MAXIMAL HARDY FIELDS

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To the memory of Michael Boshernitzan (1950–2019)

Abstract. We show that all maximal Hardy fields are elementarily equivalent as differential fields, and give various applications of this result and its proof. We also answer some questions on Hardy fields posed by Boshernitzan.

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Preface

A Hardy field is said to be *maximal* if it has no proper Hardy field extension. In these notes we show that all maximal Hardy fields are elementarily equivalent, as ordered differential fields, to the ordered differential field $\mathbb{T}$ of transseries. This is part of our main result, Theorem 6.7.22.

We shall depend heavily on our book [ADH], which contains a model-theoretic analysis of $\mathbb{T}$. Besides developing further the asymptotic differential algebra from that book we require also a good dose of analysis. These notes are divided in Parts 1–7, preceded by a somewhat lengthy Introduction including a sketch of the proof of our main result. Parts 1–4 consist of further asymptotic differential algebra and culminates in various normalization theorems for algebraic differential equations over suitable H-fields. Parts 5 and 6 are more analytic and apply the normalization theorems to Hardy fields. Part 7 consists of applications. We finish with an index and a list of symbols newly introduced in this work. (All other notation is standard or comes from [ADH].)

The present notes are probably not suitable for publication as a journal article, since we took the liberty of including extensive sections with complete proofs on classical topics such as self-adjoint linear differential operators, almost periodic functions, uniform distribution modulo 1, and Bessel functions. This was partly done for our own education, and partly to put things in a form convenient for our purpose. We also took the opportunity to develop some topics a bit further than needed for the main theorem, and in this way we could also answer in Part 5 some questions about Hardy fields raised by Boshernitzan. We have in mind further use of the material here, for example in [15] and in relation to open problems posed in [ADH]. (Our main theorem solves one of those problems.)

Readers only interested in the proof of our main result can skip Sections 1.3, 2.4, 5.4, as well as several subsections of other sections in Parts 1–6. These (sub)sections are marked by an asterisk (*).

The main results in these notes are really about *differentially algebraic* Hardy field extensions, especially their construction. We complement this in [14] with an account of constructing *differentially transcendental* Hardy field extensions, leading to the result that all maximal Hardy fields are η_1 in the sense of Hausdorff: in other words, given any Hardy field H and countable subsets $A < B$ in H, there is an element f in a Hardy field extension of H such that $A < f < B$. This can be used to show that all maximal Hardy fields are back-and-forth equivalent, which is considerably stronger than their elementary equivalence. We mention this here because the proof of a key ingredient in [14] makes essential use of the main result from the present notes.

We are still deliberating how to publish this material (these notes and [14]), but thought it best to make it available for now on the arxiv.

Introduction

Du Bois-Reymond's "orders of infinity" [28]–[31] were put on a firm basis by Hardy [87], leading to the notion of a Hardy field (Bourbaki [39]). A *Hardy field* is a field H of germs at $+\infty$ of differentiable real-valued functions on intervals $(a, +\infty)$ such that for any differentiable function whose germ is in H the germ of its derivative is also in H. (See Section 5.3 for more precision.) Every Hardy field is naturally a differential field, and an ordered field with the germ of f being > 0 iff $f(t) > 0$, eventually. *Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions* [170, p. 297]. The basic theory of Hardy fields was mostly developed by Boshernitzan [32]–[35] and Rosenlicht [170]–[174].

The germs of Hardy's logarithmico-exponential functions [84] furnish the classical example of a Hardy field: these functions are the real-valued functions that can be built from real constants and the identity function x, using addition, multiplication, division, taking logarithms, and exponentiating. Examples include the germs of the functions $(0, +\infty) \to \mathbb{R}$ given by x^r $(r \in \mathbb{R})$, e^{x^2}, and $\log\log x$. Other Hardy fields contain (germs of) differentially transcendental functions, such as the Riemann ζ-function and Euler's Γ-function [170], and even functions ultimately growing faster than each iterate of the exponential function [34]. One source of Hardy fields is o-minimality: every o-minimal structure on the real field naturally gives rise to a Hardy field (of germs of definable functions). This yields a wealth of examples such as those obtained from quasi-analytic Denjoy-Carleman classes [166], or containing certain transition maps of plane analytic vector fields [110], and explains the role of Hardy fields in model theory and its applications to real analytic geometry and dynamical systems [8, 22, 139]. Hardy fields have also found applications in computer algebra [176, 177, 185], ergodic theory (see, e.g., [20, 37, 73, 115]), and other areas of mathematics [19, 42, 44, 68, 80].

In the remainder of this introduction, H is a Hardy field. Then $H(\mathbb{R})$ (obtained by adjoining the germs of the constant functions) is also a Hardy field, and for any $h \in H$, the germ e^h generates a Hardy field $H(\mathrm{e}^h)$ over H, and so does any differentiable germ with derivative h. Moreover, H has a unique Hardy field extension that is algebraic over H and real closed. (See [32, 165, 171] or Section 5.3 below.) Our main result is Theorem 6.7.22, and it yields what appears to be the ultimate fact about differentially algebraic Hardy field extensions:

Theorem A. *Let $P(Y)$ be a differential polynomial in a single differential indeterminate Y over H, and let $f < g$ in H be such that $P(f) < 0 < P(g)$. Then there is a y in a Hardy field extension of H such that $f < y < g$ and $P(y) = 0$.*

By Zorn, every Hardy field extends to a maximal Hardy field, so by the theorem above, maximal Hardy fields have the intermediate value property for differential polynomials. (In [14] we show there are very many maximal Hardy fields, namely $2^{\mathfrak{c}}$ many, where $\mathfrak{c}$ is the cardinality of the continuum.) By the results mentioned earlier, maximal Hardy fields are also Liouville closed H-fields in the sense of [6]; thus they contain the germs of all logarithmico-exponential functions. Hiding behind the intermediate value property of Theorem A are two more fundamental properties, ω-*freeness* and *newtonianity,* which are central in our book [ADH]. (Roughly speaking, ω-freeness controls the solvability of second-order homogeneous differential equations, and newtonianity is a strong version of differential-henselianity.) We show that any Hardy field has an $\upomega$-free Hardy field extension (Theorem 5.6.2), and next the much harder result that any $\upomega$-free Hardy field extends to a newtonian $\upomega$-free Hardy field: Theorem 6.7.22, which is really the main result of this paper. It follows that every maximal Hardy field is, in the terminology of [12], an *H-closed field* with small derivation. Now the elementary theory T_H of H-closed fields with small derivation (denoted by $T^{\text{nl}}_{\text{small}}$ in [ADH]) is *complete*, by [ADH, 16.6.3]. This means in particular that any two maximal Hardy fields are indistinguishable as to their elementary properties:

Corollary 1. *If H_1 and H_2 are maximal Hardy fields, then H_1 and H_2 are elementarily equivalent as ordered differential fields.*

To derive Theorem A we use also the key results from the book [103] to the effect that $\mathbb{T}_{\text{g}}$, the ordered differential field of grid-based transseries, is H-closed with small derivation and the intermediate value property for differential polynomials. In particular, it is a model of the complete theory T_H. Thus maximal Hardy fields have the intermediate value property for differential polynomials as well, and this amounts to Theorem A, obtained here as a byproduct of more fundamental results. (A more detailed account of the differential intermediate value property for H-fields is in [13].) We sketch the proof of our main result (Theorem 6.7.22) later in this introduction, after describing further consequences.

Further consequences of our main result. In [ADH] we prove more than completeness of T_H: a certain natural extension by definitions of T_H has quantifier elimination. This leads to a strengthening of Corollary 1 by allowing parameters from a common Hardy subfield of H_1 and H_2. To fully appreciate this statement requires more knowledge of model theory, as in [ADH, Appendix B], which we do not assume for this introduction. However, we can explain a special case in a direct way, in terms of solvability of systems of algebraic differential equations, inequalities, and asymptotic inequalities. Here we find it convenient to use the notation for asymptotic relations introduced by du Bois-Reymond and Hardy instead of Bachmann-Landau's O-notation: for germs f, g in a Hardy field set

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad f = O(g) \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some real } c > 0, \\ f \prec g \quad &:\Longleftrightarrow \quad f = o(g) \quad :\Longleftrightarrow \quad |f| < c|g| \text{ for all real } c > 0. \end{aligned}$$

Let now $Y = (Y_1, \dots, Y_n)$ be a tuple of distinct (differential) indeterminates, and consider a system of the following form:

$$(*) \qquad \left\{ \begin{array}{ccc} P_1(Y) & \varrho_1 & Q_1(Y) \\ \vdots & \vdots & \vdots \\ P_k(Y) & \varrho_k & Q_k(Y) \end{array} \right.$$

Here each P_i, Q_i is a differential polynomial in Y (that is, a polynomial in the indeterminates Y_j and their formal derivatives $Y_j', Y_j'', \dots$) with coefficients in our Hardy field H, and each ϱ_i is one of the symbols $=$, $\neq$, $\leqslant$, $<$, $\preccurlyeq$, $\prec$. Given a Hardy field $E \supseteq H$, a *solution* of $(*)$ in E is an n-tuple $y = (y_1, \dots, y_n) \in E^n$ such that for $i = 1, \dots, k$, the relation $P_i(y)\, \varrho_i\, Q_i(y)$ holds in E. Here is a Hardy field analogue of the "Tarski Principle" of real algebraic geometry [ADH, B.12.14]:

Corollary 2. *If the system $(*)$ has a solution in* some *Hardy field extension of H, then $(*)$ has a solution in* every *maximal Hardy field extension of H.*

(The symbols $\neq$, $\leqslant$, $<$, $\preccurlyeq$ in $(*)$ are for convenience only: their occurrences can be eliminated at the cost of increasing m, n. But $\prec$ is essential; see [ADH, 16.2.6].) Besides the quantifier elimination alluded to, Corollary 2 depends on Lemma 7.1.1, which says that for any Hardy field H all maximal Hardy field extensions of H induce the same $\Lambda\Omega$-cut on H, as defined in [ADH, 16.3].

In particular, taking for H the smallest Hardy field $\mathbb{Q}$, we see that a system $(*)$ with a solution in some Hardy field has a solution in *every* maximal Hardy field, thus recovering a special case of our Corollary 1. Call such a system $(*)$ over $\mathbb{Q}$ *consistent.* For example, with X, Y, Z denoting here single distinct differential indeterminates, the system

$$Y'Z \ \preccurlyeq\ Z', \qquad Y \preccurlyeq 1, \qquad 1 \prec Z$$

is inconsistent, whereas for any $Q \in \mathbb{Q}\{Y\}$ and $n \geqslant 2$ the system

$$X^nY' \ =\ Q(Y), \qquad X' = 1, \quad Y \prec 1$$

is consistent. As a consequence of the completeness of T_H we obtain the existence of an algorithm (albeit a very impractical one) for deciding whether a system $(*)$ over $\mathbb{Q}$ is consistent, and this opens up the possibility of automating a substantial part of asymptotic analysis in Hardy fields. We remark that Singer [188] proved the existence of an algorithm for deciding whether a given system $(*)$ over $\mathbb{Q}$ without occurrences of $\preccurlyeq$ or $\prec$ has a solution in *some* ordered differential field (and then it will have a solution in the ordered differential field of germs of real meromorphic functions at 0); but there are such systems, like

$$X' \ =\ 1, \qquad XY^2 \ =\ 1 - X,$$

which are solvable in an ordered differential field, but not in a Hardy field. Also, algorithmically deciding the solvability of a system $(*)$ over $\mathbb{Q}$ in a *given* Hardy field H may be impossible when H is "too small": e.g., if $H = \mathbb{R}(x)$, by [55].

As these results suggest, the aforementioned quantifier elimination for T_H yields a kind of "resultant" for systems $(*)$ that allows one to make explicit within H itself for which choices of coefficients of the differential polynomials P_i, Q_i the system $(*)$ has a solution in a Hardy field extension of H. Without going into details, we only mention here some attractive consequences for systems $(*)$ depending on parameters. For this, let $X_1, \dots, X_m, Y_1, \dots, Y_n$ be distinct indeterminates and $X = (X_1, \dots, X_m)$, $Y = (Y_1, \dots, Y_n)$, and consider a system

$$(**) \qquad \left\{\begin{array}{ccc} P_1(X,Y) & \varrho_1 & Q_1(X,Y) \\ \vdots & \vdots & \vdots \\ P_k(X,Y) & \varrho_k & Q_k(X,Y) \end{array}\right.$$

where P_i, Q_i are now differential polynomials in (X, Y) over H, and the ϱ_i are as before. Specializing X to $c \in \mathbb{R}^m$ then yields a system

$$(*c) \qquad \left\{\begin{array}{ccc} P_1(c,Y) & \varrho_1 & Q_1(c,Y) \\ \vdots & \vdots & \vdots \\ P_k(c,Y) & \varrho_k & Q_k(c,Y) \end{array}\right.$$

where $P_i(c, Y)$, $Q_i(c, Y)$ are differential polynomials in Y with coefficients in the Hardy field $H(\mathbb{R})$. (We only substitute real constants, so may assume that the P_i, Q_i

are *polynomial* in X, that is, none of the derivatives $X_j', X_j'', \ldots$ occur in the P_i, Q_i.) Using [ADH, 16.0.2(ii)] we obtain:

Corollary 3. *The set of all $c \in \mathbb{R}^m$ such that the system $(*c)$ has a solution in some Hardy field extension of H is semialgebraic.*

Recall: a subset of $\mathbb{R}^m$ is said to be *semialgebraic* if it is a finite union of sets

$$\big\{c \in \mathbb{R}^m : \ p(c) = 0,\ q_1(c) > 0, \ldots, q_l(c) > 0\big\}$$

where $p, q_1, \ldots, q_l \in \mathbb{R}[X]$ are ordinary polynomials. (The topological and geometric properties of semialgebraic sets have been studied extensively [24]. For example, it is well-known that a semialgebraic set can have only have finitely many connected components, and that each such component is itself semialgebraic.)

In connection with Corollary 3 we mention that the asymptotics of Hardy field solutions to algebraic differential equations $Q(Y) = 0$, where Q is a differential polynomial with constant real coefficients, has been investigated by Hardy [85] and Fowler [72] in cases where $\operatorname{order} Q \leqslant 2$ (see [18, Chapter 5]), and later by Shackell [175, 183, 184] in general. Special case of our corollary: for any differential polynomial $P(X, Y)$ with constant real coefficients, the set of parameters $c \in \mathbb{R}^m$ such that the differential equation $P(c, Y) = 0$ has a solution y in some Hardy field, in addition possibly also satisfying given asymptotic side conditions (such as $y \prec 1$), is semialgebraic. Example: the set of real parameters $(c_1, \ldots, c_m) \in \mathbb{R}^m$ for which the homogeneous linear differential equation

$$y^{(m)} + c_1 y^{(m-1)} + \cdots + c_m y \ = \ 0$$

has a nonzero solution $y \prec 1$ in a Hardy field is semialgebraic; in fact, it is the set of all $(c_1, \ldots, c_m) \in \mathbb{R}^m$ such that the polynomial $Y^m + c_1 Y^{m-1} + \cdots + c_m \in \mathbb{R}[Y]$ has a negative real zero. (Below we discuss more general linear differential equations over Hardy fields.) Nonlinear example: for $g_2, g_3 \in \mathbb{R}$ the differential equation

$$(Y')^2 \ = \ 4Y^3 - g_2 Y - g_3$$

has a nonconstant solution in a Hardy field iff $g_2^3 = 27 g_3^2$ and $g_3 \leqslant 0$. In both cases, the Hardy field solutions are germs of logarithmico-exponential functions. But the class of differentially algebraic germs in Hardy fields is much more extensive; for example, the antiderivatives of e^{x^2} are not logarithmico-exponential (Liouville).

Instead of $c \in \mathbb{R}^m$, substitute $h \in H^m$ for X in $(**)$, resulting in a system

$$(*h) \qquad \left\{ \begin{array}{ccc} P_1(h, Y) & \varrho_1 & Q_1(h, Y) \\ \vdots & \vdots & \vdots \\ P_k(h, Y) & \varrho_k & Q_k(h, Y) \end{array} \right.$$

where $P_i(h, Y)$, $Q_i(h, Y)$ are now differential polynomials in Y with coefficients in H. It is well-known that for any semialgebraic set $S \subseteq \mathbb{R}^{m+1}$ there is a natural number $B = B(S)$ such that for every $c \in \mathbb{R}^m$, if the section $\big\{y \in \mathbb{R} : (c, y) \in S\big\}$ has $> B$ elements, then this section has nonempty interior in $\mathbb{R}$. In contrast, the set of solutions of $(*h)$ for $n = 1$ in a maximal H can be simultaneously infinite and discrete in the order topology of H: this happens precisely if some nonzero one-variable differential polynomial over H vanishes on this solution set [ADH, 16.6.11]. (Consider the example of the single algebraic differential equation $Y' = 0$, which has solution set $\mathbb{R}$ in each maximal Hardy field.) Nevertheless, we have the following uniform finiteness principle for solutions of $(*h)$; its proof is considerably deeper than Corollary 3 and also draws on results from [10].

Corollary 4. *There is a natural number $B = B(**)$ such that for all $h \in H^m$: if the system $(*h)$ has $> B$ solutions in some Hardy field extension of H, then $(*h)$ has continuum many solutions in every maximal Hardy field extension of H.*

Next we turn to issues of smoothness and analyticity in Corollary 2. By definition, a Hardy field is a differential subfield of the differential ring $\mathcal{C}^{<\infty}$ consisting of the germs of functions $(a, +\infty) \to \mathbb{R}$ ($a \in \mathbb{R}$) which are, for each n, eventually n-times continuously differentiable. Now $\mathcal{C}^{<\infty}$ has the differential subring $\mathcal{C}^{\infty}$ whose elements are the germs that are eventually $\mathcal{C}^{\infty}$. A $\mathcal{C}^{\infty}$-*Hardy field* is a Hardy field $H \subseteq \mathcal{C}^{\infty}$. (See [77] for an example of a Hardy field $H \not\subseteq \mathcal{C}^{\infty}$.) A $\mathcal{C}^{\infty}$-Hardy field is said to be $\mathcal{C}^{\infty}$-*maximal* if it has no proper $\mathcal{C}^{\infty}$-Hardy field extension. Now $\mathcal{C}^{\infty}$ in turn has the differential subring $\mathcal{C}^{\omega}$ whose elements are the germs that are eventually real analytic, and so we define likewise $\mathcal{C}^{\omega}$-Hardy fields ($\mathcal{C}^{\omega}$-maximal Hardy fields, respectively). Our main theorems go through in the $\mathcal{C}^{\infty}$- and $\mathcal{C}^{\omega}$-settings; combined with model completeness of T_H shown in [ADH, 16.2] this ensures the existence of solutions with appropriate smoothness in Corollary 2:

Corollary 5. *If $H \subseteq \mathcal{C}^{\infty}$ and the system $(*)$ has a solution in some Hardy field extension of H, then $(*)$ has a solution in every $\mathcal{C}^{\infty}$-maximal Hardy field extension of H. In particular, if H is $\mathcal{C}^{\infty}$-maximal and $(*)$ has a solution in a Hardy field extension of H, then it has a solution in H. (Likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.)*

We already mentioned $\mathbb{T}_{\mathrm{g}}$ as a quintessential example of an H-closed field. Its cousin $\mathbb{T}$, the ordered differential field of transseries, extends $\mathbb{T}_{\mathrm{g}}$ and is also H-closed with constant field $\mathbb{R}$ [ADH, 15.0.2]. The elements of $\mathbb{T}$ are certain generalized series (in the sense of Hahn) in an indeterminate $x > \mathbb{R}$ with real coefficients, involving exponential and logarithmic terms, such as

$$f \;=\; \mathrm{e}^{\frac{1}{2}\mathrm{e}^{x}} - 5\,\mathrm{e}^{x^2} + \mathrm{e}^{x^{-1}+2x^{-2}+\cdots} + \sqrt[3]{2}\log x - x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-2x} + \cdots + 5\,\mathrm{e}^{-x^{3/2}}.$$

Mathematically significant examples are the more simply structured transseries

$$\mathrm{Ai} \;=\; \frac{\mathrm{e}^{-\xi}}{2\sqrt{\pi}x^{1/4}}\sum_n (-1)^n c_n \xi^{-n}, \quad \mathrm{Bi} \;=\; \frac{\mathrm{e}^{\xi}}{\sqrt{\pi}x^{-1/4}}\sum_n c_n \xi^{-n},$$
$$\text{where } c_n \;=\; \frac{(2n+1)(2n+3)\cdots(6n-1)}{(216)^n n!} \text{ and } \xi \;=\; \frac{2}{3}x^{3/2},$$

which are $\mathbb{R}$-linearly independent solutions of the Airy equation $Y'' = xY$ [144, Chapter 11, (1.07)]. For information about $\mathbb{T}$ see [ADH, Appendix A] or [62, 103]. We just mention here that like each H-field, $\mathbb{T}$ comes equipped with its own versions of the asymptotic relations $\preccurlyeq$, $\prec$, defined as for H above. The asymptotic rules valid in all Hardy fields, such as

$$f \preccurlyeq 1 \;\Rightarrow\; f' \prec 1, \qquad f \preccurlyeq g \prec 1 \;\Rightarrow\; f' \preccurlyeq g', \qquad f' = f \neq 0 \;\Rightarrow\; f \succ x^n$$

also hold in $\mathbb{T}$. Here x denotes, depending on the context, the germ of the identity function on $\mathbb{R}$, as well as the element $x \in \mathbb{T}$. (We make this precise in Section 7.3, where we also give a finite axiomatization of these rules.)

Now suppose that we are given an embedding $\iota\colon H \to \mathbb{T}$ of ordered differential fields. We may view such an embedding as a *formal expansion operator* and its inverse as a *summation operator*. (See Section 7.3 below for an example of a Hardy field, arising from a fairly rich o-minimal structure, which admits such an embedding.) From $(*)$ we obtain a system

$$(\iota *) \qquad \begin{cases} \iota(P_1)(Y) & \varrho_1 & \iota(Q_1)(Y) \\ \vdots & \vdots & \vdots \\ \iota(P_k)(Y) & \varrho_m & \iota(Q_k)(Y) \end{cases}$$

of algebraic differential equations and (asymptotic) inequalities over $\mathbb{T}$, where $\iota(P_i)$, $\iota(Q_i)$ denote the differential polynomials over $\mathbb{T}$ obtained by applying ι to the coefficients of P_i, Q_i, respectively. A *solution* of $(\iota *)$ is a tuple $y = (y_1, \dots, y_n) \in \mathbb{T}^n$ such that $\iota(P_i)(y)\, \varrho_i\, \iota(Q_i)(y)$ holds in $\mathbb{T}$, for $i = 1, \dots, m$. Differential-difference equations in $\mathbb{T}$ are sometimes amenable to functional-analytic techniques like fixed point theorems or small (compact-like) operators [102], and the formal nature of transseries also makes it possible to solve algebraic differential equations in $\mathbb{T}$ by quasi-algorithmic methods [100, 103]. The simple example of the Euler equation

$$Y' + Y \;=\; x^{-1}$$

is instructive: its solutions in $\mathcal{C}^{<\infty}$ are given by the germs of

$$t \mapsto \mathrm{e}^{-t} \int_1^t \frac{\mathrm{e}^s}{s}\, ds + c\, \mathrm{e}^{-t} \colon (1, +\infty) \to \mathbb{R} \qquad (c \in \mathbb{R}),$$

all contained in a common Hardy field extension of $\mathbb{R}(x)$. The solutions of this differential equation in $\mathbb{T}$ are

$$\sum_n n!\, x^{-(n+1)} + c\, \mathrm{e}^{-x} \qquad (c \in \mathbb{R}),$$

where the particular solution $\sum_n n!\, x^{-(n+1)}$ is obtained as the unique fixed point of the operator $f \mapsto x^{-1} - f'$ on the differential subfield $\mathbb{R}(\!(x^{-1})\!)$ of $\mathbb{T}$ (cf. [ADH, 2.2.13]). (Note: $\sum_n n!\, t^{-(n+1)}$ diverges for each $t > 0$.) In general, the existence of a solution of $(\iota *)$ in $\mathbb{T}$ entails the existence of a solution of $(*)$ in some Hardy field extension of H and vice versa; more precisely:

Corollary 6. *The system $(\iota *)$ has a solution in $\mathbb{T}$ iff $(*)$ has a solution in some Hardy field extension of H. In this case, we can choose a solution of $(*)$ in a Hardy field extension E of H for which ι extends to an embedding of ordered differential fields $E \to \mathbb{T}$.*

In particular, a system $(*)$ over $\mathbb{Q}$ is consistent if and only if it has a solution in $\mathbb{T}$. (The "if" direction already follows from [ADH, Chapter 16] and [104]; the latter constructs a summation operator on the ordered differential subfield $\mathbb{T}^{\mathrm{da}} \subseteq \mathbb{T}$ of differentially algebraic transseries.)

It may seem remarkable that a result about differential polynomials in one differentiable indeterminate, like Theorem A (or Theorem B below), yields similar facts about *systems* of algebraic differential equations and asymptotic inequalities in several indeterminates over Hardy fields as in the corollaries above; we owe this to the strength of the model-theoretic methods employed in [ADH]. But our theorem in combination with [ADH] already has interesting consequences for one-variable differential polynomials over H and over its "complexification" $K := H[\mathrm{i}]$ (where $\mathrm{i}^2 = -1$), which is a differential subfield of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$.

Some of these facts are analogous to familiar properties of ordinary one-variable polynomials over the real or complex numbers. First, it follows from Theorem A that every differential polynomial in a single differential indeterminate over H of odd degree has a zero in a Hardy field extension of H. (See Corollary 7.1.20.) For example, a differential polynomial like

$$(Y'')^5 + \sqrt{2}\,\mathrm{e}^x (Y'')^4 Y''' - x^{-1} \log x\, Y^2 Y'' + YY' - \Gamma$$

has a zero in every maximal Hardy field extension of the Hardy field $\mathbb{R}\langle \mathrm{e}^x, \log x, \Gamma\rangle$. Passing to $K = H[\mathrm{i}]$ we have:

Corollary 7. *For each differential polynomial $P \notin K$ in a single differential indeterminate with coefficients in K there are f, g in a Hardy field extension of H such that $P(f + g\mathrm{i}) = 0$.*

In particular, each linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y \;=\; b \qquad (a_1, \dots, a_n, b \in K)$$

has a solution $y = f + g\mathrm{i}$ where f, g lie in some Hardy field extension of H. (Of course, if $b = 0$, then we may take here the trivial solution $y = 0$.) Although this special case of Corollary 7 concerns differential polynomials of degree 1, it seems hard to obtain this result without recourse to our more general extension theorems: a solution y of a linear differential equation of order n over K as above may simultaneously be a zero of a non-linear differential polynomial P over K of order $< n$, and the structure of the differential field extension of K generated by y is governed by P (when taken of minimal complexity in the sense of [ADH, 4.3]).

Turning now to homogeneous linear differential equations over Hardy fields, we first introduce some notation and terminology. Let $R[\partial]$ be the ring of linear differential operators over a differential ring R: this ring is a free left R-module with basis ∂^n ($n \in \mathbb{N}$) such that $\partial^0 = 1$ and $\partial \cdot f = f\partial + f'$ for $f \in R$, where $\partial := \partial^1$. (See [ADH, 5.1] or [158, 2.1].) Any operator $A \in R[\partial]$ gives rise to an additive map $y \mapsto A(y)\colon R \to R$, with $\partial^n(y) = y^{(n)}$ (the nth derivative of y in R) and $r(y) = ry$ for $r = r \cdot 1 \in R \subseteq R[\partial]$. The elements of $\partial^n + R\partial^{n-1} + \cdots + R \subseteq R[\partial]$ are said to be *monic* of order n. It is well-known [43, 136, 137] that for $R = \mathcal{C}^{<\infty}[\mathrm{i}]$, each monic $A \in R[\partial]$ factors as a product of monic operators of order 1 in $R[\partial]$; if $A \in K[\partial]$, then such a factorization already happens over the complexification of some Hardy field extension of H:

Corollary 8. *If H is maximal, then each monic operator in $K[\partial]$ is a product of monic operators of order 1 in $K[\partial]$.*

This follows quite easily from Corollary 7 using the Riccati transform [ADH, 5.8]. *In the remainder of this subsection we let $A \in K[\partial]$ be monic of order n, and we fix a maximal Hardy field extension E of H.* The factorization result in Corollary 8 gives rise to a description of a fundamental system of solutions for the homogeneous linear differential equation $A(y) = 0$ in terms of Hardy field germs. Here, of course, complex exponential terms naturally appear, but only in a controlled way: the $\mathbb{C}$-linear space consisting of all $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $A(y) = 0$ has a basis of the form

$$f_1\,\mathrm{e}^{\phi_1 \mathrm{i}},\ \dots,\ f_n\,\mathrm{e}^{\phi_n \mathrm{i}}$$

where $f_j \in E[\mathrm{i}]$ and $\phi_j \in E$ with $\phi_j = 0$ or $|\phi_j| > \mathbb{R}$ for $j = 1, \dots, n$. We can arrange here that for $i, j = 1, \dots, n$ we have $\phi_i = \phi_j$ or $|\phi_i - \phi_j| > \mathbb{R}$. (Note that for ϕ in a Hardy field we have $\phi > \mathbb{R}$ iff $\phi(t) \to +\infty$ as $t \to +\infty$.) In this case, the basis elements $f_i \mathrm{e}^{\phi_i \mathrm{i}}$ for distinct frequencies ϕ_i are pairwise orthogonal in a sense made precise in Section 7.4.

Example. If $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ is *holonomic,* that is, $L(y) = 0$ for some monic $L \in \mathbb{C}(x)[\partial]$, then y is a $\mathbb{C}$-linear combination of germs $f\,\mathrm{e}^{\phi \mathrm{i}}$ where $f \in E[\mathrm{i}]$, $\phi \in E$, and $\phi = 0$ or $|\phi| > \mathbb{R}$. Here, more information about the f, ϕ is available (see, e.g., [70, VIII.7], [204, §19.1]). Many special functions are holonomic [70, B.4].

By the usual correspondence between linear differential operators and matrix differential equations (see, e.g., [ADH, 5.5]), our results about zeros of linear differential operators also yield facts about systems $y' = Ny$ of linear differential equations over Hardy fields. If the matrix N has suitable symmetry, we can even guarantee the existence of a nonzero solution y which lies in $E[\mathrm{i}]^n$ (and thus does not exhibit oscillating behavior). A sample result, also shown in Section 7.4: every matrix differential equation $y' = Ny$, where N is an $n \times n$ matrix over K ($n \geqslant 1$), has a nonzero solution $y \in E[\mathrm{i}]^n$ provided n is *odd* and N is *skew-symmetric.* (The study of such matrix differential equations, for $n = 3$, goes back at least to Darboux [54, Livre I, Chapitre II].) For example, for each $c, d \in \mathbb{C}$ there is a nonzero $y \in E[\mathrm{i}]^3$ such that

$$y' \;=\; \begin{pmatrix} 0 & -c & -\dfrac{d}{\mathrm{e}^x + \mathrm{e}^{-x}} \\ c & 0 & \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}} \\ \dfrac{d}{\mathrm{e}^x + \mathrm{e}^{-x}} & \dfrac{\mathrm{e}^{-x} - \mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^{-x}} & 0 \end{pmatrix} y.$$

(For $c = 0$, $d = 2\sqrt{2}$, this equation is studied in [2].)

We now return to the operator setting and focus on the case where A is real, that is, $A \in H[\partial]$. Mammana [137] conjectured that each monic operator in $\mathcal{C}^{<\infty}[\partial]$ of odd order has a monic factor of order 1; this is false in general [178] but holds in the Hardy field world, thanks to our "real" version of Corollary 8:

Corollary 9. *Suppose $A \in H[\partial]$. Then A is a product of monic operators in $E[\partial]$, each of order 1 or irreducible of order 2.*

As a consequence, the $\mathbb{R}$-linear space of zeros of $A \in H[\partial]$ in $\mathcal{C}^{<\infty}$ has a basis

$$g_1 \cos\phi_1,\ g_1 \sin\phi_1,\ \dots,\ g_r \cos\phi_r,\ g_r \sin\phi_r,\ h_1,\ \dots,\ h_s \qquad (2r + s = n)$$

where $g_j, \phi_j \in E$ with $\phi_j > \mathbb{R}$ for $j = 1, \dots, r$ and $h_k \in E$ for $k = 1, \dots, s$. In particular, if n is odd, then $A(y) = 0$ for some nonzero $y \in E$.

A function $y\colon [a, +\infty) \to \mathbb{R}$ ($a \in \mathbb{R}$) is *non-oscillating* if $\operatorname{sign} y(t)$ is eventually constant (and otherwise y *oscillates*). Similarly we define (non-) oscillation of germs. No germ in a Hardy field oscillates. The following corollary characterizes when A in Corollary 9 is a product of monic operators of order 1 in $E[\partial]$:

Corollary 10. *The (monic) operator $A \in H[\partial]$ is a product of monic operators of order 1 in $E[\partial]$ iff no zero of A in $\mathcal{C}^{<\infty}$ oscillates. In this case E contains a basis $y_1 \prec \dots \prec y_n$ of the $\mathbb{R}$-linear space of zeros of A in $\mathcal{C}^{<\infty}$, and*

$$A \;=\; (\partial - a_n) \cdots (\partial - a_1)$$

for a unique tuple $(a_1, \dots, a_n) \in E^n$ such that for all sufficiently small $f > \mathbb{R}$ in E we have $a_j + (f''/f') < a_{j+1}$ for $j = 1, \dots, n-1$.

Factorizations of linear differential operators as in Corollary 10 are closely connected to the classical topic of *disconjugacy.* We recall the definition, which arose from the calculus of variations [212]. Let $f_1,\dots,f_n\colon I\to\mathbb{R}$ be continuous, where $I=[a,+\infty)$, $a\in\mathbb{R}$. The linear differential equation

$$\text{(L)}\qquad y^{(n)}+f_1y^{(n-1)}+\cdots+f_ny \;=\; 0$$

on I is said to be *disconjugate* if every nonzero solution $y\in\mathcal{C}^n(I)$ of (L) has at most $n-1$ zeros, counted with their multiplicities. (For example, $y^{(n)}=0$, on any such I, is disconjugate.) The solutions of disconjugate linear differential equations are suitable for approximation and interpolation purposes; see [52, Chapter 3] and [56, Chapter 3, §11]. We also say that (L) is *eventually disconjugate* if for some $b\geqslant a$ the linear differential equation on $J:=[b,+\infty)$ obtained from (L) by restricting $f_1,\dots,f_n$ to J is disconjugate. If (L) is eventually disconjugate, then it has no oscillating solutions in $\mathcal{C}^n(I)$. The converse of this implication holds when $n\leqslant 2$ but fails for each $n>2$ [81]. There is an extensive literature, mostly dating back to the 1970s, which develops sufficient conditions for (eventual) disconjugacy of linear differential equations (see, e.g. [52, 63, 64, 78, 198]), often by restricting the growth of the f_i; for example, (L) is disconjugate if $\int_a^\infty|f_i(t)|(t-a)^{i-1}\,dt<\infty$ for $i=1,\dots,n$ (cf. [209]). Corollary 10 allows us to contribute another natural criterion for eventual disconjugacy:

Corollary 11. *If the germs of $f_1,\dots,f_n$ lie in a Hardy field and* (L) *has no oscillating solutions in $\mathcal{C}^n(I)$, then* (L) *is eventually disconjugate.*

A fundamental property of disconjugate linear differential operators is the existence of a canonical factorization discovered by Trench [200]. (See also Proposition 5.2.42 below.) Corollary 10 can also be used to strengthen this factorization in the situation of Corollary 11. See Corollary 7.4.58 for the details.

We finish with discussing the instructive case of an operator $A\in H[\partial]$ of order 2. If such A has a non-oscillating zero $y\neq 0$ in $\mathcal{C}^{<\infty}$, then by Sturm's Oscillation Theorem all zeros of A in $\mathcal{C}^{<\infty}$ are non-oscillating and hence contained in every maximal Hardy field, by Corollary 5.5.7 or [33, Theorem 16.7], [171, Corollary 2]. For example, the germs of the $\mathbb{R}$-linearly independent solutions $\operatorname{Ai},\operatorname{Bi}\colon\mathbb{R}\to\mathbb{R}$ of the Airy equation $Y''-xY=0$ given by

$$\begin{aligned}\operatorname{Ai}(t) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{s^3}{3}+st\right)ds,\\ \operatorname{Bi}(t) &= \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{s^3}{3}+st\right)+\sin\left(\frac{s^3}{3}+st\right)\right]ds\end{aligned}$$

lie in each maximal Hardy field, with $\operatorname{Ai}\prec 1\prec\operatorname{Bi}$. In the oscillating case, we have:

Corollary 12. *If $A\in H[\partial]$ of order 2 has an oscillating zero in $\mathcal{C}^{<\infty}$, then there are $g,\phi\in E$ with $\phi>\mathbb{R}$ such that the zeros of A in $\mathcal{C}^{<\infty}$ are exactly the germs $cg\cos(\phi+d)$ $(c,d\in\mathbb{R})$.*

This corollary was announced by Boshernitzan as part of [35, Theorem 5.4], but apparently a proof of this theorem never appeared in print. (See also [33, Conjecture 4 in §20].) In Section 7.5 below we state and prove a strengthening of his theorem; this includes a criterion for the uniqueness of the germs g, ϕ. For every $\phi>\mathbb{R}$ in E there is at most one $g\in E$ (up to multiplication by a nonzero constant) such that the conclusion of Corollary 12 holds; in general, the pair (g, ϕ) is not unique (up to multiplication of g by a nonzero constant and addition of a constant to ϕ), but it is if the coefficients of A are differentially algebraic (over $\mathbb{R}$). As a final example, consider the Bessel equation of order $\nu \in \mathbb{R}$ (see, e.g., [66, VII], [144, I, §9], [205]):

$$x^2Y'' + xY' + (x^2 - \nu^2)Y \;=\; 0.$$

It is well-known that each solution $y \in \mathcal{C}^{<\infty}$ of this equation satisfies

$$y \;=\; cx^{-1/2}\cos(x+d) + o(x^{-1/2}) \quad \text{for some } c, d \in \mathbb{R}.$$

(See [18, Chapter 6, §18], [91, Corollary XI.8.1], [204, Example 13.2].) We give a similar parametrization using germs in Hardy fields. More precisely, we show that there is a unique germ $\phi = \phi_\nu$ in a Hardy field with $\phi - x \preccurlyeq x^{-1}$ such that every solution $y \in \mathcal{C}^{<\infty}$ of the Bessel equation has the form

$$y \;=\; cx^{-1/2}g\cos(\phi + d) \quad \text{for some } c, d \in \mathbb{R} \text{ and } g := 1/\sqrt{\phi'}.$$

This explains the phenomenon, observed in [95, 96], that the Bessel equation admits a non-oscillating phase function. Knowing that ϕ lives in a Hardy field allows one to reprove a number of classical results about Bessel functions in a short transparent way. Remarkably, every maximal Hardy field contains the phase function ϕ, and ϕ has an asymptotic expansion

$$\phi \sim x + \frac{\mu - 1}{8}x^{-1} + \frac{\mu^2 - 26\mu + 25}{384}x^{-3} + \frac{\mu^3 - 115\mu^2 + 1187\mu - 1073}{5120}x^{-5} + \cdots$$

where $\mu = 4\nu^2$. Only for special choices of ν is the germ ϕ contained in the Liouville closure of $\mathbb{R}(x)$, and hence easily obtainable by the classical extension results for Hardy fields from [32, 84, 171]: by results of Liouville [132] this holds precisely if $\nu \in \frac{1}{2} + \mathbb{Z}$; see Section 7.6 for a proof.

Michael Boshernitzan's papers [32]–[36] on Hardy fields have been a frequent source of inspiration for us, and we dedicate this work to his memory. He paid particular attention to the germs in $\mathcal{C}^{<\infty}$, such as ϕ_ν above, that lie in *every* maximal Hardy field. They form a Liouville closed Hardy field E properly containing Hardy's differential field of logarithmico-exponential functions. In the course of our work below we prove Conjecture 1 from [32, §10] and Conjecture 4 from [33, §20] about E. We also prove Conjecture 17.11 from [33, §17] and answer Question 4 from [34, §7]. (See Corollaries 7.2.14, Theorems 7.5.1 and 5.5.38, and Proposition 5.6.6, respectively.) Section 7.7 contains some additional observations which may eventually help to shed further light on the nature of the Hardy field E.

Synopsis of the proof of our main theorem. In the rest of the paper we assume familiarity with the terminology and concepts of asymptotic differential algebra from our book [ADH]. (We review some of this in the last subsection of the introduction below.) The proof of our main result requires, besides differential-algebraic and valuation-theoretic tools from [ADH], also analytic arguments in an essential way. Some of our analytic machinery is obtained by adapting material from [104] to a more general setting. As explained earlier, our main Theorem A is a consequence of the following extension theorem:

Theorem B. *Every ω-free Hardy field has a newtonian Hardy field extension.*

The proof of this is long, so it may be useful to outline the strategy behind it.

Holes and slots. For now, let K be an H-asymptotic field with rational asymptotic integration. In Section 3.2 below we introduce the apparatus of *holes* in K as a means to systematize the study of solutions of algebraic differential equations over K in immediate asymptotic extensions of K: such a hole in K is a triple $(P, \mathfrak{m}, \widehat{f})$ where P is a differential polynomial in a single differential indeterminate Y with coefficients in K, $P \neq 0$, $0 \neq \mathfrak{m} \in K$, and $\widehat{f} \notin K$ lies an immediate asymptotic extension of K with $P(\widehat{f}) = 0$ and $\widehat{f} \prec \mathfrak{m}$. It is sometimes technically convenient to work with the more flexible concept of a *slot* in K, where instead of $P(\widehat{f}) = 0$ we only require P to vanish at $(K, \widehat{f})$ in the sense of [ADH, 11.4]. The *complexity* of a slot $(P, \mathfrak{m}, \widehat{f})$ is the complexity of the differential polynomial P as in [ADH, p. 216]. Now if K is $\boldsymbol{\omega}$-free, then by Lemma 3.2.1,

$$K \text{ is newtonian} \quad \Longleftrightarrow \quad K \text{ has no hole.}$$

This equivalence suggests an opening move for proving Theorem B by induction on complexity as follows: Let $H \supseteq \mathbb{R}$ be an $\boldsymbol{\omega}$-free Liouville closed Hardy field, and suppose H is not newtonian; it is enough to show that then H has a proper Hardy field extension. By the above equivalence, H has a hole $(P, \mathfrak{m}, \widehat{f})$, and we can take here $(P, \mathfrak{m}, \widehat{f})$ to be of minimal complexity among holes in H. This minimality has consequences that are important for us; for example $r := \operatorname{order} P \geqslant 1$, P is a minimal annihilator of $\widehat{f}$ over H, and H is $(r-1)$-newtonian as defined in [ADH, 14.2]. We arrange $\mathfrak{m} = 1$ by replacing $(P, \mathfrak{m}, \widehat{f})$ with the hole $(P_{\times\mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$ in H.

Solving algebraic differential equations over Hardy fields. For Theorem B it is enough to show that under these conditions P is a minimal annihilator of some germ $f \in \mathcal{C}^{<\infty}$ that generates a (necessarily proper) Hardy field extension $H\langle f\rangle$ of H. So at a minimum, we need to find a solution in $\mathcal{C}^{<\infty}$ to the algebraic differential equation $P(Y) = 0$. For this, it is natural to use fixed point techniques as in [104]. Notation: for $a \in \mathbb{R}$, let $\mathcal{C}^n_a$ be the $\mathbb{R}$-linear space of functions $[a, +\infty) \to \mathbb{R}$ which extend to an n-times continuously differentiable function $U \to \mathbb{R}$ on an open subset $U \supseteq [a, +\infty)$ of $\mathbb{R}$. For any a and n, each germ in $\mathcal{C}^{<\infty}$ has representatives in $\mathcal{C}^n_a$.

A fixed point theorem. Let $L := L_P \in H[\partial]$ be the linear part of P. Replacing $(P, 1, \widehat{f})$ with another minimal hole in H we arrange $\operatorname{order} L = r$. Representing the coefficients of P (and thus of L) by functions in $\mathcal{C}^0_a$ we obtain an $\mathbb{R}$-linear operator $y \mapsto L(y)\colon \mathcal{C}^r_a \to \mathcal{C}^0_a$. For now we make the bold assumption that $L \in H[\partial]$ splits over H. Using such a splitting and increasing a if necessary, r-fold integration yields an $\mathbb{R}$-linear operator $L^{-1}\colon \mathcal{C}^0_a \to \mathcal{C}^r_a$ which is a *right-inverse* of $L\colon \mathcal{C}^r_a \to \mathcal{C}^0_a$, that is, $L\big(L^{-1}(y)\big) = y$ for all $y \in \mathcal{C}^0_a$. Consider the (generally non-linear) operator

$$f \mapsto \Phi(f) := L^{-1}\big(R(f)\big)$$

on $\mathcal{C}^r_a$; here $P = P_1 - R$ where P_1 is the homogeneous part of degree 1 of P. We try to show that Φ restricts to a contractive operator on a closed ball of an appropriate subspace of $\mathcal{C}^r_a$ equipped with a suitable complete norm, whose fixed points are then solutions to $P(Y) = 0$; this may also involve increasing a again and replacing the coefficient functions of P by their corresponding restrictions. To obtain such contractivity, we would need to ensure that R is asymptotically small compared to P_1 in a certain sense. This can indeed be achieved by transforming $(P, 1, \widehat{f})$

into a certain normal form through successive *refinements* and (*additive*, *multiplicative*, and *compositional*) *conjugations* of the hole $(P,1,\widehat{f})$. This normalization is done under more general algebraic assumptions in Section 3.3. The analytic arguments leading to fixed points are in Sections 6.1–6.3. Developments below involve the algebraic closure $K := H[\mathrm{i}]$ of H and we work more generally with a decomposition $P = \widetilde{P}_1 - R$ where $\widetilde{P}_1 \in K\{Y\}$ is homogeneous of degree 1, not necessarily $\widetilde{P}_1 = P_1$, such that $L_{\widetilde{P}_1} \in K[\partial]$ splits and R is "small" compared to $\widetilde{P}_1$.

Passing to the complex realm. In general we are not so lucky that L splits over H. The minimality of our hole $(P,1,\widehat{f})$ does not even ensure that L splits over K. At this point we recall from [ADH, 11.7.23] that K is ω-free because H is. We can also draw hope from the fact that every nonzero linear differential operator over K would split over K if H were newtonian [ADH, 14.5.8]. Although H is not newtonian, it is $(r-1)$-newtonian, and L is only of order r, so we optimistically restart our attempt, and instead of a hole of minimal complexity in H, we now let $(P,\mathfrak{m},\widehat{f})$ be a hole of minimal complexity in K. Again it follows that $r := \operatorname{order} P \geqslant 1$, P is a minimal annihilator of $\widehat{f}$ over K, and K is $(r-1)$-newtonian. As before we arrange that $\mathfrak{m} = 1$ and the linear part $L_P \in K[\partial]$ of P has order r. We can also arrange $\widehat{f} \in \widehat{K} = \widehat{H}[\mathrm{i}]$ where $\widehat{H}$ is an immediate asymptotic extension of H. So $\widehat{f} = \widehat{g} + \widehat{h}\mathrm{i}$ where $\widehat{g}, \widehat{h} \in \widehat{H}$ satisfy $\widehat{g}, \widehat{h} \prec 1$, and $\widehat{g} \notin H$ or $\widehat{h} \notin H$, say $\widehat{g} \notin H$. Now minimality of $(P,1,\widehat{f})$ and algebraic closedness of K give that K is r-linearly closed, that is, every nonzero $A \in K[\partial]$ of order $\leqslant r$ splits over K (Corollary 3.2.4). Then L_P splits over K as desired, and a version of the above fixed point construction with $\mathcal{C}^r_a[\mathrm{i}]$ in place of $\mathcal{C}^r_a$ can be carried out successfully to solve $P(Y) = 0$ in the differential ring extension $\mathcal{C}^{<\infty}[\mathrm{i}]$ of $\mathcal{C}^{<\infty}$.

Return to the real world. But at this point we face another obstacle: even once we have our hands on a zero $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ of P, it is not clear why $g := \operatorname{Re} f$ should generate a proper Hardy field extension of H: Let Q be a minimal annihilator of $\widehat{g}$ over H; we cannot expect that $Q(g) = 0$. If $L_Q \in H[\partial]$ splits over K, then we can try to apply fixed point arguments like the ones above, with $(P,1,\widehat{f})$ replaced by the hole $(Q,1,\widehat{g})$ in H, to find a zero $y \in \mathcal{C}^{<\infty}$ of Q. (We do need to take care that constructed zero is real.) Unfortunately we can only ascertain that $1 \leqslant s \leqslant 2r$ for $s := \operatorname{order} Q$, and since we may have $s > r$, we cannot leverage the minimality of $(P,1,\widehat{f})$ anymore to ensure that L_Q splits over K, or to normalize $(Q,1,\widehat{g})$ in the same way as indicated above for $(P,1,\widehat{f})$. This situation seems hopeless, but now a purely differential-algebraic observation comes to the rescue: although the linear part $L_{Q_{+\widehat{g}}} \in \widehat{H}[\partial]$ of the differential polynomial $Q_{+\widehat{g}} \in \widehat{H}\{Y\}$ also has order s (which may be $> r$), *if $\widehat{K}$ is r-linearly closed, then $L_{Q_{+\widehat{g}}}$ does split over $\widehat{K}$*; see [ADH, 5.1.37]. If moreover $g \in H$ is sufficiently close to $\widehat{g}$, then the linear part $L_{Q_{+g}} \in H[\partial]$ of $Q_{+g} \in H\{Y\}$ is close to an operator in $H[\partial]$ that does split over $K = H[\mathrm{i}]$, and so using $(Q_{+g}, 1, \widehat{g} - g)$ instead of $(Q,1,\widehat{g})$ may offer a way out of this impasse.

Approximating $\widehat{g}$. Suppose for a moment that H is (valuation) dense in $\widehat{H}$. Then by extending $\widehat{H}$ we arrange that $\widehat{H}$ is the completion of H, and $\widehat{K}$ of K (as in [ADH, 4.4]). In this case $\widehat{K}$ inherits from K the property of being r-linearly closed, by results in Section 1.8, and the desired approximation of $\widehat{g}$ by $g \in H$ can be achieved.

We cannot in general expect H to be dense in $\widehat{H}$. But we are saved by Section 1.6 to the effect that $\widehat{g}$ can be made *special* over H in the sense of [ADH, 3.4], that is, some nontrivial convex subgroup Δ of the value group of H is cofinal in $v(\widehat{g}-H)$. Then passing to the Δ-specializations of the various valued differential fields encountered above (see [ADH, 9.4]) we regain density and this allows us to implement the desired approximation. The technical details are involved, and are carried out in the first three sections of Part 4. A minor obstacle to obtain the necessary specialness of $\widehat{g}$ is that the hole $(Q,1,\widehat{g})$ in H may not be of minimal complexity. This can be ameliorated by using a differential polynomial of minimal complexity vanishing at $(H,\widehat{g})$ instead of Q, in the process replacing the hole $(Q,1,\widehat{g})$ in H by a slot in H, which we then aim to approximate by a *strongly split-normal* slot in H; see Definition 4.5.32. Another caveat: to carry out our approximation scheme we require $\deg P > 1$. Fortunately, if $\deg P = 1$, then necessarily $r = \operatorname{order} P = 1$, and this case can be dealt with through separate arguments: see Section 6.7 where we finish the proof of Theorem B.

Enlarging the Hardy field. Now suppose we have finally arranged things so that our Fixed Point Theorem applies: it delivers $g \in \mathcal{C}^{<\infty}$ such that $Q(g) = 0$ and $g \prec 1$. (Notation: for a germ $\phi \in \mathcal{C}^{<\infty}[i]$ and $0 \neq \mathfrak{n} \in H$ we write $\phi \prec \mathfrak{n}$ if $\phi(t)/\mathfrak{n}(t) \to 0$ as $t \to +\infty$.) However, in order that g generates a proper Hardy field extension $H\langle g\rangle$ of H isomorphic to $H\langle\widehat{g}\rangle$ by an isomorphism over H sending g to $\widehat{g}$ requires that g and $\widehat{g}$ have similar asymptotic properties with respect to the elements of H. For example, suppose $h, \mathfrak{n} \in H$ and $\widehat{g} - h \prec \mathfrak{n} \preccurlyeq 1$; then we must show $g - h \prec \mathfrak{n}$. (Of course, we need to show much more about the asymptotic behavior of g, and this is expressed using the notion of *asymptotic similarity*: see Sections 6.6 and 6.7.) Now the germ $(g-h)/\mathfrak{n} \in \mathcal{C}^{<\infty}$ is a zero of the conjugated differential polynomial $Q_{+h,\times\mathfrak{n}} \in H\{Y\}$, as is the element $(\widehat{g}-h)/\mathfrak{n} \prec 1$ of $\widehat{H}$. The Fixed Point Theorem can also be used to produce a zero $y \prec 1$ of $Q_{+h,\times\mathfrak{n}}$ in $\mathcal{C}^{<\infty}$. Set $g_1 := y\mathfrak{n} + h$; then $Q(g) = Q(g_1) = 0$ and $g, g_1 \prec 1$. We are thus naturally lead to consider the difference $g - g_1$ between the solutions $g, g_1 \in \mathcal{C}^{<\infty}$ of the differential equation (with asymptotic side condition)

$$Q(Y) = 0, \qquad Y \prec 1. \tag{E}$$

If we manage to show $g - g_1 \prec \mathfrak{n}$, then $g - h = (g - g_1) - y\mathfrak{n} \prec \mathfrak{n}$ as required. Simple estimates coming out of the proof of the Fixed Point Theorem are not good enough for this (cf. Lemma 6.2.5). We need a generalization of the Fixed Point Theorem for *weighted norms* with (the germ of) the relevant weight function given by $\mathfrak{n}$, shown in Section 6.5. To render this generalized version useful, we also have to make the construction of the right-inverse A^{-1} of the linear differential operator $A \in H[\partial]$, which splits over K and approximates L_Q as postulated by strong split-normality, and which is central for the definition of the contractive operator used in the Fixed Point Theorem, in some sense uniform in $\mathfrak{n}$. This is carried out in Section 4.5, refining our approximation arguments by improving strong split-normality to *strong repulsive-normality* as defined in 4.5.32.

Exponential sums. Just for this discussion, call $\phi \in \mathcal{C}^{<\infty}[i]$ *small* if $\phi \prec \mathfrak{n}$ for all $\mathfrak{n} \in H$ with $v\mathfrak{n} \in v(\widehat{g} - H)$. Thus our aim is to show that differences between solutions of (E) in $\mathcal{C}^{<\infty}$ are small in this sense. We show that each such difference gives rise to a zero $z \in \mathcal{C}^{<\infty}[i]$ of A with $z \prec 1$ whose smallness would imply the smallness of the difference under consideration. To ensure that every zero $z \prec 1$

of A is indeed small requires us to have performed beforehand yet another (rather unproblematic) normalization procedure on our slot, transforming it into *ultimate* shape. (See Section 4.4.) Recall the special fundamental systems of solutions to linear differential equations over maximal Hardy fields explained after Corollary 8: since A splits over K, our zero z of A is a $\mathbb{C}$-linear combination of exponential terms. As a tool for systematically dealing with such exponential sums over K in a formal way, we introduce the concept of the *universal exponential extension* of a differential field. Finally, from conditions like $z \prec 1$ we need to be able to obtain asymptotic information about the summands of z when expressed as an exponential sum in a certain canonical way. For this we are able to exploit facts about uniform distribution mod 1 for germs in Hardy fields due to Boshernitzan [36]; see Sections 5.8–5.10.

Organization of the manuscript. Part 1 has preliminaries on linear differential operators and differential polynomials, on the group of logarithmic derivatives, on special elements, and on differential-henselianity and newtonianity. In Part 2 we define the universal exponential extension of a differential field, and we consider the eigenvalues of linear differential operators and their connections to splittings. Part 3 then introduces holes and slots, and proves the Normalization Theorem hinted at earlier in this introduction. In Part 4 we focus on slots in H-fields and their algebraic closures, and implement the approximation arguments for obtaining (strongly) split-normal or repulsive-normal slots. In Part 5 we begin the analytic part of the paper, introducing Hardy fields, showing that maximal Hardy fields are ω-free, and investigating the universal exponential extensions of Hardy fields. In the final act (Part 6) we prove our Fixed Point Theorem and give the proof of Theorem B. We finish with a coda (Part 7) consisting of applications, including the proof of Theorem A and the corollaries above. We refer to the introduction of each part for more details about their respective contents.

Previous work. Theorem A for P of order 1 is in [59]. By [104] there exists a Hardy field $H \supseteq \mathbb{R}$ isomorphic as an ordered differential field to $\mathbb{T}_{\mathrm{g}}$, so by [103] this H has the intermediate value property for all differential polynomials over it. We announced the ω-freeness of maximal Hardy fields already in [12].

Notations and terminology. We freely use the notations and conventions from our book [ADH], and recall here a few. Throughout, m, n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$. Given an additively written abelian group A we let $A^{\neq} := A \setminus \{0\}$. Rings (usually, but not always, commutative) are associative with identity 1. For a ring R we let $R^{\times}$ be the multiplicative group of units of R (consisting of the $a \in R$ such that $ab = ba = 1$ for some $b \in R$).

A *differential ring* is a commutative ring R containing (an isomorphic copy of) $\mathbb{Q}$ as a subring and equipped with a derivation $\partial\colon R \to R$, in which case $C_R := \ker \partial$ is a subring of R, called the ring of constants of R, and $\mathbb{Q} \subseteq C_R$. A *differential field* is a differential ring K whose underlying ring is a field. In this case C_K as a subfield of K, and if K is understood from the context we often write C instead of C_K. An *ordered differential field* is an ordered field equipped with a derivation on its underlying field; such an ordered differential field is in particular a differential ring.

Often we are given a differential field H in which -1 is not a square, and then $H[\mathrm{i}]$ is a differential field extension with $\mathrm{i}^2 = -1$. Then for $z \in H[\mathrm{i}]$, $z = a + b\mathrm{i}$, $a, b \in H$

we set $\operatorname{Re} z := a$, $\operatorname{Im} z := b$, and $\overline{z} := a - bi$. Hence $z \mapsto \overline{z}$ is an automorphism of the differential field H. If in addition there is given a differential field extension F of H in which -1 is not a square, we always tacitly arrange i to be such that $H[i]$ is a differential subfield of the differential field extension $F[i]$ of F.

Let R be a differential ring and $a \in R$. When its derivation ∂ is clear from the context we denote $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$ by $a', a'', \dots, a^{(n)}, \dots$, and if $a \in R^\times$, then $a^\dagger := a'/a$ denotes the logarithmic derivative of a, so $(ab)^\dagger = a^\dagger + b^\dagger$ for all $a, b \in R^\times$. We have the differential ring $R\{Y\} = R[Y, Y', Y'', \dots]$ of differential polynomials in a differential indeterminate Y over R. Given $P = P(Y) \in R\{Y\}$, the smallest $r \in \mathbb{N}$ such that $P \in R[Y, Y', \dots, Y^{(r)}]$ is called the order of P, denoted by $r = \operatorname{order}(P)$; if P has order r, then $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$, as in [ADH, 4.2], with $\boldsymbol{i}$ ranging over tuples $(i_0, \dots, i_r) \in \mathbb{N}^{1+r}$, $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(r)})^{i_r}$, coefficients $P_{\boldsymbol{i}}$ in R, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. For $P \in R\{Y\}$ and $a \in R$ we let $P_{+a}(Y) := P(a+Y)$ and $P_{\times a}(Y) := P(aY)$ be the *additive conjugate* and the *multiplicative conjugate* of P by a, respectively. For $\phi \in R^\times$ we also let R^ϕ be the *compositional conjugate of R by ϕ*: the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ (usually denoted by δ) instead of ∂. We have an R-algebra isomorphism $P \mapsto P^\phi \colon R\{Y\} \to R^\phi\{Y\}$ such that $P^\phi(y) = P(y)$ for all $y \in R$; see [ADH, 5.7].

For a field K we have $K^\times = K^{\neq}$, and a (Krull) valuation on K is a surjective map $v \colon K^\times \to \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v \colon K \to \Gamma_\infty := \Gamma \cup \{\infty\}$ by $v(0) = \infty$, where the ordering on Γ is extended to a total ordering on Γ_∞ by $\gamma < \infty$ for all $\gamma \in \Gamma$. A *valued field* K is a field (also denoted by K) together with a valuation ring $\mathcal{O}$ of that field, and the corresponding valuation $v \colon K^\times \to \Gamma$ on the underlying field is such that $\mathcal{O} = \{a \in K : va \geqslant 0\}$ as explained in [ADH, 3.1].

Let K be a valued field with valuation ring $\mathcal{O}_K$ and valuation $v \colon K^\times \to \Gamma_K$. Then $\mathcal{O}_K$ is a local ring with maximal ideal $\mathcal{o}_K = \{a \in K : va > 0\}$ and residue field $\operatorname{res}(K) = \mathcal{O}_K/\mathcal{o}_K$. If $\operatorname{res}(K)$ has characteristic zero, then K is said to be of equicharacteristic zero. When, as here, we use the capital K for the valued field under consideration, then we denote Γ_K, $\mathcal{O}_K$, $\mathcal{o}_K$, by Γ, $\mathcal{O}$, $\mathcal{o}$, respectively. A very handy alternative notation system in connection with the valuation is as follows. With a, b ranging over K, set

$$\begin{aligned} a \asymp b \ &:\Leftrightarrow\ va = vb, & a \preccurlyeq b \ &:\Leftrightarrow\ va \geqslant vb, & a \prec b \ &:\Leftrightarrow\ va > vb, \\ a \succcurlyeq b \ &:\Leftrightarrow\ b \preccurlyeq a, & a \succ b \ &:\Leftrightarrow\ b \prec a, & a \sim b \ &:\Leftrightarrow\ a - b \prec a. \end{aligned}$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$ and $a \asymp b$, and that $\sim$ is an equivalence relation on $K^\times$. Given a valued field extension L of K, we identify in the usual way $\operatorname{res}(K)$ with a subfield of $\operatorname{res}(L)$, and Γ with an ordered subgroup of Γ_L. We use *pc-sequence* to abbreviate *pseudocauchy sequence*, and $a_\rho \leadsto a$ indicates that (a_ρ) is a pc-sequence pseudoconverging to a; here the a_ρ and a lie in a valued field understood from the context, see [ADH, 2.2, 3.2].

As in [ADH], a *valued differential field* is a valued field of equicharacteristic zero together with a derivation, generally denoted by ∂, on the underlying field. (Unlike [11] we do not assume in this definition that ∂ is continuous with respect to the valuation topology.) A valued differential field K is said to have *small derivation* if $\partial\smallo \subseteq \smallo$; then also $\partial\mathcal{O} \subseteq \mathcal{O}$ [ADH, 4.4.2], and so ∂ induces a derivation on $\operatorname{res}(K)$ making the residue morphism $\mathcal{O} \to \operatorname{res}(K)$ into a morphism of differential rings.

We shall also consider various special classes of valued differential fields introduced in [ADH], such as the class of *asymptotic fields* (and their relatives, *H-asymptotic fields*) and its subclass of *pre*-d-*valued fields*, which in turn contains the class of d-*valued fields* [ADH, 9.1, 10.1]. (As usual in [ADH], the prefix "d" abbreviates "differential".) Every asymptotic field K has its associated asymptotic couple (Γ, ψ), where $\psi\colon \Gamma^{\neq} \to \Gamma$ satisfies $\psi(vg) = v(g^\dagger)$ for $g \in K^\times$ with $vg \neq 0$. See [ADH, 9.1, 9.2] for more on asymptotic couples, in particular the taxonomy of asymptotic fields introduced via their asymptotic couples: having a *gap,* being *grounded,* having *asymptotic integration,* and having *rational asymptotic integration.*

An *ordered valued differential field* is a valued differential field K equipped with an ordering on K making K an ordered field. An ordered differential field K is called an *H-field* if for all $f \in K$ with if $f \succ 1$ we have $f^\dagger > 0$, and $\mathcal{O} = C + \smallo$ where $\mathcal{O} = \big\{g \in K : |g| \leqslant c \text{ for some } c \in C\big\}$ and $\smallo$ is the maximal ideal of the convex subring $\mathcal{O}$ of K. Thus K equipped with its valuation ring $\mathcal{O}$ is an ordered valued differential field. *Pre-H-fields* are the ordered valued differential subfields of H-fields. See [ADH, 10.5] for basic facts about (pre-) H-fields. An H-field K is said to be *Liouville closed* if K is real closed and for all $f, g \in K$ there exists $y \in K^\times$ with $y' + fy = g$. Every H-field extends to a Liouville closed one; see [ADH, 10.6].

We alert the reader that in a few places we refer to the Liouville closed H-field $\mathbb{T}_{\mathrm{g}}$ of grid-based transseries from [103], which is denoted there by $\mathbb{T}$. Here we adopt the notation of [ADH] where $\mathbb{T}$ is the larger field of logarithmic-exponential series.

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Part 1. Preliminaries

After generalities on linear differential operators and differential polynomials in Section 1.1, we investigate the group of logarithmic derivatives in valued differential fields of various kinds (Section 1.2) and recall how iterated logarithmic derivatives can be used to study the asymptotic behavior of differential polynomials over such valued differential fields for "large" arguments (Section 1.3). We also assemble some basic preservation theorems for $\uplambda$-*freeness* and $\upomega$-*freeness* (Section 1.4) and continue the study of linear differential operators over H-asymptotic fields initiated in [ADH, 5.6, 14.2] (Section 1.5). In our analysis of the solutions of algebraic differential equations over H-asymptotic fields in Part 3, special pc-sequences in the sense of [ADH, 3.4] play an important role; Section 1.6 explains why. A cornerstone of [ADH] is the concept of *newtonianity*, an analogue of henselianity appropriate for H-asymptotic fields with asymptotic integration [ADH, Chapter 14]. Related to this is *differential-henselianity* [ADH, Chapter 7], which makes sense for a broader class of valued differential fields. In Sections 1.7 and 1.8 we further explore these notions. Among other things, we study their persistence under taking the completion of a valued differential field with small derivation (as defined in [ADH, 4.4]).

1.1. Linear Differential Operators and Differential Polynomials

This section gathers miscellaneous facts of a general nature about linear differential operators and differential polynomials, sometimes in a valued differential setting. We first discuss splittings and least common left multiples of linear differential operators, then recall the complexity and the separant of differential polynomials, and finally deduce some useful estimates for derivatives of exponential terms.

Splittings. In this subsection K is a differential field. Let $A \in K[\partial]^{\neq}$ be monic of order $r \geqslant 1$. A **splitting of** A **over** K is a tuple $(g_1, \dots, g_r) \in K^r$ such that $A = (\partial - g_1)\cdots(\partial - g_r)$. If $(g_1, \dots, g_r)$ is a splitting of A over K and $\mathfrak{n} \in K^\times$, then $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is a splitting of $A_{\ltimes\mathfrak{n}} = \mathfrak{n}^{-1}A\mathfrak{n}$ over K.

Suppose $A = A_1 \cdots A_m$ where every $A_i \in K[\partial]$ is monic of positive order r_i (so $r = r_1 + \cdots + r_m$). Given any splittings

$$(g_{11}, \dots, g_{1r_1}),\ \dots,\ (g_{m1}, \dots, g_{mr_m})$$

of $A_1, \dots, A_m$, respectively, we obtain a splitting

$$\big(g_{11}, \dots, g_{1r_1},\ \dots,\ g_{m1}, \dots, g_{mr_m}\big)$$

of A by concatenating the given splittings of $A_1, \dots, A_m$ in the order indicated, and call it a splitting of A **induced** by the factorization $A = A_1 \cdots A_m$. For $B \in K[\partial]$ of order $r \geqslant 1$ we have $B = bA$ with $b \in K^\times$ and monic $A \in K[\partial]$, and then a **splitting of** B **over** K is by definition a splitting of A over K. A splitting of B over K remains a splitting of aB over K, for any $a \in K^\times$. Thus:

Lemma 1.1.1. *If $B \in K[\partial]$ has order $r \geqslant 1$, and $(g_1, \dots, g_r)$ is a splitting of B over K and $\mathfrak{n} \in K^\times$, then $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is a splitting of $B_{\ltimes\mathfrak{n}}$ over K and a splitting of $B\mathfrak{n}$ over K.*

From [ADH, 5.1, 5.7] we know that if $A \in K[\partial]$ splits over K, then for any $\phi \in K^\times$ the operator $A^\phi \in K^\phi[\updelta]$ splits over K^ϕ; here is how a splitting of A over K transforms into a splitting of A^ϕ over K^ϕ:

Lemma 1.1.2. *Let $\phi \in K^\times$ and*

$$A \;=\; c(\partial - a_1)\cdots(\partial - a_r) \quad \text{with } c \in K^\times \text{ and } a_1,\dots,a_r \in K.$$

Then in $K^\phi[\delta]$ we have

$$A^\phi \;=\; c\phi^r(\delta - b_1)\cdots(\delta - b_r) \quad \text{where } b_j := \phi^{-1}\big(a_j - (r-j)\phi^\dagger\big)\ (j=1,\dots,r).$$

Proof. Induction on r. The case $r=0$ being obvious, suppose $r \geqslant 1$, and set $B := (\partial - a_2)\cdots(\partial - a_r)$. By inductive hypothesis

$$B^\phi = \phi^{r-1}(\delta - b_2)\cdots(\delta - b_r) \quad \text{where } b_j := \phi^{-1}\big(a_j - (r-j)\phi^\dagger\big) \text{ for } j=2,\dots,r.$$

Then

$$A^\phi \;=\; c\phi\big(\delta - (a_1/\phi)\big)\, B^\phi \;=\; c\phi^r\,\big(\delta - (a_1/\phi)\big)_{\ltimes\phi^{r-1}}\,(\delta - b_2)\cdots(\delta - b_r)$$

with

$$\big(\delta - (a_1/\phi)\big)_{\ltimes\phi^{r-1}} = \delta - (a_1/\phi) + (r-1)\phi^\dagger/\phi$$

by [ADH, p. 243]. □

A different kind of factorization, see for example [156], reduces the process of solving the differential equation $A(y)=0$ to repeated multiplication and integration:

Lemma 1.1.3. *Let $A \in K[\partial]^{\neq}$ be monic of order $r \geqslant 1$. If $b_1,\dots,b_r \in K^\times$ and*

$$A \;=\; b_1\cdots b_{r-1}b_r(\partial b_r^{-1})(\partial b_{r-1}^{-1})\cdots(\partial b_1^{-1}),$$

then $(a_r,\dots,a_1)$, where $a_j := (b_1\cdots b_j)^\dagger$ for $j=1,\dots,r$, is a splitting of A over K. Conversely, if $(a_r,\dots,a_1)$ is a splitting of A over K and $b_1,\dots,b_r \in K^\times$ are such that $b_j^\dagger = a_j - a_{j-1}$ for $j=1,\dots,r$ with $a_0 := 0$, then A is as in the display.

This follows easily by induction on r.

Real splittings. Let H be a differential field in which -1 is not a square. Then we let i denote an element in a differential field extension of H with $i^2=-1$, and consider the differential field $K = H[i]$. Suppose $A \in H[\partial]$ is monic of order 2 and splits over K, so

$$A \;=\; (\partial - f)(\partial - g), \qquad f,g \in K.$$

Then

$$A \;=\; \partial^2 - (f+g)\partial + fg - g',$$

and thus $f \in H$ iff $g \in H$. One checks easily that if $g \notin H$, then there are unique $a,b \in H$ with $b \neq 0$ such that

$$f \;=\; a - bi + b^\dagger, \qquad g \;=\; a + bi,$$

and thus

$$A \;=\; \partial^2 - (2a + b^\dagger)\partial + a^2 + b^2 - a' + ab^\dagger.$$

Conversely, if $a,b \in H$ and $b \neq 0$, then for $f := a - bi + b^\dagger$ and $g := a + bi$ we have $(\partial - f)(\partial - g) \in H[\partial]$.

Let now $A \in H[\partial]$ be monic of order $r \geqslant 1$.

Lemma 1.1.4. *Suppose A splits over K. Then $A = A_1\cdots A_m$ for some $A_1,\dots,A_m$ in $H[\partial]$ that are monic and irreducible of order 1 or 2 and split over K.*

Proof. By [ADH, 5.1.35], $A = A_1\cdots A_m$, where every $A_i \in H[\partial]$ is monic and irreducible of order 1 or 2. By [ADH, 5.1.22], such A_i split over K. □

Definition 1.1.5. A **real splitting of** A (over K) is a splitting of A over K that is induced by a factorization $A = A_1 \cdots A_m$ where every $A_i \in H[\partial]$ is monic of order 1 or 2 and splits over K. (Note that we do not require the A_i of order 2 to be irreducible in $H[\partial]$.)

Thus if A splits over K, then A has a real splitting over K by Lemma 1.1.4. Note that if $(g_1, \dots, g_r)$ is a real splitting of A and $\mathfrak{n} \in H^\times$, then $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is a real splitting of $A_{\ltimes \mathfrak{n}}$.

It is convenient to extend the above slightly: for $B \in H[\partial]$ of order $r \geqslant 1$ we have $B = bA$ with $b \in H^\times$ and monic $A \in H[\partial]$, and then a **real splitting of** B (over K) is by definition a real splitting of A (over K).

In later use, H is a valued differential field with small derivation such that -1 is not a square in the differential residue field $\operatorname{res}(H)$. For such H, let $\mathcal{O}$ be the valuation ring of H. We make K a valued differential field extension of H with small derivation by taking $\mathcal{O}_K = \mathcal{O} + \mathcal{O}i$ as the valuation ring of K. We have the residue map $a \mapsto \operatorname{res} a \colon \mathcal{O}_K \to \operatorname{res}(K)$, so $\operatorname{res}(K) = \operatorname{res}(H)[i]$, writing here i for $\operatorname{res} i$. We extend this map to a ring morphism $B \mapsto \operatorname{res} B \colon \mathcal{O}_K[\partial] \to \operatorname{res}(K)[\partial]$ by sending $\partial \in \mathcal{O}[\partial]$ to $\partial \in \operatorname{res}(K)[\partial]$.

Lemma 1.1.6. *Suppose $(g_1, \dots, g_r) \in \operatorname{res}(K)^r$ is a real splitting of a monic operator $D \in \operatorname{res}(H)[\partial]$ of order $r \geqslant 1$. Then there are $b_1, \dots, b_r \in \mathcal{O}_K$ such that*

$$B \ := \ (\partial - b_1) \cdots (\partial - b_r) \in \mathcal{O}[\partial],$$

$(b_1, \dots, b_r)$ is a real splitting of B, and $\operatorname{res} b_j = g_j$ for $j = 1, \dots, r$.

Proof. We can assume $r \in \{1, 2\}$. The case $r = 1$ is obvious, so let $r = 2$. Then the case where $g_1, g_2 \in \operatorname{res}(H)$ is again obvious, so let $g_1 = \operatorname{res}(a) - \operatorname{res}(b)i + (\operatorname{res} b)^\dagger$, $g_2 = \operatorname{res}(a) + \operatorname{res}(b)i$ where $a, b \in \mathcal{O}$, $\operatorname{res} b \neq 0$. Set $b_1 := a - bi + b^\dagger$, $b_2 := a + bi$. Then $b_1, b_2 \in \mathcal{O}_K$ with $\operatorname{res} b_1 = g_1$, $\operatorname{res} b_2 = g_2$, and $B := (\partial - b_1)(\partial - b_2) \in \mathcal{O}[\partial]$ have the desired properties. $\square$

Least common left multiples and complex conjugation. In this subsection H is a differential field. Recall from [ADH, 5.1] the definition of the *least common left multiple* $\operatorname{lclm}(A_1, \dots, A_m)$ of operators $A_1, \dots, A_m \in H[\partial]^{\neq}$, $m \geqslant 1$: this is the monic operator $A \in H[\partial]$ such that $H[\partial]A_1 \cap \cdots \cap H[\partial]A_m = H[\partial]A$. For $A, B \in H[\partial]^{\neq}$ we have:

$$\max\big\{\operatorname{order}(A), \operatorname{order}(B)\big\} \ \leqslant \ \operatorname{order}\big(\operatorname{lclm}(A, B)\big) \ \leqslant \ \operatorname{order}(A) + \operatorname{order}(B).$$

For the inequality on the right, note that the natural $H[\partial]$-module morphism

$$H[\partial] \ \to \ \big(H[\partial]/H[\partial]A\big) \times \big(H[\partial]/H[\partial]B\big)$$

has kernel $H[\partial] \operatorname{lclm}(A, B)$, and for $D \in H[\partial]^{\neq}$, the H-linear space $H[\partial]/K[\partial]D$ has dimension $\operatorname{order} D$.

We now assume that -1 is not a square in H; then we have a differential field extension $H[i]$ where $i^2 = -1$. The automorphism $a + bi \mapsto \overline{a + bi} := a - bi$ ($a, b \in H$) of the differential field $H[i]$ extends uniquely to an automorphism $A \mapsto \overline{A}$ of the ring $H[i][\partial]$ with $\overline{\partial} = \partial$. Let $A \in H[i][\partial]$; then $\overline{A} = A \Longleftrightarrow A \in H[\partial]$. Hence if $A \neq 0$ is monic, then $L := \operatorname{lclm}(A, \overline{A}) \in H[\partial]$ and thus $L = BA = \overline{B}\,\overline{A}$ where $B \in H[i][\partial]$.

Example 1.1.7. Let $A = \partial - a$ where $a \in H[i]$. If $a \in H$, then $\operatorname{lclm}(A, \overline{A}) = A$, and if $a \notin H$, then $\operatorname{lclm}(A, \overline{A}) = (\partial - b)(\partial - a) = (\partial - \overline{b})(\partial - \overline{a})$ where $b \in H[i] \setminus H$.

Let now F be a differential field extension of H in which -1 is not a square; we assume that i is an element of a differential ring extension of F.

Lemma 1.1.8. *Let $A \in H[\mathrm{i}][\partial]^{\neq}$ be monic, $b \in H[\mathrm{i}]$, and $f \in F[\mathrm{i}]$ such that $A(f) = b$. Let $B \in H[\mathrm{i}][\partial]$ be such that $L := \operatorname{lclm}(A, \overline{A}) = BA$. Then $L(f) = B(b)$ and hence $L\big(\operatorname{Re}(f)\big) = \operatorname{Re}\big(B(b)\big)$ and $L\big(\operatorname{Im}(f)\big) = \operatorname{Im}\big(B(b)\big)$.*

In Sections 6.4 and 6.7 we need a slight extension of this lemma:

Remark 1.1.9. Let F be a differential ring extension of H in which -1 is not a square and let i be an element of a commutative ring extension of F such that $i^2 = -1$ and the F-algebra $F[\mathrm{i}] = F + Fi$ is a free F-module with basis $1, i$. For $f = g + hi \in F[\mathrm{i}]$ with $g, h \in F$ we set $\operatorname{Re}(f) := g$ and $\operatorname{Im}(f) := h$. We make $F[\mathrm{i}]$ into a differential ring extension of F in the only way possible (which has $i' = 0$). Then Lemma 1.1.8 goes through.

Complexity and the separant. We recall some definitions and observations from [ADH, 4.3]. Let K be a differential field and $P \in K\{Y\}$, $P \notin K$, and set $r = \operatorname{order} P$, $s = \deg_{Y^{(r)}} P$, and $t = \deg P$. Then the *complexity* of P is the triple $\mathrm{c}(P) = (r, s, t) \in \mathbb{N}^3$; we order $\mathbb{N}^3$ lexicographically. Let $a \in K$. Then $\mathrm{c}(P_{+a}) = \mathrm{c}(P)$, and $\mathrm{c}(P_{\times a}) = \mathrm{c}(P)$ if $a \neq 0$. The differential polynomial $S_P := \frac{\partial P}{\partial Y^{(r)}}$ is called the *separant* of P; thus $\mathrm{c}(S_P) < \mathrm{c}(P)$ (giving complexity $(0,0,0)$ to elements of K), and $S_{aP} = aS_P$ if $a \neq 0$. Moreover:

Lemma 1.1.10. *We have*

$$S_{P_{+a}} = (S_P)_{+a}, \quad S_{P_{\times a}} = a \cdot (S_P)_{\times a}, \quad \textit{and} \quad S_{P^\phi} = \phi^r (S_P)^\phi \textit{ for } \phi \in K^\times.$$

Proof. For $S_{P_{+a}}$ and $S_{P_{\times a}}$ this is from [ADH, p. 216]; for S_{P^ϕ}, express P as a polynomial in $Y^{(r)}$ and use $(Y^{(r)})^\phi = \phi^r Y^{(r)} + \text{lower order terms}$. $\square$

Some transformation formulas. In this subsection K is a differential ring. Let $u \in K^\times$. Then in $K[\partial]$ we have

$$\begin{aligned}
(\partial - u^\dagger)^0 &= 1, \\
(\partial - u^\dagger)^1 &= \partial - u' u^{-1}, \\
(\partial - u^\dagger)^2 &= \partial^2 - 2u'u^{-1}\partial + \big(2(u')^2 - u''u\big)u^{-2}.
\end{aligned}$$

More generally:

Lemma 1.1.11. *There are differential polynomials $Q^n_k(X) \in \mathbb{Q}\{X\}$ $(0 \leqslant k \leqslant n)$, independent of K and u, such that $Q^n_n = 1$ and*

$$(\partial - u^\dagger)^n = Q^n_n(u)\partial^n + Q^n_{n-1}(u)u^{-1}\partial^{n-1} + \cdots + Q^n_0(u)u^{-n}.$$

Setting $Q^n_{-1} := 0$, we have

$$Q^{n+1}_k(X) = Q^n_k(X)'X + Q^n_k(X)(k - n - 1)X' + Q^n_{k-1}(X) \qquad (0 \leqslant k \leqslant n).$$

Hence every Q^n_k with $0 \leqslant k \leqslant n$ has integer coefficients and is homogeneous of degree $n - k$ and isobaric of weight $n - k$.

Proof. By induction on n. The case $n=0$ is obvious. Suppose for a certain n we have Q^n_k for $0 \leqslant k \leqslant n$ as above. Then

$$\begin{aligned}
(\partial - u^\dagger)^{n+1} &= (\partial - u^\dagger)\sum_{k=0}^{n} Q^n_k(u)u^{k-n}\partial^k \\
&= \sum_{k=0}^{n}\Big(\big(Q^n_k(u)u^{k-n}\big)' - u^\dagger Q^n_k(u)u^{k-n}\Big)\partial^k + \sum_{k=0}^{n} Q^n_k(u)u^{k-n}\partial^{k+1} \\
&= \sum_{k=0}^{n}\Big(Q^n_k(u)'u + Q^n_k(u)(k-n-1)u'\Big)u^{k-(n+1)}\partial^k + \\
&\ \sum_{k=1}^{n+1} Q^n_{k-1}(u)u^{k-(n+1)}\partial^k,
\end{aligned}$$

and this yields the inductive step. $\square$

For $f \in K$ we have

$$(fu^{-1})^{(n)} = (\partial^n)_{\ltimes u^{-1}}(f)u^{-1} = (\partial_{\ltimes u^{-1}})^n(f)u^{-1} = (\partial - u^\dagger)^n(f)u^{-1}$$

and hence:

Corollary 1.1.12. *Let $f \in K$; then*

$$(fu^{-1})^{(n)} = Q^n_n(u)f^{(n)}u^{-1} + Q^n_{n-1}(u)f^{(n-1)}u^{-2} + \cdots + Q^n_0(u)fu^{-(n+1)}.$$

Estimates for derivatives of exponential terms. In this subsection K is an asymptotic differential field with small derivation, and $\phi \in K$. We also fix $\mathfrak{m} \in K^\times$ with $\mathfrak{m} \prec 1$. Recall from [ADH, 4.2] that for $P \in K\{Y\}^{\neq}$ the *multiplicity of P at* 0 is $\operatorname{mul}(P) = \min\{d \in \mathbb{N} : P_d \neq 0\}$, where P_d denotes the homogeneous part of degree d of P. Here is a useful bound:

Lemma 1.1.13. *Let $r \in \mathbb{N}$ and $y \in K$ satisfy $y \prec \mathfrak{m}^{r+m} \prec 1$. Then $P(y) \prec \mathfrak{m}^{m\mu}P$ for all $P \in K\{Y\}^{\neq}$ of order at most r with $\mu = \operatorname{mul}(P) \geqslant 1$.*

Proof. Note that $0 \neq \mathfrak{m} \prec 1$ and $r+m \geqslant 1$. Hence

$$y' \prec (\mathfrak{m}^{r+m})' = (r+m)\mathfrak{m}^{r+m-1}\mathfrak{m}' \prec \mathfrak{m}^{r-1+m},$$

so by induction $y^{(i)} \prec \mathfrak{m}^{r-i+m}$ for $i = 0,\dots,r$. Hence $y^{\boldsymbol{i}} \prec \mathfrak{m}^{(r+m)|\boldsymbol{i}| - \|\boldsymbol{i}\|} \preccurlyeq \mathfrak{m}^{m|\boldsymbol{i}|}$ for nonzero $\boldsymbol{i} = (i_0,\dots,i_r) \in \mathbb{N}^{1+r}$, which yields the lemma. $\square$

Corollary 1.1.14. *If $f \in K$ and $f \prec \mathfrak{m}^n$, then $f^{(k)} \prec \mathfrak{m}^{n-k}$ for $k = 0,\dots,n$.*

Proof. This is a special case of Lemma 1.1.13. $\square$

Corollary 1.1.15. *Let $f \in K^\times$ and $n \geqslant 1$ be such that $f \preccurlyeq \mathfrak{m}^n$. Then $f^{(k)} \prec \mathfrak{m}^{n-k}$ for $k = 1,\dots,n$.*

Proof. Note that $\mathfrak{m}^n \neq 0$, so $f' \preccurlyeq (\mathfrak{m}^n)' = n\mathfrak{m}^{n-1}\mathfrak{m}' \prec \mathfrak{m}^{n-1}$ [ADH, 9.1.3]. Now apply Corollary 1.1.14 with f', $n-1$ in place of f, n. $\square$

In the remainder of this subsection we let $\xi \in K^\times$ and assume $\xi \succ 1$ and $\zeta := \xi^\dagger \succcurlyeq 1$.

Lemma 1.1.16. *The elements $\xi, \zeta \in K$ have the following asymptotic properties:*

(i) *$\zeta^n \prec \xi$ for all n;*
(ii) *$\zeta^{(n)} \preccurlyeq \zeta^2$ for all n.*

Thus for each $P \in \mathcal{O}\{Z\}$ there is an $N \in \mathbb{N}$ with $P(\zeta) \preccurlyeq \zeta^N$, and hence $P(\zeta) \prec \xi$.

Proof. Part (i) follows from [ADH, 9.2.10(iv)] for $\gamma = v(\xi)$. As to (ii), if $\zeta' \preccurlyeq \zeta$, then $\zeta^{(n)} \preccurlyeq \zeta$ by [ADH, 4.5.3], and we are done. Suppose $\zeta' \succ \zeta$ and set $\gamma := v(\zeta)$. Then $\gamma, \gamma^\dagger < 0$, so $\gamma^\dagger = o(\gamma)$ by [ADH, 9.2.10(iv)] and hence $v(\zeta^{(n)}) = \gamma + n\gamma^\dagger > 2\gamma = v(\zeta^2)$ by [ADH, 6.4.1(iv)]. $\square$

Recall from [ADH, 5.8] that for a homogeneous differential polynomial $P \in K\{Y\}$ of degree $d \in \mathbb{N}$ the *Riccati transform* $\operatorname{Ri}(P) \in K\{Z\}$ of P satisfies

$$\operatorname{Ri}(P)(z) = P(y)/y^d \quad \text{for } y \in K^\times,\ z = y^\dagger.$$

In the next two corollaries, $l \in \mathbb{Z}$, $\xi = \phi'$, and e^ϕ denotes a unit of a differential ring extension of K with multiplicative inverse $\mathrm{e}^{-\phi}$ and such that $(\mathrm{e}^\phi)' = \phi' \mathrm{e}^\phi$.

Corollary 1.1.17. $(\xi^l \mathrm{e}^\phi)^{(n)} \ = \ \xi^{l+n}(1+\varepsilon)\,\mathrm{e}^\phi$ *where* $\varepsilon \in K$, $\varepsilon \prec 1$.

Proof. By Lemma 1.1.16(i) we have $l\zeta + \xi \sim \xi \succ 1$. Now use $(\xi^l \mathrm{e}^\phi)^{(n)}/(\xi^l \mathrm{e}^\phi) = R_n(l\zeta + \xi)$ for $R_n = \operatorname{Ri}(Y^{(n)})$ in combination with [ADH, 11.1.5]. $\square$

Applying the corollary above with ϕ, ξ replaced by $-\phi$, $-\xi$, respectively, we obtain:

Corollary 1.1.18. $(\xi^l \mathrm{e}^{-\phi})^{(n)} \ = \ (-1)^n \xi^{l+n}(1+\delta)\,\mathrm{e}^{-\phi}$ *where* $\delta \in K$, $\delta \prec 1$.

Estimates for Riccati transforms. In this subsection K is a valued differential field with small derivation. For later use we prove variants of [ADH, 11.1.5].

Lemma 1.1.19. *If* $z \in K^{\succ 1}$, *then* $R_n(z) = z^n(1+\varepsilon)$ *with* $v\varepsilon \geqslant v(z^{-1}) + o(vz) > 0$.

Proof. This is clear for $n = 0$ and $n = 1$. Suppose $z \succ 1$, $n \geqslant 1$, and $R_n(z) = z^n(1+\varepsilon)$ with ε as in the lemma. As in the proof of [ADH, 11.1.5],

$$R_{n+1}(z) \ = \ z^{n+1}\left(1 + \varepsilon + n\frac{z^\dagger}{z}(1+\varepsilon) + \frac{\varepsilon'}{z}\right).$$

Now $v(z^\dagger) \geqslant o(vz)$: this is obvious if $z^\dagger \preccurlyeq 1$, and follows from $\nabla(\gamma) = o(\gamma)$ for $\gamma \neq 0$ if $z^\dagger \succ 1$ [ADH, 6.4.1(iii)]. This gives the desired result in view of $\varepsilon' \prec 1$. $\square$

Lemma 1.1.20. *Suppose* $\partial\mathcal{O} \subseteq \scriptstyle\mathcal{O}$. *If* $z \in K^{\succcurlyeq 1}$, *then* $R_n(z) = z^n(1+\varepsilon)$ *with* $\varepsilon \prec 1$.

