(\hat{\gamma})$. Since R_{P_i} has lower complexity than P, there exists a $g \in \mathcal{T}$ with $R_{P_i}(g) = 0$ modulo $o(\hat{\gamma})$. Since $\mathcal{T}$ is truncation closed we may take $g \in \mathcal{T}_{\succ \hat{\gamma}}$. But then $f \asymp \mathrm{e}^{\int g} \in \mathcal{T} \cap \mathfrak{T}$. This contradiction proves that we cannot have $\tilde{f} \in \mathcal{T}$.

Let us now consider the case when $\tilde{f} \notin \mathcal{T}$. Since $\deg_{\prec \operatorname{supp} \tilde{f}} P_{+\tilde{f}} > 0$, there exists a root $\varphi \trianglerighteq \tilde{f}$ of P in the set of well-based transseries with complex coefficients. But P admits only grid-based solutions, whence $\tilde{f} \in \mathbb{T}$. By construction, f and $\tilde{f}$ are asymptotically equivalent over $\mathcal{T}$. Let $\mathfrak{v} \in \mathcal{T} \cap \mathfrak{T}$ be such that $\operatorname{uspan} \tilde{f} \asymp\!\!\asymp \mathfrak{v}$. Modulo an additive and a multiplicative conjugation we may assume without loss of generality that $\tilde{f}$ is a normal cut. In case **C2**, we notice that $\operatorname{supp} \tilde{f} \succ \mathfrak{m}\,\hat{\gamma}$, whence $\mathfrak{m} \prec_{\mathfrak{v}}^{*} 1$, since $\operatorname{uspan} \tilde{f} = \mathfrak{v}$. Consequently, we always have $P_{\underline{\preccurlyeq\!\!\preccurlyeq} \mathfrak{v}}(\tilde{f}) = 0$.

We claim that the cuts f and $\tilde{f}$ are differentially equivalent over $\mathcal{T}$. Assume the contrary and let $Q \in \mathcal{T}_{\underline{\preccurlyeq\!\!\preccurlyeq} \mathfrak{v}}\{F\}$ be a minimal annihilator of $\tilde{f}$. By lemma 2.15 and modulo an additive and multiplicative conjugation, we may assume without loss of generality that $\tilde{f} \prec_{\mathfrak{v}} 1$ and that Q is normal. Since $\mathcal{H}$ is differentially Henselian, it follows that Q admits a root $g \prec_{\mathfrak{v}} 1$ in $\mathcal{H}$. Now $\chi_Q < \chi_P$ in case **C1** and $\chi_Q \leqslant \chi_P$ in case **C2**, so this root is already in $\mathcal{T}$, by the induction hypothesis. But Q admits at most one solution in $\mathbb{T}_{\underline{\preccurlyeq\!\!\preccurlyeq} \mathfrak{v}}$, whence $\tilde{f} = g_{\underline{\preccurlyeq\!\!\preccurlyeq} \mathfrak{v}} \in \mathcal{T}$. This contradiction completes the proof of our claim. By lemma 3.10, we conclude that $\mathcal{T}\langle f\rangle$ carries the structure of a transserial Hardy field extension of $\mathcal{T}$. $\square$

COROLLARY 5.13. *Let $\mathcal{T}$ be a transserial Hardy field and $\mathcal{H}$ a differentially algebraic Hardy field extension of $\mathcal{T}$, such that $\mathcal{H}$ is differentially Henselian. Assume that $\mathcal{H}$ admits no non-trivial algebraically differential Hardy field extensions. Then $\mathcal{H}$ satisfies the differential intermediate value property.*

Proof. The fact that $\mathcal{H}$ admits no non-trivial algebraically differential Hardy field extensions implies that $\mathcal{H}$ is stable under exponentiation. By theorem 5.12, we may give $\mathcal{H}$ the structure of a transserial Hardy field. By theorem 5.4, we also have $\mathcal{T}^{\mathrm{dalg}} = \mathcal{T}$. We conclude in a similar way as in the proof of corollary 5.5. □

It is quite possible that there exist maximal Hardy fields whose differentially algebraic parts are not differentially Henselian, although we have not searched hard for such examples yet. The differentially algebraic part of the intersection of all maximal Hardy fields is definitely not differentially Henselian (and therefore does not satisfy the differential intermediate value property), due to the following result [Bos87, Proposition 3.7]:

THEOREM 5.14. *Any solution of the equation*

$$f'' + f = \mathrm{e}^{x^2}$$

is contained in a Hardy field. However, none of these solutions is contained in the intersection of all maximal Hardy fields.

