Majorants for formal power series

Joris van der Hoeven

Dépt. de Mathématiques (bât. 425)
Université Paris-Sud
91405 Orsay Cedex
France

December 9, 2008

In previous papers, we have started to develop a fully effective complex analysis. The aim of this theory is to evaluate constructible analytic functions to any desired precision and to continue such functions analytically whenever possible. In order to guarantee that the desired precision is indeed obtained, bound computations are an important part of this program. In this paper we will recall or show how the classical majorant technique can be used in order to obtain many such bounds.

Keywords: majorant equations, power series, computer algebra, partial differential equations, singular differential equations, convolution products

A.M.S. subject classification: 35A10, 13F25, 44A35

1. Introduction

In [vdH03, vdH99, vdH01a, vdH02], we have started to develop a fully effective complex analysis. The aim of this theory is to evaluate constructible analytic functions to any desired precision and to continue such functions analytically whenever possible. In order to guarantee that the desired precision is indeed obtained, bound computations are an important part of this program. A key tool for doing this is Cauchy-Kovalevskaya's classical technique of majorant equations [vK75, Pet50, Car61].

In this paper, we will study this technique in a quite detailed way. Although none of the results is fundamentally new or involved, we nevertheless felt the necessity to write this paper for several reasons:

- The need for a more abstract treatment in terms of "majorant relations" $\trianglelefteq$.
- The need for explicit majorants which can be used in effective complex analysis.
- Our wish to extend the technique to singular differential equations.
- Our wish to obtain a better control over the precision of the majorants.
- The need for majorants in the contexts of integral transformations and convolution equations.

The first two points are dealt with in sections 2, 3 and 4. In section 2.2, we isolate a few abstract properties of "majorant relations" $\trianglelefteq$. It might be interesting to pursue this abstract study in more general contexts like the one from [vdH01b]. In sections 2.3 and 2.4, we also mention some simple, but useful explicit majorants. In sections 3 and 4 we give a detailed account of the Cauchy-Kovalevskaya theorem, with a strong emphasis on "majorant-theoretic properties". The obtained majorants are quite precise, so that they can be applied to effective complex analysis.

We present some new results in section 5. In sections 5.1 and 5.2, we show how to use the majorant technique in the case of regular singular equations. This improves the treatment in [vdH01a]. In section 5.3 we consider the problem of finding "good" majorants for solutions to algebraic differential equations, in the sense that the radius of convergence of the majorant should be close to the radius of convergence of the actual solution. This problem admits a fully adequate solution in the linear case (see sections 3.4 and 3.5), but becomes much harder in the non-linear setting:

THEOREM 1.1. [DL89] *Given a power series $f=\sum f_n \zeta^n$ with rational coefficients, which is the unique solution of an algebraic differential equation*

$$P(\zeta, f, \ldots, f^{(l)}) = 0,$$

with rational coefficients and rational initial conditions, one cannot in general decide whether the radius of convergence $\rho(f)$ of f is <1 or $\geqslant 1$.

In section 5.3 we will nevertheless show how to compute majorants whose radii of convergence approximate the radius of convergence of the actual solution up to any precision (see section 5.3), but *without* controlling this precision. This result is analogous to theorem 11 in [vdH03].

Our final motivation for this paper was to publish some of the majorants we found while developing a multivariate theory of resurgent functions. In this context, a central problem is the resolution of convolution equations and the analytic continuation of the solutions. At the moment, this study is still at a very embryonary stage, because there do not exist natural isotropic equivalents for majors and minors, and we could not yet prove all necessary bounds in order to construct a general multivariate resummation theory.

Nevertheless, in a fixed Cartesian system of coordinates, multivariate convolution products are naturally defined and in section 6 we prove several explicit majorants. If one does not merely want to study convolution equations at the origin, but also wants to consider the analytic continuation of the solutions, then it is useful to have uniform majorants on the paths where the convolution integrals are computed. Such uniform majorants are studied in section 7 and an application is given. We have also tried to consider convolution integrals in other coordinate systems. In that setting, we rather recommend to study integral operators of the form $g \mapsto f * g$. Some majorants for such (and more general) operators are proved in section 8.

2. MAJORANTS

2.1. Notations

Throughout this paper, vectors and matrices will be written in bold. We will consider vectors as column matrices or n-tuples and systematically use the following notations:

$$\begin{aligned}
\mathbf{0} &= (0, \ldots, 0) \\
\mathbf{1} &= (1, \ldots, 1) \\
|\boldsymbol{\alpha}| &= |\alpha_1| + \cdots + |\alpha_n| \\
\boldsymbol{\alpha}^{\boldsymbol{k}} &= \alpha_1^{k_1} \cdots \alpha_n^{k_n} \\
\boldsymbol{k}! &= k_1! \cdots k_n! \\
\binom{\boldsymbol{k}}{\boldsymbol{l}} &= \binom{k_1}{l_1} \cdots \binom{k_n}{l_n} \\
\boldsymbol{\alpha} \cdot \boldsymbol{\beta} &= \alpha_1 \beta_1 + \cdots + \alpha_n \beta_n \\
\boldsymbol{\alpha} \times \boldsymbol{\beta} &= (\alpha_1 \beta_1, \ldots, \alpha_n \beta_n) \\
\boldsymbol{\alpha} \div \boldsymbol{\beta} &= (\alpha_1 / \beta_1, \ldots, \alpha_n / \beta_n) \\
\operatorname{abs}(\boldsymbol{\alpha}) &= (|\alpha_1|, \ldots, |\alpha_n|) \\
\boldsymbol{\alpha} \leqslant \boldsymbol{\beta} &\Leftrightarrow \alpha_1 \leqslant \beta_1 \wedge \cdots \wedge \alpha_n \leqslant \beta_n
\end{aligned}$$

We will denote by $\mathbb{C}[[\boldsymbol{\zeta}]] = \mathbb{C}[[\zeta_1, \ldots, \zeta_n]]$ the set of power series in $\zeta_1, \ldots, \zeta_n$ with coefficients in $\mathbb{C}$. We will sometimes consider other sets of coefficients, like $\mathbb{R}^{\geqslant} = \{x \in \mathbb{R}: x \geqslant 0\}$ or $R^{>} = \{x \in \mathbb{R}: x > 0\}$. Given $f \in \mathbb{C}[[\boldsymbol{\zeta}]]$, the coefficient of $\boldsymbol{\zeta}^{\boldsymbol{k}}$ in f will be denoted by $f_{\boldsymbol{k}}$. We define $\partial_i = \frac{\partial}{\partial \zeta_i}$ to be the partial derivation with respect to ζ_i and

$$\int_i : f(\zeta_1, \ldots, \zeta_n) \longmapsto \int_0^{\zeta_i} f(\zeta_1, \ldots, \zeta_{i-1}, \xi_i, \zeta_{i+1}, \ldots, \zeta_n) \, \mathrm{d}\xi_i$$

its *distinguished right inverse* $(i = 1, \ldots, n)$.

2.2. Basic algebraic properties of majorants

Given $f, \bar{f} \in \mathbb{C}[[\boldsymbol{\zeta}]]$, we say that f is *majored* by $\bar{f}$, and we write $f \trianglelefteq \bar{f}$, if $g \in \mathbb{R}^{\geqslant}[[\boldsymbol{\zeta}]]$ and

$$|f_{\boldsymbol{k}}| \leqslant \bar{f}_{\boldsymbol{k}}$$

for all $\boldsymbol{k} \in \mathbb{N}^n$. More generally, if $\boldsymbol{f} = (f_1, \ldots, f_r) \in \mathbb{C}[[\boldsymbol{\zeta}]]^r$ and $\bar{\boldsymbol{f}} = (\bar{f}_1, \ldots, \bar{f}_r) \in \mathbb{R}^{\geqslant}[[\boldsymbol{\zeta}]]^r$, then we write $\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}}$ if $f_i \trianglelefteq \bar{f}_i$ for all $i \in \{1, \ldots, r\}$.

PROPOSITION 2.1. *We have*

$$-\mathbf{1} \trianglelefteq \mathbf{1}; \tag{2.1}$$
$$\mathbf{0} \trianglelefteq \lambda \mathbf{1} \quad (\lambda \in \mathbb{R}^{\geqslant}). \tag{2.2}$$

For all $\boldsymbol{f}, \bar{\boldsymbol{f}}, \bar{\bar{\boldsymbol{f}}} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r$:

$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \implies \bar{\boldsymbol{f}} \trianglelefteq \bar{\boldsymbol{f}}; \tag{2.3}$$
$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \wedge \bar{\boldsymbol{f}} \trianglelefteq \bar{\bar{\boldsymbol{f}}} \implies \boldsymbol{f} \trianglelefteq \bar{\bar{\boldsymbol{f}}}; \tag{2.4}$$
$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \wedge \bar{\boldsymbol{f}} \trianglelefteq \boldsymbol{f} \implies \boldsymbol{f} = \bar{\boldsymbol{f}}. \tag{2.5}$$

PROPOSITION 2.2. *Let $\boldsymbol{f}, \boldsymbol{g}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{g}} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r$. Then*

$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \wedge \boldsymbol{g} \trianglelefteq \bar{\boldsymbol{g}} \implies \boldsymbol{f} + \boldsymbol{g} \trianglelefteq \bar{\boldsymbol{f}} + \bar{\boldsymbol{g}}; \tag{2.6}$$
$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \wedge \boldsymbol{g} \trianglelefteq \bar{\boldsymbol{g}} \implies \boldsymbol{f} \times \boldsymbol{g} \trianglelefteq \bar{\boldsymbol{f}} \times \bar{\boldsymbol{g}}; \tag{2.7}$$
$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \implies \partial_i \boldsymbol{f} \trianglelefteq \partial_i \bar{\boldsymbol{f}} \quad (i = 1, \ldots, n); \tag{2.8}$$
$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \implies \int_i \boldsymbol{f} \trianglelefteq \int_i \bar{\boldsymbol{f}} \quad (i = 1, \ldots, n). \tag{2.9}$$

If $\boldsymbol{f}, \bar{\boldsymbol{f}} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r$ and $\boldsymbol{g}, \bar{\boldsymbol{g}} \in \mathbb{C}[[\boldsymbol{\xi}]]^s$ are such that $\boldsymbol{g} \circ \boldsymbol{f}$ and $\bar{\boldsymbol{g}} \circ \bar{\boldsymbol{f}}$ are defined (this is so if $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r)$ and $\boldsymbol{f}_{\mathbf{0}} = \bar{\boldsymbol{f}}_{\mathbf{0}}$), then

$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \wedge \boldsymbol{g} \trianglelefteq \bar{\boldsymbol{g}} \implies \boldsymbol{g} \circ \boldsymbol{f} \trianglelefteq \bar{\boldsymbol{g}} \circ \bar{\boldsymbol{f}}. \tag{2.10}$$

PROOF. This is a direct consequence of the fact that the coefficients of $\boldsymbol{f} + \boldsymbol{g}$, $\boldsymbol{f} \times \boldsymbol{g}$, $\partial_i \boldsymbol{f}$, $\int_i \boldsymbol{f}$ and $\boldsymbol{f} \circ \boldsymbol{g}$ can all be expressed as polynomials in the coefficients of $\boldsymbol{f}$ and $\boldsymbol{g}$ with positive coefficients. □

2.3. Basic explicit majorants

We will often seek for majorations of the form $f \trianglelefteq \mathbb{b}_{\boldsymbol{\alpha}}^p$, where

$$\mathbb{b}_{\boldsymbol{\alpha}} = \frac{1}{1 - \mathbb{z}_{\boldsymbol{\alpha}}};$$
$$\mathbb{z}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \cdot \boldsymbol{\zeta},$$

$\boldsymbol{\alpha} \in (\mathbb{R}^{\geqslant})^n$ and $p \in \mathbb{N}^{>}$. For simplicity, we set $\mathbb{b} = \mathbb{b}_{\mathbf{1}}$ and $\mathbb{z} = \mathbb{z}_{\mathbf{1}}$.

PROPOSITION 2.3. *For* $\boldsymbol{\alpha} \in (\mathbb{R}^>)^n$, $i \in \{1, \ldots, n\}$ *and* $p > 0$, *we have*

$$\zeta_i \, \mathbb{b}_{\boldsymbol{\alpha}}^p \trianglelefteqslant \frac{1}{\alpha_i} \, \mathbb{b}_{\boldsymbol{\alpha}}^p,$$

whence

$$f \, \mathbb{b}_{\boldsymbol{\alpha}}^p \trianglelefteqslant f(\mathbf{1} \div \boldsymbol{\alpha}) \, \mathbb{b}_{\boldsymbol{\alpha}}^p$$

for every power series f *with positive coefficients which converges at* $\mathbf{1} \div \boldsymbol{\alpha}$.

PROOF. This is a trivial consequence from the fact that the coefficients of $\mathbb{b}_{\mathbf{1}}$ are increasing. □

PROPOSITION 2.4. *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ *be such that* $\beta_1 < \alpha_1, \ldots, \beta_n < \alpha_n$. *Then*

$$\mathbb{b}_{\boldsymbol{\beta}} \, \mathbb{b}_{\boldsymbol{\alpha}} \;\trianglelefteqslant\; (\mathbf{1} - \boldsymbol{\beta} \div \boldsymbol{\alpha})^{-\mathbf{1}} \, \mathbb{b}_{\boldsymbol{\alpha}}.$$

PROOF. For $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^n$ we first notice that

$$\binom{i_1 + j_1}{i_1} \cdots \binom{i_n + j_n}{i_n} \leqslant \binom{i_1 + j_1 + \cdots + i_n + j_n}{i_1 + \cdots + i_n}.$$

Indeed, this inequality follows from the combinatorial fact that each tuple of n choices of i_l persons among $i_l + j_l$ $(l = 1, \ldots, n)$, determines a unique choice of $i_1 + \cdots + i_n$ persons among $i_1 + j_1 + \cdots + i_n + j_n$. Hence

$$\begin{aligned}
(\mathbb{b}_{\boldsymbol{\beta}} \, \mathbb{b}_{\boldsymbol{\alpha}})_{\boldsymbol{k}} &= \sum_{\boldsymbol{i} + \boldsymbol{j} = \boldsymbol{k}} \boldsymbol{\beta}^{\boldsymbol{i}} \, \boldsymbol{\alpha}^{\boldsymbol{j}} \, \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!} \, \frac{(j_1 + \cdots + j_n)!}{j_1! \cdots j_n!} \\
&\leqslant \sum_{\boldsymbol{i} + \boldsymbol{j} = \boldsymbol{k}} \boldsymbol{\beta}^{\boldsymbol{i}} \, \boldsymbol{\alpha}^{\boldsymbol{j}} \, \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} \\
&\leqslant (\mathbf{1} - \boldsymbol{\beta} \div \boldsymbol{\alpha})^{-\mathbf{1}} \, \boldsymbol{\alpha}^{\boldsymbol{k}} \, \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} \\
&= ((\mathbf{1} - \boldsymbol{\beta} \div \boldsymbol{\alpha})^{-\mathbf{1}} \, \mathbb{b}_{\boldsymbol{\alpha}})_{\boldsymbol{k}}.
\end{aligned}$$

for all $\boldsymbol{k} \in \mathbb{N}^n$. □

COROLLARY 2.5. *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ *and* $p, q \in \mathbb{N}^>$ *be such that* $\beta_1 < \alpha_1, \ldots, \beta_n < \alpha_n$. *Then*

$$\mathbb{b}_{\boldsymbol{\beta}}^q \, \mathbb{b}_{\boldsymbol{\alpha}}^p \;\trianglelefteqslant\; (\mathbf{1} - \boldsymbol{\beta} \div \boldsymbol{\alpha})^{-q\mathbf{1}} \, \mathbb{b}_{\boldsymbol{\alpha}}^p.$$

PROOF. Apply the above proposition q times for $p = 1$. Next multiply the majoration by $\mathbb{b}_{\boldsymbol{\alpha}}^{p-1}$ on both sides. □

2.4. Majorant spaces

For fixed $p > 0$ and $\boldsymbol{\alpha} \in (\mathbb{R}^>)^n$, let $\mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha}, p}$ be the space of all analytic power series f, such that there exists a majoration $f \trianglelefteqslant c \, \mathbb{b}_{\boldsymbol{\alpha}}^p$ for some $c > 0$. We call $\mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha}, p}$ a *majorant space* for the *majorant norm* $\| \cdot \|_{\boldsymbol{\alpha}, p}$ given by

$$\begin{aligned}
\| f \|_{\boldsymbol{\alpha}, p} &= \min \{ c \in \mathbb{R}^{\geqslant} : f \trianglelefteqslant c \, \mathbb{b}_{\boldsymbol{\alpha}}^p \} \\
&= \sup \left\{ \frac{f_{\boldsymbol{k}}}{(\mathbb{b}_{\boldsymbol{\alpha}}^p)_{\boldsymbol{k}}} : \boldsymbol{k} \in \mathbb{N}^n \right\}.
\end{aligned}$$

PROPOSITION 2.6.

a) *For all* $f, g \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha}, p}$ *we have*

$$\| f + g \|_{\boldsymbol{\alpha}, p} \leqslant \| f \|_{\boldsymbol{\alpha}, p} + \| g \|_{\boldsymbol{\alpha}, p}. \tag{2.11}$$

b) For all $f \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha},p}$ *and* $g \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\beta},q}$ *with* $\beta_1 < \alpha_1, \ldots, \beta_n < \alpha_n$ *we have*

$$\| f g \|_{\boldsymbol{\alpha},p} \leqslant (\mathbf{1} - \boldsymbol{\beta} \div \boldsymbol{\alpha})^{-q\mathbf{1}} \| f \|_{\boldsymbol{\alpha},p} \| g \|_{\boldsymbol{\beta},q}. \tag{2.12}$$

c) For all $f \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha},p}$ *and* $g \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha},q}$ *we have*

$$\| f g \|_{\boldsymbol{\alpha},p+q} \leqslant \| f \|_{\boldsymbol{\alpha},p} \| g \|_{\boldsymbol{\alpha},q}. \tag{2.13}$$

d) For all $f \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha},p}$ *and* $i \in \{1, \ldots, n\}$ *we have*

$$\| \partial_i f \|_{\boldsymbol{\alpha},p+1} \leqslant p \, \alpha_i \, \| f \|_{\boldsymbol{\alpha},p}. \tag{2.14}$$

e) For all $f \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\boldsymbol{\alpha},p}$ *and* $i \in \{1, \ldots, n\}$*, such that* $p > 1$*, we have*

$$\| \textstyle\int_i f \|_{\boldsymbol{\alpha},p-1} \leqslant \frac{1}{(p-1)\,\alpha_i} \| f \|_{\boldsymbol{\alpha},p}. \tag{2.15}$$

PROOF. Part (b) follows from proposition 2.5. The other properties are easy. □

2.5. Majorants of Gevrey type

Divergent power series solutions to ordinary or partial differential equations usually admit majorants of "Gevrey type". In order to compute such majorants, it may be interesting to consider more general majorant spaces as the ones from the previous section. Given $p \in \mathbb{Q}^{\geqslant}$, $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$ and $\boldsymbol{\tau} \in (\mathbb{Q}^{\geqslant})^n$, we define

$$\mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},p} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \frac{((\boldsymbol{\tau} + \mathbf{1}) \cdot \boldsymbol{k} + p)!}{\boldsymbol{k}!} (\boldsymbol{\alpha} \times \boldsymbol{\zeta})^{\boldsymbol{k}}.$$

Then we notice that

$$\begin{aligned} \partial_i \, \mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},p} &= \alpha_i \, \mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},p+\tau_i+1}; \\ \textstyle\int_i \mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},p} &\trianglelefteq \alpha_i^{-1} \, \mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},p-\tau_i-1}. \end{aligned}$$

Furthermore, for all $\boldsymbol{\tau}$, $\boldsymbol{\alpha}$, p and q, there exist constants $c_{p,q,\boldsymbol{\tau}}$ and $r_{p,q,\boldsymbol{\tau}}$ with

$$\mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},p} \, \mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},q} \trianglelefteq c_{p,q,\boldsymbol{\tau}} \, \mathbb{d}_{\boldsymbol{\tau},\boldsymbol{\alpha},r_{p,q,\boldsymbol{\tau}}}.$$

Since divergent power series will not be studied in the sequel of this paper, we will not perform the actual computation of sharp values for $c_{p,q,\boldsymbol{\tau}}$ and $r_{p,q,\boldsymbol{\tau}}$ here.