Proof. The case $z \succ 1$ follows from Lemma 1.1.19. For $z \asymp 1$, proceed as in the proof of that lemma, using $\partial\mathcal{O} \subseteq \scriptstyle\mathcal{O}$. $\square$

By [ADH, 9.1.3 (iv)] the condition $\partial\mathcal{O} \subseteq \scriptstyle\mathcal{O}$ is satisfied if K is d-valued, or asymptotic with $\Psi \cap \Gamma^> \neq \emptyset$.

Lemma 1.1.21. *Suppose* K *is asymptotic, and* $z \in K$ *with* $0 \neq z \preccurlyeq z' \prec 1$. *Then* $R_n(z) \sim z^{(n-1)}$ *for* $n \geqslant 1$.

Proof. Induction on n gives $z \preccurlyeq z' \preccurlyeq \cdots \preccurlyeq z^{(n)} \prec 1$ for all n. We now show $R_n(z) \sim z^{(n-1)}$ for $n \geqslant 1$, also by induction. The case $n = 1$ is clear from $R_1 = Z$, so suppose $n \geqslant 1$ and $R_n(z) \sim z^{(n-1)}$. Then

$$R_{n+1}(z) \ = \ zR_n(z) + R_n(z)'$$

where $R_n(z)' \sim z^{(n)}$ by [ADH, 9.1.4(ii)] and $zR_n(z) \asymp zz^{(n-1)} \prec z^{(n-1)} \preccurlyeq z^{(n)}$. Hence $R_{n+1}(z) \sim z^{(n)}$. $\square$

Valued differential fields with very small derivation (*)**.** The generalities in this subsection will be used in Section 7.3. Let K be a valued differential field with derivation ∂. Recall that if K has small derivation (that is, $\partial\smallo \subseteq \smallo$), then also $\partial\mathcal{O} \subseteq \mathcal{O}$ by [ADH, 4.4.2], so we have a unique derivation on the residue field $\boldsymbol{k} := \mathcal{O}/\smallo$ that makes the residue morphism $\mathcal{O} \to \boldsymbol{k}$ into a morphism of differential rings (and we call $\boldsymbol{k}$ with this induced derivation the differential residue field of K). We say that ∂ is **very small** if $\partial\mathcal{O} \subseteq \smallo$. So K has very small derivation iff K has small derivation and the induced derivation on $\boldsymbol{k}$ is trivial. If K has small derivation and $\mathcal{O} = C + \smallo$, then K has very small derivation. If K has very small derivation, then so does every valued differential subfield of K, and if L is a valued differential field extension of K with small derivation and $\boldsymbol{k}_L = \boldsymbol{k}$, then L has very small derivation. Moreover:

Lemma 1.1.22. *Let L be a valued differential field extension of K, algebraic over K, and suppose K has very small derivation. Then L also has very small derivation.*

Proof. By [ADH, 6.2.1], L has small derivation. The derivation of $\boldsymbol{k}$ is trivial and $\boldsymbol{k}_L$ is algebraic over $\boldsymbol{k}$ [ADH, 3.1.9], so the derivation of $\boldsymbol{k}_L$ is also trivial. $\square$

Next we focus on pre-d-valued fields with very small derivation. First an easy observation about asymptotic couples:

Lemma 1.1.23. *Let (Γ, ψ) be an asymptotic couple; then*

$$(\Gamma,\psi) \text{ has gap } 0 \iff (\Gamma,\psi) \text{ has small derivation and } \Psi \subseteq \Gamma^{<}.$$

In particular, if (Γ, ψ) has small derivation and does not have gap 0, then each asymptotic couple extending (Γ, ψ) has small derivation.

Corollary 1.1.24. *Suppose K is pre-*d*-valued with small derivation, and suppose 0 is not a gap in K. Then K has very small derivation.*

Proof. The previous lemma gives $g \in K^\times$ with $g \not\asymp 1$ and $g^\dagger \preccurlyeq 1$. Then for each $f \in K$ with $f \preccurlyeq 1$ we have $f' \prec g^\dagger \preccurlyeq 1$. $\square$

Corollary 1.1.25. *Suppose K is pre-*d*-valued of H-type with very small derivation. Then the* d*-valued hull* $\operatorname{dv}(K)$ *of K has small derivation.*

Proof. By Lemma 1.1.23, if 0 is not a gap in K, then every pre-d-valued field extension of K has small derivation. If 0 is a gap in K, then no $b \asymp 1$ in K satisfies $b' \asymp 1$, since K has very small derivation. Thus $\Gamma_{\operatorname{dv}(K)} = \Gamma$ by [ADH, 10.3.2(ii)], so 0 remains a gap in $\operatorname{dv}(K)$. In both cases, $\operatorname{dv}(K)$ has small derivation. $\square$

If K is pre-d-valued and ungrounded, then for each $\phi \in K$ which is active in K, the pre-d-valued field K^ϕ (with derivation $\delta = \phi^{-1}\partial$) has very small derivation.

Now a fact about $A \in F[\partial]$, where F is any differential field. For the definition of $A^{(n)}$, see [ADH, p. 243]. Recall that $\operatorname{Ri}(A) \in F\{Z\}$. For $P \in F\{Z\}$ we denote by $P_{[0]}$ the isobaric part of P of weight 0, as in [ADH, p. 212], so $P \in F[Z]$.

Lemma 1.1.26. *For $P := \operatorname{Ri}(A)_{[0]}$ we have $\operatorname{Ri}(A^{(n)})_{[0]} = P^{(n)}$ for all n.*

Proof. We treat the case $n = 1$; the general case then follows by induction on n. By F-linearity of $A \mapsto \operatorname{Ri}(A)$ we reduce to the case $A = \partial^m$, $m \geqslant 1$, so $P = Z^m$. Then $A' = m\partial^{m-1}$, so $\operatorname{Ri}(A') = mR_{m-1}$ and hence $\operatorname{Ri}(A')_{[0]} = mZ^{m-1} = P'$. $\square$

We need this for the next lemma, which in turn is used in proving Corollary 1.8.50. As usual, we extend the residue map $a \mapsto \operatorname{res} a\colon \mathcal{O} \to \boldsymbol{k}$ to the ring morphism

$$P \mapsto \operatorname{res} P \ :\ \mathcal{O}[Y] \to \boldsymbol{k}[Y], \qquad Y \mapsto Y.$$

Lemma 1.1.27. *Let K have very small derivation, $A \in \mathcal{O}[\partial]$, $R := \operatorname{Ri}(A)$, so $R \in \mathcal{O}\{Z\}$, and $P := R_{[0]} \in \mathcal{O}[Z]$. Let $a \in \mathcal{O}$, so $Q := (R_{+a})_{[0]} \in \mathcal{O}[Z]$. Then*

$$(\operatorname{res} P)_{+\operatorname{res} a} \ =\ \operatorname{res} Q.$$

Proof. It suffices to show $P_{+a} - Q \prec 1$. We have $R(a) \equiv R_{[0]}(a) \bmod \smallo$, as K has very small derivation. Applying this to $\operatorname{Ri}(A^{(n)})$ in place of $R = \operatorname{Ri}(A)$ and using Lemma 1.1.26 yields $\operatorname{Ri}(A^{(n)})(a) \equiv P^{(n)}(a) \bmod \smallo$ for all n. Now use $P_{+a} = \sum_n \frac{1}{n!} P^{(n)}(a)\, Z^n$ by Taylor expansion and $R_{+a} = \sum_n \frac{1}{n!} \operatorname{Ri}(A^{(n)})(a)\, R_n$ by [ADH, p. 301], so $Q = \sum_n \frac{1}{n!} \operatorname{Ri}(A^{(n)})(a)\, Z^n$. □

Rosenlicht's proof of a result of Kolchin (*)**.** Corollary 1.1.30 below will be used in Section 7.4. Let K be a differential field and $m \geqslant 1$, $P \in K\{Y\}$, $\deg P < m$.

Lemma 1.1.28. *Let L be a differential field extension of K and $t \in L$, $t' \in K + tK$, and suppose L is algebraic over $K(t)$. If $y^m = P(y)$ for some $y \in L$, then $z^m = P(z)$ for some z in a differential field extension of K which is algebraic over K.*

Proof. We may assume t is transcendental over K. View $K(t)$ as a subfield of the Laurent series field $F = K(\!(t^{-1})\!)$. We equip F with the valuation ring $\mathcal{O}_F = K[[t^{-1}]]$ and the continuous derivation extending that of $K(t)$, cf. [ADH, p. 226]. Then the valued differential field F is monotone. Hence the valued differential subfield $K(t)$ of F is also monotone. We equip L with a valuation ring $\mathcal{O}_L$ lying over $\mathcal{O}_F \cap K(t)$. Then L is monotone by [ADH, 6.3.10]. We identify K with its image under the residue morphism $a \mapsto \operatorname{res} a\colon \mathcal{O}_L \to \boldsymbol{k}_L := \operatorname{res}(L)$. Then K is a differential subfield of the differential residue field $\boldsymbol{k}_L$ of L, and $\boldsymbol{k}_L$ is algebraic over K. Let now $y \in L$ with $y^m = P(y)$, and towards a contradiction suppose $y \succ 1$. Then $y^\dagger \preccurlyeq 1$, thus $y^{(n)} = y\, R_n(y^\dagger) \preccurlyeq y$ for all n, and hence $y^m = P(y) \preccurlyeq y^d$ where $d = \deg P < m$, a contradiction. Thus $y \preccurlyeq 1$, and $z := \operatorname{res} y \in \boldsymbol{k}_L$ has the required property. □

We recall from [ADH, p. 462] that a *Liouville extension* of K is a differential field extension E of K such that C_E is algebraic over C and for each $t \in E$ there are $t_1, \dots, t_n \in L$ such that $t \in K(t_1, \dots, t_n)$ and for $i = 1, \dots, n$:

(1) t_i is algebraic over $K(t_1, \dots, t_{i-1})$, or
(2) $t_i' \in K(t_1, \dots, t_{i-1})$, or
(3) $t_i' \in t_i K(t_1, \dots, t_{i-1})$.

Proposition 1.1.29 (Rosenlicht [167, p. 371])**.** *Suppose $y^m = P(y)$ for some y in a Liouville extension of K. Then $z^m = P(z)$ for some z in a differential field extension of K which is algebraic over K.*

Proof. A *Liouville sequence* over K is by definition a sequence $(t_1, \dots, t_n)$ of elements of a differential field extension E of K such that for $i = 1, \dots, n$:

(1) t_i is algebraic over $K(t_1, \dots, t_{i-1})$, or
(2) $t_i' \in K(t_1, \dots, t_{i-1})$, or
(3) $t_i' \in t_i K(t_1, \dots, t_{i-1})$.

Note that then $K_i := K(t_1,\dots,t_i)$ is a differential subfield of E for $i = 1,\dots,n$. By induction on $d \in \mathbb{N}$ we now show: if $(t_1,\dots,t_n)$ is a Liouville sequence over K with $\operatorname{trdeg}\big(K(t_1,\dots,t_n)|K\big) = d$ and $y^m = P(y)$ for some $y \in K(t_1,\dots,t_n)$, then the conclusion of the proposition holds. This is obvious for $d = 0$, so let $(t_1,\dots,t_n)$ be a Liouville sequence over K with $\operatorname{trdeg}\big(K(t_1,\dots,t_n)|K\big) = d \geqslant 1$ and $y^m = P(y)$ for some $y \in K(t_1,\dots,t_n)$. Take $i \in \{1,\dots,n\}$ maximal such that t_i is transcendental over $K_{i-1} = K(t_1,\dots,t_{i-1})$. Then $t_i' \in K_{i-1} + t_i K_{i-1}$, and $K(t_1,\dots,t_n)$ is algebraic over $K(t_1,\dots,t_i)$. Applying Lemma 1.1.28 to K_{i-1}, t_i in the role of K, t yields a z in an algebraic differential field extension of K_{i-1} with $z^m = P(z)$. Now apply the inductive hypothesis to the Liouville sequence $(t_1,\dots,t_{i-1},z)$ over K. □

Corollary 1.1.30 (Kolchin)**.** *Let $A \in K[\partial]^{\neq}$, and suppose $A(y) = 0$ for some $y \neq 0$ in a Liouville extension of K. Then $A(y) = 0$ for some $y \neq 0$ in a differential field extension of K with $y^\dagger$ algebraic over K.*

Proof. Let $m = \operatorname{order} A$ and note that $\operatorname{Ri}(A) = Z^m + Q$ with $\deg Q < m$ [ADH, p. 299]. Now apply the proposition above. □

Remark. Corollary 1.1.30 goes back to Liouville [131] in an analytic setting for A of order 2 and $K = \mathbb{C}(x)$ with $C = \mathbb{C}$, $x' = 1$.

Results of Srinivasan (*)**.** Corollary 1.1.35 and Lemma 1.1.36 below will be used in Section 7.6. In this subsection K is a differential field and $a_2,\dots,a_n \in K$, $n \geqslant 2$, $a_n \neq 0$. We investigate the solutions of the differential equation

$$y' \;=\; a_2y^2 + a_3y^3 + \cdots + a_ny^n \tag{1.1.1}$$

in Liouville extensions of K. For $n = 3$ this is a special case of Abel's differential equation of the first kind, first studied by Abel [1] (cf. [111, §A.4.10]). In the next three lemmas and in Proposition 1.1.34 we let y be an element of a differential field extension L of K satisfying (1.1.1). At various places we consider a field $E(\!(t)\!)$ of Laurent series over a field E; it is to be viewed as a valued field in the usual way.

Lemma 1.1.31. *Suppose y is transcendental over K. Then $K\langle y\rangle^\dagger \cap K = K^\dagger$.*

Proof. We view $K\langle y\rangle = K(y)$ as a differential subfield of $K(\!(y)\!)$ equipped with the unique continuous derivation extending that of $K(y)$. Let $f = \sum_{j\geqslant k} f_j y^j \in K(\!(y)\!)$ with $k \in \mathbb{Z}$, all $f_j \in K$, and $f_k \neq 0$. Then

$$f' = f_k'y^k + (f_{k+1}' + kf_ka_2)y^{k+1} + \big(f_{k+2}' + kf_ka_3 + (k+1)f_{k+1}a_2\big)y^{k+2} + \cdots.$$

Hence if $f' = af$ where $a \in K$, then $f_k' = af_k$ and so $a \in K^\dagger$. □

Lemma 1.1.32. *Suppose y transcendental over K and R is a differential subring of K with $C \subseteq R = \partial(R)$ and $a_2,\dots,a_n \in R$. Then $\partial(K\langle y\rangle) \cap K = \partial(K)$.*

Proof. Let $K(\!(y)\!)$ and f be as in the proof of Lemma 1.1.31. Then

$$g := f' = \sum_{j\geqslant k} g_jy^j \quad \text{where } g_j = f_j' + \sum_{i=1}^{n-1}(j-i)f_{j-i}a_{i+1} \text{ and } f_l := 0 \text{ for } l < k.$$

Towards a contradiction, suppose $f' = a \in K \setminus \partial(K)$. By induction on $j \geqslant k$ we show that then $f_j \in R$ and $g_j = 0$. We have $g_k = f_k'$, and if $f_k' \neq 0$, then $k = 0$ and $a = f_k' \in \partial(K)$, a contradiction. Therefore $f_k \in C^\times$ and $g_k = 0$. Suppose $j \geqslant k+1$ and $f_k,\dots,f_{j-1} \in R$. Take $h \in R$ with $h' = \sum_{i=1}^{n-1}(j-i)f_{j-i}a_{i+1}$. Now $g_j = (f_j + h)' \neq a$ since $a \notin \partial(K)$, hence $g_j = 0$ and thus $f_j \in -h + C \subseteq R$. □

Lemma 1.1.33. *Let $t \in L^\times$ and suppose L is algebraic over $K(t)$. If*

(i) *$t' \in K \setminus \partial(K)$ and there is a differential subring of K with $C \subseteq R = \partial(R)$ and $a_2, \dots, a_n \in R$, or*

(ii) *$t^\dagger \in K \setminus \mathbb{Q}K^\dagger$,*

then y is algebraic over K.

Proof. We may assume that t is transcendental over K. Suppose y is transcendental over K. Then t is algebraic over $K(y) = K\langle y\rangle$. If $t' \in K$, then with $R = K\langle y\rangle$, $r = t'$, and $x = t$ in [ADH, 4.6.10] we obtain $t' \in \partial(K\langle y\rangle)$. If $t^\dagger \in K$, then with $R = K\langle y\rangle$, $r = t^\dagger$, and $x = t$ in [ADH, 4.6.11] we get $t^\dagger \in \mathbb{Q}K\langle y\rangle^\dagger$. Thus (i) contradicts Lemma 1.1.32 and (ii) contradicts Lemma 1.1.31. □

Proposition 1.1.34. *Suppose C is algebraically closed, R is a differential subring of K with $C \subseteq R = \partial(R)$ and $a_2, \dots, a_n \in R$, and L is a Liouville extension of K. Then y is algebraic over K.*

Proof. By induction on $d \in \mathbb{N}$ we show: if $(t_1, \dots, t_m) \in L^m$ is a Liouville sequence over K with $\operatorname{trdeg}\big(K(t_1, \dots, t_m)|K\big) = d$ and $y \in K(t_1, \dots, t_m)$, then y is algebraic over K. This is clear for $d = 0$, so let $(t_1, \dots, t_m) \in L^m$ be a Liouville sequence over K with $\operatorname{trdeg}\big(K(t_1, \dots, t_m)|K\big) = d \geqslant 1$ and $y \in K(t_1, \dots, t_m)$. Take $i \in \{1, \dots, m\}$ maximal such that t_i is transcendental over $K_{i-1} := K(t_1, \dots, t_{i-1})$. Then $t_i' \in K_{i-1} \setminus \partial(K_{i-1})$ or $t_i^\dagger \in K_{i-1} \setminus \mathbb{Q}K_{i-1}^\dagger$. Hence y is algebraic over K_{i-1} by Lemma 1.1.33 applied to K_{i-1}, t_i, $K(t_1, \dots, t_m)$ in the role of K, t, L. Now apply the (tacit) inductive hypothesis to the Liouville sequence $(t_1, \dots, t_{i-1}, y)$ over K. □

In the remainder of this subsection C is algebraically closed, $x \in K$, $x' = 1$ (so x is transcendental over C), and $a_2, \dots, a_n \in C[x]$. Applying Proposition 1.1.34 with $C(x)$, $C[x]$ in place of K, R, respectively, yields [196, Proposition 4.1] with a shorter proof:

Corollary 1.1.35 (Srinivasan)**.** *Any y in any Liouville extension of $C(x)$ satisfying* (1.1.1) *is algebraic over $C(x)$.*

We now assume $a_2, \dots, a_n \in C$ and put $P := a_2Y^2 + \cdots + a_nY^n \in C[Y]$. We equip $C(Y)$ with the derivation that is trivial on C and satisfies $Y' = 1$. Thus the field isomorphism $C(x) \to C(Y)$ over C with $x \mapsto Y$ is an isomorphism between the differential subfield $C(x)$ of K and $C(Y)$. Next a special case of [194, Proposition 3.1]:

Lemma 1.1.36 (Srinivasan)**.** *The following are equivalent:*

(i) *there is a non-constant y in a differential field extension of $C(x)$ such that y is algebraic over $C(x)$ and $y' = P(y)$;*

(ii) *there exists $Q \in C(Y)$ such that $Q' = 1/P$.*

Proof. Let $y \notin C$ be algebraic over $C(x)$ with $y' = P(y)$. Then y is transcendental over C, hence x is algebraic over $C(y)$ and so $x \in C(y)$ by [ADH, 4.6.10] applied to $R = C(y)$. Take $Q \in C(Y)$ such that $x = Q(y)$. Then $1 = x' = Q(y)' = Q'(y)P(y)$ and thus $Q' = 1/P$. This shows (i) $\Rightarrow$ (ii). Conversely, let $Q \in C(Y)$ be such that $Q' = 1/P$, and let $Q = A/B$ with $A, B \in C[Y]$, $B \neq 0$. By [ADH, 4.6.14] we have y in a differential field extension of $C(x)$ with constant field C such that $y' = P(y)$ and $B(y) \neq 0$. Then $Q(y)' = Q'(y)y' = (1/P(y))P(y) = 1$, so $Q(y) = x + c \in C(y)$ with $c \in C$. Then $y \notin C$, hence y is transcendental over C and y is algebraic over $C(x)$. $\square$

Here are two applications of Lemma 1.1.36. In the proofs we extend the derivation of $C(Y)$ to the continuous C-linear derivation on $C(\!(Y)\!)$ with $Y' = 1$.

Corollary 1.1.37. *Suppose $n \geqslant 3$, $a_2, a_3 \neq 0$, and y in a Liouville extension of $C(x)$ satisfies $y' = P(y)$. Then $y \in C$.*

Proof. In $C(\!(Y)\!)$ we have $1/P = (1/a_2)Y^{-2} - (a_3/a_2^2)Y^{-1} + \cdots$ and hence $Q' \neq 1/P$ for all $Q \in C(\!(Y)\!)$, so $y \in C$ by Corollary 1.1.35 and Lemma 1.1.36. $\square$

Corollary 1.1.38. *Suppose P has a simple zero in C and y in a Liouville extension of $C(x)$ satisfies $y' = P(y)$. Then $y \in C$.*

Proof. Let $c \in C$ with $P(c) = 0$, $P'(c) \neq 0$. Then in $C(\!(Y)\!)$ we have $1/P(Y + c) \in aY^{-1} + C[[Y]]$ where $a \in C^\times$, hence $Q' \neq 1/P$ for all $Q \in C(\!(Y)\!)$. Thus $y \in C$ by Corollary 1.1.35 and Lemma 1.1.36. $\square$

1.2. The Group of Logarithmic Derivatives

Let K be a differential field. The map $y \mapsto y^\dagger \colon K^\times \to K$ is a morphism from the multiplicative group of K to the additive group of K, with kernel $C^\times$. Its image

$$(K^\times)^\dagger \ = \ \{y^\dagger : y \in K^\times\}$$

is an additive subgroup of K, which we call the **group of logarithmic derivatives** of K. The morphism $y \mapsto y^\dagger$ induces an isomorphism $K^\times/C^\times \to (K^\times)^\dagger$. To shorten notation, set $0^\dagger := 0$, so $K^\dagger = (K^\times)^\dagger$. For $\phi \in K^\times$ we have $\phi(K^\phi)^\dagger = K^\dagger$. The group $K^\times$ is divisible iff both $C^\times$ and $K^\dagger$ are divisible. If K is algebraically closed, then $K^\times$ and hence $K^\dagger$ are divisible, making $K^\dagger$ a $\mathbb{Q}$-linear subspace of K. Likewise, if K is real closed, then the multiplicative subgroup $K^>$ of $K^\times$ is divisible, so $K^\dagger = (K^>)^\dagger$ is a $\mathbb{Q}$-linear subspace of K.

Lemma 1.2.1. *Suppose $K^\dagger$ is divisible, L is a differential field extension of K with $L^\dagger \cap K = K^\dagger$, and M is a differential field extension of L and algebraic over L. Then $M^\dagger \cap K = K^\dagger$.*

Proof. Let $f \in M^\times$ be such that $f^\dagger \in K$. Then $f^\dagger \in L$, so for $n := [L(f) : L]$,

$$nf^\dagger \ = \ \operatorname{tr}_{L(f)|L}(f^\dagger) \ = \ \mathrm{N}_{L(f)|L}(f)^\dagger \in L^\dagger$$

by an identity in [ADH, 4.4]. Hence $nf^\dagger \in K^\dagger$, and thus $f^\dagger \in K^\dagger$. $\square$

In particular, if $K^\dagger$ is divisible and M is a differential field extension of K and algebraic over K, then $M^\dagger \cap K = K^\dagger$.

In the next two lemmas $a, b \in K$; distinguishing whether or not $a \in K^\dagger$ helps to describe the solutions to the differential equation $y' + ay = b$:

Lemma 1.2.2. *Suppose $\partial K = K$, and let L be differential field extension of K with $C_L = C$. Suppose also $a \in K^\dagger$. Then for some $y_0 \in K^\times$ and $y_1 \in K$,*

$$\{y \in L : y' + ay = b\} \ = \ \{y \in K : y' + ay = b\} \ = \ Cy_0 + y_1.$$

Proof. Take $y_0 \in K^\times$ with $y_0^\dagger = -a$, so $y_0' + ay_0 = 0$. Twisting $\partial + a \in K[\partial]$ by y_0 (see [ADH, p. 243]) transforms the equation $y' + ay = b$ into $z' = y_0^{-1}b$. This gives $y_1 \in K$ with $y_1' + ay_1 = b$. Using $C_L = C$, these y_0, y_1 have the desired properties. $\square$

Lemma 1.2.3. *Let L be a differential field extension of K with $L^\dagger \cap K = K^\dagger$. Assume $a \notin K^\dagger$. Then there is at most one $y \in L$ with $y' + ay = b$.*

Proof. If y_1, y_2 are distinct solutions in L of the equation $y' + ay = b$, then we have $-a = (y_1 - y_2)^\dagger \in L^\dagger \cap K = K^\dagger$, contradicting $a \notin K^\dagger$. $\square$

Logarithmic derivatives under algebraic closure. *In this subsection K is a differential field.* We describe for real closed K how $K^\dagger$ changes if we pass from K to its algebraic closure. More generally, suppose the underlying field of K is euclidean; in particular, -1 is not a square in K. We equip K with the unique ordering making K an ordered field. For $y = a + b\mathrm{i} \in K[\mathrm{i}]$ $(a, b \in K)$ we let $|y| \in K^{\geqslant}$ be such that $|y|^2 = a^2 + b^2$. Then $y \mapsto |y| \colon K[\mathrm{i}] \to K^{\geqslant}$ is an absolute value on $K[\mathrm{i}]$, i.e., for all $x, y \in K[\mathrm{i}]$,

$$|x| \ = \ 0 \iff x = 0, \qquad |xy| \ = \ |x||y|, \qquad |x + y| \ \leqslant \ |x| + |y|.$$

For $a \in K$ we have $|a| = \max\{a, -a\}$. We have the subgroup

$$S \ := \ \big\{y \in K[\mathrm{i}] : |y| = 1\big\} \ = \ \big\{a + b\mathrm{i} : a, b \in K,\ a^2 + b^2 = 1\big\}$$

of the multiplicative group $K[\mathrm{i}]^\times$. By an easy computation all elements of $K[\mathrm{i}]$ are squares in $K[\mathrm{i}]$; hence $K[\mathrm{i}]^\dagger$ is 2-divisible. The next lemma describes $K[\mathrm{i}]^\dagger$; it partly generalizes [ADH, 10.7.8].

Lemma 1.2.4. *We have $K[\mathrm{i}]^\times = K^> \cdot S$ with $K^> \cap S = \{1\}$, and*

$$K[\mathrm{i}]^\dagger \ = \ K^\dagger \oplus S^\dagger \qquad (\textit{internal direct sum of subgroups of } K[\mathrm{i}]^\dagger).$$

For $a, b \in K$ with $a + b\mathrm{i} \in S$ we have $(a + b\mathrm{i})^\dagger = \mathrm{wr}(a, b)\mathrm{i}$. Thus $K[\mathrm{i}]^\dagger \cap K = K^\dagger$.

Proof. Let $y = a + b\mathrm{i} \in K[\mathrm{i}]^\times$ $(a, b \in K)$, and take $r \in K^>$ with $r^2 = a^2 + b^2$; then $y = r \cdot (y/r)$ with $y/r \in S$. Thus $K[\mathrm{i}]^\times = K^> \cdot S$, and clearly $K^> \cap S = \{1\}$. Hence $K[\mathrm{i}]^\dagger = K^\dagger + S^\dagger$. Suppose $a \in K^\times$, $s \in S$, and $a^\dagger = s^\dagger$; then $a = cs$ with $c \in C_{K[\mathrm{i}]}$, and $C_{K[\mathrm{i}]} = C[\mathrm{i}]$ by [ADH, 4.6.20] and hence $\max\{a, -a\} = |a| = |c| \in C$, so $a \in C$ and thus $a^\dagger = s^\dagger = 0$; therefore the sum is direct. Now if $a, b \in K$ and $|a + b\mathrm{i}| = 1$, then

$$\begin{aligned}(a + b\mathrm{i})^\dagger \ &= \ (a' + b'\mathrm{i})(a - b\mathrm{i}) \\ &= \ (aa' + bb') + (ab' - a'b)\mathrm{i} \\ &= \ \tfrac{1}{2}\big(a^2 + b^2\big)' + (ab' - a'b)\mathrm{i} \ = \ (ab' - a'b)\mathrm{i} \ = \ \mathrm{wr}(a, b)\mathrm{i}. \qquad \square\end{aligned}$$

Corollary 1.2.5. *For $y \in K[\mathrm{i}]^\times$ we have $\mathrm{Re}(y^\dagger) = |y|^\dagger$, and the group morphism $y \mapsto \mathrm{Re}(y^\dagger) \colon K[\mathrm{i}]^\times \to K$ has kernel $C^> S$.*

If K is real closed and $\mathcal{O}$ a convex valuation ring of K, then $\mathcal{O}[\mathrm{i}] = \mathcal{O} + \mathcal{O}\mathrm{i}$ is the unique valuation ring of $K[\mathrm{i}]$ that lies over $\mathcal{O}$, and so $S \subseteq \mathcal{O}[\mathrm{i}]^\times$, hence $y \asymp |y|$ for all $y \in K[\mathrm{i}]^\times$. Thus by [ADH, 10.5.2(i)] and Corollary 1.2.5:

Corollary 1.2.6. *If K is a real closed pre-H-field, then for all $y, z \in K[\mathrm{i}]^\times$,*

$$y \prec z \quad \Longrightarrow \quad \mathrm{Re}(y^\dagger) < \mathrm{Re}(z^\dagger).$$

We also have a useful decomposition for S:

Corollary 1.2.7. *Suppose K is a real closed H-field. Then*

$$S \ = \ S_C \cdot \big(S \cap (1 + o_{K[\mathrm{i}]})\big)$$

where $S_C := S \cap C[\mathrm{i}]^\times$ and $S \cap (1 + o_{K[\mathrm{i}]})$ are subgroups of $\mathcal{O}[\mathrm{i}]^\times$.

Proof. The inclusion $\supseteq$ is clear. For the reverse inclusion, let $a, b \in K$, $a^2 + b^2 = 1$ and take the unique $c, d \in C$ with $a - c \prec 1$ and $b - d \prec 1$. Then $c^2 + d^2 = 1$ and $a + b\mathrm{i} \sim c + d\mathrm{i}$, and so $(a + b\mathrm{i})/(c + d\mathrm{i}) \in S \cap (1 + \smallo_{K[\mathrm{i}]})$. $\square$

Logarithmic derivatives in asymptotic fields. *Let K be an asymptotic field.* If K is henselian and $\boldsymbol{k} := \operatorname{res} K$, then by [ADH, remark before 3.3.33], $K^\times$ is divisible iff the groups $\boldsymbol{k}^\times$ and Γ are both divisible. Recall that in [ADH, 14.2] we defined the $\mathcal{O}$-submodule

$$\mathrm{I}(K) \;=\; \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}$$

of K. We have $\partial\mathcal{O} \subseteq \mathrm{I}(K)$, hence $(1 + \smallo)^\dagger \subseteq (\mathcal{O}^\times)^\dagger \subseteq \mathrm{I}(K)$. One easily verifies:

Lemma 1.2.8. *Suppose K is pre-*d*-valued. If $\mathrm{I}(K) \subseteq \partial K$, then $\mathrm{I}(K) = \partial\mathcal{O}$. If $\mathrm{I}(K) \subseteq K^\dagger$, then $\mathrm{I}(K) = (\mathcal{O}^\times)^\dagger$, with $\mathrm{I}(K) = (1 + \smallo)^\dagger$ if K is* d*-valued.*

If K is d-valued or K is pre-d-valued without a gap, then

$$\mathrm{I}(K) \;=\; \{y \in K : y \preccurlyeq f' \text{ for some } f \in \smallo\}.$$

For $\phi \in K^\times$ we have $\phi\,\mathrm{I}(K^\phi) = \mathrm{I}(K)$. If K has asymptotic integration and L is an asymptotic extension of K, then $\mathrm{I}(K) = \mathrm{I}(L) \cap K$. The following is [ADH, 14.2.5]:

Lemma 1.2.9. *If K is H-asymptotic, has asymptotic integration, and is 1-linearly newtonian, then it is* d*-valued and $\partial\mathcal{O} = \mathrm{I}(K) = (1 + \smallo)^\dagger$.*

We now turn our attention to the condition $\mathrm{I}(K) \subseteq K^\dagger$. If $\mathrm{I}(K) \subseteq K^\dagger$, then also $\mathrm{I}(K^\phi) \subseteq (K^\phi)^\dagger$ for $\phi \in K^\times$, where

$$(K^\phi)^\dagger \;:=\; \{\phi^{-1} f'/f : f \in K^\times\} \;=\; \phi^{-1} K^\dagger.$$

By [ADH, Section 9.5 and 10.4.3]:

Lemma 1.2.10. *Let K be of H-type. If K is* d*-valued, or pre-*d*-valued without a gap, then K has an immediate henselian asymptotic extension L with $\mathrm{I}(L) \subseteq L^\dagger$.*

Corollary 1.2.11. *Suppose K has asymptotic integration. Let L be an asymptotic field extension of K such that $L^\times = K^\times C_L^\times (1 + \smallo_L)$. Then $L^\dagger = K^\dagger + (1 + \smallo_L)^\dagger$, and if $\mathrm{I}(K) \subseteq K^\dagger$, then $L^\dagger \cap K = K^\dagger$.*

Proof. Let $f \in L^\times$, and take $b \in K^\times$, $c \in C_L^\times$, $g \in \smallo_L$ with $f = bc(1 + g)$; then $f^\dagger = b^\dagger + (1 + g)^\dagger$, showing $L^\dagger = K^\dagger + (1 + \smallo_L)^\dagger$. Next, suppose $\mathrm{I}(K) \subseteq K^\dagger$, let b, c, f, g be as before, and assume $a := f^\dagger \in K$; then

$$a - b^\dagger \in (1 + \smallo_L)^\dagger \cap K \;\subseteq\; \mathrm{I}(L) \cap K \;=\; \mathrm{I}(K) \;\subseteq\; K^\dagger$$

and hence $a \in K^\dagger$. This shows $L^\dagger \cap K = K^\dagger$. $\square$

Two cases where the assumption on L in Corollary 1.2.11 is satisfied: (1) L is an immediate asymptotic field extension of K, because then $L^\times = K^\times(1 + \smallo_L)$; and (2) L is a d-valued field extension of K with $\Gamma = \Gamma_L$.

If F is a henselian valued field of residue characteristic 0, then clearly the subgroup $1 + \smallo_F$ of $F^\times$ is divisible. Hence, if K and L are as in Corollary 1.2.11 and in addition $K^\dagger$ is divisible and L is henselian, then $L^\dagger$ is divisible.

Example 1.2.12. Let C be a field of characteristic 0 and Q be a subgroup of $\mathbb{Q}$ with $1 \in Q$. The Hahn field $C(\!(t^Q)\!) = C[[x^Q]]$, with $x = t^{-1}$, is given the natural derivation with $c' = 0$ for all $c \in C$ and $x' = 1$: this derivation is defined by

$$\Big(\sum_{q \in Q} c_q x^q\Big)' \ := \ \sum_{q\in Q} q c_q x^{q-1} \qquad (\text{all } c_q \in C).$$

Then $C(\!(t^Q)\!)$ has constant field C, and is d-valued of H-type. Thus $K := C(\!(t^Q)\!)$ satisfies $\mathrm{I}(K) \subseteq K^\dagger$ by Lemma 1.2.10. Hence by Lemma 1.2.8,

$$\mathrm{I}(K) \ = \ (1+\mathcal{o})^\dagger \ = \ \big\{f \in K : f \prec x^\dagger = t\big\} \ = \ \mathcal{o}\, t.$$

It follows easily that $K^\dagger = Qt \oplus \mathrm{I}(K)$ (internal direct sum of subgroups of $K^\dagger$) and thus $(K^t)^\dagger = Q \oplus \mathcal{o} \subseteq \mathcal{O}$. In particular, if $Q = \mathbb{Z}$ (so $K = C(\!(t)\!)$), then $(K^t)^\dagger = \mathbb{Z} \oplus tC[[t]]$. Moreover, if $L := \mathrm{P}(C) \subseteq C(\!(t^{\mathbb{Q}})\!)$ is the differential field of Puiseux series over C, then $(L^t)^\dagger = \mathbb{Q} \oplus \mathcal{o}_L$.

In the next three corollaries we continue with the d-valued Hahn field $K = C(\!(t^Q)\!)$ from the example above. So $CK^\dagger = Ct \oplus \mathrm{I}(K)$ (internal direct sum of C-linear subspaces of K) where $\mathrm{I}(K) = \mathcal{o}t$, hence $CK^\dagger = \mathcal{O}t$. For $f = \sum_{q\in Q} f_q x^q \in K$ (all $f_q \in C$) we have the "residue" f_{-1} of f, and we observe that $f \mapsto f_{-1} \colon K \to C$ is C-linear with kernel $\partial(K)$. Thus:

Corollary 1.2.13. $\partial(K) \cap CK^\dagger = \mathrm{I}(K)$.

This yields a fact needed in Section 7.6:

Corollary 1.2.14. *Let $F := C(x) \subseteq K$. Then $\partial(F) \cap CF^\dagger = \{0\}$.*

Proof. We arrange that C is algebraically closed. Let $f \in \partial(F) \cap CF^\dagger$. Then $f \in \mathrm{I}(K) = \mathcal{o}t$ by Corollary 1.2.13, so it suffices to show $f \in C[x]$. For $c \in C$, let $v_c \colon F^\times \to \mathbb{Z}$ be the valuation on F with $v(C^\times) = \{0\}$ and $v(x-c) = 1$. Then $v_c = v \circ \sigma_c$ where σ_c is the C-linear automorphism of the field F with $x \mapsto c+t$. Hence it suffices that $\sigma_c(f) \preccurlyeq 1$ for all $c \in C$. For $c \in C$, $g \in F$ we have $\sigma_c(g)' = -t^2\sigma_c(g')$, so $-t^2\sigma_c(f) \in \partial(F) \cap CF^\dagger \subseteq \mathcal{o}t$, hence $\sigma_c(f) \prec x$ and thus $\sigma_c(f) \preccurlyeq 1$. □

For the next corollary, compare [86, p. 14] and think of y_j as $\log(x - c_j)$.

Corollary 1.2.15 (Linear independence of logarithms)**.** *Let $c_1, \dots, c_n \in C$ be distinct, and let $y_1, \dots, y_n$ in a common differential field extension of $C(x)$ be such that $y_j' = (x - c_j)^{-1}$ for $j = 1, \dots, n$. Then for all $a_1, \dots, a_n \in C$,*

$$a_1 y_1 + \cdots + a_n y_n \in C(x) \implies a_1 = \cdots = a_n = 0.$$

Proof. Set $F := C(x)$ and suppose $a_1, \dots, a_n \in C$ and $f := a_1y_1 + \cdots + a_ny_n \in F$. Then $f' = a_1(x-c_1)^{-1} + \cdots + a_n(x-c_n)^{-1} \in \partial(F) \cap CF^\dagger$, so by Corollary 1.2.14,

$$a_1(x-c_1)^{-1} + \cdots + a_n(x-c_n)^{-1} \ = \ 0.$$

Multiplying both sides of this equality by $\prod_{j=1}^n (x - c_j)$ and substituting c_j for x yields $a_j = 0$, for $j = 1, \dots, n$. □

The real closed case. *In this subsection H is a real closed asymptotic field whose valuation ring $\mathcal{O}$ is convex with respect to the ordering of H.* (In later use H is often a Hardy field, which is why we use the letter H here.) The valuation ring of the asymptotic field extension $K = H[\mathrm{i}]$ of H is then $\mathcal{O}_K = \mathcal{O} + \mathcal{O}\mathrm{i}$, from which we obtain $\mathrm{I}(K) = \mathrm{I}(H) \oplus \mathrm{I}(H)\mathrm{i}$. Let

$$S \ := \ \big\{y \in K : \, |y| = 1\big\}, \qquad W \ := \ \big\{\mathrm{wr}(a,b) : \, a,b \in H, \ a^2+b^2=1\big\},$$

so S is a subgroup of $\mathcal{O}_K^\times$ with $S^\dagger = W\mathrm{i}$ and $K^\dagger = H^\dagger \oplus W\mathrm{i}$ by Lemma 1.2.4. Since $\partial\mathcal{O} \subseteq \mathrm{I}(H)$, we have $W \subseteq \mathrm{I}(H)$, and thus: $W = \mathrm{I}(H) \iff \mathrm{I}(H)\mathrm{i} \subseteq K^\dagger$.

Lemma 1.2.16. *The following are equivalent:*

(i) $\mathrm{I}(K) \subseteq K^\dagger$*;*

(ii) $W = \mathrm{I}(H) \subseteq H^\dagger$*.*

Proof. Assume (i). Then $\mathrm{I}(H)\,\mathrm{i} \subseteq \mathrm{I}(K) \subseteq K^\dagger$, so $W = \mathrm{I}(H)$ by the equivalence preceding the lemma. Also $\mathrm{I}(H) \subseteq \mathrm{I}(K)$ and $K^\dagger \cap H = H^\dagger$ (by Lemma 1.2.4), hence $\mathrm{I}(H) \subseteq H^\dagger$, so (ii) holds. For the converse, assume (ii). Then

$$\mathrm{I}(K) \ = \ \mathrm{I}(H) \oplus \mathrm{I}(H)\mathrm{i} \ \subseteq \ H^\dagger \oplus W\mathrm{i} \ = \ K^\dagger. \qquad \square$$

Applying now Lemma 1.2.9 we obtain:

Corollary 1.2.17. *If H is H-asymptotic and has asymptotic integration, and K is* 1*-linearly newtonian, then K is* d*-valued and $\mathrm{I}(K) \subseteq K^\dagger$; in particular, $W = \mathrm{I}(H)$.*

Corollary 1.2.18. *Suppose H has asymptotic integration and $W = \mathrm{I}(H)$. Let F be a real closed asymptotic extension of H whose valuation ring is convex. Then*

$$F[\mathrm{i}]^\dagger \cap K \ = \ (F^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i}.$$

If in addition $H^\dagger = H$, then $F[\mathrm{i}]^\dagger \cap K = H \oplus \mathrm{I}(H)\mathrm{i} = K^\dagger$.

Proof. We have

$$F^\dagger \cap H \subseteq F[\mathrm{i}]^\dagger \cap K \quad \text{ and } \quad \mathrm{I}(H)\mathrm{i} = W\mathrm{i} \subseteq K^\dagger \cap H\mathrm{i} \subseteq F[\mathrm{i}]^\dagger \cap K,$$

so $(F^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i} \subseteq F[\mathrm{i}]^\dagger \cap K$. For the reverse inclusion, $F[\mathrm{i}]^\dagger = F^\dagger \oplus W_F\mathrm{i}$, with

$$W_F \ := \ \big\{\mathrm{wr}(a,b) : \, a,b \in F, \ a^2+b^2=1\big\} \ \subseteq \ \mathrm{I}(F),$$

hence

$$\begin{aligned} F[\mathrm{i}]^\dagger \cap K \ &= \ (F^\dagger \cap H) \oplus (W_F \cap H)\mathrm{i} \\ &\subseteq \ (F^\dagger \cap H) \oplus \big(\mathrm{I}(F) \cap H\big)\mathrm{i} \ = \ (F^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i}, \end{aligned}$$

using $\mathrm{I}(F) \cap H = \mathrm{I}(H)$, a consequence of H having asymptotic integration. If $H^\dagger = H$ then clearly $F^\dagger \cap H = H$, hence $F[\mathrm{i}]^\dagger \cap K = K^\dagger$. $\square$

Trigonometric closure. In this subsection H is a real closed H-field. Let $\mathcal{O}$ be its valuation ring and $\smallo$ the maximal ideal of $\mathcal{O}$. The algebraic closure $K = H[\mathrm{i}]$ of H is a d-valued H-asymptotic extension with valuation ring $\mathcal{O}_K = \mathcal{O} + \mathcal{O}\mathrm{i}$. We have the "complex conjugation" automorphism $z = a + b\mathrm{i} \mapsto \overline{z} = a - b\mathrm{i}$ $(a,b \in H)$ of the valued differential field K. For such z, a, b we have

$$|z| \ = \ \sqrt{z\overline{z}} \ = \ \sqrt{a^2+b^2} \ \in \ H^{\geqslant}.$$

Lemma 1.2.19. *Suppose $\theta \in H$ and $\theta'\mathrm{i} \in K^\dagger$. Then $\theta' \in \partial\smallo$, and there is a unique $y \sim 1$ in K such that $y^\dagger = \theta'\mathrm{i}$. For this y we have $|y| = 1$, so $y^{-1} = \overline{y}$.*

Proof. From $\theta'\mathrm{i} \in K^\dagger$ we get $\theta' \in W \subseteq \mathrm{I}(H)$, so $\theta \preccurlyeq 1$, hence $\theta' \in \partial\mathcal{O} = \partial\smallo$. Let $z \in K^\times$ and $z^\dagger = \theta'\mathrm{i}$. Then $\operatorname{Re} z^\dagger = 0$, so by Corollaries 1.2.5 and 1.2.7 we have $z = cy$ with $c \in C_K^\times$ and $y \in S \cap (1 + \smallo_K)$ where $S = \{a \in K : |a| = 1\}$. Hence $y \sim 1$, $|y| = 1$, and $y^\dagger = \theta'\mathrm{i}$. If also $y_1 \in K$ and $y_1 \sim 1$, $y_1^\dagger = \theta'\mathrm{i}$, then $y_1 = c_1 y$ with $c_1 \in C_K^\times$, so $c_1 = 1$ in view of $y \sim y_1$. $\square$

By [ADH, 10.4.3], if y in an H-asymptotic extension L of K satisfies $y \sim 1$ and $y^\dagger \in \partial\smallo_K$, then the asymptotic field $K(y) \subseteq L$ is an immediate extension of K, and so is any algebraic asymptotic extension of $K(y)$.

Call H **trigonometrically closed** if for all $\theta \prec 1$ in H there is a (necessarily unique) $y \in K$ such that $y \sim 1$ and $y^\dagger = \theta'\mathrm{i}$. (By convention "trigonometrically closed" includes "real closed".) For such θ and y we think of y as $\mathrm{e}^{\mathrm{i}\theta}$ and accordingly of the elements $\frac{y+\overline{y}}{2} = \frac{y+y^{-1}}{2}$ and $\frac{y-\overline{y}}{2\mathrm{i}} = \frac{y-y^{-1}}{2\mathrm{i}}$ of H as $\cos\theta$ and $\sin\theta$; this explains the terminology. By Lemma 1.2.19 the restrictions $\theta \prec 1$ and $y \sim 1$ are harmless. Our aim in this subsection is to construct a canonical trigonometric closure of H.

Note that if $\mathrm{I}(K) \subseteq K^\dagger$, then H is trigonometrically closed. As a partial converse, if $\mathrm{I}(H) \subseteq H^\dagger \cap \partial H$ and H is trigonometrically closed, then $\mathrm{I}(K) \subseteq K^\dagger$; this is an easy consequence of $\mathrm{I}(K) = \mathrm{I}(H) + \mathrm{I}(H)\mathrm{i}$. Thus for Liouville closed H we have:

$$H \text{ is trigonometrically closed} \iff \mathrm{I}(K) \subseteq K^\dagger.$$

Note also that for trigonometrically closed H there is no y in any H-asymptotic extension of K such that $y \notin K$, $y \sim 1$, and $y^\dagger \in (\partial\smallo)\mathrm{i}$.

If H is Schwarz closed, then H is trigonometrically closed by the next lemma:

Lemma 1.2.20. *Suppose H is Liouville closed and $\omega(H)$ is downward closed. Then H is trigonometrically closed.*

Proof. Let $0 \neq \theta \prec 1$ in H. By Lemma 1.2.19 it suffices to show that then $\theta'\mathrm{i} \in K^\dagger$. Note that $h := \theta' \in \mathrm{I}(H)^{\neq}$; we arrange $h > 0$. Now

$$f \;:=\; \omega(-h^\dagger) + 4h^2 \;=\; \sigma(2h), \qquad 2h \in H^> \cap \mathrm{I}(H),$$

hence $2h \in H^> \setminus \Gamma(H)$ by [ADH, 11.8.19]. So $f \in \omega(H)^\downarrow = \omega(H)$ by [ADH, 11.8.31], and thus $\dim_{C_H} \ker 4\partial^2 + f \geqslant 1$ by [ADH, p. 258]. Put $A := \partial^2 - h^\dagger\partial + h^2 \in H[\partial]$. The isomorphism $y \mapsto y\sqrt{h}\colon \ker(4\partial^2 + f) \to \ker A$ of C_H-linear spaces [ADH, 5.1.13] then yields an element of $\ker^{\neq} A$ that for suggestiveness we denote by $\cos\theta$. Put $\sin\theta := -(\cos\theta)'/h$. Then

$$\begin{aligned}(\sin\theta)' \;&=\; -(\cos\theta)''/h + (\cos\theta)'h^\dagger/h \\ &=\; \big(-h^\dagger(\cos\theta)' + h^2\cos\theta\big)/h + (\cos\theta)'h^\dagger/h \;=\; h\cos\theta\end{aligned}$$

and thus $y^\dagger = \theta'\mathrm{i}$ for $y := \cos\theta + \mathrm{i}\sin\theta \in K^\times$. $\square$

If H is H-closed, then H is Schwarz closed by [ADH, 14.2.20], and thus trigonometrically closed. Using also Lemma 1.2.16 and remarks preceding it this yields:

Corollary 1.2.21. *If H is H-closed, then* $\mathrm{I}(K) \subseteq K^\dagger = H \oplus \mathrm{I}(H)\mathrm{i}$.

Suppose now that H is *not* trigonometrically closed; so we have $\theta \prec 1$ in H with $\theta'\mathrm{i} \notin K^\dagger$. Then [ADH, 10.4.3] provides an immediate asymptotic extension $K(y)$ of K with $y \sim 1$ and $y^\dagger = \theta'\mathrm{i}$. To simplify notation and for suggestiveness we set

$$\cos\theta \;:=\; \frac{y+y^{-1}}{2}, \qquad \sin\theta \;:=\; \frac{y-y^{-1}}{2\mathrm{i}},$$

so $y = \cos\theta + i\sin\theta$ and $(\cos\theta)^2 + (\sin\theta)^2 = 1$. Moreover $(\cos\theta)' = -\theta'\sin\theta$ and $(\sin\theta)' = \theta'\cos\theta$. It follows that $H^+ := H(\cos\theta, \sin\theta)$ is a differential subfield of $K(y)$ with $K(y) = H^+[i]$, and thus H^+, as a valued differential subfield of $H(y)$, is an asymptotic extension of H.

Lemma 1.2.22. *H^+ is an immediate extension of H.*

Proof. Since $(y^{-1})^\dagger = -\theta' i$, the uniqueness property stated in [ADH, 10.4.3] allows us to extend the complex conjugation automorphism of K (which is the identity on H and sends i to $-i$) to an automorphism σ of the valued differential field $K(y)$ such that $\sigma(y) = y^{-1}$. Then $\sigma(\cos\theta) = \cos\theta$ and $\sigma(\sin\theta) = \sin\theta$, so $H^+ = \operatorname{Fix}(\sigma)$. Let $\boldsymbol{k}$ be the residue field of H; so $\boldsymbol{k}[\operatorname{res} i]$ is the residue field of K and of its immediate extension $K(y)$. Now $\sigma(\mathcal{O}_{K(y)}) = \mathcal{O}_{K(y)}$, so σ induces an automorphism of this residue field $\boldsymbol{k}[\operatorname{res} i]$ which is the identity on $\boldsymbol{k}$ and sends $\operatorname{res} i$ to $-\operatorname{res} i$. Hence $\operatorname{res} i$ does not lie in the residue field of H^+, so this residue field is just $\boldsymbol{k}$. $\square$

Equip H^+ with the unique field ordering making it an ordered field extension of H in which $\mathcal{O}_{H^+}$ is convex; see [ADH, 10.5.8]. Then H^+ is an H-field, and its real closure is an immediate real closed H-field extension of H.

Lemma 1.2.23. *The H-field H^+ embeds uniquely over H into any trigonometrically closed H-field extension of H.*

Proof. Let H^* be a trigonometrically closed H-field extension of H. Take the unique $z \sim 1$ in H^* such that $z^\dagger = \theta' i$. Then any H-field embedding $H^+ \to H^*$ over H extends to a valued differential field embedding $H^+[i] = K(y) \to H^*[i]$ sending $i \in K$ to $i \in H^*[i]$, and this extension must send y to z. Hence there is at most one H-field embedding $H^+ \to H^*$ over H. For the existence of such an embedding, the uniqueness properties from [ADH, 10.4.3] yield a valued differential field embedding $K(y) \to H^*[i]$ over H sending $i \in K$ to $i \in H^*[i]$ and y to z. This embedding maps H^+ into H^*. The uniqueness property of the ordering on H^+ shows that this embedding restricts to an H-field embedding $H^+ \to H^*$. $\square$

By iterating the extension step that leads from H to H^+, alternating it with taking real closures, and taking unions at limit stages we obtain:

Proposition 1.2.24. *H has a trigonometrically closed H-field extension H^{trig} that embeds uniquely over H into any trigonometrically closed H-field extension of H.*

This is an easy consequence of Lemma 1.2.23. Note that the universal property stated in Proposition 1.2.24 determines H^{trig} up-to-unique-isomorphism of H-fields over H. We refer to such H^{trig} as the **trigonometric closure** of H. Note that H^{trig} is an immediate extension of H, by Lemma 1.2.22, and that $H^{\text{trig}}[i]$ is a Liouville extension of K and thus of H.

A *trigonometric extension* of H is a real closed H-field extension E of H such that for all $a \in E$ there are real closed H-subfields $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n$ of E such that

1. $H_0 = H$ and $a \in H_n$;
2. for $j = 0, \dots, n-1$ there are $\theta_j \in H_j$ and $y_j \in H_{j+1}[i] \subseteq E[i]$ such that $y_j \sim 1$, $\theta_j' i = y_j^\dagger$, and $H_{j+1}[i]$ is algebraic over $H_j[i](y_j)$.

If E is a trigonometric extension of H, then E is an immediate extension of H and $E[i]$ is an immediate Liouville extension of K and thus of H. The next lemma states some further easy consequences of the definition above:

Lemma 1.2.25. *If E is a trigonometric extension of H, then E is a trigonometric extension of any real closed H-subfield $F \supseteq H$ of E. If H is trigonometrically closed, then H has no proper trigonometric extension.*

Induction on m shows that if E is a trigonometric extension of H, then for any $a_1, \dots, a_m \in E$ there are real closed H-subfields $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n$ of E such that $H_0 = H$, $a_1, \dots, a_m \in H_n$ and (2) above holds. This helps in proving:

Corollary 1.2.26. *A trigonometric extension of a trigonometric extension of H is a trigonometric extension of H, and H^{trig} is a trigonometric extension of H.*

Asymptotic fields of Hardy type. Let (Γ, ψ) be an asymptotic couple, $\Psi := \psi(\Gamma^{\neq})$, and let γ, δ range over Γ. Recall that $[\gamma]$ denotes the archimedean class of γ [ADH, 2.4]. Following [169, Section 3] we say that (Γ, ψ) is of **Hardy type** if for all $\gamma, \delta \neq 0$ we have $[\gamma] \leqslant [\delta] \Longleftrightarrow \psi(\gamma) \geqslant \psi(\delta)$. Note that then (Γ, ψ) is of H-type, and ψ induces an order-reversing bijection $[\Gamma^{\neq}] \to \Psi$. If Γ is archimedean, then (Γ, ψ) is of Hardy type. If (Γ, ψ) is of Hardy type, then so is $(\Gamma, \psi + \delta)$ for each δ. We also say that an asymptotic field is of Hardy type if its asymptotic couple is. Every asymptotic subfield and every compositional conjugate of an asymptotic field of Hardy type is also of Hardy type. Moreover, every Hardy field is of Hardy type [ADH, 9.1.11]. Let now Δ be a convex subgroup of Γ. Note that Δ contains the archimedean class $[\delta]$ of each $\delta \in \Delta$. Hence, if $\delta \in \Delta^{\neq}$ and $\gamma \notin \Delta$, then $[\delta] < [\gamma]$ and thus:

Lemma 1.2.27. *If (Γ, ψ) is of Hardy type and $\gamma \notin \Delta$, $\delta \in \Delta^{\neq}$, then $\psi(\gamma) < \psi(\delta)$.*

Corollary 1.2.28. *Suppose (Γ, ψ) is of Hardy type with small derivation, $\gamma, \delta \neq 0$, $\psi(\delta) \leqslant 0$, and $[\gamma'] > [\delta]$. Then $\psi(\gamma) < \psi(\delta)$.*

Proof. Let Δ be the smallest convex subgroup of Γ with $\delta \in \Delta$; then $\gamma' \notin \Delta$, and $\psi(\delta) \in \Delta$ by [ADH, 9.2.10(iv)]. Thus $\gamma \notin \Delta$ by [ADH, 9.2.25]. □

In [7, Section 7] we say that an H-field H is *closed under powers* if for all $c \in C$ and $f \in H^\times$ there is a $y \in H^\times$ with $y^\dagger = cf^\dagger$. (Think of y as f^c.) Thus if H is Liouville closed, then H is closed under powers. *In the rest of this subsection we let H be an H-field closed under powers, with asymptotic couple (Γ, ψ) and constant field C.* We recall some basic facts from [7, Section 7]. First, we can make the value group Γ into an ordered vector space over the constant field C:

Lemma 1.2.29. *For all $c \in C$ and $\gamma = vf$ with $f \in H^\times$ and each $y \in H^\times$ with $y^\dagger = cf^\dagger$, the element $vy \in \Gamma$ only depends on (c, γ) (not on the choice of f and y), and is denoted by $c \cdot \gamma$. The scalar multiplication $(c, \gamma) \mapsto c \cdot \gamma \colon C \times \Gamma \to \Gamma$ makes Γ into an ordered vector space over the ordered field C.*

Let G be an ordered vector space over the ordered field C. From [ADH, 2.4] recall that the C-archimedean class of $a \in G$ is defined as

$$[a]_C := \big\{ b \in G : \tfrac{1}{c}|a| \leqslant |b| \leqslant c|a| \text{ for some } c \in C^> \big\}.$$

Thus if $C = \mathbb{Q}$, then $[a]_{\mathbb{Q}}$ is just the archimedean class $[a]$ of $a \in G$. Moreover, if C^* is an ordered subfield of C, then $[a]_{C^*} \subseteq [a]_C$ for each $a \in G$, with equality if C^* is cofinal in C. Hence if C is archimedean, then $[a] = [a]_C$ for all $a \in G$. Put $[G]_C := \big\{ [a]_C : a \in G \big\}$ and linearly order $[G]_C$ by

$$[a]_C < [b]_C \quad :\Longleftrightarrow \quad [a]_C \neq [b]_C \text{ and } |a| < |b|.$$

Thus $[G]_C$ has smallest element $[0]_C = \{0\}$. We also set $[G^{\neq}]_C := [G]_C \setminus \{[0]_C\}$. From [7, Proposition 7.5] we have:

Proposition 1.2.30. *For all $\gamma, \delta \neq 0$ we have*

$$[\gamma]_C \leqslant [\delta]_C \quad \Longleftrightarrow \quad \psi(\gamma) \geqslant \psi(\delta).$$

Hence ψ induces an order-reversing bijection $[\Gamma^{\neq}]_C \to \Psi = \psi(\Gamma^{\neq})$.

Proposition 1.2.30 yields:

Corollary 1.2.31. *H is of Hardy type $\Longleftrightarrow$ $[\gamma] = [\gamma]_C$ for all γ. Hence if C is archimedean, then H is of Hardy type; if $\Gamma \neq \{0\}$, then the converse also holds.*

1.3. The Valuation of Differential Polynomials at Infinity (*)

Our goal in this work is to solve certain kinds of algebraic differential equations in Hardy fields. In this section we review some general facts about the asymptotic behavior of solutions of algebraic differential equations in H-asymptotic fields. We will not need these results in order to achieve our main objective, but they will be used at a few points for applications and corollaries; see Section 5.4 and Corollary 7.1.20. *Throughout this section K is an H-asymptotic field, and f, g range over K.*

Iterated logarithmic derivatives. Let (Γ, ψ) be an H-asymptotic couple. As usual we introduce a new symbol $\infty \notin \Gamma$, extend the ordering of Γ to an ordering on $\Gamma_\infty = \Gamma \cup \{\infty\}$ such that $\infty > \Gamma$, and extend $\psi\colon \Gamma^{\neq} \to \Gamma$ to a map $\Gamma_\infty \to \Gamma_\infty$ by setting $\psi(0) := \psi(\infty) := \infty$. (See [ADH, 6.5].) We let γ range over Γ, and we define $\gamma^{\langle n \rangle} \in \Gamma_\infty$ inductively by $\gamma^{\langle 0 \rangle} := \gamma$ and $\gamma^{\langle n+1 \rangle} := \psi(\gamma^{\langle n \rangle})$. The following is [5, Lemma 5.2]; for the convenience of the reader we include a proof:

Lemma 1.3.1. *Suppose that $0 \in (\Gamma^{<})'$, $\gamma \neq 0$, and $n \geqslant 1$. If $\gamma^{\langle n \rangle} < 0$, then $\gamma^{\langle i \rangle} < 0$ for $i = 1, \dots, n$ and $[\gamma] > [\gamma^{\dagger}] > \cdots > [\gamma^{\langle n-1 \rangle}] > [\gamma^{\langle n \rangle}]$.*

Proof. By [ADH, 9.2.9], (Γ, ψ) has small derivation, hence the case $n = 1$ follows from [ADH, 9.2.10(iv)]. Assume inductively that the lemma holds for a certain value of $n \geqslant 1$, and suppose $\gamma^{\langle n+1 \rangle} < 0$. Then $\gamma^{\langle n \rangle} \neq 0$, so we can apply the case $n = 1$ to $\gamma^{\langle n \rangle}$ instead of γ and get $[\gamma^{\langle n \rangle}] > [\gamma^{\langle n+1 \rangle}]$. By the inductive assumption the remaining inequalities will follow from $\gamma^{\langle n \rangle} < 0$. From $0 \in (\Gamma^{<})'$ we obtain an element 1 of $\Gamma^{>}$ with $0 = (-1)' = -1 + 1^{\dagger}$. Suppose $\gamma^{\langle n \rangle} \geqslant 0$. Then $\gamma^{\langle n \rangle} \in \Psi$, thus $0 < \gamma^{\langle n \rangle} < 1 + 1^{\dagger} = 1 + 1$ and so $[\gamma^{\langle n \rangle}] \leqslant [1]$. Hence $0 > \gamma^{\langle n+1 \rangle} \geqslant 1^{\dagger} = 1$, a contradiction. □

Suppose now that (Γ, ψ) is the asymptotic couple of K. If $y \in K^{\times}$ and $(vy)^{\langle n \rangle} \neq \infty$, then the nth iterated logarithmic derivative $y^{\langle n \rangle}$ of y is defined (see [ADH, 4.2]), and $v(y^{\langle n \rangle}) = (vy)^{\langle n \rangle} \in \Gamma$. Recall from [ADH, p. 383] that for $f, g \neq 0$,

$$f \prec\!\!\!\prec g \;:\Leftrightarrow\; f^{\dagger} \prec g^{\dagger}, \quad f \preccurlyeq\!\!\!\preccurlyeq g \;:\Leftrightarrow\; f^{\dagger} \preccurlyeq g^{\dagger}, \quad f \asymp\!\!\!\asymp g \;:\Leftrightarrow\; f^{\dagger} \asymp g^{\dagger},$$

hence, assuming also $f, g \not\asymp 1$,

$$f \prec\!\!\!\prec g \;\Rightarrow\; [vf] < [vg], \qquad [vf] \leqslant [vg] \;\Rightarrow\; f \preccurlyeq\!\!\!\preccurlyeq g.$$

In the rest of this section we are given $x \succ 1$ in K with $x' \asymp 1$. Then $0 \in (\Gamma^{<})'$, so from the previous lemma we obtain:

Corollary 1.3.2. *If $y \in K^\times$, $y \not\asymp 1$, $n \geqslant 1$, and $(vy)^{\langle n\rangle} < 0$, then $y^{\langle i\rangle} \succ 1$ for $i = 1,\dots,n$ and $[vy] > \big[v(y^\dagger)\big] > \cdots > \big[v(y^{\langle n-1\rangle})\big] > \big[v(y^{\langle n\rangle})\big]$.*

Let $\boldsymbol{i} = (i_0,\dots,i_n) \in \mathbb{Z}^{1+n}$ and $y \in K^\times$ be such that $y^{\langle n\rangle}$ is defined; we put

$$y^{\langle \boldsymbol{i}\rangle} \ :=\ (y^{\langle 0\rangle})^{i_0}\cdots(y^{\langle n\rangle})^{i_n} \in K.$$

If $y^{\langle n\rangle} \neq 0$, then $\boldsymbol{i} \mapsto y^{\langle \boldsymbol{i}\rangle}\colon \mathbb{Z}^{1+n} \to K^\times$ is a group morphism. Suppose now that $y \in K^\times$, $(vy)^{\langle n\rangle} < 0$, and $\boldsymbol{i} = (i_0,\dots,i_n) \in \mathbb{Z}^{1+n}$, $\boldsymbol{i} \neq 0$, and $m \in \{0,\dots,n\}$ is minimal with $i_m \neq 0$. Then by Corollary 1.3.2, $\big[v(y^{\langle \boldsymbol{i}\rangle})\big] = \big[v(y^{\langle m\rangle})\big]$. Thus if $y \succ 1$, we have the equivalence $y^{\langle \boldsymbol{i}\rangle} \succ 1 \ \Leftrightarrow\ i_m \geqslant 1$. If K is equipped with an ordering making it a pre-H-field and $y \succ 1$, then $y^\dagger > 0$, so $y^{\langle i\rangle} > 0$ for $i = 1,\dots,n$, and thus $\operatorname{sign} y^{\langle \boldsymbol{i}\rangle} = \operatorname{sign} y^{i_0}$.

Iterated exponentials. *In this subsection we assume that Ψ is downward closed.* For $f \succ 1$ we have $f' \succ f^\dagger$, so we can and do choose $\mathrm{E}(f) \in K^\times$ such that $\mathrm{E}(f) \succ 1$ and $\mathrm{E}(f)^\dagger \asymp f'$, hence $f \prec \mathrm{E}(f)$ and $f \prec\!\!\prec \mathrm{E}(f)$. Moreover, if $f, g \succ 1$, then

$$f \prec g \quad\Longleftrightarrow\quad \mathrm{E}(f) \prec\!\!\prec \mathrm{E}(g).$$

For $f \succ 1$ define $\mathrm{E}_n(f) \in K^{\succ 1}$ inductively by

$$\mathrm{E}_0(f) \ :=\ f, \qquad \mathrm{E}_{n+1}(f) \ :=\ \mathrm{E}\big(\mathrm{E}_n(f)\big),$$

and thus by induction

$$\mathrm{E}_n(f) \ \prec\ \mathrm{E}_{n+1}(f) \quad \text{and} \quad \mathrm{E}_n(f) \ \prec\!\!\prec\ \mathrm{E}_{n+1}(f) \qquad \text{for all } n.$$

In the rest of this subsection $f \succcurlyeq x$, and y ranges over elements of H-asymptotic extensions of K. The proof of the next lemma is like that of [7, Lemma 1.3(2)].

Lemma 1.3.3. *If $y \succcurlyeq \mathrm{E}_{n+1}(f)$, $n \geqslant 1$, then $y \neq 0$ and $y^\dagger \succcurlyeq \mathrm{E}_n(f)$.*

Proof. If $y \succcurlyeq \mathrm{E}_2(f)$, then $y \neq 0$, and using $\mathrm{E}_2(f) \succ 1$ we obtain

$$y^\dagger \ \succcurlyeq\ \mathrm{E}_2(f)^\dagger \ \asymp\ \mathrm{E}(f)' \ =\ \mathrm{E}(f)\,\mathrm{E}(f)^\dagger \ \asymp\ \mathrm{E}(f)f' \ \succcurlyeq\ \mathrm{E}(f),$$

Thus the lemma holds for $n = 1$. In general, $\mathrm{E}_{n-1}(f) \succcurlyeq f \succcurlyeq x$, hence the lemma follows from the case $n = 1$ applied to $\mathrm{E}_{n-1}(f)$ in place of f. □

An obvious induction on n using Lemma 1.3.3 shows: if $y \succcurlyeq \mathrm{E}_n(f)$, then $(vy)^{\langle n\rangle} \leqslant vf < 0$. We shall use this fact without further reference.

Lemma 1.3.4. *If $y \succcurlyeq \mathrm{E}_{n+1}(f)$, then $y^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} \succcurlyeq \mathrm{E}(f)$.*

Proof. First note that if $y \neq 0$, $n \geqslant 1$, and $(y^\dagger)^{\langle n-1\rangle}$ is defined, then $y^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} = (y^\dagger)^{\langle n-1\rangle}$. Now use induction on n and Lemma 1.3.3. □

Lemma 1.3.5. *If $y \succcurlyeq \mathrm{E}_n(f^2)$, then $y^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} \succcurlyeq f$, with $y^{\langle n\rangle} \succ f$ if $f \succ x$.*

Proof. This is clear if $n = 0$, so suppose $y \succcurlyeq \mathrm{E}_{n+1}(f^2)$. Then by Lemma 1.3.4 (applied with f^2 in place of f) we have $y^{\langle n\rangle} \succcurlyeq \mathrm{E}(f^2) \succ 1$, so

$$y^{\langle n+1\rangle} \ =\ (y^{\langle n\rangle})^\dagger \ \succcurlyeq\ \mathrm{E}(f^2)^\dagger \ \asymp\ (f^2)' \ =\ 2ff' \ \succcurlyeq\ f,$$

with $y^{\langle n+1\rangle} \succ f$ if $f \succ x$, as required. □

Corollary 1.3.6. *Suppose $y \succcurlyeq \mathrm{E}_n(f^2)$, and let $\boldsymbol{i} \in \mathbb{Z}^{1+n}$ be such that $\boldsymbol{i} > 0$ lexicographically. Then $y^{\langle n\rangle}$ is defined and $y^{\langle \boldsymbol{i}\rangle} \succcurlyeq f$, with $y^{\langle \boldsymbol{i}\rangle} \succ f$ if $f \succ x$.*

Proof. By Lemma 1.3.5, $y^{\langle n\rangle}$ is defined with $y^{\langle n\rangle} \succcurlyeq f$, and $y^{\langle n\rangle} \succ f$ if $f \succ x$. Let $m \in \{0,\dots,n\}$ be minimal such that $i_m \neq 0$; so $i_m \geqslant 1$. If $m = n$ then $y^{\langle \boldsymbol{i}\rangle} = (y^{\langle n\rangle})^{i_n} \succcurlyeq y^{\langle n\rangle}$, hence $y^{\langle \boldsymbol{i}\rangle} \succcurlyeq f$, with $y^{\langle \boldsymbol{i}\rangle} \succ f$ if $f \succ x$. Suppose $m < n$. Then $y \succcurlyeq \mathrm{E}_{m+1}(f^2)$ and hence $y^{\langle m\rangle} \succcurlyeq \mathrm{E}(f^2)$ by Lemma 1.3.4. Also, $f \preccurlyeq f^2 \prec\!\!\prec \mathrm{E}(f^2)$, thus $y^{\langle m\rangle} \succ\!\!\succ f$. The remarks following Corollary 1.3.2 now yield $y^{\langle \boldsymbol{i}\rangle} \succ f$. $\square$

Asymptotic behavior of $P(y)$ for large y. In this subsection $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ range over $\mathbb{N}^{1+n}$. Let $P_{\langle \boldsymbol{i}\rangle} \in K$ be such that $P_{\langle \boldsymbol{i}\rangle} = 0$ for all but finitely many $\boldsymbol{i}$ and $P_{\langle \boldsymbol{i}\rangle} \neq 0$ for some $\boldsymbol{i}$, and set $P := \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} Y^{\langle \boldsymbol{i}\rangle} \in K\langle Y\rangle$. So if $P \in K\{Y\}$, then $P = \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} Y^{\langle \boldsymbol{i}\rangle}$ is the logarithmic decomposition of the differential polynomial P as defined in [ADH, 4.2]. If y is an element in a differential field extension L of K such that $y^{\langle n\rangle}$ is defined, then we put $P(y) := \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle} \in L$ (and for $P \in K\{Y\}$ this has the usual value). Let $\boldsymbol{j}$ be lexicographically maximal such that $P_{\langle \boldsymbol{j}\rangle} \neq 0$, and choose $\boldsymbol{k}$ so that $P_{\langle \boldsymbol{k}\rangle}$ has minimal valuation. If $P_{\langle \boldsymbol{k}\rangle}/P_{\langle \boldsymbol{j}\rangle} \succ x$, set $f := P_{\langle \boldsymbol{k}\rangle}/P_{\langle \boldsymbol{j}\rangle}$; otherwise set $f := x^2$. Then $f \succ x$ and $f \succcurlyeq P_{\langle \boldsymbol{i}\rangle}/P_{\langle \boldsymbol{j}\rangle}$ for all $\boldsymbol{i}$. The following is a more precise version of [ADH, 16.6.10] and [103, (8.8)]:

Proposition 1.3.7. *Suppose Ψ is downward closed, and y in an H-asymptotic extension of K satisfies $y \succcurlyeq \mathrm{E}_n(f^2)$. Then $y^{\langle n\rangle}$ is defined and $P(y) \sim P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle}$.*

Proof. Let $\boldsymbol{i} < \boldsymbol{j}$. We have $f \succ x$, so $y^{\langle \boldsymbol{j}-\boldsymbol{i}\rangle} \succ f \succcurlyeq P_{\langle \boldsymbol{i}\rangle}/P_{\langle \boldsymbol{j}\rangle}$ by Corollary 1.3.6. Hence $P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle} \succ P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle}$. $\square$

From Corollary 1.3.2, Lemma 1.3.5, and Proposition 1.3.7 we obtain:

Corollary 1.3.8. *Suppose Ψ is downward closed and y in an H-asymptotic extension of K satisfies $y \succ K$. Then y is* d*-transcendental over K, and for all n, $y^{\langle n\rangle}$ is defined, $y^{\langle n\rangle} \succ K$, and $y^{\langle n+1\rangle} \prec\!\!\prec y^{\langle n\rangle}$. The H-asymptotic extension $K\langle y\rangle$ of K has residue field* $\operatorname{res} K\langle y\rangle = \operatorname{res} K$ *and value group $\Gamma_{K\langle y\rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}v(y^{\langle n\rangle})$ (internal direct sum), and $\Gamma_{K\langle y\rangle}$ contains Γ as a convex subgroup.*

Suppose now that K is equipped with an ordering making it a pre-H-field. From Proposition 1.3.7 we recover [7, Theorem 3.4] in slightly stronger form:

Corollary 1.3.9. *Suppose y lies in a Liouville closed H-field extension of K. If $y \succcurlyeq \mathrm{E}_n(f^2)$, then $y^{\langle n\rangle}$ is defined and* $\operatorname{sign} P(y) = \operatorname{sign} P_{\langle \boldsymbol{j}\rangle} y^{j_0}$. *In particular, if $y^{\langle n\rangle}$ is defined and $P(y) = 0$, then $y \prec \mathrm{E}_n(f^2)$.*

Example. Suppose $P \in K\{Y\}$. Using [ADH, 4.2, subsection on logarithmic decomposition] we obtain $j_0 = \deg P$, and the logarithmic decomposition

$$P(-Y) \ = \ \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle}(-1)^{i_0} Y^{\langle \boldsymbol{i}\rangle}.$$

If $\deg P$ is odd, and $y > 0$ lies in a Liouville closed H-field extension of K such that $y \succcurlyeq \mathrm{E}_n(f^2)$, then

$$\operatorname{sign} P(y) \ = \ \operatorname{sign} P_{\langle \boldsymbol{j}\rangle}, \qquad \operatorname{sign} P(-y) \ = \ -\operatorname{sign} P_{\langle \boldsymbol{j}\rangle} \ = \ -\operatorname{sign} P(y).$$

1.4. λ-freeness and ω-freeness

This section contains preservation results for the important properties of λ-*freeness* and ω-*freeness* from [ADH]. Let K be an ungrounded H-asymptotic field such that $\Gamma \neq \{0\}$, and as in [ADH, 11.5], fix a logarithmic sequence (ℓ_ρ) for K and define the pc-sequences $(\lambda_\rho) = (-\ell_\rho^{\dagger\dagger})$ and $(\omega_\rho) = (\omega(\lambda_\rho))$ in K, where $\omega(z) := -2z' - z^2$.

Recall that K is *λ-free* iff (λ_ρ) does not have a pseudolimit in K, and K is *ω-free* iff (ω_ρ) does not have a pseudolimit in K. If K is ω-free, then K is λ-free. We refer to [ADH, 11.6, 11.7] for this and other basic facts about λ-freeness and ω-freeness used below. (For ω-free Hardy fields, see also Section 5.6.) As in [ADH], L being λ-free or ω-free includes L being an ungrounded H-asymptotic field with $\Gamma_L \neq \{0\}$.

Preserving λ-freeness and ω-freeness. *In this subsection K is an ungrounded H-asymptotic field with $\Gamma \neq \{0\}$, and (ℓ_ρ), (λ_ρ), (ω_ρ) are as above.* If K has a λ-free H-asymptotic field extension L such that $\Gamma^<$ is cofinal in $\Gamma_L^<$, then K is λ-free, and similarly with "ω-free" in place of "λ-free" [ADH, remarks after 11.6.4, 11.7.19]. The property of ω-freeness is very robust; indeed, by [ADH, 13.6.1]:

Theorem 1.4.1. *If K is ω-free and L is a pre-*d*-valued* d*-algebraic H-asymptotic extension of K, then L is ω-free and $\Gamma^<$ is cofinal in $\Gamma_L^<$.*

In contrast, λ-freeness is more delicate: Theorem 1.4.1 fails with "λ-free" in place of "ω-free", as the next example shows.

Example 1.4.2. The H-field $K = \mathbb{R}\langle\omega\rangle$ from [ADH, 13.9.1] is λ-free, but its H-field extension $L = \mathbb{R}\langle\lambda\rangle$ is not, and this extension is d-algebraic: $2\lambda' + \lambda^2 + \omega = 0$.

In the rest of this subsection we consider cases where parts of Theorem 1.4.1 do hold. Recall from [ADH, 11.6.8] that if K is λ-free, then K has (rational) asymptotic integration, and K is λ-free iff its algebraic closure is λ-free. Moreover, λ-freeness is preserved under adjunction of constants:

Proposition 1.4.3. *Suppose K is λ-free and $L = K(D)$ is an H-asymptotic extension of K with $D \supseteq C$ a subfield of C_L. Then L is λ-free with $\Gamma_L = \Gamma$.*

We are going to deduce this from the next three lemmas. Recall that K is pre-d-valued, by [ADH, 10.1.3]. Let $\operatorname{dv}(K)$ be the d-valued hull of K (see [ADH, 10.3]).

Lemma 1.4.4. *Suppose K is λ-free. Then $L := \operatorname{dv}(K)$ is λ-free and $\Gamma_L = \Gamma$.*

Proof. The first statement is [75, Theorem 10.2], and the second statement follows from [ADH, 10.3.2(i)]. □

If $L = K(D)$ is a differential field extension of K with $D \supseteq C$ a subfield of C_L, then $D = C_L$, and K and D are linearly disjoint over C [ADH, 4.6.20]. If K is d-valued and $L = K(D)$ is an H-asymptotic extension of K with $D \supseteq C$ a subfield of C_L, then L is d-valued and $\Gamma_L = \Gamma$ [ADH, 10.5.15].

Lemma 1.4.5. *Suppose K is* d*-valued and λ-free, and $L = K(D)$ is an H-asymptotic extension of K with $D \supseteq C$ a subfield of C_L. Then L is λ-free.*

Proof. First, (λ_ρ) is of transcendental type over K: otherwise, [ADH, 3.2.7] would give an algebraic extension of K that is not λ-free. Next, our logarithmic sequence (ℓ_ρ) for K remains a logarithmic sequence for L.

Zorn and the $\forall\exists$-form of the λ-freeness axiom [ADH, 1.6.1(ii)] reduce us to the case $D = C(d)$, $d \notin C$, d transcendental over K, so $L = K(d)$. Suppose L is not λ-free. Then $\lambda_\rho \rightsquigarrow \lambda \in L$, and such λ is transcendental over K and gives an immediate extension $K(\lambda)$ of K by [ADH, 3.2.6]. Hence L is algebraic over $K(\lambda)$, so $\operatorname{res} L$ is algebraic over $\operatorname{res} K(\lambda) = \operatorname{res} K \cong C$ and thus d is algebraic over C, a contradiction. □

Lemma 1.4.6. *Suppose K is λ-free and L is an H-asymptotic extension of K, where $L = K(d)$ with $d \in C_L$. Then L is pre-*d*-valued.*

Proof. Let L^{a} be an algebraic closure of the H-asymptotic field L, and let K^{a} be the algebraic closure of K inside L^{a}. Then K^{a} is pre-d-valued by [ADH, 10.1.22]. Replacing K, L by K^{a}, $K^{\mathrm{a}}(d)$ we arrange that K is algebraically closed. We may assume $d \notin C$, so d is transcendental over K by [ADH, 4.1.1, 4.1.2].

Suppose first that $\operatorname{res}(d) \in \operatorname{res}(K) \subseteq \operatorname{res}(L)$, and take $b \in \mathcal{O}$ such that $y := b - d \prec 1$. Then $b' \notin \partial\smallo$: otherwise $y' = b' = \delta'$ with $\delta \in \smallo$, so $y = \delta \in K$ and hence $d \in K$, a contradiction. Also $vb' \in (\Gamma^{>})'$: otherwise $vb' < (\Gamma^{>})'$, by [ADH, 9.2.14], and vb' would be a gap in K, contradicting λ-freeness of K. Hence $L = K(y)$ is pre-d-valued by [ADH, 10.2.4, 10.2.5(iii)] applied to $s := b'$.

If $\operatorname{res}(d) \notin \operatorname{res}(K)$, then $\operatorname{res}(d)$ is transcendental over $\operatorname{res}(K)$ by [ADH, 3.1.17], hence $\Gamma_L = \Gamma$ by [ADH, 3.1.11], and so L has asymptotic integration and thus is pre-d-valued by [ADH, 10.1.3]. $\square$

Proof of Proposition 1.4.3. By Zorn we reduce to the case $L = K(d)$ with $d \in C_L$. Then L is pre-d-valued by Lemma 1.4.6. By Lemma 1.4.4, the d-valued hull $K_1 := \operatorname{dv}(K)$ of K is λ-free with $\Gamma_{K_1} = \Gamma$, and by the universal property of d-valued hulls we may arrange that K_1 is a d-valued subfield of $L_1 := \operatorname{dv}(L)$ [ADH, 10.3.1]. The proof of [ADH, 10.3.1] gives $L_1 = L(E)$ where $E = C_{L_1}$, and so $L_1 = K_1(E)$. Hence by Lemma 1.4.5 and the remarks preceding it, L_1 is λ-free with $\Gamma_{L_1} = \Gamma_{K_1} = \Gamma$. Thus L is λ-free with $\Gamma_L = \Gamma$. $\square$

Lemma 1.4.7. *Let H be a λ-free real closed H-field. Then the trigonometric closure H^{trig} of H is λ-free.*

Proof. We show that H^{+} as in Lemma 1.2.22 is λ-free. There $H^{+}[\mathrm{i}] = K(y)$ where K is the H-asymptotic extension $H[\mathrm{i}]$ of H and $y \sim 1$, $y^{\dagger} \notin K^{\dagger}$, $y^{\dagger} \in \mathrm{i}\partial\smallo_H$. Then K is λ-free, so $K(y)$ is λ-free by [75, Proposition 7.2], hence H^{+} is λ-free. $\square$

In Example 1.4.2 we have a λ-free K and an H-asymptotic extension L of K that is not λ-free, with $\operatorname{trdeg}(L|K) = 1$. The next proposition shows that the second part of the conclusion of Theorem 1.4.1 nevertheless holds for such K, L.

Proposition 1.4.8. *The following are equivalent:*

(i) *K has rational asymptotic integration;*

(ii) *for every H-asymptotic extension L of K with $\operatorname{trdeg}(L|K) \leqslant 1$ we have that $\Gamma^{<}$ is cofinal in $\Gamma_L^{<}$.*

Proof. For (i) $\Rightarrow$ (ii), assume (i), and let L be an H-asymptotic extension of K with $\operatorname{trdeg}(L|K) \leqslant 1$. Towards showing that $\Gamma^{<}$ is cofinal in $\Gamma_L^{<}$ we can arrange that K and L are algebraically closed. Suppose towards a contradiction that $\gamma \in \Gamma_L$ and $\Gamma^{<} < \gamma < 0$. Then $\Psi < \gamma' < (\Gamma^{>})'$, and so Γ is dense in $\Gamma + \mathbb{Q}\gamma'$ by [ADH, 2.4.16, 2.4.17], in particular, $\gamma \notin \Gamma + \mathbb{Q}\gamma'$. Thus γ, γ' are $\mathbb{Q}$-linearly independent over Γ, which contradicts $\operatorname{trdeg}(L|K) \leqslant 1$ by [ADH, 3.1.11].

As to (ii) $\Rightarrow$ (i), we prove the contrapositive, so assume K does not have rational asymptotic integration. We arrange again that K is algebraically closed. Then K has a gap vs with $s \in K^{\times}$, and so [ADH, 10.2.1 and its proof] gives an H-asymptotic extension $K(y)$ of K with $y' = s$ and $0 < vy < \Gamma^{>}$. $\square$

Recall from [ADH, 11.6] that Liouville closed H-fields are λ-free. To prove the next result we also use Gehret's theorem [75, Theorem 12.1(1)] that an H-field H has up to isomorphism over H exactly one Liouville closure iff H is grounded or λ-free. Here *isomorphism* means of course *isomorphism of H-fields*, and likewise with the embeddings referred to in the next result:

Proposition 1.4.9. *Let H be a grounded or λ-free H-field. Then H has a trigonometrically closed and Liouville closed H-field extension H^{tl} that embeds over H into any trigonometrically closed Liouville closed H-field extension of H.*

Proof. We build real closed H-fields $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ as follows: H_0 is a real closure of H, and, recursively, H_{2n+1} is a Liouville closure of H_{2n}, and $H_{2n+2} := H_{2n+1}^{\text{trig}}$ is the trigonometric closure of H_{2n+1}. Then $H^* := \bigcup_n H_n$ is a trigonometrically closed Liouville closed H-field extension of H. Induction using Lemma 1.4.7 shows that all H_n with $n \geqslant 1$ are λ-free, and that H_{2n} has for all n up to isomorphism over H a unique Liouville closure. Given any trigonometrically closed Liouville closed H-field extension E of H we then use the embedding properties of *Liouville closure* and *trigonometric closure* to construct by a similar recursion embeddings $H_n \to E$ that extend to an embedding $H^* \to E$ over H. $\square$

For H as in Proposition 1.4.9, the H^* constructed in its proof is minimal: Let $E \supseteq H$ be any trigonometrically closed Liouville closed H-subfield of H^*. Then induction on n yields $H_n \subseteq E$ for all n, so $E = H^*$. It follows that any H^{tl} as in Proposition 1.4.9 is isomorphic over H to H^*, and we refer to such H^{tl} as a **trigonometric-Liouville closure** of H. Here are some useful facts about H^{tl}:

Corollary 1.4.10. *Let H be a λ-free H-field. Then $C_{H^{\text{tl}}}$ is a real closure of C_H, the H-asymptotic extension $K^{\text{tl}} := H^{\text{tl}}[i]$ of H^{tl} is a Liouville extension of H with $\mathrm{I}(K^{\text{tl}}) \subseteq (K^{\text{tl}})^{\dagger}$, and $\Gamma_H^{<}$ is cofinal in $\Gamma_{H^{\text{tl}}}^{<}$. Moreover,*

$$H \text{ is } \omega\text{-free} \iff H^{\text{tl}} \text{ is } \omega\text{-free.}$$

Proof. The construction of H^* in the proof of Proposition 1.4.9 gives that C_{H^*} is a real closure of C_H, and that the H-asymptotic extension $K^* := H^*[i]$ of H^* is a Liouville extension of H with $\mathrm{I}(K^*) \subseteq (K^*)^{\dagger}$. Induction using Lemma 1.4.7 and Proposition 1.4.8 shows that H_n is λ-free and $\Gamma_H^{<}$ is cofinal in $\Gamma_{H_n}^{<}$, for all n, so $\Gamma_H^{<}$ is cofinal in $\Gamma_{H^*}^{<}$.