Glossary

$f \preccurlyeq g$	f is dominated by g	4
$f \prec g$	f is negligible w.r.t. g	4
$f \asymp g$	f is asymptotic to g	4
$f \sim g$	f is asymptotically similar to g	4
$f \mathrel{\preceq\!\!\!\preceq} g$	f is flatter than or as flat as g	4
$f \mathrel{\prec\!\!\!\prec} g$	f is flatter than g	4
$f \mathrel{\asymp\!\!\!\asymp} g$	f is as flat as g	4
$f \approxeq g$	f and g are similar modulo flatness	4
$f \preccurlyeq_{\mathfrak{v}} g$	$f \preccurlyeq g$ modulo elements flatter than $\mathfrak{v}$	4
$f \prec_{\mathfrak{v}} g$	$f \prec g$ modulo elements flatter than $\mathfrak{v}$	4
$f \preccurlyeq_{\mathfrak{v}}^{*} g$	$f \preccurlyeq g$ modulo elements flatter than or as flat as $\mathfrak{v}$	4
$f \prec_{\mathfrak{v}}^{*} g$	$f \prec g$ modulo elements flatter than or as flat as $\mathfrak{v}$	4
$\mathbb{T}^{>}$	shorthand for $\{f \in \mathbb{T}: f > 0\}$	4
$\mathbb{T}^{\neq}$	shorthand for $\{f \in \mathbb{T}: f \neq 0\}$	4
$\mathbb{T}^{\succ}$	shorthand for $\{f \in \mathbb{T}: f \succ 1\}$	4
$f_{\succ}$	infinite part of f	4
$f_{\prec\!\!\!\prec \mathfrak{v}}$	part of f which is flatter than $\mathfrak{v}$	4
$\mathbb{T}_{\succ}$	shorthand for $\{f_{\succ}: f \in \mathbb{T}\}$	4
$\mathbb{T}_{\prec\!\!\!\prec \mathfrak{v}}$	shorthand for $\{f_{\prec\!\!\!\prec \mathfrak{v}}: f \in \mathbb{T}\}$	4
∂	derivation with respect to x	4
$\int$	integration with respect to x	4
$f^{\dagger}$	logarithmic derivative of f	4
$\uparrow$	upward shifting	4
$\downarrow$	downward shifting	4
$f \trianglelefteq g$	f is a truncation of g	4
$\operatorname{span} f$	canonical span of f	4
$\operatorname{uspan} f$	ultimate canonical span of f	4
$\hat{\mathcal{T}}$	completion of $\mathcal{T}$ with serial cuts	5
$\mathcal{T}\{F\}$	ring of differential polynomials in F over $\mathcal{T}$	5
$\mathcal{T}\langle F\rangle$	quotient field of $\mathcal{T}\{F\}$	5
L_P	linear part of P as an operator	5
r_P	order of P	5
s_P	degree of P in its leader	5
t_P	total degree of P	5
χ_P	complexity of P	5
I_P	initial of P	5
S_P	separant of P	5
H_P	the product $I_P S_P$	5
χ_f	complexity of f over $\mathcal{T}$	5
r_f	order of f over $\mathcal{T}$	5
$P_{+\varphi}$	additive conjugation of P by φ	6
$P_{\times\varphi}$	multiplicative conjugation of P by φ	6
$L_{\times\psi}$	multiplicative conjugate of L by ψ	7
$L_{\ltimes\psi}$	twist of L by ψ	7
$\mathfrak{H}_L$	set of dominant monomials of solutions to $Lh = 0$	9
$\mathcal{G}$	ring of infinitely differentiable germs at infinity	12
$f \sim \hat{f}$	f is asymptotically similar to $\hat{f}$ over $\mathcal{T}$	13
$\mathcal{T}^{\mathrm{rcl}}$	real closure of $\mathcal{T}$	15
$\deg_{\preccurlyeq\psi} P$	Newton degree of P modulo $O(\psi)$	16
$\Vert f \Vert_{x_0}$	norm of f for $x \geqslant x_0$	18
$\mathcal{G}_{x_0;r}^{\preccurlyeq}$	shorthand for $\{f \in \mathcal{G}_{x_0}: f, \ldots, f^{(r)} \preccurlyeq 1\}$	18
$\Vert f \Vert_{x_0;r}$	norm of f and its first r derivatives for $x \geqslant x_0$	18
$\vert\vert\vert K \vert\vert\vert_{x_0}$	operator norm for $K: \mathcal{G}_{x_0} \rightarrow \mathcal{G}_{x_0}$	18
$\vert\vert\vert K \vert\vert\vert_{x_0;r}$	operator norm for $K: \mathcal{G}_{x_0} \rightarrow \mathcal{G}_{x_0;r}$	18
$\mathcal{T}^{\mathrm{fo}}$	first order differential closure of $\mathcal{T}$ in $\mathbb{T}$	23
$\mathcal{T}^{\mathrm{dalg}}$	differentially algebraic closure of $\mathcal{T}$ in $\mathbb{T}$	24

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