3. MAJORANT EQUATIONS AND APPLICATIONS

3.1. Noetherian operators

Given coordinates $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ and a subset $\mathfrak{R}$ of $\{1, \ldots, r\} \times \mathbb{N}^n$, we denote

$$\mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r = \{ \boldsymbol{f} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r : f_{i,\boldsymbol{k}} \neq 0 \Rightarrow (i, \boldsymbol{k}) \in \mathfrak{R} \}.$$

Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$ and $\mathfrak{S} \subseteq \{1, \ldots, s\} \times \mathbb{N}^p$. An operator $\boldsymbol{\Phi}: \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r \rightarrow \mathbb{C}[[\boldsymbol{\xi}]]_{\mathfrak{S}}^s$ is said to be *Noetherian* if for each $(j, \boldsymbol{l}) \in \mathfrak{S}$, there exists a finite subset $\mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}$ of $\mathfrak{R}$ and a polynomial $P_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} \in \mathbb{C}[\mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}]$, such that

$$\boldsymbol{\Phi}(\boldsymbol{f})_{j,\boldsymbol{l}} = P_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}(f_{i,\boldsymbol{k}} : (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}})$$

for every $\boldsymbol{f} \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$.

REMARK 3.1. It can be checked that this notion of Noetherian operator coincides with the one introduced in a more general setting in [vdH01b]. In fact, some of the results of this paper can be generalized to this setting.

Given two Noetherian operators $\mathbf{\Phi}$, $\bar{\mathbf{\Phi}}$: $\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}} \to \mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}}$, we say that $\mathbf{\Phi}$ is *majored* by $\mathbf{\Psi}$, and we write $\mathbf{\Phi} \trianglelefteq \bar{\mathbf{\Phi}}$, if

$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \implies \mathbf{\Phi}(\boldsymbol{f}) \trianglelefteq \bar{\mathbf{\Phi}}(\bar{\boldsymbol{f}})$$

for all $\boldsymbol{f}, \bar{\boldsymbol{f}} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}$. Notice that this implies in particular that $\mathbf{0} \trianglelefteq \bar{\mathbf{\Phi}}$. It also implies that $\bar{\mathbf{\Phi}}$ is *real*, i.e. $P^{\bar{\mathbf{\Phi}}}_{j,\boldsymbol{l}} \in \mathbb{R}[\mathcal{F}^{\bar{\mathbf{\Phi}}}_{j,\boldsymbol{l}}]$ for all $(j, \boldsymbol{l}) \in \mathfrak{S}$. If $\mathbf{\Phi} \trianglelefteq \mathbf{\Phi}$, then we say that $\mathbf{\Phi}$ is a *majorating* Noetherian operator. We say that $\mathbf{\Phi}$ is *strongly majored* by $\bar{\mathbf{\Phi}}$, and we write $\mathbf{\Phi} \trianglelefteq^* \bar{\mathbf{\Phi}}$, if

$$P^{\mathbf{\Phi}}_{j,\boldsymbol{l}} \trianglelefteq P^{\bar{\mathbf{\Phi}}}_{j,\boldsymbol{l}}$$

for all $(j, \boldsymbol{l}) \in \mathfrak{S}$. For this majoration, we interpret $P^{\mathbf{\Phi}}_{j,\boldsymbol{l}}$ and $P^{\bar{\mathbf{\Phi}}}_{j,\boldsymbol{l}}$ as a power series in $\mathbb{C}[[\mathcal{F}^{\mathbf{\Phi}}_{j,\boldsymbol{l}} \cup \mathcal{F}^{\bar{\mathbf{\Phi}}}_{j,\boldsymbol{l}}]]$. Clearly, $\mathbf{\Phi} \trianglelefteq^* \bar{\mathbf{\Phi}}$ implies $\mathbf{\Phi} \trianglelefteq \bar{\mathbf{\Phi}}$, as well as $\mathbf{0} \trianglelefteq^* \bar{\mathbf{\Phi}}$ and $\bar{\mathbf{\Phi}} \trianglelefteq^* \bar{\mathbf{\Phi}}$.

REMARK 3.2. In general, we do not have $\mathbf{0} \trianglelefteq \mathbf{\Phi} \Rightarrow \mathbf{0} \trianglelefteq^* \mathbf{\Phi}$. A counterexample is the operator Φ: $\mathbb{C}[[z]] \to \mathbb{C}[[z]]$ with $\Phi(f) = (f_0 - f_1)^2$. For strongly linear operators $\mathbf{\Phi}$ (see [vdH01b]), we do have $\mathbf{0} \trianglelefteq \mathbf{\Phi} \Leftrightarrow \mathbf{0} \trianglelefteq^* \mathbf{\Phi}$.

The following proposition, which can be regarded as the operator analogue of proposition 2.2, is a again direct consequence of the classical formulas for the coefficients of $\boldsymbol{f} + \boldsymbol{g}$, $\boldsymbol{f} \times \boldsymbol{g}$, $\boldsymbol{f} * \boldsymbol{g}$, $\partial_i \boldsymbol{f}$, $\int_i \boldsymbol{f}$ and $\boldsymbol{f} \circ \boldsymbol{g}$:

PROPOSITION 3.3.

a) *The addition* $+$: $\mathbb{C}[[\boldsymbol{\zeta}]]^{2r} \cong (\mathbb{C}[[\boldsymbol{\zeta}]]^r)^2 \to \mathbb{C}[[\boldsymbol{\zeta}]]^r$ *is a Noetherian operator and* $\mathbf{0} \trianglelefteq^* +$.
b) *The componentwise multiplication* $\times$: $\mathbb{C}[[\boldsymbol{\zeta}]]^{2r} \to \mathbb{C}[[\boldsymbol{\zeta}]]^r$ *is Noetherian and* $\mathbf{0} \trianglelefteq^* \times$.
c) *The partial derivation* ∂_i: $\mathbb{C}[[\boldsymbol{\zeta}]]^r \to \mathbb{C}[[\boldsymbol{\zeta}]]^r$ *is Noetherian for each* i *and* $\mathbf{0} \trianglelefteq^* \partial_i$.
d) *The integration* $\int_i$: $\mathbb{C}[[\boldsymbol{\zeta}]]^r \to \mathbb{C}[[\boldsymbol{\zeta}]]^r$ *is Noetherian for each* i *and* $\mathbf{0} \trianglelefteq^* \int_i$.
e) *The composition* $\circ$: $\mathbb{C}[[\boldsymbol{\xi}]]^s \times (\mathbb{C}[[\boldsymbol{\zeta}]]^{\downarrow})^r \to \mathbb{C}[[\boldsymbol{\zeta}]]^s$ *is Noetherian and* $\mathbf{0} \trianglelefteq^* \circ$. *Here* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r)$ *and* $\mathbb{C}[[\boldsymbol{\zeta}]]^{\downarrow}$ *denotes the set of* $f \in \mathbb{C}[[\boldsymbol{\zeta}]]$ *with* $f_{\mathbf{0}} = 0$.

The composition of two Noetherian operators is again Noetherian and we have:

PROPOSITION 3.4. *Let* $\mathbf{\Phi}, \bar{\mathbf{\Phi}}$: $\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}} \to \mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}}$ *and* $\mathbf{\Psi}, \bar{\mathbf{\Psi}}$: $\mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}} \to \mathbb{C}[[\boldsymbol{\chi}]]^t_{\mathfrak{T}}$ *be Noetherian operators. Then*

$$\mathbf{\Phi} \trianglelefteq \bar{\mathbf{\Phi}} \wedge \mathbf{\Psi} \trianglelefteq \bar{\mathbf{\Psi}} \Rightarrow \mathbf{\Psi} \circ \mathbf{\Phi} \trianglelefteq \bar{\mathbf{\Psi}} \circ \bar{\mathbf{\Phi}}. \tag{3.1}$$

3.2. Majorant equations

A *fixed point operator* is a Noetherian operator $\mathbf{\Phi}$: $\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}} \to \mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}$, such that there exists a well-ordering $\leqslant^{\mathrm{wo}}$ on $\mathfrak{R}$ with

$$\forall (j, \boldsymbol{l}) \in \mathfrak{R}: \forall (i, \boldsymbol{k}) \in \mathcal{F}^{\mathbf{\Phi}}_{j,\boldsymbol{l}}: (i, \boldsymbol{k}) <^{\mathrm{wo}} (j, \boldsymbol{l}). \tag{3.2}$$

In general, the i-th component of the total order $\leqslant^{\mathrm{wo}}$ will be compatible with the addition on $\mathbb{N}^r$ for each $i \in \{1, \ldots, r\}$.

PROPOSITION 3.5. *Let* $\mathbf{\Phi}$: $\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}} \to \mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}$ *be a fixed point operator. Then the equation*

$$\boldsymbol{f} = \mathbf{\Phi}(\boldsymbol{f}) \tag{3.3}$$

admits a unique solution $\boldsymbol{f} = \operatorname{Fix} \boldsymbol{\Phi}$ *in* $\mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$.

PROOF. We claim that the sequence $\boldsymbol{f}^0 = \boldsymbol{0}$, $\boldsymbol{f}^1 = \boldsymbol{\Phi}(\boldsymbol{f}^0)$, $\boldsymbol{f}^2 = \boldsymbol{\Phi}(\boldsymbol{f}^1)$, ... admits a limit, which is a solution to the equation. Here a limit of a sequence $\boldsymbol{f}^0, \boldsymbol{f}^1, \ldots$ is a series $\boldsymbol{f}$ such that for all $(i, \boldsymbol{k})$, we have $f_{i,\boldsymbol{k}} = f_{i,\boldsymbol{k}}^p$ for all sufficiently large p.

Assume for contradiction that the sequence $\boldsymbol{f}^0$, $\boldsymbol{f}^1$, ... does not admit a limit and let $(j, \boldsymbol{l})$ be minimal for $\leqslant^{\mathrm{wo}}$, such that $f_{j,\boldsymbol{l}}^0, f_{j,\boldsymbol{l}}^1, \ldots$ is not ultimately constant. By (3.2), the sequence $f_{i,\boldsymbol{k}}^0, f_{i,\boldsymbol{k}}^1, \ldots$ is ultimately constant for every $(i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}$. Consequently, the sequence $f_{j,\boldsymbol{l}}^0 = P_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}(f_{i,\boldsymbol{k}}^0 \colon (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}})$, $f_{j,\boldsymbol{l}}^1 = P_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}(f_{i,\boldsymbol{k}}^1 \colon (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}})$, ... is ultimately constant. This contradiction implies our claim.

Similarly, assume that there exists a second solution $\boldsymbol{g} \neq \boldsymbol{f}$ and let $(j, \boldsymbol{l})$ be minimal for $\leqslant^{\mathrm{wo}}$ such that $f_{j,\boldsymbol{l}} \neq g_{j,\boldsymbol{l}}$. Then $f_{i,\boldsymbol{k}} = g_{i,\boldsymbol{k}}$ for all $(i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}$, by (3.2). Therefore, $f_{j,\boldsymbol{l}} = P_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}(f_{i,\boldsymbol{k}} \colon (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}) = P_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}(g_{i,\boldsymbol{k}} \colon (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}) = g_{j,\boldsymbol{l}}$, and this contraction proves that $\boldsymbol{f}$ is the unique solution to (3.3). □

Let $\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}} \colon \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r \rightarrow \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$ be two Noetherian operators and assume that $\boldsymbol{\Phi} \trianglelefteqslant \bar{\boldsymbol{\Phi}}$. Then the equation

$$\boldsymbol{f} = \bar{\boldsymbol{\Phi}}(\boldsymbol{f}) \tag{3.4}$$

is called a *majorant equation* of (3.3).

PROPOSITION 3.6. *Let* $\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}} \colon \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r \rightarrow \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$ *be two fixed point operators, such that* (3.4) *is a majorant equation of* (3.3). *Then* $\operatorname{Fix} \boldsymbol{\Phi} \trianglelefteqslant \operatorname{Fix} \bar{\boldsymbol{\Phi}}$.

PROOF. We have seen in the previous proof that $\operatorname{Fix} \boldsymbol{\Phi} = \lim_{p \rightarrow \infty} \boldsymbol{\Phi}^{\circ p}(\boldsymbol{0})$ and $\operatorname{Fix} \bar{\boldsymbol{\Phi}} = \lim_{p \rightarrow \infty} \bar{\boldsymbol{\Phi}}^{\circ p}(\boldsymbol{0})$. Using induction on p, we observe that $\boldsymbol{\Phi} \trianglelefteqslant \bar{\boldsymbol{\Phi}}$ implies $\boldsymbol{\Phi}^{\circ p}(\boldsymbol{0}) \trianglelefteqslant \bar{\boldsymbol{\Phi}}^{\circ p}(\boldsymbol{0})$. Consequently, $\operatorname{Fix} \boldsymbol{\Phi} \trianglelefteqslant \operatorname{Fix} \bar{\boldsymbol{\Phi}}$. □

For complicated equations, it can be hard to find an explicit solution to the majorant equation (like (3.8)). In that case, one may use

PROPOSITION 3.7. *Let* $\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}} \colon \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r \rightarrow \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$ *be two fixed point operators, such that* (3.4) *is a majorant equation of* (3.3). *Assume that* $\boldsymbol{f} \in \mathbb{C}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$ *is such that*

$$\bar{\boldsymbol{\Phi}}(\boldsymbol{f}) \trianglelefteqslant \boldsymbol{f}.$$

Then $\operatorname{Fix} \boldsymbol{\Phi} \trianglelefteqslant \boldsymbol{f}$.

PROOF. Since $\bar{\boldsymbol{\Phi}}$ is real, we have $\boldsymbol{\delta} = \boldsymbol{f} - \bar{\boldsymbol{\Phi}}(\boldsymbol{f}) \in \mathbb{R}^{\geqslant}[[\boldsymbol{\zeta}]]_{\mathfrak{R}}^r$. The fixed point operator $\bar{\bar{\boldsymbol{\Phi}}} \colon \boldsymbol{g} \mapsto \boldsymbol{\Psi}(\boldsymbol{g}) + \boldsymbol{\delta}$ therefore satisfies $\bar{\boldsymbol{\Phi}} \trianglelefteqslant \bar{\bar{\boldsymbol{\Phi}}}$. We conclude that $\operatorname{Fix} \boldsymbol{\Phi} \trianglelefteqslant \boldsymbol{f} = \operatorname{Fix} \bar{\bar{\boldsymbol{\Phi}}}$. □

3.3. A classical application of the majorant technique

In this section, we will prove the classical Cauchy-Kovalevskaya theorem in the case of ordinary differential equations. We will consider partial differential equations in section 4. Let $\boldsymbol{g} \in \mathbb{C}\{\{\boldsymbol{\xi}\}\}^r$ be a system of convergent power series with $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r)$ and consider the equation

$$\boldsymbol{f}' = \boldsymbol{g} \circ \boldsymbol{f} \tag{3.5}$$

in $\boldsymbol{f} \in \mathbb{C}[[\zeta]]^r$, with the initial condition $\boldsymbol{f}_0 = \boldsymbol{0}$. This equation may be rewritten as a fixed point equation

$$\boldsymbol{f} = \int \boldsymbol{g} \circ \boldsymbol{f}, \tag{3.6}$$

which has a unique solution $\boldsymbol{f} \in (\mathbb{C}[[\zeta]]^{\uparrow})^r$. Since the g_i are convergent, there exist $M \in \mathbb{R}^{\geqslant}$ and $\boldsymbol{\alpha} \in (\mathbb{R}^{\geqslant})^r$ with

$$\boldsymbol{g} \trianglelefteq M \, \mathbb{b}_{\boldsymbol{\alpha}} \mathbf{1}. \tag{3.7}$$

By propositions (3.3) and (3.4), the equation

$$\bar{\boldsymbol{f}} = \int (M \, \mathbb{b}_{\boldsymbol{\alpha}} \mathbf{1}) \circ \bar{\boldsymbol{f}} \tag{3.8}$$

is a majorant equation for (3.6). But this equation is symmetric in the $\bar{f}_i$, so we have

$$(1 - |\boldsymbol{\alpha}| \, \bar{f}_i) \, \mathrm{d} \, \bar{f}_i = M \, \mathrm{d} \zeta$$

for each $i \in \{1, ..., r\}$. The unique solution $\bar{\boldsymbol{f}}$ to (3.8) is therefore given by

$$\bar{\boldsymbol{f}} = \frac{1 - \sqrt{1 - 2 \, |\boldsymbol{\alpha}| \, M \zeta}}{|\boldsymbol{\alpha}|} \mathbf{1}.$$

From proposition 3.6 we now deduce

THEOREM 3.8. *Let $\boldsymbol{f}$ be the unique solution to* (3.5) *with $\boldsymbol{f}_0 = \mathbf{0}$ and assume* (3.7)*. Then*

$$\boldsymbol{f} \trianglelefteq \frac{1 - \sqrt{1 - 2 \, |\boldsymbol{\alpha}| \, M \zeta}}{|\boldsymbol{\alpha}|} \mathbf{1}.$$

REMARK 3.9. Notice that many ordinary differential equations can be reduced to (3.5). For instance, equations of the form $\boldsymbol{f}' = \boldsymbol{g} \circ (\boldsymbol{f}, z)$ can be solved by adding z as an unknown to $\boldsymbol{f}$, together with the equation $z' = 1$. Similarly, higher order equations can be dealt with through the introduction of new unknowns for the derivatives of unknowns. Modulo substitutions of the form $\boldsymbol{f} \mapsto \boldsymbol{\nu} + \tilde{\boldsymbol{f}}$ with $\boldsymbol{\nu} \in \mathbb{C}^r$, one may also consider more general initial conditions.

3.4. First order systems of linear differential equations

It is good not to treat linear differential equations as a special case of arbitrary linear differential equations, because the radius of convergence of the computed solution may be far from optimal. So let us study the system of linear differential equations

$$\boldsymbol{f}' = \boldsymbol{M} \boldsymbol{f}, \tag{3.9}$$

in $\boldsymbol{f} \in \mathbb{C}[[\zeta]]^r$, with initial conditions $\boldsymbol{f}_0 = \boldsymbol{\nu} \in \mathbb{C}^r$, where $\boldsymbol{M}$ is an r by r matrix with entries in $\mathbb{C}[[\zeta]]$. Assume that

$$\boldsymbol{M} \trianglelefteq K \, \mathbb{b}_{\alpha} \boldsymbol{J}, \tag{3.10}$$

where $\boldsymbol{J}$ denotes the matrix whose coefficients are all 1, and let $C = \max \{|\nu_1|, ..., |\nu_r|\}$. Then the equation

$$\bar{\boldsymbol{f}} = C \, \mathbf{1} + \int K \, \mathbb{b}_{\alpha} \boldsymbol{J} \bar{\boldsymbol{f}}, \tag{3.11}$$

is a majorant equation of

$$\boldsymbol{f} = \boldsymbol{\nu} + \int \boldsymbol{M} \boldsymbol{f}. \tag{3.12}$$

Now the equation (3.11) is again symmetric in the $\bar{f}_i$, and the fixed point of the equation

$$\bar{f}_i = C + \int K \, r \, \mathbb{b}_{\alpha} \bar{f}_i$$

is given by

$$\bar{f}_i = C \left(\frac{1}{1 - \alpha \zeta} \right)^{\frac{K r}{\alpha}}.$$

Proposition 3.6 now implies

THEOREM 3.10. *Let $\boldsymbol{f}$ be the unique solution to* (3.9) *with* $\boldsymbol{f}_0 = \boldsymbol{\nu}$ *and assume* (3.10). *Then*

$$\boldsymbol{f} \trianglelefteq \max \{|\nu_1|, \ldots, |\nu_r|\} \left(\frac{1}{1 - \alpha \zeta} \right)^{\frac{K r}{\alpha}}.$$

3.5. Higher order linear differential equations

The exponent $\frac{K r}{\alpha}$ in the majorant from theorem 3.10 is not always optimal. Assume for instance that we have a linear differential equation

$$f^{(r)} = L_{r-1} f^{(r-1)} + \cdots + L_0 f, \tag{3.13}$$

with initial conditions $f(0) = \nu_0, \ldots, f^{(r-1)}(0) = \nu_{r-1}$, and where the L_i satisfy

$$L_i \trianglelefteq M \, \mathbb{b}_\alpha. \tag{3.14}$$

It is not easy to find a closed form solution for (3.13). For this reason, we will apply the technique from proposition 3.7.