The final equivalence follows from Theorem 1.4.1 and a remark preceding it. $\square$

Proposition 1.4.8 and [ADH, remarks after 11.6.4 and after 11.7.19] yield:

Corollary 1.4.11. *Suppose K has rational asymptotic integration, and let L be an H-asymptotic extension of K with $\operatorname{trdeg}(L|K) \leqslant 1$. If L is λ-free, then so is K, and if L is ω-free, then so is K.*

We also have a similar characterization of λ-freeness:

Proposition 1.4.12. *The following are equivalent:*

(i) *K is λ-free;*
(ii) *every H-asymptotic extension L of K with $\operatorname{trdeg}(L|K) \leqslant 1$ has asymptotic integration.*

Proof. Assume K is λ-free; let L be an H-asymptotic extension of K such that $\operatorname{trdeg}(L|K) \leqslant 1$. By Proposition 1.4.8, $\Gamma^{<}$ is cofinal in $\Gamma_L^{<}$, so L is ungrounded. Towards a contradiction, suppose vf $(f \in L^{\times})$ is a gap in L. Passing to algebraic closures we arrange that K and L are algebraically closed. Set $\lambda := -f^{\dagger}$. Then for all active a in L we have $\lambda + a^\dagger \prec a$ by [ADH, 11.5.9] and hence $\lambda_\rho \rightsquigarrow \lambda$ by [ADH, 11.5.6]. By λ-freeness of K and [ADH, 3.2.6, 3.2.7], the valued field extension $K(\lambda) \supseteq K$ is immediate of transcendence degree 1, so $L \supseteq K(\lambda)$ is algebraic and $\Gamma = \Gamma_L$. Hence vf is a gap in K, a contradiction. This shows (i) $\Rightarrow$ (ii).

To show the contrapositive of (ii) $\Rightarrow$ (i), suppose $\lambda \in K$ is a pseudolimit of (λ_ρ). If the algebraic closure K^{a} of K does not have asymptotic integration, then clearly (ii) fails. If K^{a} has asymptotic integration, then $-\lambda$ creates a gap over K by [ADH, 11.5.14] applied to K^{a} in place of K, hence (ii) also fails. $\square$

The next two lemmas include converses to Lemmas 1.4.4 and 1.4.5.

Lemma 1.4.13. *Let E be a pre-*d*-valued H-asymptotic field. Then:*

(i) *if E is not λ-free, then* $\operatorname{dv}(E)$ *is not λ-free;*
(ii) *if E is not ω-free, then* $\operatorname{dv}(E)$ *is not ω-free.*

Proof. This is clear if E has no rational asymptotic integration, because then $\operatorname{dv}(E)$ has no rational asymptotic integration either, by [ADH, 10.3.2]. Assume E has rational asymptotic integration. Then $\operatorname{dv}(E)$ is an immediate extension of E by [ADH, 10.3.2], and then (i) and (ii) follow from the characterizations of λ-freeness and ω-freeness in terms of nonexistence of certain pseudolimits. $\square$

Lemma 1.4.14. *Let E be a* d*-valued H-asymptotic field and F an H-asymptotic extension of E such that $F = E(C_F)$. Then:*

(i) *if E is not λ-free, then F is not λ-free;*
(ii) *if E is not ω-free, then F is not ω-free.*

Proof. By [ADH, 10.5.15] E and F have the same value group. The rest of the proof is like that for the previous lemma, with F instead of $\operatorname{dv}(E)$. $\square$

In the rest of this subsection K is in addition a pre-H-field and L a pre-H-field extension of K. The following is shown in the proof of [75, Lemma 12.5]:

Proposition 1.4.15 (Gehret)**.** *Suppose K is a λ-free H-field and L is a Liouville H-field extension of K. Then L is λ-free and $\Gamma^<$ is cofinal in $\Gamma_L^<$.*

Example 1.4.16. Let $K = \mathbb{R}\langle\omega\rangle$ be the λ-free but non-ω-free H-field from [ADH, 13.9.1]. Then K has a unique Liouville closure L, up to isomorphism over K, by [75, Theorem 12.1(1)]. By Proposition 1.4.15, L is not ω-free; [9] has another proof of this fact. By [ADH, 13.9.5] we can take here K to be a Hardy field, and then L is isomorphic over K to a Hardy field extension of K [ADH, 10.6.11].

Applying Corollary 1.4.10 to $H := \mathbb{R}\langle\omega\rangle$ yields a Liouville closed H-field H^{tl} that is not ω-free but does satisfy $\mathrm{I}(K^{\mathrm{tl}}) \subseteq (K^{\mathrm{tl}})^\dagger$ for $K^{\mathrm{tl}} := H^{\mathrm{tl}}[i]$.

For a pre-H-field H we singled out in [ADH, p. 520] the following subsets:

$$\Gamma(H) := (H^{\succ 1})^\dagger, \quad \Lambda(H) := -(H^{\succ 1})^{\dagger\dagger}, \quad \Delta(H) := -(H^{\neq, \prec 1})'^\dagger.$$

Lemma 1.4.17. *Suppose K is λ-free, $\lambda \in \Lambda(L)^\downarrow$, $\omega := \omega(\lambda) \in K$, and suppose $\omega\big(\Lambda(K)\big) < \omega < \sigma\big(\Gamma(K)\big)$. Then $\lambda_\rho \rightsquigarrow \lambda$, and the pre-$H$-subfield $K\langle\lambda\rangle = K(\lambda)$ of L is an immediate extension of K (and so $K\langle\lambda\rangle$ is not λ-free).*

Proof. From $\Lambda(L) < \Delta(L)$ [ADH, p. 522] and $\Delta(K) \subseteq \Delta(L)$ we obtain $\lambda < \Delta(K)$. The restriction of ω to $\Lambda(L)^\downarrow$ is strictly increasing [ADH, p. 526] and $\Lambda(K) \subseteq \Lambda(L)$, so $\omega\big(\Lambda(K)\big) < \omega = \omega(\lambda)$ gives $\Lambda(K) < \lambda$. Hence $\lambda_\rho \rightsquigarrow \lambda$ by [ADH, 11.8.16]. Also $\omega_\rho \rightsquigarrow \omega$ by [ADH, 11.8.30]. Thus $K\langle\lambda\rangle$ is an immediate extension of K by [ADH, 11.7.13]. $\square$

Achieving ω-freeness for pre-H-fields. *In the rest of this section H is a pre-H-field and L is a Liouville closed* d*-algebraic H-field extension of H.* Thus if H is ω-free, then so is L, by Theorem 1.4.1.

The lemmas below give conditions guaranteeing that L is ω-free, while H is not.

Lemma 1.4.18. *Suppose H is grounded or has a gap. Then L is ω-free.*

Proof. Suppose H is grounded. Let H_ω be the ω-free pre-H-field extension of H introduced in connection with [ADH,11.7.17] (where we use the letter F instead of H). Identifying H_ω with its image in L under an embedding $H_\omega \to L$ over H of pre-H-fields, we apply Theorem 1.4.1 to $K := H_\omega$ to conclude that L is ω-free.

Next, suppose H has a gap $\beta = vb$, $b \in H^\times$. Take $a \in L$ with $a' = b$ and $a \not\preccurlyeq 1$. Then $\alpha := va$ satisfies $\alpha' = \beta$, and so the pre-H-field $H(a) \subseteq L$ is grounded, by [ADH, 9.8.2 and remarks following its proof]. Now apply the previous case to $H(a)$ in place of H. $\square$

Lemma 1.4.19. *Suppose H has asymptotic integration and divisible value group, and $s \in H$ creates a gap over H. Then L is ω-free.*

Proof. Take $f \in L^\times$ with $f^\dagger = s$. Then by [ADH, remark after 11.5.14], vf is a gap in $H\langle f\rangle = H(f)$, so L is ω-free by Lemma 1.4.18 applied to $H\langle f\rangle$ in place of H. $\square$

Lemma 1.4.20. *Suppose H is not λ-free. Then L is ω-free.*

Proof. By [ADH, 11.6.8], the real closure H^{rc} of H inside L is not λ-free, hence replacing H by H^{rc} we arrange that H is real closed. If H does not have asymptotic integration, then we are done by Lemma 1.4.18. So suppose H has asymptotic integration. Then some $s \in H$ creates a gap over H, by [ADH, 11.6.1], so L is ω-free by Lemma 1.4.19. $\square$

Corollary 1.4.21. *Suppose H is λ-free and $\lambda \in \Lambda(L)^\downarrow$ is such that $\omega := \omega(\lambda) \in H$ and $\omega\big(\Lambda(H)\big) < \omega < \sigma\big(\Gamma(H)\big)$. Then L is ω-free.*

Proof. By Lemma 1.4.17, the pre-H-subfield $H\langle\lambda\rangle = H(\lambda)$ of L is an immediate non-λ-free extension of H. Now apply Lemma 1.4.20 to $H\langle\lambda\rangle$ in place of H. $\square$

1.5. Complements on Linear Differential Operators

In this section we tie up loose ends from the material on linear differential operators in [ADH, 14.2] and [11, Section 8]. *Throughout K is an ungrounded asymptotic field, a, b, f, g, h range over arbitrary elements of K, and ϕ over those active in K, in particular, $\phi \neq 0$.* Recall from [ADH, p. 479] our use of the term "eventually": a property $S(\phi)$ of elements ϕ is said to hold *eventually* if for some active ϕ_0 in K, $S(\phi)$ holds for all $\phi \preccurlyeq \phi_0$.

We shall consider linear differential operators $A \in K[\partial]^{\neq}$ and set $r := \operatorname{order}(A)$. In [ADH, Section 11.1] we introduced the set

$$\mathscr{E}^{\mathrm{e}}(A) \;=\; \mathscr{E}^{\mathrm{e}}_K(A) \;:=\; \big\{\gamma \in \Gamma : \operatorname{nwt}_A(\gamma) \geqslant 1\big\} \;=\; \bigcap_\phi \mathscr{E}(A^\phi)$$

of *eventual exceptional values of A*. For $a \neq 0$ we have $\mathscr{E}^{\mathrm{e}}(aA) = \mathscr{E}^{\mathrm{e}}(A)$ and $\mathscr{E}^{\mathrm{e}}(Aa) = \mathscr{E}^{\mathrm{e}}(A) - va$. An easy consequence of the definitions: $\mathscr{E}^{\mathrm{e}}(A^f) = \mathscr{E}^{\mathrm{e}}(A)$ for $f \neq 0$. A key fact about $\mathscr{E}^{\mathrm{e}}(A)$ is that if $y \in K^\times$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, then $A(y) \asymp A^\phi y$, eventually. Since $A^\phi y \neq 0$ for $y \in K^\times$, this gives $v(\ker^{\neq} A) \subseteq \mathscr{E}^{\mathrm{e}}(A)$.

Lemma 1.5.1. *If L is an ungrounded asymptotic extension of K, then $\mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma \subseteq \mathscr{E}^{\mathrm{e}}(A)$, with equality if Ψ is cofinal in Ψ_L.*

Proof. For the inclusion, use that $\operatorname{dwt}(A^{\phi})$ decreases as $v\phi$ strictly increases [ADH, 11.1.12]. Thus its eventual value $\operatorname{nwt}(A)$, evaluated in K, cannot strictly increase when evaluated in an ungrounded asymptotic extension of K. □

In the rest of this section we assume in addition that K is H-asymptotic with asymptotic integration. Then by [ADH, 14.2.8]:

Proposition 1.5.2. *If K is r-linearly newtonian, then $v(\ker^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$.*

Remark 1.5.3. If K is d-valued, then $|v(\ker^{\neq} A)| = \dim_C \ker A \leqslant r$ by [ADH, 5.6.6], using a reduction to the case of "small derivation" by compositional conjugation.

Corollary 1.5.4. *Suppose K is* d*-valued, $\mathscr{E}^{\mathrm{e}}(A) = v(\ker^{\neq} A)$, and $0 \neq f \in A(K)$. Then $A(y) = f$ for some $y \in K$ with $vy \notin \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. Let $y \in K$, $A(y) = f$, with vy maximal. Then $vy \notin \mathscr{E}^{\mathrm{e}}(A)$: otherwise we have $z \in \ker A$ with $z \sim y$, so $A(y - z) = f$ and $v(y - z) > vy$. □

Corollary 1.5.5. *Suppose K is ω-free. Then $\sum_{\gamma \in \Gamma} \operatorname{nwt}_A(\gamma) = |\mathscr{E}^{\mathrm{e}}(A)| \leqslant r$.*

Proof. The remarks following [ADH, 14.0.1] give an immediate asymptotic extension L of K that is newtonian. Then L is d-valued by Lemma 1.2.9, hence $|\mathscr{E}^{\mathrm{e}}(A)| = |\mathscr{E}^{\mathrm{e}}_L(A)| \leqslant r$ by Proposition 1.5.2 and Remark 1.5.3. By [ADH, 13.7.10] we have $\operatorname{nwt}_A(\gamma) \leqslant 1$ for all $\gamma \in \Gamma$, thus $\sum_{\gamma \in \Gamma} \operatorname{nwt}_A(\gamma) = |\mathscr{E}^{\mathrm{e}}(A)|$. □

In [ADH, Section 11.1] we defined $v^{\mathrm{e}}_A \colon \Gamma \to \Gamma$ by requiring that for all $\gamma \in \Gamma$:

$$v_{A^{\phi}}(\gamma) \ = \ v^{\mathrm{e}}_A(\gamma) + \operatorname{nwt}_A(\gamma) v\phi, \qquad \text{eventually.} \tag{1.5.1}$$

We recall from that reference that for $a \neq 0$ and $\gamma \in \Gamma$ we have

$$v^{\mathrm{e}}_{aA}(\gamma) \ = \ va + v^{\mathrm{e}}_A(\gamma), \qquad v^{\mathrm{e}}_{Aa}(\gamma) = v^{\mathrm{e}}_A(va + \gamma).$$

As an example from [ADH, p. 481], $v^{\mathrm{e}}_{\partial}(\gamma) = \gamma + \psi(\gamma)$ for $\gamma \in \Gamma \setminus \{0\}$ and $v^{\mathrm{e}}_{\partial}(0) = 0$. By [ADH, 14.2.7 and the remark preceding it] we have:

Lemma 1.5.6. *The restriction of v^{e}_A to a function $\Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma$ is strictly increasing, and $v\big(A(y)\big) = v^{\mathrm{e}}_A(vy)$ for all $y \in K$ with $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$. Moreover, if K is ω-free, then $v^{\mathrm{e}}_A\big(\Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)\big) = \Gamma$.*

The following is [ADH, 14.2.10] without the hypothesis of ω-freeness:

Corollary 1.5.7. *Suppose K is r-linearly newtonian. Then for each $f \neq 0$ there exists $y \in K^{\times}$ such that $A(y) = f$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v^{\mathrm{e}}_A(vy) = vf$.*

Proof. If $r = 0$, then $\mathscr{E}^{\mathrm{e}}(A) = \emptyset$ and our claim is obviously valid. Suppose $r \geqslant 1$. Then K is d-valued by Lemma 1.2.9, and $v(\ker^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$ by Proposition 1.5.2, Moreover, by [ADH, 14.2.2], K is r-linearly surjective, hence $f \in A(K)$. Now Corollary 1.5.4 yields $y \in K^{\times}$ with $A(y) = f$ and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. By Lemma 1.5.6 we have $v^{\mathrm{e}}_A(vy) = v\big(A(y)\big) = vf$. □

From the proof of [ADH, 14.2.10] we extract the following:

Corollary 1.5.8. *Suppose K is r-linearly newtonian with small derivation, and $A \in \mathcal{O}[\partial]$ with $a_0 := A(1) \asymp 1$, and $f \asymp^{\flat} 1$. Then there is $y \in K^{\times}$ such that $A(y) = f$ and $y \sim f/a_0$. For any such y we have $vy \notin \mathscr{E}^{\mathrm{e}}(A)$ and $v^{\mathrm{e}}_A(vy) = vf$.*

Proof. The case $r=0$ is trivial. Assume $r \geqslant 1$, so K is d-valued by Lemma 1.2.9. Hence $f^\dagger \prec 1$, that is, $f' \prec f$, so $f^{(n)} \prec f$ for all $n \geqslant 1$ by [ADH, 4.4.2]. Then $Af \preccurlyeq f$ by [ADH, (5.1.3), (5.1.2)], and $A(f) \sim a_0 f$, so $A_{\ltimes f} \in \mathcal{O}[\partial]$ and $A_{\ltimes f}(1) \sim a_0$. Thus we may replace A, f by $A_{\ltimes f}$, 1 to arrange $f=1$. Now $a_0 \asymp 1$ gives $\operatorname{dwm}(A)=0$, so $\operatorname{dwt}(A^\phi)=0$ eventually, by [ADH, 11.1.11(ii)], that is, $\operatorname{nwt}(A)=0$. Also $A^\phi(1)=A(1)=a_0 \asymp 1$, so $v^{\mathrm{e}}(A)=0$. Arguing as in the proof of [ADH, 14.2.10] we obtain $y \in K^\times$ with $A(y)=1$ and $y \sim 1/a_0$. It is clear that $vy = 0 \notin \mathscr{E}^{\mathrm{e}}(A)$ and $v^{\mathrm{e}}_A(vy)=v^{\mathrm{e}}(A)=0=vf$ for any such y. □

In the next few subsections below we consider more closely the case of order $r=1$, and in the last subsection the case of arbitrary order.

First-order operators. *In this subsection* $A=\partial - g$. By [ADH, p. 481],

$$\mathscr{E}^{\mathrm{e}}(A) \;=\; \mathscr{E}^{\mathrm{e}}_K(A) \;=\; \big\{vy :\, y \in K^\times,\ v(g-y^\dagger) > \Psi\big\}$$

has at most one element. We also have $|v(\ker^{\neq} A)| = \dim_C \ker A \leqslant 1$ in view of $C^\times \subseteq \mathcal{O}^\times$. Proposition 1.5.2 holds under a weaker assumption on K for $r=1$:

Lemma 1.5.9. *Suppose* $\mathrm{I}(K) \subseteq K^\dagger$. *Then* $v(\ker^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$.

Proof. It remains to show "$\supseteq$". Suppose $\mathscr{E}^{\mathrm{e}}(A)=\{0\}$. Then $g-y^\dagger \in \mathrm{I}(K)$ with $y \asymp 1$ in K, hence $g \in \mathrm{I}(K) \subseteq K^\dagger$, so $g=h^\dagger$ with $h \asymp 1$, and thus $0 = vh \in v(\ker^{\neq} A)$. The general case reduces to the case $\mathscr{E}^{\mathrm{e}}(A)=\{0\}$ by twisting. □

Lemma 1.5.10. *Suppose L is an ungrounded H-asymptotic extension of K. Then* $\mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma = \mathscr{E}^{\mathrm{e}}(A)$.

Proof. Lemma 1.5.1 gives $\mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma \subseteq \mathscr{E}^{\mathrm{e}}(A)$. Next, let $vy \in \mathscr{E}^{\mathrm{e}}(A)$, $y \in K^\times$. Then $v(g-y^\dagger) > \Psi$ and so $v(g-y^\dagger) \in (\Gamma^>)'$ since K has asymptotic integration. Hence $v(g-y^\dagger) > \Psi_L$ and thus $vy \in \mathscr{E}^{\mathrm{e}}_L(A)$, by [ADH, p. 481]. □

Recall also from [ADH, 9.7] that for an ordered abelian group G and $U \subseteq G$, a function $\eta\colon U \to G$ is said to be *slowly varying* if $\eta(\alpha)-\eta(\beta) = o(\alpha-\beta)$ for all $\alpha \neq \beta$ in U; then the function $\gamma \mapsto \gamma + \eta(\gamma)\colon U \to G$ is strictly increasing. The quintessential example of a slowly varying function is $\psi\colon \Gamma^{\neq} \to \Gamma$ [ADH, 6.5.4(ii)].

Proposition 1.5.11. *There is a unique slowly varying function* $\psi_A\colon \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma$ *such that for all* $y \in K^\times$ *with* $vy \notin \mathscr{E}^{\mathrm{e}}(A)$ *we have* $v\big(A(y)\big) = vy + \psi_A(vy)$.

Proof. For d-valued K, use [11, 8.4]. In general, pass to the d-valued hull $L := \operatorname{dv}(K)$ of K from [ADH, 10.3] and use $\Gamma_L = \Gamma$ [ADH, 10.3.2]. □

If $b \neq 0$, then $\mathscr{E}^{\mathrm{e}}(A_{\ltimes b}) = \mathscr{E}^{\mathrm{e}}(A) - vb$ and $\psi_{A_{\ltimes b}}(\gamma) = \psi_A(\gamma + vb)$ for $\gamma \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A_{\ltimes b})$.

Example. We have $\mathscr{E}^{\mathrm{e}}(\partial) = \{0\}$ and $\psi_\partial = \psi$. More generally, if $g = b^\dagger$, $b \neq 0$, then $A_{\ltimes b} = \partial$ and so $\mathscr{E}^{\mathrm{e}}(A) = \{vb\}$ and $\psi_A(\gamma) = \psi(\gamma - vb)$ for $\gamma \in \Gamma \setminus \{vb\}$.

If Γ is divisible, then $\Gamma \setminus v\big(A(K)\big)$ has at most one element by [ADH, 11.6.16]. Also, K is $\uplambda$-free iff $v\big(A(K)\big) = \Gamma_\infty$ for all $A = \partial - g$ by [ADH, 11.6.17].

Lemma 1.5.12. *Suppose K is $\uplambda$-free and $f \neq 0$. Then for some $y \in K^\times$ with $vy \notin \mathscr{E}^{\mathrm{e}}(A)$ we have $A(y) \asymp f$. (Hence $\gamma \mapsto \gamma + \psi_A(\gamma)\colon \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma$ is surjective.)*

Proof. [ADH, 11.6.17] gives $y \in K^\times$ with $A^\phi y \asymp f$ eventually. Now

$$A^\phi y \;=\; \phi y \delta - (g - y^\dagger) y \;\text{ in } K^\phi[\delta], \qquad \delta := \phi^{-1}\partial.$$

Since $v(A^\phi y) = vf$ eventually, this forces $g - y^\dagger \succ \phi$ eventually, so $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. $\square$

Call A **steep** if $g \succ^\flat 1$, that is, $g \succ 1$ and $g^\dagger \succcurlyeq 1$. If K has small derivation and A is steep, then $g^\dagger \prec g$ by [ADH, 9.2.10].

Lemma 1.5.13. *Suppose K has small derivation, A is steep, and $y \in K^\times$ such that $A(y) = f \neq 0$, $g \succ f^\dagger$, and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. Then $y \sim -f/g$.*

Proof. We have

$$(f/g)^\dagger - g = f^\dagger - g^\dagger - g \sim -g \succ g^\dagger,$$

hence $v(f/g) \notin \mathscr{E}^{\mathrm{e}}(A)$, and

$$A(f/g) \;=\; (f/g)' - (f/g)g \;=\; (f/g)\cdot\big(f^\dagger - g^\dagger - g\big) \;\sim\; (f/g)\cdot(-g) \;=\; -f.$$

Since $A(y) = f \sim A(-f/g)$ and $vy, v(f/g) \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, this gives $y = u \cdot f/g$ where $u \asymp 1$, by Proposition 1.5.11. Now $u^\dagger \prec 1 \prec g$ and $(f/g)^\dagger = f^\dagger - g^\dagger \prec g$, hence $y^\dagger \prec g$ and so

$$f = A(y) = y\cdot(y^\dagger - g) \sim -yg.$$

Therefore $y \sim -f/g$. $\square$

Lemma 1.5.14. *Suppose K has small derivation and $y \in K^\times$ is such that $A(y) = f \neq 0$, $g - f^\dagger \succ^\flat 1$ and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. Then $y \sim f/(f^\dagger - g)$.*

Proof. From $g - f^\dagger \succ 1$ we get $vf \notin \mathscr{E}^{\mathrm{e}}(A)$. Now $A(y) = f \prec f(f^\dagger - g) = A(f)$, so $y \prec f$ by [ADH, 5.6.8], and $v(y/f) \notin \mathscr{E}^{\mathrm{e}}(A_{\ltimes f}) = \mathscr{E}^{\mathrm{e}}(A) - vf$. Since $A_{\ltimes f} = \partial - (g - f^\dagger)$ is steep, Lemma 1.5.13 applies to $A_{\ltimes f}$, y/f, 1 in the role of A, y, f. $\square$

Suppose K is λ-free and $f \neq 0$. Then [ADH, 11.6.1] gives an active ϕ_0 in K with $f^\dagger - g - \phi^\dagger \succcurlyeq \phi_0$ for all $\phi \prec \phi_0$. The convex subgroups $\Gamma^\flat_\phi$ of Γ become arbitrarily small as we let $v\phi$ increase cofinally in $\Psi^\downarrow$, so $\phi \prec^\flat_\phi \phi_0$ eventually, and hence $f^\dagger - g - \phi^\dagger \succ^\flat_\phi \phi$ eventually, that is, $\phi^{-1}(f/\phi)^\dagger - g/\phi \succ^\flat_\phi 1$ eventually. So replacing K by K^ϕ, A by $\phi^{-1}A^\phi = \delta - (g/\phi)$ in $K^\phi[\delta]$, and f and g by f/ϕ and g/ϕ, for suitable ϕ, we arrange $f^\dagger - g \succ^\flat 1$. Thus by Lemma 1.5.14:

Corollary 1.5.15. *If K is λ-free, $y \in K^\times$, $A(y) = f \neq 0$, and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, then $y \sim f/\big((f/\phi)^\dagger - g\big)$, eventually.*

Example. If K is λ-free and $y \in K$, $y' = f \neq 0$ with $y \not\asymp 1$, then $y \sim f/(f/\phi)^\dagger$, eventually.

From K to $K[\mathrm{i}]$. *In this subsection K is a real closed H-field.* Then $K[\mathrm{i}]$ with $\mathrm{i}^2 = -1$ is an H-asymptotic extension of K, with $\Gamma_{K[\mathrm{i}]} = \Gamma$. Consider a linear differential operator $B = \partial - (g + h\mathrm{i})$ over $K[\mathrm{i}]$. Note that $g + h\mathrm{i} \in K[\mathrm{i}]^\dagger$ iff $g \in K^\dagger$ and $h\mathrm{i} \in K[\mathrm{i}]^\dagger$, by Lemma 1.2.4. Under further assumptions on K, the next two results give explicit descriptions of ψ_B when $g \in K^\dagger$.

Proposition 1.5.16. *Suppose $K[\mathrm{i}]$ is* 1*-linearly newtonian and $g \in K^\dagger$. Then:*

(i) *if $h\mathrm{i} \in K[\mathrm{i}]^\dagger$, then for some $\beta \in \Gamma$ we have*

$$\mathscr{E}^{\mathrm{e}}(B) \;=\; \{\beta\}, \qquad \psi_B(\gamma) \;=\; \psi(\gamma - \beta) \;\text{ for all } \gamma \in \Gamma \setminus \{\beta\};$$

(ii) *if* $hi \notin K[\mathrm{i}]^\dagger$ *and* $g = b^\dagger$, $b \neq 0$, *then*

$$\mathscr{E}^{\mathrm{e}}(B) \ = \ \emptyset, \qquad \psi_B(\gamma) \ = \ \min\bigl(\psi(\gamma - vb), vh\bigr) \ \text{ for all } \gamma \in \Gamma.$$

Proof. As to (i), apply the example following Proposition 1.5.11 to $K[\mathrm{i}]$, B, $g+hi$ in the roles of K, A, g. For (ii), assume $hi \notin K[\mathrm{i}]^\dagger$, $g = b^\dagger$, $b \neq 0$. Replacing B by $B_{\ltimes b}$ we arrange $g = 0$, $b = 1$, $B = \partial - hi$. Corollary 1.2.17 gives $K[\mathrm{i}]^\dagger = K^\dagger \oplus \mathrm{I}(K)\mathrm{i}$, so $h \notin \mathrm{I}(K)$, and thus $vh \in \Psi^\downarrow$. Let $y \in K[\mathrm{i}]^\times$, and take $z \in K^\times$ and $s \in \mathrm{I}(K)$ with $y^\dagger = z^\dagger + si$. Then $vh < vs$, hence

$$v(y^\dagger - hi) \ = \ \min\bigl(v(z^\dagger), v(s-h)\bigr) \ = \ \min\bigl(v(z^\dagger), vs, vh\bigr) \ = \ \min\bigl(v(y^\dagger), vh\bigr),$$

where the last equality uses $v(y^\dagger) = \min\bigl(v(z^\dagger), vs\bigr)$. Thus $v(y^\dagger - hi) \in \Psi^\downarrow$ and

$$v\bigl(B(y)\bigr) - vy \ = \ v(y^\dagger - hi) \ = \ \min\bigl(v(y^\dagger), vh\bigr) \ = \ \min\bigl(\psi(vy), vh\bigr),$$

which gives the desired result. $\square$

Corollary 1.5.17. *Suppose K is $\upomega$-free, $g \in K^\dagger$, $g = b^\dagger$, $b \neq 0$. Then either for some $\beta \in \Gamma$ we have $\mathscr{E}^{\mathrm{e}}(B) = \{\beta\}$ and $\psi_B(\gamma) = \psi(\gamma - \beta)$ for all $\gamma \in \Gamma \setminus \{\beta\}$, or $\mathscr{E}^{\mathrm{e}}(B) = \emptyset$ and $\psi_B(\gamma) = \min\bigl(\psi(\gamma - vb), vh\bigr)$ for all $\gamma \in \Gamma$.*

Proof. By [ADH, 14.0.1 and remarks following it] we have an immediate newtonian extension L of K. Then L is still a real closed H-field [ADH, 10.5.8, 3.5.19], and $L[\mathrm{i}]$ is newtonian [ADH, 14.5.7], so Proposition 1.5.16 applies to L in place of K. $\square$

Higher-order operators. We begin with the following observation:

Lemma 1.5.18. *Let $B \in K[\partial]^{\neq}$ and $\gamma \in \Gamma$. Then $\operatorname{nwt}_{AB}(\gamma) \geqslant \operatorname{nwt}_B(\gamma)$, and*

$$\gamma \notin \mathscr{E}^{\mathrm{e}}(B) \implies \operatorname{nwt}_{AB}(\gamma) \ = \ \operatorname{nwt}_A\bigl(v^{\mathrm{e}}_B(\gamma)\bigr) \text{ and } v^{\mathrm{e}}_{AB}(\gamma) \ = \ v^{\mathrm{e}}_A\bigl(v^{\mathrm{e}}_B(\gamma)\bigr).$$

Proof. We have $\operatorname{nwt}_{AB}(\gamma) = \operatorname{dwt}_{(AB)^\phi}(\gamma)$ eventually, and $(AB)^\phi = A^\phi B^\phi$. Hence by [ADH, Section 5.6] and the definition of $v^{\mathrm{e}}_B(\gamma)$ in (1.5.1):

$$\begin{aligned} \operatorname{nwt}_{AB}(\gamma) \ &= \ \operatorname{dwt}_{A^\phi}\bigl(v_{B^\phi}(\gamma)\bigr) + \operatorname{dwt}_{B^\phi}(\gamma) \\ &= \ \operatorname{dwt}_{A^\phi}\bigl(v^{\mathrm{e}}_B(\gamma) + \operatorname{nwt}_B(\gamma)v\phi\bigr) + \operatorname{nwt}_B(\gamma), \text{ eventually,} \end{aligned}$$

so $\operatorname{nwt}_{AB}(\gamma) \geqslant \operatorname{nwt}_B(\gamma)$. Now suppose $\gamma \notin \mathscr{E}^{\mathrm{e}}(B)$. Then $\operatorname{nwt}_B(\gamma) = 0$, so

$$\operatorname{nwt}_{AB}(\gamma) = \operatorname{dwt}_{A^\phi}\bigl(v^{\mathrm{e}}_B(\gamma)\bigr) = \operatorname{nwt}_A\bigl(v^{\mathrm{e}}_B(\gamma)\bigr), \qquad \text{eventually.}$$

Moreover, $v_{(AB)^\phi} = v_{A^\phi B^\phi} = v_{A^\phi} \circ v_{B^\phi}$, hence using (1.5.1):

$$v_{(AB)^\phi}(\gamma) \ = \ v_{A^\phi}\bigl(v_{B^\phi}(\gamma)\bigr) \ = \ v_{A^\phi}\bigl(v^{\mathrm{e}}_B(\gamma)\bigr), \text{ eventually,}$$

and thus eventually

$$\begin{aligned} v^{\mathrm{e}}_{AB}(\gamma) \ &= \ v_{(AB)^\phi}(\gamma) - \operatorname{nwt}_{AB}(\gamma)v\phi \\ &= \ v_{A^\phi}\bigl(v^{\mathrm{e}}_B(\gamma)\bigr) - \operatorname{nwt}_A\bigl(v^{\mathrm{e}}_B(\gamma)\bigr)v\phi \ = \ v^{\mathrm{e}}_A\bigl(v^{\mathrm{e}}_B(\gamma)\bigr). \end{aligned}$$

$\square$

Lemmas 1.5.6 and 1.5.18 yield:

Corollary 1.5.19. *Let $B \in K[\partial]^{\neq}$. Then*

$$\mathscr{E}^{\mathrm{e}}(AB) \ = \ (v^{\mathrm{e}}_B)^{-1}\bigl(\mathscr{E}^{\mathrm{e}}(A)\bigr) \cup \mathscr{E}^{\mathrm{e}}(B)$$

and hence $|\mathscr{E}^{\mathrm{e}}(AB)| \leqslant |\mathscr{E}^{\mathrm{e}}(A)| + |\mathscr{E}^{\mathrm{e}}(B)|$, with equality if $v^{\mathrm{e}}_B\bigl(\Gamma \setminus \mathscr{E}^{\mathrm{e}}(B)\bigr) = \Gamma$.

As an easy consequence we have a variant of Corollary 1.5.5:

Corollary 1.5.20. *If A splits over K, then $|\mathscr{E}^{\mathrm{e}}(A)| \leqslant r$.*

To study v^{e}_A in more detail we introduce the function

$$\psi_A\ :\ \Gamma\setminus\mathscr{E}^{\mathrm{e}}(A)\to\Gamma,\qquad \gamma\mapsto v^{\mathrm{e}}_A(\gamma)-\gamma.$$

For monic A of order 1 this agrees with ψ_A as defined in Proposition 1.5.11. For $A = a$ $(a\neq 0)$ we have $\mathscr{E}^{\mathrm{e}}(A)=\emptyset$ and $\psi_A(\gamma)=va$ for all $\gamma\in\Gamma$.

Lemma 1.5.21. *Let $B\in K[\partial]^{\neq}$ and $\gamma\in\Gamma\setminus\mathscr{E}^{\mathrm{e}}(AB)$. Then*

$$\psi_{AB}(\gamma)\ =\ \psi_A\big(v^{\mathrm{e}}_B(\gamma)\big)+\psi_B(\gamma).$$

Proof. We have $\gamma\notin\mathscr{E}^{\mathrm{e}}(B)$ and $v^{\mathrm{e}}_B(\gamma)\notin\mathscr{E}^{\mathrm{e}}(A)$ by Corollary 1.5.19, hence

$$\psi_{AB}(\gamma)\ =\ v^{\mathrm{e}}_A\big(v^{\mathrm{e}}_B(\gamma)\big)-\gamma\ =\ v^{\mathrm{e}}_B(\gamma)+\psi_A\big(v^{\mathrm{e}}_B(\gamma)\big)-\gamma\ =\ \psi_A\big(v^{\mathrm{e}}_B(\gamma)\big)+\psi_B(\gamma)$$

by Lemma 1.5.18. □

Thus for $a\neq 0$ and $\gamma\in\Gamma$ we have

$$\psi_{aA}(\gamma)=va+\psi_A(\gamma)\text{ if }\gamma\notin\mathscr{E}^{\mathrm{e}}(A),\quad \psi_{Aa}(\gamma)=\psi_A(va+\gamma)+va\text{ if }\gamma\notin\mathscr{E}^{\mathrm{e}}(A)-va.$$

Example. Suppose K has small derivation and $x\in K$, $x'\asymp 1$. Then $vx<0$ and $\mathscr{E}^{\mathrm{e}}(\partial^2)=\{vx,0\}$, and $\psi_{\partial^2}(\gamma)=\psi\big(\gamma+\psi(\gamma)\big)+\psi(\gamma)$ for $\gamma\in\Gamma\setminus\mathscr{E}^{\mathrm{e}}(\partial^2)$.

Lemma 1.5.22. *Suppose ψ_A is slowly varying. Let Δ be a convex subgroup of Γ and let $y,z\in K^\times$ be such that $vy,vz\notin\mathscr{E}^{\mathrm{e}}(A)$. Then*

$$v_\Delta(y)<v_\Delta(z)\iff v_\Delta\big(A(y)\big)<v_\Delta\big(A(z)\big).$$

Proof. By Lemma 1.5.6 we have

$$v\big(A(y)\big)-v\big(A(z)\big)\ =\ v^{\mathrm{e}}_A(vy)-v^{\mathrm{e}}_A(vz)\ =\ vy-vz+\psi_A(vy)-\psi_A(vz)$$

and $\psi_A(vy)-\psi_A(vz)=o(vy-vz)$ if $vy\neq vz$. □

Call A **asymptotically surjective** if $v^{\mathrm{e}}_A\big(\Gamma\setminus\mathscr{E}^{\mathrm{e}}(A)\big)=\Gamma$ and ψ_A is slowly varying. If A is asymptotically surjective, then so are aA and Aa for $a\neq 0$, and if A has order 0, then A is asymptotically surjective. If K is $\uplambda$-free and A has order 1, then A is asymptotically surjective, thanks to Proposition 1.5.11 and Lemma 1.5.12.
The next lemma has an obvious proof.

Lemma 1.5.23. *Let G be an ordered abelian group and $U,V\subseteq G$. If $\eta_1,\eta_2\colon U\to G$ are slowly varying, then so is $\eta_1+\eta_2$. If $\eta\colon U\to G$ and $\zeta\colon V\to G$ are slowly varying and $\gamma+\zeta(\gamma)\in U$ for all $\gamma\in V$, then the function $\gamma\mapsto\eta\big(\gamma+\zeta(\gamma)\big)\colon V\to G$ is also slowly varying.*

Lemma 1.5.24. *If A and $B\in K[\partial]^{\neq}$ are asymptotically surjective, then so is AB.*

Proof. Let A, B be asymptotically surjective and $\gamma\in\Gamma$. This gives $\alpha\in\Gamma\setminus\mathscr{E}^{\mathrm{e}}(A)$ with $v^{\mathrm{e}}_A(\alpha)=\gamma$ and $\beta\in\Gamma\setminus\mathscr{E}^{\mathrm{e}}(B)$ with $v^{\mathrm{e}}_B(\beta)=\alpha$. Then $\beta\notin\mathscr{E}^{\mathrm{e}}(AB)$ by Corollary 1.5.19, and $v^{\mathrm{e}}_{AB}(\beta)=\gamma$ by Lemma 1.5.18. Moreover, ψ_{AB} is slowly varying by Lemmas 1.5.21 and 1.5.23. □

A straightforward induction on r using this lemma yields:

Corollary 1.5.25. *If K is $\uplambda$-free and A splits over K, then A is asymptotically surjective.*

We can now add to Lemma 1.5.6:

Corollary 1.5.26. *Suppose K is $\upomega$-free. Then A is asymptotically surjective.*

Proof. By the second part of Lemma 1.5.6 it is enough to show that ψ_A is slowly varying. For this we may replace K by any ω-free H-asymptotic extension L of K with Ψ cofinal in Ψ_L. Thus we can arrange by [ADH, 14.5.7, remarks following 14.0.1] that K is newtonian, and by passing to the algebraic closure, algebraically closed. Then A splits over K by [ADH, 5.8.9, 14.5.3], and so A is asymptotically surjective by Corollary 1.5.25. $\square$

1.6. Special Elements

Let K be a valued field and let $\widehat{a}$ be an element of an immediate extension of K with $\widehat{a} \notin K$. Recall that

$$v(\widehat{a} - K) \ = \ \big\{v(\widehat{a} - a) : a \in K\big\}$$

is a nonempty downward closed subset of $\Gamma := v(K^\times)$ without a largest element. Call $\widehat{a}$ *special* over K if some nontrivial convex subgroup of Γ is cofinal in $v(\widehat{a} - K)$ [ADH, p. 167]. In this case $v(\widehat{a} - K) \cap \Gamma^> \neq \emptyset$, and there is a unique such nontrivial convex subgroup Δ of Γ, namely

$$\Delta \ = \ \big\{\delta \in \Gamma : |\delta| \in v(\widehat{a} - K)\big\}.$$

We also call $\widehat{a}$ *almost special* over K if $\widehat{a}/\mathfrak{m}$ is special over K for some $\mathfrak{m} \in K^\times$. If $\Gamma \neq \{0\}$ is archimedean, then $\widehat{a}$ is special over K iff $v(\widehat{a} - K) = \Gamma$, iff $\widehat{a}$ is the limit of a divergent c-sequence in K. (Recall that "c-sequence" abbreviates "cauchy sequence" [ADH, p. 82].) In the next lemma a ranges over K and $\mathfrak{m}$, $\mathfrak{n}$ over $K^\times$.

Lemma 1.6.1. *Suppose $\widehat{a} \prec \mathfrak{m}$ and $\widehat{a}/\mathfrak{m}$ is special over K. Then for all a, $\mathfrak{n}$, if $\widehat{a} - a \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, then $(\widehat{a} - a)/\mathfrak{n}$ is special over K.*

Proof. Replacing $\widehat{a}$, a, $\mathfrak{m}$, $\mathfrak{n}$ by $\widehat{a}/\mathfrak{m}$, $a/\mathfrak{m}$, 1, $\mathfrak{n}/\mathfrak{m}$, respectively, we arrange $\mathfrak{m} = 1$. So let $\widehat{a}$ be special over K with $\widehat{a} \prec 1$. It is enough to show: (1) $\widehat{a} - a$ is special over K, for all a; (2) for all $\mathfrak{n}$, if $\widehat{a} \prec \mathfrak{n} \preccurlyeq 1$, then $\widehat{a}/\mathfrak{n}$ is special over K. Here (1) follows from $v(\widehat{a} - a - K) = v(\widehat{a} - K)$. For (2), note that if $\widehat{a} \prec \mathfrak{n} \preccurlyeq 1$, then $v\mathfrak{n} \in \Delta$ with Δ as above, and so $v(\widehat{a}/\mathfrak{n} - K) = v(\widehat{a} - K) - v\mathfrak{n} = v(\widehat{a} - K)$. $\square$

The remainder of this section is devoted to showing that (almost) special elements arise naturally in the analysis of certain immediate d-algebraic extensions of valued differential fields. We first treat the case of asymptotic fields with small derivation, and then focus on the linearly newtonian H-asymptotic case.

We recall some notation: for an ordered abelian group Γ and $\alpha \in \Gamma_\infty$, $\beta \in \Gamma$, $\gamma \in \Gamma^>$ we mean by "$\alpha \geqslant \beta + o(\gamma)$" that $\alpha \geqslant \beta - (1/n)\gamma$ for all $n \geqslant 1$, while "$\alpha < \beta + o(\gamma)$" is its negation, that is, $\alpha < \beta - (1/n)\gamma$ for some $n \geqslant 1$; see [ADH, p. 312]. Here and later inequalities are in the sense of the ordered divisible hull $\mathbb{Q}\Gamma$ of the relevant Γ.

A source of special elements. We recall that a differential field F is said to be r-linearly surjective ($r \in \mathbb{N}$) if $A(F) = F$ for every nonzero $A \in F[\partial]$ of order at most r. *In this subsection K is an asymptotic field with small derivation, value group $\Gamma = v(K^\times) \neq \{0\}$, and differential residue field $\boldsymbol{k}$; we also let $r \in \mathbb{N}^{\geqslant 1}$.* Below we use the notion *neatly surjective* from [ADH, 5.6]: $A \in K[\partial]^{\neq}$ is neatly surjective iff for all $b \in K^\times$ there exists $a \in K^\times$ such that $A(a) = b$ and $v_A(va) = vb$. We often let $\widehat{f}$ be an element in an immediate asymptotic extension $\widehat{K}$ of K, but in the statement of the next lemma we take $\widehat{f} \in K$:

Lemma 1.6.2. *Assume $\boldsymbol{k}$ is r-linearly surjective, $A \in K[\partial]^{\neq}$ of order $\leqslant r$ is neatly surjective, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{f} \in K^{\times}$, and $v\big(A(\widehat{f})\big) \geqslant v(A\widehat{f}) + \gamma$. Then $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$ for some $f \in K$.*

Proof. Set $B := g^{-1}A\widehat{f}$, where we take $g \in K^{\times}$ such that $vg = v(A\widehat{f})$. Then $B \asymp 1$, B is still neatly surjective, and $B(1) = g^{-1}A(\widehat{f})$, $v\big(B(1)\big) \geqslant \gamma$. It suffices to find $y \in K$ such that $B(y) = 0$ and $v(y-1) \geqslant \gamma + o(\gamma)$, because then $f := \widehat{f}y$ has the desired property. If $B(1) = 0$, then $y = 1$ works, so assume $B(1) \neq 0$. By [ADH, 7.2.7] we have an immediate extension $\widehat{K}$ of K that is r-differential henselian. Then $\widehat{K}$ is asymptotic by [ADH, 9.4.2 and 9.4.5]. Set $R(Z) := \operatorname{Ri}(B) \in K\{Z\}$. Then the proof of [ADH, 7.5.1] applied to $\widehat{K}$ and B in the roles of K and A yields $z \prec 1$ in $\widehat{K}$ with $R(z) = 0$. Now $R(0) = B(1)$, hence by [ADH, 7.2.2] we can take such z with $v(z) \geqslant \beta + o(\beta)$ where $\beta := v\big(B(1)\big) \geqslant \gamma$. As in the proof of [ADH, 7.5.1] we next take $y \in \widehat{K}$ with $v(y-1) > 0$ and $y^{\dagger} = z$ to get $B(y) = 0$, and observe that then $v(y-1) \geqslant \beta + o(\beta)$, by [ADH, 9.2.10(iv)], hence $v(y-1) \geqslant \gamma + o(\gamma)$. It remains to note that $y \in K$ by [ADH, 7.5.7]. $\square$

By a remark following the proof of [ADH, 7.5.1] the assumption that $\boldsymbol{k}$ is r-linearly surjective in the lemma above can be replaced for $r \geqslant 2$ by the assumption that $\boldsymbol{k}$ is $(r-1)$-linearly surjective.

Next we establish a version of the above with $\widehat{f}$ in an immediate asymptotic extension of K. Recall that an asymptotic extension of K with the same value group as K has small derivation, by [ADH, 9.4.1].

Lemma 1.6.3. *Assume $\boldsymbol{k}$ is r-linearly surjective, $A \in K[\partial]^{\neq}$ of order $\leqslant r$ is neatly surjective, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{K}$ is an immediate asymptotic extension of K, $\widehat{f} \in \widehat{K}^{\times}$, and $v\big(A(\widehat{f})\big) \geqslant v(A\widehat{f}) + \gamma$. Then for some $f \in K$ we have*

$$A(f) \;=\; 0, \qquad v(\widehat{f} - f) \;\geqslant\; v(\widehat{f}) + \gamma + o(\gamma).$$

Proof. By extending $\widehat{K}$ we can arrange that $\widehat{K}$ is r-differential henselian, so A remains neatly surjective as an element of $\widehat{K}[\partial]$, by [ADH, 7.1.8]. Then by Lemma 1.6.2 with $\widehat{K}$ in the role of K we get $f \in \widehat{K}$ such that $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$. It remains to note that $f \in K$ by [ADH, 7.5.7]. $\square$

We actually need an inhomogeneous variant of the above:

Lemma 1.6.4. *Assume $\boldsymbol{k}$ is r-linearly surjective, $A \in K[\partial]^{\neq}$ of order $\leqslant r$ is neatly surjective, $b \in K$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $v(A) = o(\gamma)$, $v(b) \geqslant o(\gamma)$, $\widehat{K}$ is an immediate asymptotic extension of K, $\widehat{f} \in \widehat{K}$, $\widehat{f} \preccurlyeq 1$, and $v\big(A(\widehat{f}) - b\big) \geqslant \gamma + o(\gamma)$. Then*

$$A(f) \;=\; b, \qquad v(\widehat{f} - f) \;\geqslant\; (1/2)\gamma + o(\gamma)$$

for some $f \in K$.

Proof. Take $y \in K$ with $A(y) = b$ and $v(y) \geqslant o(\gamma)$. Then $A(\widehat{g}) = A(\widehat{f}) - b$ for $\widehat{g} := \widehat{f} - y$, so $v\big(A(\widehat{g})\big) \geqslant \gamma + o(\gamma)$ and $v(\widehat{g}) \geqslant o(\gamma)$. We distinguish two cases:

(1) $v(\widehat{g}) \geqslant (1/2)\gamma + o(\gamma)$. Then $v(\widehat{f} - y) \geqslant (1/2)\gamma + o(\gamma)$, so $f := y$ works.

(2) $v(\widehat{g}) < (1/2)\gamma + o(\gamma)$. Then by [ADH, 6.1.3],

$$v(A\widehat{g}) \;<\; (1/2)\gamma + o(\gamma), \qquad v\big(A(\widehat{g})\big) \;\geqslant\; \gamma + o(\gamma),$$

so $v\big(A(\widehat{g})\big) \geqslant v(A\widehat{g}) + (1/2)\gamma$. Then Lemma 1.6.3 gives an element $g \in K$ such that $A(g) = 0$ and $v(\widehat{g} - g) \geqslant (1/2)\gamma + o(\gamma)$. Hence $f := y + g$ works. $\square$

Recall from [ADH, 7.2] that $\mathcal{O}$ is said to be *r-linearly surjective* if for every A in $K[\partial]^{\neq}$ of order r with $v(A) = 0$ there exists $y \in \mathcal{O}$ with $A(y) = 1$.

Proposition 1.6.5. *Assume $\mathcal{O}$ is r-linearly surjective, $P \in K\{Y\}$, $\operatorname{order}(P) \leqslant r$, $\operatorname{ddeg} P = 1$, and $P(\widehat{a}) = 0$, where $\widehat{a} \preccurlyeq 1$ lies in an immediate asymptotic extension of K and $\widehat{a} \notin K$. Then $\widehat{a}$ is special over K.*

Proof. The hypothesis on $\mathcal{O}$ yields: $\boldsymbol{k}$ is r-linearly surjective and all $A \in K[\partial]^{\neq}$ of order $\leqslant r$ are neatly surjective. Let $0 < \gamma \in v(\widehat{a} - K)$; we claim that $v(\widehat{a} - K)$ has an element $\geqslant (4/3)\gamma$. We arrange $P \asymp 1$. Take $a \in K$ with $v(\widehat{a} - a) = \gamma$. Then $P_{+a} \asymp 1$, $\operatorname{ddeg} P_{+a} = 1$, so

$$P_{+a,1} \ \asymp\ 1, \quad P_{+a,>1} \ \prec\ 1, \quad P_{+a} \ =\ P(a) + P_{+a,1} + P_{+a,>1}$$

and

$$0 \ =\ P(\widehat{a}) \ =\ P_{+a}(\widehat{a} - a) \ =\ P(a) + P_{+a,1}(\widehat{a} - a) + P_{+a,>1}(\widehat{a} - a),$$

with

$$v\big(P_{+a,1}(\widehat{a} - a) + P_{+a,>1}(\widehat{a} - a)\big) \geqslant \gamma + o(\gamma),$$

and thus $v(P(a)) \geqslant \gamma + o(\gamma)$. Take $g \in K^{\times}$ with $vg = \gamma$ and set $Q := g^{-1}P_{+a,\times g}$, so $Q = Q_0 + Q_1 + Q_{>1}$ with

$$Q_0 \ =\ Q(0) \ =\ g^{-1}P(a), \quad Q_1 \ =\ g^{-1}(P_{+a,1})_{\times g}, \quad Q_{>1} \ =\ g^{-1}(P_{+a,>1})_{\times g},$$

hence

$$v(Q_0) \geqslant o(\gamma), \quad v(Q_1) = o(\gamma), \quad v(Q_{>1}) \geqslant \gamma + o(\gamma).$$

We set $\widehat{f} := g^{-1}(\widehat{a} - a)$, so $Q(\widehat{f}) = 0$ and $\widehat{f} \asymp 1$, and $A := L_Q \in K[\partial]$. Then $Q(\widehat{f}) = 0$ gives

$$Q_0 + A(\widehat{f}) \ =\ Q_0 + Q_1(\widehat{f}) \ =\ -Q_{>1}(\widehat{f}), \quad \text{with } v\big(Q_{>1}(\widehat{f})\big) \ \geqslant\ \gamma + o(\gamma),$$

so $v\big(Q_0 + A(\widehat{f})\big) \geqslant \gamma + o(\gamma)$. Since $v(A) = v(Q_1) = o(\gamma)$, Lemma 1.6.4 then gives $f \in K$ with $v(\widehat{f} - f) \geqslant (1/3)\gamma$. In view of $\widehat{a} - a = g\widehat{f}$, this yields

$$v\big(\widehat{a} - (a + gf)\big) \ =\ \gamma + v(\widehat{f} - f) \ \geqslant\ (4/3)\gamma,$$

which proves our claim. It gives the desired result. $\square$

A source of almost special elements. *In this subsection K, Γ, $\boldsymbol{k}$, and r are as in the previous subsection, and we assume that $\mathcal{O}$ is r-linearly surjective.* (So $\boldsymbol{k}$ is r-linearly surjective, and $\sup \Psi = 0$ by [ADH, 9.4.2].) Let $\widehat{a}$ be an element in an immediate asymptotic extension of K such that $\widehat{a} \notin K$ and $K\langle\widehat{a}\rangle$ has transcendence degree $\leqslant r$ over K. We shall use Proposition 1.6.5 to show:

Proposition 1.6.6. *If Γ is divisible, then $\widehat{a}$ is almost special over K.*

Towards the proof we first note that $\widehat{a}$ has a minimal annihilator $P(Y)$ over K of order $\leqslant r$. We also fix a divergent pc-sequence (a_ρ) in K such that $a_\rho \rightsquigarrow \widehat{a}$. We next show how to improve $\widehat{a}$ and P (without assuming divisibility of Γ):

Lemma 1.6.7. *For some $\widehat{b}$ in an immediate asymptotic extension of K we have:*

(i) $v(\widehat{a} - K) = v(\widehat{b} - K)$;

(ii) (a_ρ) *has a minimal differential polynomial* Q *over* K *of order* $\leqslant r$ *such that* Q *is also a minimal annihilator of* $\widehat{b}$ *over* K.

Proof. By [ADH, 6.8.1, 6.9.2], (a_ρ) is of d-algebraic type over K with a minimal differential polynomial $Q(Y)$ over K such that $\operatorname{order} Q \leqslant \operatorname{order} P \leqslant r$. By [ADH, 6.9.3, 9.4.5] this gives an element $\widehat{b}$ in an immediate asymptotic extension of K such that Q is a minimal annihilator of $\widehat{b}$ over K and $a_\rho \rightsquigarrow \widehat{b}$. Then Q and $\widehat{b}$ have the desired properties. $\square$

Proof of Proposition 1.6.6. Replace $\widehat{a}$ and P by $\widehat{b}$ and Q from Lemma 1.6.7 (and rename) to arrange that P is a minimal differential polynomial of (a_ρ) over K. Now assuming Γ is divisible, [160, Proposition 3.1] gives $a \in K$ and $g \in K^\times$ such that $\widehat{a} - a \asymp g$ and $\operatorname{ddeg} P_{+a,\times g} = 1$.

Set $F := P_{+a,\times g}$ and $\widehat{f} := (\widehat{a} - a)/g$. Then $\operatorname{ddeg} F = 1$, $F(\widehat{f}) = 0$, and $\widehat{f} \preccurlyeq 1$. Applying Proposition 1.6.5 to F and $\widehat{f}$ in the role of P and $\widehat{a}$ yields a nontrivial convex subgroup Δ of Γ that is cofinal in $v(\widehat{f} - K)$. Setting $\alpha := vg$, it follows that $\alpha + \Delta$ is cofinal in $v\big((\widehat{a} - a) - K\big) = v(\widehat{a} - K)$. $\square$

We can trade the divisibility assumption in Proposition 1.6.6 against a stronger hypothesis on K, the proof using [160, 3.3] instead of [160, 3.1]:

Corollary 1.6.8. *If* K *is henselian and* $\boldsymbol{k}$ *is linearly surjective, then* $\widehat{a}$ *is almost special over* K.

The linearly newtonian setting. *In this subsection* K *is an* ω*-free* r*-linearly newtonian* H*-asymptotic field,* $r \geqslant 1$. Thus K is d-valued by Lemma 1.2.9. We let ϕ range over the elements active in K. We now mimick the material in the previous two subsections. Note that for $A \in K[\partial]^{\neq}$ and any element $\widehat{f}$ in an asymptotic extension of K we have $A(\widehat{f}) \preccurlyeq A^\phi \widehat{f}$, since $A(\widehat{f}) = A^\phi(\widehat{f})$.

Lemma 1.6.9. *Assume that* $A \in K[\partial]^{\neq}$ *has order* $\leqslant r$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{f} \in K^\times$, *and* $v\big(A(\widehat{f})\big) \geqslant v(A^\phi \widehat{f}) + \gamma$, *eventually. Then there exists an* $f \in K$ *such that* $A(f) = 0$ *and* $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$.

Proof. Take ϕ such that $v\phi \geqslant \gamma^\dagger$ and $v\big(A(\widehat{f})\big) \geqslant v(A^\phi \widehat{f}) + \gamma$. Next, take $\beta \in \Gamma$ such that $\beta \geqslant \gamma$ and $v\big(A(\widehat{f})\big) \geqslant v(A^\phi \widehat{f}) + \beta$. Then $v\phi \geqslant \beta^\dagger$, so $\beta > \Gamma^\flat_\phi$, hence the valuation ring of the flattening $(K^\phi, v^\flat_\phi)$ is r-linearly surjective, by [ADH, 14.2.1]. We now apply Lemma 1.6.2 to

$$(K^\phi, v^\flat_\phi), \quad A^\phi, \quad \dot\beta := \beta + \Gamma^\flat_\phi$$

in the role of K, A, γ to give $f \in K$ with $A(f) = 0$ and $v^\flat_\phi(\widehat{f} - f) \geqslant v^\flat_\phi(\widehat{f}) + \dot\beta + o(\dot\beta)$. Then also $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \beta + o(\beta)$, and thus $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$. $\square$

Lemma 1.6.10. *Assume* $A \in K[\partial]^{\neq}$ *has order* $\leqslant r$, $\widehat{K}$ *is an immediate* d*-algebraic asymptotic extension of* K, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{f} \in \widehat{K}^\times$, *and* $v\big(A(\widehat{f})\big) \geqslant v(A^\phi \widehat{f}) + \gamma$ *eventually. Then* $A(f) = 0$ *and* $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$ *for some* $f \in K$.

Proof. Since K is ω-free, so is $\widehat{K}$ by Theorem 1.4.1. By [ADH, 14.0.1 and subsequent remarks] we can extend $\widehat{K}$ to arrange that $\widehat{K}$ is also newtonian. Then by Lemma 1.6.9 with $\widehat{K}$ in the role of K we get $f \in \widehat{K}$ with $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$. Now use that $f \in K$ by [ADH, line before 14.2.10]. $\square$

Lemma 1.6.11. *Assume $A \in K[\partial]^{\neq}$ has order $\leqslant r$, $b \in K$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{K}$ is an immediate* d*-algebraic asymptotic extension of K, and $\widehat{f} \in \widehat{K}$, $v(\widehat{f}) \geqslant o(\gamma)$. Assume also that eventually $v(b) \geqslant v(A^{\phi})+o(\gamma)$ and $v\big(A(\widehat{f})-b\big) \geqslant v(A^{\phi})+\gamma+o(\gamma)$. Then for some $f \in K$ we have $A(f) = b$ and $v(\widehat{f}-f) \geqslant (1/2)\gamma + o(\gamma)$.*

Proof. We take $y \in K$ with $A(y) = b$ as follows: If $b = 0$, then $y := 0$. If $b \neq 0$, then Corollary 1.5.7 yields $y \in K^{\times}$ such that $A(y) = b$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v^{\mathrm{e}}_A(vy) = vb$. In any case, $vy \geqslant o(\gamma)$: when $b \neq 0$, the sentence preceding [ADH, 14.2.7] gives $v_{A^{\phi}}(vy) = vb$, eventually, to which we apply [ADH, 6.1.3].

Now $A(\widehat{g}) = A(\widehat{f}) - b$ for $\widehat{g} := \widehat{f} - y$, so $v(\widehat{g}) \geqslant o(\gamma)$, and eventually $v\big(A(\widehat{g})\big) \geqslant v(A^{\phi}) + \gamma + o(\gamma)$. We distinguish two cases:

(1) $v(\widehat{g}) \geqslant (1/2)\gamma + o(\gamma)$. Then $v(\widehat{f}-y) \geqslant (1/2)\gamma + o(\gamma)$, so $f := y$ works.

(2) $v(\widehat{g}) < (1/2)\gamma + o(\gamma)$. Then by [ADH, 6.1.3] we have eventually

$$v(A^{\phi}\widehat{g}) \ < \ v(A^{\phi}) + (1/2)\gamma + o(\gamma), \qquad v\big(A(\widehat{g})\big) \ \geqslant \ v(A^{\phi}) + \gamma + o(\gamma),$$

so $v(A(\widehat{g})) \geqslant v(A^{\phi}\widehat{g}) + (1/2)\gamma$, eventually. Lemma 1.6.10 gives an element $g \in K$ with $A(g) = 0$ and $v(\widehat{g}-g) \geqslant (1/2)\gamma + o(\gamma)$. Hence $f := y + g$ works. □

Proposition 1.6.12. *Suppose that $P \in K\{Y\}$,* $\operatorname{order} P \leqslant r$, $\operatorname{ndeg} P = 1$*, and $P(\widehat{a}) = 0$, where $\widehat{a} \preccurlyeq 1$ lies in an immediate asymptotic extension of K and $\widehat{a} \notin K$. Then $\widehat{a}$ is special over K.*

The proof is like that of Proposition 1.6.5, but there are some differences that call for further details.

Proof. Given $0 < \gamma \in v(\widehat{a} - K)$, we claim that $v(\widehat{a} - K)$ has an element $\geqslant (4/3)\gamma$. Take $a \in K$ with $v(\widehat{a} - a) = \gamma$. Then $\operatorname{ndeg} P_{+a} = 1$ by [ADH, 11.2.3(i)], so eventually we have

$$P(a) \ \preccurlyeq \ P^{\phi}_{+a,1} \ \succ \ P^{\phi}_{+a,>1}, \quad P^{\phi}_{+a} \ = \ P(a) + P^{\phi}_{+a,1} + P^{\phi}_{+a,>1}$$

and

$$\begin{aligned} 0 \ = \ P(\widehat{a}) \ &= \ P^{\phi}_{+a}(\widehat{a}-a) \\ &= \ P(a) + P^{\phi}_{+a,1}(\widehat{a}-a) + P^{\phi}_{+a,>1}(\widehat{a}-a), \\ &\qquad v\big(P^{\phi}_{+a,1}(\widehat{a}-a) + P^{\phi}_{+a,>1}(\widehat{a}-a)\big) \ \geqslant \ v(P^{\phi}_{+a,1}) + \gamma + o(\gamma), \end{aligned}$$

and thus eventually $v\big(P(a)\big) \ \geqslant \ v(P^{\phi}_{+a,1}) + \gamma + o(\gamma)$. Take $g \in K^{\times}$ with $vg = \gamma$ and set $Q := g^{-1}P_{+a,\times g}$, so $Q = Q_0 + Q_1 + Q_{>1}$ with

$$Q_0 \ = \ Q(0) \ = \ g^{-1}P(a), \quad Q_1 \ = \ g^{-1}(P_{+a,1})_{\times g}, \quad Q_{>1} \ = \ g^{-1}(P_{+a,>1})_{\times g}.$$

Then $v(Q_0) = v\big(P(a)\big) - \gamma \geqslant v(P^{\phi}_{+a,1}) + o(\gamma)$, eventually. By [ADH, 6.1.3],

$$v(Q^{\phi}_1) \ = \ v(P^{\phi}_{+a,1}) + o(\gamma), \qquad v(Q^{\phi}_{>1}) \ \geqslant \ v(P^{\phi}_{+a,>1}) + \gamma + o(\gamma)$$

for all ϕ. Since $P^{\phi}_{+a,>1} \preccurlyeq P^{\phi}_{+a,1}$, eventually, the last two displayed inequalities give $v(Q^{\phi}_{>1}) \geqslant v(Q^{\phi}_1) + \gamma + o(\gamma)$, eventually. We set $\widehat{f} := g^{-1}(\widehat{a}-a)$, so $Q(\widehat{f}) = 0$ and $\widehat{f} \asymp 1$. Set $A := L_Q \in K[\partial]$. Then $Q(\widehat{f}) = 0$ gives

$$Q_0 + A(\widehat{f}) \ = \ Q_0 + Q_1(\widehat{f}) \ = \ -Q^{\phi}_{>1}(\widehat{f}),$$

with $v\big(Q^{\phi}_{>1}(\widehat{f})\big) \geqslant v(Q^{\phi}_1) + \gamma + o(\gamma)$, eventually, so

$$v\big(Q_0 + A(\widehat{f})\big) \ \geqslant\ v(A^{\phi}) + \gamma + o(\gamma), \quad \text{eventually.}$$

Moreover, $v(Q_0) \geqslant v(A^{\phi}) + o(\gamma)$, eventually. Lemma 1.6.11 then gives $f \in K$ with $v(\widehat{f} - f) \geqslant (1/3)\gamma$. In view of $\widehat{a} - a = g\widehat{f}$, this yields

$$v\big(\widehat{a} - (a + gf)\big) \ =\ \gamma + v(\widehat{f} - f) \ \geqslant\ (4/3)\gamma,$$

which proves our claim. □

In the rest of this subsection we assume that $\widehat{a} \notin K$ lies in an immediate asymptotic extension of K and $K\langle\widehat{a}\rangle$ has transcendence degree $\leqslant r$ over K.

Proposition 1.6.13. *If Γ is divisible, then $\widehat{a}$ is almost special over K.*

Towards the proof, we fix a minimal annihilator $P(Y)$ of $\widehat{a}$ over K, so $\operatorname{order} P \leqslant r$. We also fix a divergent pc-sequence (a_ρ) in K such that $a_\rho \rightsquigarrow \widehat{a}$. We next show how to improve $\widehat{a}$ and P if necessary:

Lemma 1.6.14. *For some $\widehat{b}$ in an immediate asymptotic extension of K we have:*

(i) *$v(\widehat{a} - a) = v(\widehat{b} - a)$ for all $a \in K$;*

(ii) *(a_ρ) has a minimal differential polynomial Q over K of order $\leqslant r$ such that Q is also a minimal annihilator of $\widehat{b}$ over K.*

Proof. By the remarks following the proof of [ADH, 11.4.3] we have $P \in Z(K, \widehat{a})$. Take $Q \in Z(K, \widehat{a})$ of minimal complexity. Then $\operatorname{order} Q \leqslant \operatorname{order} P \leqslant r$, and Q is a minimal differential polynomial of (a_ρ) over K by [ADH, 11.4.13]. By [ADH, 11.4.8 and its proof] this gives an element $\widehat{b}$ in an immediate asymptotic extension of K such that (i) holds and Q is a minimal annihilator of $\widehat{b}$ over K. Then Q and $\widehat{b}$ have the desired properties. □

Proof of Proposition 1.6.13. Assume Γ is divisible. Replace $\widehat{a}$, P by $\widehat{b}$, Q from Lemma 1.6.14 and rename to arrange that P is a minimal differential polynomial of (a_ρ) over K. By [ADH, 14.5.1] we have $a \in K$ and $g \in K^{\times}$ such that $\widehat{a} - a \asymp g$ and $\operatorname{ndeg} P_{+a,\times g} = 1$. Set $F := P_{+a,\times g}$ and $\widehat{f} := (\widehat{a} - a)/g$. Then $\operatorname{ndeg} F = 1$, $F(\widehat{f}) = 0$, and $\widehat{f} \preccurlyeq 1$. Applying Proposition 1.6.12 to F and $\widehat{f}$ in the role of P and $\widehat{a}$ yields a nontrivial convex subgroup Δ of Γ that is cofinal in $v(\widehat{f} - K)$. Setting $\alpha := vg$, it follows that $\alpha + \Delta$ is cofinal in $v\big((\widehat{a} - a) - K\big) = v(\widehat{a} - K)$. □

Corollary 1.6.15. *If K is henselian, then $\widehat{a}$ is almost special over K.*

The proof is like that of Proposition 1.6.13, using [159, 3.3] instead of [ADH, 14.5.1].

The case of order 1. We show here that Proposition 1.6.12 goes through in the case of order 1 under weaker assumptions: *in this subsection K is a 1-linearly newtonian H-asymptotic field with asymptotic integration.* Then K is d-valued with $\mathrm{I}(K) \subseteq K^{\dagger}$, by Lemma 1.2.9, and λ-free, by [ADH, 14.2.3]. We let ϕ range over elements active in K. In the next two lemmas $A \in K[\partial]^{\neq}$ has order $\leqslant 1$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, and $\widehat{K}$ is an immediate asymptotic extension of K.

Lemma 1.6.16. *Let $\widehat{f} \in \widehat{K}^{\times}$ be such that $v\big(A(\widehat{f})\big) \geqslant v(A^{\phi}\widehat{f}) + \gamma$ eventually. Then there exists $f \in K$ such that $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma$.*

Proof. Note that $\operatorname{order}(A) = 1$; we arrange $A = \partial - g$ ($g \in K$). If $A(\widehat{f}) = 0$, then $\widehat{f}$ is in K [ADH, line before 14.2.10], and $f := \widehat{f}$ works. Assume $A(\widehat{f}) \neq 0$. Then

$$v\big(A^{\phi}(\widehat{f})\big) = v\big(A(\widehat{f})\big) \geqslant v(A^{\phi}\widehat{f}) + \gamma > v(A^{\phi}\widehat{f}), \quad \text{eventually,}$$

so $v(\widehat{f}) \in \mathscr{E}^{\mathrm{e}}(A)$, and Lemma 1.5.9 yields an $f \in K$ with $f \sim \widehat{f}$ and $A(f) = 0$. We claim that this f has the desired property. Set $b := A(\widehat{f})$. By the remarks preceding Corollary 1.5.15 we can replace K, $\widehat{K}$, A, b by K^{ϕ}, $\widehat{K}^{\phi}$, $\phi^{-1}A^{\phi}$, $\phi^{-1}b$, respectively, for suitable ϕ, to arrange that K has small derivation and $b^{\dagger} - g \succ^{\flat} 1$. Using the hypothesis of the lemma we also arrange $vb \geqslant v(A\widehat{f}) + \gamma$. It remains to show that for $\widehat{g} := \widehat{f} - f \neq 0$ we have $v(\widehat{g}) \geqslant v(\widehat{f}) + \gamma$. Now $A(\widehat{g}) = b$ with $v(\widehat{g}) \notin \mathscr{E}^{\mathrm{e}}(A)$, hence $\widehat{g} \sim b/(b^{\dagger} - g) \prec^{\flat} b$ by Lemma 1.5.14, and thus $v(\widehat{g}) > vb \geqslant v(A\widehat{f}) + \gamma$, so it is enough to show $v(A\widehat{f}) \geqslant v(\widehat{f})$. Now $b = A(\widehat{f}) = \widehat{f}(\widehat{f}^{\dagger} - g)$ and $A\widehat{f} = \widehat{f}\big(\partial + \widehat{f}^{\dagger} - g\big)$. As $vb \geqslant v(A\widehat{f}) + \gamma > v(A\widehat{f})$, this yields $v(\widehat{f}^{\dagger} - g) > 0$, so $v(A\widehat{f}) = v(\widehat{f})$. □

Lemma 1.6.17. *Let $b \in K$ and $\widehat{f} \in \widehat{K}$ with $v(\widehat{f}) \geqslant o(\gamma)$. Assume also that eventually $v(b) \geqslant v(A^{\phi}) + o(\gamma)$ and $v\big(A(\widehat{f}) - b\big) \geqslant v(A^{\phi}) + \gamma + o(\gamma)$. Then for some $f \in K$ we have $A(f) = b$ and $v(\widehat{f} - f) \geqslant (1/2)\gamma + o(\gamma)$.*

The proof is like that of Lemma 1.6.11, using Lemma 1.6.16 instead of Lemma 1.6.10. In the same way Lemma 1.6.11 gave Proposition 1.6.12, Lemma 1.6.17 now yields:

Proposition 1.6.18. *If $P \in K\{Y\}$, $\operatorname{order} P \leqslant 1$, $\operatorname{ndeg} P = 1$, and $P(\widehat{a}) = 0$, where $\widehat{a} \preccurlyeq 1$ lies in an immediate asymptotic extension of K and $\widehat{a} \notin K$, then $\widehat{a}$ is special over K.*

Remark. Proposition 1.6.13 does not hold for $r = 1$ under present assumptions. To see this, let K be a Liouville closed H-field which is not ω-free, as in Example 1.4.16 or [9]. Then K is 1-linearly newtonian by Corollary 1.8.29 below. Consider the pc-sequences (λ_{ρ}) and (ω_{ρ}) in K as in [ADH, 11.7], let $\omega \in K$ with $\omega_{\rho} \rightsquigarrow \omega$, and $P = 2Y' + Y^2 + \omega$. Then [ADH, 11.7.13] gives an element λ in an immediate asymptotic extension of K but not in K with $\lambda_{\rho} \rightsquigarrow \lambda$ and $P(\lambda) = 0$. However, λ is not almost special over K [ADH, 3.4.13, 11.5.2].

Relating $Z(K, \widehat{a})$ and $v(\widehat{a} - K)$ for special $\widehat{a}$. *In this subsection K is a valued differential field with small derivation $\partial \neq 0$ such that $\Gamma \neq \{0\}$ and $\Gamma^{>}$ has no least element.* We recall from [11] that a valued differential field extension L of K is said to be *strict* if for all $\phi \in K^{\times}$,

$$\partial\mathcal{o} \subseteq \phi\mathcal{o} \;\Rightarrow\; \partial\mathcal{o}_L \subseteq \phi\mathcal{o}_L, \qquad \partial\mathcal{O} \subseteq \phi\mathcal{o} \;\Rightarrow\; \partial\mathcal{O}_L \subseteq \phi\mathcal{o}_L.$$

(If K is asymptotic, then any immediate asymptotic extension of K is automatically strict, by [11, 1.11].) Let $\widehat{a}$ lie in an immediate strict extension of K such that $\widehat{a} \preccurlyeq 1$, $\widehat{a} \notin K$, and $\widehat{a}$ is special over K. We adopt from [11, Sections 2, 4] the definitions of $\operatorname{ndeg} P$ for $P \in K\{Y\}^{\neq}$ and of the set $Z(K, \widehat{a}) \subseteq K\{Y\}^{\neq}$. Also recall that $\Gamma(\partial) := \{v\phi : \phi \in K^{\times}, \partial\mathcal{o} \subseteq \phi\mathcal{o}\}$.

Lemma 1.6.19. *Let $P \in Z(K, \widehat{a})$ and $P \asymp 1$. Then $v\big(P(\widehat{a})\big) > v(\widehat{a} - K)$.*

Proof. Take a divergent pc-sequence (a_{ρ}) in $\mathcal{O}$ with $a_{\rho} \rightsquigarrow \widehat{a}$, and as in [ADH, 11.2] let $\boldsymbol{a} := c_K(a_{\rho})$. Then $\operatorname{ndeg}_{\boldsymbol{a}} P \geqslant 1$ by [11, 4.7]. We arrange $\gamma_{\rho} := v(\widehat{a} - a_{\rho})$ to be strictly increasing as a function of ρ, with $0 < 2\gamma_{\rho} < \gamma_{s(\rho)}$ for all ρ. Take $g_{\rho} \in \mathcal{o}$

with $g_\rho \asymp \widehat{a}-a_\rho$; then $1 \leqslant d := \operatorname{ndeg}_{\boldsymbol{a}} P = \operatorname{ndeg} P_{+a_\rho,\times g_\rho}$ for all sufficiently large ρ, and we arrange that this holds for all ρ. We have $\widehat{a} = a_\rho + g_\rho y_\rho$ with $y_\rho \asymp 1$, and

$$P(\widehat{a}) \ = \ P_{+a_\rho,\times g_\rho}(y_\rho) \ = \ \sum_i (P_{+a_\rho,\times g_\rho})_i(y_\rho).$$

Pick for every ρ an element $\phi_\rho \in K^\times$ such that $0 \leqslant v(\phi_\rho) \in \Gamma(\partial)$ and $(P^{\phi_\rho}_{+a_\rho,\times g_\rho})_i \preccurlyeq (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_d$ for all i. Then for all ρ and i,

$$\begin{aligned} &(P_{+a_\rho,\times g_\rho})_i(y_\rho) \ = (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_i(y_\rho) \ \preccurlyeq \ (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_i \ \preccurlyeq \ (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_d \ \text{ with} \\ &v\big((P^{\phi_\rho}_{+a_\rho,\times g_\rho})_d\big) \ \geqslant \ d\gamma_\rho + o(\gamma_\rho) \ \geqslant \ \gamma_\rho + o(\gamma_\rho), \end{aligned}$$

where for the next to last inequality we use [ADH, 11.1.1, 5.7.1, 5.7.5, 6.1.3]. Hence $v\big(P(\widehat{a})\big) \geqslant \gamma_\rho + o(\gamma_\rho)$ for all ρ, and thus $v\big(P(\widehat{a})\big) > v(\widehat{a}-K)$. $\square$

We also have a converse under extra assumptions:

Lemma 1.6.20. *Assume K is asymptotic and $\Psi \subseteq v(\widehat{a}-K)$. Let $P \in K\{Y\}$ be such that $P \asymp 1$ and $v\big(P(\widehat{a})\big) > v(\widehat{a}-K)$. Then $P \in Z(K,\widehat{a})$.*

Proof. Let Δ be the nontrivial convex subgroup of Γ that is cofinal in $v(\widehat{a}-K)$. Let $\kappa := \operatorname{cf}(\Delta)$. Take a divergent pc-sequence $(a_\rho)_{\rho<\kappa}$ in K such that $a_\rho \rightsquigarrow \widehat{a}$. We arrange $\gamma_\rho := v(\widehat{a}-a_\rho)$ is strictly increasing as a function of ρ, with $\gamma_\rho > 0$ for all ρ; thus $a_\rho \preccurlyeq 1$ for all ρ. Consider the Δ-coarsening $\dot{v} = v_\Delta$ of the valuation v of K; it has valuation ring $\dot{\mathcal{O}}$ with differential residue field $\dot{K}$. Consider likewise the Δ-coarsening of the valuation of the immediate extension $L = K\langle\widehat{a}\rangle$ of K. Let a^* be the image of $\widehat{a}$ in the differential residue field $\dot{L}$ of $(L,\dot{v})$. Note that $\dot{L}$ is an immediate extension of $\dot{K}$. The pc-sequence (a_ρ) then yields a sequence $(\dot{a}_\rho)$ in $\dot{K}$ with $v(a^* - \dot{a}_\rho) = \gamma_\rho$ for all ρ. Thus $(\dot{a}_\rho)$ is a c-sequence in $\dot{K}$ with $\dot{a}_\rho \to a^*$, so $\dot{P}(\dot{a}_\rho) \to \dot{P}(a^*)$ by [ADH, 4.4.5]. From $v\big(P(\widehat{a})\big) > \Delta$ we obtain $\dot{P}(a^*) = 0$, and so $\dot{P}(\dot{a}_\rho) \to 0$. So far we have not used our assumption that K is asymptotic and $\Psi \subseteq v(\widehat{a}-K)$. Using this now, we note that for $\alpha \in \Delta^>$ we have $0 < \alpha' = \alpha + \alpha^\dagger$, so $\alpha' \in \Delta$, hence the derivation of $\dot{K}$ is nontrivial. Thus we can apply [ADH, 4.4.10] to $\dot{K}$ and modify the a_ρ without changing $\gamma_\rho = v(a^* - \dot{a}_\rho)$ to arrange that in addition $\dot{P}(\dot{a}_\rho) \neq 0$ for all ρ. Since $\kappa = \operatorname{cf}(\Delta)$, we can replace (a_ρ) by a cofinal subsequence so that $P(a_\rho) \rightsquigarrow 0$, hence $P \in Z(K,\widehat{a})$ by [11, 4.6]. $\square$

To elaborate on this, let Δ be a convex subgroup of Γ and $\dot{K}$ the valued differential residue field of the Δ-coarsening v_Δ of the valuation v of K. We view $\dot{K}$ in the usual way as a valued differential subfield of the valued differential residue field $\dot{\widehat{K}}$ of the Δ-coarsening of the valuation of $\widehat{K}$ by Δ; see [ADH, pp. 159–160 and 4.4.4].

Corollary 1.6.21. *Suppose K is asymptotic, $\Psi \subseteq v(\widehat{a}-K)$, and Δ is cofinal in $v(\widehat{a}-K)$. Let $P \in K\{Y\}$ with $P \asymp 1$. Then $P \in Z(K,\widehat{a})$ if and only if $\dot{P}(\dot{\widehat{a}}) = 0$ in $\dot{\widehat{K}}$. Also, P is an element of $Z(K,\widehat{a})$ of minimal complexity if and only if $\dot{P}$ is a minimal annihilator of $\dot{\widehat{a}}$ over $\dot{K}$ and $\dot{P}$ has the same complexity as P.*

Proof. The first statement is immediate from Lemmas 1.6.19 and 1.6.20. For the rest use that for $R \in \dot{\mathcal{O}}\{Y\}$ we have $\operatorname{c}(\dot{R}) \leqslant \operatorname{c}(R)$ and that for all $Q \in \dot{K}\{Y\}$ there is an $R \in \dot{\mathcal{O}}\{Y\}$ with $Q = \dot{R}$ and $Q_{\boldsymbol{i}} \neq 0$ iff $R_{\boldsymbol{i}} \neq 0$ for all $\boldsymbol{i}$, so $\operatorname{c}(\dot{R}) = \operatorname{c}(R)$. $\square$

1.7. Differential Henselianity of the Completion

Let K be a valued differential field with small derivation. We let $\Gamma := v(K^\times)$ be the value group of K and $\boldsymbol{k} := \operatorname{res}(K)$ be the differential residue field of K, and we let $r \in \mathbb{N}$. The following summarizes [ADH, 7.1.1, 7.2.1]:

Lemma 1.7.1. *The valued differential field K is r-*d*-henselian iff for each P in $K\{Y\}$ of order $\leqslant r$ with* $\operatorname{ddeg} P = 1$ *there is a $y \in \mathcal{O}$ with $P(y) = 0$.*

Recall that the derivation of K being small, it is continuous [ADH, 4.4.6], and hence extends uniquely to a continuous derivation on the completion K^{c} of the valued field K [ADH, 4.4.11]. We equip K^{c} with this derivation, which remains small [ADH, 4.4.12], so K^{c} is an immediate valued differential field extension of K with small derivation, in particular, $\boldsymbol{k} = \operatorname{res}(K^{\mathrm{c}})$.

Below we characterize in a first-order way when K^{c} is r-d-henselian. We shall use tacitly that for $P \in K\{Y\}$ we have $P(g) \preccurlyeq P_{\times g}$ for all $g \in K$; to see this, replace P by $P_{\times g}$ to reduce to $g = 1$, and observe that $P(1) = \sum_{\|\boldsymbol{\sigma}\|=0} P_{[\boldsymbol{\sigma}]} \preccurlyeq P$.

Lemma 1.7.2. *Let $P \in K^{\mathrm{c}}\{Y\}$, $b \in K^{\mathrm{c}}$ with $b \preccurlyeq 1$ and $P(b) = 0$, and $\gamma \in \Gamma^{>}$. Then there is an $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$.*

Proof. To find an a as claimed we take $f \in K$ satisfying $f \asymp P$ and replace P, γ by $f^{-1}P$, $\gamma - vf$, respectively, to arrange $P \asymp 1$ and thus $P_{+b} \asymp 1$. We also assume $b \neq 0$. Since K is dense in K^{c} we can take $a \in K$ such that $a \sim b$ (so $a \in \mathcal{O}$) and $v(a-b) > 2\gamma$. Then with $g := a-b$, using [ADH, 4.5.1(i) and 6.1.4] we conclude

$$v\big(P(a)\big) = v\big(P_{+b}(g)\big) \geqslant v\big((P_{+b})_{\times g}\big) \geqslant v(P_{+b}) + vg + o(vg) = vg + o(vg) > \gamma$$

as required. □

Recall that if K is asymptotic, then so is K^{c} by [ADH, 9.1.6].

Lemma 1.7.3. *Suppose K is asymptotic, $\Gamma \neq \{0\}$, and for every $P \in K\{Y\}$ of order at most r with* $\operatorname{ddeg} P = 1$ *and every $\gamma \in \Gamma^{>}$ there is an $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$. Then K^{c} is r-*d*-henselian.*

Proof. The hypothesis applied to $P \in \mathcal{O}\{Y\}$ of order $\leqslant r$ with $\operatorname{ddeg} P = \deg P = 1$ yields that $\boldsymbol{k}$ is r-linearly surjective. Let now $P \in K^{\mathrm{c}}\{Y\}$ be of order $\leqslant r$ with $\operatorname{ddeg} P = 1$. We need to show that there exists $b \in K^{\mathrm{c}}$ with $b \preccurlyeq 1$ and $P(b) = 0$. First we arrange $P \asymp 1$. By [ADH, remarks after 9.4.11] we can take $b \preccurlyeq 1$ in an immediate d-henselian asymptotic field extension L of K^{c} with $P(b) = 0$. We prove below that $b \in K^{\mathrm{c}}$. We may assume $b \notin K$, so $v(b-K)$ has no largest element, since $L \supseteq K$ is immediate. Note also that $\operatorname{ddeg} P_{+b} = 1$ by [ADH, 6.6.5(i)]; since $P(b) = 0$ we thus have $\operatorname{ddeg} P_{+b,\times g} = 1$ for all $g \preccurlyeq 1$ in $L^\times$ by [ADH, 6.6.7].

Claim: *Let $\gamma \in \Gamma^{>}$ and $a \in K$ with $v(b-a) \geqslant 0$. There is a $y \in \mathcal{O}$ such that $v\big(P(y)\big) > \gamma$ and $v(b-y) \geqslant v(b-a)$.*

To prove this claim, take $g \in K^\times$ with $vg = v(b-a)$. Then by [ADH, 6.6.6] and the observation preceding the claim we have $\operatorname{ddeg} P_{+a,\times g} = \operatorname{ddeg} P_{+b,\times g} = 1$. Thanks to density of K in K^{c} we may take $Q \in K\{Y\}$ of order $\leqslant r$ with $P_{+a,\times g} \sim Q$ and $v(P_{+a,\times g} - Q) > \gamma$. Then $\operatorname{ddeg} Q = 1$, so by the hypothesis of the lemma we have $z \in \mathcal{O}$ with $v\big(Q(z)\big) > \gamma$. Set $y := a+gz \in \mathcal{O}$; then $v\big(P(y)\big) = v\big(P_{+a,\times g}(z)\big) > \gamma$ and $v(b-y) = v(b-a-gz) \geqslant v(b-a) = vg$ as claimed.

Let now $\gamma \in \Gamma^{>}$; to show that $b \in K^{\mathrm{c}}$, it is enough by [ADH, 3.2.15, 3.2.16] to show that then $v(a-b) > \gamma$ for some $a \in K$. Let $A := L_{P_{+b}} \in L[\partial]$; then $A \asymp 1$. Since $|\mathscr{E}_L(A)| \leqslant r$ by [ADH, 7.5.3], the claim gives an $a \in \mathcal{O}$ with $v\big(P(a)\big) > 2\gamma$ and $0 < v(b-a) \notin \mathscr{E}_L(A)$. Put $g := a-b$ and $R := (P_{+b})_{>1}$. Then $R \prec 1$ and

$$P(a) \ = \ P_{+b}(g) \ = \ A(g) + R(g)$$

where by the definition of $\mathscr{E}_L(A)$ and [ADH, 6.4.1(iii), 6.4.3] we have in $\mathbb{Q}\Gamma$:

$$v\big(A(g)\big) \ = \ v_A(vg) \ = \ vg + o(vg) \ < \ vR + (3/2)vg \ \leqslant \ v_R(vg) \ \leqslant \ v\big(R(g)\big)$$

and so $v\big(P(a)\big) = vg + o(vg) > 2\gamma$. Therefore $v(a-b) = vg > \gamma$ as required. $\square$

The last two lemmas yield an analogue of [ADH, 3.3.7] for r-d-henselianity and a partial generalization of [ADH, 7.2.15]:

Corollary 1.7.4. *Suppose K is asymptotic and $\Gamma \neq \{0\}$. Then the following are equivalent:*

(i) *K^{c} is r-d-henselian;*
(ii) *for every $P \in K\{Y\}$ of order at most r with $\operatorname{ddeg} P = 1$ and every $\gamma \in \Gamma^{>}$ there exists $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$.*

In particular, if K is r-d-henselian, then so is K^{c}.

1.8. Complements on Newtonianity

In this section K is an ungrounded H-asymptotic field with $\Gamma = v(K^{\times}) \neq \{0\}$. Note that then K^{c} is also H-asymptotic. We let r range over $\mathbb{N}$ and ϕ over the active elements of K. Our first aim is a newtonian analogue of Corollary 1.7.4:

Proposition 1.8.1. *For d-valued and ω-free K, the following are equivalent:*

(i) *K^{c} is r-newtonian;*
(ii) *for every $P \in K\{Y\}$ of order at most r with $\operatorname{ndeg} P = 1$ and every $\gamma \in \Gamma^{>}$ there is an $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$.*

If K is d-valued, ω-free, and r-newtonian, then so is K^{c}.

The final statement in this proposition extends [ADH, 14.1.5]. Towards the proof we first state a variant of [ADH, 13.2.2] which follows easily from [ADH, 11.1.4]:

Lemma 1.8.2. *Assume K has small derivation and let $P, Q \in K\{Y\}^{\neq}$ and $\phi \preccurlyeq 1$. Then $P^{\phi} \asymp^{\flat} P$, and so we have the three implications*

$$P \preccurlyeq^{\flat} Q \implies P^{\phi} \preccurlyeq^{\flat} Q^{\phi}, \quad P \prec^{\flat} Q \implies P^{\phi} \prec^{\flat} Q^{\phi}, \quad P \sim^{\flat} Q \implies P^{\phi} \sim^{\flat} Q^{\phi}.$$

The last implication gives: $P \sim^{\flat} Q \implies \operatorname{ndeg} P = \operatorname{ndeg} Q$ and $\operatorname{nmul} P = \operatorname{nmul} Q$.

For $P^{\phi} \asymp^{\flat} P$ and the subsequent three implications in the lemma above we can drop the assumption that K is ungrounded.