The series f is the unique solution to the fixed point equation

$$f = \left(\nu_0 + \int \right) \cdots \left(\nu_{r-1} + \int \right) \left(L_{r-1} f^{(r-1)} + \cdots + L_0 f \right). \tag{3.15}$$

For all $\bar{\nu}_0, \ldots, \bar{\nu}_{r-1} \in \mathbb{R}^{\geqslant}$ and $R \in \mathbb{R}^{\geqslant}[[z]]$, such that $|\nu_0| \leqslant \bar{\nu}_0, \ldots, |\nu_{r-1}| \leqslant \bar{\nu}_{r-1}$, the equation

$$\bar{f} = \left(\bar{\nu}_0 + \int \right) \cdots \left(\bar{\nu}_{r-1} + \int \right) \left(M \, \mathbb{b}_\alpha \bar{f}^{(r-1)} + \cdots + M \, \mathbb{b}_\alpha \bar{f} \right) + R \tag{3.16}$$

is a majorant equation of (3.15). Let

$$h = \mathbb{b}_\alpha^{(M+1)/\alpha}.$$

We take $\hat{\nu}_i = C h^{(i)}(0)$ for all $i \in \{0, \ldots, r-1\}$, where

$$C = \max \{|\nu_0|, \ldots, |\nu_{r-1}|\} \geqslant \max \left\{ \frac{|\nu_0|}{h^{(0)}(0)}, \ldots, \frac{|\nu_{r-1}|}{h^{(r-1)}(0)} \right\}.$$

We have

$$\begin{aligned} & M \, \mathbb{b}_\alpha h^{(r-1)} + \cdots + M \, \mathbb{b}_\alpha h \\ \trianglelefteq \; & M \left[(M+1) \cdots (M + (r-2) \alpha + 1) \, \mathbb{b}_\alpha^{(M + r\alpha + 1)/\alpha} + \cdots + \mathbb{b}_\alpha^{(M + \alpha + 1)/\alpha} \right] \\ \trianglelefteq \; & (M+1) \cdots (M + (r-1) \alpha + 1) \, \mathbb{b}_\alpha^{(M + r\alpha + 1)/\alpha} \\ = \; & h^{(r)} \end{aligned}$$

Therefore, we may take

$$R = \left(\bar{\nu}_0 + \int \right) \cdots \left(\bar{\nu}_{r-1} + \int \right) \left(M \, \mathbb{b}_\alpha C h^{(r-1)} + \cdots + M \, \mathbb{b}_\alpha C h \right) - C h \in \mathbb{R}^{\geqslant}[[z]].$$

This choice ensures that (3.16) has the particularly simple solution $C h$. Proposition 3.6 therefore implies:

THEOREM 3.11. *Let f be the unique solution to* (3.13) *with* $f(0)=\nu_0,\ldots,f^{(r-1)}(0)=\nu_{r-1}$ *and assume that we have* (3.14) *for* $i\in\{0,\ldots,r-1\}$*. Then*

$$\boldsymbol{f}\trianglelefteq\max\{|\nu_0|,\ldots,|\nu_{r-1}|\}\left(\frac{1}{1-\alpha\,\zeta}\right)^{\frac{M+1}{\alpha}}.$$

4. PARTIAL DIFFERENTIAL EQUATIONS

4.1. Indirect majorant equations and reduction of dimension

In order to obtain majorants for solutions of partial differential equations, it is sometimes possible to generalize Cauchy-Kovalevskaya's technique from section 3.3. However, the more the type of the original equation becomes complex, the harder the explicit resolution of the corresponding majorant equation may become. For this reason, we will introduce a technique, which allows the reduction of the majorant equation to an ordinary differential equation.

Given $\boldsymbol{f}\in\mathbb{C}[[\boldsymbol{\zeta}]]^r$ and $\boldsymbol{\alpha}\in(\mathbb{R}^>)^r$, the idea is to systematically search for majorants of the form $\boldsymbol{f}\trianglelefteq\bar{\boldsymbol{f}}\circ\mathbb{z}_{\boldsymbol{\alpha}}$, where $\bar{\boldsymbol{f}}\in\mathbb{C}[[\xi]]$. The choice of $\boldsymbol{\alpha}$ depends on the region near the origin where we want a bound for $\boldsymbol{f}$. The ring monomorphism $\boldsymbol{f}\mapsto\boldsymbol{f}\circ\mathbb{z}_{\boldsymbol{\alpha}}$ satisfies the following properties:

PROPOSITION 4.1. *Let* $\boldsymbol{\alpha}\in(\mathbb{R}^>)^r$ *and* $\boldsymbol{f},\bar{\boldsymbol{f}}\in\mathbb{C}[[\xi]]^r$*. Then*

$$\begin{aligned}\boldsymbol{f}\trianglelefteq\bar{\boldsymbol{f}}\ &\Leftrightarrow\ \boldsymbol{f}\circ\mathbb{z}_{\boldsymbol{\alpha}}\trianglelefteq\boldsymbol{g}\circ\mathbb{z}_{\boldsymbol{\alpha}} &(4.1)\\ \partial_i(\boldsymbol{f}\circ\mathbb{z}_{\boldsymbol{\alpha}})\ &=\ (\alpha_i\boldsymbol{f}')\circ\mathbb{z}_{\boldsymbol{\alpha}}\quad(i=1,\ldots,n)\,; &(4.2)\\ \textstyle\int_i(\boldsymbol{f}\circ\mathbb{z}_{\boldsymbol{\alpha}})\ &\trianglelefteq\ (\alpha_i^{-1}\textstyle\int_i\boldsymbol{f})\circ\mathbb{z}_{\boldsymbol{\alpha}}\quad(i=1,\ldots,n). &(4.3)\end{aligned}$$

The next idea is to extend the majorant technique as follows: given a majorating mapping $\mathbf{M}\colon\mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}}\to\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}$, and two fixed point operators $\bar{\boldsymbol{\Phi}}\colon\mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}}\to\mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}}$ and $\boldsymbol{\Phi}\colon\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}\to\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}$, we say that the equation

$$\bar{\boldsymbol{f}}=\bar{\boldsymbol{\Phi}}(\bar{\boldsymbol{f}})\tag{4.4}$$

is an *indirect majorant equation* of

$$\boldsymbol{f}=\boldsymbol{\Phi}(\boldsymbol{f})\tag{4.5}$$

if for all $\boldsymbol{f}\in\mathbb{C}[[\boldsymbol{\zeta}]]^r_{\mathfrak{R}}$ and $\bar{\boldsymbol{f}}\in\mathbb{C}[[\boldsymbol{\xi}]]^s_{\mathfrak{S}}$ we have

$$\boldsymbol{f}\trianglelefteq\mathbf{M}(\boldsymbol{g})\implies\boldsymbol{\Phi}(\boldsymbol{f})\trianglelefteq\mathbf{M}(\bar{\boldsymbol{\Phi}}(\boldsymbol{g}))$$

We will attempt to apply the following generalization of proposition 3.6 for $\mathbf{M}\colon\hat{\boldsymbol{f}}\mapsto\hat{\boldsymbol{f}}\circ\mathbb{z}_{\boldsymbol{\alpha}}$.

PROPOSITION 4.2. *Let* $\boldsymbol{\Phi}$ *and* $\bar{\boldsymbol{\Phi}}$ *be two fixed point operators, such that* (4.4) *is an indirect majorant equation of* (4.5)*. Then* $\operatorname{Fix}\boldsymbol{\Phi}\trianglelefteq\mathbf{M}(\operatorname{Fix}\bar{\boldsymbol{\Phi}})$.

PROOF. Similar to the proof of proposition 3.6. □

4.2. The Cauchy-Kovalevskaya theorem: equations in normal form

Consider the system of partial differential equations

$$\partial_n^p\boldsymbol{f}=\boldsymbol{\varphi}\circ((\boldsymbol{\partial}^{\boldsymbol{k}}\boldsymbol{f})_{|\boldsymbol{k}|<p},\boldsymbol{\zeta})\tag{4.6}$$

with initial conditions

$$\begin{cases} \boldsymbol{f}(\zeta_1, \ldots, \zeta_{n-1}, 0) = \mathbf{0}\,; \\ \vdots \\ \boldsymbol{f}^{(p-1)}(\zeta_1, \ldots, \zeta_{n-1}, 0) = \mathbf{0}, \end{cases}$$

where $\boldsymbol{\varphi} \in \mathbb{C}\{\{\boldsymbol{\xi}\}\}^r$ is a tuple of convergent power series in $m = r\binom{n+p}{p} + n$ variables. We may rewrite (4.6) as a fixed point equation

$$\boldsymbol{f} = \int_n \overset{p\times}{\cdots} \int_n \boldsymbol{\varphi} \circ ((\boldsymbol{\partial}^{\boldsymbol{k}} \boldsymbol{f})_{|\boldsymbol{k}|<p}, \boldsymbol{\zeta}), \tag{4.7}$$

Let $M \in \mathbb{R}^{\geqslant}$, $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$ and $\boldsymbol{\beta} \in (\mathbb{R}^{>})^m$ be such that $\boldsymbol{\varphi} \trianglelefteq M\, \mathbb{b}_{\boldsymbol{\beta}} \mathbf{1}$. Then

$$\bar{\boldsymbol{f}} = M \alpha_i^{-p} \int \overset{p\times}{\cdots} \int (\mathbb{b}_{\boldsymbol{\beta}} \mathbf{1}) \circ ((\boldsymbol{\alpha}^{\boldsymbol{k}} \bar{\boldsymbol{f}}^{(|\boldsymbol{k}|)})_{|\boldsymbol{k}|<p}, (\mathbf{1} \div \boldsymbol{\alpha})\, \xi) \tag{4.8}$$

with initial conditions $\bar{\boldsymbol{f}}(0) = \cdots = \bar{\boldsymbol{f}}^{(p-1)} = \mathbf{0}$ is an indirect majorant equation of (4.11). This equation can be reinterpreted as an ordinary differential equation of the form (3.5) in $r\,p + 1$ unknowns $\bar{\boldsymbol{f}}, \ldots, \bar{\boldsymbol{f}}^{(p-1)}, \xi$. By theorem 3.8, the unique solution $\bar{\boldsymbol{f}}$ to (4.8) is convergent. By theorem 4.2, the unique solution $\boldsymbol{f}$ to (4.6) is therefore convergent, since $\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \circ \mathbb{z}_{\boldsymbol{\alpha}}$. We have proved the following theorem:

THEOREM 4.3. *The system* (4.6) *admits a unique solution* $\boldsymbol{f}$ *in* $\mathbb{C}[[\boldsymbol{\zeta}]]^r$ *and* $\boldsymbol{f}$ *is convergent.*

REMARK 4.4. Like in remark 3.9, we notice that many systems of partial differential equations can be seen as variants of (4.6). For instance, modulo substitutions $\boldsymbol{f} \mapsto \boldsymbol{\nu}_0 + \cdots + \boldsymbol{\nu}_{p-1}\, \zeta_n^{p-1} + \tilde{\boldsymbol{f}}$ with $\boldsymbol{\nu}_0, \ldots, \boldsymbol{\nu}_{p-1} \in \mathbb{C}\{\{\zeta_1, \ldots, \zeta_{n-1}\}\}^r$, we may consider more general types of convergent initial conditions. Also, if the f_i are allowed to satisfy differential equations

$$\partial_n^{p_i} f_i = \varphi_i \circ ((\boldsymbol{\partial}^{\boldsymbol{k}} \boldsymbol{f})_{|\boldsymbol{k}|<p_i}, \boldsymbol{\zeta})$$

of different orders, then it suffices to differentiate these equations $\max\{p_1, \ldots, p_r\} - p_i$ times and compute the corresponding initial conditions, in order to reduce this more general case to the case when all equations have the same order.

4.3. Further generalizations of the majorant technique

We did not compute an explicit majorant for $\boldsymbol{f}$, because theorem 4.3 is not the result we are really after. In fact, the right-hand side of (4.6) may also depend on all partial derivatives $\boldsymbol{\partial}^{\boldsymbol{k}} \boldsymbol{f}$ with $|\boldsymbol{k}| = p$, except for $\partial_n^p \boldsymbol{f}$. However, the corresponding indirect majorant equation would no longer be a fixed point equation in the sense of section 3.2.

Consider an operator $\boldsymbol{\Phi}: \mathcal{R} \to \mathcal{S}$ where $\mathcal{R} \subseteq \mathbb{C}[[\boldsymbol{\zeta}]]^r$ and $\mathcal{S} \subseteq \mathbb{C}[[\boldsymbol{\xi}]]^s$. We say that $\boldsymbol{\Phi}$ is *Noetherian* in the generalized sense, if for each $(j, \boldsymbol{l}) \in \{1, \ldots, s\} \times \mathbb{N}^p$, there exists a finite subset $\mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}$ of $\mathfrak{R}$ and a convergent power series $g_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} \in \mathbb{C}\{\{\mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}\}\}$, such that

$$\boldsymbol{\Phi}(\boldsymbol{f})_{j,\boldsymbol{l}} = g_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} \circ (f_{i,\boldsymbol{k}}: (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}})$$

for every $\boldsymbol{f} \in \mathcal{R}$. All previously defined concepts naturally generalize to this setting.

For instance, given two Noetherian operators $\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}}: \mathcal{R} \to \mathcal{S}$ in the generalized sense, we say that $\boldsymbol{\Phi}$ is *majored* by $\bar{\boldsymbol{\Phi}}$, and we write $\boldsymbol{\Phi} \trianglelefteq \bar{\boldsymbol{\Phi}}$, if

$$\boldsymbol{f} \trianglelefteq \bar{\boldsymbol{f}} \Longrightarrow \boldsymbol{\Phi}(\boldsymbol{f}) \trianglelefteq \bar{\boldsymbol{\Phi}}(\bar{\boldsymbol{f}})$$

for all $\boldsymbol{f} \in \mathcal{R}$. This is in particular so, if $g_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} \trianglelefteq g_{j,\boldsymbol{l}}^{\bar{\boldsymbol{\Phi}}}$ for all $(j, \boldsymbol{l}) \in \{1, \ldots, s\} \times \mathbb{N}^p$, in which case we say that $\boldsymbol{\Phi}$ is *strongly majored* by $\bar{\boldsymbol{\Phi}}$. The concepts of majorating Noetherian operators and indirect majorant equations can be generalized in a similar way.

PROPOSITION 4.5. *Let $\boldsymbol{\Phi}$ be a fixed point operator in the usual sense and $\bar{\boldsymbol{\Phi}}$ a Noetherian operator in the generalized sense, such that* (4.9) *is an indirect majorant equation of* (4.10)*. If* (4.9) *admits a solution $\bar{\boldsymbol{f}}$ with $\mathbf{0} \trianglelefteq \boldsymbol{f}$, then* $\operatorname{Fix} \boldsymbol{\Phi} \trianglelefteq \mathbf{M}(\bar{\boldsymbol{f}})$.

PROOF. By induction on p, we observe that $\boldsymbol{\Phi}^{\circ p}(\mathbf{0}) \trianglelefteq \mathbf{M}(\bar{\boldsymbol{f}})$ for all $p \in \mathbb{N}$. Consequently, $\operatorname{Fix} \boldsymbol{\Phi} = \lim_{p \to \infty} \boldsymbol{\Phi}^{\circ p}(\mathbf{0}) \trianglelefteq \mathbf{M}(\bar{\boldsymbol{f}})$. □

REMARK 4.6. Notice that we no longer require (4.9) to be a fixed point equation in proposition 4.5. Nevertheless, it is possible to generalize proposition 3.5 to fixed point operators in a more general sense. For instance, one may replace (3.2) by

$$\forall (j, \boldsymbol{l}) \colon \forall (i, \boldsymbol{k}) \in \mathcal{F}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} \colon (i, \boldsymbol{k}) \leqslant^{\mathrm{wo}} (j, \boldsymbol{l}) \tag{4.9}$$

and the requirement that $g_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} = \alpha_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}} f_{j,\boldsymbol{l}} + \tilde{g}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}$ for all $(j, \boldsymbol{l})$, where $|\alpha_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}| < 1$ and $\tilde{g}_{j,\boldsymbol{l}}^{\boldsymbol{\Phi}}$ does not depend on $f_{j,\boldsymbol{l}}$. This kind of fixed point operators will be encountered naturally in the next section (modulo a change of variables). More generally, one may consider operators $\boldsymbol{\Phi}$, whose "traces for truncated series to initial segments" are contracting.

4.4. The Cauchy-Kovalevskaya theorem: the general case

Consider the system of partial differential equations

$$\partial_n \boldsymbol{f} = \boldsymbol{\varphi} \circ (\partial_1 \boldsymbol{f}, \ldots, \partial_{n-1} \boldsymbol{f}, \boldsymbol{f}, \boldsymbol{\zeta}) \tag{4.10}$$

with initial condition

$$\boldsymbol{f}(\zeta_1, \ldots, \zeta_{n-1}, 0) = \mathbf{0},$$

where $\boldsymbol{\varphi} \in \mathbb{C}\{\{\boldsymbol{\chi}\}\}^r$ is a tuple of convergent power series in $m = n\,(r+1)$ variables. We may rewrite this equation as a fixed point equation

$$\boldsymbol{f} = \int_n \boldsymbol{\varphi} \circ (\partial_1 \boldsymbol{f}, \ldots, \partial_{n-1} \boldsymbol{f}, \boldsymbol{f}, \boldsymbol{\zeta}). \tag{4.11}$$

Let $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$ and $\boldsymbol{\beta} \in (\mathbb{R}^{>})^m$ be such that $\boldsymbol{\varphi} \trianglelefteq M\, \mathbb{b}_{\boldsymbol{\beta}} \mathbf{1}$ and define

$$\begin{aligned} A &= \alpha_1 (\beta_1 + \cdots + \beta_r) + \cdots + \alpha_{n-1} (\beta_{(n-2)r+1} + \cdots + \beta_{(n-1)r}); \\ B &= \beta_{(n-1)r+1} + \cdots + \beta_{nr}; \\ C &= \beta_{nr+1}/\alpha_1 + \cdots + \beta_{nr+n}/\alpha_n. \end{aligned}$$

Choosing $\alpha_1, \ldots, \alpha_{n-1}$ sufficiently small, we may assume that

$$4\, A\, M < \alpha_n. \tag{4.12}$$

Then we claim that

$$\bar{\boldsymbol{f}} = M\, \alpha_n{}^{-1} \int (\mathbb{b}_{\boldsymbol{\beta}} \mathbf{1}) \circ (\alpha_1 \bar{\boldsymbol{f}}', \ldots, \alpha_{n-1} \bar{\boldsymbol{f}}', \bar{\boldsymbol{f}}, (\mathbf{1} \div \boldsymbol{\alpha})\, \xi) \tag{4.13}$$

is an indirect majorant equation of (4.11) for $\mathbf{M} = \cdot \circ \mathbb{z}_{\boldsymbol{\alpha}}$ and the initial conditions $\bar{\boldsymbol{f}}(0) = 0$, $\bar{\boldsymbol{f}}'(0) = \nu\, \mathbf{1}$, where

$$\nu = \frac{1 - \sqrt{1 - \frac{4\,M\,A}{\alpha_n}}}{2\,A}.$$

Indeed, the majorant equation (4.13) is symmetric in the components of $\bar{\boldsymbol{f}}$, so each $\bar{f}_i$ satisfies the equation

$$\bar{f}_i = \bar{\Phi}(f_i) = M\,\alpha_n^{-1} \int \frac{1}{1 - A\,\bar{f}_i' - B\,\bar{f}_i - C\,\xi} \tag{4.14}$$

with the initial conditions $\bar{f}_i(0)=0$, $\bar{f}_i'(0)=\nu$. The choice of ν now guarantees that $\bar{\Phi}$ maps $\{\bar{f} \in \mathbb{C}[[\xi]]: \bar{f}(0)=0, \bar{f}'(0)=\nu\}$ into itself.

The equation (4.14), which was obtained mechanically from (4.11) using reduction of dimension, does not admit a simple closed form solution. Therefore, it is convenient to major it a second time by the equation

$$\bar{\bar{f}}_i = M\,\alpha_n^{-1} \int \frac{1}{1 - A\,\bar{\bar{f}}_i' - B\,\xi\,\bar{\bar{f}}_i' - C\,\xi} \tag{4.15}$$

with the same initial conditions. This latter equation can be rewritten as an equation

$$(1 - A\,\bar{\bar{f}}_i' - B\,\xi\,\bar{\bar{f}}_i' - C\,\xi)\,\bar{\bar{f}}_i' = M\,\alpha_n^{-1}$$

of second degree in $\bar{\bar{f}}_i'$. This leads to the simple closed form majorant for $\bar{\bar{f}}_i$:

$$\bar{\bar{f}}_i \trianglelefteq \xi\,\bar{\bar{f}}_i' = \frac{\xi - C\,\xi^2 - \xi\,\sqrt{1 - \frac{4\,A\,M}{\alpha_n} - \frac{4\,B\,M\,\xi}{\alpha_n} - 2\,C\,\xi + C^2\,\xi^2}}{2\,(A + B\,\xi)}. \tag{4.16}$$

We have proved:

THEOREM 4.7. *Let $\boldsymbol{\varphi} \in \mathbb{C}[[\boldsymbol{\zeta}]]^{n(r+1)}$ be such that $\boldsymbol{\varphi} \trianglelefteq M\,\mathbb{b}_{\boldsymbol{\beta}}\,\mathbf{1}$, for $M \in \mathbb{R}^{\geqslant}$, $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$ and $\boldsymbol{\beta} \in (\mathbb{R}^{>})^m$ which satisfy* (4.12)*. Then the system* (4.10) *admits a unique solution $\boldsymbol{f} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r$ with $\boldsymbol{f}(\zeta_1, \ldots, \zeta_{n-1}, 0) = \mathbf{0}$. Moreover, this solution satisfies*

$$\boldsymbol{f} \trianglelefteq \frac{\mathbb{z}_{\boldsymbol{\alpha}} - C\,\mathbb{z}_{\boldsymbol{\alpha}}^2 - \mathbb{z}_{\boldsymbol{\alpha}}\,\sqrt{1 - \frac{4\,A\,M}{\alpha_n} - \frac{4\,B\,M\,\mathbb{z}_{\boldsymbol{\alpha}}}{\alpha_n} - 2\,C\,\mathbb{z}_{\boldsymbol{\alpha}} + C^2\,\mathbb{z}_{\boldsymbol{\alpha}}^2}}{2\,(A + B\,\mathbb{z}_{\boldsymbol{\alpha}})}\,\mathbf{1}.$$

REMARK 4.8. The original Cauchy-Kovalevskaya theorem, which deals with equations of arbitrary order, directly follows from the first order case. Indeed, consider a system

$$\left\{ \begin{array}{lll} \partial_n^p\,f_1 & = & \varphi_1 \circ ((\boldsymbol{\partial}^{\boldsymbol{k}}\,\boldsymbol{f})_{|\boldsymbol{k}| \leqslant p, \boldsymbol{k} \neq p\boldsymbol{e}_n}, \boldsymbol{\zeta}) \\ & \vdots & \\ \partial_n^p\,f_r & = & \varphi_r \circ ((\boldsymbol{\partial}^{\boldsymbol{k}}\,\boldsymbol{f})_{|\boldsymbol{k}| \leqslant p, \boldsymbol{k} \neq p\boldsymbol{e}_n}, \boldsymbol{\zeta}) \end{array} \right. \tag{4.17}$$

Introduce the unknowns $g_{i,\boldsymbol{k}} = \boldsymbol{\partial}^{\boldsymbol{k}}\,f_i$ for all i and $\boldsymbol{k}$ with $|\boldsymbol{k}| < p$. We have equations $\partial_n\,g_{i,\boldsymbol{k}} = g_{i,\boldsymbol{k}+\boldsymbol{e}_n}$ for all $\boldsymbol{k}$ with $|\boldsymbol{k}| < p-1$. For each $\boldsymbol{k} \neq (p-1)\,\boldsymbol{e}_n$ with $|\boldsymbol{k}| = p-1$, we also have $\partial_n\,g_{i,\boldsymbol{k}} = \partial_m\,g_{i,\boldsymbol{k}+\boldsymbol{e}_n-\boldsymbol{e}_m}$ for some $m < n$ with $k_m \neq 0$. Finally, the equations (4.17) express each $\partial_n\,g_{i,(p-1)\boldsymbol{e}_n}$ in terms of the $g_{i,\boldsymbol{k}}$ with $|\boldsymbol{k}| < p$ and $\partial_m\,g_{i,\boldsymbol{k}}$ with $m < n$ and $|\boldsymbol{k}| = p-1$. In a similar way as in remark 4.4, one may also deal with different orders p_i for each i.

5. COMPLEMENTS TO THE MAJORANT TECHNIQUE

5.1. Regular singular systems of ordinary differential equations

Consider the first order system

$$\delta\,\boldsymbol{f} = \boldsymbol{\varphi} \circ (\boldsymbol{f}, \zeta), \tag{5.1}$$

where $\delta = \frac{\zeta \partial}{\partial \zeta}$ and $\boldsymbol{\varphi}$ is a tuple of convergent power series in $r+1$ variables with $\boldsymbol{\varphi}_{\mathbf{0}} = \mathbf{0}$. We search solutions $\boldsymbol{f} \in \mathbb{C}[[\boldsymbol{\zeta}]]^r$ to this system with $\boldsymbol{f}_0 = \mathbf{0}$. Let $\Phi_{i,j}$ be the coefficient of f_j in φ_i. Then extraction of the coefficient of ζ^k in (5.1) leads to the relation

$$(k - \boldsymbol{\Phi})\, \boldsymbol{f}_k = \boldsymbol{P}_k(\boldsymbol{f}_{k-1}, \ldots, \boldsymbol{f}_0), \tag{5.2}$$

where $\boldsymbol{P}$ is a polynomial. The matrix $k - \boldsymbol{\Phi}$ is invertible for all but a finite number of values $\lambda_1, \ldots, \lambda_r$ of k, so that (5.2) is a recurrence relation for the coefficients of $\boldsymbol{f}$ whenever $k \notin \{\lambda_1, \ldots, \lambda_r\}$. For $k \in \{\lambda_1, \ldots, \lambda_r\} \cap \mathbb{N}$, we may see $\boldsymbol{f}_k$ as an initial condition and the relation $\boldsymbol{P}_k(\boldsymbol{f}_{k-1}, \ldots, \boldsymbol{f}_0)$ as an additional requirement on $\boldsymbol{f}_0, \ldots, \boldsymbol{f}_{k-1}$.

Using the notations from (5.2), the original equation (5.1) can be rewritten as a fixed point equation

$$\boldsymbol{f} = (\delta - \boldsymbol{\Phi})^{-1} (\boldsymbol{\psi} \circ (\boldsymbol{f}, \zeta)), \tag{5.3}$$

where $\boldsymbol{\psi}$ is a tuple of convergent power series in $r+1$ variables with $\boldsymbol{\psi}_{\mathbf{0}} = \mathbf{0}$ and such that the coefficient of f_j in ψ_i vanishes for all i and j. Given $l \in \mathbb{N}$, let us now consider the change of variables

$$\boldsymbol{f} = \boldsymbol{f}_0 + \cdots + \boldsymbol{f}_{l-1} \zeta^{l-1} + \tilde{\boldsymbol{f}},$$

where $\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_{l-1}$ are computed by the recurrence relation (5.2). Then (5.3) transforms into a new equation of the form

$$\tilde{\boldsymbol{f}} = (\delta - \boldsymbol{\Phi})^{-1} (\tilde{\boldsymbol{\psi}} \circ (\tilde{\boldsymbol{f}}, \zeta)), \tag{5.4}$$

such that the coefficients of $1, \ldots, \zeta^{l-1}$ and $\tilde{\boldsymbol{f}}$ in $\tilde{\boldsymbol{\psi}}$ vanish.

Now choose l sufficiently large, such that $l > \Lambda = \max \{|\lambda_1|, \ldots, |\lambda_n|\}$. Then we may compute a $\kappa > 0$ with

$$\max_{1 \leqslant i,j \leqslant r} |((k - \boldsymbol{\Phi})^{-1})_{i,j}| \leqslant \kappa \tag{5.5}$$

for all $k \geqslant l$. Choose a majorant for $\tilde{\boldsymbol{\psi}}$ of the form

$$\tilde{\boldsymbol{\psi}} \circ (\tilde{\boldsymbol{f}}, \zeta) \trianglelefteqslant M \left(\frac{1}{(1 - \boldsymbol{\alpha} \cdot \tilde{\boldsymbol{f}})\,(1 - \beta \zeta)} - \frac{1 - (\beta \zeta)^l}{1 - \beta \zeta} - \boldsymbol{\alpha} \cdot \tilde{\boldsymbol{f}} \right) \mathbf{1},$$

where $\boldsymbol{\alpha} = \alpha\, \mathbf{1}$. Then the equation

$$\bar{\boldsymbol{f}} = M \kappa r \left(\frac{1}{(1 - \boldsymbol{\alpha} \cdot \bar{\boldsymbol{f}})\,(1 - \beta \zeta)} - \frac{1 - (\beta \zeta)^l}{1 - \beta \zeta} - \boldsymbol{\alpha} \cdot \bar{\boldsymbol{f}} \right) \mathbf{1}. \tag{5.6}$$

is a majorant equation of (5.3) for $\tilde{\boldsymbol{f}}, \bar{\boldsymbol{f}} \in \mathbb{C}[[\zeta]]^r_{\geqslant l}$, where we denote

$$\mathbb{C}[[\zeta]]_{\geqslant l} = \{ f \in \mathbb{C}[[\zeta]] : f_0 = \cdots = f_{l-1} = 0 \}.$$

The equation (5.6) admits a rather long closed form solution. In order to simplify the majorant, we notice that (5.6) is majored by the same equation with $l = 1$. This leads to the majorant

$$\bar{\boldsymbol{f}} \trianglelefteqslant \frac{1 - \sqrt{1 - 4\, M \alpha \kappa r^2 \,(1 + M \alpha \kappa r^2)\, \frac{\beta \zeta}{1 - \beta \zeta}}}{2\, r \alpha \,(1 + M \alpha \kappa r^2)} \mathbf{1}.$$

We have proved:

THEOREM 5.1. *Let $\boldsymbol{\Phi}$ be a matrix with coefficients in $\mathbb{C}$, such that $k - \boldsymbol{\Phi}$ is invertible for all $k \geqslant l$ and let $\kappa > 0$ be such that we have* (5.5)*. Consider the equation*

$$(\delta - \boldsymbol{\Phi})\, \boldsymbol{f} = \boldsymbol{\varphi} \circ (\boldsymbol{f}, \zeta),$$

where $\boldsymbol{\varphi}$ satisfies the majoration

$$\boldsymbol{\varphi} \circ (\boldsymbol{f}, \zeta) \trianglelefteq M \left(\frac{1}{(1 - \boldsymbol{\alpha} \cdot \boldsymbol{f})(1 - \beta \zeta)} - \frac{1 - (\beta \zeta)^l}{1 - \beta \zeta} - \boldsymbol{\alpha} \cdot \boldsymbol{f} \right) \mathbf{1},$$

for $\boldsymbol{\alpha} = \alpha \mathbf{1}$ and $\alpha, \beta, M > 0$. Then this equation admits a unique solution $\boldsymbol{f} \in \mathbb{C}[[\zeta]]^r_{\geqslant l}$ and we have

$$\boldsymbol{f} \trianglelefteq \frac{1 - \sqrt{1 - 4 M \alpha \kappa r^2 (1 + M \alpha \kappa r^2) \frac{\beta \zeta}{1 - \beta \zeta}}}{2 r \alpha (1 + M \alpha \kappa r^2)} \mathbf{1}. \tag{5.7}$$

5.2. Regular singular linear differential equations

In the case of linear differential equations, the majorant (5.7) can be further improved. Consider a regular singular system of linear differential equations

$$(\delta - \boldsymbol{\Phi}) \boldsymbol{f} = \boldsymbol{L} \boldsymbol{f} + \boldsymbol{g}, \tag{5.8}$$

where $\boldsymbol{\Phi}$ is an $r \times r$ matrix with coefficients in $\mathbb{C}$, $\boldsymbol{L}$ an $r \times r$ matrix with coefficients in $\zeta \mathbb{C}\{\{\zeta\}\}$, and $\boldsymbol{g} \in \mathbb{C}\{\{\zeta\}\}^r$. We observe that the form of the equation is invariant under substitutions of the form $\boldsymbol{f} = \boldsymbol{\varphi} + \tilde{\boldsymbol{f}}$. Consequently, after the computation of the first l coefficients of the solution (if such a solution exists), and modulo the change of variables $\boldsymbol{f} = \boldsymbol{f}_0 + \cdots + \boldsymbol{f}_{l-1} \zeta^{l-1} + \tilde{\boldsymbol{f}}$, we may assume without loss of generality that $\boldsymbol{g} \in \mathbb{C}[[\zeta]]^r_{\geqslant l}$.

Now take $l > \Lambda$ with the notations from the previous section. Let $M, N \geqslant 0$ and $\alpha > 0$ be such that $\boldsymbol{L} \trianglelefteq M \zeta \mathbb{b}_\alpha \boldsymbol{J}$ and $\boldsymbol{g} \trianglelefteq N \mathbb{b}_\alpha \mathbf{1}$. Here $\boldsymbol{J}$ denotes the matrix whose entries are all 1. Then, for $\kappa > 0$ satisfying (5.5), the equation

$$\boldsymbol{f} = (\delta - \boldsymbol{\Phi})^{-1} (\boldsymbol{L} \boldsymbol{f} + \boldsymbol{g}),$$

admits the majorant equation

$$\bar{\boldsymbol{f}} = \kappa \left(M \zeta \mathbb{b}_\alpha \boldsymbol{J} \bar{\boldsymbol{f}} + N \mathbb{b}_\alpha \mathbf{1} \right).$$

This latter equation has the unique solution

$$\bar{\boldsymbol{f}} = \frac{\kappa N \mathbb{b}_\alpha}{1 - r \kappa M \zeta \mathbb{b}_\alpha} \mathbf{1} = \frac{\kappa N}{1 - (\alpha + r \kappa M) \zeta} \mathbf{1} = \kappa N \mathbb{b}_{\alpha + r\kappa M} \mathbf{1}.$$

We have proved:

THEOREM 5.2. *Let $\boldsymbol{\Phi}$ be a matrix with coefficients in $\mathbb{C}$, such that $k - \boldsymbol{\Phi}$ is invertible for all $k \geqslant l$ and let $\kappa > 0$ be such that we have (5.5). Consider the equation*

$$(\delta - \boldsymbol{\Phi}) \boldsymbol{f} = \boldsymbol{L} \boldsymbol{f} + \boldsymbol{g},$$

where $\boldsymbol{L}$ is a matrix with coefficients in $\mathbb{C}\{\{\zeta\}\}_{\geqslant 1}$ and $\boldsymbol{g} \in \mathbb{C}\{\{\zeta\}\}_{\geqslant l}$. This equation admits a unique solution $\boldsymbol{f} \in \mathbb{C}[[\zeta]]^r_{\geqslant l}$ and, assuming that $\boldsymbol{L} \trianglelefteq M \zeta \mathbb{b}_\alpha \boldsymbol{J}$ and $\boldsymbol{g} \trianglelefteq N \mathbb{b}_\alpha \mathbf{1}$, we have

$$\boldsymbol{f} \trianglelefteq \kappa N \mathbb{b}_{\alpha + r\kappa M} \mathbf{1}.$$

REMARK 5.3. Since substitutions of the form $\boldsymbol{f} = \boldsymbol{f}_0 + \cdots + \boldsymbol{f}_{l-1} \zeta^{l-1} + \tilde{\boldsymbol{f}}$ do not influence the radius of convergence of $\boldsymbol{g}$, it is important to notice that the constant κ may be chosen arbitrarily small, when taking l large enough. Consequently, we may compute majorants of the form $\boldsymbol{f} \trianglelefteq K \mathbb{b}_\beta \mathbf{1}$, with $\beta > \alpha$ as close to α as we wish.

5.3. Controlling the radius of convergence

When applying the majorant technique in a straightforward way, the convergence radius of the solution of the majorant equation is usually strictly smaller than the convergence radius of the actual solution. Theorem 1.1 implies that this cannot be avoided in general in the case of non-linear differential equations. Nevertheless, if the radius of convergence of the solution is empirically known, then arbitrarily good majorants can be obtained.

Consider an algebraic differential equation

$$\boldsymbol{f}' = \boldsymbol{P}(\boldsymbol{f}), \tag{5.9}$$

with $\boldsymbol{P} \in \mathbb{C}[f_1, \ldots, f_r]$ and initial condition $\boldsymbol{f}_0 = \boldsymbol{0}$. Given k, let $\boldsymbol{\varphi} = \boldsymbol{f}_{<k} = \boldsymbol{f}_0 + \cdots + \boldsymbol{f}_{k-1}\,\zeta^{k-1}$ and $\boldsymbol{\varepsilon} = \boldsymbol{f}_{\geqslant k} = \boldsymbol{f} - \boldsymbol{f}_{<k}$. We may rewrite (5.9) as an equation in $f_{\geqslant k}$.

$$\boldsymbol{\varepsilon}' = \boldsymbol{P}_{+\boldsymbol{\varphi}}(\boldsymbol{\varepsilon}),$$

where $\boldsymbol{P}_{+\boldsymbol{\varphi}} \in \mathbb{C}[\zeta][\varepsilon_1, \ldots, \varepsilon_r]$ and its coefficients are given by

$$\begin{aligned}
(\boldsymbol{P}_{+\boldsymbol{f}})_{\boldsymbol{0}} &= \boldsymbol{P}(\boldsymbol{\varphi}) - \boldsymbol{\varphi}' \\
(\boldsymbol{P}_{+\boldsymbol{f}})_{\boldsymbol{i}} &= \frac{1}{\boldsymbol{i}!}\frac{\partial^{\boldsymbol{i}}\boldsymbol{P}}{(\partial\boldsymbol{f})^{\boldsymbol{i}}}(\boldsymbol{\varphi}) \qquad (\boldsymbol{i} \neq \boldsymbol{0})
\end{aligned}$$

For each $\alpha \in \mathbb{R}^{>}$, let $M_\alpha \in \mathbb{R}^{\geqslant}$ be minimal such that

$$(\boldsymbol{P}_{+\boldsymbol{\varphi}})_{\boldsymbol{i}} \trianglelefteq \frac{M_\alpha}{1 - \alpha\,\zeta}\,\boldsymbol{1}$$

for all $\boldsymbol{i}$. The function $\alpha \mapsto \log M_\alpha$ is piecewise linear. For each α, the equation

$$\bar{\boldsymbol{\varepsilon}} = \frac{M_\alpha\,\zeta}{k\,(1 - \alpha\,\zeta)\,(1 - \boldsymbol{1} \cdot \bar{\boldsymbol{\varepsilon}})}\,\boldsymbol{1} \tag{5.10}$$

is a majorant equation for

$$\boldsymbol{\varepsilon} = \int \boldsymbol{P}_{+\boldsymbol{\varphi}}(\boldsymbol{\varepsilon})$$

on $\mathbb{C}[[\zeta]]^r_{\geqslant k}$. The solution to (5.10) is given by

$$\bar{\boldsymbol{\varepsilon}} = \frac{1 - \sqrt{1 - \frac{4\,r\,M_\alpha\,\zeta}{k\,(1 - \alpha\,\zeta)}}}{2\,r}\,\boldsymbol{1}.$$

This solution has radius of convergence

$$\rho_{\alpha,k} = \left(\alpha + \frac{4\,r\,M_\alpha}{k}\right)^{-1}.$$

Let α be such that $\rho_k = \rho_{\alpha,k}$ is maximal. We claim that ρ_k tends to the radius of convergence ρ of $\boldsymbol{f}$ when $k \to \infty$.