Lemma 1.8.3. *Suppose K is d-valued, ω-free, and for every $P \in K\{Y\}$ of order at most r with $\operatorname{ndeg} P = 1$ and every $\gamma \in \Gamma^{>}$ there is an $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$. Then K^{c} is d-valued, ω-free, and r-newtonian.*

Proof. By [ADH, 9.1.6 and 11.7.20], K^{c} is d-valued and ω-free. Let $P \in K^{\mathrm{c}}\{Y\}$ be of order $\leqslant r$ with $\operatorname{ndeg} P = 1$. We need to show that $P(b) = 0$ for some $b \preccurlyeq 1$ in K^{c}. To find b we may replace K, P by K^{ϕ}, P^{ϕ}; in particular we may assume that K has small derivation and $\Gamma^{\flat} \neq \Gamma$. By [ADH, 14.0.1 and the remarks following it] we can take $b \preccurlyeq 1$ in an immediate newtonian extension L of K^{c} such that $P(b) = 0$. We claim that $b \in K^{\mathrm{c}}$. To show this we may assume $b \notin K$, so $v(b - K)$ does not have a largest element. By [ADH, 11.2.3(i)] we have $\operatorname{ndeg} P_{+b} = 1$ and so $\operatorname{ndeg} P_{+b,\times g} = 1$ for all $g \preccurlyeq 1$ in $L^{\times}$ by [ADH, 11.2.5], in view of $P(b) = 0$.

Claim: *Let $\gamma \in \Gamma^{>}$ and $a \in K$ with $v(b - a) \geqslant 0$. There is a $y \in \mathcal{O}$ such that $v\big(P(y)\big) > \gamma$ and $v(b - y) \geqslant v(b - a)$.*

The proof is similar to that of the claim in the proof of Lemma 1.7.3: Take $g \in K^{\times}$ with $vg = v(b-a)$. Then $\operatorname{ndeg} P_{+a,\times g} = \operatorname{ndeg} P_{+b,\times g} = 1$ by [ADH, 11.2.4] and the observation preceding the claim. Density of K in K^{c} yields $Q \in K\{Y\}$ of order $\leqslant r$ with $v(P_{+a,\times g} - Q) > \gamma$ and $P_{+a,\times g} \sim^{\flat} Q$, the latter using $\Gamma^{\flat} \neq \Gamma$. Then $\operatorname{ndeg} Q = \operatorname{ndeg} P_{+a,\times g} = 1$ by Lemma 1.8.2, so the hypothesis of the lemma gives $z \in \mathcal{O}$ with $v\big(Q(z)\big) > \gamma$. Setting $y := a + gz \in \mathcal{O}$ we have $v\big(P(y)\big) = v\big(P_{+a,\times g}(z)\big) > \gamma$ and $v(b - y) = v(b - a - gz) \geqslant vg = v(b - a)$.

Let $\gamma \in \Gamma^{>}$; to get $b \in K^{\mathrm{c}}$, it is enough to show that then $v(a - b) > \gamma$ for some $a \in K$. Let $A := L_{P_{+b}} \in L[\partial]$. Since $|\mathscr{E}^{\mathrm{e}}_L(A)| \leqslant r$ by [ADH, 14.2.9], by the claim we can take $a \in \mathcal{O}$ with $v\big(P(a)\big) > 2\gamma$ and $0 < v(b - a) \notin \mathscr{E}^{\mathrm{e}}_L(A)$. Now put $g := a - b$ and take ϕ with $vg \notin \mathscr{E}_{L^{\phi}}(A^{\phi})$; note that then $A^{\phi} = L_{P^{\phi}_{+b}}$. Replacing K, L, P by K^{ϕ}, L^{ϕ}, P^{ϕ} we arrange $vg \notin \mathscr{E}_L(A)$, and (changing ϕ if necessary) $\operatorname{ddeg} P_{+b} = 1$. We also arrange $P_{+b} \asymp 1$, and then $(P_{+b})_{>1} \prec 1$. As in the proof of Lemma 1.7.3 above we now derive $v(a - b) = vg > \gamma$. $\square$

Combining Lemmas 1.7.2 and 1.8.3 now yields Proposition 1.8.1. $\square$

To show that newtonianity is preserved under specialization, we assume below that $\Psi \cap \Gamma^{>} \neq \emptyset$, so K has small derivation. Let $\Delta \neq \{0\}$ be a convex subgroup of Γ with $\psi(\Delta^{\neq}) \subseteq \Delta$. Then $1 \in \Delta$ where 1 denotes the unique positive element of Γ fixed by the function ψ: use that $\psi(\gamma) \geqslant 1$ for $0 < \gamma < 1$. (Conversely, any convex subgroup G of Γ with $1 \in G$ satisfies $\psi(G^{\neq}) \subseteq G$.) Let $\dot{v}$ be the coarsening of the valuation v of K by Δ, with valuation ring $\dot{\mathcal{O}}$, maximal ideal $\dot{o}$ of $\dot{\mathcal{O}}$, and residue field $\dot{K} = \dot{\mathcal{O}}/\dot{o}$. The derivation of K is small with respect to $\dot{v}$, and $\dot{K}$ with the induced valuation $v\colon \dot{K}^{\times} \to \Delta$ and induced derivation as in [ADH, p. 405] is an asymptotic field with asymptotic couple $(\Delta, \psi|\Delta^{\neq})$, and so is of H-type with small derivation. If K is d-valued, then so is $\dot{K}$ by [ADH, 10.1.8], and if K is ω-free, then so is $\dot{K}$ by [ADH, 11.7.24]. The residue map $a \mapsto \dot{a} := a + \dot{o}\colon \dot{\mathcal{O}} \to \dot{K}$ is a differential ring morphism, extends to a differential ring morphism $P \mapsto \dot{P}\colon \dot{\mathcal{O}}\{Y\} \to \dot{K}\{Y\}$ of differential rings sending Y to Y, and $\operatorname{ddeg} P = \operatorname{ddeg} \dot{P}$ for $P \in \dot{\mathcal{O}}\{Y\}$ with $\dot{P} \neq 0$.

We now restrict ϕ to range over active elements of $\mathcal{O}$. Then $v\phi \leqslant 1 + 1$, so $v\phi \in \Delta$, and hence ϕ is a unit of $\dot{\mathcal{O}}$. It follows that $\dot{\phi}$ is active in $\dot{K}$, and every active element of $\dot{K}$ lying in its valuation ring is of this form. Moreover, the differential subrings $\dot{\mathcal{O}}$ of K and $\dot{\mathcal{O}}^{\phi} := (\dot{\mathcal{O}})^{\phi}$ of K^{ϕ} have the same underlying ring, and the derivation of K^{ϕ} is small with respect to $\dot{v}$. Thus the differential residue fields $\dot{K} = \dot{\mathcal{O}}/\dot{o}$ and $\dot{K}^{\phi} := \dot{\mathcal{O}}^{\phi}/\dot{o}$ have the same underlying field and are related as follows:

$$\dot{K}^{\phi} \;=\; (\dot{K})^{\dot{\phi}}.$$

For $P \in \dot{\mathcal{O}}\{Y\}$ we have $P^{\phi} \in \dot{\mathcal{O}}^{\phi}\{Y\}$, and the image of P^{ϕ} under the residue map $\dot{\mathcal{O}}^{\phi}\{Y\} \to \dot{K}^{\phi}\{Y\}$ equals $\dot{P}^{\dot{\phi}}$; hence $\operatorname{ndeg} P = \operatorname{ndeg} \dot{P}$ for $P \in \dot{\mathcal{O}}\{Y\}$ satisfying $\dot{P} \neq 0$. These remarks imply:

Lemma 1.8.4. *If K is r-newtonian, then $\dot{K}$ is r-newtonian.*

Combining Proposition 1.8.1 and Lemmas 1.8.3 and 1.8.4 yields:

Corollary 1.8.5. *Suppose K is* d*-valued,* ω*-free, and r-newtonian. Then $\dot{K}$ and its completion are* d*-valued,* ω*-free, and r-newtonian.*

We finish with a newtonian analogue of [ADH, 7.1.7]:

Lemma 1.8.6. *Suppose $(K, \dot{\mathcal{O}})$ is r-*d*-henselian and $\dot{K}$ is r-newtonian. Then K is r-newtonian.*

Proof. Let $P \in K\{Y\}$ be of order $\leqslant r$ and $\operatorname{ndeg} P = 1$; we need to show the existence of $b \in \mathcal{O}$ with $P(b) = 0$. Replacing K and P by K^ϕ and P^ϕ for suitable ϕ (and renaming) we arrange $\operatorname{ddeg} P = 1$; this also uses [ADH, section 7.3, subsection on compositional conjugation]. We can further assume that $P \asymp 1$. Put $Q := \dot{P} \in \dot{K}\{Y\}$, so $\operatorname{ndeg} Q = 1$, and thus r-newtonianity of $\dot{K}$ yields an $a \in \mathcal{O}$ with $Q(\dot{a}) = 0$. Then $P(a) \mathrel{\dot{\prec}} 1$, $P_{+a,1} \sim P_1 \asymp 1$, and $P_{+a,>1} \prec 1$. Since $(K, \dot{\mathcal{O}})$ is r-d-henselian, this gives $y \in \dot{o}$ with $P_{+a}(y) = 0$, and then $P(b) = 0$ for $b := a + y \in \mathcal{O}$. $\square$

Lemmas 1.8.4, 1.8.6, and [ADH, 14.1.2] yield:

Corollary 1.8.7. *K is r-newtonian iff $(K, \dot{\mathcal{O}})$ is r-*d*-henselian and $\dot{K}$ is r-newtonian.*

Invariance of Newton quantities. *In this subsection $P \in K\{Y\}^{\neq}$.* In [ADH, 11.1] we associated to P its Newton weight $\operatorname{nwt} P$, Newton degree $\operatorname{ndeg} P$, and Newton multiplicity $\operatorname{nmul} P$ at 0, all elements of $\mathbb{N}$, as well as the element $v^{\mathrm{e}}(P)$ of Γ; these quantities do not change when passing to an H-asymptotic extension L of K with Ψ cofinal in Ψ_L, cf. [ADH, p. 480], where the assumptions on K, L are slightly weaker. Thus by Theorem 1.4.1, these quantities do not change for ω-free K in passing to an H-asymptotic pre-d-valued d-algebraic extension of K. Below we improve on this in several ways. First, for $\operatorname{order} P \leqslant 1$ we can drop Ψ being cofinal in Ψ_L by a strengthening of [ADH, 11.2.13]:

Lemma 1.8.8. *Suppose K is H-asymptotic with rational asymptotic integration and $P \in K[Y, Y']^{\neq}$. Then there are $w \in \mathbb{N}$, $\alpha \in \Gamma^{>}$, $A \in K[Y]^{\neq}$, and an active ϕ_0 in K such that for every asymptotic extension L of K and active $f \preccurlyeq \phi_0$ in L,*

$$P^f \;=\; f^w A(Y)(Y')^w + R_f, \quad R_f \in L^f[Y, Y'], \quad v(R_f) \;\geqslant\; v(P^f) + \alpha.$$

For such w, A we have for any ungrounded H-asymptotic extension L of K,

$$\operatorname{nwt}_L P \;=\; w, \quad \operatorname{ndeg}_L P \;=\; \deg A + w, \quad \operatorname{nmul}_L P \;=\; \operatorname{mul} A + w, \quad v^{\mathrm{e}}_L(P) = v(A).$$

Proof. Let w be as in the proof of [ADH, 11.2.13]. Using its notations, this proof yields an active ϕ_0 in K such that

$$w\gamma + v(A_w) \;<\; j\gamma + v(A_j) \tag{1.8.1}$$

for all $\gamma \geqslant v(\phi_0)$ in $\Psi^{\downarrow}$ and $j \in J \setminus \{w\}$. This gives $\beta \in \mathbb{Q}\Gamma$ such that $\beta > \Psi$ and (1.8.1) remains true for all $\gamma \in \Gamma$ with $v(\phi_0) \leqslant \gamma < \beta$. Since $(\mathbb{Q}\Gamma, \psi)$ has asymptotic integration, β is not a gap in $(\mathbb{Q}\Gamma, \psi)$, so $\beta > \beta_0 > \Psi$ with $\beta_0 \in \mathbb{Q}\Gamma$. This yields an element $\alpha \in (\mathbb{Q}\Gamma)^{>}$ such that for all $\gamma \in \mathbb{Q}\Gamma$ with $v(\phi_0) \leqslant \gamma \leqslant \beta_0$ we have

$$w\gamma + v(A_w) + \alpha \;\leqslant\; j\gamma + v(A_j) \tag{1.8.2}$$

Since Γ has no least positive element, we can decrease α to arrange $\alpha \in \Gamma^{>}$. Now (1.8.2) remains true for all elements γ of any divisible ordered abelian group extending $\mathbb{Q}\Gamma$ with $v(\phi_0) \leqslant \gamma \leqslant \beta_0$. Thus w, α, $A = A_w$, and ϕ_0 are as required.

For any ungrounded H-asymptotic extension L of K we obtain for active $f \preccurlyeq \phi_0$ in L that $v(P^f) = v(A) + wv(f)$, so $v_L^{\mathrm{e}}(P) = v(A)$ in view of the identity in [ADH, 11.1.15] defining $v_L^{\mathrm{e}}(P)$. □

For quasilinear P we have:

Lemma 1.8.9. *Suppose K is λ-free and* $\operatorname{ndeg} P = 1$. *Then there are active ϕ_0 in K and $a, b \in K$ with $a \preccurlyeq b \neq 0$ such that either* (i) *or* (ii) *below holds:*

(i) *$P^f \sim^{\flat}_{\phi_0} a + bY$ for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K;*
(ii) *$P^f \sim^{\flat}_{\phi_0} \frac{f}{\phi_0} bY'$ for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K.*

In particular, for each ungrounded H-asymptotic extension L of K,

$$\operatorname{nwt}_L P = \operatorname{nwt} P \leqslant 1, \quad \operatorname{ndeg}_L P = 1, \quad \operatorname{nmul}_L P = \operatorname{nmul} P, \quad v_L^{\mathrm{e}}(P) = v^{\mathrm{e}}(P).$$

Proof. From [ADH, 13.7.10] we obtain an active ϕ_0 in K and $a, b \in K$ with $a \preccurlyeq b$ such that in $K^{\phi_0}\{Y\}$, either $P^{\phi_0} \sim^{\flat}_{\phi_0} a + bY$ or $P^{\phi_0} \sim^{\flat}_{\phi_0} bY'$ (so $b \neq 0$). In the first case, set $A(Y) := a + bY$, $w := 0$; in the second case, set $A(Y) := bY'$, $w := 1$. Then $P^{\phi_0} = A + R$ where $R \prec^{\flat}_{\phi_0} b \asymp P^{\phi_0}$ in $K^{\phi_0}\{Y\}$.

Let L be an H-asymptotic extension of K. Then $R \prec^{\flat}_{\phi_0} P^{\phi_0}$ remains true in $L^{\phi_0}\{Y\}$, and if $f \preccurlyeq \phi_0$ is active in L, then $P^f = (P^{\phi_0})^{f/\phi_0} = (f/\phi_0)^w A + R^{f/\phi_0}$ where $R^{f/\phi_0} \prec^{\flat}_{\phi_0} P^f$ by Lemma 1.8.2 and the remark following its proof. As to $v_L^{\mathrm{e}}(P) = v^{\mathrm{e}}(P)$ for ungrounded L, the identity from [ADH, 11.1.15] defining these quantities shows that both are vb in case (i), and $v(b) - v(\phi_0)$ in case (ii). □

Lemma 1.8.9 has the following consequence, partly generalizing Corollary 1.5.5:

Corollary 1.8.10. *Suppose K is λ-free, $A \in K[\partial]^{\neq}$ and L is an ungrounded H-asymptotic extension of K. Then for $\gamma \in \Gamma$ the quantities* $\operatorname{nwt}_A(\gamma) \leqslant 1$ *and* $v_A^{\mathrm{e}}(\gamma)$ *do not change when passing from K to L; in particular,*

$$\mathscr{E}^{\mathrm{e}}(A) \;=\; \big\{\gamma \in \Gamma : \operatorname{nwt}_A(\gamma) = 1\big\} \;=\; \mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma.$$

This leads to a variant of Corollary 1.5.20:

Corollary 1.8.11. *Suppose K is λ-free. Then* $|\mathscr{E}^{\mathrm{e}}(A)| \leqslant \operatorname{order} A$ *for all* $A \in K[\partial]^{\neq}$.

Proof. By [ADH, 10.1.3], K is pre-d-valued, hence by [ADH, 11.7.18] it has an ω-free H-asymptotic extension. It remains to appeal to Corollaries 1.5.5 and 1.8.10. □

For completeness we next state a version of Lemma 1.8.9 for $\operatorname{ndeg} P = 0$; the proof using [ADH, 13.7.9] is similar, but simpler, and hence omitted.

Lemma 1.8.12. *Suppose K is λ-free and* $\operatorname{ndeg} P = 0$. *Then there are an active ϕ_0 in K and $a \in K^{\times}$ such that $P^f \sim^{\flat}_{\phi_0} a$ for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K.*

In particular, for K, P as in Lemma 1.8.12, no H-asymptotic extension of K contains any $y \preccurlyeq 1$ such that $P(y) = 0$.

For general P and ω-free K we can still do better than stated earlier:

Lemma 1.8.13. *Suppose K is $\upomega$-free. Then there are $w \in \mathbb{N}$, $A \in K[Y]^{\neq}$, and an active ϕ_0 in K such that for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K,*

$$P^f \ \sim^{\flat}_{\phi_0}\ (f/\phi_0)^w A(Y)(Y')^w.$$

For such w, A, ϕ_0 we have for any ungrounded H-asymptotic extension L of K,

$$\begin{aligned} \operatorname{nwt}_L P &= w, & \operatorname{ndeg}_L P &= \deg A + w,\\ \operatorname{nmul}_L P &= \operatorname{mul} A + w, & v_L^{\mathrm{e}}(P) &= v(A) - wv(\phi_0). \end{aligned}$$

Proof. By [ADH, 13.6.11] we have active ϕ_0 in K and $A \in K[Y]^{\neq}$ such that

$$P^{\phi_0} \ =\ A\cdot(Y')^w + R, \quad w := \operatorname{nwt} P, \quad R \in K^{\phi_0}\{Y\},\ R \prec^{\flat}_{\phi_0} P^{\phi_0}.$$

(Here ϕ_0 and A are the e and aA in [ADH, 13.6.11].) The rest of the argument is just like in the second part of the proof of Lemma 1.8.9. $\square$

Remarks on newton position. For the next lemma we put ourselves in the setting of [ADH, 14.3]: K is $\upomega$-free, $P \in K\{Y\}^{\neq}$, and a ranges over K. Recall that P is said to be in *newton position at a* if $\operatorname{nmul} P_{+a} = 1$.

Suppose P is in newton position at a; then $A := L_{P_{+a}} \in K[\partial]^{\neq}$. Recall the definition of $v^{\mathrm{e}}(P,a) = v^{\mathrm{e}}_K(P,a) \in \Gamma_\infty$: if $P(a)=0$, then $v^{\mathrm{e}}(P,a)=\infty$; if $P(a)\neq 0$, then $v^{\mathrm{e}}(P,a) = vg$ where $g \in K^\times$ satisfies $P(a) \asymp (P_{+a})^{\phi}_{1,\times g}$ eventually, that is, $v_{A^\phi}(vg) = v\big(P(a)\big)$ eventually. In the latter case $\operatorname{nwt}_A(vg)=0$, that is, $vg \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v^{\mathrm{e}}_A(vg) = v\big(P(a)\big)$, since $v_{A^\phi}(vg) = v^{\mathrm{e}}_A(vg) + \operatorname{nwt}_A(vg)v\phi$ eventually. For any $f \in K^\times$, P^f is also in newton position at a, and $v^{\mathrm{e}}(P^f,a) = v^{\mathrm{e}}(P,a)$. Note also that P_{+a} is in newton position at 0 and $v^{\mathrm{e}}(P_{+a},0) = v^{\mathrm{e}}(P,a)$. Moreover, in passing from K to an $\upomega$-free extension, P remains in newton position at a and $v^{\mathrm{e}}(P,a)$ does not change, by Lemma 1.8.13.

In the rest of this subsection P is in newton position at a, and $\widehat{a}$ is an element of an H-asymptotic extension $\widehat{K}$ of K such that $P(\widehat{a})=0$. (We allow $\widehat{a}\in K$.) We first generalize part of [ADH, 14.3.1], with a similar proof:

Lemma 1.8.14. *$v^{\mathrm{e}}(P,a) > 0$ and $v(\widehat{a}-a) \leqslant v^{\mathrm{e}}(P,a)$.*

Proof. This is clear if $P(a)=0$. Assume $P(a)\neq 0$. Replace P, $\widehat{a}$, a by P_{+a}, $\widehat{a}-a$, 0, respectively, to arrange $a=0$. Recall that K^ϕ has small derivation. Set $\gamma := v^{\mathrm{e}}(P,0) \in \Gamma$ and take $g \in K$ with $vg=\gamma$. Now $(P_1^\phi)_{\times g} \asymp P_0$, eventually, and $\operatorname{nmul} P = 1$ gives $P(0) \prec P_1^\phi$, eventually, hence $g \prec 1$. Moreover, for $j \geqslant 2$, $P_1^\phi \succcurlyeq P_j^\phi$, eventually, so $(P_1^\phi)_{\times g} \succ (P_j^\phi)_{\times g}$, eventually, by [ADH, 6.1.3]. Thus for $j\geqslant 1$ we have $(P^\phi_{\times g})_j = (P_j^\phi)_{\times g} \preccurlyeq P(0)$, eventually; in particular, there is no $y \prec 1$ in any H-asymptotic extension of K with $P_{\times g}(y)=0$. Since $P(\widehat{a})=0$, this yields $v(\widehat{a}) \leqslant \gamma = v^{\mathrm{e}}(P,0)$. $\square$

Here is a situation where $v(\widehat{a}-a) = v^{\mathrm{e}}(P,a)$:

Lemma 1.8.15. *Suppose Ψ is cofinal in $\Psi_{\widehat{K}}$, $\widehat{a}-a \prec 1$, and $v(\widehat{a}-a) \notin \mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ where $A := L_{P_{+a}}$. Then $v(\widehat{a}-a) = v^{\mathrm{e}}(P,a)$.*

Proof. Note that $\widehat{K}$ is ungrounded, so $\mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ is defined, and $\widehat{K}$ is pre-d-valued. As in the proof of Lemma 1.8.14 we arrange $a=0$. As an asymptotic subfield of $\widehat{K}$, $K\langle\widehat{a}\rangle$ is pre-d-valued. Hence $K\langle\widehat{a}\rangle$ is $\upomega$-free by Theorem 1.4.1. The remarks preceding Lemma 1.8.14 then allow us to replace K by $K\langle\widehat{a}\rangle$ to arrange $\widehat{a}\in K$.

The case $\widehat{a}=0$ is trivial, so assume $0\neq\widehat{a}\prec 1$. Now $\operatorname{nmul}P=1$ gives for $j\geqslant 2$ that $P_1^{\phi}\succcurlyeq P_j^{\phi}$, eventually, hence $(P_1^{\phi})_{\times\widehat{a}}\succ(P_j^{\phi})_{\times\widehat{a}}$, eventually, by [ADH, 6.1.3]. Moreover, $P_1(\widehat{a})=A(\widehat{a})=A^{\phi}(\widehat{a})\asymp A^{\phi}\widehat{a}$, eventually, using $v(\widehat{a})\notin\mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ in the last step, so for $j\geqslant 2$, eventually

$$P_1(\widehat{a})\ \asymp\ (P_1^{\phi})_{\times\widehat{a}}\ \succ\ (P_j^{\phi})_{\times\widehat{a}}\ \succcurlyeq P_j^{\phi}(\widehat{a})\ =\ P_j(\widehat{a}).$$

Also $P_1(\widehat{a})\neq 0$, since $A^{\phi}\widehat{a}\neq 0$. Then $P(\widehat{a})=0$ gives $P(0)\asymp P_1(\widehat{a})$. Thus $v\big(P(0)\big)=v_{A^{\phi}}\big(v(\widehat{a})\big)$, eventually, so $v^{\mathrm{e}}(P,0)=v(\widehat{a})$ by the definition of $v^{\mathrm{e}}(P,0)$. □

Corollary 1.8.16. *Suppose $\widehat{K}$ is ungrounded and equipped with an ordering making it a pre-H-field, and assume $\widehat{a}-a\prec 1$ and $v(\widehat{a}-a)\notin\mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ where $A:=L_{P_{+a}}$. Then $v(\widehat{a}-a)=v^{\mathrm{e}}(P,a)$.*

Proof. In view of Lemma 1.5.1 and using [ADH, 14.5.11] we can extend $\widehat{K}$ to arrange that it is an ω-free newtonian Liouville closed H-field. Next, let H be the real closure of the H-field hull of $K\langle\widehat{a}\rangle$, all inside $\widehat{K}$. Then H is ω-free, by Theorem 1.4.1, and hence has a Newton-Liouville closure L inside $\widehat{K}$ [ADH, 14.5]. Since $L\preccurlyeq\widehat{K}$ by [ADH, 16.2.5], we have $v(\widehat{a}-a)\notin\mathscr{E}^{\mathrm{e}}_{L}(A)$. Now L is d-algebraic over K by [ADH, 14.5.9], so Ψ is cofinal in Ψ_L by Theorem 1.4.1. It remains to apply Lemma 1.8.15. □

Newton position in the order 1 case. *In this subsection K is λ-free, $P\in K\{Y\}$ has order 1, and $a\in K$.* We basically copy here a definition and two lemmas from [ADH, 14.3] with the ω-free assumption there replaced by the weaker λ-freeness, at the cost of restricting P to have order 1.

Suppose $\operatorname{nmul}P=1$, $P_0\neq 0$. Then [ADH, 11.6.17] yields $g\in K^{\times}$ with $P_0\asymp P^{\phi}_{1,\times g}$, eventually. Since $P_0\prec P_1^{\phi}$, eventually, we have $g\prec 1$. Moreover, if $i\geqslant 2$, then $P_1^{\phi}\succcurlyeq P_i^{\phi}$, eventually, hence $P^{\phi}_{1,\times g}\succ P^{\phi}_{i,\times g}$, eventually. Thus $\operatorname{ndeg}P_{\times g}=1$.

Define P to be in **newton position at** a if $\operatorname{nmul}P_{+a}=1$. Suppose P is in newton position at a; set $Q:=P_{+a}$, so $Q(0)=P(a)$. If $P(a)\neq 0$, then the above yields $g\in K^{\times}$ with $P(a)=Q(0)\asymp Q^{\phi}_{1,\times g}$, eventually; as vg does not depend on the choice of such g, we set $v^{\mathrm{e}}(P,a):=vg$. If $P(a)=0$, then we set $v^{\mathrm{e}}(P,a):=\infty\in\Gamma_{\infty}$. In passing from K to a λ-free extension, P remains in newton position at a and $v^{\mathrm{e}}(P,a)$ does not change, by Lemma 1.8.8. *In the rest of this subsection we assume P is in newton position at a.*

Lemma 1.8.17. *If $P(a)\neq 0$, then there exists $b\in K$ with the following properties:*

(i) *P is in newton position at b, $v(a-b)=v^{\mathrm{e}}(P,a)$, and $P(b)\prec P(a)$;*
(ii) *for all $b^*\in K$ with $v(a-b^*)\geqslant v^{\mathrm{e}}(P,a)$: $P(b^*)\prec P(a)\Leftrightarrow a-b\sim a-b^*$;*
(iii) *for all $b^*\in K$, if $a-b\sim a-b^*$, then P is in newton position at b^* and $v^{\mathrm{e}}(P,b^*)>v^{\mathrm{e}}(P,a)$.*

This is shown as in [ADH, 14.3.2]. Next an analogue of [ADH, 14.3.3], with the same proof, but using Lemma 1.8.17 in place of [ADH, 14.3.2]:

Lemma 1.8.18. *If there is no b with $P(b)=0$ and $v(a-b)=v^{\mathrm{e}}(P,a)$, then there is a divergent pc-sequence $(a_{\rho})_{\rho<\lambda}$ in K, indexed by all ordinals ρ smaller than some infinite limit ordinal λ, such that $a_0=a$, $v(a_{\rho}-a_{\rho'})=v^{\mathrm{e}}(P,a_{\rho})$ for all $\rho<\rho'<\lambda$, and $P(a_{\rho})\rightsquigarrow 0$.*

The next result is proved just like Lemma 1.8.14:

Lemma 1.8.19. *If $P(\widehat{a}) = 0$ with $\widehat{a}$ in an H-asymptotic extension of K, then $v^{\mathrm{e}}(P,a) > 0$ and $v(\widehat{a}-a) \leqslant v^{\mathrm{e}}(P,a)$.*

Next an analogue of Lemma 1.8.15 using Propositions 1.4.8 and 1.4.12 in its proof:

Lemma 1.8.20. *Suppose $\widehat{a}$ in an ungrounded H-asymptotic extension $\widehat{K}$ of K satisfies $P(\widehat{a})=0$, $\widehat{a}-a \prec 1$, and $v(\widehat{a}-a) \notin \mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$, where $A := L_{P_{+a}}$. Then $v(\widehat{a}-a) = v^{\mathrm{e}}(P,a)$.*

Proof. We arrange $a = 0$ and assume $\widehat{a} \neq 0$. Then $L := K\langle\widehat{a}\rangle$ has asymptotic integration, by Proposition 1.4.12, and $v(\widehat{a}) \notin \mathscr{E}^{\mathrm{e}}_L(A)$ by Lemma 1.5.10 (applied with L, $\widehat{K}$ in place of K, L). Moreover, Ψ is cofinal in Ψ_L by Proposition 1.4.8. As in the proof of Lemma 1.8.15 this leads to $P_1(\widehat{a}) = A(\widehat{a}) = A^{\phi}(\widehat{a}) \asymp A^{\phi}\widehat{a}$, eventually, and then as in the rest of that proof we derive $v^{\mathrm{e}}(P,0) = v(\widehat{a})$. $\square$

Zeros of differential polynomials of order and degree 1. *In this subsection K has asymptotic integration.* We fix a differential polynomial

$$P(Y) \;=\; a(Y' + gY - u) \in K\{Y\} \qquad (a,g,u \in K,\ a \neq 0),$$

and set $A := L_P = a(\partial + g) \in K[\partial]$. Section 1.2 gives for $y \in K$ the equivalence $y \in \mathrm{I}(K) \Leftrightarrow vy > \Psi$, so by Section 1.5, $\mathscr{E}^{\mathrm{e}}(A) = \emptyset \Leftrightarrow g \notin \mathrm{I}(K) + K^{\dagger}$, and $v(\ker^{\neq}_{\widehat{K}} A) \subseteq \mathscr{E}^{\mathrm{e}}(A)$ for each immediate H-asymptotic field extension $\widehat{K}$ of K. Thus:

Lemma 1.8.21. *If $g \notin \mathrm{I}(K) + K^{\dagger}$, then each immediate H-asymptotic extension of K contains at most one y such that $P(y) = 0$.*

If $\partial K = K$ and $g \in K^{\dagger}$, then $P(y) = 0$ for some $y \in K$, and if moreover K is d-valued, then any y in any immediate H-asymptotic extension of K with $P(y) = 0$ lies in K. (Lemma 1.2.2.) If $y \prec 1$ in an immediate H-asymptotic extension of K satisfies $P(y) = 0$, then by [ADH, 11.2.3(ii), 11.2.1] we have

$$\operatorname{nmul} P \;=\; \operatorname{nmul} P_{+y} \;=\; \operatorname{mul} P_{+y} \;=\; 1.$$

Lemma 1.8.18 yields the following partial converse (a variant of [11, Lemma 8.5]):

Corollary 1.8.22. *Suppose K is $\uplambda$-free and $\operatorname{nmul} P = 1$. Then there is a $y \prec 1$ in an immediate H-asymptotic extension of K with $P(y) = 0$.*

Proof. Replacing K by its henselization and using [ADH, 11.6.7], we arrange that K is henselian. Suppose that P has no zero in $\mathcal{O}$. Then P is in newton position at 0, and so Lemma 1.8.18 yields a divergent pc-sequence $(a_\rho)_{\rho<\lambda}$ in K, indexed by all ordinals ρ smaller than some infinite limit ordinal λ, with $a_0 = 0$, $v(a_\rho - a_{\rho'}) = v^{\mathrm{e}}(P,a_\rho)$ for all $\rho < \rho' < \lambda$, and $P(a_\rho) \rightsquigarrow 0$. Since $\deg P = \operatorname{order} P = 1$ and K is henselian, P is a minimal differential polynomial of (a_ρ) over K, and $v(a_\rho) = v^{\mathrm{e}}(P,0) > 0$ for all $\rho > 0$. Hence [ADH, 9.7.6] yields a pseudolimit y of (a_ρ) in an immediate asymptotic extension of K with $P(y) = 0$ and $y \prec 1$, as required. $\square$

We say that P is **proper** if $u \neq 0$ and $g + u^{\dagger} \succ^{\flat} 1$. If P is proper, then so is bP for each $b \in K^{\times}$. For $\mathfrak{m} \in K^{\times}$ we have

$$P_{\times\mathfrak{m}} \;=\; a\mathfrak{m}\big(Y' + (g + \mathfrak{m}^{\dagger})Y - u\mathfrak{m}^{-1}\big),$$

hence if P is proper, then so is $P_{\times\mathfrak{m}}$. If $u \neq 0$, then P is proper iff $a^{-1}A_{\ltimes u} = \partial + (g + u^{\dagger})$ is steep, as defined in Section 1.5. Note that

$$P^{\phi} \;=\; a\phi\big(Y' + (g/\phi)Y - (u/\phi)\big).$$

Lemma 1.8.23. *Suppose K has small derivation, and P is proper. Then P^{ϕ} is proper (with respect to K^{ϕ}) for all $\phi \preccurlyeq 1$.*

Proof. Let $\phi \preccurlyeq 1$. Then we have $\phi \asymp^{\flat} 1$ and hence $\phi^{\dagger} \asymp^{\flat} \phi' \preccurlyeq 1 \prec^{\flat} g + u^{\dagger}$. Thus

$$g + (u/\phi)^{\dagger} = (g + u^{\dagger}) - \phi^{\dagger} \sim^{\flat} g + u^{\dagger} \succ^{\flat} 1 \succcurlyeq \phi,$$

hence $(g/\phi) + \phi^{-1}(u/\phi)^{\dagger} \succ^{\flat} 1$ and so $(g/\phi) + \phi^{-1}(u/\phi)^{\dagger} \succ^{\flat}_{\phi} 1$. Therefore P^{ϕ} is proper (with respect to K^{ϕ}). □

Lemma 1.8.24. *Suppose K is $\boldsymbol{\lambda}$-free and $u \neq 0$. Then there is an active ϕ_0 in K such that for all $\phi \prec \phi_0$, P^{ϕ} is proper with $g + (u/\phi)^{\dagger} \sim g + (u/\phi_0)^{\dagger}$.*

Proof. The argument before Corollary 1.5.15 yields an active ϕ_0 in K such that $u^{\dagger} + g - \phi^{\dagger} \succcurlyeq \phi_0$ for all $\phi \prec \phi_0$. For such ϕ we have $\phi^{\dagger} - \phi_0^{\dagger} \prec \phi_0$ as noted just before [ADH, 11.5.3], and so $(u/\phi)^{\dagger} + g \sim (u/\phi_0)^{\dagger} + g$. The argument before Corollary 1.5.15 also gives $\phi^{-1}(u/\phi)^{\dagger} + g/\phi \succ^{\flat}_{\phi} 1$ eventually, and if $\phi^{-1}(u/\phi)^{\dagger} + g/\phi \succ^{\flat}_{\phi} 1$, then P^{ϕ} is proper. □

Lemma 1.8.25. *We have* $\operatorname{nmul} P = 1$ *iff $u \prec g$ or $u \in \mathrm{I}(K)$. Moreover, if K is $\boldsymbol{\lambda}$-free,* $\operatorname{nmul} P = 1$, *and $u \neq 0$, then $u \prec^{\flat}_{\phi} g + (u/\phi)^{\dagger}$, eventually.*

Proof. For the equivalence, note that the identity above for P^{ϕ} yields:

$$\operatorname{nmul} P = 0 \iff u \succcurlyeq g, \text{ and } u/\phi \succcurlyeq 1 \text{ eventually.}$$

Suppose K is $\boldsymbol{\lambda}$-free, $\operatorname{nmul} P = 1$, and $u \neq 0$. If $u \in \mathrm{I}(K)$, then $u \prec \phi \prec^{\flat}_{\phi} g + (u/\phi)^{\dagger}$, eventually, by Lemma 1.8.24. Suppose $u \notin \mathrm{I}(K)$. Then $v(u) \in \Psi^{\downarrow}$ and $u \prec g$. Hence by [ADH, 9.2.11] we have $(u/\phi)^{\dagger} \prec u \prec g$, eventually, and thus $u \prec g \sim g + (u/\phi)^{\dagger}$, eventually. Thus $u \prec^{\flat}_{\phi} g + (u/\phi)^{\dagger}$, eventually. □

Assume now $P(y) = 0$ with y in an immediate H-asymptotic extension of K; so $A(y) = u$. Note: if $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, then $u \neq 0$. From Lemma 1.5.14 we get:

Lemma 1.8.26. *If K has small derivation, P is proper, and $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, then $y \sim u/(g + u^{\dagger})$.*

By Lemmas 1.8.24 and 1.8.26, and using Lemma 1.8.25 for the last part:

Corollary 1.8.27. *If K is $\boldsymbol{\lambda}$-free and $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, then*

$$y \sim u/\big(g + (u/\phi)^{\dagger}\big) \qquad \textit{eventually.}$$

If in addition $\operatorname{nmul} P = 1$, *then $y \prec 1$.*

A characterization of 1-linear newtonianity. *In this subsection K has asymptotic integration.* We first expand [ADH, 14.2.4]:

Proposition 1.8.28. *The following are equivalent:*

(i) *K is 1-linearly newtonian;*
(ii) *every $P \in K\{Y\}$ with* $\operatorname{nmul} P = \deg P = 1$ *and* $\operatorname{order} P \leqslant 1$ *has a zero in $\mathcal{O}$;*
(iii) *K is* d*-valued, $\boldsymbol{\lambda}$-free, and 1-linearly surjective, with $\mathrm{I}(K) \subseteq K^{\dagger}$.*

Proof. The equivalence of (i) and (ii) is [ADH, 14.2.4], and the implication (i) $\Rightarrow$ (iii) follows from [ADH, 14.2.2, 14.2.3, 14.2.5]. To show (iii) $\Rightarrow$ (ii), suppose (iii) holds, and let $g, u \in K$ and $P = Y' + gY - u$ with $\operatorname{nmul} P = 1$. We need to find $y \in \smallo$ such that $P(y) = 0$. Corollary 1.8.22 gives an element $y \prec 1$ in an immediate H-asymptotic extension L of K with $P(y) = 0$. It suffices to show that then $y \in K$ (and thus $y \in \smallo$). If $g \notin K^\dagger$, then this follows from Lemma 1.8.21, using $\mathrm{I}(K) \subseteq K^\dagger$ and 1-linear surjectivity of K; if $g \in K^\dagger$, then this follows from Lemma 1.2.2 and $\partial K = K$. □

By the next corollary, each Liouville closed H-field is 1-linearly newtonian:

Corollary 1.8.29. *Suppose $K^\dagger = K$. Then the following are equivalent:*

(i) *K is 1-linearly newtonian;*
(ii) *K is* d*-valued and 1-linearly surjective;*
(iii) *K is* d*-valued and $\partial K = K$.*

Proof. Note that K is $\boldsymbol{\lambda}$-free by [ADH, remarks following 11.6.2]. Hence the equivalence of (i) and (ii) follows from Proposition 1.8.28. For the equivalence of (ii) with (iii), see [ADH, example following 5.5.22]. □

Linear newtonianity descends. *In this subsection H is* d*-valued with valuation ring $\mathcal{O}$ and constant field C. Let $r \in \mathbb{N}^{\geqslant 1}$.* If H is $\boldsymbol{\omega}$-free, Γ is divisible, and H has a newtonian algebraic extension $K = H(C_K)$, then H is also newtonian, by [ADH, 14.5.6]. Here is an analogue of this for r-linear newtonianity:

Lemma 1.8.30. *Let $K = H(C_K)$ be an algebraic asymptotic extension of H which is r-linearly newtonian. Then H is r-linearly newtonian.*

Proof. Take a basis B of the C-linear space C_K with $1 \in B$, and let b range over B. We have $H(C_K) = H[C_K]$, and H is linearly disjoint from C_K over C [ADH, 4.6.16], so B is a basis of the H-linear space $H[C_K]$. Let $P \in H\{Y\}$ with $\deg P = 1$ and $\operatorname{order}(P) \leqslant r$ be quasilinear; then P as element of $K\{Y\}$ remains quasilinear, since $\Gamma_K = \Gamma$ by [ADH, 10.5.15]. Let $y \in \mathcal{O}_K$ be a zero of P. Take $y_b \in H$ $(b \in B)$ with $y_b = 0$ for all but finitely many b and $y = \sum_b y_b\, b$. Then $y_b \in \mathcal{O}$ for all b, and

$$0 \;=\; P(y) \;=\; P_0 + P_1(y) \;=\; P_0 + \sum_b P_1(y_b)b,$$

so $P(y_1) = P_0 + P_1(y_1) = 0$. □

Thus if $H[\mathrm{i}]$ with $\mathrm{i}^2 = -1$ is r-linearly newtonian, then H is r-linearly newtonian.

Cases of bounded order. *In the rest of this section $r \in \mathbb{N}^{\geqslant 1}$.* Define K to be **strongly r-newtonian** if K is r-newtonian and for each divergent pc-sequence (a_ρ) in K with minimal differential polynomial $G(Y)$ over K of order $\leqslant r$ we have $\operatorname{ndeg}_{\boldsymbol{a}} G = 1$, where $\boldsymbol{a} := c_K(a_\rho)$. Given $P \in K\{Y\}^{\neq}$, a **K-external zero of P** is an element $\widehat{a}$ of some immediate asymptotic extension $\widehat{K}$ of K such that $P(\widehat{a}) = 0$ and $\widehat{a} \notin K$. Now [ADH, 14.1.11] extends as follows with the same proof:

Lemma 1.8.31. *Suppose K has rational asymptotic integration and K is strongly r-newtonian. Then no $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ can have a K-external zero.*

The following is important in certain inductions on the order.

Lemma 1.8.32. *Suppose K has asymptotic integration, is 1-linearly newtonian, and r-linearly closed. Then K is r-linearly newtonian.*

Proof. Note that K is $\uplambda$-free and d-valued by Proposition 1.8.28. Let $P \in K\{Y\}$ be such that $\operatorname{nmul} P = \deg P = 1$ and $\operatorname{order} P \leqslant r$; by [ADH, 14.2.6] it suffices to show that then P has a zero in $\smallo$. By [ADH, proof of 13.7.10] we can compositionally conjugate, pass to an elementary extension, and multiply by an element of $K^\times$ to arrange that K has small derivation, $P_0 \prec^\flat 1$, and $P_1 \asymp 1$. Let $A := L_P$. The valuation ring of the flattening $(K, v^\flat)$ is 1-linearly surjective by [ADH, 14.2.1], so all operators in $K[\partial]$ of order 1 are neatly surjective in the sense of $(K, v^\flat)$. Since A splits over K, we obtain from [ADH, 5.6.10(ii)] that A is neatly surjective in the sense of $(K, v^\flat)$. As $v^\flat(A) = 0$ and $v^\flat(P_0) > 0$, this gives $y \in K$ with $v^\flat(y) > 0$ such that $P_0 + A(y) = 0$, that is, $P(y) = 0$. $\square$

Using the terminology of K-external zeros, we can add another item to the list of equivalent statements in Proposition 1.8.28:

Lemma 1.8.33. *Suppose K has asymptotic integration. Then we have:*

$$K \textit{ is 1-linearly newtonian} \iff K \textit{ is } \uplambda\textit{-free and no } P \in K\{Y\} \textit{ with } \deg P = 1 \textit{ and } \operatorname{order} P = 1 \textit{ has a } K\textit{-external zero.}$$

Proof. Suppose K is 1-linearly newtonian. Then by (i) $\Rightarrow$ (iii) in Proposition 1.8.28, K is $\uplambda$-free, d-valued, 1-linearly surjective, and $\mathrm{I}(K) \subseteq K^\dagger$. Let $P \in K\{Y\}$ where $\deg P = \operatorname{order} P = 1$ and y in an immediate asymptotic extension L of K with $P(y) = 0$. Then [ADH, 9.1.2] and Corollary 1.2.11 give $L^\dagger \cap K = K^\dagger$, so $y \in K$ by Lemmas 1.2.2 and 1.2.3. This gives the direction $\Rightarrow$. The converse follows from Corollary 1.8.22 and (ii) $\Rightarrow$ (i) in Proposition 1.8.28. $\square$

Here is a higher-order version of Lemma 1.8.33:

Lemma 1.8.34. *Suppose K is $\upomega$-free. Then*

$$K \textit{ is } r\textit{-linearly newtonian} \iff \textit{no } P \in K\{Y\} \textit{ with } \deg P = 1 \textit{ and } \operatorname{order} P \leqslant r \textit{ has a } K\textit{-external zero.}$$

Proof. Suppose K is r-linearly newtonian. Then K is d-valued by Lemma 1.2.9. Let $P \in K\{Y\}$ be of degree 1 and order $\leqslant r$, and let y be in an immediate asymptotic extension L of K with $P(y) = 0$. Then $A(y) = b$ for $A := L_P \in K[\partial]$, $b := -P(0) \in K$. By [ADH, 14.2.2] there is also a $z \in K$ with $A(z) = b$, hence $y - z \in \ker_L A = \ker A$ by [ADH, remarks after 14.2.9] and so $y \in K$. This gives the direction $\Rightarrow$. For the converse note that every quasilinear $P \in K\{Y\}$ has a zero $\widehat{a} \preccurlyeq 1$ in an immediate asymptotic extension of K by [ADH, 14.0.1 and subsequent remarks]. $\square$

We also have the following r-version of [ADH, 14.0.1]:

Proposition 1.8.35. *If K is $\uplambda$-free and no $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero, then K is $\upomega$-free and r-newtonian.*

Proof. The $\upomega$-freeness follows as before from [ADH, 11.7.13]. The rest of the proof is as in [ADH, p. 653] with P restricted to have order $\leqslant r$. $\square$

Application to solving asymptotic equations. *Here K is* d*-valued,* $\upomega$*-free, with small derivation, and $\mathfrak{M}$ is a monomial group of K.* We let a, b, y range over K. In addition we fix a $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ and a $\preccurlyeq$-closed set $\mathcal{E} \subseteq K^{\times}$. (Recall that $r \geqslant 1$.) This gives the asymptotic equation

$$P(Y) = 0, \qquad Y \in \mathcal{E}. \tag{E}$$

This gives the following r-version of [ADH, 13.8.8], with basically the same proof:

Proposition 1.8.36. *Suppose Γ is divisible, no $Q \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero, $d := \operatorname{ndeg}_{\mathcal{E}} P \geqslant 1$, and there is no $f \in \mathcal{E} \cup \{0\}$ with $\operatorname{mul} P_{+f} = d$. Then* (E) *has an unraveler.*

Here is an r-version of [ADH, 14.3.4] with the same proof:

Lemma 1.8.37. *Suppose K is r-newtonian. Let $g \in K^{\times}$ be an approximate zero of P with $\operatorname{ndeg} P_{\times g} = 1$. Then there exists $y \sim g$ such that $P(y) = 0$.*

For the next three results we assume the following:

C is algebraically closed, Γ is divisible, and no $Q \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero.

These three results are r-versions of [ADH, 14.3.5, 14.3.6, 14.3.7] with the same proofs, using Propositions 1.8.35 and 1.8.36 instead of [ADH, 14.0.1, 13.8.8]:

Proposition 1.8.38. *If $\operatorname{ndeg}_{\mathcal{E}} P > \operatorname{mul}(P) = 0$, then* (E) *has a solution.*

Corollary 1.8.39. *K is weakly r-differentially closed.*

Corollary 1.8.40. *Suppose $g \in K^{\times}$ is an approximate zero of P. Then $P(y) = 0$ for some $y \sim g$.*

A useful equivalence. *Suppose K is* $\upomega$*-free.* (No small derivation or monomial group assumed.) Recall that $r \geqslant 1$. Here is an r-version of [159, 3.4]:

Corollary 1.8.41. *The following are equivalent:*

(i) *K is r-newtonian;*
(ii) *K is strongly r-newtonian;*
(iii) *no $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero.*

Proof. Since K is $\upomega$-free it has rational asymptotic integration [ADH, p. 515]. Also, if K is 1-newtonian, then K is henselian [ADH, p. 645] and d-valued [ADH, 14.2.5]. For (i) $\Rightarrow$ (ii), use [159, 3.3], for (ii) $\Rightarrow$ (iii), use Lemma 1.8.31, and for (iii) $\Rightarrow$ (i), use Proposition 1.8.35. $\square$

Next an r-version of [ADH, 14.5.3]:

Corollary 1.8.42. *Suppose K is r-newtonian, Γ is divisible, and C is algebraically closed. Then K is weakly r-differentially closed, so K is $(r+1)$-linearly closed and thus $(r+1)$-linearly newtonian.*

Proof. To show that K is weakly r-differentially closed we arrange by compositional conjugation and passing to a suitable elementary extension that K has small derivation and K has a monomial group. Then K is weakly r-differentially closed by Corollaries 1.8.39 and 1.8.41. The rest uses [ADH, 5.8.9] and Lemma 1.8.32. $\square$

Complementing [ADH, 14.2.12] (*)**.** In this subsection $P(Y) \in \mathcal{O}\{Y\}$ has order at most $r \in \mathbb{N}^{\geqslant 1}$.

Lemma 1.8.43. *Let $y \in K^\times$, $y' \preccurlyeq y \prec 1$, and $P(0) = P(y)$. Then $L_P(y) \prec y$.*

Proof. Induction on n gives $y^{(n)} \preccurlyeq y^{(n-1)} \preccurlyeq \cdots \preccurlyeq y \prec 1$ for all n. Hence if $\boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}$, $|\boldsymbol{i}| \geqslant 2$, then $y^{\boldsymbol{i}} = y^{i_0}(y')^{i_1} \cdots (y^{(r)})^{i_r} \preccurlyeq y^{|\boldsymbol{i}|} \prec y$. Now

$$P(y) \ = \ P(0) + L_P(y) + \sum_{|\boldsymbol{i}| \geqslant 2} P_{\boldsymbol{i}}\, y^{\boldsymbol{i}},$$

so $L_P(y) + \sum_{|\boldsymbol{i}|\geqslant 2} P_{\boldsymbol{i}}\, y^{\boldsymbol{i}} = 0$, and thus $L_P(y) \prec y$. ☐

We extend the residue map $a \mapsto \operatorname{res} a \colon \mathcal{O} \to \boldsymbol{k} := \operatorname{res}(K)$ to the ring morphism

$$p \mapsto \operatorname{res} p \ : \ \mathcal{O}[Y] \to \boldsymbol{k}[Y], \qquad Y \mapsto Y.$$

For $w \in \mathbb{N}$ we let $P_{[w]}$ be the isobaric part of P of weight w, as in [ADH, 4.2]. Thus $p := P_{[0]} \in \mathcal{O}[Y]$.

Corollary 1.8.44. *Suppose the derivation of K is very small, and let $a \in \mathcal{O}$, $y \in \smallo$ with $P(a) = P(a+y)$ and $(\operatorname{res} p)'(\operatorname{res} a) \neq 0$. Then $y' \succcurlyeq y$.*

Proof. Put $R := \sum_{w \geqslant 1} P_{[w]} = P - p$. Now $a^{(n)} \prec 1$ for all $n \geqslant 1$, so $\big(\frac{\partial R}{\partial Y}\big)(a) \prec 1$. Towards a contradiction, assume $y' \prec y$. Then $L_{P_{+a}}(y) \prec y$ by Lemma 1.8.43 applied to P_{+a} in place of P. Induction on n gives $y^{(n)} \prec y^{(n-1)} \prec \cdots \prec y \prec 1$ for all n and so $L_{R_{+a}}(y) = \sum_n \big(\frac{\partial R}{\partial Y^{(n)}}\big)(a) y^{(n)} \prec y$. Together with $L_{P_{+a}}(y) = p'(a)y + L_{R_{+a}}(y)$ and $p'(a) \asymp 1$ this yields the desired contradiction. ☐

In the next corollary we assume that K has asymptotic integration. We let ϕ range over active elements of K, and we let $\smallo_\phi^\flat = \{f \in \smallo : f' \succcurlyeq f\phi\}$ be the maximal ideal of the flattened valuation ring of K^ϕ; cf. [ADH, pp. 406–407].

Corollary 1.8.45. *Suppose K is r-newtonian. Let $u \in \mathcal{O}$ and $A \in \boldsymbol{k}[Y]$ be such that $A(\operatorname{res} u) = 0$, $A'(\operatorname{res} u) \neq 0$, and $D_{P^\phi} \in \boldsymbol{k}^\times A$, eventually. Then P has a zero in $u + \smallo$, and for all zeros $a, b \in u + \smallo$ of P we have: $a - b \in \smallo_\phi^\flat$, eventually.*

Proof. For the first claim, see [ADH, 14.2.12]. Suppose $D_{P^\phi} \in \boldsymbol{k}^\times A$ and take $\mathfrak{m} \in K^\times$ with $\mathfrak{m} \asymp P^\phi$, so $Q := \mathfrak{m}^{-1} P^\phi \in K^\phi\{Y\}$ and $q := Q_{[0]}$ satisfy $vQ = 0$ and $\operatorname{res} q \in \boldsymbol{k}^\times A$. Note that K is d-valued by [ADH, 14.2.5]; hence K^ϕ has very small derivation. Let $a, b \in u + \smallo$ and $P(a) = P(b) = 0$; then $y := b - a \in \smallo$, and so Corollary 1.8.44 applied to K^ϕ, Q in place of K, P yields $y' \succcurlyeq y\phi$. ☐

Newton polynomials of Riccati transforms (*)**.** *In this subsection we assume that K has small derivation and asymptotic integration.* Let

$$A \ = \ a_0 + a_1\partial + \cdots + a_r\partial^r \in K[\partial] \qquad \text{where } a_0, \dots, a_r \in K,\ a_r \neq 0,$$

with Riccati transform

$$R \ := \ \operatorname{Ri}(A) \ = \ a_0 R_0(Z) + a_1 R_1(Z) + \cdots + a_r R_r(Z) \in K\{Z\},$$

and set

$$P \ := \ a_0 + a_1 Z + \cdots + a_r Z^r \in K[Z].$$

We equip the differential fraction field $K\langle Z\rangle$ of $K\{Z\}$ with the gaussian extension of the valuation of K and likewise with K^ϕ instead of K. Then $K^\phi\langle Z\rangle$ is a valued differential field with small derivation by [ADH, 6.3]. (Although K is asymptotic, $K\langle Z\rangle$ is not, by [ADH, 9.4.6].)

Lemma 1.8.46. *Eventually, $R^\phi \sim P$.*

Proof. It is enough to show that $R_n(Z)^\phi \sim Z^n$ eventually. For $n = 0, 1$ we have $R_n(Z) = Z^n$. Now $R_{n+1}(Z) = ZR_n(Z) + \partial\big(R_n(Z)\big)$, so by [ADH, 5.7.1],

$$R_{n+1}(Z)^\phi \;=\; ZR_n(Z)^\phi + \phi\delta\big(R_n(Z)^\phi\big), \qquad \delta \;:=\; \phi^{-1}\partial.$$

Assuming $R_n(Z)^\phi \sim Z^n$ eventually, this yields $R_{n+1}(Z)^\phi \sim Z^{n+1}$ eventually . $\square$

Remark. Suppose K is d-valued and equipped with a monomial group. In [ADH, 13.0.1] we associate to $Q \in K\{Z\}^{\neq}$ its Newton polynomial $N_Q \in C\{Z\}$ such that $D_{Q^\phi} = N_Q$, eventually. Then $N_R = D_P \in C[Z]$ by Lemma 1.8.46.

Next an application of Lemma 1.8.46. For simplicity, assume $vA = 0$, so $P \in \mathcal{O}[Z]$, $vP = 0$. We also let $Q \mapsto \operatorname{res} Q\colon \mathcal{O}[Z] \to \boldsymbol{k}[Z]$ be the extension of the residue map $a \mapsto \operatorname{res} a\colon \mathcal{O} \to \boldsymbol{k}$ to a ring morphism with $Z \mapsto Z$.

Corollary 1.8.47. *Suppose K is $(r-1)$-newtonian, $r \geqslant 1$. Then for all $\alpha \in \boldsymbol{k}$ with $\operatorname{res} P(\alpha) = 0$ and $(\operatorname{res} P)'(\alpha) \neq 0$ there is $a \in \mathcal{O}$ with $R(a) = 0$ and $\operatorname{res} a = \alpha$.*

Proof. If $r = 1$, use $R = P = a_0 + a_1 Z$. Assume $r \geqslant 2$. By Lemma 1.8.46 we have $D_{R^\phi} \in \boldsymbol{k}^\times \cdot \operatorname{res} P$, eventually, so we can apply [ADH, 14.2.12] to R, $\operatorname{res} P$ in the role of P, A there. $\square$

In the rest of this subsection we assume $A \in \mathcal{O}[\partial]$ is monic. To what extent is the zero a of R in Corollary 1.8.47 unique? Corollaries 1.8.49 and 1.8.50 below give answers to this question.

Lemma 1.8.48. *Let $a, b \in \mathcal{O}$ be such that $R(a) = R(b) = 0$ and $y := b - a \prec 1$. Then $y' \preccurlyeq y$.*

Proof. Replace R by R_{+a} to arrange $a = 0$, $b = y$, so $a_0 = 0$. Note that $r \geqslant 1$. Towards a contradiction, assume $y \prec y'$. Then $R_n(y) \sim y^{(n-1)}$ for all $n \geqslant 1$ by Lemma 1.1.21, and $y \prec y' \prec \cdots \prec y^{(r-1)}$, so

$$R(y) \;=\; a_1 R_1(y) + \cdots + a_{r-1} R_{r-1}(y) + R_r(y) \;\sim\; y^{(r-1)},$$

hence $R(y) \neq 0$, a contradiction. $\square$

Corollary 1.8.49. *Suppose K has very small derivation, $\alpha \in \boldsymbol{k}$ is a simple zero of $\operatorname{res} P$, and $a, b \in \mathcal{O}$, $R(a) = R(b) = 0$, and $\operatorname{res} a = \operatorname{res} b = \alpha$. Then for $y := b - a$ we have $y' \asymp y$.*

Proof. We have $y' \preccurlyeq y$ by Lemma 1.8.48, and $y' \succcurlyeq y$ by Corollary 1.8.44 applied to R in place of P, using $R_{[0]} = P$. $\square$

In the next result we assume $K = H[i]$ where H is a real closed differential subfield of K such that the valuation ring $\mathcal{O}_H := \mathcal{O} \cap H$ of H is convex with respect to the ordering of H and $\mathcal{O}_H = C_H + o_H$. So $C = C_H[i]$ and $\mathcal{O} = C + o$ (cf. remarks after Corollary 1.2.5). We identify C with $\boldsymbol{k}$ via the residue morphism $\mathcal{O} \to \boldsymbol{k}$.

Corollary 1.8.50. *Let $\alpha \in C$ be a simple zero of $\operatorname{res} P \in C[Z]$ such that for all zeros $\beta \in C$ of $\operatorname{res} P$ we have $\operatorname{Re} \alpha \leqslant \operatorname{Re} \beta$. Then there is at most one $a \in \mathcal{O}$ with $R(a) = 0$ and $\operatorname{res} a = \alpha$.*

Proof. Let $a \in \mathcal{O}$, $R(a) = 0$, and $\operatorname{res} a = \alpha$. Towards a contradiction, suppose $b \in \mathcal{O}$, $b \neq a$, $R(b) = 0$, and $\operatorname{res} b = \alpha$. By Lemma 1.1.27 we may replace a, R by 0, R_{+a} to arrange $a = 0$. Then $0 \neq b \prec 1$ and $a_0 = R(0) = 0$, so $P = QZ$ where $Q \in \mathcal{O}[Z]$ and all zeros of $\operatorname{res} Q$ in C have nonnegative real part. Moreover $b' \asymp b$ by Corollary 1.8.49. Take $c \in C^\times$ with $b' \sim bc$. By Lemma 1.1.21 and [ADH, 9.1.4(ii)] we get $R_n(b) \sim b^{(n-1)} \sim bc^{n-1}$ for $n \geqslant 1$. Now $R = a_1 R_1 + \cdots + a_r R_r$ gives $Q = a_1 + \cdots + a_r Z^{r-1}$, so $R(b) \in b \cdot \big(Q(c) + \smallo\big)$, hence $(\operatorname{res} Q)(c) = 0$ in view of $R(b) = 0$, and thus $\operatorname{Re} c \geqslant 0$. On the other hand, $b \prec 1$ and Corollary 1.2.6 give $\operatorname{Re}(b^\dagger) < 0$, so $\operatorname{Re} c < 0$, a contradiction. $\square$

Part 2. The Universal Exponential Extension

Let K be an algebraically closed differential field. In Section 2.2 below we extend K in a canonical way to a differential integral domain $\mathrm{U} = \mathrm{U}_K$ whose differential fraction field has the same constant field C as K, called the *universal exponential extension* of K. (The universal exponential extension of $\mathbb{T}[i]$ appeared in [103] in the guise of "oscillating transseries"; we explain the connection at the end of Section 2.5.) The underlying ring of U is a group ring of a certain abelian group over K, and we therefore first review some relevant basic facts about such group rings in Section 2.1. The main feature of U is that if K is 1-linearly surjective, then each $A \in K[\partial]$ of order $r \in \mathbb{N}$ which splits over K has r many C-linearly independent zeros in U. This is explained in Section 2.5, after some differential-algebraic preliminaries in Sections 2.3 and 2.4, where we consider a novel kind of *spectrum* of a linear differential operator over a differential field. In Section 2.6 we introduce for H-asymptotic K with small derivation and asymptotic integration the *ultimate exceptional values* of a given linear differential operator $A \in K[\partial]^{\neq}$. These help to isolate the zeros of A in U much like the exceptional values of A help to locate the zeros of A in immediate asymptotic extensions of K as in Section 1.5. In Section 5.10 below we discuss the analytic meaning of U when K is the algebraic closure of a Liouville closed Hardy field containing $\mathbb{R}$ as a subfield.

2.1. Some Facts about Group Rings

In this section G is a torsion-free abelian group, written multiplicatively, K is a field, and γ, δ range over G. For use in Section 2.2 below we recall some facts about the group ring $K[G]$: a commutative K-algebra with $1 \neq 0$ that contains G as a subgroup of its multiplicative group $K[G]^\times$ and which, as a K-linear space, decomposes as

$$K[G] \;=\; \bigoplus_\gamma K\gamma \qquad \text{(internal direct sum).}$$

Hence for any $f \in K[G]$ we have a unique family (f_γ) of elements of K, with $f_\gamma = 0$ for all but finitely many γ, such that

$$f \;=\; \sum_\gamma f_\gamma \gamma. \tag{2.1.1}$$

We define the support of $f \in K[G]$ as above by

$$\operatorname{supp}(f) \;:=\; \{\gamma : f_\gamma \neq 0\} \;\subseteq\; G.$$

In the rest of this section f, g, h range over $K[G]$. For any K-algebra R, every group morphism $G \to R^\times$ extends uniquely to a K-algebra morphism $K[G] \to R$.

Clearly $K[G]^\times \supseteq K^\times G$; in fact:

Lemma 2.1.1. *The ring $K[G]$ is an integral domain and $K[G]^\times = K^\times G$.*

Proof. We take an ordering of G making G into an ordered abelian group; see [ADH, 2.4]. Let $f, g \neq 0$ and set

$$\gamma^- := \min\operatorname{supp}(f),\ \ \gamma^+ := \max\operatorname{supp}(f), \quad \delta^- := \min\operatorname{supp}(g),\ \ \delta^+ := \max\operatorname{supp}(g);$$

so $\gamma^- \leqslant \gamma^+$ and $\delta^- \leqslant \delta^+$. We have $(fg)_{\gamma^-\delta^-} = f_{\gamma^-}g_{\delta^-} \neq 0$, and likewise with γ^+, δ^+ in place of γ^-, δ^-. In particular, $fg \neq 0$, showing that $K[G]$ is an integral domain. Now suppose $fg = 1$. Then $\operatorname{supp}(fg) = \{1\}$, hence $\gamma^-\delta^- = 1 = \gamma^+\delta^+$, so $\gamma^- = \gamma^+$, and thus $f \in K^\times G$. $\square$

Lemma 2.1.2. *Suppose K has characteristic 0 and $G \neq \{1\}$. Then the fraction field Ω of $K[G]$ is not algebraically closed.*

Proof. Let $\gamma \in G\setminus\{1\}$ and $n \geqslant 1$. We claim that there is no $y \in \Omega$ with $y^2 = 1-\gamma^n$. For this, first replace G by its divisible hull to arrange that G is divisible. Towards a contradiction, suppose $f, g \in K[G]^{\neq}$ and $f^2 = g^2(1-\gamma^n)$. Take a divisible subgroup H of G that is complementary to the smallest divisible subgroup $\gamma^{\mathbb{Q}}$ of G containing γ, so $G = H\gamma^{\mathbb{Q}}$ and $G \cap \gamma^{\mathbb{Q}} = \{1\}$. Then $K[G] \subseteq K(H)[\gamma^{\mathbb{Q}}]$ (inside Ω), so we may replace K, G by $K(H)$, $\gamma^{\mathbb{Q}}$ to arrange $G = \gamma^{\mathbb{Q}}$. For suitable $m \geqslant 1$ we apply the K-algebra automorphism of $K[G]$ given by $\gamma \mapsto \gamma^m$ to arrange $f, g \in K[\gamma, \gamma^{-1}]$ (replacing n by mn). Then replace f, g by $\gamma^m f$, $\gamma^m g$ for suitable $m \geqslant 1$ to arrange $f, g \in K[\gamma]$. Now use that $1-\gamma$ is a prime divisor of $1-\gamma^n$ of multiplicity 1 in the UFD $K[\gamma]$ to get a contradiction. $\square$

The K-linear map
$$f \mapsto \operatorname{tr}(f) := f_1 \ :\ K[G] \to K$$
is called the **trace** of $K[G]$. Thus
$$\operatorname{tr}(fg) \ = \ \sum_\gamma f_\gamma g_{\gamma^{-1}}.$$
We claim that $\operatorname{tr} \circ \sigma = \operatorname{tr}$ for every automorphism σ of the K-algebra $K[G]$. This invariance comes from an intrinsic description of $\operatorname{tr}(f)$ as follows: given f we have a unique finite set $U \subseteq K[G]^\times = K^\times G$ such that $f = \sum_{u \in U} u$ and $u_1/u_2 \notin K^\times$ for all distinct $u_1, u_2 \in U$; if $U \cap K^\times = \{c\}$, then $\operatorname{tr}(f) = c$; if $U \cap K^\times = \emptyset$, then $\operatorname{tr}(f) = 0$. If G_0 is a subgroup of G and K_0 is a subfield of K, then $K_0[G_0]$ is a subring of $K[G]$, and the trace of $K[G]$ extends the trace of $K_0[G_0]$.

The automorphisms of $K[G]$. For a commutative group H, written multiplicatively, $\operatorname{Hom}(G, H)$ denotes the set of group morphisms $G \to H$, made into a group by pointwise multiplication. Any $\chi \in \operatorname{Hom}(G, K^\times)$—sometimes called a *character*—gives a K-algebra automorphism $f \mapsto f_\chi$ of $K[G]$ defined by

$$f_\chi \ := \ \sum_\gamma f_\gamma \chi(\gamma)\gamma. \tag{2.1.2}$$

This yields a group action of $\operatorname{Hom}(G, K^\times)$ on $K[G]$ by K-algebra automorphisms:
$$\operatorname{Hom}(G, K^\times) \times K[G] \to K[G], \qquad (\chi, f) \mapsto f_\chi.$$
Sending $\chi \in \operatorname{Hom}(G, K^\times)$ to $f \mapsto f_\chi$ yields an embedding of the group $\operatorname{Hom}(G, K^\times)$ into the group $\operatorname{Aut}(K[G]|K)$ of automorphisms of the K-algebra $K[G]$; its image is the (commutative) subgroup of $\operatorname{Aut}(K[G]|K)$ consisting of the K-algebra automorphisms σ of $K[G]$ such that $\sigma(\gamma)/\gamma \in K^\times$ for all γ. Identify $\operatorname{Hom}(G, K^\times)$ with its image under this embedding. From $K[G]^\times = K^\times G$ we obtain $\sigma(K^\times G) = K^\times G$ for all $\sigma \in \operatorname{Aut}(K[G]|K)$, and using this one verifies easily that $\operatorname{Hom}(G, K^\times)$ is a normal subgroup of $\operatorname{Aut}(K[G]|K)$. We also have the group embedding
$$\operatorname{Aut}(G) \ \to \ \operatorname{Aut}(K[G]|K)$$

assigning to each $\sigma \in \mathrm{Aut}(G)$ the unique automorphism of the K-algebra $K[G]$ extending σ. Identifying $\mathrm{Aut}(G)$ with its image in $\mathrm{Aut}(K[G]|K)$ via this embedding we have $\mathrm{Hom}(G, K^\times) \cap \mathrm{Aut}(G) = \{\mathrm{id}\}$ and $\mathrm{Hom}(G, K^\times) \cdot \mathrm{Aut}(G) = \mathrm{Aut}(K[G], |K)$ inside $\mathrm{Aut}(K[G]|K)$, and thus $\mathrm{Aut}(K[G]|K) = \mathrm{Hom}(G, K^\times) \rtimes \mathrm{Aut}(G)$, an internal semidirect product of subgroups of $\mathrm{Aut}(K[G]|K)$.

The gaussian extension. *In this subsection $v\colon K^\times \to \Gamma$ is a valuation on the field K.* We extend v to a map $v_{\mathrm{g}}\colon K[G]^{\neq} \to \Gamma$ by setting

$$(2.1.3) \qquad v_{\mathrm{g}} f \ := \ \min_\gamma v f_\gamma \qquad (f \in K[G]^{\neq} \text{ as in (2.1.1)}).$$

Proposition 2.1.3. *The map $v_{\mathrm{g}}\colon K[G]^{\neq} \to \Gamma$ is a valuation on the domain $K[G]$.*

Proof. We can reduce to the case that G is finitely generated, since $K[G]$ is the directed union of its subrings $K[G_0]$ with G_0 a finitely generated subgroup of G. Then we have a group isomorphism $G \to \mathbb{Z}^n$ inducing a K-algebra isomorphism $K[G] \to K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ (with distinct indeterminates $X_1, \dots, X_n$) under which v_{g} corresponds to the gaussian extension of the valuation of K to $K(X)$ restricted to its subring $K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$; see [ADH, Section 3.1]. $\square$

We call v_{g} the **gaussian extension** of the valuation of K to $K[G]$. We denote by $\preccurlyeq_{\mathrm{g}}$ the dominance relation on $\Omega := \mathrm{Frac}(K[G])$ associated to the extension of v_{g} to a valuation on the field Ω [ADH, (3.1.1)], with corresponding asymptotic relations $\asymp_{\mathrm{g}}$ and $\prec_{\mathrm{g}}$. For the subring $\mathcal{O}[G]$ of $K[G]$ generated by G over $\mathcal{O}$ we have

$$\mathcal{O}[G] \ = \ \{f : f \preccurlyeq_{\mathrm{g}} 1\}.$$

The residue morphism $\mathcal{O} \to \boldsymbol{k} := \mathcal{O}/\smallo$ extends to a surjective ring morphism $\mathcal{O}[G] \to \boldsymbol{k}[G]$ with $\gamma \mapsto \gamma$ for all γ and whose kernel is the ideal

$$\smallo[G] := \ \{f : f \prec_{\mathrm{g}} 1\}$$

of $\mathcal{O}[G]$. Hence this ring morphism induces an isomorphism $\mathcal{O}[G]/\smallo[G] \cong \boldsymbol{k}[G]$. If G_0 is subgroup of G and K_0 is a valued subfield of K, then the restriction of v_{g} to a valuation on $K_0[G_0]$ is the gaussian extension of the valuation of K_0 to $K_0[G_0]$.

An inner product and two norms. *In the rest of this section H is a real closed subfield of K such that $K = H[\mathrm{i}]$ where $\mathrm{i}^2 = -1$.* In later use H will be a Hardy field, which is why we use the letter H here. Note that the only nontrivial automorphism of the (algebraically closed) field K over H is *complex conjugation*:

$$z = a + b\mathrm{i} \mapsto \overline{z} := a - b\mathrm{i} \qquad (a, b \in H).$$

For f as in (2.1.1) we set

$$f^* \ := \ \sum_\gamma \overline{f_\gamma}\gamma^{-1},$$

so $(f^*)^* = f$, and $f \mapsto f^*$ lies in $\mathrm{Aut}\big(K[G]|H\big)$. We define the function

$$(f, g) \mapsto \langle f, g \rangle \ : \ K[G] \times K[G] \to K$$

by

$$\langle f, g \rangle \ := \ \mathrm{tr}\big(fg^*\big) \ = \ \sum_\gamma f_\gamma \overline{g_\gamma}.$$

One verifies easily that this is a "positive definite hermitian form" on the K-linear space $K[G]$: it is additive on the left and on the right, and for all f, g and all $\lambda \in K$: $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$, $\langle g, f \rangle = \overline{\langle f, g \rangle}$, $\langle f, f \rangle \in H^{\geqslant}$, and $\langle f, f \rangle = 0 \Leftrightarrow f = 0$, and thus also $\langle f, \lambda g\rangle = \overline{\lambda}\langle f, g\rangle$. (Hermitian forms are usually defined only on $\mathbb{C}$-linear spaces and are $\mathbb{C}$-valued, which is why we used quote marks, as we do below for *norm* and *orthonormal basis*; see [122, Chapter XV, §5] for the more general case.) Note:

$$\langle f, gh\rangle \;=\; \operatorname{tr}\big(f(gh)^*\big) \;=\; \big\langle fg^*, h\big\rangle.$$

Lemma 2.1.4. *Let $u, w \in K[G]^\times$. If $u \notin K^\times w$, then $\langle u, w\rangle = 0$, and if $u \in K^\times w$, then $\langle u, w\rangle = uw^*$.*

Proof. Take $a, b \in K^\times$ and γ, δ such that $u = a\gamma$, $w = b\delta$. If $u \notin K^\times w$, then $\gamma \neq \delta$, so $\langle u, w\rangle = 0$. If $u \in K^\times w$, then $\gamma = \delta$, hence $\langle u, w\rangle = a\overline{b} = uw^*$. □

For $z \in K$ we set $|z| := \sqrt{z\overline{z}} \in H^{\geqslant}$, and then define $\|\cdot\|\colon K[G] \to H^{\geqslant}$ by

$$\|f\|^2 \;=\; \langle f, f\rangle \;=\; \sum_\gamma |f_\gamma|^2.$$

As in the case $H = \mathbb{R}$ and $K = \mathbb{C}$ one derives the Cauchy-Schwarz Inequality:

$$|\langle f, g\rangle| \;\leqslant\; \|f\| \cdot \|g\|.$$

Thus $\|\cdot\|$ is a "norm" on the K-linear space $K[G]$: for all f, g and all $\lambda \in K$,

$$\|f + g\| \leqslant \|f\| + \|g\|, \quad \|\lambda f\| = |\lambda| \cdot \|f\|, \quad \|f\| = 0 \Leftrightarrow f = 0.$$

Note that G is an "orthonormal basis" of $K[G]$ with respect to $\langle\ ,\ \rangle$, and $f_\gamma = \langle f, \gamma\rangle$. We also use the function $\|\cdot\|_1\colon K[G] \to H^{\geqslant}$ given by

$$\|f\|_1 \;:=\; \sum_\gamma |f_\gamma|,$$

which is a "norm" on $K[G]$ in the sense of obeying the same laws as we mentioned for $\|\cdot\|$. The two "norms" are in some sense equivalent:

$$\|f\| \;\leqslant\; \|f\|_1 \;\leqslant\; \sqrt{n}\|f\| \qquad (n := |\operatorname{supp}(f)|).$$

where the first inequality follows from the triangle inequality for $\|\cdot\|$ and the second is of Cauchy-Schwarz type. Moreover:

Lemma 2.1.5. *Let $u \in K[G]^\times$. Then $\|fu\| = \|f\|\,\|u\|$ and $\|fu\|_1 = \|f\|_1\,\|u\|_1$.*

Proof. We have

$$\|f\gamma\| \;=\; \langle f\gamma, f\gamma\rangle \;=\; \big\langle f\gamma\gamma^*, f\big\rangle \;=\; \big\langle f, f\big\rangle \;=\; \|f\|$$

using $\gamma^* = \gamma^{-1}$. Together with $K[G]^\times = K^\times G$ this yields the first claim; the second claim follows easily from the definition of $\|\cdot\|_1$. □

Corollary 2.1.6. $\|fg\| \leqslant \|f\| \cdot \|g\|_1$ *and* $\|fg\|_1 \leqslant \|f\|_1 \cdot \|g\|_1$.

Proof. By the triangle inequality for $\|\cdot\|$ and the previous lemma,

$$\|fg\| \;\leqslant\; \sum_\gamma \|fg_\gamma\gamma\| \;=\; \sum_\gamma \|f\|\,\|g_\gamma\gamma\| \;=\; \|f\|\sum_\gamma |g_\gamma| \;=\; \|f\|\,\|g\|_1.$$

The inequality involving $\|fg\|_1$ follows likewise. □

In the next lemma we let $\chi \in \operatorname{Hom}(G, K^\times)$; recall from (2.1.2) the automorphism $f \mapsto f_\chi$ of the K-algebra $K[G]$.

Lemma 2.1.7. $(f_\chi)^* = (f^*)_\chi$ *iff* $|\chi(\gamma)| = 1$ *for all* $\gamma \in \operatorname{supp}(f)$.

Proof. Let $a \in K$; then $\big((a\gamma)_\chi\big)^* = \overline{a\chi(\gamma)}\gamma^{-1}$ and $\big((a\gamma)^*\big)_\chi = \overline{a}\chi(\gamma)^{-1}\gamma^{-1}$. □

Corollary 2.1.8. *Let $\chi \in \operatorname{Hom}(G, K^\times)$ with $|\chi(\gamma)| = 1$ for all γ. Then $\langle f_\chi, g_\chi\rangle = \langle f, g\rangle$ for all f, g, and hence $\|f_\chi\| = \|f\|$ for all f.*

Proof. Since $\operatorname{tr}\circ\sigma = \operatorname{tr}$ for every automorphism σ of the K-algebra $K[G]$,

$$\langle f_\chi, g_\chi\rangle \;=\; \operatorname{tr}\big(f_\chi (g_\chi)^*\big) \;=\; \operatorname{tr}\big((fg^*)_\chi\big) \;=\; \operatorname{tr}(fg^*) \;=\; \langle f, g\rangle,$$

where we use Lemma 2.1.7 for the second equality. □

Valuation and norm. Let $v\colon H^\times \to \Gamma$ be a convex valuation on the ordered field H, extended uniquely to a valuation $v\colon K^\times \to \Gamma$ on the field $K = H[\mathrm{i}]$, so $a \asymp |a|$ for $a \in K$. (See the remarks before Corollary 1.2.6.) Let $v_{\mathrm{g}}\colon K[G]^{\neq} \to \Gamma$ be the gaussian extension of v, given by (2.1.3).

Lemma 2.1.9. $\|f\|_1 \preccurlyeq 1 \Leftrightarrow f \preccurlyeq_{\mathrm{g}} 1$, *and* $\|f\|_1 \prec 1 \Leftrightarrow f \prec_{\mathrm{g}} 1$.

Proof. Using that the valuation ring of H is convex we have

$$\|f\|_1 = \sum_\gamma |f_\gamma| \preccurlyeq 1 \iff |f_\gamma| \preccurlyeq 1 \text{ for all } \gamma \iff f_\gamma \preccurlyeq 1 \text{ for all } \gamma \iff f \preccurlyeq_{\mathrm{g}} 1.$$

Likewise one shows: $\|f\|_1 \prec 1 \Leftrightarrow f \prec_{\mathrm{g}} 1$. □

Corollary 2.1.10. $\|f\| \asymp \|f\|_1 \asymp_{\mathrm{g}} f$.

Proof. This is trivial for $f = 0$, so assume $f \neq 0$. Take $a \in H^>$ with $a \asymp_{\mathrm{g}} f$, and replace f by f/a, to arrange $f \asymp_{\mathrm{g}} 1$. Then $\|f\| \asymp \|f\|_1 \asymp_{\mathrm{g}} 1$ by Lemma 2.1.9. □

2.2. The Universal Exponential Extension

As in [ADH, 5.9], given a differential ring K, a *differential K-algebra* is a differential ring R with a morphism $K \to R$ of differential rings. If R is a differential ring extension of a differential ring K we consider R as a differential K-algebra via the inclusion $K \to R$.

Exponential extensions. *In this subsection R is a differential ring and K is a differential subring of R.* Call $a \in R$ **exponential over** K if $a' \in aK$. Note that if $a \in R$ is exponential over K, then $K[a]$ is a differential subring of R. If $a \in R$ is exponential over K and $\phi \in K^\times$, then a, as element of the differential ring extension R^ϕ of K^ϕ, is exponential over K^ϕ. Every $c \in C_R$ is exponential over K, and every $u \in K^\times$ is exponential over K. If $a, b \in R$ are exponential over K, then so is ab, and if $a \in R^\times$ is exponential over K, then so is a^{-1}. Hence the units of R that are exponential over K form a subgroup E of the group $R^\times$ of units of R with $E \supseteq C_R^\times \cdot K^\times$; if $R = K[E]$, then we call R **exponential over** K. An **exponential extension of** K is a differential ring extension of K that is exponential over K. If $R = K[E]$ where E is a set of elements of $R^\times$ which are exponential over K, then R is exponential over K. If R is an exponential extension of K and $\phi \in K^\times$, then R^ϕ is an exponential extension of K^ϕ. The following lemma is extracted from the proof of [168, Theorem 1]:

Lemma 2.2.1 (Rosenlicht)**.** *Suppose K is a field and R is an integral domain with differential fraction field F. Let $I \neq R$ be a differential ideal of R, and let $u_1, \dots, u_n \in R^\times$ $(n \geqslant 1)$ be exponential over K with $u_i \notin u_j C_F^\times K^\times$ for $i \neq j$. Then $\sum_i u_i \notin I$.*

Proof. Suppose $u_1, \dots, u_n$ is a counterexample with minimal $n \geqslant 1$. Then $n \geqslant 2$ and $\sum_i u_i' \in I$, so

$$\sum_i u_i' - u_1^\dagger \sum_i u_i \;=\; \sum_{i>1} (u_i/u_1)^\dagger u_i \in I.$$

Hence $(u_i/u_1)^\dagger = 0$ and thus $u_i/u_1 \in C_F^\times$, for all $i > 1$, a contradiction. $\square$

Corollary 2.2.2. *Suppose K is a field and $F = K(E)$ is a differential field extension of K with $C_F = C$, where E is a subgroup of $F^\times$ whose elements are exponential over K. Then $\{y \in F^\times : y \text{ is exponential over } K\} = K^\times E$.*

Proof. Let $y \in F^\times$ be exponential over K. Take K-linearly independent $u_1, \dots, u_n$ in E and $a_1, \dots, a_n, b_1, \dots, b_n \in K$ with $b_j \neq 0$ for some j, such that

$$y \;=\; \Big(\textstyle\sum_i a_i u_i\Big) \Big/ \Big(\textstyle\sum_j b_j u_j\Big).$$

Then $\sum_j b_j y u_j - \sum_i a_i u_i = 0$, and so Lemma 2.2.1 applied with $R = F$, $I = \{0\}$ gives $b_j y u_j \in a_i u_i K^\times$ for some i, j with $a_i, b_j \neq 0$, and thus $y \in K^\times E$. $\square$

Remark. In the context of Corollary 2.2.2, see [168, Theorem 1] for the structure of the group of elements of $F^\times$ exponential over K, for finitely generated E.

Lemma 2.2.3. *Suppose $C_R^\times$ is divisible and E is a subgroup of $R^\times$ containing $C_R^\times$. Then there is a group morphism $e\colon E^\dagger \to E$ such that $e(b)^\dagger = b$ for all $b \in E^\dagger$.*

Proof. We have a short exact sequence of commutative groups

$$1 \to C_R^\times \xrightarrow{\;\iota\;} E \xrightarrow{\;\ell\;} E^\dagger \to 0,$$

where ι is the natural inclusion and $\ell(a) := a^\dagger$ for $a \in E$. Since $C_R^\times$ is divisible, this sequence splits, which is what we claimed. $\square$

Let E, e, R be as in the previous lemma. Then e is injective, and its image is a complement of $C_R^\times$ in E. Moreover, given also a group morphism $\widetilde{e}\colon E^\dagger \to E$ such that $\widetilde{e}(b)^\dagger = b$ for all $b \in E^\dagger$, the map $b \mapsto e(b)\widetilde{e}(b)^{-1}$ is a group morphism $E^\dagger \to C_R^\times$.

In the rest of this section K is a differential field with algebraically closed constant field C and divisible group $K^\dagger$ of logarithmic derivatives. (These conditions are satisfied if K is an algebraically closed differential field.) In the next subsection we show that up to isomorphism over K there is a unique exponential extension R of K satisfying $C_R = C$ and $(R^\times)^\dagger = K$. By Lemma 2.2.3 we must then have a group embedding $e\colon K \to R^\times$ such that $e(b)^\dagger = b$ for all $b \in K$; this motivates the construction below.

The universal exponential extension. We first describe a certain exponential extension of K. For this, take a **complement** Λ of $K^\dagger$, that is, a $\mathbb{Q}$-linear subspace of K such that $K = K^\dagger \oplus \Lambda$ (internal direct sum of $\mathbb{Q}$-linear subspaces of K). Below λ ranges over Λ. Let $\mathrm{e}(\Lambda)$ be a multiplicatively written abelian group, isomorphic to the additive subgroup Λ of K, with isomorphism $\lambda \mapsto \mathrm{e}(\lambda)\colon \Lambda \to \mathrm{e}(\Lambda)$. Put

$$\mathrm{U} \;:=\; K\big[\mathrm{e}(\Lambda)\big],$$

the group ring of $\mathrm{e}(\Lambda)$ over K, an integral domain. As K-linear space,

$$\mathrm{U} \;=\; \bigoplus_\lambda K\,\mathrm{e}(\lambda) \qquad \text{(an internal direct sum of } K\text{-linear subspaces).}$$

For every $f \in \mathrm{U}$ we have a unique family (f_λ) in K such that

$$f \;=\; \sum_\lambda f_\lambda \,\mathrm{e}(\lambda),$$

with $f_\lambda = 0$ for all but finitely many λ; we call (f_λ) the **spectral decomposition** of f (with respect to Λ). We turn U into a differential ring extension of K by

$$\mathrm{e}(\lambda)' \;=\; \lambda\,\mathrm{e}(\lambda) \qquad \text{for all } \lambda.$$

(Think of $\mathrm{e}(\lambda)$ as $\exp(\int \lambda)$.) Thus for $f \in \mathrm{U}$ with spectral decomposition (f_λ),

$$f' \;=\; \sum_\lambda \big(f_\lambda' + \lambda f_\lambda\big)\,\mathrm{e}(\lambda),$$

so f' has spectral decomposition $(f_\lambda' + \lambda f_\lambda)$. Note that U is exponential over K by Lemma 2.1.1: $\mathrm{U}^\times = K^\times \mathrm{e}(\Lambda)$, so $(\mathrm{U}^\times)^\dagger = K^\dagger + \Lambda = K$.

Example 2.2.4. Let $K = C(\!(t^{\mathbb{Q}})\!)$ be as in Example 1.2.12, so $K^\dagger = (\mathbb{Q} \oplus \smallo)t$. Take a $\mathbb{Q}$-linear subspace Λ_{c} of C with $C = \mathbb{Q} \oplus \Lambda_{\mathrm{c}}$ (internal direct sum of $\mathbb{Q}$-linear subspaces of C), and let

$$K_\succ \;:=\; \big\{f \in K:\ \operatorname{supp}(f) \succ 1\big\},$$

a C-linear subspace of K. Then $\Lambda := (K_\succ \oplus \Lambda_{\mathrm{c}})t$ is a complement to $K^\dagger$, and hence $t^{-1}\Lambda = K_\succ \oplus \Lambda_{\mathrm{c}}$ is a complement to $(K^t)^\dagger$ in K^t. Moreover, if $L := \mathrm{P}(C) \subseteq K$ is the differential field of Puiseux series over C and $L_\succ := K_\succ \cap L$, then $L_\succ \oplus \Lambda_{\mathrm{c}}$ is a complement to $(L^t)^\dagger$.

A subgroup Λ_0 of Λ yields a differential subring $K\big[\mathrm{e}(\Lambda_0)\big]$ of U that is exponential over K as well. These differential subrings have a useful property. Recall from [ADH, 4.6] that a differential ring is said to be *simple* if $\{0\}$ is its only proper differential ideal.