THEOREM 5.4. *With the above notations, the improved lower bounds ρ_k for the radius of convergence ρ of $\boldsymbol{f}$ tend to ρ when $k \to \infty$.*

PROOF. Given $\rho' < \rho$ and $\beta = (\rho^{-1} + (\rho')^{-1})/2$, there exists a constant M with

$$\boldsymbol{f} \trianglelefteq M\,\mathbb{b}_\beta\,\boldsymbol{1}.$$

In particular,

$$\boldsymbol{\varepsilon} \trianglelefteq M\,\mathbb{b}_\beta\,\boldsymbol{1}.$$

Now the radius of convergence of $(\boldsymbol{P}_{+\boldsymbol{f}})_{\boldsymbol{i}}$ is also $\geqslant \rho$ for all $\boldsymbol{i}$. Consequently, there exists a constant N with

$$(\boldsymbol{P}_{+\boldsymbol{f}})_{\boldsymbol{i}} \trianglelefteq N\, \mathbb{b}_\beta\, \mathbf{1}$$

for all $\boldsymbol{i}$. Since we also have

$$\begin{aligned}(\boldsymbol{P}_{+\boldsymbol{\varphi}})_{\mathbf{0}} &= \boldsymbol{P}_{+\boldsymbol{f}}(-\boldsymbol{\varepsilon}) + \boldsymbol{\varepsilon}' \\ (\boldsymbol{P}_{+\boldsymbol{\varphi}})_{\boldsymbol{i}} &= \frac{1}{\boldsymbol{i}!} \frac{\partial^{\boldsymbol{i}} \boldsymbol{P}_{+\boldsymbol{f}}}{(\partial \boldsymbol{f})^{\boldsymbol{i}}}(-\boldsymbol{\varepsilon}) \qquad (\boldsymbol{i} \neq \mathbf{0})\end{aligned}$$

it follows that

$$(\boldsymbol{P}_{+\boldsymbol{\varphi}})_{\boldsymbol{i}} \trianglelefteq \left[\sum_{1 \leqslant |\boldsymbol{j}| \leqslant d} \binom{\boldsymbol{j}}{\boldsymbol{i}} N\, \mathbb{b}_\beta\, (M\, \mathbb{b}_\beta)^{|\boldsymbol{j}|} + (M\, \mathbb{b}_\beta)'\right] \mathbf{1},$$

where d denotes the degree of $\boldsymbol{P}$. In other words, for a suitable constant K, which does not depend on k, we have

$$(\boldsymbol{P}_{+\boldsymbol{\varphi}})_{\boldsymbol{i}} \trianglelefteq K\, \mathbb{b}_\beta^{d+1}.$$

For $\beta < \alpha < (\rho')^{-1}$, it follows that

$$M_\alpha \leqslant \frac{K}{(1 - \beta/\alpha)^{d+1}}.$$

In particular, for all sufficiently large k, we have $\rho_{\alpha,k}^{-1} = \alpha + \frac{4\, r\, M_\alpha}{k} < (\rho')^{-1}$. Consequently, for all sufficiently large k, we have $\rho_k > \rho'$. Since is true for all $\rho' < \rho$, we conclude that ρ_k tends to ρ. □

6. CONVOLUTION PRODUCTS

6.1. Sharp power series

A *sharp power series* is a sum

$$f = \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} f_{;\boldsymbol{\epsilon}}\, \delta_{;\boldsymbol{\epsilon}},$$

where the $f_{;\boldsymbol{\epsilon}} \in \mathbb{C}[[\boldsymbol{\zeta}]]$ are such that $f_{;\boldsymbol{\epsilon}}$ does not depend on ζ_i whenever $\epsilon_i = 1$ and where $\delta_{;\boldsymbol{\epsilon}}$ is the Dirac distribution on those ζ_i with $\epsilon_i = 1$. We denote by $\mathbb{C}[[\boldsymbol{\zeta}]]^\sharp$ the set of sharp power series. The majorant relation $\trianglelefteq$ naturally extends to sharp power series $f, g \in \mathbb{C}[[\boldsymbol{\zeta}]]^\sharp$, by setting $f \trianglelefteq g$ if and only if $f_{;\boldsymbol{\epsilon}} \trianglelefteq g_{;\boldsymbol{\epsilon}}$ for all $\boldsymbol{\epsilon} \in \{0,1\}^n$. Given $\boldsymbol{\alpha} \in (\mathbb{R}^>)^n$, we define

$$\begin{aligned}\mathbb{b}_{\boldsymbol{\alpha}}^\sharp &= \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} \mathbb{b}_{\bar{\boldsymbol{\epsilon}} \times \boldsymbol{\alpha}}\, \delta_{;\boldsymbol{\epsilon}}; \\ \mathbb{z}_{\boldsymbol{\alpha}}^\sharp &= \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} \mathbb{z}_{\bar{\boldsymbol{\epsilon}} \times \boldsymbol{\alpha}}\, \delta_{;\boldsymbol{\epsilon}}.\end{aligned}$$

We will also write $\mathbb{b}^\sharp = \mathbb{b}_{\mathbf{1}}^\sharp$ and $\mathbb{z}^\sharp = \mathbb{z}_{\mathbf{1}}^\sharp$.

Let $\boldsymbol{\epsilon} \in \{0,1\}^n$ and denote $\bar{\boldsymbol{\epsilon}} = \mathbf{1} - \boldsymbol{\epsilon}$. Given $f \in \mathbb{C}[[\boldsymbol{\zeta}]]$, we will denote by $f_{\boldsymbol{\epsilon}}$ the result of the substitution of 0 for each ζ_i with $\epsilon_i = 0$ in f. In particular, if f is a sharp power series, then $f_{;\boldsymbol{\epsilon},\boldsymbol{\epsilon}} \in \mathbb{C}$ and $f_{;\boldsymbol{\epsilon},\bar{\boldsymbol{\epsilon}}} = f_{;\boldsymbol{\epsilon}}$. Given $\boldsymbol{\zeta} \in \mathbb{C}^n$ and $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_l \in \{0,1\}^n$ such that $\boldsymbol{\epsilon}_1 + \cdots + \boldsymbol{\epsilon}_l = \mathbf{1}$, we will often abbreviate $\boldsymbol{\zeta}_i = \boldsymbol{\zeta} \times \boldsymbol{\epsilon}_i$ and $\boldsymbol{\zeta}_{\bar{\imath}} = \boldsymbol{\zeta} \times \bar{\boldsymbol{\epsilon}_i}$ for $i = 1, \ldots, l$ (if the $\boldsymbol{\epsilon}_i$ are clear from the context). In particular, $\boldsymbol{\zeta} = \boldsymbol{\zeta}_1 + \cdots + \boldsymbol{\zeta}_l$. One should be careful not confuse $\boldsymbol{\zeta}_i$ with ζ_i. If $l = n$ and $\boldsymbol{\epsilon}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{\epsilon}_n = \boldsymbol{e}_n$, then $\boldsymbol{\zeta}_i = \zeta_i\, \boldsymbol{e}_i$ for all i.

6.2. Convolution products

For convergent power series f and g, the n-dimensional convolution product is defined by

$$(f * g)(\boldsymbol{\zeta}) = \int_{\mathbf{0}}^{\boldsymbol{\zeta}} f(\boldsymbol{\zeta} - \boldsymbol{\xi})\, g(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{6.1}$$

where

$$\int_{\mathbf{0}}^{\boldsymbol{\zeta}} = \int_0^{\zeta_1} \cdots \int_0^{\zeta_n}.$$

In particular, for all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}^n$ we have

$$\boldsymbol{\zeta}^{\boldsymbol{k}} * \boldsymbol{\zeta}^{\boldsymbol{l}} = (\boldsymbol{k} + \boldsymbol{l} + \mathbf{1})^{-\mathbf{1}} \binom{\boldsymbol{k} + \boldsymbol{l}}{\boldsymbol{k}}^{-1} \boldsymbol{\zeta}^{\boldsymbol{k} + \boldsymbol{l} + \mathbf{1}}.$$

This relation allows us to extend the definition of $*$ to $\mathbb{C}[[\boldsymbol{\zeta}]]$ in a coefficient-wise way:

$$(f * g)_{\boldsymbol{m}} = \sum_{\boldsymbol{k} + \boldsymbol{l} = \boldsymbol{m} - \mathbf{1}} f_{\boldsymbol{k}}\, g_{\boldsymbol{l}}\, \boldsymbol{m}^{-\mathbf{1}} \binom{\boldsymbol{m} - \mathbf{1}}{\boldsymbol{k}}^{-1}. \tag{6.2}$$

Given a function which is analytic at a point $\boldsymbol{\chi}$, we denote by $f_{+\boldsymbol{\chi}}$ the translate of f at $\boldsymbol{\chi}$, i.e. $f_{+\boldsymbol{\chi}}(\boldsymbol{\zeta}) = f(\boldsymbol{\chi} + \boldsymbol{\zeta})$. A convolution integral at a translated point $\boldsymbol{\chi} + \boldsymbol{\zeta}$ can be decomposed in 2^n parts:

$$\begin{aligned}(f * g)(\boldsymbol{\chi} + \boldsymbol{\zeta}) &= \sum_{\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \mathbf{1}} \int_{\boldsymbol{\chi}_1}^{\boldsymbol{\chi} + \boldsymbol{\zeta}_1} f(\boldsymbol{\chi} + \boldsymbol{\zeta} - \boldsymbol{\xi})\, g(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \\ &= \sum_{\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \mathbf{1}} (f_{+\boldsymbol{\zeta}_2} * g_{+\boldsymbol{\chi}_1})(\boldsymbol{\chi}_2 + \boldsymbol{\zeta}_1).\end{aligned} \tag{6.3}$$

REMARK 6.1. The formula (6.3) can be reinterpreted using shuffle products by interpreting the n-dimensional vector $\boldsymbol{\zeta}$ as a shuffle of $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ (and similarly for $\boldsymbol{\chi}$). Using "shuffle notation", the equation (6.3) becomes

$$(f * g)(\boldsymbol{\chi} + \boldsymbol{\zeta}) = \sum_{\binom{\boldsymbol{\chi}_1}{\boldsymbol{\zeta}_1} ш \binom{\boldsymbol{\chi}_2}{\boldsymbol{\zeta}_2} = \binom{\boldsymbol{\chi}}{\boldsymbol{\zeta}}} (f_{+\boldsymbol{\zeta}_2} * g_{+\boldsymbol{\chi}_1})(\boldsymbol{\chi}_2 + \boldsymbol{\zeta}_1).$$

More generally, given $\boldsymbol{\epsilon} \in \{0, 1\}$ and taking $\boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}_2 = \bar{\boldsymbol{\epsilon}}$, we define the *degenerated convolution* product $*_{\boldsymbol{\epsilon}}$ by

$$(f *_{\boldsymbol{\epsilon}} g)(\boldsymbol{\zeta}) = \int_{\mathbf{0}}^{\boldsymbol{\zeta}_1} f_{+\boldsymbol{\zeta}_2}(\boldsymbol{\zeta}_1 - \boldsymbol{\xi}_1)\, g_{+\boldsymbol{\zeta}_2}(\boldsymbol{\xi}_1)\, \mathrm{d}\boldsymbol{\xi}_1, \tag{6.4}$$

where the integral is taken over the $|\boldsymbol{\epsilon}|$-dimensional block $[\mathbf{0}, \boldsymbol{\zeta}_1]$. In particular, $\cdot = *_{\mathbf{0}}$ and $* = *_{\mathbf{1}}$. We have the following analogue for (6.3):

$$(f *_{\boldsymbol{\epsilon}} g)(\boldsymbol{\chi} + \boldsymbol{\zeta}) = \sum_{\substack{\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_3 = \mathbf{1} \\ \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}}} (f_{+\boldsymbol{\zeta}_2} *_{\boldsymbol{\epsilon}} g_{+\boldsymbol{\chi}_1})(\boldsymbol{\chi}_2 + \boldsymbol{\zeta}_1 + \boldsymbol{\chi}_3 + \boldsymbol{\zeta}_3). \tag{6.5}$$

The definitions of the convolution product and degenerate convolution products extend to the case when $f \in \mathbb{C}[[\boldsymbol{\zeta}]]^\sharp$ and $g \in \mathbb{C}[[\boldsymbol{\zeta}]]$:

$$f * g = \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} f_{;\boldsymbol{\epsilon}} *_{\bar{\boldsymbol{\epsilon}}} g_{\bar{\boldsymbol{\epsilon}}}. \tag{6.6}$$

For $\boldsymbol{\epsilon} \in \{0, 1\}^n$:

$$f *_{\boldsymbol{\epsilon}} g = \sum_{\boldsymbol{\epsilon}_2 \leqslant \bar{\boldsymbol{\epsilon}}} \left[\sum_{\boldsymbol{\epsilon}_1 \leqslant \boldsymbol{\epsilon}} f_{;\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2} *_{\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_1} g_{\bar{\boldsymbol{\epsilon}}_1} \right] \delta_{\boldsymbol{\epsilon}_2}. \tag{6.7}$$

6.3. Majorants for convolution products

Propositions 2.2 and 3.3 still hold if one replaces the componentwise product by any of the componentwise convolution products $*_{\boldsymbol{\epsilon}}$. More interesting explicit majorants follow below.

PROPOSITION 6.2. *Let $f, g \in \mathbb{R}^{\geqslant}[[\xi]]$ be two univariate power series. Then*

$$(f \circ \mathbb{z}) * (g \circ \mathbb{z}) \trianglelefteq (\xi^{n-1} (f * g)) \circ \mathbb{z}. \tag{6.8}$$

PROOF. By strong bilinearity, it suffices to prove (6.8) in the case when $f = \xi^p$ and $g = \xi^q$. Then (6.2) implies that

$$(\mathbb{z}^p * \mathbb{z}^q)_{\boldsymbol{k}+\boldsymbol{1}} = \sum_{\substack{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{k}\\ |\boldsymbol{i}|=p \wedge |\boldsymbol{j}|=q}} \frac{p!\, q!}{(\boldsymbol{k}+\boldsymbol{1})!}.$$

We also have

$$(\mathbb{z}^{p+q+n})_{\boldsymbol{k}+\boldsymbol{1}} = \frac{(p+q+n)!}{(\boldsymbol{k}+\boldsymbol{1})!}.$$

Since the cardinality of $\{(\boldsymbol{i}, \boldsymbol{j}) \colon \boldsymbol{i}+\boldsymbol{j}=\boldsymbol{k} \wedge |\boldsymbol{i}| = p \wedge |\boldsymbol{j}| = q\}$ is bounded by $\frac{(p+q+n)!}{(p+q+1)!}$, it follows that

$$\frac{(\mathbb{z}^p * \mathbb{z}^q)_{\boldsymbol{k}+\boldsymbol{1}}}{(\mathbb{z}^{p+q+n})_{\boldsymbol{k}+\boldsymbol{1}}} = \frac{p!\, q!}{(p+q+1)!}.$$

Since $\xi^p * \xi^q = \frac{p!\, q!}{(p+q+1)!}\, \xi^{p+q+1}$, it follows that

$$\mathbb{z}^p * \mathbb{z}^q \trianglelefteq (\xi^{n-1} (\xi^p * \xi^q)) \circ (\mathbb{z}),$$

as desired. □

PROPOSITION 6.3. *For each $p > 0$ there exists a constant K_p such that*

$$\mathbb{b}_{\boldsymbol{\alpha}} * \mathbb{b}_{\boldsymbol{\alpha}} \trianglelefteq \frac{K_1}{\boldsymbol{\alpha}^{\boldsymbol{1}}} \log (1 - \mathbb{z}_{\boldsymbol{\alpha}})^{-1}, \tag{6.9}$$

$$\mathbb{b}_{\boldsymbol{\alpha}}^p * \mathbb{b}_{\boldsymbol{\alpha}}^p \trianglelefteq \frac{K_p}{\boldsymbol{\alpha}^{\boldsymbol{1}}} \mathbb{b}_{\boldsymbol{\alpha}}^{p-1} \quad (p > 1) \tag{6.10}$$

for all $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$.

PROOF. We first observe that the general case reduces to the case when $\boldsymbol{\alpha} = \boldsymbol{1}$ using

$$[f \circ (\alpha_1 \zeta_1, \ldots, \alpha_n \zeta_n)] * [g \circ (\alpha_1 \zeta_1, \ldots, \alpha_n \zeta_n)] = \frac{1}{\boldsymbol{\alpha}^{\boldsymbol{1}}} (f * g) \circ (\alpha_1 \zeta_1, \ldots, \alpha_n \zeta_n).$$

Furthermore, setting $b^{[0]} = \log (1 - \xi)^{-1}$, $b^{[p]} = 1/(1 - \xi)^p$ for $p > 1$ and letting $k \to \infty$, we have

$$\begin{aligned}
(b^{[p]} * b^{[p]})_k &= \frac{1}{k+1} \sum_{i+j=k} \binom{k}{i}^{-1} \binom{i+p-1}{i} \binom{j+p-1}{j} \\
&\leqslant \frac{2}{k+1} \binom{k+p-1}{k} \sum_{i=0}^{\lceil k/2 \rceil} \binom{k}{i}^{-1} \binom{i+p-1}{i} \\
&= O(k^{p-2}) \left(1 + \frac{p}{k} + \frac{p(p+1)}{k(k-1)} + \cdots + \frac{p(p+1)\cdots(p+\lceil k/2 \rceil)}{k(k-1)\cdots(k-\lceil k/2 \rceil)}\right) \\
&= O(k^{p-2}).
\end{aligned}$$

Since $(k-n+1)^{p-2} \sim k^{p-2}$, $k^{-1} \sim b_k^{[0]}$ and $k^{p-2} \sim \frac{1}{(p-2)!} b_k^{[p-1]}$ for all $p > 1$, it follows that

$$(\xi^{n-1} b^{[p]} * b^{[p]})_k = O((b^{[p-1]})_k).$$

We conclude by (6.8). □

PROPOSITION 6.4. *For all $p, q > 0$ and $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$ we have*

$$\mathbb{b}_{\boldsymbol{\alpha}}^p * \mathbb{b}_{\boldsymbol{\alpha}}^q \trianglelefteqslant \frac{1}{\boldsymbol{\alpha}^{\mathbf{1}}} \mathbb{b}_{\boldsymbol{\alpha}}^{p+q-1} \tag{6.11}$$

PROOF. The final majoration (6.11) again only needs to be proved for $\boldsymbol{\alpha} = \mathbf{1}$. With the notations from the previous proposition, we have

$$\begin{aligned}\frac{(b^{[p]} * b^{[q]})_k}{b_k^{[p+q-1]}} &= \frac{1}{k+1} \sum_{i+j=k} \binom{k}{i}^{-1} \binom{i+p-1}{i} \binom{j+q-1}{j} \binom{k+p+q-2}{k}^{-1} \\ &= \frac{1}{k+1} \sum_{i+j=k} \binom{p+q-2}{p-1} \Big/ \binom{k+p+q-2}{i+p-1} \\ &\leqslant 1\end{aligned}$$