Lemma 2.2.5. *Let Λ_0 be a subgroup of Λ. Then the differential subring $K\big[\mathrm{e}(\Lambda_0)\big]$ of* U *is simple. In particular, the differential ring* U *is simple.*

Proof. Let $I \neq R$ be a differential ideal of $R := K\big[\mathrm{e}(\Lambda_0)\big]$. Let $f_1, \dots, f_n \in K^\times$ and let $\lambda_1, \dots, \lambda_n \in \Lambda_0$ be distinct such that $f = \sum_{i=1}^n f_i \,\mathrm{e}(\lambda_i) \in I$. If $n \geqslant 1$, then Lemma 2.2.1 yields $i \neq j$ with $\mathrm{e}(\lambda_i)/\mathrm{e}(\lambda_j) = cg$ for some constant c in the differential fraction field of U and some $g \in K^\times$, so by taking logarithmic derivatives, $\lambda_i - \lambda_j \in K^\dagger$ and thus $\lambda_i = \lambda_j$, a contradiction. Thus $f = 0$. $\square$

Corollary 2.2.6. *Any morphism $K\big[\mathrm{e}(\Lambda_0)\big] \to R$ of differential K-algebras, with Λ_0 a subgroup of Λ and R a differential ring extension of K, is injective.*

The differential ring U is the directed union of its differential subrings of the form $\mathrm{U}_0 = K\big[\mathrm{e}(\Lambda_0)\big]$ where Λ_0 is a finitely generated subgroup of Λ. These U_0 are simple by Lemma 2.2.5 and finitely generated as a K-algebra, hence their differential fraction fields have constant field C by [ADH, 4.6.12]. Thus the differential fraction field of U has constant field C.

Lemma 2.2.7. *Suppose R is an exponential extension of K and R_0 is a differential subring of R with $C_R^\times \subseteq C_{R_0}$ and $K \subseteq (R_0^\times)^\dagger$. Then $R_0 = R$.*

Proof. Let E be the group of units of R that are exponential over K; so $R = K[E]$. Given $u \in E$ we have $u^\dagger \in K \subseteq (R_0^\times)^\dagger$, hence we have $u_0 \in R_0^\times$ with $u^\dagger = u_0^\dagger$, so $u = cu_0$ with $c \in C_R^\times \subseteq C_{R_0}$. Thus $E \subseteq R_0$ and so $R_0 = R$. $\square$

Corollary 2.2.8. *Every endomorphism of the differential K-algebra* U *is an automorphism.*

Proof. Injectivity holds by Corollary 2.2.6, and surjectivity by Lemma 2.2.7. □

Every exponential extension of K with constant field C embeds into U, and hence is an integral domain. More precisely:

Lemma 2.2.9. *Let R be an exponential extension of K such that $C_R^\times$ is divisible, and set $\Lambda_0 := \Lambda \cap (R^\times)^\dagger$, a subgroup of Λ. Then there exists a morphism $K\big[\mathrm{e}(\Lambda_0)\big] \to R$ of differential K-algebras. Any such morphism is injective, and if $C_R = C$, then any such morphism is an isomorphism.*

Proof. Let E be as in the proof of Lemma 2.2.7, and let $e_E\colon E^\dagger \to E$ be the map e from Lemma 2.2.3. Since $E^\dagger = K^\dagger + \Lambda_0$ we have

$$E \;=\; C_R^\times\, e_E(E^\dagger) \;=\; C_R^\times\, e_E(K^\dagger)\, e_E(\Lambda_0) \;=\; C_R^\times\, K^\times\, e_E(\Lambda_0). \tag{2.2.1}$$

The group morphism $\mathrm{e}(\lambda_0) \mapsto e_E(\lambda_0)\colon \mathrm{e}(\Lambda_0) \to E$ ($\lambda_0 \in \Lambda_0$) extends uniquely to a K-algebra morphism $\iota\colon K\big[\mathrm{e}(\Lambda_0)\big] \to R = K[E]$. One verifies easily that ι is a differential ring morphism. The injectivity claim follows from Corollary 2.2.6. If $C_R = C$, then $E = K^\times e_E(\Lambda_0)$ by (2.2.1), whence surjectivity. □

Recall that U is an exponential extension of K with $C_{\mathrm{U}} = C$ and $(\mathrm{U}^\times)^\dagger = K$. By Lemma 2.2.9, this property characterizes U up to isomorphism:

Corollary 2.2.10. *If U is an exponential extension of K such that $C_U = C$ and $K \subseteq (U^\times)^\dagger$, then U is isomorphic to* U *as a differential K-algebra.*

Now U is also an exponential extension of K with $C_{\mathrm{U}} = C$ and with the property that every exponential extension R of K with $C_R = C$ embeds into U as a differential K-algebra. This property determines U up to isomorphism as well:

Corollary 2.2.11. *Suppose U is an exponential extension of K with $C_U = C$ such that every exponential extension R of K with $C_R = C$ embeds into U as a differential K-algebra. Then U is isomorphic to* U *as a differential K-algebra.*

Proof. Any embedding $\mathrm{U} \to U$ of differential K-algebras gives $K \subseteq (U^\times)^\dagger$. □

The results above show to what extent U is independent of the choice of Λ. We call U the **universal exponential extension of** K. If we need to indicate the dependence of U on K we denote it by U_K. By [ADH, 5.1.40] every $y \in \mathrm{U} = K\{\mathrm{e}(\Lambda)\}$ satisfies a linear differential equation $A(y) = 0$ where $A \in K[\partial]^{\neq}$; in the next section we isolate conditions on K which ensure that every $A \in K[\partial]^{\neq}$ has a zero $y \in \mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$.

Corollary 2.2.10 gives for $\phi \in K^\times$ an isomorphism $\mathrm{U}_{K^\phi} \cong (\mathrm{U}_K)^\phi$ of differential K^ϕ-algebras. Next we investigate how U_K behaves when passing from K to a differential field extension. Therefore, *in the rest of this subsection L is a differential field extension of K with algebraically closed constant field C_L, and $L^\dagger$ is divisible.* The next lemma relates the universal exponential extension U_L of L to U_K:

Lemma 2.2.12. *The inclusion $K \to L$ extends to an embedding $\iota\colon \mathrm{U}_K \to \mathrm{U}_L$ of differential rings. The image of any such embedding ι is contained in $K[E]$ where $E := \{u \in \mathrm{U}_L^\times : u^\dagger \in K\}$, and if $C_L = C$, then $\iota(\mathrm{U}_K) = K[E]$.*

Proof. The differential subring $R := K[E]$ of U_L is exponential over K with $(R^\times)^\dagger = K$, hence Lemma 2.2.9 gives an embedding $\mathrm{U}_K \to R$ of differential K-algebras. Let $\iota\colon \mathrm{U}_K \to \mathrm{U}_L$ be any embedding of differential K-algebras. Then $\iota\big(\mathrm{e}(\Lambda)\big) \subseteq E$, so $\iota(\mathrm{U}_K) \subseteq R$; if $C_L = C$, then $\iota(\mathrm{U}_K) = R$ by Lemma 2.2.7. $\square$

Corollary 2.2.13. *If $L^\dagger \cap K = K^\dagger$ and $\iota\colon \mathrm{U}_K \to \mathrm{U}_L$ is an embedding of differential K-algebras, then $L^\times \cap \iota(\mathrm{U}_K^\times) = K^\times$.*

Proof. Assume $L^\dagger \cap K = K^\dagger$ and identify U_K with a differential K-subalgebra of U_L via an embedding $\mathrm{U}_K \to \mathrm{U}_L$ of differential K-algebras. Let $a \in L^\times \cap \mathrm{U}_K^\times$; then $a^\dagger \in L^\dagger \cap K = K^\dagger$, so $a = bc$ where $c \in C_L^\times$, $b \in K^\times$. Now $c = a/b \in C_L^\times \cap \mathrm{U}_K^\times = C^\times$, since U_K has ring of constants C. So $a \in K^\times$ as required. $\square$

Suppose $L^\dagger \cap K = K^\dagger$. Then the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L has a complement $\Lambda_L \supseteq \Lambda$. We fix such Λ_L and extend $\mathrm{e}\colon \Lambda \to \mathrm{e}(\Lambda)$ to a group isomorphism $\Lambda_L \to \mathrm{e}(\Lambda_L)$, also denoted by e, with $\mathrm{e}(\Lambda_L)$ a multiplicatively written commutative group extending $\mathrm{e}(\Lambda)$. Let $\mathrm{U}_L := L\big[\mathrm{e}(\Lambda_L)\big]$ be the corresponding universal exponential extension of L. Then the natural inclusion $\mathrm{U}_K \to \mathrm{U}_L$ is an embedding of differential K-algebras.

Automorphisms of U. These are easy to describe: the beginning of Section 2.1 gives a group embedding

$$\chi \mapsto \sigma_\chi\colon \operatorname{Hom}(\Lambda, K^\times) \to \operatorname{Aut}\big(K[\mathrm{e}(\Lambda)]|K\big)$$

into the group of K-algebra automorphisms of $K\big[\mathrm{e}(\Lambda)\big]$, given by

$$\sigma_\chi(f) \ := f_\chi \ = \ \sum_\lambda f_\lambda \chi(\lambda)\,\mathrm{e}(\lambda) \qquad (\chi \in \operatorname{Hom}(\Lambda, K^\times),\ f \in K[\mathrm{e}(\Lambda)]).$$

It is easy to check that if $\chi \in \operatorname{Hom}(\Lambda, C^\times) \subseteq \operatorname{Hom}(\Lambda, K^\times)$, then $\sigma_\chi \in \operatorname{Aut}_\partial(\mathrm{U}|K)$, that is, σ_χ is a differential K-algebra automorphism of U. Moreover:

Lemma 2.2.14. *The map $\chi \mapsto \sigma_\chi\colon \operatorname{Hom}(\Lambda, C^\times) \to \operatorname{Aut}_\partial(\mathrm{U}|K)$ is a group isomorphism. Its inverse assigns to any $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$ the function $\chi\colon \Lambda \to C^\times$ given by $\chi(\lambda) := \sigma\big(\mathrm{e}(\lambda)\big)\,\mathrm{e}(-\lambda)$. In particular, $\operatorname{Aut}_\partial(\mathrm{U}|K)$ is commutative.*

Proof. Let $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$ and let $\chi\colon \Lambda \to \mathrm{U}^\times$ be given by $\chi(\lambda) := \sigma\big(\mathrm{e}(\lambda)\big)\,\mathrm{e}(-\lambda)$. Then $\chi(\lambda)^\dagger = 0$ for all λ. It follows easily that $\chi \in \operatorname{Hom}(\Lambda, C^\times)$ and $\sigma_\chi = \sigma$. $\square$

The proof of the next result uses that the additive group $\mathbb{Q}$ embeds into $C^\times$.

Corollary 2.2.15. *If $f \in \mathrm{U}$ and $\sigma(f) = f$ for all $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$, then $f \in K$.*

Proof. Suppose $f \in U$ and $\sigma(f) = f$ for all $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$. For $\chi \in \operatorname{Hom}(\Lambda, C^\times)$ we have $f_\chi = f$, that is, $f_\lambda \chi(\lambda) = f_\lambda$ for all λ, so $\chi(\lambda) = 1$ whenever $f_\lambda \neq 0$. Now use that for $\lambda \neq 0$ there exists $\chi \in \operatorname{Hom}(\Lambda, C^\times)$ such that $\chi(\lambda) \neq 1$, so $f_\lambda = 0$. $\square$

Corollary 2.2.16. *Every automorphism of the differential field K extends to an automorphism of the differential ring* U.

Proof. Lemma 2.2.3 yields a group morphism $\mu\colon K \to \mathrm{U}^\times$ such that $\mu(a)^\dagger = a$ for all $a \in K$. Let $\sigma \in \operatorname{Aut}_\partial(K)$. Then σ extends to an endomorphism, denoted also by σ, of the ring U, such that $\sigma\big(\mathrm{e}(\lambda)\big) = \mu\big(\sigma(\lambda)\big)$ for all λ. Then

$$\sigma\big(\mathrm{e}(\lambda)'\big) \ = \ \sigma\big(\lambda\,\mathrm{e}(\lambda)\big) \ = \ \sigma(\lambda)\mu\big(\sigma(\lambda)\big) \ = \ \mu\big(\sigma(\lambda)\big)' \ = \ \sigma\big(\mathrm{e}(\lambda)\big)',$$

hence σ is an endomorphism of the differential ring U. By Lemma 2.2.5, σ is injective, and by Lemma 2.2.7, σ is surjective. $\square$

The real case. *In this subsection $K = H[\mathrm{i}]$ where H is a real closed differential subfield of K and $\mathrm{i}^2 = -1$.* Set $S_C := \{c \in C : |c| = 1\}$, a subgroup of $C^\times$. Then by Lemmas 2.1.7 and 2.2.14:

Corollary 2.2.17. *For $\sigma \in \mathrm{Aut}_\partial(\mathrm{U}|K)$ we have the equivalence*

$$\sigma(f^*) = \sigma(f)^* \text{ for all } f \in \mathrm{U} \quad \Longleftrightarrow \quad \sigma = \sigma_\chi \text{ for some } \chi \in \mathrm{Hom}(\Lambda, S_C).$$

Corollaries 2.2.17 and 2.1.8 together give:

Corollary 2.2.18. *Let $\sigma \in \mathrm{Aut}_\partial(\mathrm{U}|K)$ satisfy $\sigma(f^*) = \sigma(f)^*$ for all $f \in \mathrm{U}$. Then $\langle \sigma(f), \sigma(g) \rangle = \langle f, g \rangle$ for all $f, g \in \mathrm{U}$, hence $\|\sigma(f)\| = \|f\|$ for all $f \in \mathrm{U}$.*

Next we consider the subgroup

$$S := \{a + bi : a, b \in H, \ a^2 + b^2 = 1\}$$

of $K^\times$, which is divisible, hence so is the subgroup $S^\dagger$ of $K^\dagger$. Lemma 1.2.4 yields $K^\dagger = H^\dagger \oplus S^\dagger$ (internal direct sum of $\mathbb{Q}$-linear subspaces of K) and $S^\dagger \subseteq Hi$. Thus we can (and do) take the complement Λ of $K^\dagger$ in K so that $\Lambda = \Lambda_\mathrm{r} + \Lambda_\mathrm{i} i$ where $\Lambda_\mathrm{r}, \Lambda_\mathrm{i}$ are subspaces of the $\mathbb{Q}$-linear space H with Λ_r a complement of $H^\dagger$ in H and $\Lambda_\mathrm{i} i$ a complement of $S^\dagger$ in Hi. The automorphism $a + bi \mapsto \overline{a + bi} := a - bi$ $(a, b \in H)$ of the differential field K now satisfies in $\mathrm{U} = K[\mathrm{e}(\Lambda)]$ the identity

$$\mathrm{e}(\overline{\lambda + \mu}) = \mathrm{e}(\overline{\lambda})\, \mathrm{e}(\overline{\mu}) \qquad (\lambda, \mu \in \Lambda),$$

so it extends to an automorphism $f \mapsto \overline{f}$ of the ring U as follows: for $f \in \mathrm{U}$ with spectral decomposition (f_λ), set

$$\overline{f} := \sum_\lambda \overline{f_\lambda}\, \mathrm{e}(\overline{\lambda}) = \sum_\lambda \overline{f_{\overline{\lambda}}}\, \mathrm{e}(\lambda),$$

so $\overline{\mathrm{e}(\lambda)} = \mathrm{e}(\overline{\lambda})$, and $\overline{f}$ has spectral decomposition $(\overline{f_{\overline{\lambda}}})$. We have $\overline{\overline{f}} = f$ for $f \in \mathrm{U}$, and $f \mapsto \overline{f}$ lies in $\mathrm{Aut}_\partial(\mathrm{U}|H)$. If $H^\dagger = H$, then $\Lambda_r = \{0\}$ and hence $\overline{f} = f^*$ for $f \in \mathrm{U}$, where f^* is as defined in Section 2.1. For $f \in \mathrm{U}$ we set

$$\mathrm{Re}\, f := \tfrac{1}{2}(f + \overline{f}), \qquad \mathrm{Im}\, f := \tfrac{1}{2\mathrm{i}}(f - \overline{f}).$$

(For $f \in K$ these agree with the usual real and imaginary parts of f as an element of $H[\mathrm{i}]$.) Consider the differential H-subalgebra

$$\mathrm{U}_\mathrm{r} := \{f \in \mathrm{U} : \overline{f} = f\}$$

of U. For $f \in \mathrm{U}$ with spectral decomposition (f_λ) we have $f \in \mathrm{U}_\mathrm{r}$ iff $f_{\overline{\lambda}} = \overline{f_\lambda}$ for all λ; in particular $\mathrm{U}_\mathrm{r} \cap K = H$. For $f \in \mathrm{U}$ we have $f = (\mathrm{Re}\, f) + (\mathrm{Im}\, f)i$ with $\mathrm{Re}\, f, \mathrm{Im}\, f \in \mathrm{U}_\mathrm{r}$, hence

$$\mathrm{U} = \mathrm{U}_\mathrm{r} \oplus \mathrm{U}_\mathrm{r} i \quad \text{(internal direct sum of } H\text{-linear subspaces).}$$

Let D be a subfield of H (not necessarily the constant field of H), so $D[\mathrm{i}]$ is a subfield of K. Let V be a $D[\mathrm{i}]$-linear subspace of U; then $V_\mathrm{r} := V \cap \mathrm{U}_\mathrm{r}$ is a D-linear subspace of V. If $\overline{V} = V$ (that is, V is closed under $f \mapsto \overline{f}$), then $\mathrm{Re}\, f, \mathrm{Im}\, f \in V_\mathrm{r}$ for all $f \in V$, hence $V = V_\mathrm{r} \oplus V_\mathrm{r} i$ (internal direct sum of D-linear subspaces of V), so any basis of the D-linear space V_r is a basis of the $D[\mathrm{i}]$-linear space V.

Suppose now that $V = \bigoplus_\lambda V_\lambda$ (internal direct sum of subspaces of V) where V_λ is for each λ a $D[\mathrm{i}]$-linear subspace of $K\,\mathrm{e}(\lambda)$. Then $\overline{V} = V$ iff $V_{\overline{\lambda}} = \overline{V_\lambda}$ for all λ. Moreover:

Lemma 2.2.19. *Assume $H = H^\dagger$, $V_0 = \{0\}$, and $\overline{V} = V$. Let $\mathcal{V} \subseteq \mathrm{U}^\times$ be a basis of the subspace $\sum_{\operatorname{Im}\lambda>0} V_\lambda$ of V. Then the maps $v \mapsto \operatorname{Re} v$, $v \mapsto \operatorname{Im} v \colon \mathcal{V} \to V_{\mathrm{r}}$ are injective, $\operatorname{Re}\mathcal{V}$ and $\operatorname{Im}\mathcal{V}$ are disjoint, and $\operatorname{Re}\mathcal{V} \cup \operatorname{Im}\mathcal{V}$ is a basis of V_{r}.*

Proof. Note that $\Lambda = \Lambda_{\mathrm{i}} i$. Let μ range over $\Lambda_{\mathrm{i}}^{>}$ and set $\mathcal{V}_\mu = \mathcal{V} \cap K^\times\,\mathrm{e}(\mu i)$, a basis of the $D[\mathrm{i}]$-linear space $V_{\mu\mathrm{i}}$. Then $\mathcal{V} = \bigcup_\mu \mathcal{V}_\mu$, a disjoint union. For $v \in \mathcal{V}_\mu$ we have $v = a\,\mathrm{e}(\mu i)$ with $a = a_v \in K^\times$, so

$$\operatorname{Re} v \;=\; \tfrac{a}{2}\,\mathrm{e}(\mu i) + \tfrac{\overline{a}}{2}\,\mathrm{e}(-\mu i), \qquad \operatorname{Im} v \;=\; \tfrac{a}{2i}\,\mathrm{e}(\mu i) - \tfrac{\overline{a}}{2\mathrm{i}}\,\mathrm{e}(-\mu i),$$

from which it is clear that the two maps $\mathcal{V} \to V_{\mathrm{r}}$ in the statement of the lemma are injective. It is also easy to check that $\operatorname{Re}\mathcal{V}$ and $\operatorname{Im}\mathcal{V}$ are disjoint.

As $\mathcal{V}$ is a basis of the $D[\mathrm{i}]$-linear space $\sum_\mu V_{\mu\mathrm{i}} = \sum_{\operatorname{Im}\lambda>0} V_\lambda$, its set of conjugates $\overline{\mathcal{V}}$ is a basis of the $D[\mathrm{i}]$-linear space $\sum_\mu \overline{V_{\mu\mathrm{i}}} = \sum_\mu V_{-\mu\mathrm{i}} = \sum_{\operatorname{Im}\lambda<0} V_\lambda$, and so $\mathcal{V} \cup \overline{\mathcal{V}}$ (a disjoint union) is a basis of V. Thus $\operatorname{Re}\mathcal{V} \cup \operatorname{Im}\mathcal{V}$ is a basis of V as well. As $\operatorname{Re}\mathcal{V} \cup \operatorname{Im}\mathcal{V}$ is contained in V_{r}, it is a basis of the D-linear space V_{r}. □

If $H = H^\dagger$, then $V := \sum_{\lambda\neq 0} K\,\mathrm{e}(\lambda)$ gives $\overline{V} = V$, so Lemma 2.2.19 gives then for $D := H$ the basis of the H-linear space V_{r} consisting of the elements

$$\operatorname{Re}\big(\mathrm{e}(\lambda)\big) \;=\; \tfrac{1}{2}\big(\mathrm{e}(\lambda) + \mathrm{e}(\overline{\lambda})\big), \qquad \operatorname{Im}\big(\mathrm{e}(\lambda)\big) \;=\; \tfrac{1}{2\mathrm{i}}\big(\mathrm{e}(\lambda) - \mathrm{e}(\overline{\lambda})\big) \qquad (\operatorname{Im}\lambda > 0).$$

Corollary 2.2.20. *Suppose $H = H^\dagger$. Set $\mathrm{c}(\lambda) := \operatorname{Re}\big(\mathrm{e}(\lambda)\big)$ and $\mathrm{s}(\lambda) := \operatorname{Im}\big(\mathrm{e}(\lambda)\big)$, for $\operatorname{Im}\lambda > 0$. Then for $V := \sum_{\lambda\neq 0} K\,\mathrm{e}(\lambda)$ we have $\mathrm{U}_{\mathrm{r}} = H + V_{\mathrm{r}}$, so*

$$\mathrm{U}_{\mathrm{r}} \;=\; H \oplus \bigoplus_{\operatorname{Im}\lambda>0} \big(H\,\mathrm{c}(\lambda) \oplus H\,\mathrm{s}(\lambda)\big) \quad (\textit{internal direct sum of } H\textit{-linear subspaces}),$$

and thus $\mathrm{U}_{\mathrm{r}} = H\big[\,\mathrm{c}(\Lambda_{\mathrm{i}}^{>} i) \cup \mathrm{s}(\Lambda_{\mathrm{i}}^{>} i)\big]$.

2.3. The Spectrum of a Differential Operator

In this section K is a differential field, a, b range over K, and A, B over $K[\partial]$. This and the next two sections are mainly differential-algebraic in nature, and deal with splittings of linear differential operators. In the present section we introduce the concept of *eigenvalue* of A and the *spectrum* of A (the collection of its eigenvalues). In Section 2.4 we give criteria for A to have eigenvalue 0, and in Section 2.5 we show how the eigenvalues of A relate to the behavior of A over the universal exponential extension of K.

Twisting. Let L be a differential field extension of K with $L^\dagger \supseteq K$. Let $u \in L^\times$ be such that $u^\dagger = a \in K$. Then the twist $A_{\ltimes u} = u^{-1}Au$ of A by u has the same order as A and coefficients in K [ADH, 5.8.8], and only depends on a, not on u or L; in fact, $\operatorname{Ri}(A_{\ltimes u}) = \operatorname{Ri}(A)_{+a}$ [ADH, 5.8.5]. Hence for each a we may define

$$A_a \;:=\; A_{\ltimes u} \;=\; u^{-1}Au \in K[\partial]$$

where $u \in L^\times$ is arbitrary with $u^\dagger = a$. The map $A \mapsto A_{\ltimes u}$ is an automorphism of the ring $K[\partial]$ that is the identity on K (with inverse $B \mapsto B_{\ltimes u^{-1}}$); so $A \mapsto A_a$ is an automorphism of the ring $K[\partial]$ that is the identity on K (with inverse $B \mapsto B_{-a}$). Note that $\partial_a = \partial + a$, and that

$$(a, A) \mapsto A_a \ : \ K \times K[\partial] \to K[\partial]$$

is an action of the additive group of K on the set $K[\partial]$, in particular, $A_a = A$ for $a = 0$. For $b \neq 0$ we have $(A_a)_{\ltimes b} = A_{a+b^\dagger}$.

Eigenvalues. *In the rest of this section $A \neq 0$ and $r := \operatorname{order}(A)$.* We call

$$\operatorname{mult}_a(A) \ := \ \dim_C \ker_K A_a \in \{0, \dots, r\}$$

the **multiplicity** of A at a. If $B \neq 0$, then $\operatorname{mult}_a(B) \leqslant \operatorname{mult}_a(AB)$, as well as

$$\operatorname{mult}_a(AB) \ \leqslant \ \operatorname{mult}_a(A) + \operatorname{mult}_a(B), \tag{2.3.1}$$

with equality if and only if $B_a(K) \supseteq \ker_K A_a$; see [ADH, remarks before 5.1.12]. For $u \in K^\times$ we have an isomorphism

$$y \mapsto yu : \ \ker_K A_{\ltimes u} \to \ker_K A$$

of C-linear spaces, hence

$$\operatorname{mult}_a(A) \ = \ \operatorname{mult}_b(A) \qquad \text{whenever } a - b \in K^\dagger.$$

Thus we may define the **multiplicity** of A at the element $[a] := a + K^\dagger$ of $K/K^\dagger$ as $\operatorname{mult}_{[a]}(A) := \operatorname{mult}_a(A)$.

In the rest of this section α ranges over $K/K^\dagger$. We say that α is an **eigenvalue** of A if $\operatorname{mult}_\alpha(A) \geqslant 1$. Thus for $B \neq 0$: if α is an eigenvalue of B of multiplicity μ, then α is an eigenvalue of AB of multiplicity $\geqslant \mu$; if α is an eigenvalue of AB, then it is an eigenvalue of A or of B; and if $B_a(K) \supseteq \ker_K(A_a)$, then $\alpha = [a]$ is an eigenvalue of AB if and only if it is an eigenvalue of A or of B.

Example 2.3.1. Suppose $A = \partial - a$. Then for each element $u \neq 0$ in a differential field extension of K with $b := u^\dagger \in K$ we have $A_b = A_{\ltimes u} = \partial - (a - b)$, so $\operatorname{mult}_b(A) \geqslant 1$ iff $a - b \in K^\dagger$. Hence the only eigenvalue of A is $[a]$.

The **spectrum** of A is the set $\Sigma(A) = \Sigma_K(A)$ of its eigenvalues. Thus $\Sigma(A) = \emptyset$ if $r = 0$, and for $b \neq 0$ we have $\operatorname{mult}_a(A) = \operatorname{mult}_a(bA) = \operatorname{mult}_a(A_{\ltimes b})$, so A, bA, and $Ab = bA_{\ltimes b}$ all have the same spectrum. By [ADH, 5.1.21] we have

$$\Sigma(A) = \big\{\alpha : A \in K[\partial](\partial - a) \text{ for some } a \text{ with } [a] = \alpha\big\}. \tag{2.3.2}$$

Hence for irreducible A: $\ \Sigma(A) \neq \emptyset \ \Leftrightarrow \ r = 1$. From (2.3.1) we obtain:

Lemma 2.3.2. *Suppose $B \neq 0$ and set $s := \operatorname{order} B$. Then*

$$\operatorname{mult}_\alpha(B) \ \leqslant \ \operatorname{mult}_\alpha(AB) \ \leqslant \ \operatorname{mult}_\alpha(A) + \operatorname{mult}_\alpha(B),$$

where the second inequality is an equality if K is s-linearly surjective. Hence

$$\Sigma(B) \ \subseteq \ \Sigma(AB) \ \subseteq \ \Sigma(A) \cup \Sigma(B).$$

If K is s-linearly surjective, then $\Sigma(AB) = \Sigma(A) \cup \Sigma(B)$.

Example. For $n \geqslant 1$ we have $\Sigma\big((\partial - a)^n\big) = \big\{[a]\big\}$. (By induction on n, using Example 2.3.1 and Lemma 2.3.2.)

It follows from Lemma 2.3.2 that A has at most r eigenvalues. More precisely:

Lemma 2.3.3. *We have $\sum_\alpha \operatorname{mult}_\alpha(A) \leqslant r$. If $\sum_\alpha \operatorname{mult}_\alpha(A) = r$, then A splits over K; the converse holds if $r = 1$ or K is 1-linearly surjective.*

Proof. By induction on r. The case $r=0$ is obvious, so suppose $r>0$. We may also assume $\Sigma(A)\neq\emptyset$: otherwise $\sum_\alpha \operatorname{mult}_\alpha(A)=0$ and A does not split over K. Now (2.3.2) gives a, B with $A=B(\partial-a)$. By Example 2.3.1 we have $\Sigma(\partial-a)=\big\{[a]\big\}$ and $\operatorname{mult}_a(\partial-a)=1$. By the inductive hypothesis applied to B and the second inequality in Lemma 2.3.2 we thus get $\sum_\alpha \operatorname{mult}_\alpha(A)\leqslant r$.

Suppose that $\sum_\alpha \operatorname{mult}_\alpha(A)=r$. Then $\sum_\alpha \operatorname{mult}_\alpha(B)=r-1$ by Lemma 2.3.2 and the inductive hypothesis applied to B. Therefore B splits over K, again by the inductive hypothesis, and so does A. Finally, if K is 1-linearly surjective and A splits over K, then we arrange that B splits over K, so $\sum_\alpha \operatorname{mult}_\alpha(B)=r-1$ by the inductive hypothesis, hence $\sum_\alpha \operatorname{mult}_\alpha(A)=r$ by Lemma 2.3.2. □

Section 2.5 gives a more explicit proof of Lemma 2.3.3, under additional hypotheses on K. Next, let L be a differential field extension of K. Then $\operatorname{mult}_a(A)$ does not strictly decrease in passing from K to L [ADH, 4.1.13]. Hence the group morphism

$$a+K^\dagger\mapsto a+L^\dagger\colon K/K^\dagger\to L/L^\dagger$$

restricts to a map $\Sigma_K(A)\to\Sigma_L(A)$; in particular, if $\Sigma_K(A)\neq\emptyset$, then $\Sigma_L(A)\neq\emptyset$. If $L^\dagger\cap K=K^\dagger$, then $|\Sigma_K(A)|\leqslant|\Sigma_L(A)|$, and $\sum_\alpha \operatorname{mult}_\alpha(A)$ also does not strictly decrease if K is replaced by L.

Lemma 2.3.4. *Let $a_1,\dots,a_r\in K$ and*

$$A\ =\ (\partial-a_r)\cdots(\partial-a_1),\qquad \sum_\alpha \operatorname{mult}_\alpha(A)\ =\ r.$$

Then the spectrum of A is $\big\{[a_1],\dots,[a_r]\big\}$, and for all α,

$$\operatorname{mult}_\alpha(A)\ =\ \big|\big\{i\in\{1,\dots,r\}:\ \alpha=[a_i]\big\}\big|.$$

Proof. Let i range over $\{1,\dots,r\}$. By Lemma 2.3.2 and Example 2.3.1,

$$\operatorname{mult}_\alpha(A)\ \leqslant\ \sum_i \operatorname{mult}_\alpha(\partial-a_i)\ =\ \big|\big\{i:\alpha=[a_i]\big\}\big|$$

and hence

$$r\ =\ \sum_\alpha \operatorname{mult}_\alpha(A)\ \leqslant\ \sum_\alpha\big|\big\{i:\alpha=[a_i]\big\}\big|\ =\ r.$$

Thus for each α we have $\operatorname{mult}_\alpha(A)=\big|\big\{i:\alpha=[a_i]\big\}\big|$ as required. □

Recall from [ADH, 5.1.8] that $D^*\in K[\partial]$ denotes the *adjoint* of $D\in K[\partial]$, and that the map $D\mapsto D^*$ is an involution of the ring $K[\partial]$ with $a^*=a$ for all a and $\partial^*=-\partial$. If A splits over K, then so does A^*. Furthermore, $(A_a)^*=(A^*)_{-a}$ for all a. By Lemmas 2.3.3 and 2.3.4:

Corollary 2.3.5. *Suppose K is 1-linearly surjective and $\sum_\alpha \operatorname{mult}_\alpha(A)=r$. Then $\operatorname{mult}_\alpha(A)=\operatorname{mult}_{-\alpha}(A^*)$ for all α. In particular, the map $\alpha\mapsto-\alpha$ restricts to a bijection $\Sigma(A)\to\Sigma(A^*)$.*

Let $\phi\in K^\times$. Then $(A^\phi)_a=(A_{\phi a})^\phi$ and hence

$$\operatorname{mult}_a(A^\phi)\ =\ \operatorname{mult}_{\phi a}(A),$$

so the group isomorphism

$$[a]\mapsto[\phi a]\ :\ K^\phi/\phi^{-1}K^\dagger\to K/K^\dagger \tag{2.3.3}$$

maps $\Sigma(A^\phi)$ onto $\Sigma(A)$.

Note that $K[\partial]/K\partial]A$ as a K-linear space has dimension $r = \operatorname{order} A$. Recall from [ADH, 5.1] that A and $B \neq 0$ are said to *have the same type* if the (left) $K[\partial]$-modules $K[\partial]/K[\partial]A$ and $K[\partial]/K[\partial]B$ are isomorphic (and so $\operatorname{order} B = r$). By [ADH, 5.1.19]:

Lemma 2.3.6. *The operators A and $B \neq 0$ have the same type iff* $\operatorname{order} B = r$ *and there is $R \in K[\partial]$ of order $< r$ with $1 \in K[\partial]R + K[\partial]A$ and $BR \in K[\partial]A$.*

Hence if A, B have the same type, then they also have the same type as elements of $L[\partial]$, for any differential field extension L of K. Since $B \mapsto B_a$ is an automorphism of the ring $K[\partial]$, Lemma 2.3.6 and [ADH, 5.1.20] yield:

Lemma 2.3.7. *If A and $B \neq 0$ have the same type, then so do A_a, B_a, for all a, and thus A, B have the same eigenvalues, with same multiplicity.*

By this lemma the spectrum of A depends only on the type of A, that is, on the isomorphism type of the $K[\partial]$-module $K[\partial]/K[\partial]A$, suggesting one might try to associate a spectrum to each differential module over K. (Recall from [ADH, 5.5] that a differential module over K is a $K[\partial]$-module of finite dimension as K-linear space.) Although our focus is on differential operators, we carry this out in the next subsection: it motivates the terminology of "eigenvalues" originating in the case of the differential field of Puiseux series over $\mathbb{C}$ treated in [158]. This point of view will be further developed in the projected second volume of [ADH].

The spectrum of a differential module (*). *In this subsection M is a differential module over K and $r = \dim_K M$.* For each B we let $\ker_M B$ denote the kernel of the C-linear map $y \mapsto By \colon M \to M$. For $M = K$ as horizontal differential module over K [ADH, 5.5.2], this agrees with the C-linear subspace

$$\ker_K B \ = \ \ker B \ = \ \big\{y \in K : B(y) = 0\big\}$$

of K. Also, for $B = \partial$ we obtain the C-linear subspace $\ker_M \partial$ of horizontal elements of M. We define the **spectrum** of M to be the set

$$\Sigma(M) := \big\{\alpha : \ker_M(\partial - a) \neq \{0\} \text{ for some } a \text{ with } [a] = \alpha\big\}.$$

The elements of $\Sigma(M)$ are called **eigenvalues** of M. If $M = \{0\}$, then $\Sigma(M) = \emptyset$. Isomorphic differential modules over K have clearly the same spectrum.

Let $\phi \in K^\times$ and $\delta = \phi^{-1}\partial$. Then $K[\partial] = K^\phi[\delta]$ as rings, hence M is also a differential module over K^ϕ with $\phi^{-1}\partial_M$ instead of ∂_M as its derivation; we denote it by M^ϕ and call it the **compositional conjugate** of M by ϕ. Every cyclic vector of M is also a cyclic vector of M^ϕ. The group isomorphism (2.3.3) maps $\Sigma(M^\phi)$ onto $\Sigma(M)$.

In the next lemma we assume $r \geqslant 1$ and denote the $r \times r$ identity matrix over K by I_r. Below P is also an $r \times r$ matrix over K. The C-linear space of solutions to the matrix differential equation $y' = Py$ over K is the set of all column vectors $e \in K^r$ such that $e' = Pe$, and is denoted by $\operatorname{sol}(P)$ [ADH, p. 276]. Recall that a is said to be an eigenvalue of P over K if $Pe = ae$ for some nonzero column vector $e \in K^r$. Recall also from [ADH, p. 277] that we associate to P the differential module M_P having the space K^r of column vectors as its underlying K-linear space and satisfying $\partial e = e' - Pe$ for all $e \in K^r$. Thus by [ADH, 5.4.8]:

Lemma 2.3.8. *Let $M = M_{-P}$ be the differential module over K associated to $-P$. Then* $\ker_M(\partial - a) = \operatorname{sol}(aI_r - P)$, *so* $\dim_C \ker_M(\partial - a) \leqslant r$ *and*

$$\Sigma(M) = \big\{\alpha : \operatorname{sol}(aI_r - P) \neq \{0\} \textit{ for some } a \textit{ with } [a] = \alpha\big\}.$$

We define $\mathrm{mult}_a(M) := \dim_C \ker_M(\partial - a)$; thus $\mathrm{mult}_a(M) \in \{0,\dots,r\}$ by the previous lemma. For $b \neq 0$ we have a C-linear isomorphism

$$y \mapsto by\colon\ \ker_M(\partial - a) \to \ker_M(\partial - a - b^\dagger).$$

This observation allows us to define the **multiplicity** $\mathrm{mult}_\alpha(M)$ of M at α as the quantity $\mathrm{mult}_a(M)$ where a with $[a] = \alpha$ is arbitrary. Clearly isomorphic differential modules over K have the same multiplicity at a given α.

Lemma 2.3.9. *The following are equivalent:*

(i) *K is r-linearly surjective;*
(ii) *for each differential module N over K with $\dim_K N = r$ and every a, the C-linear map $y \mapsto (\partial - a)y\colon N \to N$ is surjective;*
(iii) *for each differential module N over K with $\dim_K N \leqslant r$ we have $\partial N = N$;*
(iv) *for $n = 1,\dots,r$, each matrix differential equation $y' = Fy + g$ with F an $n \times n$ matrix over K and $g \in K^n$ has a solution in K.*

Proof. For (i) $\Rightarrow$ (ii), let K be r-linearly surjective. The case $r = 0$ being trivial, let $r \geqslant 1$, so $C \neq K$. Let N be a differential module over K with $\dim_K N = r$. Towards proving that $y \mapsto (\partial - a)y\colon N \to N$ is surjective, we can assume by [ADH, 5.5.3] that $N = K[\partial]/K[\partial]A$ with A of order r. Let a, B be given, and let y range over K. It suffices to find $R \in K[\partial]$ and y such that $(\partial - a)R = yA - B$, that is, $yA - B \in (\partial - a)K[\partial]$ for some y, equivalently, $yA_a - B_a \in \partial K[\partial]$ for some y. Taking adjoints this amounts to finding y such that $A_a^* y - B_a^* \in K[\partial]\partial$, that is, $A_a^*(y) = B_a^*(1)$. Such y exists because K is r-linearly surjective.

For (ii) $\Rightarrow$ (iii), use that by [ADH, 5.5.2] each differential module over K of dimension $\leqslant r$ is a direct summand of a differential module over K of dimension r. For (iii) $\Rightarrow$ (iv), note that for an $n \times n$ matrix F over K ($n \geqslant 1$), with associated differential module M_F over K, and $g, y \in K^n$, we have $y' = Fy + g$ iff $\partial y = g$ in M_F [ADH, p. 277]. For (iv) $\Rightarrow$ (i), use [ADH, remarks before 5.4.3]. ☐

The previous lemma refines [ADH, 5.4.2], and leads to a more precise version of [ADH, 5.4.3] with a similar proof:

Corollary 2.3.10. *Suppose K is mn-linearly surjective and L is a differential field extension of K with $[L : K] = m$. Then L is n-linearly surjective.*

Proof. Let F be an $n \times n$ matrix over L, $n \geqslant 1$, and $g \in L^n$; by (iv) $\Rightarrow$ (i) in Lemma 2.3.9 with L in place of K it is enough to show that the equation $y' + Fy = g$ has a solution in L. For this, take a basis $e_1,\dots,e_m$ of the K-linear space L. As in the proof of [ADH, 5.4.3] we obtain an $mn \times mn$ matrix $F^\diamond$ over K and a column vector $g^\diamond \in K^{mn}$ such that any solution of $z' = F^\diamond z + g^\diamond$ in K yields a solution of $y' = Fy + g$ in L. Such a solution z exists by (i) $\Rightarrow$ (iv) in Lemma 2.3.9. ☐

Let $0 \to M_1 \xrightarrow{\iota} M \xrightarrow{\pi} M_2 \to 0$ be a short exact sequence of differential modules over K, where for notational simplicity we assume that M_1 is a submodule of M and ι is the natural inclusion. By restriction we obtain a sequence

$$0 \to \ker_{M_1}(\partial - a) \xrightarrow{\iota_a} \ker_M(\partial - a) \xrightarrow{\pi_a} \ker_{M_2}(\partial - a) \to 0, \tag{2.3.4}$$

of C-linear maps, not necessarily exact, but with $\mathrm{im}\,\iota_a = \ker \pi_a$. Hence

$$\mathrm{mult}_a(M) \ \leqslant\ \mathrm{mult}_a(M_1) + \mathrm{mult}_a(M_2). \tag{2.3.5}$$

Therefore $\Sigma(M) \subseteq \Sigma(M_1) \cup \Sigma(M_2)$. If $(\partial - a)M \cap M_1 = (\partial - a)M_1$, then π_a is surjective, so the sequence of C-linear maps (2.3.4) is exact, hence the inequality (2.3.5) is an equality. Thus the next corollary whose hypothesis holds if $M = M_1 \oplus M_2$ (internal direct sum of submodules of M) and ι and π are the natural morphisms, and also if K is r_1-linearly surjective for $r_1 := \dim_K M_1$, by Lemma 2.3.9:

Corollary 2.3.11. *Suppose* $(\partial - a)M \cap M_1 = (\partial - a)M_1$ *for each* a. *Then*

$$\operatorname{mult}_\alpha(M) = \operatorname{mult}_\alpha(M_1) + \operatorname{mult}_\alpha(M_2) \quad \textit{for each } \alpha;$$

in particular, $\Sigma(M) = \Sigma(M_1) \cup \Sigma(M_2)$.

Let $E := \operatorname{End}_C(M)$ be the C-algebra of endomorphisms of the C-linear space M. We have a ring morphism $K[\partial] \to E$ which assigns to $B \in K[\partial]$ the element $y \mapsto By$ of E, and we view E accordingly as $K[\partial]$-module: $(Bf)(y) := B \cdot f(y)$ for $f \in E$, $y \in M$. In the next corollary of Lemma 2.3.9 we let $\partial - a$ stand for the image of $\partial - a \in K[\partial]$ under the above ring morphism $K[\partial] \to E$.

Corollary 2.3.12. *If K is r-linearly surjective where $r = \dim_K M$, then*

$$\Sigma(M) = \big\{\alpha : (\partial - a) \notin E^\times \textit{ for some } a \textit{ with } [a] = \alpha\big\}.$$

Remark. The description of $\Sigma(M)$ in the previous corollary is reminiscent of the definition of the *spectrum* of an element x of an arbitrary K-algebra E with unit as the set of all a such that $(x - a) \notin E^\times$, as given in [41, §1]. (If $C = K$, then $K^\dagger = \{0\}$, and identifying $K/K^\dagger$ with K in the natural way, $\Sigma(M)$ is the spectrum of $\partial \in E$ in this sense.)

Let now N be a differential module over K and $s := \dim_K N$. From [ADH, p. 279] recall that the K-linear space $\operatorname{Hom}_K(M,N)$ of all K-linear maps $M \to N$ (of dimension $\dim_K \operatorname{Hom}_K(M,N) = rs$) is a differential module over K with

$$(\partial\phi)(f) \ := \ \partial(\phi f) - \phi(\partial f) \qquad \text{for } \phi \in \operatorname{Hom}_K(M,N) \text{ and } f \in M.$$

Given a $K[\partial]$-linear map $\theta\colon N \to P$ into a differential module P over K, this yields a $K[\partial]$-linear map $\operatorname{Hom}_K(M,\theta)\colon \operatorname{Hom}_K(M,N) \to \operatorname{Hom}_K(M,P)$ which sends any ϕ in $\operatorname{Hom}_K(M,N)$ to $\theta \circ \phi \in \operatorname{Hom}_K(M,P)$. The horizontal elements of $\operatorname{Hom}_K(M,N)$ are the $K[\partial]$-module morphisms $M \to N$; they are the elements of a finite-dimensional C-linear subspace $\operatorname{Hom}_{K[\partial]}(M,N)$ of $\operatorname{Hom}_K(M,N)$:

Lemma 2.3.13. *We have*

$$\dim_C \operatorname{Hom}_{K[\partial]}(M,N) \ \leqslant \ \dim_K \operatorname{Hom}_K(M,N),$$

with equality iff $\operatorname{Hom}_K(M,N)$ *is horizontal.*

Proof. By [ADH, 5.4.8 and remarks before 5.5.2], the dimension of the C-linear space of horizontal elements of M is at most $\dim_K M$, with equality iff M is horizontal. Now apply this with $\operatorname{Hom}_K(M,N)$ in place of M. $\square$

Recall: $M^* := \operatorname{Hom}_K(M,K)$ is the *dual* of M; see [ADH, 5.5]. By Lemma 2.3.13, the dimension of the C-linear subspace $\operatorname{Hom}_{K[\partial]}(M,K) = \ker_{M^*}\partial$ of M^* is at most $\dim_K M$. For the differential module $M = K[\partial]/K[\partial]A$ we can say more:

Lemma 2.3.14. *Suppose $M = K[\partial]/K[\partial]A$ and $e := 1 + K[\partial]A \in M$. Then for all $\phi \in \operatorname{Hom}_{K[\partial]}(M,K)$ we have $\phi(e) \in \ker A$, and the map*

$$\phi \mapsto \phi(e)\colon \operatorname{Hom}_{K[\partial]}(M,K)\partial \to \ker A$$

is an isomorphism of C-linear spaces.

Proof. The first claim follows from $A(\phi(e)) = \phi(Ae) = 0$, as $Ae = A + K[\partial]A$ is the zero element of M. This yields a C-linear map as displayed. To show that it is surjective, let $y \in \ker A$ be given. Then $B \mapsto B(y)\colon K[\partial] \to K$ is $K[\partial]$-linear with $K[\partial]A$ contained in its kernel, and thus yields $\phi \in \operatorname{Hom}_{K[\partial]}(M, K)$ with $\phi(e) = y$. Injectivity is clear since $M = K[\partial]e$. □

Given a, the map $\partial - a\colon M \to M$ is a ∂-compatible derivation on the K-linear space M [ADH, 5.5]. Let M_a be the K-linear space M equipped with this ∂-compatible derivation. Thus M_a is a differential module over K with

$$\dim_K M_a = \dim_K M = r, \quad \ker_{M^*}(\partial - a) = \operatorname{Hom}_{K[\partial]}(M_a, K).$$

Moreover, if e is a cyclic vector of M with $Ae = 0$, then e is a cyclic vector of M_a with $A_a e = 0$. Hence by the previous lemma:

Corollary 2.3.15. *Let A, e, M be as in Lemma 2.3.14. Then for each $\phi \in \ker_{M^*}(\partial - a)$ we have $\phi(e) \in \ker A_a$, and the map*

$$\phi \mapsto \phi(e)\colon \ker_{M^*}(\partial - a) \to \ker A_a$$

is an isomorphism of C-linear spaces. In particular, $\operatorname{mult}_\alpha(M^*) = \operatorname{mult}_\alpha(A)$*, so α is an eigenvalue of M^* iff α is an eigenvalue of A.*

Recall that every differential module M has finite length, denoted by $\ell(M)$ [ADH, pp. 36–38, 251], with $\ell(M) \leqslant \dim_K M = r$. We say that M **splits** if $\ell(M) = r$. By [ADH, 5.1.25], $M = K[\partial]/K[\partial]A$ splits iff A splits over K. By additivity of $\ell(-)$ and $\dim_K(-)$ on short exact sequences (see [ADH, 1.2]) we have:

Lemma 2.3.16. *Let N be a differential submodule of M. Then M splits iff both N and M/N split.*

Hence if N is a differential module over K, then $M \oplus N$ splits iff M and N split. Thus the least common left multiple of $A_1, \dots, A_m \in K[\partial]^{\neq}$, $m \geqslant 1$, splits over K iff $A_1, \dots, A_m$ split over K: use that the differential module

$$K[\partial]/K[\partial] \operatorname{lclm}(A_1, \dots, A_m)$$

over K is isomorphic to the image of the natural (diagonal) $K[\partial]$-linear map

$$K[\partial] \to \big(K[\partial]/K[\partial]A_1\big) \times \cdots \times \big(K[\partial]/K[\partial]A_m\big).$$

A $K[\partial]$-linear map $M \to N$ into a differential module N over K induces a $K[\partial]$-linear map $\phi^*\colon N^* \to M^*$ given by $\phi^*(f) = f \circ \phi$, and if ϕ is surjective, then ϕ^* is injective. This gives a contravariant functor $(-)^*$ from the category of differential modules over K to itself; the morphisms of this category are the $K[\partial]$-linear maps between differential modules over K. Using $\dim_K M = \dim_K M^* < \infty$ it follows easily from these facts that if $\phi\colon M \to N$ is an injective $K[\partial]$-linear map into a differential module N, then $\phi^*\colon N^* \to M^*$ is surjective.

Lemma 2.3.17. *$\ell(M) = \ell(M^*)$, so if M splits, then M^* splits as well.*

Proof. Induction on $\ell(M)$ using the canonical $K[\partial]$-linear isomorphism $M \cong M^{**}$ and what was said about the functor $(-)^*$ shows $\ell(M) = \ell(M^*)$. □

Let L be a differential field extension of K. Recall from [ADH, 5.9.2] that the base change $L \otimes_K M$ of M to L is a differential module over L with $\dim_L L \otimes_K M =$

$\dim_K M$. A $K[\partial]$-linear map $M \to N$ into a differential module N over K induces an $L[\partial]$-linear map

$$L \otimes_K \phi : \ L \otimes_K M \to L \otimes_K N, \qquad \lambda \otimes x \mapsto \lambda \otimes \phi(x),$$

and this yields a covariant functor $L \otimes_K -$ from the category of differential modules over K to the category of differential modules over L. Using the above invariance of dimension one checks easily that this functor transforms short exact sequences in the first category to short exact sequences in the second category. Hence, by an easy induction on $\ell(M)$ we have $\ell(M) \leqslant \ell(L \otimes_K M)$.

We say that M **splits over** L if $L \otimes_K M$ splits. The $K[\partial]$-linear isomorphism

$$x \mapsto 1 \otimes x \ : \ M \to K \otimes_K M$$

shows that M splits iff M splits over K, and then M splits over each differential field extension of K. Let E be a differential field extension of L. Then we have an $E[\partial]$-linear isomorphism

$$E \otimes_K M \to E \otimes_L (L \otimes_K M), \quad e \otimes x \mapsto e \otimes (1 \otimes x),$$

so if M splits over L, then M also splits over E. If N is a differential submodule of M, then M splits over L iff both N and M/N split over L.

Lemma 2.3.18. *If M splits over L, then so does M^*.*

Proof. For $\phi \in M^*$ we have the L-linear map

$$\mathrm{id}_L \otimes \phi \ : \ L \otimes M \to L \otimes_K K, \quad s \otimes y \mapsto s \otimes \phi(y) \quad (\lambda, s \in L, \phi \in M^*, y \in M).$$

We also have the $L[\partial]$-linear isomorphism $i_L \colon L \otimes_K K \to L$ given by $i_L(s \otimes 1) = s$ for $s \in L$. It is straightforward to check that this yields an L-linear isomorphism

$$L \otimes_K M^* \to (L \otimes_K M)^*, \qquad 1 \otimes \phi \mapsto i_L \circ (\mathrm{id}_L \otimes \phi) \quad (\phi \in M^*),$$

and that this map is even $L[\partial]$-linear. Now use Lemma 2.3.17. ☐

Call M **cyclic** if it has a cyclic vector, equivalently, for some A we have

$$M \ \cong \ K[\partial]/K[\partial]A, \text{ as } K[\partial]\text{-modules.}$$

Corollary 2.3.19. *We have $\sum_\alpha \mathrm{mult}_\alpha(M) \leqslant r$, hence $|\Sigma(M)| \leqslant r$. If moreover $\sum_\alpha \mathrm{mult}_\alpha(M) = r$, then M splits. Conversely, if K is 1-linearly surjective and M splits, then $\sum_\alpha \mathrm{mult}_\alpha(M) = r$.*

Proof. We prove this for M^* instead of M. (Then by various results above and the natural $K[\partial]$-linear isomorphism $M \cong M^{**}$ it also follows for M.) If M is cyclic, then $\sum_\alpha \mathrm{mult}_\alpha(M^*) \leqslant r$ by Lemma 2.3.3 and Corollary 2.3.15, and the rest follows using also Lemma 2.3.17 and remarks following Corollary 2.3.15. Thus we are done if $C \neq K$, by [ADH, 5.5.3].

If $C = K$, then a differential module over K is just a finite-dimensional vector space M over K equipped with a K-linear map $\partial \colon M \to M$, so we can use the well-known internal direct sum decomposition $\sum_a \ker_M(\partial - a) = \bigoplus_a \ker_M(\partial - a)$. ☐

Likewise we obtain from Corollary 2.3.5 and [ADH, 5.5.8]:

Corollary 2.3.20. *Suppose K is 1-linearly surjective and $\sum_\alpha \mathrm{mult}_\alpha(M) = r$. Then the map $\alpha \mapsto -\alpha$ restricts to a bijection $\Sigma(M) \to \Sigma(M^*)$ with $\mathrm{mult}_\alpha(M) = \mathrm{mult}_{-\alpha}(M^*)$ for each α.*

We now aim for a variant of Corollary 2.3.20: Corollary 2.3.23 below.

Lemma 2.3.21. *Suppose* $\dim_C \ker A = r$. *Then* $\dim_C \ker A^* = r$.

Proof. The case $r = 0$ being trivial, suppose $r \geqslant 1$ and set $M := K[\partial]/K[\partial]A$, so $M \neq \{0\}$. By Lemma 2.3.14, M^* is horizontal, hence M is also horizontal by [ADH, remark after 5.5.5], and therefore $\dim_C \ker A^* = \dim_C \ker_M \partial = r$ by Lemma 2.3.14 and [ADH, 5.5.8]. $\square$

Corollary 2.3.22. *Let* $d := \dim_C \ker A$ *and suppose* K *is* $(r-d)$*-linearly surjective. Then* $\dim_C \ker A^* = d$.

Proof. It suffices to show $\dim_C \ker A^* \geqslant d$, since then the reverse inequality follows by interchanging the role of A and A^*. Let $y_1, \dots, y_d$ be a basis of the C-linear space $\ker A$. Then

$$L(Y) := \operatorname{wr}(Y, y_1, \dots, y_d) \in K\{Y\}$$

is homogeneous of degree 1 and order d with zero set $Z(L) = \ker A$ [ADH, 4.1.13]. So with $B \in K[\partial]$ the linear part of L we have $A = DB$ where $D \in K[\partial]$ has order $r - d$, by [ADH, 5.1.15(i)], so $A^* = B^*D^*$ and $D^*(K) = K$. Hence

$$\dim_C \ker A^* \ = \ \dim_C \ker B^* + \dim_C \ker D^* \ \geqslant \ \dim_C \ker B^* \ = \ \dim_C \ker B \ = \ d$$

where we used [ADH, remark before 5.1.12] for the first equality and the previous lemma (applied to B in place of A) for the second equality. $\square$

Suppose now that $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Then Corollary 2.3.22 and $A^{**} = A$ give $\dim_C \ker A = \dim_C \ker A^*$ (even when $\dim_C \ker A = 0$). Hence for all a we have $\dim_C \ker A_a = \dim_C \ker (A^*)_{-a}$. This leads to:

Corollary 2.3.23. *If* $r \geqslant 1$ *and* K *is* $(r-1)$*-linearly surjective, then we have a bijection* $\alpha \mapsto -\alpha \colon \Sigma(M) \to \Sigma(M^*)$, *and* $\operatorname{mult}_\alpha(M) \ = \ \operatorname{mult}_{-\alpha}(M^*)$ *for all* α.

Complex conjugation (*)**.** *In this subsection* $K = H[\mathrm{i}]$ *where* H *is a differential subfield of* K, $\mathrm{i}^2 = -1$, *and* $\mathrm{i} \notin H$. Then $C = C_H[\mathrm{i}]$. Recall that $A \in K[\partial]^{\neq}$ has order r. The complex conjugation automorphism $z = g + h\mathrm{i} \mapsto \overline{z} := g - h\mathrm{i}$ $(g, h \in H)$ of the differential field K induces an automorphism $\alpha \mapsto \overline{\alpha}$ of the group $K/K^\dagger$ with $\overline{\alpha} = [\overline{a}]$ for $\alpha = [a]$, $a \in K$. The automorphism $z \mapsto \overline{z}$ of K extends uniquely to an automorphism $D \mapsto \overline{D}$ of the ring $K[\partial]$ with $\overline{\partial} = \partial$. If A and $B \neq 0$ have the same type, then so do $\overline{A}$ and $\overline{B}$. (Lemma 2.3.6.) Now $\overline{A(f)} = \overline{A}(\overline{f})$ for $f \in K$, so $\dim_C \ker_K A = \dim_C \ker_K \overline{A}$. Moreover, $\overline{A_a} = \overline{A}_{\overline{a}}$, hence $\operatorname{mult}_\alpha(A) = \operatorname{mult}_{\overline{\alpha}}(\overline{A})$ for all α; so α is an eigenvalue of A iff $\overline{\alpha}$ is an eigenvalue of $\overline{A}$. Note that $\overline{A^*} = \overline{A}^*$. We call $\overline{A^*}$ the **conjugate adjoint** of A. Corollary 2.3.5 yields:

Corollary 2.3.24. *If* K *is* 1*-linearly surjective and* $\sum_\alpha \operatorname{mult}_\alpha(A) = r$, *then we have a bijection* $\alpha \mapsto -\overline{\alpha} \colon \Sigma(A) \to \Sigma(\overline{A^*})$, *with* $\operatorname{mult}_\alpha(A) = \operatorname{mult}_{-\overline{\alpha}}(\overline{A^*})$ *for all* α.

Next, let M be a (left) $K[\partial]$-module. Then we define $\overline{M}$ as the $K[\partial]$-module arising from M by replacing its scalar multiplication $(A, f) \mapsto Af \colon K[\partial] \times M \to M$ with

$$(A, f) \mapsto \overline{A}f \ : \ K[\partial] \times \overline{M} \to \overline{M}.$$

We call $\overline{M}$ the **complex conjugate** of M. Note that $\overline{\overline{M}} = M$. If $\phi \colon M \to N$ is a morphism of $K[\partial]$-modules, then ϕ is also a morphism of $K[\partial]$-modules $\overline{M} \to \overline{N}$, which we denote by $\overline{\phi}$. Hence we have a covariant functor $\overline{(\,\cdot\,)}$ from the category of $K[\partial]$-modules to itself. We have $\dim_K M = \dim_K \overline{M}$, hence if M is a differential module over K, then so is $\overline{M}$. Thus $\overline{(\,\cdot\,)}$ restricts to a functor from the category of differential modules over K to itself. If P is an $r \times r$ matrix over K and $M = M_P$ is the differential module associated to P [ADH, p. 277], then the differential modules $\overline{M}$ and $M_{\overline{P}}$ over K both have underlying additive group K^r, and the map $e \mapsto \overline{e}\colon \overline{M} \to M_{\overline{P}}$ is an isomorphism of differential modules over K.

Example 2.3.25. For $M = K[\partial]$ we have an isomorphism $B \mapsto \overline{B}\colon M \to \overline{M}$ of $K[\partial]$-modules. For $N = K[\partial]/K[\partial]A$ we have an isomorphism

$$B + K[\partial]A \mapsto \overline{B} + K[\partial]\overline{A} \,:\, \overline{N} \to K[\partial]/K[\partial]\overline{A}$$

of differential modules over K.

Below M is a differential module over K and $r = \dim_K M$. Then for each B we have $\ker_M B = \ker_{\overline{M}} \overline{B}$. Hence $\operatorname{mult}_\alpha(M) = \operatorname{mult}_{\overline{\alpha}}(\overline{M})$ for all α, so α is an eigenvalue of M iff $\overline{\alpha}$ is an eigenvalue of $\overline{M}$.

Next, let N be a differential module over K. A map $\phi\colon M \to N$ is K-linear if $\phi\colon \overline{M} \to \overline{N}$ is K-linear, so $\operatorname{Hom}_K(M,N)$ and $\operatorname{Hom}_K(\overline{M},\overline{N})$ have the same underlying additive group. It is easy to check that for the differential module $P := \operatorname{Hom}_K(M,N)$ we have $\overline{P} = \operatorname{Hom}_K(\overline{M},\overline{N})$. Thus $\overline{M^*} = \operatorname{Hom}_K(\overline{M},\overline{K})$. In view of the isomorphism $z \mapsto \overline{z}\colon K \to \overline{K}$ of differential modules over K this yields an isomorphism $\overline{M^*} \cong \overline{M}^*$ of differential modules over K. We call $\overline{M^*}$ the **conjugate dual** of M. From Corollaries 2.3.20 and 2.3.23 we obtain:

Corollary 2.3.26. *Suppose K is 1-linearly surjective and $\sum_\alpha \operatorname{mult}_\alpha(M) = r$, or $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Then the map $\alpha \mapsto -\overline{\alpha}$ restricts to a bijection $\Sigma(M) \to \Sigma(\overline{M^*})$ with $\operatorname{mult}_\alpha(M) = \operatorname{mult}_{-\overline{\alpha}}(\overline{M^*})$ for all α.*

In the remainder of this section we discuss eigenvalues of differential modules over K in the presence of a valuation on K. This is only used for the proof of Lemma 7.4.27 in Section 7.4. In preparation for this, we first study lattices over valued fields.

Lattices (*)**.** *In this subsection F is a valued field with valuation ring R.* Let L be an R-module, with its torsion submodule

$$L_{\text{tor}} = \{y \in L : ry = 0 \text{ for some } r \in R^{\neq}\}.$$

Call L *torsion-free* if $L_{\text{tor}} = \{0\}$, and a *torsion module* if $L_{\text{tor}} = L$. For the following basic fact, cf. [40, VI, §4, Lemme 1]:

Lemma 2.3.27. *Every finitely generated torsion-free R-module is free.*

Proof. Let L be a finitely generated torsion-free R-module. Let $x_1,\dots,x_m \in L$ be distinct such that $\{x_1,\dots,x_m\}$ is a minimal set of generators of L [ADH, p. 44]. Towards a contradiction, suppose $r_1x_1 + \cdots + r_mx_m = 0$ with $r_1,\dots,r_m$ in R not all zero. By reordering, arrange $r_j \in r_1R$ for $j = 2,\dots,m$. Torsion-freeness of L yields $x_1 + s_2x_2 + \cdots + s_mx_m = 0$ where $s_j := r_j/r_1$ for $j = 2,\dots,m$. Hence $\{x_2,\dots,x_m\}$ is also a set of generators of L, contradicting the minimality of $\{x_1,\dots,x_m\}$. Thus $x_1,\dots,x_m$ are R-linearly independent. □

Let now M be a finite-dimensional F-linear space and $m := \dim_F M$.

Lemma 2.3.28. *Let L be a finitely generated R-submodule of M. Then L is free of rank $\leqslant m$, and the following are equivalent:*

(i) *L has rank m;*
(ii) *L has a basis which is also a basis of the F-linear space M;*

(iii) L *contains a basis of* M;
(iv) L *contains a generating set of* M;
(v) *the* R*-module* M/L *is a torsion module.*

Proof. Freeness of L follows from Lemma 2.3.27. Every set of R-linearly independent elements of L is F-linearly independent, so $\operatorname{rank}(L) \leqslant m$. Let $y_1, \dots, y_n$ be a basis of L. Assuming $n < m$ yields $z \in M$ such that $y_1, \dots, y_n, z$ are F-linearly independent, so M/L is not a torsion module. This shows (v) $\Rightarrow$ (i), and (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear. $\square$

A finitely generated R-submodule L of M is called an R-**lattice** in M, or just a **lattice** in M if R is understood from the context, if it satisfies one of the equivalent conditions (i)–(v) in Lemma 2.3.28. If L is a lattice in M, then every basis of L is a basis of the F-linear space M, and every lattice of M is of the form $\sigma(L)$ for some automorphism σ of M. Next some easy consequences of Lemma 2.3.28:

Corollary 2.3.29. *If* $\pi\colon M \to M'$ *is a surjective morphism of* F*-linear spaces and* L *a lattice in* M*, then* $L' := \pi(L)$ *is a lattice in* M'.

Corollary 2.3.30. *Let* N *be an* F*-linear subspace of* M *and* L *a lattice in* M*. Then* $L \cap N$ *is a lattice in* N.

Proof. Take a basis $x_1, \dots, x_n$ of N. Lemma 2.3.28(v) gives $r_1, \dots, r_n \in R^{\neq}$ with $r_1x_1, \dots, r_nx_n \in L$. Then $r_1x_1, \dots, r_nx_n$ is a basis of N contained in $L \cap N$. Now apply Lemma 2.3.28(iii) to N, $L \cap N$ in place of M, L. $\square$

Corollary 2.3.31. *If* L *is a lattice in* M *and* E *a valued field extension of* F *with valuation ring* S*, then the* E*-linear space* $M_E := E \otimes_F M$ *has dimension* m*, and the* S*-submodule* L_E *of* M_E *generated by the image of* L *under the* F*-linear embedding* $y \mapsto 1 \otimes y\colon M \to M_E$ *is an* S*-lattice in* M_E.

For $i = 1, 2$ let M_i be a F-linear space with $m_i := \dim_K M_i < \infty$ and L_i be a lattice in M_i. Then $L_1 \oplus L_2$ is a lattice in $M_1 \oplus M_2$, and the R-submodule of $M_1 \otimes_F M_2$ generated by the elements $y_1 \otimes y_2$ ($y_1 \in L_1$, $y_2 \in L_2$) is a lattice in $M_1 \otimes_F M_2$. The F-linear space $\operatorname{Hom}(M_1, M_2) = \operatorname{Hom}_F(M_1, M_2)$ of F-linear maps $M_1 \to M_2$ has dimension m_1m_2, and the R-module $\operatorname{Hom}(L_1, L_2)$ of R-linear maps $L_1 \to L_2$ is free of rank m_1m_2. Each R-linear map $\phi\colon L_1 \to L_2$ extends uniquely to an F-linear map $\widehat{\phi}\colon M_1 \to M_2$, and $\phi \mapsto \widehat{\phi}$ is an embedding of $\operatorname{Hom}(L_1, L_2)$ into $\operatorname{Hom}(M_1, M_2)$ viewed as R-module. We identify $\operatorname{Hom}(L_1, L_2)$ via this embedding with its image in $\operatorname{Hom}(M_1, M_2)$; then $\operatorname{Hom}(L_1, L_2)$ is a lattice in $\operatorname{Hom}(M_1, M_2)$. In particular, if L is a lattice in M, then $L^* = \operatorname{Hom}(L, R)$ is a lattice in $M^* = \operatorname{Hom}(M, F)$.

Lattices in differential modules (*). *In the rest of this section* K *is equipped with a valuation ring* $\mathcal{O}$ *making* K *a valued differential field with small derivation. We also let* M *be a differential module over* K. Thus M is a $K[\partial]$-module which is finite-dimensional as K-linear space. We have the subring $\mathcal{O}[\partial]$ of $K[\partial]$. A **lattice** in M is an $\mathcal{O}[\partial]$-submodule of M that is also an $\mathcal{O}$-lattice in the K-linear space M. An $\mathcal{O}$-lattice L in the K-linear space M is a lattice in the differential module M

iff $\partial L \subseteq L$, iff there is a generating set S of the $\mathcal{O}$-module L with $\partial S \subseteq L$. If $a \neq 0$ and $a^\dagger \preccurlyeq 1$ and L is a lattice in M, then aL is also a lattice in M.

Examples 2.3.32.

(1) Suppose $M = M_N$ where N is an $n \times n$ matrix over $\mathcal{O}$ ($n \geqslant 1$). The underlying K-linear space of M is K^n, and for each $e \in M$ we have $\partial e = e' - Ne$ [ADH, p. 277], so $L := \mathcal{O}^n$ is a lattice in M.

(2) Suppose $M \cong K[\partial]/K[\partial]A$ where $A \in \mathcal{O}[\partial]$ is monic. Let e be a cyclic vector of M with $Ae = 0$. Then the K-linear space M has basis $e, \partial e, \dots, \partial^{r-1}e$ and $L := \mathcal{O}e + \mathcal{O}\partial e + \cdots + \mathcal{O}\partial^{r-1}e$ is a lattice in M.

For $i = 1, 2$ let M_i be a differential module over K and L_i be a lattice in M_i. Then $L_1 \oplus L_2$ is a lattice in the differential module $M_1 \oplus M_2$ over K, and the $\mathcal{O}$-submodule of $M_1 \otimes_K M_2$ generated by the elements $y_1 \otimes y_2$ ($y_i \in L_i$, $i = 1, 2$) is a lattice in the differential module $M_1 \otimes_K M_2$ over K. Also, $\operatorname{Hom}(L_1, L_2)$ is a lattice in the differential module $\operatorname{Hom}_K(M_1, M_2)$ over K.

Let L be a lattice in M. If $\pi\colon M \to N$ is a surjective morphism of differential modules over K, then $\pi(L)$ is a lattice in N. If N is a differential submodule of M, then $L \cap N$ is a lattice in N. Using the notation from Corollary 2.3.31 we have:

Lemma 2.3.33. *Let L be a lattice in M and E be a valued differential field extension of K with small derivation. Then L_E is a lattice in the base change $M_E = E \otimes_K M$ of the differential module M over K to a differential module over E.*

If $A \in \mathcal{O}[\partial]$ is monic (of order r by our convention), then $\mathcal{O}[\partial]A$ is a left ideal of the ring $\mathcal{O}[\partial]$, and the resulting left $\mathcal{O}[\partial]$-module $\mathcal{O}[\partial]/\mathcal{O}[\partial]A$ is free on $e, \partial e, \dots, \partial^{r-1}e$ for $e := 1 + \mathcal{O}[\partial]A$, as is easily verified. Conversely, if L is a left $\mathcal{O}[\partial]$-module free on $e, \partial e, \dots, \partial^{r-1}e$, $e \in L$, then the unique monic $A \in \mathcal{O}[\partial]$ (of order r) such that $Ae = 0$ yields an isomorphism $\mathcal{O}[\partial]/\mathcal{O}[\partial]A \to L$ sending $1 + \mathcal{O}[\partial]A$ to e.

Next, let $\boldsymbol{k} = \mathcal{O}/\smallo$ be the differential residue field of K; cf. [ADH, 4.4]. Here is a version of the cyclic vector theorem [ADH, 5.5.3] for lattices:

Proposition 2.3.34. *Suppose the derivation on $\boldsymbol{k}$ is nontrivial and L is a lattice in M and* $\dim_K M = r$*. Then $L \cong \mathcal{O}[\partial]/\mathcal{O}[\partial]A$ for some monic $A \in \mathcal{O}[\partial]$.*

The case $r = 0$ being trivial, we assume for the proof below that $r \geqslant 1$. We now introduce a tuple $Y = (Y_0, \dots, Y_{r-1})$ of distinct differential indeterminates over K, and let i, j, k, l range over $\{0, \dots, r-1\}$.

Lemma 2.3.35. *For all i, j, let $P_{ij} \in Y_i^{(j)} + \sum_{k<r,\ l<j} K\, Y_k^{(l)}$. Then the coefficient of $Y_0 Y_1' \cdots Y_{r-1}^{(r-1)}$ in* $\det(P_{ij}) \in K\{Y\}$ *is* 1*.*

Proof. For $p = 0, \dots, r-1$ we prove by induction on p that the coefficient of $Y_0 Y_1' \cdots Y_p^{(p)}$ in $\det(P_{ij})_{i,j \leqslant p} \in K\{Y\}$ is 1 (which for $p = r-1$ gives the desired result). The case $p = 0$ is clear, so assume $p \geqslant 1$. Then $Y_p^{(p)}$ occurs in the matrix $(P_{ij})_{i,j \leqslant p}$ only in the (p,p)-entry $P_{pp} \in Y_p^{(p)} + \sum_{k<r,\ l<p} K\, Y_l^{(l)}$, and so the coefficient of $Y_0 Y_1' \cdots Y_p^{(p)}$ in $\det(P_{ij})_{i,j \leqslant p}$ is the coefficient of $Y_0 Y_1' \cdots Y_{p-1}^{(p-1)}$ in $\det(P_{ij})_{i,j \leqslant p-1}$, and the latter is 1 by inductive assumption. $\square$

Proof of Proposition 2.3.34. Let $z_0, \dots, z_{r-1}$ be a basis of the $\mathcal{O}$-module L. Consider the base change $\widehat{M} := K\{Y\} \otimes_K M$ of M to the differential K-algebra $K\{Y\}$, cf. [ADH, p. 304]. So $\widehat{M}$ is a left $K\{Y\}[\partial]$-module and the $K\{Y\}$-module $\widehat{M}$ is free on $1\otimes z_0,\dots,1\otimes z_{r-1}$. Set

$$\widehat{e} \ := \ Y_0\otimes z_0+\cdots+Y_{r-1}\otimes z_{r-1}\in \widehat{M}$$

and let $P_{ij}\in K\{Y\}$ be such that

$$\partial^j\widehat{e} \ = \ P_{0j}\otimes z_0+\cdots+P_{r-1,j}\otimes z_{r-1}.$$

An easy induction on j using $\partial L\subseteq L$ shows that $P_{ij}\in Y_i^{(j)}+\sum_{k<r,\ l<j}\mathcal{O}\,Y_k^{(l)}$. Put $P:=\det(P_{ij})\in\mathcal{O}\{Y\}$; then $v(P)=0$ by the lemma above, so [ADH, 4.2.1] applied to $\boldsymbol{k}$ and the image of P under the natural morphism $\mathcal{O}\{Y\}\to\boldsymbol{k}\{Y\}$ in place of K and P, respectively, yields an $a\in\mathcal{O}^r$ such that $P(a)\asymp 1$. We obtain a $K[\partial]$-module morphism

$$\phi\colon \widehat{M}\to M \quad \text{ with } \quad \phi(Q\otimes z)=Q(a)z \text{ for } Q\in K\{Y\} \text{ and } z\in M.$$

Put $R:=\mathcal{O}\{Y\}$, a differential subring of $K\{Y\}$, and let $\widehat{L}$ be the $R[\partial]$-submodule of $\widehat{M}$ generated by $1\otimes z_0,\dots,1\otimes z_{r-1}$, so

$$\widehat{L} \ = \ \{Q_0\otimes z_0+\cdots+Q_{r-1}\otimes z_{r-1}:\ Q_0,\dots,Q_{r-1}\in\mathcal{O}\{Y\}\}.$$

Then $\partial^j\widehat{e}\in\widehat{L}$ for all j and $\phi(\widehat{L})=L$. With $e:=\phi(\widehat{e})$ we have

$$\partial^j e=\phi(\partial^j\widehat{e})=P_{0j}(a)z_0+\cdots+P_{r-1,j}(a)z_{r-1}$$

and $\det(P_{ij}(a))=P(a)\in\mathcal{O}^\times$, so $L=\mathcal{O}e+\mathcal{O}\partial e+\cdots+\mathcal{O}\partial^{r-1}e$. By a remark preceding Proposition 2.3.34, this concludes its proof. □

Remark. Taking $K=\mathcal{O}$, Proposition 2.3.34 yields another proof of [ADH, 5.5.3], in the spirit of [51]; cf. [48]. Note also that e as constructed in the proof of Proposition 2.3.34 is a cyclic vector of M, and so yields the isomorphism $M\cong K[\partial]/K[\partial]A$ sending e to $1+K[\partial]A$, with monic $A\in\mathcal{O}$ (of order r by convention) determined by the requirement $Ae=0$.

Eigenvalues of bounded operators (*). *In this subsection $A\in\mathcal{O}[\partial]$ is monic, of order r by earlier convention.* Recall that $[a]:=a+K^\dagger$ for $a\in K$. Put

$$[\mathcal{O}]:=(\mathcal{O}+K^\dagger)/K^\dagger=\big\{[a]:a\in\mathcal{O}\big\} \qquad \text{(a divisible subgroup of } K/K^\dagger).$$

Thus $\Sigma(A)\subseteq[\mathcal{O}]$ by (2.3.2) and [ADH, 5.6.3]. More precisely, with $\boldsymbol{k}$ denoting the differential residue field of K, recall from [ADH, 5.6] that the residue map $a\mapsto\operatorname{res}a\colon\mathcal{O}\to\boldsymbol{k}$ extends to a ring morphism $B\mapsto\operatorname{res}B\colon\mathcal{O}[\partial]\to\boldsymbol{k}[\partial]$ with $\partial\mapsto\partial$. For each $B\in\mathcal{O}[\partial]$ and $y\in\mathcal{O}$ we have $B(y)\in\mathcal{O}$ and $\operatorname{res}\big(B(y)\big)=(\operatorname{res}B)(\operatorname{res}y)$. Also, $\operatorname{res}A\in\boldsymbol{k}[\partial]$ is monic with $\operatorname{order}\operatorname{res}A=\operatorname{order}A=r$. By [ADH, 5.6.3], if $B,D\in K[\partial]$ are monic and $A=BD$, then $B,D\in\mathcal{O}[\partial]$. In particular, all $a_1,\dots,a_r\in K$ such that $A=(\partial-a_1)\cdots(\partial-a_r)$ are in $\mathcal{O}$, and

$$\operatorname{res}A \ = \ (\partial-\operatorname{res}a_1)\cdots(\partial-\operatorname{res}a_r).$$

Moreover, using also (2.3.2), we conclude:

Lemma 2.3.36. *If $A=B(\partial-a)$, $B\in K[\partial]$, then $a\in\mathcal{O}$, $B\in\mathcal{O}[\partial]$, and $\operatorname{res}A=(\operatorname{res}B)\cdot(\partial-\operatorname{res}a)$. Hence for each $\alpha\in\Sigma(A)$ there is an $a\in\mathcal{O}$ such that $\alpha=[a]$ and $\operatorname{res}a+\boldsymbol{k}^\dagger\in\Sigma(\operatorname{res}A)$.*

Suppose now $\partial\mathcal{O} \subseteq \smallo$. Then the derivation of $\boldsymbol{k}$ is trivial, and we have a $\boldsymbol{k}$-algebra isomorphism $P(Y) \mapsto P(\partial)\colon \boldsymbol{k}[Y] \to \boldsymbol{k}[\partial]$. We let the **characteristic polynomial** of B be the $\chi_B \in \boldsymbol{k}[Y]$ satisfying $\chi_B(\partial) = \operatorname{res} B$. Then $B \mapsto \chi_B\colon \mathcal{O}[\partial] \to \boldsymbol{k}[Y]$ is a ring morphism extending the residue morphism $\mathcal{O} \to \boldsymbol{k}$ with $\partial \mapsto Y$. Identifying $\boldsymbol{k}/\boldsymbol{k}^\dagger$ with $\boldsymbol{k}$ in the natural way, the set of zeros of χ_A in $\boldsymbol{k}$ is $\Sigma(\operatorname{res} A)$. Thus by Lemma 2.3.36:

Corollary 2.3.37. *If $A \in K[\partial](\partial - a)$, then $a \in \mathcal{O}$ and $\chi_A(\operatorname{res} a) = 0$. Hence for each $\alpha \in \Sigma(A)$ there is an $a \in \mathcal{O}$ such that $\alpha = [a]$ and $\chi_A(\operatorname{res} a) = 0$.*

If $\mathcal{O} = C + \smallo$, then $\partial\mathcal{O} \subseteq \smallo$ and the residue morphism $\mathcal{O} \to \boldsymbol{k}$ restricts to an isomorphism $C \to \boldsymbol{k}$, via which we identify $\boldsymbol{k}$ with C making χ_A an element of $C[Y]$.

In the rest of this subsection $K = H[\mathrm{i}]$ where H is a real closed differential subfield of K such that the valuation ring $\mathcal{O}_H := \mathcal{O} \cap H$ of H is convex with respect to the ordering of H and $\mathcal{O}_H = C_H + \smallo_H$. Then $C = C_H[\mathrm{i}]$. A remark after Corollary 1.2.5 then yields $\mathcal{O} = \mathcal{O}_H + \mathcal{O}_H\mathrm{i} = C + \smallo$. Using that remark and Lemma 1.2.4 we have $K^\dagger \subseteq H^\dagger + \smallo_H\mathrm{i} \subseteq H^\dagger + \mathcal{O}$, and thus:

Lemma 2.3.38. $H^\dagger + \mathcal{O} \ = \ H^\dagger + C_H + C_H\mathrm{i} + \smallo \ = \ \big\{a \in K : [a] \in [\mathcal{O}]\big\}$.

Lemma 2.3.39. *Suppose that $C_H \subseteq H^\dagger$, and let $\alpha \in [\mathcal{O}]$. Then there is a unique $b \in C_H$ such that $\alpha = [b\mathrm{i} + \varepsilon]$ for some $\varepsilon \in \smallo$. For this b we have*

$$\operatorname{mult}_\alpha(A) \ \leqslant \sum_{c \in C,\ \operatorname{Im} c = b} \operatorname{mult}_c(\chi_A). \tag{2.3.6}$$

Proof. Lemma 2.3.38 and $C_H \subseteq H^\dagger$ yield the existence of $b \in C_H$ such that $\alpha = [b\mathrm{i} + \varepsilon]$ for some $\varepsilon \in \smallo$. Since $K^\dagger \subseteq H + \smallo_H\mathrm{i}$, there is at most one $b \in C_H$. We prove (2.3.6) by induction on r. The cases $r = 0$ and $\operatorname{mult}_\alpha(A) = 0$ being trivial, suppose $r \geqslant 1$ and $\alpha \in \Sigma(A)$. From (2.3.2) and [ADH, 5.6.3] we get $a \in \mathcal{O}$ and monic $B \in \mathcal{O}[\partial]$ with $[a] = \alpha$ and $A = B(\partial - a)$. Then $\operatorname{mult}_\alpha(A) \leqslant \operatorname{mult}_\alpha(B) + 1$ by Lemma 2.3.2, and with $c \in C$ such that $a - c \prec 1$ we have $b = \operatorname{Im} c$ and $\chi_A(c) = 0$. Now apply the inductive hypothesis to B in place of A. □

Remark. The inequality (2.3.6) is strict in general: H can be an H-field with an element $x \in H$ such that $x' = 1$. Then $x \succ 1$, $1/x \notin \mathrm{I}(H)$, so $\varepsilon := i/x \in \smallo \setminus K^\dagger$. Then for $A := (\partial - (\mathrm{i} + \varepsilon))(\partial - \mathrm{i})$ we have $\operatorname{mult}_{[\mathrm{i}]} A = 1$ while $\chi_A = (Y - i)^2$.

Corollary 2.3.40. *Suppose K has asymptotic integration and is $(r-1)$-newtonian, $r \geqslant 1$. Let $c_1, \dots, c_r \in C$ be the zeros of χ_A, and suppose $c_1, \dots, c_r$ are distinct and $\operatorname{Re} c_1 \geqslant \cdots \geqslant \operatorname{Re} c_r$. Then for each splitting $(a_1, \dots, a_r)$ of A over K we have $a_1, \dots, a_r \in \mathcal{O}$. Moreover, there is a unique such splitting of A over K such that $a_1 - c_1, \dots, a_r - c_r \prec 1$.*

Proof. The first claim is immediate from [ADH, 5.6.3]. We prove the second claim by induction on r. The case $r = 1$ being trivial, suppose $r > 1$. Corollary 1.8.47 yields an $a_r \in \mathcal{O}$ with $\operatorname{Ri}(A)(a_r) = 0$ and $a_r - c_r \prec 1$, and then $A = B(\partial - a_r)$ where $B \in \mathcal{O}[\partial]$ is monic, by [ADH, 5.6.3, 5.8.7]. By inductive hypothesis $B = (\partial - a_1) \cdots (\partial - a_{r-1})$ where $a_j \in \mathcal{O}$ with $a_j - c_j \prec 1$ for $j = 1, \dots, r-1$. This shows existence. Uniqueness follows in a similar way, using Corollary 1.8.50. □

Bounded differential modules (*). *In this subsection A is monic and M is a differential module over K.* We call M **bounded** if there exists a lattice in M. By remarks in an earlier subsection the class of bounded differential modules over K is quite robust: if M_1, M_2 are bounded differential modules over K, then so are the differential modules $M_1 \oplus M_2$, $M_1 \otimes_K M_2$, and $\operatorname{Hom}_K(M_1, M_2)$ over K, and if M is bounded, then so is every differential submodule of M, every image of M under a morphism of differential modules over K, and every base change of M to a valued differential field extension of K with small derivation.

Example 2.3.41. Let $u \in K$ and suppose $M = K$ with $\partial a = a' + ua$ for all a. Then for $e \in K^\times$, $\mathcal{O}e$ is a lattice in M iff $e' + ue = \partial e \in \mathcal{O}e$. Hence M is bounded iff $u \in \mathcal{O} + K^\dagger$.

If $A_{\ltimes a} \in \mathcal{O}[\partial]$ for some $a \neq 0$, then $K[\partial]/K[\partial]A$ is bounded by Example 2.3.32(2).

Lemma 2.3.42. *Suppose $M = K[\partial]/K[\partial]A$ and $r = 1$. Then*

$$M \text{ is bounded} \iff A_{\ltimes a} \in \mathcal{O}[\partial] \text{ for some } a \neq 0.$$

Proof. Let $A = \partial - u$, $u \in K$. Identifying K with M via $a \mapsto a + K[\partial]A$ we have $\partial a = a' + ua$ for all a in $K = M$, so if M is bounded, then Example 2.3.41 gives $a \neq 0$ with $u \in \mathcal{O} + a^\dagger$, hence $A_{\ltimes a} \in \mathcal{O}[\partial]$. □

Lemma 2.3.43. *Suppose the valuation ring $\mathcal{O}$ is discrete (that is, a DVR) and $M = K[\partial]/K[\partial]A$ is bounded. Then $A_{\ltimes a^{-1}} \in \mathcal{O}[\partial]$ for some $a \in \mathcal{O}^{\neq}$.*

Proof. Let L be a lattice in M and $e := 1 + K[\partial]A$, a cyclic vector of M. Since M/L is a torsion module we get $a \in \mathcal{O}^{\neq}$ with $f := ae \in L$. Because $\mathcal{O}$ is noetherian, the submodule of the finitely generated $\mathcal{O}$-module L generated by $f, \partial f, \partial^2 f, \dots$ is itself finitely generated, and this yields n with $\partial^n f \in \mathcal{O}f + \mathcal{O}\partial f + \cdots + \mathcal{O}\partial^{n-1} f$ [122, Chapter X, §1]. We obtain a monic $B \in \mathcal{O}[\partial]$ of order n with $Bf = 0$. Then $B_{\ltimes a} e = 0$, so $B \in K[\partial]A_{\ltimes a^{-1}}$, and thus $A_{\ltimes a^{-1}} \in \mathcal{O}[\partial]$ by [ADH, 5.6.3]. □

If K is monotone K, then $v(B_{\ltimes a}) = v(B)$ for all B and $a \neq 0$, by [ADH, 4.5.4]. If $\mathcal{O}$ is discrete, then K is monotone by [ADH, 6.1.2]. Hence by Lemma 2.3.43:

Corollary 2.3.44. *Suppose $\mathcal{O}$ is discrete. Then:*

$$\text{the differential module } K[\partial]/K[\partial]A \text{ over } K \text{ is bounded} \iff A \in \mathcal{O}[\partial].$$

Remark. In the case $(K, \mathcal{O}) = \big(\mathbb{C}((t)), \mathbb{C}[[t]]\big)$ and $\partial = t\frac{d}{dt}$, bounded differential modules over K are called *regular singular* in [158], and Corollary 2.3.44 in this case is implicit in the proof of [158, Proposition 3.16].

Lemma 2.3.45. *Suppose $M \cong K[\partial]/K[\partial]A$ where $A \in \mathcal{O}[\partial]$. Then $\Sigma(M) \subseteq [\mathcal{O}]$.*

Proof. The case $r = 0$ is trivial, so assume $r \geqslant 1$. Corollary 2.3.15 and remarks in the last subsection yield $\Sigma(M^*) = \Sigma(A) \subseteq [\mathcal{O}]$. By [ADH, 5.5.8] we have $M^* \cong K[\partial]/K[\partial]B$ where $B := (-1)^r A^* \in \mathcal{O}[\partial]$ is monic. Also $M \cong M^{**}$, hence $\Sigma(M) = \Sigma(M^{**}) \subseteq [\mathcal{O}]$ by the above applied to M^*, B in place of M, A. □

Corollary 2.3.46. *Suppose M is bounded. Assume also that the derivation of $\boldsymbol{k}$ is nontrivial, or the derivation of K is nontrivial and $\mathcal{O}$ is discrete. Then $\Sigma(M) \subseteq [\mathcal{O}]$.*

Proof. The remark following the proof of Proposition 2.3.34 and Lemma 2.3.43 yield $A \in \mathcal{O}[\partial]$ with $M \cong K[\partial]/K[\partial]A$, and so $\Sigma(M) \subseteq [\mathcal{O}]$ by Lemma 2.3.45. □

Corollary 2.3.47. *Suppose M splits and is bounded. Then $\Sigma(M) \subseteq [\mathcal{O}]$.*

Proof. We proceed by induction on $r = \dim_K M = \ell(M)$. If $r = 0$, then $\Sigma(M) = \emptyset$. Next suppose $r = 1$, and take $e \in M$ with $M = Ke$ and $a \in K$ with $\partial e = ae$. Then $\Sigma(M) = \big\{[a]\big\}$ by Corollary 2.3.19, and $M \cong K[\partial]/K[\partial](\partial - a)$, so $[a] \in [\mathcal{O}]$ by Lemma 2.3.42. Now suppose $r > 1$. Take a differential submodule M_1 of M with $\ell(M_1) = r - 1$, so $\ell(M_2) = 1$ for $M_2 := M/M_1$. Then M_1, M_2 split and are bounded, by Lemma 2.3.16 and the remarks before Example 2.3.41. Hence by inductive hypothesis $\Sigma(M_i) \subseteq [\mathcal{O}]$ for $i = 1, 2$, so $\Sigma(M) \subseteq [\mathcal{O}]$ by the remark after (2.3.5). □

Corollary 2.3.48. *Let H be a Liouville closed, trigonometrically closed H-field with small derivation, $K = H[\mathrm{i}]$, and suppose M is bounded. Then $\Sigma(M) \subseteq [\mathcal{O}]$.*

Proof. We have $K^\dagger = H \oplus \mathrm{I}(H)\mathrm{i}$ and so $\mathcal{O} + K^\dagger = H \oplus \mathcal{O}_H \mathrm{i}$. Take an H-closed field extension H_1 of H and set $K_1 := H_1[\mathrm{i}]$. The base change $M_1 := K_1 \otimes_K M$ of M to K_1 splits and is bounded, so $\Sigma(M_1) \subseteq [\mathcal{O}_1]$ by Corollary 2.3.47. We have $K_1^\dagger = H_1 \oplus \mathrm{I}(H_1)\mathrm{i}$ by Corollary 1.2.21, so $\mathcal{O}_1 + K_1^\dagger = H_1 \oplus \mathcal{O}_{H_1}\mathrm{i}$. This yields $K_1^\dagger \cap K = K^\dagger$ and $(\mathcal{O}_1 + K_1^\dagger) \cap K = \mathcal{O} + K^\dagger$, so identifying $K/K^\dagger$ with its image under the group embedding $a + K^\dagger \mapsto a + K_1^\dagger \colon K/K^\dagger \to K_1/K_1^\dagger$ $(a \in K)$ we have $\Sigma(M) \subseteq \Sigma(M_1)$ and $[\mathcal{O}] = [\mathcal{O}_1] \cap (K/K^\dagger)$. Thus $\Sigma(M) \subseteq [\mathcal{O}]$. □

Question. Does it follow from M being bounded and $C \neq K$ that $\Sigma(M) \subseteq [\mathcal{O}]$?

2.4. Self-Adjointness and its Variants (*)

In this section K is a differential field. We let A, B range over $K[\partial]$ with $A \neq 0$, and set $r := \operatorname{order} A$. We also let α range over $K/K^\dagger$. The material in this section elaborates on Corollaries 2.3.5 and 2.3.20 and shows how symmetries of A force it to have eigenvalue $0 \in K/K^\dagger$, mainly by making some classical results (cf. [54, Chapitre V] and [181, §23–25]) precise and putting them into our present context. It can be skipped on first reading, since it is only needed in Section 7.4 for applications of our main theorem to linear differential equations over complexified Hardy fields.

Operators of the same type. Suppose $B \neq 0$ and set $s := \operatorname{order} B$. Consider now the C-linear subspace

$$\mathcal{E}(A,B) \;:=\; \big\{R \in K[\partial] : \operatorname{order} R < r \text{ and } BR \in K[\partial]A\big\}$$

of $K[\partial]$. The next lemma and its corollary elaborate on Lemma 2.3.6.

Lemma 2.4.1. *Let $M := K[\partial]/K[\partial]A$ and $N := K[\partial]/K[\partial]B$. Then we have an isomorphism*

$$R \;\mapsto\; \phi_R \;:\; \mathcal{E}(A,B) \to \operatorname{Hom}_{K[\partial]}(N, M)$$

of C-linear spaces where

$$\phi_R(1 + K[\partial]B) \;=\; R + K[\partial]A \qquad \textit{for } R \in \mathcal{E}(A,B).$$

Proof. Let $R \in \mathcal{E}(A,B)$. Then the kernel of the $K[\partial]$-linear map $K[\partial] \to K[\partial]/K[\partial]A$ sending 1 to $R + K[\partial]A$ contains $K[\partial]B$, hence induces a $K[\partial]$-linear map

$$\phi_R \colon N = K[\partial]/K[\partial]B \to K[\partial]/K[\partial]A = M$$

as indicated. It is easy to check that $R \mapsto \phi_R$ is C-linear. If $\phi_R = 0$, then $\phi_R(1 + K[\partial]B) = K[\partial]A$ and hence $R \in K[\partial]A$, so $R = 0$, since $\operatorname{order} R < r$.

Given $\phi \in \operatorname{Hom}_{K[\partial]}(N, M)$ we have $\phi(1 + K[\partial]B) = R + K[\partial]A$ where $R \in K[\partial]$ has order $< r$; then $BR \in K[\partial]A$, so $R \in \mathcal{E}(A, B)$, and we have $\phi = \phi_R$. $\square$

In particular, $0 \leqslant \dim_C \mathcal{E}(A, B) \leqslant rs$ by Lemmas 2.3.13 and 2.4.1. Moreover:

Corollary 2.4.2. *Suppose $r = s$. Then the isomorphism $R \mapsto \phi_R$ from the previous lemma maps the subset*

$$\mathcal{E}(A, B)^{\times} \ := \ \big\{R \in \mathcal{E}(A, B) : 1 \in K[\partial]R + K[\partial]A\big\}$$

of $\mathcal{E}(A, B)$ bijectively onto the set of $K[\partial]$-linear isomorphisms $N \to M$.

Set $M := K[\partial]/K[\partial]A$. We make the C-module $\operatorname{End}_{K[\partial]}(M) := \operatorname{Hom}_{K[\partial]}(M, M)$ into a C-algebra with its ring multiplication given by composition. We equip the C-module $\mathcal{E}(A) := \mathcal{E}(A, A)$ with the ring multiplication making the map

$$R \mapsto \phi_R \ : \ \mathcal{E}(A) \to \operatorname{End}_{K[\partial]}(M)$$

an isomorphism of C-algebras. The C-algebra $\mathcal{E}(A)$ is called the **eigenring** of A; cf. [189] or [158, §2.2]. Note that if $r \geqslant 1$, then $C \subseteq \mathcal{E}(A)$. If the $K[\partial]$-module M is irreducible, then $\operatorname{End}_{K[\partial]}(M)$ is a division ring, by Schur's Lemma [122, Chapter XVII, Proposition 1.1]. Now M is irreducible iff A is irreducible [ADH, p. 251], hence:

Corollary 2.4.3. *Suppose A is irreducible. Then $\mathcal{E}(A)$ is a division algebra over C. If C is algebraically closed, then $\mathcal{E}(A) = C$.*

Proof. As to the second claim, let $e \in \mathcal{E}(A)$. The elements of C commute with e, so we have a commutative domain $C[e] \subseteq \mathcal{E}(A)$, hence e is algebraic over C in view of $\dim_C \mathcal{E}(A) \leqslant r^2$, and thus $e \in C$ if C is algebraically closed. $\square$

We may have $\mathcal{E}(A) = C$ without A being irreducible [158, Exercise 2.14]. If A, B have the same type, then the C-algebras $\mathcal{E}(A)$, $\mathcal{E}(B)$ are isomorphic. By Lemma 2.4.1 and Corollary 2.4.2 we have:

Corollary 2.4.4. *Suppose $\mathcal{E}(A) = C$ and A, B have the same type. Then for some $e \in \mathcal{E}(A, B)^{\neq}$ we have $\mathcal{E}(A, B) = Ce$, and $\mathcal{E}(A, B)^{\times} = C^{\times}e$.*

Self-duality. Let M be a differential module over K. We say that M is **self-dual** if $M \cong M^*$. If M is self-dual, then so is of course every isomorphic $K[\partial]$-module, in particular M^*. Given also a differential module N over K, we say that a K-bilinear map $[\ ,\ ]\colon M \times N \to K$ is **∂-compatible** if

$$\partial[f, g] \ = \ [\partial f, g] + [f, \partial g] \qquad \text{for all } f \in M,\ g \in N.$$

The non-degenerate K-bilinear map

$$(\phi, f) \mapsto \langle \phi, f\rangle := \phi(f) \ : \ M^* \times M \to K$$

is ∂-compatible by [ADH, (5.5.1)]. One verifies easily:

Lemma 2.4.5. *M is self-dual iff there is a non-degenerate ∂-compatible K-bilinear form on M. In more detail, any isomorphism $\iota\colon M \to M^*$ yields a non-degenerate ∂-compatible K-bilinear form $(f, g) \mapsto \langle \iota(f), g\rangle\colon M \times M \to K$, and every non-degenerate ∂-compatible K-bilinear form on M arises in this way from a unique isomorphism $\iota\colon M \to M^*$ (of differential modules over K).*

In terms of matrices, let $e_1, \dots, e_n$ be a basis of M and let $e_1^*, \dots, e_n^*$ be the dual basis of M^*. Let $\iota\colon M \to M^*$ be an isomorphism $M \to M^*$ with matrix P with respect to these bases. Then for the corresponding K-bilinear form $[\ ,\ ]$ on M from the above lemma we have $[e_i, e_j] = P_{ji} = (P^{\mathrm{t}})_{ij}$.

Next a consequence of Corollaries 2.3.20 and 2.3.23. It provides useful information about the spectrum of M, which explains our interest in self-duality.

Corollary 2.4.6. *Let $\dim_K M = r$, and suppose $\sum_\alpha \operatorname{mult}_\alpha(M) = r$ and K is 1-linearly surjective, or $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Assume also that M is self-dual. Then $\operatorname{mult}_\alpha(M) = \operatorname{mult}_{-\alpha}(M)$ for all α. Hence if additionally $K^\dagger$ is 2-divisible and $\sum_\alpha \operatorname{mult}_\alpha(M)$ is odd, then $0 \in \Sigma(M)$.*

Suppose now that $M = K[\partial]/K[\partial]A$ and $r \geqslant 1$. Then $M^* \cong K[\partial]/K[\partial]A^*$ by [ADH, 5.5.8], hence M is self-dual iff A, A^* have the same type. By Lemma 2.3.6 this is the case iff there are $R, S \in K[\partial]$ of order $< r$ with $1 \in K[\partial]R + K[\partial]A$ and $A^*R = SA$. When $\mathcal{E}(A) = C$, we can replace this with a more symmetric condition:

Lemma 2.4.7. *Suppose $\mathcal{E}(A) = C$. Then A, A^* have the same type iff for some $R \in K[\partial]$ of order $n < r$ we have $A^*R = (-1)^{n+r}R^*A$ and $1 \in K[\partial]R + K[\partial]A$.*

Proof. Suppose A, A^* have the same type. By Corollary 2.4.4 we obtain $R, S \in K[\partial]$ of order $< r$ such that $A^*R = SA$, $1 \in K[\partial]R + K[\partial]A$, and $\mathcal{E}(A, A^*)^\times = C^\times R$. Now taking adjoints yields $A^*S^* = R^*A$, so $0 \neq S^* \in \mathcal{E}(A, A^*)$, hence $S^* \in \mathcal{E}(A, A^*)^\times$ and thus $S^* = cR$ ($c \in C^\times$). Comparing coefficients of the highest order terms on both sides of $cA^*R = R^*A$ gives $c = (-1)^{n+r}$. □

We say that A is **self-dual** if A, A^* have the same type. Thus if A is self-dual, then so is A^*, and so is every operator of the same type as A. Moreover, by Lemma 2.3.7, if A is self-dual, then A, A^* have the same eigenvalues, with the same multiplicities. Combining Corollary 2.4.3 with the previous lemma yields:

Corollary 2.4.8. *Suppose A is irreducible and C is algebraically closed. Then A is self-dual iff for some $R \in K[\partial]$ of order $n < r$ we have $A^*R = (-1)^{n+r}R^*A$ and $1 \in K[\partial]R + K[\partial]A$.*

Here is the operator version of Corollary 2.4.6:

Corollary 2.4.9. *Suppose A is self-dual, and set $s := \sum_\alpha \operatorname{mult}_\alpha(A)$. Also assume K is 1-linearly surjective and $s = r$, or $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Then $\operatorname{mult}_\alpha(A) = \operatorname{mult}_{-\alpha}(A)$ for each α. Hence if in addition $K^\dagger$ is 2-divisible and s is odd, then $0 \in \Sigma(A)$.*

Let $\phi \in K^\times$, $B \neq 0$. If A, B have the same type, then so do $A^\phi, B^\phi \in K^\phi[\delta]$, by Lemma 2.3.6. Hence by the next lemma, if A is self-dual, then so is A^ϕ.

Lemma 2.4.10. $(A^\phi)^* = (A^*)^\phi_{\ltimes\phi}$.

Proof. We have

$$(\partial^\phi)^* \;=\; (\phi\delta)^* \;=\; -\delta\phi \;=\; (-\phi\delta)_{\ltimes\phi} \;=\; (-\partial)^\phi_{\ltimes\phi} \;=\; (\partial^*)^\phi_{\ltimes\phi},$$

so the lemma holds for $A = \partial$. It remains to note that $B \mapsto (B^\phi)^*$ and $B \mapsto (B^*)^\phi_{\ltimes\phi}$ are ring morphisms $K[\partial] \to K^\phi[\delta]^{\mathrm{opp}}$ that are the identity on K, where $K^\phi[\delta]^{\mathrm{opp}}$ is the opposite ring of $K^\phi[\delta]$; cf. [ADH, proof of 5.1.8]. □

If $A^* = (-1)^r A_{\ltimes a}$ ($a \in K^\times$), then A is self-dual, so there is a non-degenerate ∂-compatible K-bilinear form on $K[\partial]/K[\partial]A$. The next proposition gives more information. We say that a K-bilinear form $[\ ,\ ]$ on a K-linear space M is **$(-1)^n$-symmetric** if $[f,g] = (-1)^n [g,f]$ for all $f,g \in M$.

Proposition 2.4.11 (Bogner [25])**.** *Suppose $r \geqslant 1$, and let $M = K[\partial]/K[\partial]A$ and $e = 1 + K[\partial]A \in M$. Then the following are equivalent:*

(i) $A^* = (-1)^r A_{\ltimes a}$ *for some* $a \in K^\times$*;*

(ii) *there is a non-degenerate ∂-compatible K-bilinear form $[\ ,\ ]$ on M such that*

$$[e, \partial^j e] = 0 \quad \textit{for } j = 0, \dots, r-2.$$

Any form $[\ ,\ ]$ on M as in (ii) *is $(-1)^{r-1}$-symmetric.*

Proof. We first arrange that A is monic. Let $e_0^*, \dots, e_{r-1}^*$ be the basis of M^* dual to the basis $e, \partial e, \dots, \partial^{r-1} e$ of M, so $e^* := e_{r-1}^*$ is a cyclic vector of M^* with $A^* e^* = 0$, by [ADH, 5.5.7]. Below we let i, j range over $\{0, \dots, r-1\}$.

Suppose $A^* = (-1)^r A_{\ltimes a}$, $a \in K^\times$. Then $A^* e^* = 0$ gives $Aae^* = 0$, so we have a $K[\partial]$-linear isomorphism $\varphi \colon M \to M^*$ with $\varphi(e) = ae^*$. Let $[\ ,\ ]$ be the non-degenerate ∂-compatible K-bilinear form on M given by $[f,g] = \langle \varphi(f), g \rangle$. Then

$$[e, \partial^j e] = \langle \varphi(e), \partial^j e \rangle = a \langle e^*, \partial^j e \rangle \quad \text{ for all } j,$$

proving (ii). Suppose conversely that $[\ ,\ ]$ is as in (ii). Then $a := [e, \partial^{r-1} e] \neq 0$ since $[\ ,\ ]$ is non-degenerate. Let $\varphi \colon M \to M^*$ be the isomorphism with $\varphi(f) = [f, -]$ for all $f \in M$. Then $[e, \partial^j e] = \langle \varphi(e), \partial^j e \rangle$ for all j and thus $\varphi(e) = a\, e^*$. Hence

$$Aa\, e^* \ =\ A\varphi(e) \ =\ \varphi(Ae) \ =\ \varphi(0) \ =\ 0,$$

and this yields $A^* = (-1)^r A_{\ltimes a}$.

Let now $[\ ,\ ]$ be as in (ii) and set $a := [e, \partial^{r-1} e]$. Induction on i using ∂-compatibility of $[\ ,\ ]$ shows $[\partial^i e, \partial^j e] = 0$ for $i \leqslant r-2$, $j \leqslant r-2-i$. In particular, $[\partial^i e, e] = 0 = [e, \partial^i e]$ for $i \leqslant r-2$. Induction on i using the second display in the proof of [ADH, 5.5.7] gives

$$\begin{aligned} (\partial^*)^i e^* \ &\in\ e_{r-1-i}^* + \sum_{r-i \leqslant j \leqslant r-1} K e_j^*, \text{ and hence} \\ (-1)^{r-1} \partial^{r-1} e^* \ &\in\ e_0^* + K e_1^* + \cdots + K e_{r-1}^*. \end{aligned}$$

It follows that $[\partial^{r-1} e, e] = \langle \partial^{r-1} a e^*, e \rangle = (-1)^{r-1} a$. This covers the base case $i = 0$ of an induction on i showing $[\partial^i e, g] = (-1)^{r-1} [g, \partial^i e]$ for all $g \in M$. Suppose this identity holds for a certain $i \leqslant r-2$. Then by ∂-compatibility

$$[\partial^{i+1} e, g] = \partial[\partial^i e, g] - [\partial^i e, \partial g] = (-1)^{r-1} \big(\partial [g, \partial^i e] - [\partial g, \partial^i e] \big) = (-1)^{r-1} [g, \partial^{i+1} e]$$

as required. $\square$

See [25] for the geometric significance of operators as in this proposition when K is the differential field of germs of meromorphic functions at 0.

Let $r \geqslant 1$, and A, a be as in (i) of Proposition 2.4.11, with

$$A \ =\ \partial^r + a_{r-1} \partial^{r-1} + \cdots + a_0 \quad (a_0, \dots, a_{r-1} \in K).$$

Then $a^\dagger = -(2/r) a_{r-1}$. Set $b := a^\dagger$. Then the operator $B := A_{b/2}$ of order r satisfies $B^* = (-1)^r B$, so the cases $B^* = B$ and $B^* = -B$ deserve particular attention:

Definition 2.4.12. A linear differential operator B is said to be (formally) **self-adjoint** if $B^* = B$ and **skew-adjoint** if $B^* = -B$.

We discuss self-adjoint and skew-adjoint operators in more depth after reviewing a useful identity relating a linear differential operator and its adjoint, which is obtained by transferring [ADH, (5.5.1)] to the level of operators.

The Lagrange Identity. Let M be a differential module over K with $\dim_K M = n \geqslant 1$, and suppose e is a cyclic vector of M. Then $e_0, \dots, e_{n-1}$ with $e_i := \partial^i e$ is a basis of M. Let $e_0^*, \dots, e_{n-1}^*$ be the dual basis of M^*. Then $e^* := e_{n-1}^*$ is a cyclic vector of M^* [ADH, 5.5.7], so $e^*, \partial e^*, \dots, \partial^{n-1} e^*$ is a basis of M^*. Let $L \in K[\partial]$ be monic of order n such that $Le = 0$. Then

$$L = a_0 + a_1 \partial + \dots + a_n \partial^n \qquad \text{where } a_0, \dots, a_n \in K \text{ (so } a_n = 1).$$

By [ADH, 5.5.7] and its proof we have $L^* e^* = 0$ and

$$(2.4.1) \quad e_{n-i-1}^* = L_i e^* \quad \text{where } L_i := \sum_{j=0}^{i} (\partial^*)^{i-j} a_{n-j} \in K[\partial] \quad (i = 0, \dots, n-1).$$

Let $d_0, \dots, d_{n-1}$ be the basis of M dual to the basis $e^*, \partial e^*, \dots, \partial^{n-1} e^*$ of M^*. Then

$$(2.4.2) \quad \langle e_{n-1}^*, d_j \rangle = \delta_{0j}, \quad \langle e_{n-i-1}^*, d_{n-1} \rangle = (-1)^{n-1} \delta_{i,n-1} \quad \text{for } 1 \leqslant i \leqslant n-1$$

(Kronecker deltas) using (2.4.1). Let $y, z \in K$ and set

$$\phi := y e_0^* + y' e_1^* + \dots + y^{(n-1)} e_{n-1}^* \in M^*, \quad f := z d_0 + z' d_1 + \dots + z^{(n-1)} d_{n-1} \in M.$$

Then

$$\partial \phi = L(y) e_{n-1}^*, \qquad \partial f = (-1)^n L^*(z) d_{n-1}.$$

For the first equality, use the first display in the proof of [ADH, 5.5.7]. The second equality follows from the first by reversing the roles of M and M^* and noting that $(-1)^n L^*$ is monic of order n with $(-1)^n L^* e^* = 0$. Hence

$$\langle \partial \phi, f \rangle = L(y) \langle e^*, f \rangle = L(y) z, \qquad \langle \phi, \partial f \rangle = (-1)^n L^*(z) \langle \phi, d_{n-1} \rangle = -L^*(z) y$$

by (2.4.2) and so by the identity [ADH, (5.5.1)],

$$\partial \langle \phi, f \rangle = \langle \partial \phi, f \rangle + \langle \phi, \partial f \rangle = L(y) z - L^*(z) y.$$

Now $\langle \partial^i e^*, f \rangle = z^{(i)}$ for $i < n$, so $\langle B e^*, f \rangle = B(z)$ for all B of order $< n$, hence

$$\langle \phi, f \rangle = \sum_{i=0}^{n-1} y^{(n-i-1)} \langle L_i e^*, f \rangle = \sum_{0 \leqslant j \leqslant i < n} y^{(n-i-1)} (-1)^{i-j} (a_{n-j} z)^{(i-j)} = P_L(y, z)$$

where

$$(2.4.3) \qquad P_L(Y, Z) := \sum_{0 \leqslant i \leqslant j < n} Y^{(i)} (-1)^{j-i} (a_{j+1} Z)^{(j-i)} \in K\{Y, Z\},$$

a homogeneous differential polynomial of degree 2. These considerations show:

Proposition 2.4.13 (Lagrange Identity). *The map*

$$(y, z) \mapsto [y, z]_L := P_L(y, z) \ : \ K \times K \to K$$

is C-bilinear, and for $y, z \in K$ we have

$$(2.4.4) \qquad \partial\big([y, z]_L\big) = L(y) z - L^*(z) y.$$

We assumed here that L is monic of order $n \geqslant 1$. For an arbitrary linear differential operator $L = a_0 + a_1\partial + \cdots + a_n\partial^n \in K[\partial]$ $(a_0, \dots, a_n \in K)$ we define P_L as in (2.4.3) and set $[y,z]_L := P_L(y,z)$ for $y, z \in K$. Then Proposition 2.4.13 continues to hold; to see (2.4.4) for $L \neq 0$, reduce to the monic case, using that for all $a \in K$ and $L \in K[\partial]$ we have

(2.4.5) $\quad P_{aL}(Y,Z) = P_L(Y,aZ), \qquad$ hence $\qquad [y,z]_{aL} = [y,az]_L$ for $y, z \in K$.

The differential polynomial P_L is called the **concomitant** of L; it does not change when passing from K to a differential field extension.

Lemma 2.4.14 (Hesse)**.** *Let $L, L_1, L_2 \in K[\partial]$; then*

$$P_{L^*}(Y,Z) = -P_L(Z,Y) \quad \textit{and} \quad P_{L_1+L_2}(Y,Z) = P_{L_1}(Y,Z) + P_{L_2}(Y,Z).$$

Proof. By (2.4.4) we have $\partial\big([y,z]_{L^*}\big) = -\partial\big([z,y]_L\big)$ for all y, z in every differential field extension of K, hence $\big(P_{L^*}(Y,Z) + P_L(Z,Y)\big)' = 0$ in $K\{Y,Z\}$ and then $P_{L^*}(Y,Z) + P_L(Z,Y) = 0$, since P_{L^*}, P_L are homogeneous of degree 2. This shows the first identity. The second identity is clear by inspection of (2.4.3). □

Example. For $L = 0, 1, \partial, \partial^2$ we have

$$P_0 = P_1 = 0, \quad P_\partial = YZ, \quad P_{\partial^2} = Y'Z - YZ'$$

so for $y, z \in K$:

$$[y,z]_0 = [y,z]_1 = 0, \quad [y,z]_\partial = yz, \quad [y,z]_{\partial^2} = y'z - yz',$$

which for $L = a\partial^2 + b\partial + c$ $(a, b, c \in K)$, using (2.4.5), gives

$$P_L \;=\; aY'Z - aYZ' + (b - a')YZ, \quad [y,z]_L \;=\; ay'z - ayz' + (b-a')yz.$$

Below we use that evaluating the differential operator $L \in K[\partial]$ at the element Y of the differential ring extension $K\{Y\}$ of K results in the differential polynomial $L(Y) \in K\{Y\}$, which is homogeneous of degree 1. With this notation, we have $P_L(Y,Z)' = L(Y)Z - L^*(Z)Y$. The next result characterizes the concomitant and adjoint of a differential operator accordingly.

Lemma 2.4.15 (Frobenius)**.** *The pair (P_L, L^*) is the only pair $(P, \widetilde{L})$ with P in $K\{Y,Z\}$ homogeneous of degree 2 and $\widetilde{L} \in K[\partial]$ such that*

$$P(Y,Z)' \;=\; L(Y)Z - \widetilde{L}(Z)Y.$$

Proof. If $(P, \widetilde{L})$ is such a pair, then $P_1 := P_L - P$, $L_1 := L^* - \widetilde{L}$ gives $P_1(Y,Z)' = -L_1(Z)Y$, and from this one can derive $L_1 = 0$ and then $P_1 = 0$. □

Let now $L \in K[\partial]^{\neq}$ be of order n, and set $V := \ker L$, $W := \ker L^*$. Then for $y \in V$, $z \in W$ we have $[y,z]_L \in C$ by (2.4.4); thus $[\ ,\ ]_L$ restricts to a C-bilinear map $V \times W \to C$, also denoted by $[\ ,\ ]_L$.

Corollary 2.4.16. *Suppose $\dim_C V = n$. Then the pairing*

$$[\ ,\ ]_L \;:\; V \times W \to C$$

is non-degenerate.

Proof. By Lemma 2.3.21 we have $\dim_C W = n$. Let $y \in V^{\neq}$. Then $P_L(y,Z) \in K\{Z\}$ is homogeneous of degree 1 and order $n-1$, hence cannot vanish on the C-linear subspace W of K of dimension n [ADH, 4.1.14]. Similarly with $z \in W^{\neq}$, $P_L(Y,z) \in K\{Y\}$, V in place of y, $P_L(y,Z)$, W, respectively. □

The concomitant of operators which split over K. In this subsection we assume that $A \in K[\partial]$ and $a_1, \dots, a_r \in K$ satisfy

$$A = (\partial - a_r) \cdots (\partial - a_1), \quad \text{so} \quad A^* = (-1)^r (\partial + a_1) \cdots (\partial + a_r).$$

For $i = 0, \dots, r$ we define

$$A_i := (\partial - a_i) \cdots (\partial - a_1), \qquad B_i := (-1)^{r-i} (\partial + a_{i+1}) \cdots (\partial + a_r). \tag{2.4.6}$$

Thus A_i has order i, B_i has order $r - i$, and

$$A_0 = B_r = 1, \quad A_r = A, \quad B_0 = A^*.$$

We then have the following formula for P_A:

Lemma 2.4.17. $P_A(Y, Z) = \displaystyle\sum_{i=0}^{r-1} A_i(Y) B_{i+1}(Z)$.

Towards proving this, take $b_1, \dots, b_r \neq 0$ in a differential field extension E of K with $b_j^\dagger = a_j - a_{j-1}$ for $j = 1, \dots, r$, where $a_0 := 0$, and set $b_{r+1} := (b_1 \cdots b_r)^{-1}$. Lemma 1.1.3 then gives

$$A \ = \ b_{r+1}^{-1} (\partial b_r^{-1}) \cdots (\partial b_2^{-1})(\partial b_1^{-1}).$$

For $i = 0, \dots, r$,

$$L_i \ := \ b_{i+1}^{-1} (\partial b_i^{-1}) \cdots (\partial b_2^{-1})(\partial b_1^{-1}) \in E[\partial]$$

has order i, with $L_0 = b_1^{-1}$, $L_r = A$. Likewise we introduce for $i = 0, \dots, r$,

$$M_i \ := \ (-1)^{r-i} b_{i+1}^{-1} (\partial b_{i+2}^{-1}) \cdots (\partial b_r^{-1})(\partial b_{r+1}^{-1}) \in E[\partial]$$

of order $r - i$, so $M_0 = A^*$, $M_r = b_{r+1}^{-1}$. Note that

$$L_{i+1} \ = \ b_{i+2}^{-1} \partial L_i, \quad M_i = -b_{i+1}^{-1} \partial M_{i+1} \qquad (i = 0, \dots, r-1). \tag{2.4.7}$$

With these notations, we have:

Lemma 2.4.18 (Darboux)**.** $P_A(Y, Z) \ = \ \displaystyle\sum_{i=0}^{r-1} L_i(Y) M_{i+1}(Z)$ *in* $E\{Y, Z\}$.

Proof. The cases $r = 0$ and $r = 1$ are easy to verify directly. Assume $r \geqslant 2$. It suffices to show that the differential polynomial $P(Y, Z)$ on the right-hand side of the claimed equality satisfies $P(Y, Z)' = A(Y)Z - Y A^*(Z)$. From (2.4.7) we obtain

$$A(Y)Z \ = \ L_{r-1}(Y)' M_r(Z), \qquad -Y A^*(Z) \ = \ -Y M_0(Z) \ = \ L_0(Y) M_1(Z)'$$

and

$$\begin{aligned} \big(L_i(Y) M_{i+1}(Z)\big)' \ &= \ L_i(Y)' M_{i+1}(Z) + L_i(Y) M_{i+1}(Z)' \\ &= \ L_i(Y) M_{i+1}(Z)' - L_{i+1}(Y) M_{i+2}(Z)' \quad \text{for } i = 0, \dots, r-2. \end{aligned}$$

Now use the cancellations in $\sum_{i=0}^{r-1} \big(L_i(Y) M_{i+1}(Z)\big)'$. □

This yields Lemma 2.4.17: For $i = 0, \dots, r-1$, we have

$$L_i \ = \ (b_{i+1} b_i \cdots b_1)^{-1} (\partial - a_i) \cdots (\partial - a_1) \ =$$

and

$$M_{i+1} = (-1)^{r-i-1} (b_{i+2} \cdots b_{r+1})^{-1} (\partial + a_{i+2}) \cdots (\partial + a_r)$$

and hence $L_i(Y) M_{i+1}(Z) = A_i(Y) B_{i+1}(Z)$. □

Self-adjoint and skew-adjoint operators. If A is self-adjoint, then $r = \operatorname{order} A$ is even. Moreover, for $B \neq 0$: A is self-adjoint iff B^*AB is self-adjoint. The self-adjoint operators form a C-linear subspace of $K[\partial]$ containing K.

Lemma 2.4.19 (Jacobi)**.** *Let $s \in \mathbb{N}$ and suppose $r = 2s$. Then A is self-adjoint iff there are $b_0, \dots, b_s \in K$ such that*

$$A \;=\; \partial^s b_s \partial^s + \partial^{s-1} b_{s-1} \partial^{s-1} + \cdots + b_0.$$

Proof. If A has the displayed shape, then evidently A is self-adjoint. We show the converse by induction on s. The case $s = 0$ being trivial, suppose $s \geqslant 1$. Say $A = a_r \partial^r + \text{lower order terms}$ ($a_r \in K^\times$). Then $B = A - \partial^s a_r \partial^s$ is self-adjoint of order $< r$, hence the inductive hypothesis applies to B. □

Example. If $r = 2$, then A is self-adjoint iff $A = a\partial^2 + a'\partial + b$ ($a, b \in K$). In particular, $\partial^2 + b$ ($b \in K$) is self-adjoint.

If A is self-adjoint, then $[y,z]_A = -[z,y]_A$ for all $y, z \in K$, by Lemma 2.4.14. Thus $[y,y]_A = 0$ for $y \in K$. This fact is used in the proof of the next lemma:

Lemma 2.4.20. *Suppose A is self-adjoint and splits over K, and $r = 2s$, $s \in \mathbb{N}$. Then there are $a \in K^\times$ and $a_1, \dots, a_s \in K$ such that*

$$A \;=\; (\partial + a_1) \cdots (\partial + a_s) a (\partial - a_s) \cdots (\partial - a_1).$$

If A is monic, then $a = 1$ for any such a.

Proof. By induction on s. The case $s = 0$ being trivial, suppose $s \geqslant 1$. Let $z \neq 0$ be a zero of A in a differential field extension Ω of K with $a_1 := z^\dagger \in K$. The differential polynomial $P(Y) := P_A(Y, z)$ is homogeneous of degree 1 and order $r-1$ with $P(z) = [z,z]_A = 0$; hence by [ADH, 5.1.21] we obtain $A_0 \in K[\partial]$ with $L_P = A_0(\partial - a_1)$. By (2.4.4) we have $zA = \partial L_P = \partial A_0(\partial - a_1)$ and so

$$A = z^{-1}\partial A_0(\partial - a_1) = (\partial + a_1) A_1 (\partial - a_1) \qquad \text{where } A_1 := z^{-1} A_0 \in \Omega[\partial].$$

The inductive hypothesis applies to A_1: $A_1 \in K[\partial]$ by [ADH, 5.1.11], A_1 is self-adjoint of order $r - 2$, and A_1 splits over K by [ADH, 5.1.22]. □

This gives rise to the following corollary:

Corollary 2.4.21 (Frobenius, Jacobi)**.** *Suppose A is self-adjoint and $\dim_C \ker A = r = 2s$. Then $A = B^* b B$ where $B = \partial b_s^{-1} \cdots \partial b_1^{-1}$ with $b, b_1, \dots, b_s \in K^\times$.*

Proof. From $\dim_C \ker A = r$ we obtain that A splits over K. Hence the previous lemma yields $a_1, \dots, a_s \in K$, $a \in K^\times$ such that

$$A = (\partial + a_1) \cdots (\partial + a_s) a (\partial - a_s) \cdots (\partial - a_1),$$

and $a_1, \dots, a_s \in K^\dagger$ by Lemma 2.3.4. Lemma 1.1.3 yields $b_1, \dots, b_s \in K^\times$ with

$$(\partial - a_s) \cdots (\partial - a_1) = b_1 \cdots b_s \partial b_s^{-1} \cdots \partial b_1^{-1}.$$

Set $B := \partial b_s^{-1} \cdots \partial b_1^{-1}$. Then $A = B^* b B$ for $b := (-1)^s (b_1 \cdots b_s)^2 a$. □

Recall that A is called skew-adjoint if $A^* = -A$ (and then $r = \operatorname{order} A$ is odd). The skew-adjoint operators form a C-linear subspace of $K[\partial]$. For $B \neq 0$, the operator B^*AB ($B \neq 0$) is skew-adjoint iff A is skew-adjoint. We have a characterization of skew-adjoint operators analogous to Lemma 2.4.19:

Lemma 2.4.22. *Let $s \in \mathbb{N}$ and suppose $r = 2s+1$. Then A is skew-adjoint iff there are $b_0, \dots, b_s \in K$ such that*

$$A = (\partial^{s+1} b_s \partial^s + \partial^s b_s \partial^{s+1}) + (\partial^s b_{s-1} \partial^{s-1} + \partial^{s-1} b_{s-1} \partial^s) + \cdots + (\partial b_0 + b_0 \partial).$$

Proof. Suppose A is skew-adjoint. Say $A = a_r \partial^r +$ lower order terms ($a_r \in K^\times$), and set $b_s := a_r/2$. Then $A - (\partial^{s+1} b_s \partial^s + \partial^s b_s \partial^{s+1})$ is skew-adjoint of order $< r$. Hence the forward direction follows by induction on s. The converse is obvious. $\square$

Example. If $r = 1$, then A is skew-adjoint iff $A = a\partial + (a'/2)$ ($a \in K^\times$).

For monic A of order 3, A is skew-adjoint iff $A = \partial^3 + f\partial + (f'/2)$ for some $f \in K$. In the next lemma we consider this case; for a more general version of this lemma, see [158, Proposition 4.26(1)].

Lemma 2.4.23. *Let $f \in K$, $A = \partial^3 + f\partial + (f'/2)$, $B = 4\partial^2 + f$ and $y, z \in \ker B$. Then $yz \in \ker A$. Moreover, if y, z is a basis of the C-linear space $\ker B$, then y^2, yz, z^2 is a basis of $\ker A$.*

Proof. We have

$$(yz)' \ = \ y'z + yz', \quad (yz)'' \ = \ y''z + 2y'z' + yz'' \ = \ 2y'z' - (f/2)yz,$$

hence

$$\begin{aligned}(yz)''' \ &= \ 2y''z' + 2y'z'' - (f'/2)yz - (f/2)(yz)' \\ &= \ -(f/2)(yz' + y'z) - (f'/2)yz - (f/2)(yz)' \\ &= \ -f(yz)' - (f'/2)yz,\end{aligned}$$

and so $yz \in \ker A$. Suppose $ay^2 + byz + cz^2 = 0$ for some $a, b, c \in C$, not all zero; we claim that then y, z are C-linearly dependent. We have $a \neq 0$ or $c \neq 0$, and so we may assume $a \neq 0$, $z \neq 0$. Then $u := y/z$ satisfies $au^2 + bu + c = 0$, so $u \in C$ [ADH, 4.1.1], hence $y \in Cz$. $\square$

If A is skew-adjoint, then $P_A(Y,Z) = P_A(Z,Y)$, so

$$P_A(Y+Z, Y+Z) \ = \ P_A(Y,Y) + P_A(Z,Z) + 2P_A(Y,Z)$$

and $[y,z]_A = [z,y]_A$ for all $y, z \in K$.

Lemma 2.4.24. *Suppose $r \geqslant 1$. Then the following are equivalent:*

(i) *A is skew-adjoint;*
(ii) *there is a homogeneous differential polynomial $Q \in K\{Y\}$ of degree 2 such that $A(Y)Y = Q(Y)'$.*

Moreover, the differential polynomial Q in (ii) *is unique, and $Q(Y) = \frac{1}{2} P_A(Y,Y)$.*

Proof. For (i) $\Rightarrow$ (ii) take $Q(Y) := \frac{1}{2} P_A(Y,Y)$. For the converse let Q be as in (ii). Let Z be a differential indeterminate over K different from Y and c be a constant in a differential field extension Ω, with c transcendental over C. Then

$$A(Y+cZ)(Y+cZ) \ = \ Q(Y+cZ)' \ \text{ in } \Omega\{Y,Z\}.$$

Also, in $\Omega\{Y,Z\}$,

$$A(Y+cZ)(Y+cZ) \ = \ A(Y)Y + c\big(A(Y)Z + A(Z)Y\big) + c^2 A(Z)Z.$$

Take $P, R \in K\{Y,Z\}$ such that

$$Q(Y+cZ) \ = \ Q(Y) + cP(Y,Z) + c^2 R(Y,Z).$$

Comparing the coefficients of c now yields

$$A(Y)Z + A(Z)Y \;=\; P(Y,Z)'.$$

Now using Lemma 2.4.15 gives $A^* = -A$, $P = P_A$, proving (i). □

In the rest of this subsection A is skew-adjoint, $r \geqslant 3$, and $Q(Y) := \frac{1}{2}P_A(Y,Y)$.

Lemma 2.4.25. *If $\dim_C \ker A \geqslant 2$ and $C^\times$ is 2-divisible, then $A(z) = Q(z) = 0$ for some $z \in K^\times$.*

Proof. Apply [122, Chapter XV, Theorem 3.1] to the symmetric bilinear form

$$(y,z) \mapsto [y,z]_A \;:\; \ker A \times \ker A \to C$$

on the C-linear space $\ker A$. □

Lemma 2.4.26. *Suppose $K^\dagger$ is 2-divisible, and $z \neq 0$ lies in a differential field extension of K with $A(z) = 0$ and $z^\dagger \in K \setminus K^\dagger$. Then $Q(z) = 0$.*

Proof. From $z' \in Kz$ it follows by induction that $(Kz)^{(n)} \subseteq Kz$ for all n. Using (2.4.3) this yields $Q(z) = az^2$ for a certain $a \in K$. Also $Q(z)' = A(z)z = 0$ and so if $a \neq 0$, then $z^\dagger = -\frac{1}{2}a^\dagger \in K^\dagger$, a contradiction. □

Let $z \neq 0$ lie in a differential field extension Ω of K with $A(z) = Q(z) = 0$. The differential polynomial $P(Y) := P_A(Yz, z) \in \Omega\{Y\}$ is homogeneous of degree 1 and order $r-1$. Substitution in the identity $P_A(Y,Z)' = A(Y)Z + A(Z)Y$ gives $P(Y)' = zA(Yz)$. The coefficient of Y in P is $P(1) = P_A(z,z) = 0$, hence

$$P(Y) \;=\; A_0(Y'), \qquad A_0 \in \Omega[\partial] \text{ of order } r-2.$$

Lemma 2.4.27. *In $\Omega[\partial]$ we have $\partial A_0 \partial = zAz$, so A_0 is skew-adjoint.*

Proof. From $P(Y)' = zA(Yz)$ and $P(Y) = A_0(Y')$ we obtain $A_0(Y')' = zA(Yz)$. In terms of operators this means $\partial A_0 \partial = zAz$. □

Next we use these lemmas to prove a skew-adjoint version of Lemma 2.4.20.

Factorization of skew-adjoint operators. *In this subsection K is 1-linearly surjective, $K^\dagger$ and $C^\times$ are 2-divisible, and A is monic.*

Proposition 2.4.28. *Suppose A is skew-adjoint and splits over K. Then there are $a_1, \dots, a_s \in K$, where $r = 2s+1$, such that*

$$A \;=\; (\partial + a_1)\cdots(\partial + a_s)\partial(\partial - a_s)\cdots(\partial - a_1).$$

Proof. We proceed by induction on s. The case $s = 0$ is clear (see the example following Lemma 2.4.22), so let $s \geqslant 1$. With Q as in the previous subsection we claim that $A(z) = Q(z) = 0$ and $z^\dagger \in K$ for some $z \neq 0$ in a differential field extension Ω of K. If $\dim_C \ker A = r$, then Lemma 2.4.25 yields such a z in $\Omega = K$. Otherwise, Lemma 2.3.3 gives $a \in K \setminus K^\dagger$ with $\operatorname{mult}_a(A) \geqslant 1$, which in turn gives $z \neq 0$ in a differential field extension Ω of K with $A(z) = 0$ and $z^\dagger \in a + K^\dagger$, and thus $Q(z) = 0$ by Lemma 2.4.26. This proves the claim.

Let z and Ω be as in the claim, set $a_1 := z^\dagger$, and let $A_0 \in \Omega[\partial]$ be the skew-adjoint differential operator from the previous subsection. Then

$$A \;=\; z^{-1}\partial A_0 \partial z^{-1} \;=\; (\partial + a_1)A_1(\partial - a_1) \qquad \text{where } A_1 := z^{-1}A_0 z^{-1}.$$

By Lemma 2.4.27, A_1 is skew-adjoint of order $r-2$. By [ADH, 5.1.11, 5.1.22], $A_1 \in K[\partial]$ is monic and splits over K, so the inductive hypothesis applies to A_1. □

Corollary 2.4.29 (Darboux). *Suppose A is skew-adjoint with $\dim_C \ker A = r = 2s+1$. Then $A = B^*\partial B$ for some B. More precisely, there are $b_1, \dots, b_s \in K^\times$ such that $A = B^*\partial B$ for $B := \partial b_s^{-1} \cdots \partial b_1^{-1}$.*

Proof. Arguing as in the proof of Corollary 2.4.21, using Proposition 2.4.28 instead of Lemma 2.4.20, gives $A = (-1)^s B^* b\partial b B$ with $B = \partial b_s^{-1} \cdots \partial b_1^{-1}$, $b_1, \dots, b_s \in K^\times$, and $b = b_1 \cdots b_s$. But A is monic, so $(-1)^s b^2 = 1$, hence $b \in C$ and $A = B^*\partial B$. $\square$

Corollary 2.4.30. *Suppose $A^* = (-1)^r A_{\ltimes a}$ with $a \in K^\times$, and A splits over K. Then there are $a_1, \dots, a_r \in K$ such that*

$$A = (\partial - a_r) \cdots (\partial - a_1) \quad \textit{and} \quad a_j + a_{r+1-j} = a^\dagger \textit{ for } j = 1, \dots, r.$$

Proof. By a remark preceding Definition 2.4.12 we have $B^* = (-1)^r B$ where $B := A_{b/2}$, $b := a^\dagger$, so $B = A_{\ltimes d}$ with $d \in K^\times$, $d^2 = a$. Suppose $r = 2s$ is even. Then B is self-adjoint and Lemma 2.4.20 gives

$$B \;=\; (\partial + b_1) \cdots (\partial + b_s)(\partial - b_s) \cdots (\partial - b_1) \quad \text{with } b_1, \dots, b_s \in K.$$

Hence

$$A \;=\; B_{\ltimes d^{-1}} \;=\; (\partial + b_1 - d^\dagger) \cdots (\partial + b_s - d^\dagger)(\partial - b_s - d^\dagger) \cdots (\partial - b_1 - d^\dagger).$$

with the desired result for $a_j = b_j + d^\dagger$ and $a_{r+1-j} = -b_j + d^\dagger$, $j = 1, \dots, s$. The case of odd $r = 2s+1$ is handled in the same way, using Proposition 2.4.28 instead of Lemma 2.4.20. $\square$

Eigenrings of matrix differential equations. *In the rest of this section N, N_1, N_2, P, range over $n \times n$ matrices over K ($n \geqslant 1$).* Associated to the matrix differential equation $y' = Ny$ over K we have the differential module M_N over K with $\dim_K M = n$ [ADH, 5.5]. Recall that matrix differential equations $y' = N_1 y$ and $y' = N_2 y$ over K are said to be *equivalent* if $M_{N_1} \cong M_{N_2}$. Let $K^{n \times n}$ be the C-linear space of all $n \times n$ matrices over K, and consider the subspace

$$\mathcal{E}(N_1, N_2) \;:=\; \big\{P :\; P' = N_2 P - P N_1\big\}$$

of $K^{n \times n}$. Given a differential ring extension R of K, each $P \in \mathcal{E}(N_1, N_2)$ yields a C_R-linear map $y \mapsto Py\colon \operatorname{sol}_R(N_1) \to \operatorname{sol}_R(N_2)$. By Lemma 2.3.13 and the next lemma we have $\dim_C \mathcal{E}(N_1, N_2) \leqslant n^2$:

Lemma 2.4.31. *We have an isomorphism*

$$P \mapsto \phi_P \;:\; \mathcal{E}(N_1, N_2) \to \operatorname{Hom}_{K[\partial]}(M_{N_1}, M_{N_2})$$

of C-linear spaces given by

$$\phi_P(y) \;=\; Py \quad \textit{for } P \in \mathcal{E}(N_1, N_2) \textit{ and } y \in M_{N_1}.$$

Proof. Let $P \in \mathcal{E}(N_1, N_2)$, and define $\phi_P \in \operatorname{Hom}_K(M_{N_1}, M_{N_2})$ by $\phi_P(y) = Py$. Then for $y \in M_{N_1}$ we have

$$\phi_P(\partial y) \;=\; Py' - PN_1 y \;=\; Py' + (P'y - N_2 P y) \;=\; (Py)' - N_2 P y \;=\; \partial \phi_P(y),$$

hence $\phi_P \in \operatorname{Hom}_{K[\partial]}(M_{N_1}, M_{N_2})$. The rest follows easily. $\square$

One verifies easily that $\mathcal{E}(N) := \mathcal{E}(N, N)$ is a subalgebra of the C-algebra of $n \times n$-matrices over K and that this yields an isomorphism

$$P \mapsto \phi_P \;:\; \mathcal{E}(N) \to \operatorname{End}_{K[\partial]}(M_N)$$

of C-algebras. The C-algebra $\mathcal{E}(N)$ is called the **eigenring** of $y' = Ny$. We have $1 \leqslant \dim_C \mathcal{E}(N) \leqslant n^2$, and C is algebraically closed in K. It follows that the minimum polynomial of any $P \in \mathcal{E}(N)$ over K (that is, the monic polynomial $f(T) \in K[T]$ of least degree with $f(P) = 0$) has degree at most n^2 and has its coefficients in C. In particular, if C is algebraically closed, then the eigenvalues of any $P \in \mathcal{E}(N)$ are in C. If $y' = N_1 y$ and $y' = N_2 y$ are equivalent, then their eigenrings are isomorphic as C-algebras.

Corollary 2.4.32. *The isomorphism $P \mapsto \phi_P$ from the previous lemma restricts to a bijection between the subset*

$$\mathcal{E}(N_1, N_2)^\times \ := \ \mathrm{GL}_n(K) \cap \mathcal{E}(N_1, N_2)$$

of $\mathcal{E}(N_1, N_2)$ and the set of isomorphisms $M_{N_1} \to M_{N_2}$. If $\mathcal{E}(N_1) = C \cdot 1$ and $P \in \mathcal{E}(N_1, N_2)^\times$, then $\mathcal{E}(N_1, N_2) = C \cdot P$ and $\mathcal{E}(N_1, N_2)^\times = C^\times \cdot P$.

Hence $y' = N_1 y$ and $y' = N_2 y$ are equivalent iff $\mathcal{E}(N_1, N_2)^\times \neq \emptyset$, and in this case $y' = N_1 y$ is also called a **gauge transform** of $y' = N_2 y$.

For $P \in \mathcal{E}(N_1, N_2)^\times$ and each differential ring extension R of K we have the isomorphism

$$y \mapsto Py\colon\ \mathrm{sol}_R(N_1) \to \mathrm{sol}_R(N_2)$$

of C_R-modules, and any fundamental matrix F for $y' = N_1 y$ in R yields a fundamental matrix PF for $y' = N_2 y$ in R.

We have a right action of the group $\mathrm{GL}_n(K)$ on $K^{n \times n}$ given by

$$(N, P) \mapsto P^{-1}(N) := P^{-1} N P - P^{-1} P'.$$

For each N and $P \in \mathrm{GL}_n(K)$, we have $P \in \mathcal{E}(P^{-1}(N), N)^\times$, so the matrix differential equation $y' = P^{-1}(N) y$ is a gauge transform of $y' = Ny$.

Next we relate the eigenrings of linear differential operators introduced above with the eigenrings of matrix differential equations over K. We precede this by some generalities about differential modules: Let M, M_1, M_2 be (left) $K[\partial]$-modules. The *dual* $M^* := \mathrm{Hom}_K(M, K)$ of M is then a $K[\partial]$-module, and $\langle \phi, f \rangle := \phi(f) \in K$ for $\phi \in M^*$, $f \in M$. This yields the injective $K[\partial]$-linear map

$$\alpha \mapsto \alpha^*\colon\ \mathrm{Hom}_K(M_2, M_1) \to \mathrm{Hom}_K(M_1^*, M_2^*) \quad \text{where } \alpha^*(\phi) = \phi \circ \alpha \text{ for } \phi \in M_1^*,$$

and

$$\langle \alpha^*(\phi), f \rangle \ = \ \langle \phi, \alpha(f) \rangle \quad \text{for } \alpha \in \mathrm{Hom}_K(M_2, M_1),\ \phi \in M_1^*,\ f \in M_2.$$

If M_1, M_2 are differential modules over K, then $\alpha \mapsto \alpha^*$ is an isomorphism. Note that $\mathrm{Hom}_{K[\partial]}(M_2, M_1)$ is a C-linear subspace of $H := \mathrm{Hom}_K(M_2, M_1)$, with

$$\mathrm{Hom}_{K[\partial]}(M_2, M_1) \ = \ \ker_H \partial.$$

Hence the $K[\partial]$-module morphism $\alpha \mapsto \alpha^*$ restricts to a C-linear map

$$\mathrm{Hom}_{K[\partial]}(M_2, M_1) \to \mathrm{Hom}_{K[\partial]}(M_1^*, M_2^*),$$

which is bijective if M_1, M_2 are differential modules over K.

Let N be the companion matrix of a monic operator $A \in K[\partial]$ of order n, and set $M := K[\partial]/K[\partial]A$, a differential module over K of dimension n, with cyclic vector $e := 1 + K[\partial]A$, $Ae = 0$, and with basis $e_0, \dots, e_{n-1}$, $e_j := \partial^j e$ for $j = 0, \dots, n-1$. Then M^* has matrix N with respect to the dual basis $e_0^*, \dots, e_{n-1}^*$. Accordingly we identify M^* with M_N via the isomorphism $M^* \to M_N$ sending e_{j-1}^* to the jth standard basis vector of K^n, for $j = 1, \dots, n$.

In the following lemma N_1, N_2 are the companion matrices of monic operators $A_1, A_2 \in K[\partial]$ of order n, respectively. Set $M_1 := K[\partial]/K[\partial]A_1$, $M_2 := K[\partial]/K[\partial]A_2$ and identify M_1^*, M_2^* with M_{N_1}, M_{N_2}, as we just indicated for M. Let Φ be the isomorphism of C-linear spaces making the diagram

$$\begin{tikzcd}
\mathcal{E}(A_1, A_2) \arrow[r, "\Phi"] \arrow[d, "R\mapsto\phi_R"'] & \mathcal{E}(N_1, N_2) \arrow[d, "P\mapsto\phi_P"] \\
\operatorname{Hom}_{K[\partial]}(M_2, M_1) \arrow[r, "\alpha\mapsto\alpha^*"] & \operatorname{Hom}_{K[\partial]}(M_{N_1}, M_{N_2})
\end{tikzcd}$$

commute.

Lemma 2.4.33. *Let $R = r_0 + r_1\partial + \cdots + r_{n-1}\partial^{n-1} \in \mathcal{E}(A_1, A_2)$ $(r_0, \dots, r_{n-1} \in K)$; then the first row of the $n \times n$ matrix $\Phi(R)$ is $(r_0, r_1, \dots, r_{n-1})$.*

Proof. Set $P = \Phi(R)$; so $\phi_P = \phi_R^*$. Let $e := 1 + K[\partial]A_1 \in M_1$, and let $e_0^*, \dots, e_{n-1}^*$ be the basis of $M_{N_1} = M_1^*$ dual to the basis $e, \partial e, \dots, \partial^{n-1}e$ of M_1. Likewise, let $f := 1 + K[\partial]A_2 \in M_2$, and let $f_0^*, \dots, f_{n-1}^*$ be the basis of $M_{N_2} = M_2^*$ dual to the basis $f, \partial f, \dots, \partial^{n-1} f$ of M_2. Then for $j = 0, \dots, n-1$ we have $\phi_P(e_j^*) \in M_2^*$, and

$$\langle \phi_P(e_j^*), f \rangle \ = \ \langle \phi_R^*(e_j^*), f \rangle \ = \ \langle e_j^*, \phi_R(f) \rangle \ = \ r_j.$$

Hence the matrix P of the K-linear map ϕ_P with respect to the standard bases of $M_{N_1} = K^n$ and $M_{N_2} = K^n$ has first row $(r_0, r_1, \dots, r_{n-1})$. □

Self-dual matrix differential equations. Recall that $N^* = -N^{\mathrm{t}}$ by [ADH, 5.5.6] and $M_{N^*} \cong (M_N)^*$ by [ADH, p.279]. The *adjoint equation* of $y' = Ny$ is the matrix differential equation $y' = N^*y$ over K. We say that $y' = Ny$ is **self-dual** if it is equivalent to its adjoint equation [ADH, p. 277]. Hence $y' = Ny$ is self-dual iff the differential module M_N over K is self-dual. Thus if $y' = Ny$ is self-dual, then so is any matrix differential equation over K equivalent to $y' = Ny$, as is the adjoint equation $y' = N^*y$ of $y' = Ny$. By [ADH, 5.5.8, 5.5.9] we have:

Corollary 2.4.34. *If $C \neq K$, then every self-dual matrix differential equation $y' = N_1 y$ over K is equivalent to a matrix differential equation $y' = N_2 y$ with N_2 the companion matrix of a monic self-dual operator in $K[\partial]$.*

We set $\operatorname{mult}_\alpha(N) := \operatorname{mult}_\alpha(M_N)$ and call

$$\Sigma(N) \ := \ \Sigma(M_N) \ = \ \{\alpha : \operatorname{mult}_\alpha(N) \geqslant 1\}$$

the **spectrum** of $y' = Ny$. The elements of $\Sigma(N)$ are the **eigenvalues** of $y' = Ny$.

Lemma 2.4.35. *Suppose $B \in K[\partial]$ is monic of order n and N is the companion matrix of B. Then $\operatorname{mult}_\alpha(B) = \operatorname{mult}_\alpha(N)$ for all α. In particular, α is an eigenvalue of B iff α is an eigenvalue of $y' = Ny$.*

Proof. Use Corollary 2.3.15 and $M_N \cong M^*$ for $M := K[\partial]/K[\partial]B$ [ADH, 5.5.8]. □

From Corollary 2.4.6 we obtain:

Corollary 2.4.36. *Assume that $y' = Ny$ is self-dual. Suppose in addition that $\sum_\alpha \operatorname{mult}_\alpha(N) = n$ and K is 1-linearly surjective, or K is $(n-1)$-linearly surjective. Then $\operatorname{mult}_\alpha(N) = \operatorname{mult}_{-\alpha}(N)$ for all α. Hence, if also $K^\dagger$ is 2-divisible and $\sum_\alpha \operatorname{mult}_\alpha(N)$ is odd, then $0 \in \Sigma(N)$.*

Note that

$$\mathcal{E}(N, N^*) \ = \ \{P : P' = N^*P - PN\}, \qquad \mathcal{E}(N, N^*)^\times \ = \ \mathrm{GL}_n(K) \cap \mathcal{E}(N, N^*)$$

are both closed under matrix transpose. The matrix differential equation $y' = Ny$ is self-dual iff $\mathcal{E}(N, N^*)^\times \neq \emptyset$. Moreover, there is a $(-1)^n$-symmetric non-degenerate ∂-compatible K-bilinear form on M_N iff $\mathcal{E}(N, N^*)^\times$ contains a matrix P such that $P^{\mathrm{t}} = (-1)^n P$. One calls $y' = Ny$ **self-adjoint** if $N^* = N$, that is, N is skew-symmetric (in which case, $y' = Ny$ is self-dual) .

Example 2.4.37. Suppose $n = 3m$ and

$$N = \begin{pmatrix} & \kappa I & \\ -\kappa I & & \tau I \\ & -\tau I & \end{pmatrix}$$

where I denotes the $m \times m$ identity matrix and $\kappa, \tau \in K$. Then $y' = Ny$ is self-adjoint. Let π be the permutation of $\{1, \dots, n\}$ given for $i = 1, \dots, m$ by

$$\pi(i) \ = \ 3i - 2, \quad \pi(m+i) \ = \ 3i - 1, \quad \pi(2m+i) \ = \ 3i.$$

Then $P \in \mathrm{GL}_n(K)$ with $Pe_j = e_{\pi(j)}$ $(j = 1, \dots, n)$ gives $P' = 0 \in K^{n\times n}$, so

$$P^{-1}(N) \ = \ \mathrm{diag}(T, \dots, T) \in K^{n \times n} \quad \text{where } T := \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \in K^{3\times 3}.$$

By Corollary 2.3.11, $\mathrm{mult}_\alpha(N) = m \, \mathrm{mult}_\alpha(T)$ for all α, so $\Sigma(N) = \Sigma(T)$. If F is a fundamental matrix for $y' = Ty$, then $G := \mathrm{diag}(F, \dots, F) \in \mathrm{GL}_n(K)$ is a fundamental matrix for $y' = P^{-1}(N)y$, that is, $G' = P^{-1}NPG$, so PG is a fundamental matrix for $y' = Ny$. Suppose now that K is 1-linearly surjective, $K^\dagger$ is 2-divisible, and $\sum_\alpha \mathrm{mult}_\alpha(T) = 3$. Then $\sum_\alpha \mathrm{mult}_\alpha(N) = n$ and $\mathrm{mult}_\alpha(T) = \mathrm{mult}_{-\alpha}(T)$ for all α, so $\Sigma(N) = \Sigma(T) = \{\alpha, -\alpha, 0\}$ for some α.

Lemma 2.4.38. *Suppose $y' = Ny$ is self-adjoint and let $y, z \in \mathrm{sol}(N)$, where $y = (y_1, \dots, y_n)^{\mathrm{t}}$ and $z = (z_1, \dots, z_n)^{\mathrm{t}}$. Then $y_1 z_1 + \dots + y_n z_n \in C$.*

Proof. With $\langle \cdot\, , \cdot \rangle$ denoting the usual inner product on K^n, we have

$$\langle y, z\rangle' \ = \ \langle y', z\rangle + \langle y, z'\rangle \ = \ \langle Ny, z\rangle + \langle y, Nz\rangle \ = \ \langle y, N^{\mathrm{t}}z\rangle + \langle y, Nz\rangle \ = \ 0$$

since $N^{\mathrm{t}} = -N$. ∎

Thus if $y' = Ny$ is self-adjoint, then we have a symmetric bilinear form

$$(y, z) \mapsto \langle y, z\rangle = y_1 z_1 + \dots + y_n z_n \quad (y = (y_1, \dots, y_n)^{\mathrm{t}},\ z = (z_1, \dots, z_n)^{\mathrm{t}})$$

on the C-linear subspace $\mathrm{sol}(N)$ of K^n of dimension $\leqslant n$.

A matrix $F \in K^{n\times n}$ is said to be *orthogonal* if $FF^{\mathrm{t}} = I_n$, where I_n denotes the identity of the ring $K^{n\times n}$ of $n \times n$-matrices over K. This yields the subgroup $\mathrm{O}_n(K)$ of $\mathrm{GL}_n(K)$ consisting of the orthogonal matrices $F \in K^{n\times n}$.

Suppose $F \in \mathrm{GL}_n(K)$ is a fundamental matrix for $y' = Ny$. By [ADH, 5.5.12] this yields a a fundamental matrix $(F^{\mathrm{t}})^{-1} \in \mathrm{GL}_n(K)$ for $y' = N^*y$, so if F is orthogonal, then $y' = Ny$ is self-adjoint. Conversely:

Lemma 2.4.39. *Suppose $y' = Ny$ is self-adjoint, $\dim_C \mathrm{sol}(N) = n$, and $C^\times$ is 2-divisible. Then $\mathrm{GL}_n(K)$ contains an orthogonal fundamental matrix for $y' = Ny$.*

Proof. Take a fundamental matrix $F \in \mathrm{GL}_n(K)$ for $y' = Ny$. Then $G := (F^{\mathrm{t}})^{-1}$ is also a fundamental matrix for $y' = Ny$, so $F^{\mathrm{t}}F = G^{-1}F \in \mathrm{GL}_n(C)$ by [ADH, 5.5.11]. Now the matrix $F^{\mathrm{t}}F$ is symmetric, so [122, Chapter XV, Theorem 3.1] gives D, U in $\mathrm{GL}_n(C)$ with diagonal D such that $F^{\mathrm{t}}F = U^{\mathrm{t}}DU$. Let $V := \sqrt{D}U$ where $\sqrt{D}$ in $C^{n\times n}$ is diagonal with $(\sqrt{D})^2 = D$. Then $F^{\mathrm{t}}F = V^{\mathrm{t}}V$ and so $FV^{-1} \in \mathrm{GL}_n(K)$ is an orthogonal fundamental matrix for $y' = Ny$ by [ADH, 5.5.11]. □

The skew-symmetric $n \times n$ matrices over K form a Lie subalgebra

$$\mathfrak{so}_n(K) \ = \ \{N : \ N^* = N\}$$

of $K^{n\times n}$ equipped with the Lie bracket $[N_1, N_2] = N_1N_2 - N_2N_1$. Suppose now $n = 2m$ is even, and set $J := \left(\begin{smallmatrix} & I_m \\ -I_m & \end{smallmatrix}\right)$. Then $J^{\mathrm{t}} = J^{-1} = -J$, and

$$\mathfrak{sp}_n(K) \ = \ \{N : \ N^*J = JN\}$$

is also a Lie subalgebra of $K^{n\times n}$. The matrices in $\mathfrak{sp}_n(K)$ are called *hamiltonian*; thus N is hamiltonian iff JN is symmetric. We say that the matrix differential equation $y' = Ny$ is **hamiltonian** if $N \in \mathfrak{sp}_n(K)$; in that case $J \in \mathcal{E}(N, N^*)^\times$, so $y' = Ny$ is self-dual. A matrix $F \in K^{n\times n}$ is said to be *symplectic* if $F^{\mathrm{t}}JF = J$. The symplectic matrices form a subgroup $\mathrm{Sp}_n(K)$ of $\mathrm{GL}_n(K)$. If $y' = Ny$ has a fundamental matrix $F \in \mathrm{Sp}_n(K)$, then $y' = Ny$ is hamiltonian. In analogy with Lemma 2.4.39 we have a converse:

Lemma 2.4.40. *Suppose $y' = Ny$ is hamiltonian and $\dim_C \mathrm{sol}(N) = n$. Then $\mathrm{GL}_n(K)$ contains a symplectic fundamental matrix for $y' = Ny$.*

Proof. Take a fundamental matrix $F \in \mathrm{GL}_n(K)$ for $y' = Ny$. Then JF and $G := (F^{\mathrm{t}})^{-1}$ are fundamental matrices for $y' = N^*y$, so $F^{\mathrm{t}}JF = G^{-1}JF \in \mathrm{GL}_n(C)$. Now $F^{\mathrm{t}}JF$ is skew-symmetric, hence $F^{\mathrm{t}}JF = U^{\mathrm{t}}JU$ with $U \in \mathrm{Gl}_n(C)$ [122, Chapter XV, Corollary 8.2]; then FU^{-1} is a symplectic fundamental matrix for $y' = Ny$. □

For a hamiltonian analogue of Lemma 2.4.38, let $\langle\,\cdot\,,\cdot\,\rangle$ denote the usual inner product on K^n and let $(y, z) \mapsto \omega(y, z) := \langle y, Jz\rangle$ be the standard symplectic bilinear form on K^n.

Lemma 2.4.41. *Suppose $y' = Ny$ is hamiltonian and $y, z \in \mathrm{sol}(N)$. Then $\omega(y, z) \in C$.*

Proof. We have

$$\omega(y,z)' \ = \ \langle y', Jz\rangle + \langle y, Jz'\rangle \ = \ \langle Ny, Jz\rangle + \langle y, JNz\rangle \ = \ \langle y, N^{\mathrm{t}}Jz\rangle + \langle y, JNz\rangle \ = \ 0$$

where we used $-N^{\mathrm{t}}J = N^*J = JN$ for the last equality. □

Note also that N is hamiltonian iff $N = JN^{\mathrm{t}}J$. It follows that N is hamiltonian iff $N = \left(\begin{smallmatrix} Q & R \\ P & Q^* \end{smallmatrix}\right)$ where $P, R \in K^{m\times m}$ are symmetric and $Q \in K^{m\times m}$.

Hamiltonian matrix differential equations appear naturally in the study of more general (non-linear) Hamiltonian systems (as the variational equations along an integral curve of such a system). See, e.g., [47]. They also arise from self-adjoint linear differential operators: using Lemma 2.4.19 one can show that if A is self-adjoint with companion matrix M, then with $n := r$ there is some hamiltonian N such that $\mathcal{E}(M, N)^\times \neq \emptyset$; see [52, p. 76].

Anti-self-duality. We now continue in the setting of the subsection *Complex conjugation* in Section 2.3. Thus $K = H[\mathrm{i}]$ where H is a differential subfield of K, $\mathrm{i}^2 = -1$, and $\mathrm{i} \notin H$. The isomorphisms below are of differential modules over K. Let M be a differential module over K. We establish some analogues of results above for the conjugate dual of M instead of its dual.

Call M is **anti-self-dual** if $M \cong \overline{M^*}$. If M is anti-self-dual, then so is every isomorphic $K[\partial]$-module, in particular, $\overline{M^*}$. Here is an analogue of Corollary 2.4.6 which follows immediately from Corollary 2.3.26:

Corollary 2.4.42. *Let $\dim_K M = r$ and assume M is anti-self-dual. Suppose also that K is 1-linearly surjective and $\sum_\alpha \operatorname{mult}_\alpha(M) = r$, or $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Then $\operatorname{mult}_\alpha(M) = \operatorname{mult}_{-\overline{\alpha}}(M)$ for all α. Hence if additionally $K^\dagger$ is 2-divisible and $\sum_\alpha \operatorname{mult}_\alpha(M)$ is odd, then $[b\mathrm{i}] \in \Sigma(M)$ for some $b \in H$.*

Suppose now $M = K[\partial]/K[\partial]A$ and $r \geqslant 1$. Then $\overline{M^*} \cong K[\partial]/K[\partial]\overline{A^*}$ by [ADH, 5.5.8] and Example 2.3.25. Hence M is anti-self-dual iff A, $\overline{A^*}$ have the same type. We say that A is **anti-self-dual** if A, $\overline{A^*}$ have the same type. If A is anti-self-dual, then so are $\overline{A}$ and A^*, and so is every operator of the same type as A. If A is anti-self-dual, then A, $\overline{A^*}$ have the same eigenvalues, with the same multiplicities. The previous corollary yields:

Corollary 2.4.43. *Suppose A is anti-self-dual, and set $s := \sum_\alpha \operatorname{mult}_\alpha(A)$. Also assume K is 1-linearly surjective and $s = r$, or $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Then $\operatorname{mult}_\alpha(A) = \operatorname{mult}_{-\overline{\alpha}}(A)$ for all α. Hence if in addition $K^\dagger$ is 2-divisible and s is odd, then $[b\mathrm{i}] \in \Sigma(A)$ for some $b \in H$.*

Later H is usually a Hardy field with $H^\dagger = H$, so $\alpha = -\overline{\alpha}$ for all α. In this case Corollaries 2.4.42 and 2.4.43 are less useful than their cousins Corollaries 2.4.6 and 2.4.9. Note also that if $A \in H[\partial]$, then A is self-dual iff A is anti-self-dual.

We now consider anti-self-duality for a matrix differential equation $y' = Ny$ over K. Recall that N is an $n \times n$-matrix over K with $n \geqslant 1$, and that if $M = M_N$ is the differential module over K associated to N, then $\overline{M} \cong M_{\overline{N}}$ by the remarks preceding Example 2.3.25, and $M^* \cong M_{N^*}$ by [ADH, pp. 279–280]. We say that $y' = Ny$ is **anti-self-dual** if it is equivalent to the matrix differential equation $y' = \overline{N^*}y$ over K. (Note: $\overline{N^*} = -\overline{N}^{\mathrm{t}}$.) Hence $y' = Ny$ is anti-self-dual iff M_N is anti-self-dual. If $y' = Ny$ is anti-self-dual, then so is any matrix differential equation over K equivalent to $y' = Ny$, as are the matrix differential equations $y' = N^*y$ and $y' = \overline{N}y$ over K. If $N \in H^{n\times n}$, then the matrix differential equation $y' = Ny$ over K is self-dual iff it is anti-self-dual.

Corollary 2.4.44. *Suppose $C \neq K$ and $y' = Ny$ is anti-self-dual. Then $y' = Ny$ is equivalent to a matrix differential equation $y' = A_L y$ with A_L the companion matrix of a monic anti-self-dual $L \in K[\partial]$.*

Proof. By [ADH, 5.5.9], $y' = Ny$ is equivalent to $y' = A_L y$ where A_L is the companion matrix of a monic $L \in K[\partial]$. Then L is anti-self-dual by [ADH, 5.5.8] and Example 2.3.25. □

We say that $y' = Ny$ is **anti-self-adjoint** if $\overline{N^*} = N$, that is, $N^{\mathrm{t}} = -\overline{N}$. Then $y' = Ny$ is in particular anti-self-dual. If $N \in H^{n\times n}$, then $y' = Ny$ is anti-self-adjoint iff it is self-adjoint. To state an anti-self-adjoint analogue of Lemma 2.4.38 we use the "hermitian" inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on K^n given by $\langle y, z\rangle = y_1\overline{z}_1 + \cdots + y_n\overline{z}_n$ for $y = (y_1, \dots, y_n)^{\mathrm{t}} \in K^n$ and $z = (z_1, \dots, z_n)^{\mathrm{t}} \in K^n$.

Lemma 2.4.45. *If $y' = Ny$ is anti-self-adjoint and $y, z \in \operatorname{sol}(N)$, then $\langle y, z\rangle \in C$.*

Proof. Assume $y' = Ny$ is anti-self-adjoint. Then $\overline{N}^{\mathrm{t}} = -N$, so

$$\langle y, z\rangle' \;=\; \langle y', z\rangle + \langle y, z'\rangle \;=\; \langle Ny, z\rangle + \langle y, Nz\rangle \;=\; \langle y, \overline{N}^{\mathrm{t}} z\rangle + \langle y, Nz\rangle \;=\; 0. \quad \square$$

Thus if $y' = Ny$ is anti-self-adjoint, then we have a hermitian form

$$(y, z) \mapsto \langle y, z\rangle \;=\; y_1\overline{z_1} + \cdots + y_n\overline{z_n} \quad (y = (y_1, \dots, y_n)^{\mathrm{t}},\ z = (z_1, \dots, z_n)^{\mathrm{t}})$$

on the C-linear subspace $\operatorname{sol}(N)$ of K^n of dimension $\leqslant n$.

A matrix $U \in K^{n\times n}$ is *unitary* if $U^{\mathrm{t}}\overline{U} = I_n$, equivalently, $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in K^n$. The unitary matrices form a subgroup $\mathrm{U}_n(K)$ of $\mathrm{GL}_n(K)$. Suppose $F \in \mathrm{GL}_n(K)$ is a fundamental matrix for $y' = Ny$. Then $\overline{F}$ is a fundamental matrix for $y' = \overline{N}y$, and so $(\overline{F}^{\mathrm{t}})^{-1} \in \mathrm{GL}_n(K)$ is a fundamental matrix for $y' = \overline{N^*}y$. So if F is unitary, then $y' = Ny$ is anti-self-adjoint. Here is a converse, analogous to Lemma 2.4.39:

Lemma 2.4.46. *Suppose $y' = Ny$ is anti-self-adjoint, $\dim_C \operatorname{sol}(N) = n$, and H is real closed. Then $\mathrm{GL}_n(K)$ contains a unitary fundamental matrix for $y' = Ny$.*

Proof. Take a fundamental matrix $F \in \mathrm{GL}_n(K)$ for $y' = Ny$. Then $G := (\overline{F}^{\mathrm{t}})^{-1}$ is also a fundamental matrix for $y' = Ny$, so $\overline{F}^{\mathrm{t}}F = G^{-1}F \in \mathrm{GL}_n(C)$. Now $P := \overline{F}^{\mathrm{t}}F$ is hermitian (i.e., $\overline{P}^{\mathrm{t}} = P$), so [122, Chapter XV, §5, 6] gives a diagonal $D \in \mathrm{GL}_n(C_H)$ and a $U \in \mathrm{U}_n(C)$ with $P = \overline{U}^{\mathrm{t}}DU$. So for $x \in C^n$ and $y := U^{-1}x \in C^n$,

$$\langle Dx, x\rangle \;=\; \langle DUy, Uy\rangle \;=\; \langle Py, y\rangle \;=\; \langle Fy, Fy\rangle,$$

a sum of squares in H. As C_H is also real closed, all entries of D are squares in C_H. Take diagonal $E \in C_H^{n\times n}$ with $E^2 = D$. Then $V := EU \in \mathrm{GL}_n(C)$ and $P = \overline{V}^{\mathrm{t}}V$, so $FV^{-1} \in \mathrm{GL}_n(K)$ is a unitary fundamental matrix for $y' = Ny$. $\square$

2.5. Eigenvalues and Splittings

In this section K is a differential field such that C is algebraically closed and $K^\dagger$ is divisible. We let A, B range over $K[\partial]$, and we assume $A \neq 0$ and set $r := \operatorname{order} A$.

Spectral decomposition of differential operators. Fix a complement Λ of the subspace $K^\dagger$ of the $\mathbb{Q}$-linear space K, let $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K, let Ω be the differential fraction field of the differential K-algebra U, and let λ range over Λ. Then

$$A_\lambda \;=\; A_{\ltimes \mathrm{e}(\lambda)} \;=\; \mathrm{e}(-\lambda)A\,\mathrm{e}(\lambda) \in K[\partial].$$

Moreover, for every $a \in K$ there is a unique λ with $a - \lambda \in K^\dagger$, so $\operatorname{mult}_{[a]}(A) = \operatorname{mult}_\lambda(A)$. Call λ an **eigenvalue** of A with respect to our complement Λ of $K^\dagger$ in K if $[\lambda]$ is an eigenvalue of A; thus the group isomorphism $\lambda \mapsto [\lambda]\colon \Lambda \to K/K^\dagger$ maps the set of eigenvalues of A with respect to Λ onto the spectrum of A. For $f \in \mathrm{U}$ with spectral decomposition (f_λ) we have

$$A(f) \;=\; \sum_\lambda A_\lambda(f_\lambda)\ \mathrm{e}(\lambda),$$

so $A(\mathrm{U}^\times) \subseteq \mathrm{U}^\times \cup \{0\}$. We call the family (A_λ) the **spectral decomposition** of A (with respect to Λ). Given a C-linear subspace V of U, we set $V_\lambda := V \cap K\,\mathrm{e}(\lambda)$, a C-linear subspace of V; the sum $\sum_\lambda V_\lambda$ is direct. For $V := \mathrm{U}$ we have $\mathrm{U}_\lambda = K\,\mathrm{e}(\lambda)$, and $\mathrm{U} = \bigoplus_\lambda \mathrm{U}_\lambda$ with $A(\mathrm{U}_\lambda) \subseteq \mathrm{U}_\lambda$ for all λ. Taking $V := \ker_{\mathrm{U}} A$, we obtain $V_\lambda = (\ker_K A_\lambda)\,\mathrm{e}(\lambda)$ and hence $\dim_C V_\lambda = \mathrm{mult}_\lambda(A)$, and $V = \bigoplus_\lambda V_\lambda$. Thus

$$|\Sigma(A)| \ \leqslant\ \sum_\lambda \mathrm{mult}_\lambda(A) \ =\ \dim_C \ker_{\mathrm{U}} A \ \leqslant\ r. \tag{2.5.1}$$

Moreover:

Lemma 2.5.1. *The C-linear space* $\ker_{\mathrm{U}} A$ *has a basis contained in* $\mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$.

Example. We have a C-algebra isomorphism $P(Y) \mapsto P(\partial)\colon C[Y] \to C[\partial]$. Suppose $A \in C[\partial] \subseteq K[\partial]$, let $P(Y) \in C[Y]$, $P(\partial) = A$, and let $c_1, \dots, c_n \in C$ be the distinct zeros of P, of respective multiplicities $m_1, \dots, m_n \in \mathbb{N}^{\geqslant 1}$ (so $r = \deg P = m_1 + \cdots + m_n$). Suppose also $C \subseteq \Lambda$, and $x \in K$ satisfies $x' = 1$. (This holds in Example 2.2.4.) Then the $x^i\,\mathrm{e}(c_j) \in \mathrm{U}$ ($1 \leqslant j \leqslant n$, $0 \leqslant i < m_j$) form a basis of the C-linear space $\ker_{\mathrm{U}} A$ by [ADH, 5.1.18]. So the eigenvalues of A with respect to Λ are $c_1, \dots, c_n$, with respective multiplicities $m_1, \dots, m_n$.

Corollary 2.5.2. *Suppose* $\dim_C \ker_{\mathrm{U}} A = r \geqslant 1$ *and* $A = \partial^r + a_{r-1}\partial^{r-1} + \cdots + a_0$ *where* $a_0, \dots, a_{r-1} \in K$. *Then*

$$\sum_\lambda \mathrm{mult}_\lambda(A)\lambda \ \equiv\ -a_{r-1} \mod K^\dagger.$$

In particular, $\sum_\lambda \mathrm{mult}_\lambda(A)\lambda = 0$ *iff* $a_{r-1} \in K^\dagger$.

Proof. Take a basis $y_1, \dots, y_r$ of $\ker_{\mathrm{U}} A$ with $y_j = f_j\,\mathrm{e}(\lambda_j)$, $f_j \in K^\times$, $\lambda_j \in \Lambda$. The Wronskian matrix $\mathrm{Wr}(y_1, \dots, y_r)$ of $(y_1, \dots, y_r)$ [ADH, p. 206] equals

$$\mathrm{Wr}(y_1, \dots, y_r) \ =\ M \begin{pmatrix} \mathrm{e}(\lambda_1) & & \\ & \ddots & \\ & & \mathrm{e}(\lambda_r) \end{pmatrix} \qquad \text{where } M \in \mathrm{GL}_n(K).$$

Then $w := \mathrm{wr}(y_1, \dots, y_r) = \det \mathrm{Wr}(y_1, \dots, y_r) \neq 0$ by [ADH, 4.1.13] and

$$-a_{r-1} \ =\ w^\dagger \ =\ (\det M)^\dagger + \lambda_1 + \cdots + \lambda_r$$

where we used [ADH, 4.1.17] for the first equality. □

If A splits over K, then so does A_λ. Moreover, if $A_\lambda(K) = K$, then $A(\mathrm{U}_\lambda) = \mathrm{U}_\lambda$: for $f, g \in K$ with $A_\lambda(f) = g$ we have $A\big(f\,\mathrm{e}(\lambda)\big) = g\,\mathrm{e}(\lambda)$. Thus:

Lemma 2.5.3. *Suppose K is r-linearly surjective, or K is 1-linearly surjective and A splits over K. Then $A(\mathrm{U}_\lambda) = \mathrm{U}_\lambda$ for all λ and hence $A(\mathrm{U}) = \mathrm{U}$.*

In the next subsection we study the connection between splittings of A and bases of the C-linear space $\ker_{\mathrm{U}} A$ in more detail.

Constructing splittings and bases. Recall that $\operatorname{order} A = r \in \mathbb{N}$. Set $\mathrm{U} = \mathrm{U}_K$, so $\mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$. Let $y_1, \dots, y_r \in \mathrm{U}^\times$. We construct a sequence $A_0, \dots, A_n$ of monic operators in $K[\partial]$ with $n \leqslant r$ as follows. First, set $A_0 := 1$. Next, given $A_0, \dots, A_{i-1}$ in $K[\partial]^{\neq}$ ($1 \leqslant i \leqslant r$), set $f_i := A_{i-1}(y_i)$; if $f_i \neq 0$, then $f_i \in \mathrm{U}^\times$, so $f_i^\dagger \in K$, and the next term in the sequence is

$$A_i \ :=\ (\partial - a_i)A_{i-1}, \qquad a_i \ :=\ f_i^\dagger,$$

whereas if $f_i = 0$, then $n := i - 1$ and the construction is finished.

Lemma 2.5.4. $\ker_{\mathrm{U}} A_i = Cy_1 \oplus \cdots \oplus Cy_i$ (*internal direct sum*) *for* $i = 0, \dots, n$.

Proof. By induction on $i \leqslant n$. The case $i = 0$ being trivial, suppose $1 \leqslant i \leqslant n$ and the claim holds for $i-1$ in place of i. Then $A_{i-1}(y_i) = f_i \neq 0$, hence $y_i \notin \ker_{\mathrm{U}} A_{i-1} = Cy_1 \oplus \cdots \oplus Cy_{i-1}$, and $A_i = (\partial - f_i^\dagger)A_{i-1}$, so by [ADH, 5.1.14(i)] we have $\ker_{\mathrm{U}} A_i = \ker_{\mathrm{U}} A_{i-1} \oplus Cy_i = Cy_1 \oplus \cdots \oplus Cy_i$. $\square$

We denote the tuple $(a_1, \dots, a_n) \in K^n$ just constructed by $\operatorname{split}(y_1, \dots, y_r)$, so $A_n = (\partial - a_n) \cdots (\partial - a_1)$. Suppose $r \geqslant 1$. Then $n \geqslant 1$, $a_1 = y_1^\dagger$, $A_1 = \partial - a_1$, $A_1(y_2), \dots, A_1(y_n) \in \mathrm{U}^\times$, and we have

$$(a_2, \dots, a_n) \;=\; \operatorname{split}\bigl(A_1(y_2), \dots, A_1(y_n)\bigr).$$

By Lemma 2.5.4, $n \leqslant r$ is maximal such that $y_1, \dots, y_n$ are C-linearly independent. In particular, $y_1, \dots, y_r$ are C-linearly independent iff $n = r$.

Corollary 2.5.5. *If $A(y_i) = 0$ for $i = 1, \dots, n$, then $A \in K[\partial]A_n$. Thus if $n = r$ and $A(y_i) = 0$ for $i = 1, \dots, r$, then $A = a(\partial - a_r) \cdots (\partial - a_1)$ where $a \in K^\times$.*

This follows from [ADH, 5.1.15(i)] and Lemma 2.5.4.

Suppose that H is a differential subfield of K and $y_1^\dagger, \dots, y_r^\dagger \in H$. Then we have $\operatorname{split}(y_1, \dots, y_r) \in H^n$: use that $y' \in Hy$ with $y \in U$ gives $y^{(m)} \in Hy$ for all m, so $B(y) \in Hy$ for all $B \in H[\partial]$, hence for such B, if $f := B(y) \neq 0$, then $f^\dagger \in H$.

Corollary 2.5.6. *Suppose $\dim_C \ker_{\mathrm{U}} A = r$. Then $\ker_{\mathrm{U}} A = \ker_{\Omega} A$ and A splits over K. If $A = (\partial - a_r) \cdots (\partial - a_1)$, $a_1, \dots, a_r \in K$, then the spectrum of A is $\bigl\{[a_1], \dots, [a_r]\bigr\}$, and for all $\alpha \in K/K^\dagger$,*

$$\operatorname{mult}_\alpha(A) \;=\; \bigl|\bigl\{i \in \{1, \dots, r\} :\; \alpha = [a_i]\bigr\}\bigr|.$$

Proof. A splits over K by Lemma 2.5.1 and Corollary 2.5.5. The rest follows from Lemma 2.3.4 in view of $\sum_\lambda \operatorname{mult}_\lambda(A) = \dim_C \ker_{\mathrm{U}} A$. $\square$

Conversely, we can associate to a given splitting of A over K a basis of $\ker_{\mathrm{U}} A$ consisting of r elements of $\mathrm{U}^\times$, provided K is 1-linearly surjective when $r \geqslant 2$:

Lemma 2.5.7. *Assume K is 1-linearly surjective in case $r \geqslant 2$. Let*

$$A \;=\; (\partial - a_r) \cdots (\partial - a_1) \qquad \text{where } a_i = b_i^\dagger + \lambda_i,\ b_i \in K^\times,\ \lambda_i \in \Lambda\ (i = 1, \dots, r).$$

Then there are C-linearly independent $y_1, \dots, y_r \in \ker_{\mathrm{U}} A$ with $y_i \in K^\times \mathrm{e}(\lambda_i)$ for $i = 1, \dots, r$ and $\operatorname{split}(y_1, \dots, y_r) = (a_1, \dots, a_r)$.

Proof. By induction on r. The case $r = 0$ is trivial, and for $r = 1$ we can take $y_1 = b_1 \mathrm{e}(\lambda_1)$. Let $r \geqslant 2$ and suppose inductively that for

$$B \;:=\; (\partial - a_r) \cdots (\partial - a_2)$$

we have C-linearly independent $z_2, \dots, z_r \in \ker_{\mathrm{U}} B$ with $z_i \in K^\times \mathrm{e}(\lambda_i)$ for $i = 2, \dots, r$ and $\operatorname{split}(z_2, \dots, z_r) = (a_2, \dots, a_r)$. For $i = 2, \dots, r$, Lemma 2.5.3 gives $y_i \in K^\times \mathrm{e}(\lambda_i)$ with $(\partial - a_1)(y_i) = z_i$. Set $y_1 := b_1 \mathrm{e}(\lambda_1)$, so $\ker_{\mathrm{U}}(\partial - a_1) = Cy_1$. Then $y_1, \dots, y_r \in \ker_{\mathrm{U}} A$ are C-linearly independent with $y_i \in K^\times \mathrm{e}(\lambda_i)$ for $i = 1, \dots, r$, and one verifies easily that $\operatorname{split}(y_1, \dots, y_r) = (a_1, \dots, a_r)$. $\square$

Corollary 2.5.8. *Assume K is 1-linearly surjective when $r \geqslant 2$. Then*

$$A \text{ splits over } K \iff \dim_C \ker_{\mathrm{U}} A \;=\; r.$$

Remark. If $\dim_C \ker_{\mathrm{U}} A = r$ and $\lambda_1, \dots, \lambda_d$ are the eigenvalues of A with respect to Λ, then the differential subring $K\big[\mathrm{e}(\lambda_1), \mathrm{e}(-\lambda_1), \dots, \mathrm{e}(\lambda_d), \mathrm{e}(-\lambda_d)\big]$ of U is the Picard-Vessiot ring for A over K; see [158, Section 1.3]. If K is linearly closed and linearly surjective, then U is by Corollary 2.5.8 the universal Picard-Vessiot ring of the differential field K as defined in [158, Chapter 10]. Our construction of U above is modeled on the description of the universal Picard-Vessiot ring of the algebraic closure of $C(\!(t)\!)$ given in [158, Chapter 3].

Recalling our convention that $r = \operatorname{order} A$, here is a complement to Lemma 2.5.1:

Corollary 2.5.9. *Let V be a C-linear subspace of* U *with $r = \dim_C V$. Then there is at most one monic A with $V = \ker_{\mathrm{U}} A$. Moreover, the following are equivalent:*

(i) *$V = \ker_{\mathrm{U}} A$ for some monic A that splits over K;*
(ii) *$V = \ker_{\mathrm{U}} B$ for some $B \neq 0$;*
(iii) *$V = \sum_\lambda V_\lambda$;*
(iv) *V has a basis contained in $\mathrm{U}^\times$.*

Proof. The first claim follows from [ADH, 5.1.15] applied to the differential fraction field of U in place of K. The implication (i) ⇒ (ii) is clear, (ii) ⇒ (iii) was noted before Lemma 2.5.1, and (iii) ⇒ (iv) is obvious. For (iv) ⇒ (i), let $y_1, \dots, y_r \in \mathrm{U}^\times$ be a basis of V. Then $\operatorname{split}(y_1, \dots, y_r) = (a_1, \dots, a_r) \in K^r$, so $V = \ker_{\mathrm{U}} A$ for $A = (\partial - a_r) \cdots (\partial - a_1)$ by Lemma 2.5.4, so (i) holds. □

Let $y_1, \dots, y_r \in \mathrm{U}^\times$ and $(a_1, \dots, a_n) := \operatorname{split}(y_1, \dots, y_r)$. We finish this subsection with some remarks about $(a_1, \dots, a_n)$. Let $A_0, \dots, A_n \in K[\partial]$ be as above and recall that $n \leqslant r$ is maximal such that $y_1, \dots, y_n$ are C-linearly independent.