Hence

$$\xi^{n-1} b^{[p]} * b^{[q]} \trianglelefteq \xi^{n-1} b^{[p+q-1]} \trianglelefteq b^{[p+q-1]}$$

and the majoration again follows from (6.8). □

PROPOSITION 6.5. *Let $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3 \in \{0,1\}^n$ be such that $\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_3 = \mathbf{1}$. Then for all $p > 0$ and $\boldsymbol{\alpha} \in (\mathbb{R}^{>})^n$ we have*

$$\mathbb{b}_{\boldsymbol{\alpha}_{\bar{3}}}^p *_{\boldsymbol{\epsilon}_1} \mathbb{b}_{\boldsymbol{\alpha}_{\bar{2}}}^p \trianglelefteqslant \frac{1}{\boldsymbol{\alpha}^{\boldsymbol{\epsilon}_1}} \mathbb{b}_{\boldsymbol{\alpha}}^{p+q-1}.$$

PROOF. We may write

$$\begin{aligned}\mathbb{b}_{\boldsymbol{\alpha}_{\bar{3}}}^p &= (1 - \boldsymbol{\alpha}_1 \cdot \boldsymbol{\zeta} - \boldsymbol{\alpha}_2 \cdot \boldsymbol{\zeta})^{-p} = \sum_{k=0}^{\infty} \binom{k+p-1}{k} \mathbb{b}_{\boldsymbol{\alpha}_1}^{k+p} (\boldsymbol{\alpha}_2 \cdot \boldsymbol{\zeta})^k; \\ \mathbb{b}_{\boldsymbol{\alpha}_{\bar{2}}}^q &= (1 - \boldsymbol{\alpha}_1 \cdot \boldsymbol{\zeta} - \boldsymbol{\alpha}_3 \cdot \boldsymbol{\zeta})^{-q} = \sum_{l=0}^{\infty} \binom{l+q-1}{l} \mathbb{b}_{\boldsymbol{\alpha}_1}^{l+q} (\boldsymbol{\alpha}_3 \cdot \boldsymbol{\zeta})^l.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{b}_{\boldsymbol{\alpha}_{\bar{3}}}^p *_{\boldsymbol{\epsilon}_1} \mathbb{b}_{\boldsymbol{\alpha}_{\bar{2}}}^q &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+p-1}{k} \binom{l+q-1}{l} (\mathbb{b}_{\boldsymbol{\alpha}_1}^{k+p} *_{\boldsymbol{\epsilon}_1} \mathbb{b}_{\boldsymbol{\alpha}_1}^{l+q}) (\boldsymbol{\alpha}_2 \cdot \boldsymbol{\zeta})^k (\boldsymbol{\alpha}_3 \cdot \boldsymbol{\zeta})^l \\ &\trianglelefteqslant \frac{1}{\boldsymbol{\alpha}^{\boldsymbol{\epsilon}_1}} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+p-1}{k} \binom{l+q-1}{l} \mathbb{b}_{\boldsymbol{\alpha}_1}^{k+l+p+q-1} (\boldsymbol{\alpha}_2 \cdot \boldsymbol{\zeta})^k (\boldsymbol{\alpha}_3 \cdot \boldsymbol{\zeta})^l \\ &\trianglelefteqslant \frac{1}{\boldsymbol{\alpha}^{\boldsymbol{\epsilon}_1}} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l}{k} \binom{k+l+p+q-2}{k+l} \mathbb{b}_{\boldsymbol{\alpha}_1}^{k+l+p+q-1} (\boldsymbol{\alpha}_2 \cdot \boldsymbol{\zeta})^k (\boldsymbol{\alpha}_3 \cdot \boldsymbol{\zeta})^l \\ &= \frac{1}{\boldsymbol{\alpha}^{\boldsymbol{\epsilon}_1}} \mathbb{b}_{\boldsymbol{\alpha}}^{p+q-1},\end{aligned}$$

using (6.11) and the fact that $\binom{p+q-2}{p-1} \leqslant \binom{k+l+p+q-2}{k+p-1}$. □

7. UNIFORM MAJORANTS

7.1. Uniform majorants

Consider a compact subset $\mathcal{U}$ of $\mathbb{C}^n$ such that $[0, \chi_1] \times \cdots \times [0, \chi_n] \subseteq \mathcal{U}$ for every $\boldsymbol{\chi} \in \mathcal{U}$. Let $\boldsymbol{\rho} = \boldsymbol{\rho}_{\mathcal{U}}$ be the *multi-radius* of $\mathcal{U}$, so that each ρ_i is minimal with the property that $|\xi_i| \leqslant \rho_i$ for all $\xi \in \mathcal{U}$. Given $g \in \mathbb{R}^{\geqslant}[[\boldsymbol{\zeta}]]$, we say that f is *uniformly majored* by g on $\mathcal{U}$, and we write $f \trianglelefteqslant_{\mathcal{U}} g$, if $f_{+\boldsymbol{\chi}} \trianglelefteqslant g$ for all $\boldsymbol{\chi} \in \mathcal{U}$.

REMARK 7.1. Uniform majorants can be seen as a special form of *parametric majorants*. If $\mathcal{U}$ is a parameter space and $f(\boldsymbol{\zeta},\boldsymbol{\lambda})$ and $g(\boldsymbol{\zeta},\boldsymbol{\lambda})$ are analytic functions near $\mathbf{0}\times\mathcal{U}$, then we write $f \trianglelefteq^{\boldsymbol{\lambda}} g$, if $f(\boldsymbol{\zeta},\boldsymbol{\lambda}) \trianglelefteq g(\boldsymbol{\zeta},\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda}\in\mathcal{U}$. In the above case, we thus have

$$f(\boldsymbol{\zeta}) \trianglelefteq_{\mathcal{U}} g(\boldsymbol{\zeta}) \iff f(\boldsymbol{\chi}+\boldsymbol{\zeta}) \trianglelefteq^{\boldsymbol{\chi}} g(\boldsymbol{\zeta}).$$

PROPOSITION 7.2. *For all analytic functions f and g on $\mathcal{U}$ and $\bar{f},\bar{g}\in\mathbb{R}^{\geqslant}[[\boldsymbol{\zeta}]]$, we have*

$$f \trianglelefteq_{\mathcal{U}} \bar{f} \wedge g \trianglelefteq_{\mathcal{U}} \bar{g} \implies f+g \trianglelefteq_{\mathcal{U}} \bar{f}+\bar{g}\,; \tag{7.1}$$

$$f \trianglelefteq_{\mathcal{U}} \bar{f} \wedge g \trianglelefteq_{\mathcal{U}} \bar{g} \implies f g \trianglelefteq_{\mathcal{U}} \bar{f}\bar{g}\,; \tag{7.2}$$

$$f \trianglelefteq_{\mathcal{U}} \bar{f} \implies \partial_i f \trianglelefteq_{\mathcal{U}} \partial_i \bar{f} \quad (i=1,\ldots,n)\,; \tag{7.3}$$

$$\rho_i = 0 \wedge f \trianglelefteq_{\mathcal{U}} \bar{f} \implies \textstyle\int_i f \trianglelefteq_{\mathcal{U}} \int_i \bar{f} \quad (i=1,\ldots,n). \tag{7.4}$$

PROOF. The bounds apply pointwise. □

PROPOSITION 7.3. *For all analytic functions f and g on $\mathcal{U}$, all $\bar{f}$, $\bar{g}\in\mathbb{R}^{\geqslant}[[\boldsymbol{\zeta}]]$ and all $\boldsymbol{\epsilon}\in\{0,1\}^n$ we have*

$$f \trianglelefteq_{\mathcal{U}} \bar{f} \wedge g \trianglelefteq_{\mathcal{U}} \bar{g} \implies f *_{\boldsymbol{\epsilon}} g \trianglelefteq_{\mathcal{U}} \sum_{\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2=\boldsymbol{\epsilon}} \boldsymbol{\rho}^{\boldsymbol{\epsilon}_2}\, \bar{f} *_{\boldsymbol{\epsilon}_1} \bar{g}_{\bar{\boldsymbol{\epsilon}}_2}. \tag{7.5}$$

In particular,

$$f \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\epsilon}\times\boldsymbol{\alpha}} \wedge g \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\alpha}}^p \implies f *_{\boldsymbol{\epsilon}} g \trianglelefteq_{\mathcal{U}} (\mathbf{1}\div\boldsymbol{\alpha}+\boldsymbol{\rho})^{\boldsymbol{\epsilon}}\, \mathbb{b}_{\boldsymbol{\alpha}}^p. \tag{7.6}$$

If f is a sharp analytic function on $\mathcal{U}$, then we also have

$$f \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\alpha}}^{\sharp} \wedge g \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\alpha}}^p \implies f * g \trianglelefteq_{\mathcal{U}} (\mathbf{1}+\mathbf{1}\div\boldsymbol{\alpha}+\boldsymbol{\rho})^{\mathbf{1}}\, \mathbb{b}_{\boldsymbol{\alpha}}^p. \tag{7.7}$$

PROOF. In the equation (6.5), let $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$, $\boldsymbol{\epsilon}_3\in\{0,1\}^n$ be such that $\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2+\boldsymbol{\epsilon}_3=1$ and $\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2=\boldsymbol{\epsilon}$. Then we have

$$\begin{aligned}
&(f_{+\boldsymbol{\zeta}_2} *_{\boldsymbol{\epsilon}} g_{+\boldsymbol{\chi}_1})(\boldsymbol{\chi}_2+\boldsymbol{\zeta}_1+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3)\\
=\ & \int_{\mathbf{0}}^{\boldsymbol{\chi}_2}\int_{\mathbf{0}}^{\boldsymbol{\zeta}_1} f_{+\boldsymbol{\zeta}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\chi}_2+\boldsymbol{\zeta}_1-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\, g_{+\boldsymbol{\chi}_1+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2)\,\mathrm{d}\boldsymbol{\xi}_1\,\mathrm{d}\boldsymbol{\xi}_2\\
=\ & \int_{\mathbf{0}}^{\boldsymbol{\chi}_2}\int_{\mathbf{0}}^{\boldsymbol{\zeta}_1} f_{+\boldsymbol{\zeta}_2+\boldsymbol{\chi}_2-\boldsymbol{\xi}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\zeta}_1-\boldsymbol{\xi}_1)\, g_{+\boldsymbol{\chi}_1+\boldsymbol{\xi}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\xi}_1)\,\mathrm{d}\boldsymbol{\xi}_1\,\mathrm{d}\boldsymbol{\xi}_2
\end{aligned}$$

Now for fixed $\boldsymbol{\xi}_2$, the following majorations in $\boldsymbol{\zeta}$ hold:

$$\begin{aligned}
f_{+\boldsymbol{\zeta}_2+\boldsymbol{\chi}_2-\boldsymbol{\xi}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\zeta}_1) = f_{+\boldsymbol{\chi}_2-\boldsymbol{\xi}_2+\boldsymbol{\chi}_3}(\boldsymbol{\zeta}_1+\boldsymbol{\zeta}_2+\boldsymbol{\zeta}_3) &\trianglelefteq \bar{f}(\boldsymbol{\zeta})\,;\\
g_{+\boldsymbol{\chi}_1+\boldsymbol{\xi}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\zeta}_1) = g_{+\boldsymbol{\chi}_1+\boldsymbol{\xi}_2+\boldsymbol{\chi}_3}(\boldsymbol{\zeta}_1+\boldsymbol{\zeta}_3) &\trianglelefteq \bar{g}_{\bar{\boldsymbol{\epsilon}}_2}(\boldsymbol{\zeta}).
\end{aligned}$$

Consequently, (9.3) implies

$$\int_{\mathbf{0}}^{\boldsymbol{\zeta}_1} f_{+\boldsymbol{\zeta}_2+\boldsymbol{\chi}_2-\boldsymbol{\xi}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\zeta}_1-\boldsymbol{\xi}_1)\, g_{+\boldsymbol{\chi}_1+\boldsymbol{\xi}_2+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3}(\boldsymbol{\xi}_1) \trianglelefteq (\bar{f} *_{\boldsymbol{\epsilon}_1} \bar{g}_{\bar{\boldsymbol{\epsilon}}_2})(\boldsymbol{\zeta}).$$

Since this bound holds for all $\boldsymbol{\xi}_2\in[\mathbf{0},\boldsymbol{\chi}_2]$, we obtain

$$(f_{+\boldsymbol{\zeta}_2} *_{\boldsymbol{\epsilon}} g_{+\boldsymbol{\chi}_1})(\boldsymbol{\chi}_2+\boldsymbol{\zeta}_1+\boldsymbol{\chi}_3+\boldsymbol{\zeta}_3) \trianglelefteq (\bar{f} *_{\boldsymbol{\epsilon}_1} \bar{g}_{\bar{\boldsymbol{\epsilon}}_2})(\boldsymbol{\zeta})\,\mathrm{abs}(\boldsymbol{\chi})^{\boldsymbol{\epsilon}_2}.$$

Summing over all $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ with $\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2=\boldsymbol{\epsilon}$, this yields

$$(f *_{\boldsymbol{\epsilon}} g)(\boldsymbol{\chi}+\boldsymbol{\zeta}) \trianglelefteq \sum_{\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2=\boldsymbol{\epsilon}} (\bar{f} *_{\boldsymbol{\epsilon}_1} \bar{g}_{\bar{\boldsymbol{\epsilon}}_2})(\boldsymbol{\zeta})\,\mathrm{abs}(\boldsymbol{\chi})^{\boldsymbol{\epsilon}_2}.$$

Since $\mathrm{abs}(\boldsymbol{\xi}) \leqslant \boldsymbol{\rho}$ for every $\boldsymbol{\chi} \in \mathcal{U}$, we thus obtain

$$f *_{\boldsymbol{\epsilon}} g \trianglelefteq_{\mathcal{U}} \sum_{\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}} \boldsymbol{\rho}^{\boldsymbol{\epsilon}_2} \, \bar{f} *_{\boldsymbol{\epsilon}_1} \bar{g}_{\bar{\boldsymbol{\epsilon}}_2}.$$

This proves (7.5). From proposition 6.5, it also follows that

$$f \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\epsilon} \times \boldsymbol{\alpha}} \wedge g \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\alpha}}^p \Longrightarrow f *_{\boldsymbol{\epsilon}} g \trianglelefteq_{\mathcal{U}} \sum_{\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}} \frac{\boldsymbol{\rho}^{\boldsymbol{\epsilon}_2}}{\boldsymbol{\alpha}^{\boldsymbol{\epsilon}_1}} \mathbb{b}_{\boldsymbol{\alpha}}^p = (\mathbf{1} \div \boldsymbol{\alpha} + \boldsymbol{\rho})^{\boldsymbol{\epsilon}} \, \mathbb{b}_{\boldsymbol{\alpha}}^p.$$

This proves (7.6). We finally have

$$f * g = \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} f_{;\bar{\boldsymbol{\epsilon}}} *_{\boldsymbol{\epsilon}} g_{\boldsymbol{\epsilon}} \trianglelefteq_{\mathcal{U}} \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} (\mathbf{1} \div \boldsymbol{\alpha} + \boldsymbol{\rho})^{\boldsymbol{\epsilon}} \, \mathbb{b}_{\boldsymbol{\alpha}}^p = (\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \boldsymbol{\rho})^{\mathbf{1}} \, \mathbb{b}_{\boldsymbol{\alpha}}^p$$

for all $f \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\alpha}}^{\sharp}$ and $g \trianglelefteq_{\mathcal{U}} \mathbb{b}_{\boldsymbol{\alpha}}^p$. □

PROPOSITION 7.4. *Let* $\boldsymbol{\chi}, \boldsymbol{\delta} \in \mathbb{C}^n$, *let* $\mathcal{U} = [\mathbf{0}, \boldsymbol{\delta}]$ *and let* $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \{0, 1\}^n$ *be such that* $\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \mathbf{1}$. *Assume that* f *and* g *are analytic functions on* $\mathcal{U} + [\mathbf{0}, \boldsymbol{\chi}_2]$ *resp.* $\mathcal{U}_1 + [\boldsymbol{\chi}_1, \boldsymbol{\chi}]$, *such that*

$$\begin{aligned} f_{+\boldsymbol{\xi}_2} &\trianglelefteq_{\mathcal{U}} \bar{f}\,; \\ g_{+\boldsymbol{\chi}_1 + \boldsymbol{\xi}_2, \boldsymbol{\epsilon}_1} &\trianglelefteq_{\mathcal{U}} \bar{g}, \end{aligned}$$

for all $\boldsymbol{\xi}_2 \in [\mathbf{0}, \boldsymbol{\chi}_2]$. *Then the function*

$$h(\boldsymbol{\zeta}) = (f_{+\boldsymbol{\zeta}_2} * g_{+\boldsymbol{\chi}_1})(\boldsymbol{\zeta}_1 + \boldsymbol{\chi}_2)$$

is majored by

$$h \trianglelefteq_{\mathcal{U}} \mathrm{abs}(\boldsymbol{\chi}_2)^{\boldsymbol{\epsilon}_2} \sum_{\boldsymbol{\epsilon}_{11} + \boldsymbol{\epsilon}_{12} = \boldsymbol{\epsilon}_1} \mathrm{abs}(\boldsymbol{\delta})^{\boldsymbol{\epsilon}_{12}} \, \bar{f} *_{\boldsymbol{\epsilon}_{11}} \bar{g}_{\boldsymbol{\epsilon}_{11}}. \tag{7.8}$$

In particular, if $\bar{f} = \mathbb{b}_{\boldsymbol{\alpha}}$ *and* $\bar{g} = \mathbb{b}_{\boldsymbol{\alpha}}^p$, *then*

$$h \trianglelefteq \mathrm{abs}(\boldsymbol{\chi}_2)^{\boldsymbol{\epsilon}_2} \, (\mathbf{1} \div \boldsymbol{\alpha} + \mathrm{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_1} \, \mathbb{b}_{\boldsymbol{\alpha}}^p. \tag{7.9}$$

If f *is a sharp analytic function,* $\bar{f} = \mathbb{b}_{\boldsymbol{\alpha}}^{\sharp}$ *and* $\bar{g} = \mathbb{b}_{\boldsymbol{\alpha}}^p$, *then we also have*

$$h \trianglelefteq (\mathbf{1} + \mathrm{abs}(\boldsymbol{\chi}_2))^{\boldsymbol{\epsilon}_1} \, (\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \mathrm{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_2} \, \mathbb{b}_{\boldsymbol{\alpha}}^p. \tag{7.10}$$

PROOF. We have

$$\begin{aligned} (f_{+\boldsymbol{\zeta}_2} * g_{+\boldsymbol{\chi}_1})(\boldsymbol{\zeta}_1 + \boldsymbol{\chi}_2) &= \int_{\mathbf{0}}^{\boldsymbol{\chi}_2} \int_{\mathbf{0}}^{\boldsymbol{\zeta}_1} f_{+\boldsymbol{\zeta}_2}(\boldsymbol{\zeta}_1 - \boldsymbol{\xi}_1 + \boldsymbol{\chi}_2 - \boldsymbol{\xi}_2) \, g_{+\boldsymbol{\chi}_1}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2) \, \mathrm{d}\boldsymbol{\chi}_1 \, \mathrm{d}\boldsymbol{\chi}_2 \\ &= \int_{\mathbf{0}}^{\boldsymbol{\chi}_2} (f_{+\boldsymbol{\chi}_2 - \boldsymbol{\xi}_2} *_{\boldsymbol{\epsilon}_1} g_{+\boldsymbol{\chi}_1 + \boldsymbol{\xi}_2, \boldsymbol{\epsilon}_1})(\boldsymbol{\zeta}) \, \mathrm{d}\boldsymbol{\chi}_2. \end{aligned}$$

By our hypotheses and (7.5), the following uniform majoration holds for all $\boldsymbol{\xi}_2 \in [\mathbf{0}, \boldsymbol{\chi}_2]$:

$$f_{+\boldsymbol{\chi}_2 - \boldsymbol{\xi}_2} *_{\boldsymbol{\epsilon}_1} g_{+\boldsymbol{\chi}_1 + \boldsymbol{\xi}_2, \boldsymbol{\epsilon}_1} \trianglelefteq_{\mathcal{U}} \sum_{\boldsymbol{\epsilon}_{11} + \boldsymbol{\epsilon}_{12} = \boldsymbol{\epsilon}_1} \mathrm{abs}(\boldsymbol{\delta})^{\boldsymbol{\epsilon}_{12}} \, \bar{f} *_{\boldsymbol{\epsilon}_{11}} (\bar{g}_{\boldsymbol{\epsilon}_1})_{\overline{\boldsymbol{\epsilon}_{12}}}.$$

Since $(\bar{g}_{\boldsymbol{\epsilon}_1})_{\overline{\boldsymbol{\epsilon}_{12}}} = \bar{g}_{\boldsymbol{\epsilon}_1 \times \overline{\boldsymbol{\epsilon}_{12}}} = \bar{g}_{\boldsymbol{\epsilon}_{11}}$, the majoration (7.8) follows by integration over $\boldsymbol{\xi}_2$. The bound (7.9) follows from proposition 6.5 in a similar way as (7.6).

As to (7.10), let us fix an $\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2 \in \{0,1\}^n$ with $\hat{\boldsymbol{\epsilon}}_1 + \hat{\boldsymbol{\epsilon}}_2 = \mathbf{1}$. We first notice that

$$\begin{aligned} (f_{;\hat{\boldsymbol{\epsilon}}_1, +\boldsymbol{\zeta}_2} \delta_{;\hat{\boldsymbol{\epsilon}}_1} * g_{+\boldsymbol{\chi}_1})(\boldsymbol{\zeta}_1 + \boldsymbol{\chi}_2) &= (f_{;\hat{\boldsymbol{\epsilon}}_1, +\boldsymbol{\zeta}_2, \hat{\boldsymbol{\epsilon}}_2} *_{\hat{\boldsymbol{\epsilon}}_2} g_{+\boldsymbol{\chi}_1, \hat{\boldsymbol{\epsilon}}_2})(\boldsymbol{\zeta}_1 + \boldsymbol{\chi}_2) \\ &= (f_{;\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2, +\boldsymbol{\zeta}_{2\hat{2}}} *_{\hat{\boldsymbol{\epsilon}}_2} g_{\hat{\boldsymbol{\epsilon}}_2, +\boldsymbol{\chi}_{1\hat{2}}})(\boldsymbol{\zeta}_1 + \boldsymbol{\chi}_2) \\ &= (f_{;\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2, +\boldsymbol{\zeta}_{2\hat{2}}} *_{\hat{\boldsymbol{\epsilon}}_2} g_{\hat{\boldsymbol{\epsilon}}_2, +\boldsymbol{\chi}_{1\hat{2}}})(\boldsymbol{\zeta}_{1\hat{2}} + \boldsymbol{\chi}_{2\hat{2}}), \end{aligned}$$

where $\boldsymbol{\zeta}_{1\hat{2}} = \boldsymbol{\zeta}_{\boldsymbol{\epsilon}_1 \times \hat{\boldsymbol{\epsilon}}_2}$ and $\boldsymbol{\zeta}_{2\hat{2}} = \boldsymbol{\zeta}_{\boldsymbol{\epsilon}_2 \times \hat{\boldsymbol{\epsilon}}_2}$. Consequently, we may apply (7.9) to the lower dimensional case of series in $\boldsymbol{\zeta}_{\hat{2}} = \boldsymbol{\zeta}_{\hat{\boldsymbol{\epsilon}}_2}$. This yields

$$(f_{;\hat{\boldsymbol{\epsilon}}_1,+\boldsymbol{\zeta}_2}\, \delta_{;\hat{\boldsymbol{\epsilon}}_1} * g_{+\boldsymbol{\chi}_1})(\boldsymbol{\zeta}_1 + \boldsymbol{\chi}_2) \trianglelefteq \operatorname{abs}(\boldsymbol{\chi}_2)^{\boldsymbol{\epsilon}_{2\hat{2}}}\, (\mathbf{1} \div \boldsymbol{\alpha} + \operatorname{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_{1\hat{2}}}\, \mathbb{b}_{\boldsymbol{\alpha}}^p.$$

Summing over all $\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2 \in \{0,1\}^n$ with $\hat{\boldsymbol{\epsilon}}_1 + \hat{\boldsymbol{\epsilon}}_2 = \mathbf{1}$, we obtain

$$\begin{aligned} h &\trianglelefteq \sum_{\hat{\boldsymbol{\epsilon}}_1 + \hat{\boldsymbol{\epsilon}}_2 = \mathbf{1}} \operatorname{abs}(\boldsymbol{\chi}_2)^{\boldsymbol{\epsilon}_{2\hat{2}}}\, (\mathbf{1} \div \boldsymbol{\alpha} + \operatorname{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_{1\hat{2}}}\, \mathbb{b}_{\boldsymbol{\alpha}}^p \\ &= \sum_{\hat{\boldsymbol{\epsilon}}_{21} \leqslant \boldsymbol{\epsilon}_1} \sum_{\hat{\boldsymbol{\epsilon}}_{22} \leqslant \boldsymbol{\epsilon}_2} \operatorname{abs}(\boldsymbol{\chi}_2)^{\boldsymbol{\epsilon}_2 \times (\hat{\boldsymbol{\epsilon}}_{21} + \hat{\boldsymbol{\epsilon}}_{22})}\, (\mathbf{1} \div \boldsymbol{\alpha} + \operatorname{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_1 \times (\hat{\boldsymbol{\epsilon}}_{21} + \hat{\boldsymbol{\epsilon}}_{22})}\, \mathbb{b}_{\boldsymbol{\alpha}}^p \\ &= \sum_{\hat{\boldsymbol{\epsilon}}_{21} \leqslant \boldsymbol{\epsilon}_1} \sum_{\hat{\boldsymbol{\epsilon}}_{22} \leqslant \boldsymbol{\epsilon}_2} \operatorname{abs}(\boldsymbol{\chi}_2)^{\hat{\boldsymbol{\epsilon}}_{22}}\, (\mathbf{1} \div \boldsymbol{\alpha} + \operatorname{abs}(\boldsymbol{\delta}))^{\hat{\boldsymbol{\epsilon}}_{21}}\, \mathbb{b}_{\boldsymbol{\alpha}}^p \\ &= (\mathbf{1} + \operatorname{abs}(\boldsymbol{\chi}_2))^{\boldsymbol{\epsilon}_1}\, (\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \operatorname{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_2}\, \mathbb{b}_{\boldsymbol{\alpha}}^p. \end{aligned}$$

This proves (7.10). □

7.2. Majorant spaces

For fixed $p > 0$ and $\boldsymbol{\alpha} \in (\mathbb{R}^>)^n$, let $\operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ be the space of all analytic functions f on $\mathcal{U}$, such that there exists a majoration $f \trianglelefteq_{\mathcal{U}} c\, \mathbb{b}_{\boldsymbol{\alpha}}^p$ for some $c > 0$. We call $\operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ a *majorant space* for the *majorant norm* $\|\cdot\|_{\boldsymbol{\alpha},p} = \|\cdot\|_{\mathcal{U},\boldsymbol{\alpha},p}$ given by

$$\begin{aligned} \|f\|_{\boldsymbol{\alpha},p} &= \min\{c \in \mathbb{R}^{\geqslant} : f \trianglelefteq_{\mathcal{U}} c\, \mathbb{b}_{\boldsymbol{\alpha}}^p\} \\ &= \sup\,[f_{\boldsymbol{k}} / (\mathbb{b}_{\boldsymbol{\alpha}}^p)_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}^n\}. \end{aligned}$$

This norm may be extended to $\operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}^{\sharp} = \{f \in \operatorname{An}(\mathcal{U}) : \forall \boldsymbol{\epsilon} \in \{0,1\}^n : f_{;\boldsymbol{\epsilon}} \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}\}$ by

$$\|f\|_{\boldsymbol{\alpha},p} = \max\{\|f_{;\boldsymbol{\epsilon}}\|_{\boldsymbol{\alpha},p} : \boldsymbol{\epsilon} \in \{0,1\}^n\}.$$

The following properties directly follow from the previous propositions:

PROPOSITION 7.5.

a) *For all $f, g \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ we have*

$$\|f + g\|_{\boldsymbol{\alpha},p} \leqslant \|f\|_{\boldsymbol{\alpha},p} + \|g\|_{\boldsymbol{\alpha},p}. \tag{7.11}$$

b) *For all $f \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ and $g \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\beta},1}$ with $\beta_1 < \alpha_1, \ldots, \beta_n < \alpha_n$ we have*

$$\|f g\|_{\boldsymbol{\alpha},p} \leqslant (\mathbf{1} - \boldsymbol{\beta} \div \boldsymbol{\alpha})^{-\mathbf{1}}\, \|f\|_{\boldsymbol{\alpha},p}\, \|g\|_{\boldsymbol{\beta},1}. \tag{7.12}$$

c) *For all $f \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ and $g \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},q}$ with $q > 0$ we have*

$$\|f g\|_{\boldsymbol{\alpha},p+q} \leqslant \|f\|_{\boldsymbol{\alpha},p}\, \|g\|_{\boldsymbol{\alpha},q}. \tag{7.13}$$

d) *For all $f \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ and $i \in \{1, \ldots, n\}$ we have*

$$\|\partial_i f\|_{\boldsymbol{\alpha},p+1} \leqslant p\, \alpha_i\, \|f\|_{\boldsymbol{\alpha},p}. \tag{7.14}$$

e) *For all $f \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ and $i \in \{1, \ldots, n\}$, such that $p > 0$ and $\rho_i = 0$, we have*

$$\left\|\textstyle\int_i f\right\|_{\boldsymbol{\alpha},p-1} \leqslant \frac{1}{(p-1)\, \alpha_i}\, \|f\|_{\boldsymbol{\alpha},p}. \tag{7.15}$$

f) *For all $f \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},1}^{\sharp}$ and $g \in \operatorname{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$, we have*

$$\|f * g\|_{\boldsymbol{\alpha},p} \leqslant (\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \boldsymbol{\rho})^{\mathbf{1}}\, \|f\|_{\boldsymbol{\alpha},1}\, \|g\|_{\boldsymbol{\alpha},p}. \tag{7.16}$$

PROPOSITION 7.6. *With the notations from proposition 7.4 and allowing f to be sharp, assume that the suprema*

$$\begin{aligned} F &= \sup_{\boldsymbol{\xi}_2 \in [\mathbf{0}, \boldsymbol{\chi}_2]} \|f_{+\boldsymbol{\xi}_2}\|_{\mathcal{U},\boldsymbol{\alpha},1}; \\ G &= \sup_{\boldsymbol{\xi}_2 \in [\mathbf{0}, \boldsymbol{\chi}_2]} \|g_{+\boldsymbol{\chi}_1+\boldsymbol{\xi}_2,\bar{\boldsymbol{\epsilon}}_2}\|_{\mathcal{U},\boldsymbol{\alpha},p} \end{aligned}$$

exist. Then $h \in \mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},p}$ and

$$\|h\|_{\mathcal{U},\boldsymbol{\alpha},p} \leqslant (\mathbf{1} + \mathrm{abs}(\boldsymbol{\chi}_2))^{\boldsymbol{\epsilon}_2} (\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \mathrm{abs}(\boldsymbol{\delta}))^{\boldsymbol{\epsilon}_1} F G. \tag{7.17}$$

7.3. Local resolution of convolution p.d.e.s

Let $\mathcal{U}$ be a compact subset of the hyperplane $\zeta_n = 0$ and let $\boldsymbol{\rho} \in (\mathbb{R}^{\geqslant})^n$ be its multi-radius (so that $\rho_n = 0$). We recall that $\mathrm{An}(\mathcal{U})$ denotes the set of analytic functions on $\mathcal{U}$ (so that each such function is analytic on a small n-dimensional neighbourhood of $\mathcal{U}$). We denote by $\mathrm{An}(\mathcal{U})_{\overline{\boldsymbol{e}_n}}$ the subset of $\mathrm{An}(\mathcal{U})$ of functions which do not depend on ζ_n. For each φ and $i \in \{1, \ldots, n\}$, we also define $\varphi * \partial_i$ to be the operator which sends f to $\varphi * (\partial_i f)$.

Consider the linear convolution p.d.e.

$$(\varphi_1 \partial_1 + \cdots + \varphi_n \partial_n + \psi_1 * \partial_1 + \cdots + \psi_n * \partial_n) f = g, \tag{7.18}$$

where

- $\varphi_1, \ldots, \varphi_n, \varphi_n^{-1} \in \mathrm{An}(\mathcal{U})_{\varepsilon \boldsymbol{\alpha},1}$ for some $0 < \varepsilon < 1$.
- $\psi_1, \ldots, \psi_n \in \mathrm{An}(\mathcal{U})^\sharp_{\boldsymbol{\alpha},1}$ and $\psi_{1;\mathbf{1}} = \cdots = \psi_{n;\mathbf{1}} = 0$.
- $g \in \mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$.

We will show that (7.18) admits a unique convergent solution in $\mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$ for any convergent initial condition $f_0 \in \mathrm{An}(\mathcal{U})_{\overline{\boldsymbol{e}_n},\boldsymbol{\alpha},1}$ on $\mathcal{U}$, provided that $\boldsymbol{\alpha}$ is sufficiently large and $\boldsymbol{\rho}$ sufficiently small. More precisely:

THEOREM 7.7. *Let $\Phi_i = \|\varphi_i\|_{\varepsilon\boldsymbol{\alpha},1}$, $\Psi_i = \|\psi_i\|_{\boldsymbol{\alpha},1}$ $(i = 1, \ldots, n)$ and $\Upsilon = \|\varphi_n^{-1}\|_{\varepsilon\boldsymbol{\alpha},1}$. Denote*

$$\begin{aligned} \kappa_1 &= (1-\varepsilon)^{-n} (\alpha_1 \Phi_1 + \cdots + \alpha_{n-1} \Phi_{n-1}) \\ \kappa_2 &= ((\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \boldsymbol{\rho})^{\mathbf{1}} - 1) (\alpha_1 \Psi_1 + \cdots + \alpha_n \Psi_n) \\ \kappa &= (1-\varepsilon)^{-n} \Upsilon (\kappa_1 + \kappa_2) \end{aligned}$$

and assume that $\kappa < \alpha_n$. Then (29) admits a unique solution in $\mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$ for any initial condition $f_0 \in \mathrm{An}(\mathcal{U})_{\overline{\boldsymbol{e}_n},\boldsymbol{\alpha},1}$.

PROOF. Consider the linear operator L on $\mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$ defined by

$$L = \varphi_n^{-1} (\varphi_1 \partial_1 + \cdots + \varphi_{n-1} \partial_{n-1} + \psi_1 * \partial_1 + \cdots + \psi_n * \partial_n),$$

so that (7.18) can be rewritten as

$$\partial_n f = \varphi_n^{-1} g - L f.$$

Given $f \in \mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$ with $\|f\|_{\boldsymbol{\alpha},1} = 1$, propositions 7.5($b$) and 7.5($d$) imply

$$\begin{aligned} (\varphi_1 \partial_1 + \cdots + \varphi_{n-1} \partial_{n-1}) f &\trianglelefteqslant (\Phi_1 \mathbb{b}_{\varepsilon\boldsymbol{\alpha}}) (\alpha_1 \mathbb{b}_\alpha^2) + \cdots + (\Phi_{n-1} \mathbb{b}_{\varepsilon\boldsymbol{\alpha}}) (\alpha_{n-1} \mathbb{b}_\alpha^2) \\ &\trianglelefteqslant (1-\varepsilon)^{-n} (\alpha_1 \Phi_1 + \cdots + \alpha_{n-1} \Phi_{n-1}) \mathbb{b}_\alpha^2 \\ &= \kappa_1 \mathbb{b}_{\boldsymbol{\alpha}}^2 \end{aligned}$$

and proposition 7.5(f) entails

$$\begin{aligned}(\psi_1 * \partial_1 + \cdots + \psi_n * \partial_n) f &\trianglelefteq ((\mathbf{1} + \mathbf{1} \div \boldsymbol{\alpha} + \boldsymbol{\rho})^{\mathbf{1}} - 1)(\alpha_1 \Psi_1 + \cdots + \alpha_n \Psi_n) \mathbb{b}_{\boldsymbol{\alpha}}^2 \\ &= \kappa_2 \mathbb{b}_{\boldsymbol{\alpha}}^2,\end{aligned}$$

so that

$$L f \trianglelefteq \kappa \mathbb{b}_{\boldsymbol{\alpha}}^2.$$

Given $f \in \mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},2}$ with $\|f\|_{\boldsymbol{\alpha},2} = 1$, we also have

$$\textstyle\int_n f \trianglelefteq \frac{1}{\alpha_n} \mathbb{b}_{\boldsymbol{\alpha}}.$$

Consequently, for $f \in \mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$ we obtain

$$\|\textstyle\int_n L f\|_{\boldsymbol{\alpha},1} \leqslant \frac{\kappa}{\alpha_n} \|f\|_{\boldsymbol{\alpha},1}.$$

In other words, the operator $\int_n L$ is contracting on $\mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$, since $\kappa < \alpha_n$.

Given an initial condition $f_0 \in \mathrm{An}(\mathcal{U})_{\overline{\boldsymbol{e}_n}, \boldsymbol{\alpha},1}$, let us now consider the operator

$$K: h \longmapsto f_0 + \textstyle\int_n \varphi_n^{-1} g - \int_n L h.$$

By what precedes, the operator K is again contracting, whence it admits a unique fixed point f in $\mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$, which is the solution to (7.18), and which satisfies

$$\|f\|_{\boldsymbol{\alpha},1} \leqslant \frac{1}{1 - \frac{\kappa}{\alpha_n}} \|f_0 + \textstyle\int_n \varphi_n^{-1} g\|_{\boldsymbol{\alpha},1}.$$

Since the coefficient $(K h)_0$ of ζ_n^0 in $K h$ equals f_0 for any $h \in \mathrm{An}(\mathcal{U})_{\boldsymbol{\alpha},1}$, the solution f also satisfies the initial condition.

In order to see that f is the unique solution to (7.18) which satisfies the initial condition, let $\tilde{f}$ be another such solution and assume that $h = \tilde{f} - f \neq 0$. Let $\boldsymbol{v}$ be the lexicographic valuation of h (i.e. $v_n > 0$ is the valuation in ζ_n of h, and v_{n-1} the valuation in ζ_{n-1} of the coefficient h_{v_n} of $\zeta_n^{v_n}$ in h, and so on). Taking the coefficient of $\zeta_n^{v_n}$ in the equation $\partial_n h + L h = 0$, we obtain

$$h_{v_n} + \varphi_{n,0}^{-1} (\psi_{n; \geqslant \boldsymbol{e}_n, 0} *_{\overline{\boldsymbol{e}_n}} h_{v_n}) = 0,$$

where

$$\psi_{n; \geqslant \boldsymbol{e}_n} = \sum_{\boldsymbol{\epsilon} \geqslant \boldsymbol{e}_n} \psi_{n; \boldsymbol{\epsilon}} \delta_{; \boldsymbol{\epsilon}}.$$

But $(h_{\nu_n})_{\boldsymbol{v} - v_n \boldsymbol{e}_n} \neq 0$ and $(\varphi_{n,0}^{-1} (\psi_{n; \geqslant \boldsymbol{e}_n, 0} *_{\overline{\boldsymbol{e}_n}} h_{v_n}))_{\boldsymbol{v} - v_n \boldsymbol{e}_n} = 0$, since $\psi_{n; \mathbf{1}} = 0$ and $\boldsymbol{v}$ is the lexicographical valuation of h. This contradiction proves the uniqueness of the solution. □

8. Majorants for integral transforms

8.1. Majorants for parametric integrals

Let $\boldsymbol{T}$ be a compact subset of a real analytic variety and consider an analytic parameterization

$$\begin{aligned}\boldsymbol{\varphi}: \mathbb{C}^n \times \boldsymbol{T} &\rightarrow \mathbb{C}^n \\ (\boldsymbol{\zeta}, \boldsymbol{t}) &\mapsto \boldsymbol{\varphi}(\boldsymbol{\zeta}, \boldsymbol{t})\end{aligned}$$

which is linear in $\boldsymbol{\zeta}$. We may naturally associate a functional $f \mapsto I_{\boldsymbol{\varphi}} f$ to $\boldsymbol{\varphi}$ by

$$(I_{\boldsymbol{\varphi}} f)(\boldsymbol{\zeta}) = \int_{\boldsymbol{t} \in \boldsymbol{\mathcal{T}}} f(\boldsymbol{\varphi}(\boldsymbol{\zeta}, \boldsymbol{t}))\, \mathrm{d}\boldsymbol{t}.$$

More precisely, this functional is defined coefficient-wise on $\mathbb{C}[[\boldsymbol{\zeta}]]$ by

$$(I_{\boldsymbol{\varphi}} f)_{\boldsymbol{k}} = \int_{\boldsymbol{t} \in \boldsymbol{\mathcal{T}}} (I_{\boldsymbol{\varphi}, \boldsymbol{t}} f)_{\boldsymbol{k}}\, \mathrm{d}\boldsymbol{t}, \tag{8.1}$$

where $(I_{\boldsymbol{\varphi}, \boldsymbol{t}} f)(\boldsymbol{\zeta}) = f(\boldsymbol{\varphi}(\boldsymbol{\zeta}, \boldsymbol{t}))$. Let us study the growth rate of the coefficients of $I_{\boldsymbol{\varphi}} f$ as a function of the growth rate of the coefficients of f. We will denote

$$\begin{aligned} \boldsymbol{\mathcal{B}}_{\boldsymbol{\alpha}, r} &= \{\boldsymbol{\zeta} \in \mathbb{C}^n : \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{\zeta}) < r\}\,; \\ \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}, r} &= \{\boldsymbol{\zeta} \in \mathbb{C}^n : \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{\zeta}) \leqslant r\} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{B}}_{\boldsymbol{\alpha}} &= \boldsymbol{\mathcal{B}}_{\boldsymbol{\alpha}, 1}\,; \\ \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}} &= \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}, 1} \end{aligned}$$

for each $\boldsymbol{\alpha} \in (\mathbb{R}^{\geqslant})^n$ and $r \in \mathbb{R}^{>}$.