Lemma 2.5.10. *Assume $n = r$. Let $z_1, \dots, z_r \in \mathrm{U}^\times$. The following are equivalent:*

(i) *$z_1, \dots, z_r$ are C-linearly independent and $(a_1, \dots, a_r) = \operatorname{split}(z_1, \dots, z_r)$;*
(ii) *for $i = 1, \dots, r$ there are $c_{ii}, c_{i,i-1}, \dots, c_{i1} \in C$ such that*

$$z_i \;=\; c_{ii} y_i + c_{i,i-1} y_{i-1} + \cdots + c_{i1} y_1 \text{ and } c_{ii} \neq 0.$$

Proof. The case $r = 0$ is trivial. Let $r = 1$. If (i) holds, then $y_1^\dagger = a_1 = z_1^\dagger$, hence $z_1 \in C^\times y_1$, so (ii) holds. The converse is obvious. Let $r \geqslant 2$, and assume (i) holds. Put $\widetilde{y}_i := A_1(y_i)$ and $\widetilde{z}_i := A_1(z_i)$ for $i = 2, \dots, r$. Then

$$\operatorname{split}(\widetilde{y}_2, \dots, \widetilde{y}_r) \;=\; (a_2, \dots, a_r) \;=\; \operatorname{split}(\widetilde{z}_2, \dots, \widetilde{z}_r),$$

so we can assume inductively to have $c_{ij} \in C$ $(2 \leqslant j \leqslant i \leqslant r)$ with

$$\widetilde{z}_i \;=\; c_{ii} \widetilde{y}_i + c_{i,i-1} \widetilde{y}_{i-1} + \cdots + c_{i2} \widetilde{y}_2 \quad \text{and} \quad c_{ii} \neq 0 \qquad (2 \leqslant i \leqslant r).$$

Hence for $2 \leqslant i \leqslant r$,

$$z_i \in c_{ii} y_i + c_{i,i-1} y_{i-1} + \cdots + c_{i2} y_2 + \ker_{\mathrm{U}} A_1.$$

Now use $\ker_{\mathrm{U}} A_1 = C y_1$ to conclude (ii). For the converse, let $c_{ij} \in C$ be as in (ii). Then clearly $z_1, \dots, z_r$ are C-linearly independent. Let $(b_1, \dots, b_r) := \operatorname{split}(z_1, \dots, z_r)$ and $B_{r-1} := (\partial - b_{r-1}) \cdots (\partial - b_1)$. Then $a_r = f_r^\dagger$ where $f_r = A_{r-1}(y_r) \neq 0$, and $b_r = g_r^\dagger$ where $g_r := B_{r-1}(z_r) \neq 0$. Now inductively we have $a_j = b_j$ for $j = 1, \dots, r-1$, so $A_{r-1} = B_{r-1}$, and $A_{r-1}(y_i) = 0$ for $i = 1, \dots, r-1$ by Lemma 2.5.4. Hence $g_r = c_{rr} f_r$, and thus $a_r = b_r$. □

Lemma 2.5.11. *Let $z \in \mathrm{U}^\times$. Then $\operatorname{split}(y_1 z, \dots, y_r z) = (a_1 + z^\dagger, \dots, a_n + z^\dagger)$.*

Proof. Since for $m \leqslant r$, the units $y_1 z, \dots, y_m z$ of U are C-linearly independent iff $y_1, \dots, y_m$ are C-linearly independent, we see that the tuples $\operatorname{split}(y_1 z, \dots, y_r z)$ and $\operatorname{split}(y_1, \dots, y_r)$ have the same length n. Let $(b_1, \dots, b_n) := \operatorname{split}(y_1 z, \dots, y_r z)$; we show $(b_1, \dots, b_n) = (a_1 + z^\dagger, \dots, a_n + z^\dagger)$ by induction on n. The case $n = 0$ is obvious, so suppose $n \geqslant 1$. Then $a_1 = y_1^\dagger$ and $b_1 = (y_1 z)^\dagger = a_1 + z^\dagger$ as required. By remarks following the proof of Lemma 2.5.4 we have

$$(a_2, \dots, a_n) \ = \ \operatorname{split}\big(A_1(y_2), \dots, A_1(y_n)\big) \qquad \text{where } A_1 := \partial - a_1.$$

Now $B_1 := \partial - b_1 = (A_1)_{\ltimes z^{-1}}$, so likewise

$$(b_2, \dots, b_n) \ = \ \operatorname{split}\big(B_1(y_2 z), \dots, B_1(y_n z)\big) \ = \ \operatorname{split}\big(A_1(y_2) z, \dots, A_1(y_n) z\big).$$

Hence $b_2 = a_2 + z^\dagger, \dots, b_n = a_n + z^\dagger$ by our inductive hypothesis. $\square$

For $f \in \partial K$ we let $\int f$ denote an element of K such that $(\int f)' = f$.

Lemma 2.5.12. *Let $g_1, \dots, g_r \in K^\times$ and*

$$A \ = \ g_1 \cdots g_r (\partial g_r^{-1})(\partial g_{r-1}^{-1}) \cdots (\partial g_1^{-1}),$$

and suppose the integrals below can be chosen such that

$$y_1 \ = \ g_1, \quad y_2 \ = \ g_1 \textstyle\int g_2, \quad \dots, \quad y_r \ = \ g_1 \int \big(g_2 \int g_3 (\cdots (g_{r-1} \int g_r) \cdots)\big),$$

Then $y_1, \dots, y_r \in K^\times$, $n = r$, and $a_i = (g_1 \cdots g_i)^\dagger$ for $i = 1, \dots, r$.

Proof. Let $b_i := (g_1 \cdots g_i)^\dagger$ for $i = 1, \dots, r$. By induction on $i = 0, \dots, r$ we show $n \geqslant i$ and $(a_1, \dots, a_i) = (b_1, \dots, b_i)$. This is clear for $i = 0$, so suppose $i \in \{1, \dots, r\}$, $n \geqslant i - 1$, and $(a_1, \dots, a_{i-1}) = (b_1, \dots, b_{i-1})$. Then

$$A_{i-1} = (\partial - a_{i-1}) \cdots (\partial - a_1) = (\partial - b_{i-1}) \cdots (\partial - b_1) = g_1 \cdots g_{i-1} (\partial g_{i-1}^{-1}) \cdots (\partial g_1^{-1}),$$

using Lemma 1.1.3 for the last equality. So $A_{i-1}(y_i) = g_1 \cdots g_i \neq 0$, and thus $n \geqslant i$ and $a_i = A_{i-1}(y_i)^\dagger = b_i$. $\square$

Splittings and derivatives (*)**.** The material in this subsection is only needed for the proof of Lemma 7.5.29, and not for the proof of our main theorem. *In this subsection A is monic and $a_0 := A(1) \neq 0$.* Let A^∂ be the unique element of $K[\partial]$ such that $A^\partial \partial = \partial A - a_0^\dagger A$. Then A^∂ is monic of order r, and if $A \in H[\partial]$ for some differential subfield H of K, then also $A^\partial \in H[\partial]$.

Examples. If $\operatorname{order} A = 0$ then $A^\partial = 1$, and if $\operatorname{order} A = 1$ then $A^\partial = \partial + (a_0 - a_0^\dagger)$. Next, suppose $A = \partial^2 + a_1 \partial + a_0$ $(a_0, a_1 \in K)$; then

$$A^\partial \ = \ \partial^2 + (a_1 - a_0^\dagger)\partial + (a_1' + a_0 - a_1 a_0^\dagger).$$

If $A(y) = 0$ with y in a differential ring extension of K, then $A^\partial(y') = 0$. Also:

Lemma 2.5.13. *Let R be a differential integral domain extending K. Suppose the differential fraction field of R has constant field C, and* $\dim_C \ker_R A = r$. *Then* $\ker_R A^\partial = \{y' : y \in \ker_R A\}$ *and* $\dim_C \ker_R A^\partial = r$.

Proof. Let $y_1, \dots, y_r$ be a basis of the C-linear space $\ker_R A$, and assume towards a contradiction that $c_1 y_1' + \cdots + c_r y_r' = 0$ with $c_1, \dots, c_r \in C$ not all zero. Then $y := c_1 y_1 + \cdots + c_r y_r \in \ker_R^{\neq} A$ and $y' = 0$. Hence $a_0 y = A(y) = 0$ and thus $a_0 = 0$, a contradiction. $\square$

Let $\mathrm{U} = \mathrm{U}_K$ and $f_1\,\mathrm{e}(\lambda_1), \dots, f_r\,\mathrm{e}(\lambda_r) \in \mathrm{U}^\times$ be a basis of the C-linear space $\ker_{\mathrm{U}} A$, where $f_j \in K^\times$ and $\lambda_j \in \Lambda$ for $j = 1, \dots, r$. Then by Lemma 2.5.13,

$$(f_1' + \lambda_1 f_1)\,\mathrm{e}(\lambda_1), \dots, (f_r' + \lambda_r f_r)\,\mathrm{e}(\lambda_r) \in \mathrm{U}^\times$$

is a basis of the C-linear space $\ker_{\mathrm{U}} A^\partial$. Hence by Corollary 2.5.6:

Corollary 2.5.14. *Suppose* $\dim_C \ker_{\mathrm{U}} A = r$. *Then* $\mathrm{mult}_\alpha(A) = \mathrm{mult}_\alpha(A^\partial)$ *for all* $\alpha \in K/K^\dagger$, *so* $\Sigma(A) = \Sigma(A^\partial)$, *and both* A, A^∂ *split over* K.

Suppose now that K is 1-linearly surjective when $r \geqslant 2$, and A splits over K. Then A^∂ splits over K by Corollaries 2.5.8 and 2.5.14.

Splitting and adjoints $(^*)$**.** In this subsection $y_1, \dots, y_r \in \mathrm{U}^\times$,

$$(a_1, \dots, a_r) \ = \ \mathrm{split}(y_1, \dots, y_r), \qquad A \ = \ (\partial - a_r) \cdots (\partial - a_1).$$

So $y_1, \dots, y_r$ is a basis of the C-linear space $V := \ker_{\mathrm{U}} A = \ker_\Omega A$,

$$A^* \ = \ (-1)^r (\partial + a_1) \cdots (\partial + a_r),$$

and $\dim_C W = r$ for the C-linear space $W := \ker_{\mathrm{U}} A^* = \ker_\Omega A^*$ by Lemma 2.3.21. (Recall here that Ω denotes the differential fraction field of U.) Proposition 2.4.13 with Ω instead of K yields the C-bilinear map $[\ ,\ ]_A \colon \Omega \times \Omega \to \Omega$, which restricts to a perfect pairing $V \times W \to C$ by Corollary 2.4.16. We let j, k range over $\{1, \dots, r\}$ and take $\lambda_j \in \Lambda$ such that $y_j^\dagger \equiv \lambda_j \bmod K^\dagger$, so $y_j \in K^\times\,\mathrm{e}(\lambda_j)$.

Lemma 2.5.15. *Suppose* $z_1, \dots, z_r \in \mathrm{U}^\times$ *are* C*-linearly independent such that*

$$\mathrm{split}(z_r, \dots, z_1) \ = \ (-a_r, \dots, -a_1).$$

Then $z_1, \dots, z_r$ *is a basis of the* C*-linear space* W, $[y_j, z_k]_A = 0$ *if* $j < k$, *and* $[y_k, z_k]_A \neq 0$. *Moreover,* $z_k \in K^\times\,\mathrm{e}(-\lambda_k)$, *and if* $[y_j, z_k]_A \neq 0$, *then* $\lambda_j = \lambda_k$.

Proof. Let i range over $\{0, \dots, r\}$. As in (2.4.6), set

$$A_i \ := \ (\partial - a_i) \cdots (\partial - a_1), \qquad B_i \ := \ (-1)^{r-i} (\partial + a_{i+1}) \cdots (\partial + a_r).$$

Then by Lemma 2.5.4 we have

$$\ker_{\mathrm{U}} A_i \ = \ Cy_1 \oplus \cdots \oplus Cy_i, \qquad \ker_{\mathrm{U}} B_i \ = \ Cz_r \oplus \cdots \oplus Cz_{i+1}$$

and thus

$$A_i(y_j) \ = \ 0 \ \text{ if } i \geqslant j, \qquad B_i(z_k) \ = \ 0 \ \text{ if } i+1 \leqslant k.$$

Then Lemma 2.4.17 yields

$$[y_j, z_k]_A \ = \ \sum_{i<r} A_i(y_j) B_{i+1}(z_k) \ = \ \sum_{k-2<i<j} A_i(y_j) B_{i+1}(z_k),$$

so $[y_j, z_k]_A = 0$ whenever $j < k$. Moreover,

$$[y_k, z_k]_A \ = \ A_{k-1}(y_k) B_k(z_k) \ \neq \ 0.$$

Take $\mu_k \in \Lambda$ with $z_k^\dagger \equiv \mu_k \bmod K^\dagger$. Then $y_j \in K\,\mathrm{e}(\lambda_j)$ and $z_k \in K\,\mathrm{e}(\mu_k)$, so $[y_j, z_k]_A \in C \cap K\,\mathrm{e}(\lambda_j + \mu_k)$ by (2.4.3). Hence, if $[y_j, z_k]_A \neq 0$, then $\lambda_j + \mu_k = 0$. In particular, $\mu_k = -\lambda_k$ and so $z_k \in K^\times\,\mathrm{e}(-\lambda_k)$. □

Corollary 2.5.16. *Assume* K *is* 1*-linearly surjective if* $r \geqslant 2$. *Then there is a basis* $y_1^*, \dots, y_r^*$ *of the* C*-linear space* W *such that* $[y_j, y_k^*]_A = \delta_{jk}$ *for all* j, k, *and*

$$y_j^* \in K^\times\,\mathrm{e}(-\lambda_j) \text{ for all } j, \qquad \mathrm{split}(y_r^*, \dots, y_1^*) \ = \ (-a_r, \dots, -a_1).$$

Proof. Lemma 2.5.7 gives C-linearly independent $z_1, \dots, z_r \in \mathrm{U}^\times$ such that

$$\operatorname{split}(z_r, \dots, z_1) \;=\; (-a_r, \dots, -a_1).$$

Lemma 2.5.15 gives constants $c_k \in C^\times$ such that $[y_j, c_k z_k]_A = \delta_{jk}$ for $j \leqslant k$. We now set $y_r^* := c_r z_r$, so $[y_j, y_r^*]_A = \delta_{jr}$ for all j and $y_r^* \in K^\times \operatorname{e}(-\lambda_r)$. Let $1 < k \leqslant r$ and assume inductively that we have $y_k^*, \dots, y_r^* \in W$ such that for $i = k, \dots, r$ we have $y_i^* \in Cz_i + \cdots + Cz_r$, $[y_j, y_i^*]_A = \delta_{ji}$ for all j, and $y_i^* \in K^\times \operatorname{e}(-\lambda_i)$. Then for

$$y_{k-1}^* \;:=\; c_{k-1} z_{k-1} - \sum_{i=k}^{r} [y_i, c_{k-1} z_{k-1}]_A y_i^*$$

we have

$$y_{k-1}^* \in Cz_{k-1} + Cz_k + \cdots + Cz_r, \qquad [y_j, y_{k-1}^*]_A = \delta_{j,k-1} \text{ for all } j.$$

If $k \leqslant i \leqslant r$ and $[y_i, c_{k-1} z_{k-1}]_A \neq 0$, then $\lambda_i = \lambda_{k-1}$ by the last part of Lemma 2.5.15, so $y_{k-1}^* \in K^\times \operatorname{e}(-\lambda_{k-1})$ by the inductive assumption and $z_{k-1} \in K \operatorname{e}(-\lambda_{k-1})$.

This recursive construction yields a basis $y_1^*, \dots, y_r^*$ of the C-linear space W such that $[y_j, y_k^*] = \delta_{jk}$ for all j, k, and $y_i^* \in K^\times \operatorname{e}(-\lambda_i)$ for $i = 1, \dots, r$. It now follows from Lemma 2.5.10 that $\operatorname{split}(y_r^*, \dots, y_1^*) = (-a_r, \dots, -a_1)$. $\square$

Lemma 2.5.15 also yields:

Corollary 2.5.17. *If $(a_1, \dots, a_r) = (-a_r, \dots, -a_1)$, then $[y_j, y_{r+1-k}]_A = 0$ for all $j < k$, and $[y_k, y_{r+1-k}]_A \neq 0$ for all k.*

The case of real operators. We now continue the subsection *The real case* of Section 2.2. Thus $K = H[\mathrm{i}]$ where H is a real closed differential subfield of K and $\mathrm{i}^2 = -1$, and $\Lambda = \Lambda_{\mathrm{r}} + \Lambda_{\mathrm{i}}\mathrm{i}$ where Λ_{r}, Λ_{i} are subspaces of the $\mathbb{Q}$-linear space H. The complex conjugation automorphism $z \mapsto \overline{z}$ of the differential field K extends uniquely to an automorphism $B \mapsto \overline{B}$ of the ring $K[\partial]$ with $\overline{\partial} = \partial$. We have $\overline{A(f)} = \overline{A}(\overline{f})$ for $f \in \mathrm{U}$, from which it follows that $\dim_C \ker_K A = \dim_C \ker_K \overline{A}$, $(\overline{A})_\lambda = \overline{(A_{\overline{\lambda}})}$, $\operatorname{mult}_\lambda \overline{A} = \operatorname{mult}_{\overline{\lambda}} A$, and $f \mapsto \overline{f} \colon \mathrm{U} \to \mathrm{U}$ restricts to a C_H-linear bijection $\ker_{\mathrm{U}} A \to \ker_{\mathrm{U}} \overline{A}$.

In the rest of this subsection we assume $H = H^\dagger$ (so $\Lambda = \Lambda_{\mathrm{i}}\mathrm{i}$) and $A \in H[\partial]$ (and by earlier conventions, $A \neq 0$ and $r := \operatorname{order} A$). Then $A = \overline{A}$, hence for all λ we have $A_{\overline{\lambda}} = \overline{A_\lambda}$ and $\operatorname{mult}_\lambda A = \operatorname{mult}_{\overline{\lambda}} A$. Thus with μ ranging over $\Lambda_{\mathrm{i}}^{>}$:

$$\sum_\lambda \operatorname{mult}_\lambda(A) \;=\; \operatorname{mult}_0(A) + 2 \sum_\mu \operatorname{mult}_{\mu\mathrm{i}}(A).$$

Note that 0 is an eigenvalue of A iff $\ker_H A \neq \{0\}$.

Let $V := \ker_{\mathrm{U}} A$, a subspace of the C-linear space U with $\overline{V} = V$ and $\dim_C V \leqslant r$. Recall that we have the differential H-subalgebra $\mathrm{U}_{\mathrm{r}} = \{f \in \mathrm{U} : \overline{f} = f\}$ of U and the C_H-linear subspace $V_{\mathrm{r}} = \ker_{\mathrm{U}_{\mathrm{r}}} A$ of U_{r}. Now $V = V_{\mathrm{r}} \oplus V_{\mathrm{r}} i$ (internal direct sum of C_H-linear subspaces), so $\dim_C V = \dim_{C_H} V_{\mathrm{r}}$. Combining Lemma 2.5.1 and the remarks preceding it with Lemma 2.2.19 and its proof yields:

Corollary 2.5.18. *The C-linear space V has a basis*

$$a_1\,\mathrm{e}(\mu_1\mathrm{i}),\ \overline{a_1}\,\mathrm{e}(-\mu_1\mathrm{i}),\ \dots,\ a_m\,\mathrm{e}(\mu_m\mathrm{i}),\ \overline{a_m}\,\mathrm{e}(-\mu_m\mathrm{i}),\ h_1,\ \dots,\ h_n \qquad (2m+n\leqslant r),$$

where $a_1,\dots,a_m\in K^\times$, $\mu_1,\dots,\mu_m\in\Lambda_{\mathrm{i}}^{>}$, $h_1,\dots,h_n\in H^\times$. For such a basis,

$$\operatorname{Re}\bigl(a_1\,\mathrm{e}(\mu_1\mathrm{i})\bigr),\ \operatorname{Im}\bigl(a_1\,\mathrm{e}(\mu_1\mathrm{i})\bigr),\ \dots,\ \operatorname{Re}\bigl(a_m\,\mathrm{e}(\mu_m\mathrm{i})\bigr),\ \operatorname{Im}\bigl(a_m\,\mathrm{e}(\mu_m\mathrm{i})\bigr),\ h_1,\ \dots,\ h_n$$

is a basis of the C_H-linear space V_{r}, and $h_1,\dots,h_n$ is a basis of the C_H-linear subspace $\ker_H A = V\cap H$ of H.

Using $H=H^\dagger$, arguments as in the proof of Lemma 2.5.7 show:

Lemma 2.5.19. *Assume H is 1-linearly surjective when $r\geqslant 2$. Let $a_1,\dots,a_r\in H$ be such that $A=(\partial-a_r)\cdots(\partial-a_1)$. Then the C_H-linear space $\ker_H A$ has a basis $y_1,\dots,y_r$ such that $\operatorname{split}(y_1,\dots,y_r)=(a_1,\dots,a_r)$.*

Recall from Lemma 2.3.3 that if $r=1$ or K is 1-linearly surjective, then

$$A \text{ splits over } K \quad\Longleftrightarrow\quad \sum_\lambda \operatorname{mult}_\lambda(A)=r.$$

Now $\operatorname{mult}_\lambda(A)=\operatorname{mult}_{\overline{\lambda}}(A)$ for all λ, so if $\operatorname{mult}_\lambda(A)=r\geqslant 1$, then $\lambda=0$. Also, for $W:=V\cap K=\ker_K A$ and $W_{\mathrm{r}}:=W\cap \mathrm{U}_{\mathrm{r}}$ we have $W_{\mathrm{r}}=\ker_H A$ and

$$W \;=\; W_{\mathrm{r}}\oplus W_{\mathrm{r}}\mathrm{i} \qquad \text{(internal direct sum of C_H-linear subspaces)},$$

so $\operatorname{mult}_0(A)=\dim_C\ker_K A=\dim_{C_H}\ker_H A$. If $y_1,\dots,y_r$ is a basis of the C_H-linear space $\ker_H A$, then $\operatorname{split}(y_1,\dots,y_r)\in H^r$ in reversed order is a splitting of A over H by Corollary 2.5.5. These remarks and Lemma 2.5.19 now yield:

Corollary 2.5.20. *If $\operatorname{mult}_0(A)=r$, then A splits over H. The converse holds if H is 1-linearly surjective or $r=1$.*

Corollary 2.5.21. *Suppose $r\geqslant 1$, and K is 1-linearly surjective if $r\geqslant 2$. Then*

$$A \text{ splits over } H \quad\Longleftrightarrow\quad \operatorname{mult}_0(A)=r \quad\Longleftrightarrow\quad |\Sigma(A)|=1.$$

We now focus on the order 2 case:

Lemma 2.5.22. *Suppose $r=2$ and A splits over K but not over H. Then*

$$\dim_C \ker_{\mathrm{U}} A \;=\; 2.$$

If H is 1-linearly surjective, then A has two distinct eigenvalues.

Proof. We can assume A is monic, so $A=(\partial-f)(\partial-g)$ with $f,g\in K$ and $g=a+b\mathrm{i}$, $a,b\in H$, $b\neq 0$. Then $g=d^\dagger+\mu\mathrm{i}$ with $d\in K^\times$ and $\mu\in\Lambda_{\mathrm{i}}$, and so $d\,\mathrm{e}(\mu\mathrm{i})\in\ker_{\mathrm{U}} A$. From $A=\overline{A}$ we obtain $\overline{d}\,\mathrm{e}(-\mu\mathrm{i})\in\ker_{\mathrm{U}} A$. These two elements of $\ker_{\mathrm{U}} A$ are C-linearly independent, since

$$d\,\mathrm{e}(\mu\mathrm{i})/\overline{d}\,\mathrm{e}(-\mu\mathrm{i}) \;=\; (d/\overline{d})\,\mathrm{e}(2\mu\mathrm{i})\notin C:$$

this is clear if $\mu\neq 0$, and if $\mu=0$, then $d^\dagger=g$, so $(d/\overline{d})^\dagger=g-\overline{g}=2b\mathrm{i}\neq 0$, and hence $d/\overline{d}\notin C$. Thus $\dim_C\ker_{\mathrm{U}} A \;=\; 2$, and $\mu\mathrm{i}$, $-\mu\mathrm{i}$ are eigenvalues of A with respect to Λ. Now assume H is 1-linearly surjective. Then we claim that $\mu\neq 0$. To see this note that [ADH, 5.1.21, 5.2.10] and the assumption that A does not split over H yield $\dim_{C_H}\ker_H A=\dim_C\ker_K A=0$, hence $g\notin K^\dagger$ and thus $\mu\mathrm{i}=g-d^\dagger\neq 0$. $\square$

Combining Lemmas 2.5.19 and 2.5.22 yields:

Corollary 2.5.23. *If H is 1-linearly surjective, A has order 2, and A splits over K, then* $\dim_C \ker_{\mathrm{U}} A \;=\; 2$.

In the rest of this subsection H is 1-linearly surjective and $A = 4\partial^2 + f$, $f \in H$. Let the functions $\omega\colon H \to H$ and $\sigma\colon H^\times \to H$ be as in [ADH, 5.2]. Then we have [ADH, remarks before 5.2.1, and (5.2.1)]:

$$\begin{aligned} A \text{ splits over } H \quad &\Longleftrightarrow \quad f \in \omega(H),\\ A \text{ splits over } K \quad &\Longleftrightarrow \quad f \in \sigma(H^\times) \cup \omega(H). \end{aligned}$$

If A splits over H, then $\Sigma(A) = \{0\}$ and $\operatorname{mult}_0(A) = 2$, by Corollary 2.5.21. Suppose A splits over K but not over H, and let $y \in H^\times$ satisfy $\sigma(y) = f \notin \omega(H)$. Then by [ADH, p. 262] we have $A = 4(\partial + g)(\partial - g)$ where $g = \frac{1}{2}(-y^\dagger + y\mathrm{i})$. Hence the two distinct eigenvalues of A are $(y/2)\mathrm{i} + K^\dagger$ and $-(y/2)\mathrm{i} + K^\dagger$. We consider also the skew-adjoint differential operator

$$B \;:=\; \partial^3 + f\partial + (f'/2) \in H[\partial].$$

If $\dim_C \ker_{\mathrm{U}} A = 2$, then $\dim_C \ker_{\mathrm{U}} B = 3$ by Lemma 2.4.23. Likewise,

$$\dim_{C_H} \ker_H A = 2 \;\Longrightarrow\; \dim_{C_H} \ker_H B = 3.$$

Lemma 2.5.24. *If A splits over K, then so does B. Likewise with H instead of K.*

Proof. If A splits over K, then $\dim_C \ker_{\mathrm{U}} A = 2$ by Corollary 2.5.23 and therefore $\dim_C \ker_{\mathrm{U}} B = 3$ by the remark preceding the lemma, so B splits over K by Corollary 2.5.6. If A splits over H, then $\dim_{C_H} \ker_H A = 2$ by Lemma 2.5.19 and hence $\dim_{C_H} \ker_H B = 3$, so B splits over H by Corollary 2.5.5 and the remark following it. □

Lemma 2.5.25. *Let $y \in H^\times$ with $\sigma(y) = f \notin \omega(H)$. Then $\Sigma(B) = \{\beta, 0, -\beta\}$ where $\beta := y\mathrm{i} + K^\dagger \neq 0$, and $\dim_{C_H} \ker_H B = 1$.*

Proof. Put $g = \frac{1}{2}(-y^\dagger + y\mathrm{i})$, so $A = 4(\partial + g)(\partial - g)$, and take $d \in K^\times$ and $\mu \in \Lambda_{\mathrm{i}}$ with $g = d^\dagger + \mu i$. Then $d\,\mathrm{e}(\mu\mathrm{i})$, $\overline{d}\,\mathrm{e}(-\mu\mathrm{i})$ is a basis of $\ker_{\mathrm{U}} A$ and $\mu \neq 0$, by the argument in the proof of Lemma 2.5.22. Hence

$$d^2\,\mathrm{e}(2\mu\mathrm{i}), \quad |d|^2, \quad \overline{d}^2\,\mathrm{e}(-2\mu\mathrm{i})$$

is a basis of $\ker_{\mathrm{U}} B$ by Lemma 2.4.23, so

$$\big(d^2\,\mathrm{e}(2\mu\mathrm{i})\big)^\dagger + K^\dagger = 2\mu\mathrm{i} + K^\dagger,\ \big(|d|^2\big)^\dagger + K^\dagger = [0],\ \big(\overline{d}^2\,\mathrm{e}(-2\mu\mathrm{i})\big)^\dagger + K^\dagger = -2\mu\mathrm{i} + K^\dagger$$

are eigenvalues of B. Since $\mu i \notin K^\dagger$, these are distinct eigenvalues, and so there are no other eigenvalues. Note: $g = \frac{1}{2}(-y^\dagger + y\mathrm{i}) = d^\dagger + \mu i$ gives $y\mathrm{i} + K^\dagger = 2\mu i + K^\dagger$. Finally, $\dim_{C_H} \ker_H B = 1$ by Corollary 2.5.18. □

Factoring linear differential operators over H-fields (*). *In this subsection H is a real closed H-field with $x \in H$, $x' = 1$, $x \succ 1$, and $K = H[\mathrm{i}]$ where $\mathrm{i}^2 = -1$.* In the proof of the next lemma we use [ADH, 10.5.2(i)]:

$$y, z \in H^\times,\ y \prec z \;\Longrightarrow\; y^\dagger < z^\dagger. \tag{2.5.2}$$

Lemma 2.5.26. *Let y, z be C_H-linearly independent elements of $H^\times$ and $(a,b) :=$* $\operatorname{split}(y,z)$*. If $xy \succ z$, then $a > b$, and if $xy \prec z$, then $a < b$.*

Proof. Replacing (y,z) by $(1, z/y)$ we arrange $y = 1$, $a = 0$, by Lemma 2.5.11. Then $z' \neq 0$ and $b = z'^\dagger$. Now $x \succ z$ implies $1 \succ z'$, and $x \prec z$ implies $1 \prec z'$. It remains to use the remark preceding the lemma. □

In the next three lemmas $y_1,\dots,y_r \in H$ $(r\in\mathbb{N})$ are C_H-linearly independent and $(a_1,\dots,a_r) := \operatorname{split}(y_1,\dots,y_r)\in H^r$. We also assume that H is $\uplambda$-free.

Lemma 2.5.27. $y_1 \succ \cdots \succ y_r \ \Rightarrow\ a_1 > \cdots > a_r$.

Proof. The cases $r=0,1$ are trivial, so suppose that $r\geqslant 2$ and $y_1\succ\cdots\succ y_r$. We have $(a_1,\dots,a_{r-1}) = \operatorname{split}(y_1,\dots,y_{r-1})$. Assume $a_1>\cdots>a_{r-1}$ as inductive hypothesis. It remains to show $a_{r-1}>a_r$. Put $B := (\partial - a_{r-2})\cdots(\partial-a_1)$; so $B=A_{r-2}$ in the notation introduced before Lemma 2.5.4, and $(a_{r-1},a_r) = \operatorname{split}\big(B(y_{r-1}),B(y_r)\big)$. By Lemma 2.5.4, $y_1,\dots,y_{r-2}$ is a basis of the C_H-linear subspace $\ker_H B$ of H, and hence

$$v(\ker_H^{\neq} B) \ =\ \big\{v(y_1),\dots,v(y_{r-2})\big\}\ =\ \mathscr{E}^{\mathrm{e}}(B)$$

by Corollary 1.5.20, so $v(y_{r-1}),v(y_r)\notin\mathscr{E}^{\mathrm{e}}(B)$. Then Lemma 1.5.6 gives $B(y_{r-1})\succ B(y_r)$, so $xB(y_{r-1})\succ B(y_r)$. Now Lemma 2.5.26 yields $a_{r-1}>a_r$. □

Lemma 2.5.28. $y_1 \prec^\flat \cdots \prec^\flat y_r \ \Rightarrow\ a_1<\cdots<a_r$.

Proof. Similar to the proof of Lemma 2.5.27, using in the inductive step that B is asymptotically surjective by Corollary 1.5.25, hence if $y,z\in H^\times$, $vy,vz\notin\mathscr{E}^{\mathrm{e}}(B)$, and $y\prec^\flat z$, then $B(y)\prec^\flat B(z)$ by Lemma 1.5.22, and so $xB(y)\prec B(z)$. □

Along the lines of the proof of Lemma 2.5.27 we obtain:

Lemma 2.5.29. *Suppose $y_i\not\asymp y_j$ for all i,j with $1\leqslant i<j\leqslant r$. Then*

$$a_1\leqslant\cdots\leqslant a_r \ \Rightarrow\ y_1\prec\cdots\prec y_r.$$

Under present assumptions we can strengthen the conclusion of Lemma 2.5.19:

Lemma 2.5.30. *Assume H is Liouville closed. Let $a_1,\dots,a_r\in H$ and set*

$$A\ :=\ (\partial-a_r)\cdots(\partial-a_1).$$

Then the C_H-linear space $\ker_H A$ has a basis $y_1,\dots,y_r$ such that $\operatorname{split}(y_1,\dots,y_r) = (a_1,\dots,a_r)$ and $y_i\not\asymp y_j$ for all i,j with $1\leqslant i<j\leqslant r$.

Proof. The case $r=0$ is clear. Let $r\geqslant 1$ and assume inductively that

$$B\ :=\ (\partial-a_r)\cdots(\partial-a_2)$$

has a basis $z_2,\dots,z_r$ of $\ker_H B$ such that $\operatorname{split}(z_2,\dots,z_r)=(a_2,\dots,a_r)$ and $z_i\not\asymp z_j$ whenever $2\leqslant i<j\leqslant r$. Take $y_1\in H^\times$ with $y_1^\dagger = a_1$, so $\ker_H(\partial-a_1)=Cy_1$ and $\mathscr{E}^{\mathrm{e}}_H(\partial-a_1)=\{vy_1\}$. For $i=2,\dots,r$, Corollary 1.5.4 then gives $y_i\in H^\times$ with $(\partial-a_1)(y_i)=z_i$ and $y_i\not\asymp y_1$. Then $y_1,\dots,y_r\in\ker_H A$ and $y_i\not\asymp y_j$ for all $i\neq j$, and $\operatorname{split}(y_1,\dots,y_r)=(a_1,\dots,a_r)$. □

The valuation of H being trivial on C_H, the proof of the next lemma is obvious.

Lemma 2.5.31. *Let $g_1\prec\cdots\prec g_n$ in H and let $h_1,\dots,h_n$ be in the C_H-linear subspace spanned by $g_1,\dots,g_n$. Then the following are equivalent:*

(i) *$h_1\prec\cdots\prec h_n$;*
(ii) *for $i=1,\dots,n$ there are $c_{ii},c_{i,i-1},\dots,c_{i1}\in C_H$ such that*

$$h_i\ =\ c_{ii}g_i + c_{i,i-1}g_{i-1}+\cdots+c_{i1}g_1 \text{ and } c_{ii}\neq 0.$$

Below $A\in H[\partial]^{\neq}$ has order $r\geqslant 1$. Now the main results of this subsection:

Lemma 2.5.32. *There is at most one splitting $(a_r,\dots,a_1)$ of A over H such that $a_1 \leqslant \cdots \leqslant a_r$.*

Proof. Let $(a_r,\dots,a_1)$, $(b_r,\dots,b_1)$ be splittings of A over H with $a_1 \leqslant \cdots \leqslant a_r$ and $b_1 \leqslant \cdots \leqslant b_r$. Towards showing that $a_i = b_i$ for $i = 1,\dots,r$ we arrange that H is Liouville closed. Then Lemma 2.5.30 yields bases $y_1,\dots,y_r$ and $z_1,\dots,z_r$ of $\ker_H A$ such that $\operatorname{split}(y_1,\dots,y_r) = (a_1,\dots,a_r)$, $\operatorname{split}(z_1,\dots,z_r) = (b_1,\dots,b_r)$ and $y_i \not\asymp y_j$, $z_i \not\asymp z_j$ whenever $i \neq j$. By Lemma 2.5.29 we have $y_1 \prec \cdots \prec y_r$ and $z_1 \prec \cdots \prec z_r$, and hence by Lemmas 2.5.10 and 2.5.31,

$$(a_1,\dots,a_r) \;=\; \operatorname{split}(y_1,\dots,y_r) \;=\; \operatorname{split}(z_1,\dots,z_r) \;=\; (b_1,\dots,b_r). \qquad \square$$

Example. Let $a,b \in H$ in this example. Then

$$(\partial - b)(\partial - a) = \partial^2 - \partial a - b\partial + ab = \partial^2 - (a+b)\partial + (ab - a'),$$

so for $f,g \in H$,

$$(\partial - b)(\partial - a) = (\partial - g)(\partial - f) \quad\Longleftrightarrow\quad a+b = f+g \text{ and } ab - a' = fg - f'.$$

Now take $A = \partial^2$. Then 1, x is a basis of $\ker_H A$, and

$$\begin{aligned} A = (\partial - b)(\partial - a) \quad &\Longleftrightarrow \quad a + b = 0 \text{ and } ab - a' = 0 \\ &\Longleftrightarrow \quad a = -b = (cx+d)^\dagger \text{ for some } c,d \in C_H, \text{ not both zero.} \end{aligned}$$

Hence if (b,a) is any splitting of A over H, then $a \geqslant 0 \geqslant b$, and the only splitting (b,a) of A over H with $a \leqslant b$ is $(b,a) = (0,0)$.

We call A **scrambled** if there are $y,z \in \ker^{\neq}_H A$ with $y \not\asymp z$ and $y \asymp^\flat z$, and **unscrambled** otherwise. Hence if $r = 1$, then A is unscrambled, whereas $A = \partial^2$ is scrambled. For $a,b \in H^\times$ we have: A is scrambled $\Leftrightarrow$ aAb is scrambled. Moreover:

Lemma 2.5.33. *Assume H has asymptotic integration, and let $B \in H[\partial]$ have order $s \geqslant 1$. If B is scrambled, then so is AB. If A is scrambled, B is asymptotically surjective, $\mathscr{E}^{\mathrm{e}}(B) = v(\ker^{\neq}_H B)$, and $\ker_H A \subseteq B(H)$, then AB is scrambled.*

Proof. The first statement is clear since $\ker_H B \subseteq \ker_H AB$. Suppose B is asymptotically surjective, $\mathscr{E}^{\mathrm{e}}(B) = v(\ker^{\neq}_H B)$, and $\ker_H A \subseteq B(H)$, and let $f,g \in \ker^{\neq}_H A$ be such that $f \not\asymp g$ and $f \asymp^\flat g$. Corollary 1.5.4 yields $y,z \in H$ with $B(y) = f$, $B(z) = g$ and $vy, vz \notin \mathscr{E}^{\mathrm{e}}(B)$. Then $y,z \in \ker^{\neq}_H AB$, and $y \not\asymp z$, $y \asymp^\flat z$ by Lemmas 1.5.6 and 1.5.22. $\square$

Proposition 2.5.34. *Suppose H is Liouville closed, A splits over H, and A is unscrambled. Then there is a unique splitting $(a_r,\dots,a_1)$ of A over H such that $a_1 \leqslant \cdots \leqslant a_r$. For this splitting we have $a_1 < \cdots < a_r$.*

Proof. We first arrange that A is monic. The uniqueness part is immediate from Lemma 2.5.32. To obtain a splitting $(a_r,\dots,a_1)$ of A over H with $a_1 < \cdots < a_r$, Lemma 2.5.30 gives a basis $y_1 \prec \cdots \prec y_r$ of the C_H-linear space $\ker_H A$. Now set $(a_1,\dots,a_r) := \operatorname{split}(y_1,\dots,y_r)$. Then $(a_r,\dots,a_1)$ is a splitting of A over H, by Corollary 2.5.5. Since A is unscrambled, we have $y_1 \prec^\flat \cdots \prec^\flat y_r$, so $a_1 < \cdots < a_r$, by Lemma 2.5.28. $\square$

In [103, Exercise 7.28] it is claimed that for the Liouville closed H-field $H = \mathbb{T}_{\mathrm{g}}$ of grid-based transseries, if A splits over H, then A always has a splitting $(a_r,\dots,a_1)$ over H with $a_1 \leqslant \cdots \leqslant a_r$. The next example shows this to be incorrect for $r = 2$:

Example 2.5.35. Let $z \in H \setminus C_H$ and suppose $A = (\partial - g)\partial$ where $g := z'^{\dagger}$. Then $1, z$ is a basis of $\ker_H A$. With $a, b \in H$, this fact leads to the equivalence

$$A = (\partial - b)(\partial - a) \quad \Longleftrightarrow \quad a = g - b = (cz + d)^{\dagger} \text{ for some } c, d \in C_H, \text{ not both zero.}$$

Now take $z = x - x^{-1}$, so $z' = 1 + x^{-2}$, $z'' = -2x^{-3}$, and hence

$$g \;=\; z'^{\dagger} \;=\; \frac{z''}{z'} \;=\; -\frac{2}{x(x^2+1)} < 0.$$

Let (b, a) be a splitting of A over H. We claim that then $a \geqslant 0 > b$. This is clear if $a = 0$, so assume $a \neq 0$. For $c, d \in C_H$, $c \neq 0$ we have $(cz + d)^{\dagger} \sim x^{-1}$. Thus $a \sim x^{-1}$ and $b = g - a \sim -x^{-1}$. Hence $a > 0 > b$ as claimed.

In the rest of this subsection H has asymptotic integration, and ϕ ranges over the elements of $H^{>}$ that are active in H. Note that if A is unscrambled and $\phi \preccurlyeq 1$, then $A^{\phi} \in H^{\phi}[\delta]$ is also unscrambled. Moreover:

Lemma 2.5.36. *A^{ϕ} is unscrambled, eventually.*

Proof. By Remark 1.5.3 we have $|v(\ker^{\neq}_H A)| \leqslant r$, thus we can take ϕ so that $\gamma - \delta \notin \Gamma^{\flat}_{\phi}$ for all $\gamma \neq \delta$ in $v(\ker^{\neq}_H A)$. Now A^{ϕ} is unscrambled since $\ker_H A = \ker_{H^{\phi}} A^{\phi}$. $\square$

Lemma 2.5.37. *Let $(a_r, \dots, a_1)$ be a splitting of A over H such that $a_1 \leqslant \dots \leqslant a_r$, suppose $\phi \prec 1$, and set $b_j := \phi^{-1}\big(a_j - (j-1)\phi^{\dagger}\big)$ for $j = 1, \dots, r$. Then $(b_r, \dots, b_1)$ is a splitting of A^{ϕ} over H^{ϕ} with $b_1 < \dots < b_r$.*

Proof. By Lemma 1.1.2, $(b_r, \dots, b_1)$ is a splitting of A^{ϕ} over H^{ϕ}. Since $\phi^{\dagger} < 0$, we have $b_1 < \dots < b_r$. $\square$

Corollary 2.5.38. *Suppose H is Liouville closed and A splits over H. Then there is a unique splitting $(a_r, \dots, a_1)$ of A over H such that eventually $a_j + \phi^{\dagger} < a_{j+1}$ for $j = 1, \dots, r-1$.*

Proof. Let $(a_r, \dots, a_1)$ be a splitting of A over H and ϕ so that $a_j + \phi^{\dagger} < a_{j+1}$ for $j = 1, \dots, r-1$. Set $b_j := \phi^{-1}\big(a_j - (j-1)\phi^{\dagger}\big)$ for $j = 1, \dots, r$. Then $(b_r, \dots, b_1)$ is a splitting of A^{ϕ} over H^{ϕ} with $b_1 < \dots < b_r$. Thus by Lemma 2.5.32 there can be at most one splitting $(a_r, \dots, a_1)$ of A over H such that eventually $a_j + \phi^{\dagger} < a_{j+1}$ for $j = 1, \dots, r-1$. (Here we also use (2.5.2).)

For existence, take ϕ with unscrambled A^{ϕ} and a splitting $(b_r, \dots, b_1)$ of A^{ϕ} over H^{ϕ} with $b_1 < \dots < b_r$ as in Proposition 2.5.34. For $j = 1, \dots, r$, take $a_j \in H$ such that $b_j = \phi^{-1}\big(a_j - (j-1)\phi^{\dagger}\big)$. Then $(a_r, \dots, a_1)$ is a splitting of A over H, by Lemma 1.1.2, and $a_j + \phi^{\dagger} < a_{j+1}$ for $j = 1, \dots, r-1$. $\square$

Example. Suppose H is Liouville closed and A, g, $z = x - x^{-1}$ are as in Example 2.5.35. Then the unique splitting (b, a) of A over H such that eventually $a + \phi^{\dagger} < b$ is $(b, a) = (g, 0)$. (To see this use that eventually $\phi^{\dagger} \sim -x^{-1}$.)

We finish this subsection with a variant of Proposition 2.5.34 for Pólya-style splittings. In [ADH, 11.8] we defined $\Gamma(H) := \{h^{\dagger} : h \in H^{\succ 1}\}$. If H is Liouville closed, then $H^{>} \setminus \mathrm{I}(H) = \Gamma(H)$ [ADH, p. 520].

Proposition 2.5.39. *Suppose H is Liouville closed and A is monic and splits over H. Then there are $g_1, \dots, g_r \in H^{>}$ such that*

$$A \;=\; g_1 \cdots g_r (\partial g_r^{-1}) \cdots (\partial g_2^{-1})(\partial g_1^{-1}) \quad \textit{and} \quad g_j \in \Gamma(H) \textit{ for } j = 2, \dots, r.$$

Such $g_1, \dots, g_r$ are unique up to multiplication by positive constants.

Proof. Let $(a_r,\dots,a_1)$ be the splitting of A over H from Corollary 2.5.38. Take $g_j\in H^>$ such that $g_j^\dagger=a_j-a_{j-1}$ for $j=1,\dots,r$, where $a_0:=0$. Then

$$A=g_1\cdots g_r(\partial g_r^{-1})\cdots(\partial g_2^{-1})(\partial g_1^{-1})$$

by Lemma 1.1.3. For $j=2,\dots,r$ we have $(g_j/\phi)^\dagger>0$, eventually, hence $g_j/\phi\succcurlyeq 1$, eventually, so $g_j\in H^>\setminus \mathrm{I}(H)=\Gamma(H)$. Suppose now $h_1,\dots,h_r\in H^>$ are such that

$$A\ =\ h_1\cdots h_r(\partial h_r^{-1})\cdots(\partial h_2^{-1})(\partial h_1^{-1})\quad\text{and}\quad h_j\in\Gamma(H)\text{ for }j=2,\dots,r.$$

Set $b_j:=(h_1\cdots h_j)^\dagger$ for $j=1,\dots,r$. Then $A=(\partial-b_r)\cdots(\partial-b_1)$ by Lemma 1.1.3. Also $\int h_j\succ 1$ for $j=2,\dots,r$, for any choice of the integrals in H. Let

$$z_1:=h_1,\quad z_2:=h_1\textstyle\int h_2,\quad \dots,\quad z_r:=h_1\int\bigl(h_2\int h_3(\cdots(h_{r-1}\int h_r)\cdots)\bigr)$$

for some choice of the integrals in H. Then $z_1,\dots,z_r\in\ker^{\neq}_H A$, and induction on $j=1,\dots,r$ using [ADH, 9.1.3(iii)] gives $z_1\prec\cdots\prec z_j$. Then Lemma 2.5.12 yields $\operatorname{split}(z_1,\dots,z_r)=(b_1,\dots,b_r)$. Applied to $g_1,\dots,g_r$ instead of $h_1,\dots,h_r$, this gives $y_1\prec\cdots\prec y_r\in\ker^{\neq}_H A$ such that $\operatorname{split}(y_1,\dots,y_r)=(a_1,\dots,a_r)$. Now $y_1,\dots,y_r$ and $z_1,\dots,z_r$ are both bases of $\ker_H A$, so by Lemmas 2.5.31 and 2.5.10:

$$(a_1,\dots,a_r)\ =\ \operatorname{split}(y_1,\dots,y_r)\ =\ \operatorname{split}(z_1,\dots,z_r)\ =\ (b_1,\dots,b_r).$$

So $g_j^\dagger=a_j-a_{j-1}=b_j-b_{j-1}=h_j^\dagger$ $(b_0:=0)$, and thus $g_j\in C_H^> h_j$, $j=1,\dots,r$. ☐

The case of oscillating transseries (*)**.** We now apply the results in this section to the algebraically closed differential field $K=\mathbb{T}[i]$. Note that $\mathbb{T}[i]$ has constant field $\mathbb{C}$ and extends the (real closed) differential field $\mathbb{T}$ of transseries.

Lemma 2.5.40. $\mathbb{T}[i]$ *is linearly closed and linearly surjective.*

Proof. By [ADH, 15.0.2], $\mathbb{T}$ is newtonian, so $\mathbb{T}[i]$ is newtonian by [ADH, 14.5.7]. Hence $\mathbb{T}[i]$ is linearly closed by [ADH, 5.8.9, 14.5.3], and $\mathbb{T}[i]$ is linearly surjective by [ADH, 14.2.2]. ☐

Now applying Corollary 2.5.8 and Lemma 2.5.1 to $\mathbb{T}[i]$ gives:

Corollary 2.5.41. *For $K=\mathbb{T}[i]$, there are $\mathbb{C}$-linearly independent units $y_1,\dots,y_r$ of $\mathrm{U}_{\mathbb{T}[i]}$ with $A(y_1)=\cdots=A(y_r)=0$.*

Next we describe another incarnation of $\mathrm{U}_{\mathbb{T}[i]}$, namely as a ring of "oscillating" transseries. Towards this goal we first note that by [ADH, 11.5.1, 11.8.2] we have

$$\begin{aligned}\mathrm{I}(\mathbb{T})\ &=\ \bigl\{y\in\mathbb{T}:\ y\preccurlyeq f'\text{ for some }f\prec 1\text{ in }\mathbb{T}\bigr\}\\ &=\ \bigl\{y\in\mathbb{T}:\ y\prec 1/(\ell_0\cdots\ell_n)\text{ for all }n\bigr\},\end{aligned}$$

so a complement $\Lambda_{\mathbb{T}}$ of $\mathrm{I}(\mathbb{T})$ in $\mathbb{T}$ is given by

$$\Lambda_{\mathbb{T}}\ :=\ \bigl\{y\in\mathbb{T}:\ \operatorname{supp}(y)\succ 1/(\ell_0\cdots\ell_{n-1}\ell_n^2)\text{ for all }n\bigr\}.$$

Since $\mathbb{T}^\dagger=\mathbb{T}$ and $\mathrm{I}\bigl(\mathbb{T}[i]\bigr)\subseteq\mathbb{T}[i]^\dagger$ we have $\mathbb{T}[i]^\dagger=\mathbb{T}\oplus\mathrm{I}(\mathbb{T})i$ by Lemmas 1.2.4 and 1.2.16. We now take $\Lambda=\Lambda_{\mathbb{T}}i$ as our complement Λ of $\mathbb{T}[i]^\dagger$ in $\mathbb{T}[i]$ and explain how the universal exponential extension U of $\mathbb{T}[i]$ for this Λ was introduced in [103, Section 7.7] in a different way. Let

$$\mathbb{T}_{\succ}\ :=\ \{f\in\mathbb{T}:\operatorname{supp}f\succ 1\},$$

and similarly with $\prec$ in place of $\succ$; then $\mathbb{T}_{\prec} = \mathcal{O}_{\mathbb{T}}$ and $\mathbb{T}_{\succ}$ are $\mathbb{R}$-linear subspaces of $\mathbb{T}$, and $\mathbb{T}$ decomposes as an internal direct sum

$$\mathbb{T} \;=\; \mathbb{T}_{\succ} \oplus \mathbb{R} \oplus \mathbb{T}_{\prec} \tag{2.5.3}$$

of $\mathbb{R}$-linear subspaces of $\mathbb{T}$. Let $\mathrm{e}^{\mathrm{i}\mathbb{T}_{\succ}} = \{\mathrm{e}^{\mathrm{i}f} : f \in \mathbb{T}_{\succ}\}$ be a multiplicative copy of the additive group $\mathbb{T}_{\succ}$, with isomorphism $f \mapsto \mathrm{e}^{\mathrm{i}f}$. Then we have the group ring

$$\mathbb{O} \;:=\; K\big[\,\mathrm{e}^{\mathrm{i}\mathbb{T}_{\succ}}\,\big]$$

of $\mathrm{e}^{\mathrm{i}\mathbb{T}_{\succ}}$ over $K = \mathbb{T}[\mathrm{i}]$. We make $\mathbb{O}$ into a differential ring extension of K by

$$(\mathrm{e}^{\mathrm{i}f})' \;=\; \mathrm{i}f'\,\mathrm{e}^{\mathrm{i}f} \qquad (f \in \mathbb{T}_{\succ}).$$

Hence $\mathbb{O}$ is an exponential extension of K. The elements of $\mathbb{O}$ are called *oscillating transseries.* For each $f \in \mathbb{T}$ there is a unique $g \in \mathbb{T}$, to be denoted by $\int f$, such that $g' = f$ and g has constant term $g_1 = 0$. The injective map $\int \colon \mathbb{T} \to \mathbb{T}$ is $\mathbb{R}$-linear; we use this map to show that U and $\mathbb{O}$ are disguised versions of each other:

Proposition 2.5.42. *There is a unique isomorphism* $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big] \to \mathbb{O}$ *of differential K-algebras sending* $\mathrm{e}(h\mathrm{i})$ *to* $\mathrm{e}^{\mathrm{i}\int h}$ *for all* $h \in \Lambda_{\mathbb{T}}$.

This requires the next lemma. We assume familiarity with [ADH, Appendix A], especially with the ordered group G^{LE} (a subgroup of $\mathbb{T}^{\times}$) of logarithmic-exponential monomials and its subgroup $G^{\mathrm{E}} = \bigcup_n G_n$ of exponential monomials.

Lemma 2.5.43. *If* $\mathfrak{m} \in G^{\mathrm{LE}}$ *and* $\mathfrak{m} \succ 1$, *then* $\operatorname{supp} \mathfrak{m}' \;\subseteq\; \Lambda_{\mathbb{T}}$.

Proof. We first prove by induction on n a fact about elements of G^{E}:

$$\text{if } \mathfrak{m} \in G_n,\ \mathfrak{m} \succ 1, \text{ then } \operatorname{supp} \mathfrak{m}' \succ 1/x.$$

For $r \in \mathbb{R}^{>}$ we have $(x^r)' = rx^{r-1} \succ 1/x$, so the claim holds for $n = 0$. Suppose the claim holds for a certain n. Now $G_{n+1} = G_n \exp(A_n)$, G_n is a convex subgroup of G_{n+1}, and

$$A_n \;=\; \big\{f \in \mathbb{R}[[G_n]] :\ \operatorname{supp} f \succ G_{n-1}\big\} \qquad (\text{where } G_{-1} := \{1\}).$$

Let $\mathfrak{m} = \mathfrak{n}\exp(a) \in G_{n+1}$ where $\mathfrak{n} \in G_n$, $a \in A_n$; then

$$\mathfrak{m} \succ 1 \quad\Longleftrightarrow\quad a > 0, \text{ or } a = 0,\ \mathfrak{n} \succ 1.$$

Suppose $\mathfrak{m} \succ 1$. If $a = 0$, then $\mathfrak{m} = \mathfrak{n}$, and we are done by inductive hypothesis, so assume $a > 0$. Then $\mathfrak{m}' = (\mathfrak{n}' + \mathfrak{n}a')\exp(a)$ and $(\mathfrak{n}' + \mathfrak{n}a') \in \mathbb{R}[[G_n]]$, a differential subfield of $\mathbb{T}$, and $\exp(a) > \mathbb{R}[[G_n]]$, hence $\operatorname{supp} \mathfrak{m}' \succ 1 \succ 1/x$ as required.

Next, suppose $\mathfrak{m} \in G^{\mathrm{LE}}$ and $\mathfrak{m} \succ 1$. Take $n \geqslant 1$ such that $\mathfrak{m}{\uparrow}^n \in G^{\mathrm{E}}$. We have $(\mathfrak{m}{\uparrow}^n)' = (\mathfrak{m}' \cdot \ell_0\ell_1\cdots\ell_{n-1}){\uparrow}^n$. For $\mathfrak{n} \in \operatorname{supp} \mathfrak{m}'$ and using $\mathfrak{m}{\uparrow}^n \succ 1$ this gives

$$(\mathfrak{n} \cdot \ell_0\ell_1\cdots\ell_{n-1}){\uparrow}^n \;\succ\; 1/x$$

by what we proved for monomials in G^{E}. Applying ${\downarrow}_n$ this yields $\mathfrak{n} \succ 1/(\ell_0\ell_1\cdots\ell_n)$, hence $\mathfrak{n} \in \Lambda_{\mathbb{T}}$ as claimed. $\square$

Proof of Proposition 2.5.42. Applying ∂ to the decomposition (2.5.3) gives

$$\mathbb{T} \;=\; \partial(\mathbb{T}_{\succ}) \oplus \partial(\mathbb{T}_{\prec}).$$

Now $\partial(\mathbb{T}_{\succ}) \subseteq \Lambda_{\mathbb{T}}$ by Lemma 2.5.43, and $\partial(\mathbb{T}_{\prec}) \subseteq \mathrm{I}(\mathbb{T})$, and so these two inclusions are equalities. Thus $\int \Lambda_{\mathbb{T}} = \mathbb{T}_{\succ}$, from which the proposition follows. $\square$

Proposition 2.5.44. *There is a unique group morphism* $\exp\colon K = \mathbb{T}[i] \to \mathbb{O}^\times$ *that extends the given exponential maps* $\exp\colon \mathbb{T} \to \mathbb{T}^\times$ *and* $\exp\colon \mathbb{C} \to \mathbb{C}^\times$*, and such that* $\exp(if) = e^{if}$ *for all* $f \in \mathbb{T}_\succ$ *and* $\exp(\varepsilon) = \sum_n \frac{\varepsilon^n}{n!}$ *for all* $\varepsilon \in \mathcal{o}$*. It is surjective, has kernel* $2\pi i\mathbb{Z} \subseteq \mathbb{C}$*, and satisfies* $\exp(f)' = f' \exp(f)$ *for all* $f \in K$*.*

Proof. The first statement follows easily from the decompositions

$$K \;=\; \mathbb{T} \oplus i\mathbb{T} \;=\; \mathbb{T} \oplus i\mathbb{T}_\succ \oplus i\mathbb{R} \oplus i\mathcal{o}_\mathbb{T}, \qquad \mathbb{C} \;=\; \mathbb{R} \oplus i\mathbb{R}, \qquad \mathcal{o} \;=\; \mathcal{o}_\mathbb{T} \oplus i\mathcal{o}_\mathbb{T}$$

of K, $\mathbb{C}$, and $\mathcal{o} = \mathcal{o}_K$ as internal direct sums of $\mathbb{R}$-linear subspaces. Next,

$$\mathbb{O}^\times \;=\; K^\times e^{i\mathbb{T}_\succ} \;=\; \mathbb{T}^> \cdot S_\mathbb{C} \cdot (1+\mathcal{o}) \cdot e^{i\mathbb{T}_\succ}, \qquad S_\mathbb{C} \;:=\; \big\{z \in \mathbb{C} :\ |z| = 1\big\},$$

by Lemmas 2.1.1 and 1.2.4, and Corollary 1.2.7. Now $\mathbb{T}^> = \exp(\mathbb{T})$ and $S_\mathbb{C} = \exp(i\mathbb{R})$, so surjectivity follows from $\exp(\mathcal{o}) = 1 + \mathcal{o}$, a consequence of the well-known bijectivity of the map $\varepsilon \mapsto \sum_n \frac{\varepsilon^n}{n!} \colon \mathcal{o} \to 1 + \mathcal{o}$, whose inverse is given by

$$1 + \delta \mapsto \log(1+\delta) := \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\delta^n \qquad (\delta \in \mathcal{o}).$$

That the kernel is $2\pi i\mathbb{Z}$ follows from the initial decomposition of the additive group of K as $\mathbb{T} \oplus i\mathbb{T}_\succ \oplus i\mathbb{R} \oplus i\mathcal{o}_\mathbb{T}$. The identity $\exp(f)' = f' \exp(f)$ for $f \in K$ follows from it being satisfied for $f \in \mathbb{T}$, $f \in i\mathbb{T}_\succ$, $f \in \mathbb{C}$, and $f \in \mathcal{o}$. $\square$

To integrate oscillating transseries, note first that the $\mathbb{R}$-linear operator $\int \colon \mathbb{T} \to \mathbb{T}$ extends uniquely to a $\mathbb{C}$-linear operator $\int \colon \mathbb{T}[i] \to \mathbb{T}[i]$. This in turn extends uniquely to a $\mathbb{C}$-linear operator $\int \colon \mathbb{O} \to \mathbb{O}$ such that $(\int \Phi)' = \Phi$ for all $\Phi \in \mathbb{O}$ and $\int \mathbb{T}[i]\, e^{\phi i} \subseteq \mathbb{T}[i]\, e^{\phi i}$ for all $\phi \in \mathbb{T}_\succ$: given $\phi \in \mathbb{T}_\succ^{\neq}$ and $g \in \mathbb{T}[i]$, there is a unique $f \in \mathbb{T}[i[$ such that $(f\, e^{\phi i})' = g\, e^{\phi i}$: existence holds because $y' + y\phi' i = g$ has a solution in $\mathbb{T}[i]$, the latter being linearly surjective, and uniqueness holds by Lemma 1.2.3 applied to $K = L = \mathbb{T}[i]$, because $\phi' i \notin \mathbb{T}[i]^\dagger$ in view of remarks preceding Lemma 1.2.16. See also Corollary 7.4.45.

The operator $\int$ is a right-inverse of the linear differential operator ∂ on $\mathbb{O}$. To extend this to other linear differential operators, make the subgroup $G^\mathbb{O} := G^{\text{LE}}\, e^{i\mathbb{T}_\succ}$ of $\mathbb{O}^\times$ into an ordered group so that the ordered subgroup G^{LE} of $\mathbb{T}^>$ is a convex ordered subgroup of $G^\mathbb{O}$ and $e^{i\phi} \succ G^{\text{LE}}$ for $\phi > 0$ in $\mathbb{T}_\succ$. (Possible in only one way.) Next, extend the natural inclusion $\mathbb{T}[i] \to \mathbb{C}[[G^{\text{LE}}]]$ to a $\mathbb{C}$-algebra embedding $\mathbb{O} \to \mathbb{C}[[G^\mathbb{O}]]$ by sending $e^{i\phi} \in \mathbb{O}$ to $e^{i\phi} \in G^\mathbb{O} \subseteq \mathbb{C}[[G^\mathbb{O}]]$. Identify $\mathbb{O}$ with a subalgebra of $\mathbb{C}[[G^\mathbb{O}]]$ via this embedding, so $\operatorname{supp} f \subseteq G^\mathbb{O}$ for $f \in \mathbb{O}$. It makes the Hahn space $\mathbb{C}[[G^\mathbb{O}]]$ over $\mathbb{C}$ an immediate extension of its valued subspace $\mathbb{O}$. The latter is in particular also a Hahn space over $\mathbb{C}$.

Let $A \in \mathbb{T}[i][\partial]^{\neq}$. Then $A(\mathbb{O}) = \mathbb{O}$ by Lemmas 2.5.3, 2.5.40, and Proposition 2.5.42. The proof of [ADH, 2.3.22] now gives for each $g \in \mathbb{O}$ a unique element $f =: A^{-1}(g) \in \mathbb{O}$ with $A(f) = g$ and $\operatorname{supp}(f) \cap \mathfrak{d}\big(\ker^{\neq}_\mathbb{O} A\big) = \emptyset$. This requirement on $\operatorname{supp} A^{-1}(g)$ yields a $\mathbb{C}$-linear operator A^{-1} on $\mathbb{O}$ with $A \circ A^{-1} = \mathrm{id}_\mathbb{O}$; we call it the **distinguished** right-inverse of the operator A on $\mathbb{O}$. With this definition ∂^{-1} is the operator $\int$ on $\mathbb{O}$ specified earlier.

In the next section we explore various valuations on universal exponential extensions (such as $\mathbb{O}$) with additional properties.

In this section K is a valued differential field with algebraically closed constant field $C \subseteq \mathcal{O}$ and divisible group $K^\dagger$ of logarithmic derivatives. Then $\Gamma = v(K^\times)$ is also divisible, since we have a group isomorphism

$$va \mapsto a^\dagger + (\mathcal{O}^\times)^\dagger \ : \ \Gamma \to K^\dagger/(\mathcal{O}^\times)^\dagger \qquad (a \in K^\times).$$

Let Λ be a complement of the $\mathbb{Q}$-linear subspace $K^\dagger$ of K, let λ range over Λ, let $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K constructed in Section 2.2 and set $\Omega := \mathrm{Frac}(\mathrm{U})$. Thus Ω is a differential field with constant field C.

The gaussian extension. We equip U with the gaussian extension v_{g} of the valuation of K as defined in Section 2.1; so for $f \in \mathrm{U}$ with spectral decomposition (f_λ):

$$v_{\mathrm{g}}(f) \ = \ \min_\lambda v(f_\lambda),$$

and hence

$$v_{\mathrm{g}}(f') \ = \ \min_\lambda v(f_\lambda' + \lambda f_\lambda).$$

The field Ω with the valuation extending v_{g} is a valued differential field extension of K, but it can happen that K has small derivation, whereas Ω does not:

Example. Let $K = C(\!(t^{\mathbb{Q}})\!)$ and Λ be as in Example 2.2.4, so $t \prec 1 \prec x = t^{-1}$ and $t' = -t^2$. Then K is d-valued of H-type with small derivation, but in Ω with the above valuation,

$$t\,\mathrm{e}(x) \ \prec \ 1, \qquad \big(t\,\mathrm{e}(x)\big)' \ = \ -t^2\,\mathrm{e}(x) + \mathrm{e}(x) \ \sim \ \mathrm{e}(x) \ \asymp \ 1.$$

To obtain an example where $K = H[\mathrm{i}]$ for a Liouville closed H-field H and $\mathrm{i}^2 = -1$, take $K := \mathbb{T}[\mathrm{i}]$ and $\Lambda := \Lambda_{\mathbb{T}}i$ as at the end of Section 2.5. Now $x \in \Lambda_{\mathbb{T}}$ and in Ω equipped with the above valuation we have for $t := x^{-1}$:

$$t\,\mathrm{e}(x\mathrm{i}) \ \prec \ 1, \qquad \big(t\,\mathrm{e}(x\mathrm{i})\big)' \ = \ -t^2\,\mathrm{e}(x\mathrm{i}) + \mathrm{i}\,\mathrm{e}(x\mathrm{i}) \ \sim \ \mathrm{i}\,\mathrm{e}(x\mathrm{i}) \ \asymp \ 1,$$

so $\big(t\,\mathrm{e}(x\mathrm{i})\big)' \not\prec t^\dagger$, hence Ω is neither asymptotic nor has small derivation.

However, we show next that under certain assumptions on K with small derivation, Ω has also a valuation which does make Ω a valued differential field extension of K with small derivation. For this we rely on results from [ADH, 10.4]. Although such a valuation is less canonical than v_{g}, it is useful for harnessing the finiteness statements about the set $\mathscr{E}^{\mathrm{e}}(A)$ of eventual exceptional values of $A \in K[\partial]^{\neq}$ from Section 1.5 to obtain similar facts about the set of *ultimate exceptional values* of A introduced later in this section.

Spectral extensions. *In this subsection K is* d*-valued of H-type with $\Gamma \neq \{0\}$ and with small derivation.*

Lemma 2.6.1. *The valuation of K extends to a valuation on the field Ω that makes Ω a* d*-valued extension of K of H-type with small derivation.*

Proof. Applying [ADH, 10.4.7] to an algebraic closure of K gives a d-valued algebraically closed extension L of K of H-type with small derivation and $C_L = C$ such that $L^\dagger \supseteq K$. Let $E := \{y \in L^\times : y^\dagger \in K\}$, so E is a subgroup of $L^\times$, $E^\dagger = K$, and $K[E]$ is an exponential extension of K with $C_{K[E]} = C$. Then Corollary 2.2.10 gives an embedding $\mathrm{U} \to L$ of differential K-algebras with image $K[E]$, which extends to an embedding $\Omega \to L$ of differential fields. Using this embedding to transfer the valuation of L to Ω gives a valuation as required. $\square$

A **spectral extension** of the valuation of K to Ω is a valuation on the field Ω with the properties stated in Lemma 2.6.1. If K is ω-free, then so is Ω equipped with any spectral extension of the valuation of K, by [ADH, 13.6] (and then Ω has rational asymptotic integration by [ADH, 11.7]). We do not know whether this goes through with "λ-free" instead of "ω-free". Here is something weaker:

Lemma 2.6.2. *Suppose K is algebraically closed and λ-free. Then some spectral extension of the valuation of K to Ω makes Ω a* d*-valued field with divisible value group and asymptotic integration.*

Proof. Take L, E and an embedding $\Omega \to L$ as in the proof of Lemma 2.6.1. Use this embedding to identify Ω with a differential subfield of L, so $U = K[E]$ and $\Omega = K(E)$, and equip Ω with the spectral extension of the valuation of K obtained by restricting the valuation of L to Ω. Since L is algebraically closed, E is divisible, and $\Gamma_L = \Gamma + v(E)$ by [ADH, 10.4.7(iv)]. So $\Gamma_\Omega = \Gamma_L$ is divisible. Let $a \in K^\times$, $y \in E$. Then $K(y)$ has asymptotic integration by Proposition 1.4.12, hence $v(ay) \in (\Gamma^{\neq}_{K(y)})' \subseteq (\Gamma^{\neq}_{\Omega})'$. Thus Ω has asymptotic integration. $\square$

In the rest of this subsection Ω is equipped with a spectral extension v (with value group Γ_Ω) of the valuation of K. The proof of Lemma 2.6.1 and [ADH, 10.4.7] show that we can choose v so that $\Psi_\Omega \subseteq \Gamma$; but under suitable hypotheses on K, this is automatic:

Lemma 2.6.3. *Suppose K has asymptotic integration and $\mathrm{I}(K) \subseteq K^\dagger$. Then $\Psi_\Omega \subseteq \Gamma$, the group morphism*

$$\lambda \mapsto v\big(\mathrm{e}(\lambda)\big) \ : \ \Lambda \to \Gamma_\Omega \tag{2.6.1}$$

is injective, and Γ_Ω is divisible with $\Gamma_\Omega = \Gamma \oplus v\big(\mathrm{e}(\Lambda)\big)$ (internal direct sum of $\mathbb{Q}$-linear subspaces of Γ_Ω). Moreover, $\Psi_\Omega = \Psi^{\downarrow}$ in Γ.

Proof. For $a \in K^\times$ we have $\big(a\,\mathrm{e}(\lambda)\big)^\dagger = a^\dagger + \lambda \in K$, and if $a\,\mathrm{e}(\lambda) \asymp 1$, then

$$a^\dagger + \lambda \ = \ \big(a\,\mathrm{e}(\lambda)\big)^\dagger \in (\mathcal{O}_\Omega^\times)^\dagger \cap K \ \subseteq \ \mathrm{I}(\Omega) \cap K \ = \ \mathrm{I}(K),$$

so $\lambda \in \Lambda \cap \big(\mathrm{I}(K) + K^\dagger\big) = \Lambda \cap K^\dagger = \{0\}$ and $a \asymp 1$. Thus for $a_1, a_2 \in K^\times$ and distinct $\lambda_1, \lambda_2 \in \Lambda$ we have $a_1\,\mathrm{e}(\lambda_1) \not\asymp a_2\,\mathrm{e}(\lambda_2)$, and so for $f \in \mathrm{U}$ with spectral decomposition (f_λ) we have $vf = \min_\lambda v\big(f_\lambda\,\mathrm{e}(\lambda)\big)$. Hence

$$\Psi_\Omega \ \subseteq \big\{v(a^\dagger + \lambda) : \ a \in K^\times, \ \lambda \in \Lambda\big\} \ = \ v(K) \ = \ \Gamma_\infty,$$

the map (2.6.1) is injective and $\Gamma \cap v\big(\mathrm{e}(\Lambda)\big) = \{0\}$, and so $\Gamma_\Omega = \Gamma \oplus v\big(\mathrm{e}(\Lambda)\big)$ (internal direct sum of subgroups of Γ_Ω). Since Γ and Λ are divisible, so is Γ_Ω. Now $\Psi_\Omega = \Psi^{\downarrow}$ follows from $K = (\mathrm{U}^\times)^\dagger \subseteq \Omega^\dagger$ and K having asymptotic integration. $\square$

We can now improve on Lemma 2.5.1:

Corollary 2.6.4. *Suppose K has asymptotic integration and $\mathrm{I}(K) \subseteq K^\dagger$, and let $A \in K[\partial]^{\neq}$. Then the C-linear space $\ker_{\mathrm{U}} A$ has a basis $\mathcal{B} \subseteq \mathrm{U}^\times$ such that v is injective on $\mathcal{B}$ and $v(\mathcal{B}) = v(\ker^{\neq}_{\mathrm{U}} A)$, and thus $|v(\ker^{\neq}_{\mathrm{U}} A)| = \dim_C \ker_{\mathrm{U}} A$.*

Proof. By [ADH, 5.6.6] we have a basis $\mathcal{B}_\lambda$ of the C-linear space $\ker_K A_\lambda$ such that v is injective on $\mathcal{B}_\lambda$ and $v(\mathcal{B}_\lambda) = v(\ker^{\neq}_K A_\lambda)$. Then $\mathcal{B} := \bigcup_\lambda \mathcal{B}_\lambda\,\mathrm{e}(\lambda)$ is a basis of $\ker_{\mathrm{U}} A$. It has the desired properties by Lemma 2.6.3. $\square$

Corollary 2.6.5. *Suppose K is λ-free and $\mathrm{I}(K) \subseteq K^{\dagger}$. Then Ω has asymptotic integration, and so its H-asymptotic couple is closed by Lemma 2.6.3.*

Proof. By Lemma 2.6.3, $\Gamma_{\Omega} = \Gamma + v\big(\mathrm{e}(\Lambda)\big)$. Using Proposition 1.4.12 as in the proof of Lemma 2.6.2, with $\mathrm{e}(\Lambda)$ in place of E, shows Ω has asymptotic integration. $\square$

An application (*)**.** We use spectral extensions to prove an analogue of [ADH, 16.0.3]:

Theorem 2.6.6. *If K is an ω-free newtonian* d*-valued field, then K has no proper* d*-algebraic* d*-valued field extension L of H-type with $C_L = C$ and $L^{\dagger} \cap K = K^{\dagger}$.*

We retain of course our assumption that C is algebraically closed and $K^{\dagger}$ is divisible. In the same way that [ADH, 16.0.3] follows from [ADH, 16.1.1], Theorem 2.6.6 follows from an analogue of [ADH, 16.1.1]:

Lemma 2.6.7. *Let K be an ω-free newtonian* d*-valued field, L a* d*-valued field extension of K of H-type with $C_L = C$ and $L^{\dagger} \cap K = K^{\dagger}$, and let $f \in L \setminus K$. Suppose there is no $y \in K\langle f\rangle \setminus K$ such that $K\langle y\rangle$ is an immediate extension of K. Then the $\mathbb{Q}$-linear space $\mathbb{Q}\Gamma_{K\langle f\rangle}/\Gamma$ is infinite-dimensional.*

The proof of Lemma 2.6.7 is much like that of [ADH, 16.1.1], except where the latter uses that any b in a Liouville closed H-field equals $a^{\dagger}$ for some nonzero a in that field. This might not work with elements of K, and the remedy is to take instead for every $b \in K$ an element a in $\mathrm{U}^{\times}$ with $b = a^{\dagger}$. The relevant computation should then take place in the differential fraction field Ω_L of U_L instead of in L where Ω_L is equipped with a spectral extension of the valuation of L. For all this to make sense, we first take an active ϕ in K and replace K and L by K^{ϕ} and L^{ϕ}, arranging in this way that the derivation of L (and of K) is small. Next we replace L by its algebraic closure, so that $L^{\dagger}$ is divisible, while preserving $L^{\dagger} \cap K = K^{\dagger}$ by Lemma 1.2.1, and also preserving the other conditions on L in Lemma 2.6.7, as well as the derivation of L being small. This allows us to identify U with a differential subring of U_L as in Lemma 2.2.12, and accordingly Ω with a differential subfield of Ω_L. We equip Ω_L with a spectral extension of the valuation of L (possible by Lemma 2.6.1), and make Ω a valued subfield of L. Then the valuation of Ω is a spectral extension of the valuation of K to Ω, so we have the following inclusions of d-valued fields:

$$\begin{tikzcd} L \arrow[r] & \Omega_L \\ K \arrow[u] \arrow[r] & \Omega \arrow[u] \end{tikzcd}$$

With these preparations we can now give the proof of Lemma 2.6.7:

Proof. As we just indicated we arrange that L is algebraically closed with small derivation, and with an inclusion diagram of d-valued fields involving Ω and Ω_L, as above. (This will not be used until we arrive at the Claim below.)

By [ADH, 14.0.2], K is asymptotically d-algebraically maximal. Using this and the assumption about $K\langle f\rangle$ it follows as in the proof of [ADH, 16.1.1] that there is no divergent pc-sequence in K with a pseudolimit in $K\langle f\rangle$. Thus every y in $K\langle f\rangle \setminus K$ has a *a best approximation in K*, that is, an element $b \in K$ such that $v(y-b) = \max v(y-K)$. For such b we have $v(y-b) \notin \Gamma$, since $C_L = C$.

Now pick a best approximation b_0 in K to $f_0 := f$, and set $f_1 := (f_0 - b_0)^\dagger$. Then $f_1 \in K\langle f\rangle \setminus K$, since $L^\dagger \cap K = K^\dagger$ and $C = C_L$. Thus f_1 has a best approximation b_1 in K, and continuing this way, we obtain a sequence (f_n) in $K\langle f\rangle \setminus K$ and a sequence (b_n) in K, such that b_n is a best approximation in K to f_n and $f_{n+1} = (f_n - b_n)^\dagger$ for all n. Thus $v(f_n - b_n) \in \Gamma_{K\langle f\rangle} \setminus \Gamma$ for all n.

Claim: *$v(f_0 - b_0), v(f_1 - b_1), v(f_2 - b_2), \dots$ are $\mathbb{Q}$-linearly independent over Γ.*

To prove this claim, take $a_n \in \mathrm{U}^\times$ with $a_n^\dagger = b_n$ for $n \geqslant 1$. Then in Ω_L,

$$f_n - b_n \;=\; (f_{n-1} - b_{n-1})^\dagger - a_n^\dagger \;=\; \left(\frac{f_{n-1} - b_{n-1}}{a_n}\right)^\dagger \qquad (n \geqslant 1).$$

With $\psi := \psi_{\Omega_L}$ and $\alpha_n = v(a_n) \in \Gamma_\Omega \subseteq \Gamma_{\Omega_L}$ for $n \geqslant 1$, we get

$$\begin{aligned} v(f_n - b_n) \;&=\; \psi\big(v(f_{n-1} - b_{n-1}) - \alpha_n\big), \quad \text{so by an easy induction on } n,\\ v(f_n - b_n) \;&=\; \psi_{\alpha_1,\dots,\alpha_n}\big(v(f_0 - b_0)\big), \qquad (n \geqslant 1). \end{aligned}$$

Suppose towards a contradiction that $v(f_0 - b_0), \dots, v(f_n - b_n)$ are $\mathbb{Q}$-linearly dependent over Γ. Then we have $m < n$ and $q_1, \dots, q_{n-m} \in \mathbb{Q}$ such that

$$v(f_m - b_m) + q_1 v(f_{m+1} - b_{m+1}) + \cdots + q_{n-m} v(f_n - b_n) \in \Gamma.$$

For $\gamma := v(f_m - b_m) \in \Gamma_L \setminus \Gamma$ this gives

$$\gamma + q_1 \psi_{\alpha_{m+1}}(\gamma) + \cdots + q_{n-m} \psi_{\alpha_{m+1},\dots,\alpha_n}(\gamma) \in \Gamma.$$

By Lemma 1.2.9 we have $\mathrm{I}(K) \subseteq K^\dagger$, so the H-asymptotic couple of Ω is closed with $\Psi_\Omega \subseteq \Gamma$, by Lemma 2.6.3 and Corollary 2.6.5. Hence $\gamma \in \Gamma_\Omega$ by [ADH, 9.9.2]. Together with $\Psi_\Omega \subseteq \Gamma$ and $\alpha_{m+1}, \dots, \alpha_n \in \Gamma_\Omega$ this gives

$$\psi_{\alpha_{m+1}}(\gamma), \dots, \psi_{\alpha_{m+1},\dots,\alpha_n}(\gamma) \in \Gamma$$

and thus $\gamma \in \Gamma$, a contradiction. □

Ultimate exceptional values. *In this subsection K is H-asymptotic with small derivation and asymptotic integration.* Also $A \in K[\partial]^{\neq}$ and $r := \operatorname{order}(A)$, and γ ranges over $\Gamma = v(K^\times)$. We have $v(\ker^{\neq} A_\lambda) \subseteq \mathscr{E}^{\mathrm{e}}(A_\lambda)$, so if λ is an eigenvalue of A with respect to λ, then $\mathscr{E}^{\mathrm{e}}(A_\lambda) \neq \emptyset$. We call the elements of the set

$$\mathscr{E}^{\mathrm{u}}(A) \;=\; \mathscr{E}^{\mathrm{u}}_K(A) \;:=\; \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) \;=\; \big\{\gamma :\ \operatorname{nwt}_{A_\lambda}(\gamma) \geqslant 1 \text{ for some } \lambda\big\}$$

the **ultimate exceptional values of** A with respect to Λ. The definition of $\mathscr{E}^{\mathrm{u}}_K(A)$ involves our choice of Λ, but we are leaving this implicit to avoid complicated notation. In Section 4.4 we shall restrict K and Λ so that $\mathscr{E}^{\mathrm{u}}(A)$ does not depend any longer on the choice of Λ. There we shall use the following observation:

Lemma 2.6.8. *Let $a, b \in K$ be such that $a - b \in (\mathcal{O}^\times)^\dagger$. Then for all γ we have $\operatorname{nwt}_{A_a}(\gamma) = \operatorname{nwt}_{A_b}(\gamma)$; in particular, $\mathscr{E}^{\mathrm{e}}(A_a) = \mathscr{E}^{\mathrm{e}}(A_b)$.*

Proof. Use that if $u \in \mathcal{O}^\times$ and $a - b = u^\dagger$, then $A_a = (A_b)_{\ltimes u}$. □

Corollary 2.6.9. *Let Λ^* be a complement of the $\mathbb{Q}$-linear subspace $K^\dagger$ of K and let $\lambda \mapsto \lambda^* : \Lambda \to \Lambda^*$ be the group isomorphism with $\lambda - \lambda^* \in K^\dagger$ for all λ. If $\lambda - \lambda^* \in (\mathcal{O}^\times)^\dagger$ for all λ, then $\operatorname{nwt}_{A_\lambda}(\gamma) = \operatorname{nwt}_{A_{\lambda^*}}(\gamma)$ for all γ, so $\mathscr{E}^{\mathrm{u}}(A) = \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_{\lambda^*})$.*

Remark 2.6.10. For $a \in K^\times$ we have $\mathscr{E}^{\mathrm{u}}(aA) = \mathscr{E}^{\mathrm{u}}(A)$ and $\mathscr{E}^{\mathrm{u}}(Aa) = \mathscr{E}^{\mathrm{u}}(A) - va$. Note also that $\mathscr{E}^{\mathrm{e}}(A) = \mathscr{E}^{\mathrm{e}}(A_0) \subseteq \mathscr{E}^{\mathrm{u}}(A)$. Let $\phi \in K^\times$ be active in K, and set $\lambda^\phi := \phi^{-1}\lambda$. Then $\Lambda^\phi := \phi^{-1}\Lambda$ is a complement of the $\mathbb{Q}$-linear subspace $(K^\phi)^\dagger = \phi^{-1}K^\dagger$ of K^ϕ, and $(A^\phi)_{\lambda^\phi} = (A_\lambda)^\phi$. Hence $\mathscr{E}^{\mathrm{u}}_K(A)$ agrees with the set $\mathscr{E}^{\mathrm{u}}_{K^\phi}(A^\phi)$ of ultimate exceptional values of A^ϕ with respect to Λ^ϕ.

Remark 2.6.11. Suppose L is an H-asymptotic extension of K with asymptotic integration and algebraically closed constant field C_L such that $L^\dagger$ is divisible, and Ψ is cofinal in Ψ_L or K is $\boldsymbol{\lambda}$-free. Then $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma$, by Lemma 1.5.1 and Corollary 1.8.10. Hence if $\Lambda_L \supseteq \Lambda$ is a complement of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L, and $\mathscr{E}^{\mathrm{u}}_L(A)$ is the set of ultimate exceptional values of A (viewed as an element of $L[\partial]$) with respect to Λ_L, then $\mathscr{E}^{\mathrm{u}}(A) \subseteq \mathscr{E}^{\mathrm{u}}_L(A)$. (Note that such a complement Λ_L exists iff $L^\dagger \cap K = K^\dagger$.)

In the rest of this subsection we equip U with the gaussian extension v_{g} of the valuation of K. Recall that we have a decomposition $\ker_{\mathrm{U}} A = \bigoplus_\lambda (\ker A_\lambda)\,\mathrm{e}(\lambda)$ of the C-linear space $\ker_{\mathrm{U}} A$ as an internal direct sum of subspaces, and hence

$$v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \;=\; \bigcup_\lambda v(\ker^{\neq} A_\lambda) \;\subseteq\; \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) \;=\; \mathscr{E}^{\mathrm{u}}(A). \tag{2.6.2}$$

Here are some consequences:

Lemma 2.6.12. *Suppose K is r-linearly newtonian. Then $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$.*

Proof. By Proposition 1.5.2 we have $v(\ker^{\neq} A_\lambda) = \mathscr{E}^{\mathrm{e}}(A_\lambda)$ for each λ. Therefore $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.6.2). □

Lemma 2.6.13. *Suppose K is* d*-valued. Then $|v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)| \leqslant \dim_C \ker_{\mathrm{U}} A \leqslant r$.*

Proof. By [ADH, 5.6.6(i)] applied to A_λ in place of A we have

$$|v(\ker^{\neq} A_\lambda)| \;=\; \dim_C \ker A_\lambda \;=\; \operatorname{mult}_\lambda(A) \qquad \text{for all } \lambda$$

and thus by (2.6.2),

$$|v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)| \;\leqslant\; \sum_\lambda |v(\ker^{\neq} A_\lambda)| \;=\; \sum_\lambda \operatorname{mult}_\lambda(A) \;=\; \dim_C \ker_{\mathrm{U}} A \;\leqslant\; r$$

as claimed. □

Lemma 2.6.14. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$ and $r = 1$. Then*

$$v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \;=\; \mathscr{E}^{\mathrm{u}}(A), \qquad |\mathscr{E}^{\mathrm{u}}(A)| \;=\; 1.$$

Proof. Arrange $A = \partial - g$, $g \in K$, and take $f \in K^\times$ and λ such that $g = f^\dagger + \lambda$. Then $u := f\,\mathrm{e}(\lambda) \in \mathrm{U}^\times$ satisfies $A(u) = 0$, hence $\ker^{\neq}_{\mathrm{U}} A = Cu$ and thus $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \{vf\}$. By Lemma 1.5.9 we have $v(\ker^{\neq} A_\lambda) = \mathscr{E}^{\mathrm{e}}(A_\lambda)$ for all λ and hence $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.6.2). □

Corollary 2.6.15. *If $\mathrm{I}(K) \subseteq K^\dagger$ and $a \in K^\times$, then $\mathscr{E}^{\mathrm{e}}(\partial - a^\dagger) = \mathscr{E}^{\mathrm{u}}(\partial - a^\dagger) = \{va\}$.*

Proposition 2.6.17 below partly extends Lemma 2.6.14.

Spectral extensions and ultimate exceptional values. *In this subsection K is* d*-valued of H-type with small derivation, asymptotic integration, and* $\operatorname{I}(K) \subseteq K^\dagger$*.* Also $A \in K[\partial]^{\neq}$ has order r and γ ranges over Γ.

Suppose Ω is equipped with a spectral extension v of the valuation of K. Let $g \in K^\times$ with $vg = \gamma$. The Newton weight of $A_\lambda g \in K[\partial]$ does not change in passing from K to Ω, since Ψ is cofinal in Ψ_Ω by Lemma 2.6.3; see [ADH, 11.1]. Thus

$$\operatorname{nwt}_{A_\lambda}(\gamma) \;=\; \operatorname{nwt}(A_\lambda g) \;=\; \operatorname{nwt}\big(Ag\operatorname{e}(\lambda)\big) \;=\; \operatorname{nwt}_A\big(v(g\operatorname{e}(\lambda)\big) \;=\; \operatorname{nwt}_A\big(\gamma + v(\operatorname{e}(\lambda))\big).$$

In particular, using $\Gamma_\Omega = \Gamma \oplus v\big(\operatorname{e}(\Lambda)\big)$,

$$\mathscr{E}^{\operatorname{e}}_\Omega(A) \;=\; \bigcup_\lambda \mathscr{E}^{\operatorname{e}}(A_\lambda) + v\big(\operatorname{e}(\lambda)\big) \qquad \text{(a disjoint union).} \tag{2.6.3}$$

Thus $\mathscr{E}^{\operatorname{u}}(A) = \pi\big(\mathscr{E}^{\operatorname{e}}_\Omega(A)\big)$ where $\pi\colon \Gamma_\Omega \to \Gamma$ is given by $\pi\big(\gamma + v\big(\operatorname{e}(\lambda)\big)\big) = \gamma$.