LEMMA 8.1. *Let $\boldsymbol{\alpha}$ be such that*

$$\{\boldsymbol{\varphi}(\boldsymbol{\zeta}, \boldsymbol{t}) : \boldsymbol{\zeta} \in \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}} \wedge \boldsymbol{t} \in \boldsymbol{\mathcal{T}}\} \subseteq \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}.$$

Then

$$f \trianglelefteq \mathbb{b}_{\boldsymbol{\alpha}}^p \implies I_{\boldsymbol{\varphi}} f \trianglelefteq \operatorname{Vol}(\boldsymbol{\mathcal{T}})\, \mathbb{b}_{\boldsymbol{\alpha}}^p$$

for all $f \in \mathbb{C}[[\boldsymbol{\zeta}]]$ and $p \in \mathbb{N}$.

PROOF. Let $\boldsymbol{t} \in \boldsymbol{\mathcal{T}}$ be fixed and let us show that

$$f \trianglelefteq \mathbb{b}_{\boldsymbol{\alpha}}^p \implies I_{\boldsymbol{\varphi}, \boldsymbol{t}} f \trianglelefteq \mathbb{b}_{\boldsymbol{\alpha}}^p. \tag{8.2}$$

This will clearly imply the lemma, because of (8.1).

Since the mapping $\boldsymbol{\varphi}(\boldsymbol{\zeta}, \boldsymbol{t})$ is linear in $\boldsymbol{\zeta}$, there exists a matrix $\boldsymbol{M}$ with $\boldsymbol{\varphi}(\boldsymbol{\zeta}, \boldsymbol{t}) = \boldsymbol{M}\boldsymbol{\zeta}$. Now by the first condition of the lemma, we have

$$\{\boldsymbol{M}\boldsymbol{\zeta} : \boldsymbol{\zeta} \in \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}\} \subseteq \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}. \tag{8.3}$$

We claim that this equation is equivalent to

$$\boldsymbol{\alpha} \cdot (\operatorname{abs}(\boldsymbol{M})\, \boldsymbol{e}_i) \leqslant \alpha_i \quad (i = 1, \ldots, n). \tag{8.4}$$

Indeed, (8.3) $\Rightarrow$ (8.4) follows by taking $\boldsymbol{\zeta} = \boldsymbol{e}_i / \alpha_i$ $(i = 1, \ldots, n)$ and using the facts that $\boldsymbol{\zeta} \in \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}, r} \Leftrightarrow \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{\zeta}) \leqslant r$ and $\operatorname{abs}(\boldsymbol{M}\boldsymbol{e}_i) = \operatorname{abs}(\boldsymbol{M})\, \boldsymbol{e}_i$. Inversely, if (8.4) and $\boldsymbol{\zeta} \in \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}$, then

$$\begin{aligned} \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{M}\boldsymbol{\zeta}) &= \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{M}(\zeta_1 \boldsymbol{e}_1 + \cdots + \zeta_n \boldsymbol{e}_n)) \\ &\leqslant |\zeta_1|\, \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{M}\boldsymbol{e}_1) + \cdots + |\zeta_n|\, \boldsymbol{\alpha} \cdot \operatorname{abs}(\boldsymbol{M}\boldsymbol{e}_n) \\ &\leqslant \alpha_1 |\zeta_1| + \cdots + \alpha_n |\zeta_n| \leqslant 1, \end{aligned}$$

so that $\operatorname{abs}(\boldsymbol{M}\boldsymbol{\zeta}) \in \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}$ and $\boldsymbol{M}\boldsymbol{\zeta} \in \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}$. In particular, we notice that (8.4) whence (8.3) is satisfied for $\boldsymbol{M}$ if and only if it is satisfied for $\operatorname{abs}(\boldsymbol{M})$.

Notice also that the condition (8.4) is in it's turn equivalent to the condition

$$\boldsymbol{\alpha} \cdot (\operatorname{abs}(\boldsymbol{M})\, \boldsymbol{\zeta}) \trianglelefteq \boldsymbol{\alpha} \cdot \boldsymbol{\zeta}, \tag{8.5}$$

when interpreting $\boldsymbol{\alpha}\cdot(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta})$ and $\boldsymbol{\alpha}\cdot\boldsymbol{\zeta}$ as (linear) series in $\boldsymbol{\zeta}$. Finally, when interpreting $\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta}$ as an element of $\mathbb{C}[[\boldsymbol{\zeta}]]^n$, this latter condition is equivalent to

$$\mathbb{b}_{\boldsymbol{\alpha}}\circ(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta})\trianglelefteq\mathbb{b}_{\boldsymbol{\alpha}}, \tag{8.6}$$

since $\mathbb{b}_{\boldsymbol{\alpha}}\circ(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta})=(1-\boldsymbol{\alpha}\cdot(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta}))^{-1}$ and $(1-\boldsymbol{\alpha}\cdot\boldsymbol{\zeta})^{-1}=\mathbb{b}_{\boldsymbol{\alpha}}$.

Assume now that we have (8.6) and $f\trianglelefteq\mathbb{b}_{\boldsymbol{\alpha}}^p$. Then

$$\begin{aligned}
I_{\boldsymbol{\varphi},\boldsymbol{t}}f &= f(\boldsymbol{M}\boldsymbol{\zeta})\\
&= \sum_{\boldsymbol{k}} f_{\boldsymbol{k}}\,(\boldsymbol{M}\boldsymbol{\zeta})^{\boldsymbol{k}}\\
&\trianglelefteq \sum_{\boldsymbol{k}} |f_{\boldsymbol{k}}|\operatorname{abs}(\boldsymbol{M}\boldsymbol{\zeta})^{\boldsymbol{k}}\\
&\trianglelefteq \sum_{\boldsymbol{k}} (\mathbb{b}_{\boldsymbol{\alpha}}^p)_{\boldsymbol{k}}\operatorname{abs}(\boldsymbol{M}\boldsymbol{\zeta})^{\boldsymbol{k}}\\
&= \mathbb{b}_{\boldsymbol{\alpha}}^p\circ(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta})\\
&\trianglelefteq \mathbb{b}_{\boldsymbol{\alpha}}^p.
\end{aligned}$$

This proves (8.2) and we completed the proof of the lemma. □

8.2. Majorants for integral transforms

More generally, consider two analytic parameterizations $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$: $\mathbb{C}^n\times\boldsymbol{\mathcal{T}}\to\mathbb{C}^n$ which are linear in the first variable. Given two power series $f,g\in\mathbb{C}[[\boldsymbol{\zeta}]]$, we define $J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)$ by

$$J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)(\boldsymbol{\zeta})=\int_{\boldsymbol{t}\in\boldsymbol{\mathcal{T}}} g(\boldsymbol{\psi}(\boldsymbol{\zeta},\boldsymbol{t}))\,f(\boldsymbol{\varphi}(\boldsymbol{\zeta},\boldsymbol{t}))\,\mathrm{d}\boldsymbol{t},$$

if f and g are convergent and coefficient-wise by

$$J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)_{\boldsymbol{k}}=\int_{\boldsymbol{t}\in\boldsymbol{\mathcal{T}}} J_{\boldsymbol{\psi},\boldsymbol{\varphi},\boldsymbol{t}}(g,f)_{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{t}, \tag{8.7}$$

in the general case, where $(J_{\boldsymbol{\psi},\boldsymbol{\varphi},\boldsymbol{t}}(g,f)(\boldsymbol{\zeta})=g(\boldsymbol{\psi}(\boldsymbol{\zeta},\boldsymbol{t})\,f(\boldsymbol{\varphi}(\boldsymbol{\zeta},\boldsymbol{t}))$. For fixed g, the mapping $f\mapsto J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)$ is a linear integral transformation. We have

LEMMA 8.2. *Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be such that $\beta_1<\alpha_1,\ldots,\beta_n<\alpha_n$ and*

$$\begin{aligned}
\{\boldsymbol{\psi}(\boldsymbol{\zeta},\boldsymbol{t})\colon \boldsymbol{\zeta}\in\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\beta}}\wedge\boldsymbol{t}\in\boldsymbol{\mathcal{T}}\} &\subseteq \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\gamma}};\\
\{\boldsymbol{\varphi}(\boldsymbol{\zeta},\boldsymbol{t})\colon \boldsymbol{\zeta}\in\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}\wedge\boldsymbol{t}\in\boldsymbol{\mathcal{T}}\} &\subseteq \bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}.
\end{aligned}$$

Then

$$g\trianglelefteq\mathbb{b}_{\boldsymbol{\gamma}}\wedge f\trianglelefteq\mathbb{b}_{\boldsymbol{\alpha}}^p\implies J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)\trianglelefteq(\mathbf{1}-\boldsymbol{\beta}\div\boldsymbol{\alpha})^{-\mathbf{1}}\operatorname{Vol}(\boldsymbol{\mathcal{T}})\,\mathbb{b}_{\boldsymbol{\alpha}}^p$$

for all $f,g\in\mathbb{C}[[\boldsymbol{\zeta}]]$ and $p\in\mathbb{N}$.

PROOF. For fixed $\boldsymbol{t}\in\boldsymbol{\mathcal{T}}$, we claim that

$$g\trianglelefteq\mathbb{b}_{\boldsymbol{\gamma}}\wedge f\trianglelefteq\mathbb{b}_{\boldsymbol{\alpha}}^p\implies J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)\trianglelefteq(\mathbf{1}-\boldsymbol{\beta}\div\boldsymbol{\alpha})^{-\mathbf{1}}\,\mathbb{b}_{\boldsymbol{\alpha}}^p. \tag{8.8}$$

This will clearly implies the lemma because of (8.7).

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be such that $\boldsymbol{\varphi}(\boldsymbol{\zeta},\boldsymbol{t})=\boldsymbol{M}\boldsymbol{\zeta}$ and $\boldsymbol{\psi}(\boldsymbol{\zeta},\boldsymbol{t})=\boldsymbol{N}\boldsymbol{\zeta}$. In a similar way as in the proof of lemma 8.1, we have

$$\begin{aligned}
\mathbb{b}_{\boldsymbol{\alpha}}\circ(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta}) &\trianglelefteq \mathbb{b}_{\boldsymbol{\alpha}};\\
\mathbb{b}_{\boldsymbol{\gamma}}\circ(\operatorname{abs}(\boldsymbol{N})\,\boldsymbol{\zeta}) &\trianglelefteq \mathbb{b}_{\boldsymbol{\beta}}
\end{aligned}$$

and, by lemma 2.4,

$$\begin{array}{rcl} J_{\boldsymbol{\psi},\boldsymbol{\varphi},\boldsymbol{t}}(g,f) & = & g(\boldsymbol{N}\boldsymbol{\zeta})\,f(\boldsymbol{M}\boldsymbol{\zeta}) \\ & \trianglelefteq & [\mathbb{b}_{\boldsymbol{\gamma}}\circ(\operatorname{abs}(\boldsymbol{N})\,\boldsymbol{\zeta})]\,[\mathbb{b}_{\boldsymbol{\alpha}}^p\circ(\operatorname{abs}(\boldsymbol{M})\,\boldsymbol{\zeta})] \\ & \trianglelefteq & \mathbb{b}_{\boldsymbol{\beta}}\,\mathbb{b}_{\boldsymbol{\alpha}}^p \\ & \trianglelefteq & (\mathbf{1}-\boldsymbol{\beta}\div\boldsymbol{\alpha})^{-\mathbf{1}}\,\mathbb{b}_{\boldsymbol{\alpha}}^p, \end{array}$$

whenever $f\trianglelefteq\mathbb{b}_{\boldsymbol{\alpha}}^p$ and $g\trianglelefteq\mathbb{b}_{\boldsymbol{\gamma}}$. □

8.3. Generalized convolution products

Given a compact subset $\boldsymbol{\mathcal{M}}$ of a real affine subspace $\boldsymbol{\mathcal{A}}$ of the set of all complex $n\times n$ matrices, we define the corresponding *generalized convolution product* of convergent series f and g in $\mathbb{C}[[\boldsymbol{\zeta}]]$ by

$$(g*_{\boldsymbol{\mathcal{M}}}f)(\boldsymbol{\zeta})=\int_{\boldsymbol{M}\boldsymbol{\zeta}}g(\boldsymbol{\zeta}-\boldsymbol{\xi})\,f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{8.9}$$

This definition extends to the whole of $\mathbb{C}[[\boldsymbol{\zeta}]]^\sharp$ in a similar way as in the case of standard convolution. Notice that $*_{\boldsymbol{\mathcal{D}}}=*$ if $\boldsymbol{\mathcal{D}}$ is the set of diagonal matrices with entries in $[0,1]$.

Now choose a bijective affine parameterization $\boldsymbol{M}:\mathbb{R}^m\to\boldsymbol{\mathcal{A}};t\mapsto\boldsymbol{M}_{\boldsymbol{t}}$ of $\boldsymbol{\mathcal{A}}$ and consider the parameterizations $\boldsymbol{\varphi},\boldsymbol{\psi}:\mathbb{C}^n\times\boldsymbol{\mathcal{T}}\to\mathbb{C}^n$ with $\boldsymbol{\mathcal{T}}=\boldsymbol{M}^{-1}\boldsymbol{\mathcal{A}}$ and

$$\begin{array}{rcl} \boldsymbol{\varphi}(\boldsymbol{\zeta},\boldsymbol{t}) & = & \boldsymbol{M}_{\boldsymbol{t}}\boldsymbol{\zeta}\,; \\ \boldsymbol{\psi}(\boldsymbol{\zeta},\boldsymbol{t}) & = & \boldsymbol{\zeta}-\boldsymbol{M}_{\boldsymbol{t}}\boldsymbol{\zeta}. \end{array}$$

Then we have

$$\begin{array}{rcl} (g*_{\boldsymbol{\mathcal{M}}}f)(\boldsymbol{\zeta}) & = & \displaystyle\int_{\boldsymbol{\mathcal{T}}}g(\boldsymbol{\psi}(\boldsymbol{\zeta},\boldsymbol{t}))\,f(\boldsymbol{\varphi}(\boldsymbol{\zeta},\boldsymbol{t}))\,\mathrm{d}(\boldsymbol{M}_{\boldsymbol{t}}\boldsymbol{\zeta}) \\ & = & \Delta_{\boldsymbol{M}}(\boldsymbol{\zeta})\,J_{\boldsymbol{\psi},\boldsymbol{\varphi}}(g,f)(\boldsymbol{\zeta}), \end{array}$$

where $\Delta_{\boldsymbol{M}}(\boldsymbol{\zeta})$ is the determinant of the Jacobian of the mapping $\boldsymbol{t}\mapsto\boldsymbol{M}_{\boldsymbol{t}}\boldsymbol{\zeta}$. This determinant is a homogeneous polynomial in $\boldsymbol{\zeta}$ of degree m, since $\boldsymbol{t}\mapsto\boldsymbol{M}_{\boldsymbol{t}}$ was chosen to be affine. Notice that

$$\operatorname{Vol}(\boldsymbol{\mathcal{M}}\boldsymbol{\zeta})=\Delta_{\boldsymbol{M}}(\boldsymbol{\zeta})\operatorname{Vol}(\boldsymbol{\mathcal{T}}). \tag{8.10}$$

Consequently, we have

LEMMA 8.3. *Let $0<\varepsilon<1$ and let $\boldsymbol{\alpha}$ be such that*

$$\{\boldsymbol{M}\boldsymbol{\zeta}:\boldsymbol{M}\in\boldsymbol{\mathcal{M}}\wedge\boldsymbol{\zeta}\in\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}\}\subseteq\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}.$$

Then

$$g\trianglelefteq\mathbb{b}_{\varepsilon\boldsymbol{\alpha}/2}\wedge f\trianglelefteq\mathbb{b}_{\boldsymbol{\alpha}}^p\implies g*_{\boldsymbol{\mathcal{M}}}f\trianglelefteq\operatorname{Vol}(\boldsymbol{\mathcal{M}}\boldsymbol{\zeta})\,(1-\varepsilon)^{-n}\,\mathbb{b}_{\boldsymbol{\alpha}}^p$$

for all $f,g\in\mathbb{C}[[\boldsymbol{\zeta}]]$ and $p\in\mathbb{N}$.

PROOF. Let $\boldsymbol{\beta}=\varepsilon\,\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}=\varepsilon\,\boldsymbol{\alpha}/2$. Then $\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\beta}}=\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}/\varepsilon$, so that

$$\{\boldsymbol{M}\boldsymbol{\zeta}:\boldsymbol{M}\in\boldsymbol{\mathcal{M}}\wedge\boldsymbol{\zeta}\in\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\beta}}\}\subseteq\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\beta}},$$

and $\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\gamma}}=2\,\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\beta}}$ so that

$$\{\boldsymbol{\zeta}-\boldsymbol{M}\boldsymbol{\zeta}:\boldsymbol{M}\in\boldsymbol{\mathcal{M}}\wedge\boldsymbol{\zeta}\in\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\beta}}\}\subseteq\bar{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\gamma}}.$$

The result now follows from lemma 8.2 and (8.10). □

BIBLIOGRAPHY

[Car61] H. Cartan. *Théorie élémentaire des fonctions analytiques d'une et plusieurs variables*. Hermann, 1961.

[DL89] J. Denef and L. Lipshitz. Decision problems for differential equations. *The Journ. of Symb. Logic*, 54(3):941–950, 1989.

[Pet50] I.G. Petrovsky. *Lectures on Partial Differential Equations*. Interscience Publishers, 1950.

[vdH99] J. van der Hoeven. Fast evaluation of holonomic functions. *TCS*, 210:199–215, 1999.

[vdH01a] J. van der Hoeven. Fast evaluation of holonomic functions near and in singularities. *JSC*, 31:717–743, 2001.

[vdH01b] Joris van der Hoeven. Operators on generalized power series. *Journal of the Univ. of Illinois*, 45(4):1161–1190, 2001.

[vdH02] Joris van der Hoeven. Relax, but don't be too lazy. *JSC*, 34:479–542, 2002.

[vdH03] J. van der Hoeven. Effective complex analysis. In G.-M. Greuel A. Cohen and M.-F. Roy, editors, *Proceedings of MEGA 2003*, Kaiserslautern, 2003. To appear.

[vK75] S. von Kowalevsky. Zur theorie der partiellen differentialgleichungen. *J. Reine und Angew. Math.*, 80:1–32, 1875.