Lemma 2.6.16. *We have* $\dim_C \ker_{\operatorname{U}} A \leqslant \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)|$*, and*

$$\dim_C \ker_{\operatorname{U}} A \;=\; \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)| \iff v(\ker^{\neq} A_\lambda) \;=\; \mathscr{E}^{\operatorname{e}}(A_\lambda) \textit{ for all } \lambda.$$

Moreover, if $\dim_C \ker_{\operatorname{U}} A = \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)|$*, then* $v_{\operatorname{g}}(\ker^{\neq}_{\operatorname{U}} A) = \mathscr{E}^{\operatorname{u}}(A)$*.*

Proof. Clearly, $\dim_C \ker_{\operatorname{U}} A \leqslant \dim_C \ker_\Omega A$. Equip Ω with a spectral extension of the valuation of K. Then $\dim_C \ker_\Omega A = |v(\ker^{\neq}_\Omega A)|$ and $v(\ker^{\neq}_\Omega A) \subseteq \mathscr{E}^{\operatorname{e}}_\Omega(A)$ by [ADH, 5.6.6(i)] and [ADH, p. 481], respectively, applied to Ω in the role of K. Also $|\mathscr{E}^{\operatorname{e}}_\Omega(A)| = \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)|$ (a sum of cardinals) by the remarks preceding the lemma. This yields the first claim of the lemma.

Next, note that $v(\ker^{\neq} A_\lambda) \subseteq \mathscr{E}^{\operatorname{e}}(A_\lambda)$ for all λ. Hence from (2.6.3) and

$$v(\ker^{\neq}_{\operatorname{U}} A) \;=\; \bigcup_\lambda v(\ker^{\neq} A_\lambda) + v\big(\operatorname{e}(\lambda)\big) \qquad \text{(a disjoint union)}$$

we obtain:

$$v(\ker^{\neq}_{\operatorname{U}} A) = \mathscr{E}^{\operatorname{e}}_\Omega(A) \quad\iff\quad v(\ker^{\neq} A_\lambda) \;=\; \mathscr{E}^{\operatorname{e}}(A_\lambda) \text{ for all } \lambda.$$

Also $|v(\ker^{\neq}_{\operatorname{U}} A)| = \dim_C \ker_{\operatorname{U}} A$ by [ADH, 2.3.13], and

$$v(\ker^{\neq}_{\operatorname{U}} A) \;\subseteq\; v(\ker^{\neq}_\Omega A) \;\subseteq\; \mathscr{E}^{\operatorname{e}}_\Omega(A), \qquad |\mathscr{E}^{\operatorname{e}}_\Omega(A)| \;=\; \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)|.$$

This yields the displayed equivalence.

Suppose $\dim_C \ker_{\operatorname{U}} A = \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)|$; we need to show $v_{\operatorname{g}}(\ker^{\neq}_{\operatorname{U}} A) = \mathscr{E}^{\operatorname{u}}(A)$. We have $\pi\big(\mathscr{E}^{\operatorname{e}}_\Omega(A)\big) = \mathscr{E}^{\operatorname{u}}(A)$ for the above projection map $\pi\colon \Gamma_\Omega \to \Gamma$, so it is enough to show $\pi\big(v(\ker^{\neq}_{\operatorname{U}} A)\big) = v_{\operatorname{g}}(\ker^{\neq}_{\operatorname{U}} A)$. For that, note that for $\mathcal{B} \subseteq K^\times \operatorname{e}(\Lambda)$ in Corollary 2.6.4 we have

$$\pi\big(v(\ker^{\neq}_{\operatorname{U}} A)\big) \;=\; \pi(v\mathcal{B}) \;=\; v_{\operatorname{g}}(\mathcal{B}) \;=\; v_{\operatorname{g}}(\ker^{\neq}_{\operatorname{U}} A),$$

using for the last equality the details in the proof of Corollary 2.6.4. □

Proposition 2.6.17. *Suppose K is* $\upomega$*-free. Then* $\operatorname{nwt}_{A_\lambda}(\gamma) = 0$ *for all but finitely many pairs* (γ, λ) *and*

$$|\mathscr{E}^{\operatorname{u}}(A)| \;\leqslant\; \sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)| \;=\; \sum_{\gamma,\lambda} \operatorname{nwt}_{A_\lambda}(\gamma) \;\leqslant\; r.$$

If $\dim_C \ker_{\operatorname{U}} A = r$*, then* $\sum_\lambda |\mathscr{E}^{\operatorname{e}}(A_\lambda)| = r$ *and* $v_{\operatorname{g}}(\ker^{\neq}_{\operatorname{U}} A) = \mathscr{E}^{\operatorname{u}}(A)$*.*

Proof. Equip Ω with a spectral extension v of the valuation of K. Then Ω is ω-free, so $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = |\mathscr{E}^{\mathrm{e}}_\Omega(A)| \leqslant r$ by the remarks preceding Lemma 2.6.16 and Corollary 1.5.5 applied to Ω in place of K. These remarks also give $\operatorname{nwt}_{A_\lambda}(\gamma) = 0$ for all but finitely many pairs (γ, λ), and so

$$\sum_{\gamma,\lambda} \operatorname{nwt}_{A_\lambda}(\gamma) \ = \ \sum_{\gamma,\lambda} \operatorname{nwt}_A\big(\gamma + v(\mathrm{e}(\lambda)\big) \ = \ |\mathscr{E}^{\mathrm{e}}_\Omega(A)| \ \leqslant \ r.$$

Corollary 1.5.5 applied to A_λ in place of A yields $|\mathscr{E}^{\mathrm{e}}(A_\lambda)| = \sum_\gamma \operatorname{nwt}_{A_\lambda}(\gamma)$ and so $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = \sum_{\gamma,\lambda} \operatorname{nwt}_{A_\lambda}(\gamma)$. This proves the first part (including the display). The rest follows from this and Lemma 2.6.16. $\square$

In the next lemma (to be used in the proof of Proposition 3.1.26), as well as in Corollary 2.6.23, L is a d-valued H-asymptotic extension of K with algebraically closed constant field and asymptotic integration (so L has small derivation), such that $L^\dagger$ is divisible, $L^\dagger \cap K = K^\dagger$, and $\mathrm{I}(L) \subseteq L^\dagger$. We also fix there a complement Λ_L of the $\mathbb{Q}$-linear subspace $L^\dagger$ of L with $\Lambda \subseteq \Lambda_L$. Let $\mathrm{U}_L = L\big[\mathrm{e}(\Lambda_L)\big]$ be the corresponding universal exponential extension of L containing $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ as a differential subring, as described in the remarks following Corollary 2.2.13, with differential fraction field Ω_L.

Lemma 2.6.18. *Assume $C_L = C$. Let Ω_L be equipped with a spectral extension of the valuation of L, and take Ω as a valued subfield of Ω_L; so the valuation of Ω is a spectral extension of the valuation of K. Suppose Ψ is cofinal in Ψ_L or K is λ-free. Then $\mathscr{E}^{\mathrm{e}}_{\Omega_L}(A) \cap \Gamma_\Omega \ = \ \mathscr{E}^{\mathrm{e}}_\Omega(A)$.*

Proof. Let μ range over Λ_L. We have

$$\Gamma_{\Omega_L} \ = \ \Gamma_L \oplus v\big(\mathrm{e}(\Lambda_L)\big), \qquad \Gamma_\Omega \ = \ \Gamma \oplus v\big(\mathrm{e}(\Lambda)\big)$$

by Lemma 2.6.3 and

$$\mathscr{E}^{\mathrm{e}}_{\Omega_L} \ = \ \bigcup_\mu \mathscr{E}^{\mathrm{e}}_L(A_\mu) + v\big(\mathrm{e}(\mu)\big), \qquad \mathscr{E}^{\mathrm{e}}_\Omega \ = \ \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) + v\big(\mathrm{e}(\lambda)\big)$$

by (2.6.3). Hence

$$\mathscr{E}^{\mathrm{e}}_{\Omega_L}(A) \cap \Gamma_\Omega \ = \ \bigcup_\lambda \big(\mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma\big) + v\big(\mathrm{e}(\lambda)\big) \ = \ \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) + v\big(\mathrm{e}(\lambda)\big) \ = \ \mathscr{E}^{\mathrm{e}}_\Omega(A),$$

where we used the injectivity of $\mu \mapsto v\big(\mathrm{e}(\mu)\big)$ for the first equality and Remark 2.6.11 for the second. $\square$

Call A **terminal** with respect to Λ if $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = r$. We omit "with respect to Λ" if it is clear from the context what Λ is. In Section 4.4 we shall restrict K, Λ anyway so that this dependence on Λ disappears. Recall also that for a given spectral extension of the valuation of K to Ω we have $|\mathscr{E}_\Omega(A)| = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$. If A is terminal and $\phi \in K^\times$ is active in K, then $A^\phi \in K^\phi[\delta]$ is terminal with respect to Λ^ϕ (cf. remarks after Corollary 2.6.9). If A is terminal and $a \in K^\times$, then aA is terminal. If $r = 0$, then A is terminal.

Lemma 2.6.19. *If $r = 1$, then A is terminal.*

Proof. Assume $r = 1$. Then $\dim_C \ker_{\mathrm{U}} A = 1$, so $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \geqslant 1$ by Lemma 2.6.16. Equip Ω with a spectral extension of the valuation of K. Then Ω is ungrounded by Lemma 2.6.3, and $r = 1$ gives $|\mathscr{E}_\Omega(A)| \leqslant 1$. Now use $|\mathscr{E}_\Omega(A)| = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$. $\square$

Lemma 2.6.20. *Suppose A and $B \in K[\partial]^{\neq}$ are terminal, and each operator B_λ is asymptotically surjective. Then AB is terminal.*

Proof. Use that $(AB)_\lambda = A_\lambda B_\lambda$, and that $|\mathscr{E}^{\mathrm{e}}(A_\lambda B_\lambda)| = |\mathscr{E}^{\mathrm{e}}(A_\lambda)| + |\mathscr{E}^{\mathrm{e}}(B_\lambda)|$ by Corollary 1.5.19. □

Thus if A is terminal and $a \in K^\times$, then aA, Aa, and $A_{\ltimes a}$ are terminal. From Lemmas 2.6.19, 2.6.20, and Corollary 1.5.25 we conclude:

Corollary 2.6.21. *If K is λ-free and A splits over K, then A is terminal.*

Corollary 2.6.22. *Suppose K is ω-free and $B \in K[\partial]^{\neq}$. Then A and B are terminal iff AB is terminal.*

Proof. The "only if" part follows from Lemma 2.6.20 and Corollary 1.5.26. For the "if" part, use Corollary 1.5.19 and Proposition 2.6.17. □

Corollary 2.6.23. *Suppose A is terminal, Ψ is cofinal in Ψ_L or K is λ-free, and L is ω-free. Then, with respect to the complement Λ_L of $L^\dagger$ in L, we have:*

(i) *as an element of $L[\partial]$, A is terminal;*
(ii) *$\mathscr{E}^{\mathrm{e}}(A_\mu) = \emptyset$ for all $\mu \in \Lambda_L \setminus \Lambda$;*
(iii) *$\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for all λ; and*
(iv) *$\mathscr{E}^{\mathrm{u}}(A) = \mathscr{E}^{\mathrm{u}}_L(A)$.*

Proof. By the remarks after Corollary 2.6.9 we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) \subseteq \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for each λ, and so with μ ranging over Λ_L, by Proposition 2.6.17 applied to L in place of K, we have $r = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \leqslant \sum_\mu |\mathscr{E}^{\mathrm{e}}_L(A_\mu)| \leqslant r$. This yields (i)–(iv). □

In [15] we shall study other situations where A is terminal.

The real case. *In this subsection H is a real closed H-field with small derivation, asymptotic integration, and $H^\dagger = H$; also $K = H[\mathrm{i}]$, $\mathrm{i}^2 = -1$, for our valued differential field K. We also assume* $\mathrm{I}(H)\mathrm{i} \subseteq K^\dagger$. Then K is d-valued of H-type with small derivation, asymptotic integration, $K^\dagger = H + \mathrm{I}(H)\mathrm{i}$, and $\mathrm{I}(K) \subseteq K^\dagger$. Note that H and thus K is λ-free by [ADH, remark after 11.6.2, and 11.6.8]. Let A in $K[\partial]^{\neq}$ have order r and let γ range over Γ.

Lemma 2.6.24. *If the real closed H-asymptotic extension F of H has asymptotic integration and convex valuation ring, then $L^\dagger \cap K = K^\dagger$ for the algebraically closed H-asymptotic field extension $L := F[\mathrm{i}]$ of K.*

Proof. Use Corollary 1.2.18 and earlier remarks in the same subsection. □

Corollary 2.6.25. *The H-field H has an H-closed extension F with $C_F = C_H$, and for any such F, the algebraically closed* d*-valued field extension $L := F[\mathrm{i}]$ of H-type of K is ω-free with $C_L = C$, $\mathrm{I}(L) \subseteq L^\dagger$, and $L^\dagger \cap K = K^\dagger$.*

Proof. Use [ADH, 16.4.1, 9.1.2] to extend H to an ω-free H-field with the same constant field as H, next use [ADH, 11.7.23] to pass to its real closure, and then use [ADH, 14.5.9] to extend further to an H-closed F, still with the same constant field as H. For any such F, the d-valued field $L := F[\mathrm{i}]$ of H-type is ω-free by [ADH, 11.7.23] and newtonian by [ADH, 14.5.7]. Hence $\mathrm{I}(L) \subseteq L^\dagger$ by Lemma 1.2.9, and $L^\dagger \cap K = K^\dagger$ by Lemma 2.6.24. □

This leads to a variant of Proposition 2.6.17:

Proposition 2.6.26. *The conclusion of Proposition 2.6.17 holds. In particular:*

$$\dim_C \ker_{\mathrm{U}} A = r \implies A \text{ is terminal.}$$

Proof. Corollary 2.6.25 gives an H-closed extension F of H with $C_F = C_H$, so $L := F[\mathrm{i}]$ is $\upomega$-free, $C_L = C$, $\mathrm{I}(L) \subseteq L^\dagger$, and $L^\dagger \cap K = K^\dagger$. Take a complement $\Lambda_L \supseteq \Lambda$ of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L. By Remark 2.6.11 we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma$. Hence Proposition 2.6.17 applied to K, Λ replaced by L, Λ_L, respectively, and A viewed as element of $L[\partial]$, yields $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \leqslant r$. Corollary 1.8.10 applied to A_λ in place of A gives $|\mathscr{E}^{\mathrm{e}}(A_\lambda)| = \sum_\gamma \mathrm{nwt}_{A_\lambda}(\gamma)$. This yields the conclusion of Proposition 2.6.17 as in the proof of that proposition. $\square$

Let now F be a Liouville closed H-field extension of H and suppose $\mathrm{I}(L) \subseteq L^\dagger$ where $L := F[\mathrm{i}]$. Lemma 2.6.24 yields $L^\dagger \cap K = K^\dagger$, so L is as described just before Lemma 2.6.18, and we have a complement $\Lambda_L \supseteq \Lambda$ of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L. Note that if A splits over K, then A is terminal by Corollary 2.6.21.

Corollary 2.6.27. *Suppose A is terminal. Then, with respect to the complement Λ_L of $L^\dagger$ in L, the conclusions* (i)–(iv) *of Corollary 2.6.23 hold.*

Proof. By the remarks after Corollary 2.6.9 we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) \subseteq \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for all λ, and so with μ ranging over Λ_L, Proposition 2.6.26 applied to L in place of K, we have $r = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \leqslant \sum_\mu |\mathscr{E}^{\mathrm{e}}_L(A_\mu)| \leqslant r$. This yields (i)–(iv). $\square$

Part 3. Normalizing Holes and Slots

In this introduction K is an H-asymptotic field with small derivation and rational asymptotic integration. In Section 3.2 we introduce *holes* in K: A *hole in* K is a triple $(P, \mathfrak{m}, \widehat{a})$ with $P \in K\{Y\} \setminus K$, $\mathfrak{m} \in K^\times$, and $\widehat{a} \in \widehat{K} \setminus K$ for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$ and $P(\widehat{a}) = 0$. The main goal of Part 3 is a Normalization Theorem, namely Theorem 3.3.33, that allows us to transform under reasonable conditions a hole $(P, \mathfrak{m}, \widehat{a})$ in K into a "normal" hole; this helps to pin down the location of $\widehat{a}$ relative to K. The notion of $(P, \mathfrak{m}, \widehat{a})$ being *normal* involves the linear part of the differential polynomial $P_{\times\mathfrak{m}}$, in particular the *span* of this linear part. We introduce the span in the preliminary Section 3.1. In Section 3.4 we study *isolated* holes $(P, \mathfrak{m}, \widehat{a})$ in K, which under reasonable conditions ensure the uniqueness of the isomorphism type of $K\langle\widehat{a}\rangle$ as a valued differential field over K; see Proposition 3.4.9. In Section 3.5 we focus on holes $(P, \mathfrak{m}, \widehat{a})$ in K where $\operatorname{order} P = \deg P = 1$. For technical reasons we actually work in Part 3 also with *slots* in K, which are a bit more general than holes in K.

First some notational conventions. Let Γ be an ordered abelian group. For $\gamma, \delta \in \Gamma$ with $\gamma \neq 0$ the expression "$\delta = o(\gamma)$" means "$n|\delta| < |\gamma|$ for all $n \geqslant 1$" according to [ADH, 2.4], but here we find it convenient to extend this to $\gamma = 0$, in which case "$\delta = o(\gamma)$" means "$\delta = 0$". Suppose $\Gamma = v(E^\times)$ is the value group of a valued field E and $\mathfrak{m} \in E^\times$. Then we denote the archimedean class $[v\mathfrak{m}] \subseteq \Gamma$ of $v\mathfrak{m} \in \Gamma$ by just $[\mathfrak{m}]$. Suppose $\mathfrak{m} \not\asymp 1$. Then we have a proper convex subgroup

$$\Delta(\mathfrak{m}) \ := \ \big\{\gamma \in \Gamma : \gamma = o(v\mathfrak{m})\big\} \ = \ \big\{\gamma \in \Gamma : [\gamma] < [\mathfrak{m}]\big\},$$

of Γ. If $\mathfrak{m} \asymp_{\Delta(\mathfrak{m})} \mathfrak{n} \in E$, then $0 \neq \mathfrak{n} \not\asymp 1$ and $\Delta(\mathfrak{m}) = \Delta(\mathfrak{n})$. In particular, if $\mathfrak{m} \asymp \mathfrak{n} \in E$, then $0 \neq \mathfrak{n} \not\asymp 1$ and $\Delta(\mathfrak{m}) = \Delta(\mathfrak{n})$. Note that for $f, g \in E$ the meaning of "$f \preccurlyeq_{\Delta(\mathfrak{m})} g$" does not change in passing to a valued field extension of E, although $\Delta(\mathfrak{m})$ can increase as a subgroup of the value group of the extension.

3.1. The Span of a Linear Differential Operator

In this section K is a valued differential field with small derivation and $\Gamma := v(K^\times)$. We let a, b, sometimes subscripted, range over K, and $\mathfrak{m}$, $\mathfrak{n}$ over $K^\times$. Consider a linear differential operator

$$A \ = \ a_0 + a_1\partial + \cdots + a_r\partial^r \in K[\partial], \qquad a_r \neq 0.$$

We shall use below the quantities $\operatorname{dwm}(A)$ and $\operatorname{dwt}(A)$ defined in [ADH, 5.6]. We also introduce a measure $\mathfrak{v}(A)$ for the "lopsidedness" of A as follows:

$$\mathfrak{v}(A) \ := \ a_r/a_m \in \ K^\times \qquad \text{where } m := \operatorname{dwt}(A).$$

So $a_r \asymp \mathfrak{v}(A)A$ and $\mathfrak{v}(A) \preccurlyeq 1$, with

$$\mathfrak{v}(A) \asymp 1 \quad \Longleftrightarrow \quad \operatorname{dwt}(A) = r \quad \Longleftrightarrow \quad \mathfrak{v}(A) = 1.$$

Also note that $\mathfrak{v}(aA) = \mathfrak{v}(A)$ for $a \neq 0$. Moreover,

$$\mathfrak{v}(A_{\ltimes\mathfrak{n}})A_{\ltimes\mathfrak{n}} \ \asymp \ a_r \ \asymp \ \mathfrak{v}(A)A$$

since $A_{\ltimes\mathfrak{n}} = a_r\partial^r + \text{lower order terms in } \partial$.

Example. $\mathfrak{v}(a + \partial) = 1$ if $a \preccurlyeq 1$, and $\mathfrak{v}(a + \partial) = 1/a$ if $a \succ 1$.

We call $\mathfrak{v}(A)$ the **span** of A. We are mainly interested in the valuation of $\mathfrak{v}(A)$. This is related to the gaussian valuation $v(A)$ of A: if A is monic, then $v\big(\mathfrak{v}(A)\big) = -v(A)$. An important property of the span of A is that its valuation is not affected by small additive perturbations of A:

Lemma 3.1.1. *Suppose $B \in K[\partial]$, $\operatorname{order}(B) \leqslant r$ and $B \prec \mathfrak{v}(A)A$. Then:*

(i) $A + B \sim A$, $\operatorname{dwm}(A+B) = \operatorname{dwm}(A)$, *and* $\operatorname{dwt}(A+B) = \operatorname{dwt}(A)$*;*
(ii) $\operatorname{order}(A+B) = r$ *and* $\mathfrak{v}(A+B) \sim \mathfrak{v}(A)$.

Proof. From $B \prec \mathfrak{v}(A)A$ and $\mathfrak{v}(A) \preccurlyeq 1$ we obtain $B \prec A$, and thus (i). Set $m := \operatorname{dwt}(A)$, let i range over $\{0,\dots,r\}$, and let $B = b_0 + b_1\partial + \cdots + b_r\partial^r$. Then $a_i \preccurlyeq a_m$ and $b_i \prec \mathfrak{v}(A)A \asymp a_r \preccurlyeq a_m$. Therefore, if $a_i \asymp a_m$, then $a_i + b_i \sim a_i$, and if $a_i \prec a_m$, then $a_i + b_i \prec a_m$. Hence $\mathfrak{v}(A+B) = (a_r + b_r)/(a_m + b_m) \sim a_r/a_m = \mathfrak{v}(A)$. □

For $b \neq 0$, the valuation of $\mathfrak{v}(Ab)$ only depends on vb; it is enough to check this for $b \asymp 1$. More generally:

Lemma 3.1.2. *Let $B \in K[\partial]^{\neq}$ and $b \asymp B$. Then $\mathfrak{v}(AB) \asymp \mathfrak{v}(Ab)\mathfrak{v}(B)$.*

Proof. Let $B = b_0 + b_1\partial + \cdots + b_s\partial^s$, $b_s \neq 0$. Then

$$AB \;=\; a_r b_s \partial^{r+s} + \text{lower order terms in } \partial,$$

so by [ADH, 5.6.1(ii)] for $\gamma = 0$:

$$\begin{aligned} v\big(\mathfrak{v}(AB)\big) \;&=\; v(a_r b_s) - v(AB) \;=\; v(a_r b_s) - v(Ab) \\ &=\; v(a_r b) - v(Ab) + v(b_s) - v(B) \\ &=\; v\big(\mathfrak{v}(Ab)\mathfrak{v}(B)\big). \end{aligned}$$

□

Corollary 3.1.3. *Let $B \in K[\partial]^{\neq}$. If $\mathfrak{v}(AB) = 1$, then $\mathfrak{v}(A) = \mathfrak{v}(B) = 1$. The converse holds if B is monic.*

This is clear from from Lemma 3.1.2, and in turn gives:

Corollary 3.1.4. *Suppose $A = a(\partial - b_1)\cdots(\partial - b_r)$. Then*

$$\mathfrak{v}(A) = 1 \quad\Longleftrightarrow\quad b_1,\dots,b_r \preccurlyeq 1.$$

Remark. Suppose $K = C(\!(t)\!)$ with the t-adic valuation and derivation $\partial = t\frac{d}{dt}$. In the literature, A is called *regular singular* if $\mathfrak{v}(A) = 1$, and *irregular singular* if $\mathfrak{v}(A) \prec 1$; see [158, Definition 3.14].

Lemma 3.1.5. *Let $B \in K[\partial]^{\neq}$. Then $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(B)$, and if B is monic, then $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(A)$.*

Proof. Lemma 3.1.2 and $\mathfrak{v}(Ab) \preccurlyeq 1$ for $b \neq 0$ yields $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(B)$. Suppose B is monic, so $v(B) \leqslant 0$. To show $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(A)$ we arrange that A is also monic. Then AB is monic, and $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(A)$ is equivalent to $v(AB) \leqslant v(A)$. Now

$$v(AB) \;=\; v_{AB}(0) \;=\; v_A\big(v_B(0)\big) \;=\; v_A\big(v(B)\big) \;\leqslant\; v_A(0) \;=\; v(A)$$

by [ADH, 4.5.1(iii), 5.6.1(ii)]. □

Corollary 3.1.6. *If $A = a(\partial - b_1)\cdots(\partial - b_r)$, then $b_1,\dots,b_r \;\preccurlyeq\; \mathfrak{v}(A)^{-1}$.*

Let Δ be a convex subgroup of Γ, let $\dot{\mathcal{O}}$ be the valuation ring of the coarsening v_Δ of the valuation v of K by Δ, with maximal ideal $\dot{o}$, and $\dot{K} = \dot{\mathcal{O}}/\dot{o}$ be the valued differential residue field of v_Δ. The residue morphism $\dot{\mathcal{O}} \to \dot{K}$ extends to the ring morphism $\dot{\mathcal{O}}[\partial] \to \dot{K}[\partial]$ with $\partial \mapsto \partial$. If $A \in \dot{\mathcal{O}}[\partial]$ and $\dot{A} \neq 0$, then $\operatorname{dwm}(\dot{A}) = \operatorname{dwm}(A)$ and $\operatorname{dwt}(\dot{A}) = \operatorname{dwt}(A)$. We set $\mathfrak{v} := \mathfrak{v}(A)$.

Lemma 3.1.7. *If $A \in \dot{\mathcal{O}}[\partial]$ and $\operatorname{order}(\dot{A}) = r$, then $\mathfrak{v}(\dot{A}) = \dot{\mathfrak{v}}$.*

Behavior of the span under twisting. Recall that $o(\gamma) := 0 \in \Gamma$ for $\gamma = 0 \in \Gamma$. With this convention, here is a consequence of [ADH, 6.1.3]:

Lemma 3.1.8. *Let $B \in K[\partial]^{\neq}$. Then $v(AB) = v(A) + v(B) + o\big(v(B)\big)$.*

Proof. Take b with $b \asymp B$. Then

$$v(AB) = v_{AB}(0) = v_A\big(v_B(0)\big) = v_A(vb) = v(Ab)$$

by [ADH, 5.6.1(ii)]. Moreover, $v(Ab) = v(A) + vb + o(vb)$, by [ADH, 6.1.3]. $\square$

We have $\mathfrak{v}(A_{\ltimes\mathfrak{n}}) = \mathfrak{v}(A\mathfrak{n})$, so $v(A_{\ltimes\mathfrak{n}}) = v(A) + o(v\mathfrak{n})$ by Lemma 3.1.8. Moreover:

Lemma 3.1.9. *$v\big(\mathfrak{v}(A\mathfrak{n})\big) = v\big(\mathfrak{v}(A)\big) + o(v\mathfrak{n})$.*

Proof. Replacing A by $a_r^{-1}A$ we arrange A is monic, so $A_{\ltimes\mathfrak{n}}$ is monic, and thus

$$v\big(\mathfrak{v}(A\mathfrak{n})\big) \;=\; v\big(\mathfrak{v}(A_{\ltimes\mathfrak{n}})\big) = -v(A_{\ltimes\mathfrak{n}}) = -v(A) + o(v\mathfrak{n}) = v\big(\mathfrak{v}(A)\big) + o(v\mathfrak{n})$$

by remarks preceding the lemma. $\square$

Recall: we denote the archimedean class $[v\mathfrak{n}] \subseteq \Gamma$ by $[\mathfrak{n}]$. Lemma 3.1.9 yields:

Corollary 3.1.10. $\big[\mathfrak{v}(A)\big] < [\mathfrak{n}] \iff \big[\mathfrak{v}(A\mathfrak{n})\big] < [\mathfrak{n}]$.

Under suitable conditions on K we can say more about the valuation of $\mathfrak{v}(A_{\ltimes\mathfrak{n}})$: Lemma 3.1.12 below.

Lemma 3.1.11. *Let $\mathfrak{n}^\dagger \succcurlyeq 1$ and $\mathfrak{m}_0, \dots, \mathfrak{m}_r \in K^\times$ be such that*

$$v(\mathfrak{m}_i) + v(A) \;=\; \min_{i \leqslant j \leqslant r} v(a_j) + (j-i)v(\mathfrak{n}^\dagger).$$

Then with $m := \operatorname{dwt}(A)$ we have

$$\mathfrak{m}_0 \;\succcurlyeq\; \cdots \;\succcurlyeq\; \mathfrak{m}_r \quad \text{and} \quad (\mathfrak{n}^\dagger)^m \;\preccurlyeq\; \mathfrak{m}_0 \;\preccurlyeq\; (\mathfrak{n}^\dagger)^r.$$

(In particular, $[\mathfrak{m}_0] \leqslant [\mathfrak{n}^\dagger]$, with equality if $m > 0$.)

Proof. From $v(\mathfrak{n}^\dagger) \leqslant 0$ we obtain $v(\mathfrak{m}_0) \leqslant \cdots \leqslant v(\mathfrak{m}_r)$. We have $0 \leqslant v(a_j/a_m)$ for $j = 0, \dots, r$ and so

$$rv(\mathfrak{n}^\dagger) \;\leqslant\; \min_{0 \leqslant j \leqslant r} v(a_j/a_m) + jv(\mathfrak{n}^\dagger) \;=\; v(\mathfrak{m}_0) \;\leqslant\; mv(\mathfrak{n}^\dagger)$$

as required. $\square$

Lemma 3.1.12. *Suppose $\partial\mathcal{O} \subseteq o$. Then*

$$\mathfrak{n}^\dagger \preccurlyeq 1 \implies v(A_{\ltimes\mathfrak{n}}) = v(A), \qquad \mathfrak{n}^\dagger \succ 1 \implies |v(A_{\ltimes\mathfrak{n}}) - v(A)| \leqslant -rv(\mathfrak{n}^\dagger).$$

Proof. Let $R := \operatorname{Ri} A$. Then $v(A_{\ltimes \mathfrak{n}}) = v(R_{+\mathfrak{n}^\dagger})$ by [ADH, 5.8.11]. If $\mathfrak{n}^\dagger \preccurlyeq 1$, then $v(R_{+\mathfrak{n}^\dagger}) = v(R)$ by [ADH, 4.5.1(i)], hence $v(A_{\ltimes \mathfrak{n}}) = v(R) = v(A)$ by [ADH, 5.8.10]. Now suppose $\mathfrak{n}^\dagger \succ 1$. *Claim*: $v(A_{\ltimes \mathfrak{n}}) - v(A) \geqslant rv(\mathfrak{n}^\dagger)$. To prove this claim we replace A by $a^{-1}A$, where $a \asymp A$, to arrange $A \asymp 1$. Let i, j range over $\{0, \dots, r\}$. We have $R_{+\mathfrak{n}^\dagger} = \sum_i b_i R_i$ where

$$b_i \;=\; \sum_{j \geqslant i} \binom{j}{i} a_j R_{j-i}(\mathfrak{n}^\dagger).$$

Take $\mathfrak{m}_i \in K^\times$ as in Lemma 3.1.11. By Lemma 1.1.20 we have $R_n(\mathfrak{n}^\dagger) \sim (\mathfrak{n}^\dagger)^n$ for all n; hence $v(b_i) \geqslant v(\mathfrak{m}_i)$ for all i. Thus

$$v(A_{\ltimes \mathfrak{n}}) - v(A) \;=\; v(A_{\ltimes \mathfrak{n}}) \;=\; v(R_{+\mathfrak{n}^\dagger}) \;\geqslant\; \min_i v(b_i) \;\geqslant\; v(\mathfrak{m}_0) \;\geqslant\; rv(\mathfrak{n}^\dagger)$$

by Lemma 3.1.11, proving our claim. Applying this claim with $A_{\ltimes \mathfrak{n}}$, $\mathfrak{n}^{-1}$ in place of A, $\mathfrak{n}$ also yields $v(A_{\ltimes \mathfrak{n}}) - v(A) \leqslant -rv(\mathfrak{n}^\dagger)$, thus $|v(A_{\ltimes \mathfrak{n}}) - v(A)| \leqslant -rv(\mathfrak{n}^\dagger)$. $\square$

Remark. Suppose that $\partial \mathcal{O} \subseteq \smallo$ and $\mathfrak{n}^\dagger \succ 1$. Then Lemma 3.1.12 improves on Lemma 3.1.9, since $v(\mathfrak{n}^\dagger) = o(v\mathfrak{n})$ by [ADH, 6.4.1(iii)].

Lemma 3.1.13. *Suppose $\partial \mathcal{O} \subseteq \smallo$ and $\mathfrak{n}^\dagger \preccurlyeq \mathfrak{v}(A)^{-1}$. Let $B \in K[\partial]$ and $s \in \mathbb{N}$ be such that* $\operatorname{order}(B) \leqslant s$ *and $B \prec \mathfrak{v}(A)^{s+1} A$. Then $B_{\ltimes \mathfrak{n}} \prec \mathfrak{v}(A_{\ltimes \mathfrak{n}}) A_{\ltimes \mathfrak{n}}$.*

Proof. We may assume $B \neq 0$ and $s = \operatorname{order}(B)$. It suffices to show $B_{\ltimes \mathfrak{n}} \prec \mathfrak{v}(A)A$. If $\mathfrak{n}^\dagger \preccurlyeq 1$, then Lemma 3.1.12 applied to B in place of A yields $B_{\ltimes \mathfrak{n}} \asymp B \prec \mathfrak{v}(A)A$. Suppose $\mathfrak{n}^\dagger \succ 1$. Then Lemma 3.1.12 gives $|v(B_{\ltimes \mathfrak{n}}) - v(B)| \leqslant -sv(\mathfrak{n}^\dagger) \leqslant sv(\mathfrak{v}(A))$ and hence $B_{\ltimes \mathfrak{n}} \preccurlyeq \mathfrak{v}(A)^{-s} B \prec \mathfrak{v}(A)A$. $\square$

If $\partial \mathcal{O} \subseteq \smallo$, then we have functions $\operatorname{dwm}_A, \operatorname{dwt}_A \colon \Gamma \to \mathbb{N}$ as defined in [ADH, 5.6]. Combining Lemmas 3.1.1 and 3.1.13 yields a variant of [ADH, 6.1.7]:

Corollary 3.1.14. *Suppose $\partial \mathcal{O} \subseteq \smallo$ and $\mathfrak{n}^\dagger \preccurlyeq \mathfrak{v}(A)^{-1}$. Let $B \in K[\partial]$ be such that* $\operatorname{order}(B) \leqslant r$ *and $B \prec \mathfrak{v}(A)^{r+1} A$. Then $\operatorname{dwm}_{A+B}(v\mathfrak{n}) = \operatorname{dwm}_A(v\mathfrak{n})$ and $\operatorname{dwt}_{A+B}(v\mathfrak{n}) = \operatorname{dwt}_A(v\mathfrak{n})$. In particular,*

$$v\mathfrak{n} \in \mathscr{E}(A+B) \iff v\mathfrak{n} \in \mathscr{E}(A).$$

About $A(\mathfrak{n}^q)$ and $A\mathfrak{n}^q$. Suppose $\mathfrak{m}^l = \pm \mathfrak{n}^k$ where $k, l \in \mathbb{Z}$, $l \neq 0$. Then $\mathfrak{m}^\dagger = q\mathfrak{n}^\dagger$ with $q = k/l \in \mathbb{Q}$. In particular, if K is real closed or algebraically closed, then for any $\mathfrak{n}$ and $q \in \mathbb{Q}$ we have $\mathfrak{m}^\dagger = q\mathfrak{n}^\dagger$ for some $\mathfrak{m}$.

Below in this subsection K is d*-valued and $\mathfrak{n}$ is such that for all $q \in \mathbb{Q}^>$ we are given an element of $K^\times$, denoted by $\mathfrak{n}^q$ for suggestiveness, with $(\mathfrak{n}^q)^\dagger = q\mathfrak{n}^\dagger$.*

Let $q \in \mathbb{Q}^>$; then $v(\mathfrak{n}^q) = qv(\mathfrak{n})$: to see this we may arrange that K is algebraically closed by [ADH, 10.1.23], and hence contains an $\mathfrak{m}$ such that $v\mathfrak{m} = q\, v\mathfrak{n}$ and $\mathfrak{m}^\dagger = q\mathfrak{n}^\dagger = (\mathfrak{n}^q)^\dagger$, and thus $v(\mathfrak{n}^q) = v\mathfrak{m} = q\, v\mathfrak{n}$.

Lemma 3.1.15. *Suppose $\mathfrak{n}^\dagger \succcurlyeq 1$. Then for all but finitely many $q \in \mathbb{Q}^>$,*

$$v\big(A(\mathfrak{n}^q)\big) \;=\; v(\mathfrak{n}^q) + \min_j v(a_j) + jv(\mathfrak{n}^\dagger).$$

Proof. Let $q \in \mathbb{Q}^>$ and take $b_0, \dots, b_r \in K$ with $A\mathfrak{n}^q = b_0 + b_1\partial + \dots + b_r\partial^r$. Then

$$b_0 \;=\; A(\mathfrak{n}^q) \;=\; \mathfrak{n}^q\big(a_0 R_0(q\mathfrak{n}^\dagger) + a_1 R_1(q\mathfrak{n}^\dagger) + \dots + a_r R_r(q\mathfrak{n}^\dagger)\big).$$

Let i, j range over $\{0, \dots, r\}$. By Lemma 1.1.20, $R_i(q\mathfrak{n}^\dagger) \sim q^i(\mathfrak{n}^\dagger)^i$ for all i. Take $\mathfrak{m}$ (independent of q) such that $v(\mathfrak{m}) = \min_j v(a_j) + jv(\mathfrak{n}^\dagger)$, and let I be the nonempty set of i with $\mathfrak{m} \asymp a_i(\mathfrak{n}^\dagger)^i$. For $i \in I$ we take $c_i \in C^\times$ such that $a_i(\mathfrak{n}^\dagger)^i \sim c_i\mathfrak{m}$, and set $R := \sum_{i\in I} c_i Y^i \in C[Y]^{\neq}$. Therefore, if $R(q) \neq 0$, then

$$\sum_{i\in I} a_i R_i(q\mathfrak{n}^\dagger) \ \sim\ \mathfrak{m}R(q).$$

Assume $R(q) \neq 0$ in what follows. Then

$$\sum_{i=0}^{r} a_i R_i(q\mathfrak{n}^\dagger) \ \sim\ \sum_{i\in I} a_i R_i(q\mathfrak{n}^\dagger) \ \sim\ \mathfrak{m}R(q) \ \asymp\ \mathfrak{m},$$

hence $b_0 \asymp \mathfrak{m}\mathfrak{n}^q$, in particular, $b_0 \neq 0$. □

Lemma 3.1.16. *Assume $\mathfrak{n}^\dagger \succcurlyeq 1$ and $[\mathfrak{v}] < [\mathfrak{n}]$ for $\mathfrak{v} := \mathfrak{v}(A)$. Then $\big[\mathfrak{v}(A\mathfrak{n}^q)\big] < [\mathfrak{n}]$ for all $q \in \mathbb{Q}^>$, and for all but finitely many $q \in \mathbb{Q}^>$ we have $\mathfrak{v}(A\mathfrak{n}^q) \preccurlyeq \mathfrak{v}$, and thus $[\mathfrak{v}] \leqslant \big[\mathfrak{v}(A\mathfrak{n}^q)\big]$.*

Proof. Let $q \in \mathbb{Q}^>$. Then $[\mathfrak{v}] < [\mathfrak{n}] = [\mathfrak{n}^q]$, so $[\mathfrak{v}(A\mathfrak{n}^q)] < [\mathfrak{n}^q] = [\mathfrak{n}]$ by Corollary 3.1.10. To show the second part, let $m = \operatorname{dwt}(A)$. Replacing A by $a_m^{-1}A$ we arrange $a_m = 1$, so $a_r = \mathfrak{v}$, $A \asymp 1$. Take $b_0, \dots, b_r$ with $A\mathfrak{n}^q = b_0 + b_1\partial + \cdots + b_r\partial^r$. As in the proof of Lemma 3.1.15 we obtain an $\mathfrak{m}$ and a polynomial $R(Y) \in C[Y]^{\neq}$ (both independent of q) such that $v(\mathfrak{m}) = \min_j v(a_j) + jv(\mathfrak{n}^\dagger)$, and $b_0 \asymp \mathfrak{m}\mathfrak{n}^q$ if $R(q) \neq 0$. Assume $R(q) \neq 0$ in what follows; we show that then $\mathfrak{v}(A\mathfrak{n}^q) \preccurlyeq \mathfrak{v}$. For $n := \operatorname{dwt}(A\mathfrak{n}^q)$,

$$b_0\mathfrak{v}(A\mathfrak{n}^q) \ \preccurlyeq\ b_n\mathfrak{v}(A\mathfrak{n}^q) \ =\ b_r \ =\ \mathfrak{n}^q\mathfrak{v},$$

hence $\mathfrak{v}(A\mathfrak{n}^q) \preccurlyeq \mathfrak{v}/\mathfrak{m}$. It remains to note that $\mathfrak{m} \succcurlyeq a_m(\mathfrak{n}^\dagger)^m = (\mathfrak{n}^\dagger)^m \succcurlyeq 1$. □

Lemma 3.1.17. *Assume $\mathfrak{n}^\dagger \succcurlyeq 1$ and $\mathfrak{m}$ satisfies*

$$v\mathfrak{m} + v(A) \ =\ \min_{0\leqslant j\leqslant r} v(a_j) + jv(\mathfrak{n}^\dagger).$$

Then $[\mathfrak{m}] \leqslant [\mathfrak{n}^\dagger]$, with equality if $\operatorname{dwt}(A) > 0$, and for all but finitely many $q \in \mathbb{Q}^>$,

$$A\mathfrak{n}^q \ \asymp\ \mathfrak{m}\,\mathfrak{n}^q A, \qquad \mathfrak{v}(A)/\mathfrak{v}(A\mathfrak{n}^q) \ \asymp\ \mathfrak{m}.$$

Proof. Replacing A by $a_m^{-1}A$ where $m = \operatorname{dwt}(A)$ we arrange $a_m = 1$, so $a_r = \mathfrak{v} := \mathfrak{v}(A)$ and $A \asymp 1$. Let i, j range over $\{0, \dots, r\}$. Let $q \in \mathbb{Q}^>$, and take $b_i \in K$ such that $A\mathfrak{n}^q = \sum_i b_i\partial^i$. By [ADH, (5.1.3)] we have

$$b_i \ =\ \frac{1}{i!}A^{(i)}(\mathfrak{n}^q) \ =\ \mathfrak{n}^q\frac{1}{i!}\operatorname{Ri}(A^{(i)})(q\mathfrak{n}^\dagger) \ =\ \mathfrak{n}^q\sum_{j\geqslant i}\binom{j}{i}a_j R_{j-i}(q\mathfrak{n}^\dagger).$$

Take $\mathfrak{m}_i \in K^\times$ as in Lemma 3.1.11. Then $\mathfrak{m}_0 \asymp \mathfrak{m}$ (so $[\mathfrak{m}] \leqslant [\mathfrak{n}^\dagger]$, with equality if $m > 0$), and $\mathfrak{m}_r \asymp \mathfrak{v}$. Lemma 3.1.15 applied to $A^{(i)}/i!$ instead of A gives that for all but finitely $q \in \mathbb{Q}^>$ we have $b_i \asymp \mathfrak{m}_i\mathfrak{n}^q$ for all i. Assume that $q \in \mathbb{Q}^>$ has this property. From $v(\mathfrak{m}) = v(\mathfrak{m}_0) \leqslant \cdots \leqslant v(\mathfrak{m}_r) = v(\mathfrak{v})$ we obtain

$$v(\mathfrak{m}) + qv(\mathfrak{n}) \ =\ v(b_0) \ \leqslant\ \cdots\ \leqslant\ v(b_r) \ =\ v(\mathfrak{v}) + q\,v(\mathfrak{n}).$$

With $n = \operatorname{dwt}(A\mathfrak{n}^q)$ this gives $v(b_0) = \cdots = v(b_n) = v(A\mathfrak{n}^q)$. Thus

$$\mathfrak{v}(A\mathfrak{n}^q) \ =\ b_r/b_n \ \asymp\ b_r/b_0 \ \asymp\ (\mathfrak{n}^q\mathfrak{v})/(\mathfrak{n}^q\mathfrak{m}) \ =\ \mathfrak{v}/\mathfrak{m}$$

as claimed. □

Let $\mathfrak{v} \in K^\times$ with $\mathfrak{v} \not\asymp 1$; so we have the proper convex subgroup of Γ given by

$$\Delta(\mathfrak{v}) \;=\; \big\{\gamma \in \Gamma :\, \gamma = o(v\mathfrak{v})\big\} \;=\; \big\{\gamma \in \Gamma :\, [\gamma] < [\mathfrak{v}]\big\}.$$

If K is asymptotic of H-type, then we also have the convex subgroup

$$\Delta \;=\; \big\{\gamma \in \Gamma :\, \gamma^\dagger > v(\mathfrak{v}^\dagger)\big\}$$

of Γ with $\Delta \subseteq \Delta(\mathfrak{v})$, and $\Delta = \Delta(\mathfrak{v})$ if K is of Hardy type (cf. Section 1.2).

Corollary 3.1.18. *Suppose $\mathfrak{n}^\dagger \succcurlyeq 1$ and $[\mathfrak{n}^\dagger] < [\mathfrak{v}]$ where $\mathfrak{v} := \mathfrak{v}(A)$ (so $0 \neq \mathfrak{v} \prec 1$). Let $A_* \in K[\partial]$ and $w \geqslant r$ be such that $A_* \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^w A$. Then for all but finitely many $q \in \mathbb{Q}^>$ we have $\mathfrak{w} := \mathfrak{v}(A\mathfrak{n}^q) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ and $A_*\mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^w A\mathfrak{n}^q$.*

Proof. The case $A_* = 0$ is trivial, so assume $A_* \neq 0$. Take $\mathfrak{m}$ as in Lemma 3.1.17, and take $\mathfrak{m}_*$ likewise with A_* in place of A. By this lemma, $[\mathfrak{m}], [\mathfrak{m}_*] \leqslant [\mathfrak{n}^\dagger] < [\mathfrak{v}]$, hence $\mathfrak{m}, \mathfrak{m}_* \asymp_{\Delta(\mathfrak{v})} 1$. Moreover, for all but finitely many $q \in \mathbb{Q}^>$ we have $A\mathfrak{n}^q \asymp \mathfrak{m}\mathfrak{n}^q A$, $A_*\mathfrak{n}^q \asymp \mathfrak{m}_*\mathfrak{n}^q A_*$, and $\mathfrak{v}/\mathfrak{w} \asymp \mathfrak{m}$ where $\mathfrak{w} := \mathfrak{v}(A\mathfrak{n}^q)$; assume that $q \in \mathbb{Q}^>$ has these properties. Then $A_* \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^w A$ yields

$$A_*\mathfrak{n}^q \;\asymp\; \mathfrak{m}_*\mathfrak{n}^q A_* \;\prec_{\Delta(\mathfrak{v})} \mathfrak{m}\mathfrak{n}^q\mathfrak{v}^w A \;\asymp\; \mathfrak{v}^w A\mathfrak{n}^q.$$

Now $\mathfrak{m} \asymp_{\Delta(\mathfrak{v})} 1$ gives $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}$, hence $A_*\mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^w A\mathfrak{n}^q$. □

The behavior of the span under compositional conjugation. If K is H-asymptotic with asymptotic integration, then $\Psi \cap \Gamma^> \neq \emptyset$, but it is convenient not to require "asymptotic integration" in some lemmas below. Instead: *In this subsection K is H-asymptotic and ungrounded with* $\Psi \cap \Gamma^> \neq \emptyset$. We let ϕ, $\mathfrak{v}$ range over $K^\times$. We say that ϕ is *active* if ϕ is active in K. Recall from [ADH, pp. 290–292] that $\derive$ denotes the derivation $\phi^{-1}\partial$ of K^ϕ, and that

$$A^\phi \;=\; a_r\phi^r\derive^r + \text{lower order terms in } \derive. \tag{3.1.1}$$

Lemma 3.1.19. *Suppose $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$ and $\phi \preccurlyeq 1$ is active. Then*

$$A \;\asymp_{\Delta(\mathfrak{v})}\; A^\phi, \qquad \mathfrak{v} \;\asymp_{\Delta(\mathfrak{v})}\; \mathfrak{v}(A^\phi) \;\prec^\flat\; 1, \qquad \mathfrak{v}, \mathfrak{v}(A^\phi) \;\prec^\flat_\phi\; 1.$$

Proof. From $\phi^\dagger \prec 1 \preccurlyeq \mathfrak{v}^\dagger$ we get $[\phi] < [\mathfrak{v}]$, so $\phi \asymp_{\Delta(\mathfrak{v})} 1$. Hence $A^\phi \asymp_{\Delta(\mathfrak{v})} A$ by [ADH, 11.1.4]. For the rest we can arrange $A \asymp 1$, so $A^\phi \asymp_{\Delta(\mathfrak{v})} 1$ and $\mathfrak{v} \asymp a_r$. In view of (3.1.1) this yields $\mathfrak{v}(A^\phi) \asymp_{\Delta(\mathfrak{v})} a_r\phi^r \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. So $\mathfrak{v}(A^\phi)^\dagger \asymp \mathfrak{v}^\dagger \succcurlyeq 1$, which gives $\mathfrak{v}(A^\phi) \prec^\flat 1$, and also $\mathfrak{v}, \mathfrak{v}(A^\phi) \prec^\flat_\phi 1$. □

Lemma 3.1.20. *If* $\operatorname{nwt}(A) = r$, *then* $\mathfrak{v}(A^\phi) = 1$ *eventually, and if* $\operatorname{nwt}(A) < r$, *then* $\mathfrak{v}(A^\phi) \prec^\flat_\phi 1$ *eventually.*

Proof. Clearly, if $\operatorname{nwt}(A) = r$, then $\operatorname{dwt}(A^\phi) = r$ and so $\mathfrak{v}(A^\phi) = 1$ eventually. Suppose $\operatorname{nwt}(A) < r$. To show that $\mathfrak{v}(A^\phi) \prec^\flat_\phi 1$ eventually, we may replace A by A^{ϕ_0} for suitable active ϕ_0 and assume that $n := \operatorname{nwt}(A) = \operatorname{dwt}(A^\phi) = \operatorname{dwm}(A^\phi)$ for all active $\phi \preccurlyeq 1$. Thus $v(A^\phi) = v(A) + nv\phi$ for all active $\phi \preccurlyeq 1$ by [ADH, 11.1.11(i)]. Using (3.1.1) we therefore obtain for active $\phi \preccurlyeq 1$:

$$\mathfrak{v}(A^\phi) \;\asymp\; a_r\phi^r/a_n\phi^n \;=\; \mathfrak{v}(A)\phi^{r-n} \;\preccurlyeq\; \phi^{r-n} \;\preccurlyeq\; \phi.$$

Take $x \in K^\times$ with $x \not\asymp 1$ and $x' \asymp 1$; then $x \succ 1$, so $x^{-1} \asymp x^\dagger \prec 1$ is active. Hence for active $\phi \preccurlyeq x^{-1}$ we have $\phi \prec^\flat_\phi 1$ and thus $\mathfrak{v}(A^\phi) \prec^\flat_\phi 1$. □

Corollary 3.1.21. *The following conditions on K are equivalent:*

(i) *K is $\uplambda$-free;*

(ii) $\operatorname{nwt}(B) \leqslant 1$ *for all* $B \in K[\partial]$ *(so* $\mathfrak{v}(B^\phi) \prec^\flat_\phi 1$ *eventually);*

(iii) $\operatorname{nwt}(B) \leqslant 1$ *for all* $B \in K[\partial]$ *of order* 2*.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from [ADH, 13.7.10] and Lemma 3.1.20, and (ii) $\Rightarrow$ (iii) is clear. Suppose K is not $\uplambda$-free. Take $\uplambda \in K$ such that $\phi^\dagger + \uplambda \prec \phi$ for all active ϕ ([ADH, 11.6.1]); set $B := (\partial + \uplambda)\partial = \partial^2 + \uplambda\partial$. Then for active ϕ we have $B^\phi = \phi^2\big(\updelta^2 + (\phi^\dagger + \uplambda)\phi^{-1}\updelta\big)$, so $\operatorname{dwt}(B^\phi) = 2$. Thus (iii) $\Rightarrow$ (i). $\square$

Lemma 3.1.19 leads to an "eventual" version of Corollary 3.1.14:

Lemma 3.1.22. *Suppose K is $\uplambda$-free and $B \in K[\partial]$ is such that* $\operatorname{order}(B) \leqslant r$ *and $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1}A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{e}}(A+B) = \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. By [ADH, 10.1.3, 11.7.18] and Corollary 1.8.10 we can pass to an extension to arrange that K is $\upomega$-free. Next, by [ADH, 11.7.23, remark following 14.0.1] we extend further to arrange that K is algebraically closed and newtonian, and thus d-valued by Lemma 1.2.9. Then $\mathscr{E}^{\mathrm{e}}(A) = v(\ker^{\neq} A)$ by Proposition 1.5.2, and A splits over K by [ADH, 14.5.3, 5.8.9]. It remains to show that $\mathscr{E}^{\mathrm{e}}(A) \subseteq \mathscr{E}^{\mathrm{e}}(A+B)$: the reverse inclusion then follows by interchanging A and $A+B$, using $\mathfrak{v}(A) \sim \mathfrak{v}(A+B)$. Let $\gamma \in \mathscr{E}^{\mathrm{e}}(A)$. Take $\mathfrak{n} \in \ker^{\neq} A$ with $v\mathfrak{n} = \gamma$. Then $A \in K[\partial](\partial - \mathfrak{n}^\dagger)$ by [ADH, 5.1.21] and so $\mathfrak{n}^\dagger \preccurlyeq \mathfrak{v}^{-1}$, by [ADH, 5.1.22] and Corollary 3.1.6. Now $\mathscr{E}^{\mathrm{e}}(A) \subseteq \mathscr{E}(A)$, so $\gamma = v\mathfrak{n} \in \mathscr{E}(A+B)$ by Corollary 3.1.14. Let $\phi \preccurlyeq 1$ be active; it remains to show that then $\gamma \in \mathscr{E}\big((A+B)^\phi\big)$. By Lemma 3.1.19, $A^\phi \asymp_{\Delta(\mathfrak{v})} A$; also $B^\phi \preccurlyeq B$ by [ADH, 11.1.4]. Lemma 3.1.19 gives $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}(A^\phi)$, hence $B^\phi \prec_{\Delta(\mathfrak{v})} \mathfrak{v}(A^\phi)^{r+1}A^\phi$. Thus with K^ϕ, A^ϕ, B^ϕ in the role of K, A, B, the above argument leading to $\gamma \in \mathscr{E}(A+B)$ gives $\gamma \in \mathscr{E}(A^\phi + B^\phi) = \mathscr{E}\big((A+B)^\phi\big)$. $\square$

For $r = 1$ we can weaken the hypothesis of $\uplambda$-freeness:

Corollary 3.1.23. *Suppose K has asymptotic integration, $r = 1$, and $B \in K[\partial]$ of order $\leqslant 1$ satisfies $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^2 A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{e}}(A+B) = \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. Using Lemma 1.2.10 we replace K by an immediate extension to arrange $\mathrm{I}(K) \subseteq K^\dagger$. Then $\mathscr{E}^{\mathrm{e}}(A) = v(\ker^{\neq} A)$ by Lemma 1.5.9. Now argue as in the proof of Lemma 3.1.22. $\square$

In the next proposition and its corollary K is d*-valued with algebraically closed constant field C and divisible group $K^\dagger$ of logarithmic derivatives.* We choose a complement Λ of the $\mathbb{Q}$-linear subspace $K^\dagger$ of K. Then we have the set $\mathscr{E}^{\mathrm{u}}(A)$ of ultimate exceptional values of A with respect to Λ. The following stability result will be crucial in Section 4.4:

Proposition 3.1.24. *Suppose K is $\upomega$-free, $\mathrm{I}(K) \subseteq K^\dagger$, and $B \in K[\partial]$ of order $\leqslant r$ satisfies $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1}A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$.*

Proof. Let Ω be the differential fraction field of the universal exponential extension $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ of K from Section 2.2. Equip Ω with a spectral extension of the valuation of K; see Section 2.6. Apply Lemma 3.1.22 to Ω in place of K to get $\mathscr{E}^{\mathrm{e}}_\Omega(A+B) = \mathscr{E}^{\mathrm{e}}_\Omega(A)$. Hence $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.6.3). $\square$

In a similar manner we obtain an analogue of Corollary 3.1.23:

Corollary 3.1.25. *Suppose K has asymptotic integration, $\mathrm{I}(K) \subseteq K^\dagger$, $r = 1$, and $B \in K[\partial]$ satisfies* $\operatorname{order}(B) \leqslant 1$ *and $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^2 A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$.*

Proof. Let Ω be as in the proof of Proposition 3.1.24. Then Ω is ungrounded by Lemma 2.6.3, hence $|\mathscr{E}^{\mathrm{e}}_{\Omega}(A)| \leqslant 1$ and $v(\ker^{\neq}_{\Omega} A) \subseteq \mathscr{E}^{\mathrm{e}}_{\Omega}(A)$ by [ADH, p. 481]. But $\dim_C \ker_{\Omega} A = 1$, so $v(\ker^{\neq}_{\Omega} A) = \mathscr{E}^{\mathrm{e}}_{\Omega}(A)$. The proof of Lemma 3.1.22 with Ω in place of K now gives $\mathscr{E}^{\mathrm{e}}_{\Omega}(A+B) = \mathscr{E}^{\mathrm{e}}_{\Omega}(A)$, so $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.6.3). $\square$

In the "real" case we have the following variant of Proposition 3.1.24:

Proposition 3.1.26. *Suppose $K = H[\mathrm{i}]$, $\mathrm{i}^2 = -1$, where H is a real closed H-field with asymptotic integration such that $H^\dagger = H$ and $\mathrm{I}(H)\mathrm{i} \subseteq K^\dagger$. Let $B \in K[\partial]$ of order $\leqslant r$ be such that $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1} A$ with $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Let Λ be a complement of the subspace $K^\dagger$ of the $\mathbb{Q}$-linear space K. Then $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$, where the ultimate exceptional values are with respect to Λ.*

Proof. Take an H-closed extension F of H with $C_F = C_H$ as in Corollary 2.6.25. Then the algebraically closed d-valued H-asymptotic extension $L := F[\mathrm{i}]$ of K is $\upomega$-free, $C_L = C$, $\mathrm{I}(L) \subseteq L^\dagger$, and $L^\dagger \cap K = K^\dagger$. Take a complement $\Lambda_L \supseteq \Lambda$ of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L. Let $\mathrm{U}_L = L\big[\mathrm{e}(\Lambda_L)\big]$ be the universal exponential extension of L from Section 2.2; it has the universal exponential extension $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ of K as a differential subring. Let Ω, Ω_L be the differential fraction fields of U, U_L, respectively, and equip Ω_L with a spectral extension of the valuation of L; then the restriction of this valuation to Ω is a spectral extension of the valuation of K (see remarks preceding Lemma 2.6.18). Lemma 3.1.22 applied to Ω_L in place of K yields $\mathscr{E}^{\mathrm{e}}_{\Omega_L}(A+B) = \mathscr{E}^{\mathrm{e}}_{\Omega_L}(A)$, hence $\mathscr{E}^{\mathrm{e}}_{\Omega}(A+B) = \mathscr{E}^{\mathrm{e}}_{\Omega}(A)$ by Lemma 2.6.18 and thus $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$. $\square$

The span of the linear part of a differential polynomial. In this subsection $P \in K\{Y\}^{\neq}$ has order r. Recall from [ADH, 5.1] that the *linear part* of P is the differential operator

$$L_P \ := \ \sum_n \frac{\partial P}{\partial Y^{(n)}}(0)\,\partial^n \in K[\partial]$$

of order $\leqslant r$. We have $L_{P_{\times\mathfrak{m}}} = L_P\mathfrak{m}$ [ADH, p. 242]; hence items 3.1.9, 3.1.10 and 3.1.12 above yield information about the span of $L_{P_{\times\mathfrak{m}}}$ (provided $L_P \neq 0$). We now want to similarly investigate the span of the linear part

$$L_{P_{+a}} \ = \ \sum_n \frac{\partial P}{\partial Y^{(n)}}(a)\,\partial^n$$

of the additive conjugate P_{+a} of P by some $a \prec 1$. In the next two lemmas we assume $\operatorname{order}(L_P) = r$ (in particular, $L_P \neq 0$), $\mathfrak{v}(L_P) \prec 1$, and $a \prec 1$, we set

$$L := L_P, \quad L^+ := L_{P_{+a}}, \quad \mathfrak{v} := \mathfrak{v}(L),$$

and set $L_n := \frac{\partial P}{\partial Y^{(n)}}(0)$ and $L_n^+ := \frac{\partial P}{\partial Y^{(n)}}(a)$, so $L = \sum_n L_n \partial^n$, $L^+ = \sum_n L_n^+ \partial^n$. Recall from [ADH, 4.2] the decomposition of P into homogeneous parts: $P = \sum_d P_d$ where $P_d = \sum_{|\boldsymbol{i}|=d} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$; we set $P_{>1} := \sum_{d>1} P_d$.

Lemma 3.1.27. *Suppose $P_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v} P_1$ and $n \leqslant r$. Then*

(i) *$L_r^+ \sim_{\Delta(\mathfrak{v})} L_r$, and thus* $\operatorname{order}(L^+) = \operatorname{order}(L) = r$;
(ii) *if $L_n \asymp_{\Delta(\mathfrak{v})} L$, then $L_n^+ \sim_{\Delta(\mathfrak{v})} L_n$, and so $v(L_n^+) = v(L_n)$;*
(iii) *if $L_n \prec_{\Delta(\mathfrak{v})} L$, then $L_n^+ \prec_{\Delta(\mathfrak{v})} L$, and so $v(L_n^+) > v(L)$.*

In particular, $L^+ \sim_{\Delta(\mathfrak{v})} L$, $\operatorname{dwt} L^+ = \operatorname{dwt} L$, *and $\mathfrak{v}(L^+) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}$.*

Proof. Take $Q, R \in K\{Y\}$ with $\deg_{Y^{(n)}} Q \leqslant 0$ and $R \in Y^{(n)}K\{Y\}$, such that

$$P \;=\; Q + (L_n + R)Y^{(n)}, \qquad \text{so} \qquad \frac{\partial P}{\partial Y^{(n)}} \;=\; \frac{\partial R}{\partial Y^{(n)}} Y^{(n)} + L_n + R.$$

Now $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}P_1$, so $\frac{\partial P}{\partial Y^{(n)}} - L_n \prec_{\Delta(\mathfrak{v})} \mathfrak{v}P_1$. In $K[\partial]$ we thus have

$$L_n^+ - L_n \;=\; \frac{\partial P}{\partial Y^{(n)}}(a) - L_n \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}L \;\asymp\; L_r.$$

So $L_n^+ - L_n \prec_{\Delta(\mathfrak{v})} L$ and (taking $r = n$) $L_r^+ - L_r \prec_{\Delta(\mathfrak{v})} L_r$. This yields (i)–(iii). $\square$

Lemma 3.1.28. *Suppose $P_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1}P_1$, and let $A, B \in K[\partial]$ be such that $L = A + B$, $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1}L$. Then*

$$L^+ \;=\; A + B^+ \quad \textit{where } B^+ \in K[\partial],\ B^+ \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{m+1}L^+.$$

In particular, $L - L^+ \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1}L$.

Proof. Let $A_n, B_n \in K$ be such that $A = \sum_n A_n\partial^n$ and $B = \sum_n B_n\partial^n$, so $L_n = A_n + B_n$. Let any n (possibly $> r$) be given and take $Q, R \in K\{Y\}$ as in the proof of Lemma 3.1.27. Then $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1}P_1$. Since $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1}L$, this yields

$$\frac{\partial P}{\partial Y^{(n)}} - A_n \;=\; \frac{\partial R}{\partial Y^{(n)}} Y^{(n)} + B_n + R \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{m+1}P_1.$$

We have $L_n^+ = \frac{\partial P}{\partial Y^{(n)}}(a)$, so

$$L_n^+ - A_n \;=\; \frac{\partial P}{\partial Y^{(n)}}(a) - A_n \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{m+1}L.$$

By Lemma 3.1.27 we have $L^+ \sim_{\Delta(\mathfrak{v})} L$, hence $B^+ = L^+ - A \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1}L^+$. $\square$

3.2. Holes and Slots

Throughout this section K is an H-asymptotic field with small derivation and with rational asymptotic integration. We set $\Gamma := v(K^\times)$. So K is pre-d-valued, $\Gamma \neq \{0\}$ has no least positive element, and $\Psi \cap \Gamma^> \neq \emptyset$. We let a, b, f, g range over K, and ϕ, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{v}$, $\mathfrak{w}$ (possibly decorated) over $K^\times$. As at the end of the previous section we shorten "active in K" to "active".

Holes. A **hole** in K is a triple $(P, \mathfrak{m}, \widehat{a})$ where $P \in K\{Y\} \setminus K$ and $\widehat{a}$ is an element of $\widehat{K} \setminus K$, for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$ and $P(\widehat{a}) = 0$. (The extension $\widehat{K}$ may vary with $\widehat{a}$.) The **order**, **degree**, and **complexity** of a hole $(P, \mathfrak{m}, \widehat{a})$ in K are defined as the order, (total) degree, and complexity, respectively, of the differential polynomial P. A hole $(P, \mathfrak{m}, \widehat{a})$ in K is called **minimal** if no hole in K has smaller complexity; then P is a minimal annihilator of $\widehat{a}$ over K.

If $(P, \mathfrak{m}, \widehat{a})$ is a hole in K, then $\widehat{a}$ is a K-external zero of P, in the sense of Section 1.8. Conversely, every K-external zero $\widehat{a}$ of a differential polynomial $P \in K\{Y\}^{\neq}$ gives for every $\mathfrak{m} \succ \widehat{a}$ a hole $(P, \mathfrak{m}, \widehat{a})$ in K. By Proposition 1.8.35 and Corollary 1.8.41:

Lemma 3.2.1. *Let $r \in \mathbb{N}^{\geqslant 1}$, and suppose K is λ-free. Then*

$$K \textit{ is } \omega\textit{-free and } r\textit{-newtonian} \quad\Longleftrightarrow\quad K \textit{ has no hole of order} \leqslant r.$$

Thus for ω-free K, being newtonian is equivalent to having no holes. Recall that K being henselian is equivalent to K having no proper immediate algebraic valued field extension, and hence to K having no hole of order 0.

Minimal holes are like the "minimal counterexamples" in certain combinatorial settings, and we need to understand such holes in a rather detailed way for later use in inductive arguments. Below we also consider the more general notion of *Z-minimal hole*, which has an important role to play as well. We recall that $Z(K,\widehat{a})$ is the set of all $Q \in K\{Y\}^{\neq}$ that vanish at $(K,\widehat{a})$ as defined in [ADH, 11.4]

Lemma 3.2.2. *Let $(P,\mathfrak{m},\widehat{a})$ be a hole in K. Then $P \in Z(K,\widehat{a})$. If $(P,\mathfrak{m},\widehat{a})$ is minimal, then P is an element of minimal complexity of $Z(K,\widehat{a})$.*

Proof. Let a, $\mathfrak{v}$ with $\widehat{a}-a \prec \mathfrak{v}$. Since $\widehat{a} \notin K$ lies in an immediate extension of K we can take $\mathfrak{n}$ with $\mathfrak{n} \asymp \widehat{a}-a$. By [ADH, 11.2.1] we then have $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} \geqslant \operatorname{ndeg} P_{+a,\times\mathfrak{n}} \geqslant 1$. Hence $P \in Z(K,\widehat{a})$. Suppose P is not of minimal complexity in $Z(K,\widehat{a})$. Take $Q \in Z(K,\widehat{a})$ of minimal complexity. Then [ADH, 11.4.8] yields a K-external zero $\widehat{b}$ of Q, and any $\mathfrak{n} \succ \widehat{b}$ gives a hole $(Q,\mathfrak{n},\widehat{b})$ in K of smaller complexity than $(P,\mathfrak{m},\widehat{a})$. □

In connection with the next result, note that K being 0-newtonian just means that K is henselian as a valued field.

Corollary 3.2.3. *Suppose K is λ-free and has a minimal hole of order $r \geqslant 1$. Then K is $(r-1)$-newtonian, and ω-free if $r \geqslant 2$.*

Proof. This is clear for $r=1$ (and doesn't need λ-freeness), and for $r \geqslant 2$ follows from Lemma 3.2.1. □

Corollary 3.2.4. *Suppose K is ω-free and has a minimal hole of order $r \geqslant 2$. Assume also that C is algebraically closed and Γ is divisible. Then K is d-valued, r-linearly closed, and r-linearly newtonian.*

Proof. This follows from Lemma 1.2.9, Corollary 1.8.42, and Corollary 3.2.3. □

Here is a linear version of Lemma 3.2.1:

Lemma 3.2.5. *If K is λ-free, then*

$$K \text{ is 1-linearly newtonian} \iff K \text{ has no hole of degree 1 and order 1.}$$

If $r \in \mathbb{N}^{\geqslant 1}$ and K is ω-free, then

$$K \text{ is } r\text{-linearly newtonian} \iff K \text{ has no hole of degree 1 and order} \leqslant r.$$

Proof. The first statement follows from Lemma 1.8.33, and the second statement from Lemma 1.8.34. □

Corollary 3.2.6. *If K is ω-free and has a minimal hole in K of order r and degree > 1, then K is r-linearly newtonian.*

Lemma 3.2.7. *Suppose K has a hole $(P,\mathfrak{m},\widehat{a})$ of degree 1, and $L_P \in K[\partial]^{\neq}$ splits over K. Then K has a hole of complexity $(1,1,1)$.*

Proof. Let $(P,\mathfrak{m},\widehat{a})$ as in the hypothesis have minimal order. Then $\operatorname{order} P \geqslant 1$, so $\operatorname{order} P = \operatorname{order} L_P$. Take $A,B \in K[\partial]$ such that $\operatorname{order} A = 1$ and $L_P = AB$. If $\operatorname{order} B = 0$, then $(P,\mathfrak{m},\widehat{a})$ has complexity $(1,1,1)$. Assume $\operatorname{order} B \geqslant 1$. Then $B(\widehat{a}) \notin K$: otherwise, taking $Q \in K\{Y\}$ of degree 1 with $L_Q = B$ and $Q(0) =$

$-B(\widehat{a})$ yields a hole $(Q, \mathfrak{m}, \widehat{a})$ in K where $\deg Q = 1$ and L_Q splits over K, and $(Q, \mathfrak{m}, \widehat{a})$ has smaller order than $(P, \mathfrak{m}, \widehat{a})$. Set $\widehat{b} := B(\widehat{a})$ and take $R \in K\{Y\}$ of degree 1 with $L_R = A$ and $R(0) = P(0)$. Then

$$R(\widehat{b}) \ = \ R(0) + L_R(\widehat{b}) \ = \ P(0) + L_P(\widehat{a}) \ = \ P(\widehat{a}) \ = \ 0,$$

hence for any $\mathfrak{n} \succ \widehat{b}$, $(R, \mathfrak{n}, \widehat{b})$ is a hole in K of complexity $(1, 1, 1)$. $\square$

Corollary 3.2.8. *Suppose K is ω-free, C is algebraically closed, and Γ is divisible. Then every minimal hole in K of degree 1 has order 1. If in addition K is 1-linearly newtonian, then every minimal hole in K has degree > 1.*

Proof. The first statement follows from Corollary 3.2.4 and the preceding lemma. For the second statement, use the first and Lemma 3.2.5. $\square$

Let $(P, \mathfrak{m}, \widehat{a})$ be a hole in K. We say $(P, \mathfrak{m}, \widehat{a})$ is **Z-minimal** if P has minimal complexity in $Z(K, \widehat{a})$. Thus if $(P, \mathfrak{m}, \widehat{a})$ is minimal, then it is Z-minimal by Lemma 3.2.2. If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, then by [ADH, remarks following 11.4.3], the differential polynomial P is a minimal annihilator of $\widehat{a}$ over K. Note also that $\operatorname{ndeg} P_{\times\mathfrak{m}} \geqslant 1$ by [ADH, 11.2.1]. In more detail:

Lemma 3.2.9. *Let $(P, \mathfrak{m}, \widehat{a})$ be a hole in K. Then for all $\mathfrak{n}$ with $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$,*

$$1 \ \leqslant \ \operatorname{dmul} P_{\times\mathfrak{n}} \ \leqslant \ \operatorname{ddeg} P_{\times\mathfrak{n}} \ \leqslant \ \operatorname{ddeg} P_{\times\mathfrak{m}}.$$

In particular, $\operatorname{ddeg}_{\prec\mathfrak{m}} P \geqslant 1$.

Proof. Assume $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$. Then $\widehat{a} = \mathfrak{n}\widehat{b}$ with $\widehat{b} \prec 1$; put $Q := P_{\times\mathfrak{n}} \in K\{Y\}^{\neq}$. Then $Q(\widehat{b}) = 0$, hence $D_Q(0) = 0$ and so $\operatorname{dmul} Q = \operatorname{dmul} P_{\times\mathfrak{n}} \geqslant 1$. The rest follows from [ADH, 6.6.5(ii), 6.6.7, 6.6.9] and $\Gamma^{>}$ having no least element. $\square$

In the next lemma, (λ_ρ), (ω_ρ) are pc-sequences in K as in [ADH, 11.5, 11.7].

Lemma 3.2.10. *Suppose K is λ-free and $\omega \in K$ is such that $\omega_\rho \rightsquigarrow \omega$ (so K is not ω-free). Then we have a hole $(P, \mathfrak{m}, \lambda)$ in K where $P = 2Y' + Y^2 + \omega$ and $\lambda_\rho \rightsquigarrow \lambda$, and each such hole in K is a Z-minimal hole in K.*

Proof. From [ADH, 11.7.13] we obtain λ in an immediate asymptotic extension of K such that $\lambda_\rho \rightsquigarrow \lambda$ and $P(\lambda) = 0$. Taking any $\mathfrak{m}$ with $\lambda \prec \mathfrak{m}$ then yields a hole $(P, \mathfrak{m}, \lambda)$ in K with $\lambda_\rho \rightsquigarrow \lambda$, and each such hole in K is a Z-minimal hole in K by [ADH, 11.4.13, 11.7.12]. $\square$

Corollary 3.2.11. *If K is λ-free but not ω-free, then each minimal hole in K of positive order has complexity $(1, 1, 1)$ or complexity $(1, 1, 2)$. If K is a Liouville closed H-field and not ω-free, then $(P, \mathfrak{m}, \lambda)$ is a minimal hole of complexity $(1, 1, 2)$, where ω, P, λ, $\mathfrak{m}$ are as in Lemma 3.2.10.*

Here the second part uses Corollary 1.8.29 and Lemma 3.2.5.

Slots. In some arguments the notion of a hole in K turns out to be too stringent. Therefore we introduce a more flexible version of it:

Definition 3.2.12. A **slot** in K is a triple $(P, \mathfrak{m}, \widehat{a})$ where $P \in K\{Y\} \setminus K$ and $\widehat{a}$ is an element of $\widehat{K} \setminus K$, for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$ and $P \in Z(K, \widehat{a})$. The **order**, **degree**, and **complexity** of such a slot in K are defined to be the order, degree, and complexity of the differential polynomial P, respectively. A slot in K of degree 1 is also called a **linear** slot in K. A slot $(P,\mathfrak{m},\widehat{a})$ in K is **Z-minimal** if P is of minimal complexity among elements of $Z(K,\widehat{a})$.

Thus by Lemma 3.2.2, holes in K are slots in K, and a hole in K is Z-minimal iff it is Z-minimal as a slot in K. From [ADH, 11.4.13] we obtain:

Corollary 3.2.13. *Let $(P,\mathfrak{m},\widehat{a})$ be a Z-minimal slot in K and (a_ρ) be a divergent pc-sequence in K such that $a_\rho \rightsquigarrow \widehat{a}$. Then P is a minimal differential polynomial of (a_ρ) over K.*

We say that slots $(P,\mathfrak{m},\widehat{a})$ and $(Q,\mathfrak{n},\widehat{b})$ in K are **equivalent** if $P=Q$, $\mathfrak{m}=\mathfrak{n}$, and $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a; note that then $Z(K,\widehat{a})=Z(K,\widehat{b})$, so $(P,\mathfrak{m},\widehat{a})$ is Z-minimal iff $(P,\mathfrak{m},\widehat{b})$ is Z-minimal. Clearly this is an equivalence relation on the class of slots in K. The following lemma often allows us to pass from a Z-minimal slot to a Z-minimal hole:

Lemma 3.2.14. *Let $(P,\mathfrak{m},\widehat{a})$ be a Z-minimal slot in K. Then $(P,\mathfrak{m},\widehat{a})$ is equivalent to a Z-minimal hole in K.*

Proof. By [ADH, 11.4.8] we obtain $\widehat{b}$ in an immediate asymptotic extension of K with $P(\widehat{b})=0$ and $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a. In particular $\widehat{b}\notin K$, $\widehat{b}\prec\mathfrak{m}$, so $(P,\mathfrak{m},\widehat{b})$ is a hole in K equivalent to $(P,\mathfrak{m},\widehat{a})$. $\square$

By [ADH, 11.4.8] the extension below containing $\widehat{b}$ is not required to be immediate:

Corollary 3.2.15. *If $(P,\mathfrak{m},\widehat{a})$ is a Z-minimal hole in K and $\widehat{b}$ in an asymptotic extension of K satisfies $P(\widehat{b})=0$ and $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a, then there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

In particular, equivalent Z-minimal holes $(P,\mathfrak{m},\widehat{a})$, $(P,\mathfrak{m},\widehat{b})$ in K yield an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.

From Lemmas 3.2.1 and 3.2.14 we obtain:

Corollary 3.2.16. *Let $r\in\mathbb{N}^{\geqslant 1}$, and suppose K is ω-free. Then*

$$K \text{ is } r\text{-newtonian} \quad\Longleftrightarrow\quad K \text{ has no slot of order} \leqslant r.$$

Let $(P,\mathfrak{m},\widehat{a})$ be a slot in K. Then $(bP,\mathfrak{m},\widehat{a})$ for $b\neq 0$ is a slot in K of the same complexity as $(P,\mathfrak{m},\widehat{a})$, and if $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then so is $(bP,\mathfrak{m},\widehat{a})$; likewise with "hole in K" in place of "slot in K". For active ϕ we have the **compositional conjugate** $(P^\phi,\mathfrak{m},\widehat{a})$ by ϕ of $(P,\mathfrak{m},\widehat{a})$: it is a slot in K^ϕ of the same complexity as $(P,\mathfrak{m},\widehat{a})$, it is Z-minimal if $(P,\mathfrak{m},\widehat{a})$ is, and it is a hole (minimal hole) in K^ϕ if $(P,\mathfrak{m},\widehat{a})$ is a hole (minimal hole, respectively) in K. If the slots $(P,\mathfrak{m},\widehat{a})$, $(Q,\mathfrak{n},\widehat{b})$ in K are equivalent, then so are $(bP,\mathfrak{m},\widehat{a})$, $(bQ,\mathfrak{n},\widehat{b})$ for $b\neq 0$, as well as the slots $(P^\phi,\mathfrak{m},\widehat{a})$, $(Q^\phi,\mathfrak{n},\widehat{b})$ in K^ϕ for active ϕ.

The following conventions are in force in the rest of this section:

We let r range over natural numbers $\geqslant 1$ and let $(P,\mathfrak{m},\widehat{a})$ denote a slot in K of order r, so $P\notin K[Y]$ has order r. We set $w:=\operatorname{wt}(P)$, so $w\geqslant r\geqslant 1$.

Thus $\operatorname{wt}(P_{+a})=\operatorname{wt}(P_{\times\mathfrak{n}})=\operatorname{wt}(P^\phi)=w$.

Refinements and multiplicative conjugates. For a, $\mathfrak{n}$ such that $\widehat{a}-a \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$ we obtain a slot $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ in K of the same complexity as $(P, \mathfrak{m}, \widehat{a})$ [ADH, 4.3, 11.4]. Slots of this form are said to **refine** $(P, \mathfrak{m}, \widehat{a})$ and are called **refinements** of $(P, \mathfrak{m}, \widehat{a})$. A refinement of a refinement of $(P, \mathfrak{m}, \widehat{a})$ is itself a refinement of $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, then so is any refinement of $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is a hole in K, then so is each of its refinements, and likewise with "minimal hole" in place of "hole". For active ϕ, $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$ iff $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P^{\phi}, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$, $(P, \mathfrak{m}, \widehat{b})$ are equivalent slots in K and $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$, then $(P_{+a}, \mathfrak{n}, \widehat{b}-a)$ refines $(P, \mathfrak{m}, \widehat{b})$, and the slots $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$, $(P_{+a}, \mathfrak{n}, \widehat{b}-a)$ in K are equivalent. Conversely, if $(P, \mathfrak{m}, \widehat{a})$ and $(P, \mathfrak{m}, \widehat{b})$ are slots in K with equivalent refinements, then $(P, \mathfrak{m}, \widehat{a})$ and $(P, \mathfrak{m}, \widehat{b})$ are equivalent.

Lemma 3.2.17. *Let $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ be a slot in K. Then $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$, or $(P, \mathfrak{m}, \widehat{a})$ refines $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$.*

Proof. If $\mathfrak{n} \preccurlyeq \mathfrak{m}$, then $\widehat{a}-a \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, so $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$, whereas if $\mathfrak{m} \prec \mathfrak{n}$, then $(\widehat{a}-a)-(-a) = \widehat{a} \prec \mathfrak{m} \preccurlyeq \mathfrak{n}$, so

$$(P, \mathfrak{m}, \widehat{a}) = \big((P_{+a})_{+(-a)}, \mathfrak{m}, (\widehat{a}-a)-(-a)\big)$$

refines $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$. □

Lemma 3.2.18. *Let $Q \in K\{Y\}^{\neq}$ be such that $Q \notin Z(K, \widehat{a})$. Then there is a refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ such that $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}} = 0$ and $\widehat{a}-a \prec \mathfrak{n} \prec \widehat{a}$.*

Proof. Take b, $\mathfrak{v}$ such that $\widehat{a}-b \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+b} = 0$. We shall find an a such that $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+a} = 0$, $\widehat{a}-a \preccurlyeq \widehat{a}$, and $\widehat{a}-a \prec \mathfrak{v}$: if $\widehat{a}-b \preccurlyeq \widehat{a}$, we take $a := b$; if $\widehat{a}-b \succ \widehat{a}$, then $-b \sim \widehat{a}-b$ and so $\operatorname{ndeg}_{\prec\mathfrak{v}} Q = \operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+b} = 0$ by [ADH, 11.2.7], hence $a := 0$ works. We next arrange $\widehat{a}-a \prec \widehat{a}$: if $\widehat{a}-a \asymp \widehat{a}$, take a_1 with $\widehat{a}-a_1 \prec \widehat{a}$, so $a-a_1 \prec \mathfrak{v}$, hence $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+a_1} = \operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+a} = 0$, and thus a can be replaced by a_1. Since $\Gamma^{>}$ has no least element, we can choose $\mathfrak{n}$ with $\widehat{a}-a \prec \mathfrak{n} \prec \widehat{a}, \mathfrak{v}$, and then $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$ as desired. □

If $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$, then $D_{P_{+a,\times\mathfrak{m}}} = D_{P_{\times\mathfrak{m},+(a/\mathfrak{m})}} = D_{P_{\times\mathfrak{m}}}$ by [ADH, 6.6.5(iii)], and thus

$$\operatorname{ddeg} P_{+a,\times\mathfrak{m}} \;=\; \operatorname{ddeg} P_{\times\mathfrak{m}}, \qquad \operatorname{dmul} P_{+a,\times\mathfrak{m}} \;=\; \operatorname{dmul} P_{\times\mathfrak{m}}.$$

In combination with Lemma 3.2.9 this has some useful consequences:

Corollary 3.2.19. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a hole in K and $\operatorname{ddeg} P_{\times\mathfrak{m}} = 1$. Then $\operatorname{ddeg}_{\prec\mathfrak{m}} P = 1$, and for all $\mathfrak{n}$ with $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, $(P, \mathfrak{n}, \widehat{a})$ refines $(P, \mathfrak{m}, \widehat{a})$ with $\operatorname{ddeg} P_{\times\mathfrak{n}} = \operatorname{dmul} P_{\times\mathfrak{n}} = 1$.*

Corollary 3.2.20. *Suppose $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines the hole $(P, \mathfrak{m}, \widehat{a})$ in K. Then*

$$\operatorname{ddeg} P_{\times\mathfrak{m}} \;=\; 1 \;\Longrightarrow\; \operatorname{ddeg} P_{+a,\times\mathfrak{n}} \;=\; \operatorname{dmul} P_{+a,\times\mathfrak{n}} \;=\; 1.$$

Proof. Use

$$1 \leqslant \operatorname{dmul} P_{+a,\times\mathfrak{n}} \leqslant \operatorname{ddeg} P_{+a,\times\mathfrak{n}} \leqslant \operatorname{ddeg} P_{+a,\times\mathfrak{m}} = \operatorname{ddeg} P_{\times\mathfrak{m}},$$

where the first inequality follows from Lemma 3.2.9 applied to $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$. □

If $(P_{+a},\mathfrak{m},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, then in analogy with ddeg and dmul,

$$\operatorname{ndeg} P_{+a,\times\mathfrak{m}} \ = \ \operatorname{ndeg} P_{\times\mathfrak{m}}, \qquad \operatorname{nmul} P_{+a,\times\mathfrak{m}} \ = \ \operatorname{nmul} P_{\times\mathfrak{m}}.$$

(Use compositional conjugation by active ϕ.) Lemma 3.2.9 goes through for slots, provided we use ndeg and nmul instead of ddeg and dmul:

Lemma 3.2.21. *Suppose $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$. Then*

$$1 \ \leqslant \ \operatorname{nmul} P_{\times\mathfrak{n}} \ \leqslant \ \operatorname{ndeg} P_{\times\mathfrak{n}} \ \leqslant \ \operatorname{ndeg} P_{\times\mathfrak{m}}.$$

Proof. By [ADH, 11.2.3(iii), 11.2.5] it is enough to show $\operatorname{nmul} P_{\times\mathfrak{n}}\geqslant 1$. Replacing $(P,\mathfrak{m},\widehat{a})$ by its refinement $(P,\mathfrak{n},\widehat{a})$ we arrange $\mathfrak{m}=\mathfrak{n}$. Now $\Gamma^{>}$ has no smallest element, so by definition of $Z(K,\widehat{a})$ and [ADH, p. 483] we have

$$1 \ \leqslant \ \operatorname{ndeg}_{\prec\mathfrak{m}} P \ = \ \max\big\{\operatorname{nmul} P_{\times\mathfrak{v}} : \mathfrak{v}\prec\mathfrak{m}\big\}.$$

Thus by [ADH, 11.2.5] we can take $\mathfrak{v}$ with $\widehat{a}\prec\mathfrak{v}\prec\mathfrak{m}$ with $\operatorname{nmul} P_{\times\mathfrak{v}}\geqslant 1$, and hence $\operatorname{nmul} P_{\times\mathfrak{m}}\geqslant 1$, again by [ADH, 11.2.5]. $\square$

Lemma 3.2.21 yields results analogous to Corollaries 3.2.19 and 3.2.20 above:

Corollary 3.2.22. *If* $\operatorname{ndeg} P_{\times\mathfrak{m}}=1$*, then for all* $\mathfrak{n}$ *with* $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$*,* $(P,\mathfrak{n},\widehat{a})$ *refines* $(P,\mathfrak{m},\widehat{a})$ *and* $\operatorname{ndeg} P_{\times\mathfrak{n}}=\operatorname{nmul} P_{\times\mathfrak{n}}=1$*.*

Corollary 3.2.23. *If* $(P_{+a},\mathfrak{n},\widehat{a}-a)$ *refines* $(P,\mathfrak{m},\widehat{a})$*, then*

$$\operatorname{ndeg} P_{\times\mathfrak{m}} \ = \ 1 \implies \operatorname{ndeg} P_{+a,\times\mathfrak{n}} \ = \ \operatorname{nmul} P_{+a,\times\mathfrak{n}} \ = \ 1.$$

Any triple $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$ is also a slot in K, with the same complexity as $(P,\mathfrak{m},\widehat{a})$; it is called the **multiplicative conjugate** of $(P,\mathfrak{m},\widehat{a})$ by $\mathfrak{n}$. If $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then so is any multiplicative conjugate. If $(P,\mathfrak{m},\widehat{a})$ is a hole in K, then so is any multiplicative conjugate; likewise with "minimal hole" in place of "hole". If two slots in K are equivalent, then so are their multiplicative conjugates by $\mathfrak{n}$.

Refinements and multiplicative conjugates interact in the following way: Suppose $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$. Multiplicative conjugation of the slot $(P_{+a},\mathfrak{n},\widehat{a}-a)$ in K by $\mathfrak{v}$ then results in the slot $(P_{+a,\times\mathfrak{v}},\mathfrak{n}/\mathfrak{v},(\widehat{a}-a)/\mathfrak{v})$ in K. On the other hand, first taking the multiplicative conjugate $(P_{\times\mathfrak{v}},\mathfrak{m}/\mathfrak{v},\widehat{a}/\mathfrak{v})$ of $(P,\mathfrak{m},\widehat{a})$ by $\mathfrak{v}$ and then refining to $(P_{\times\mathfrak{v},+a/\mathfrak{v}},\mathfrak{n}/\mathfrak{v},\widehat{a}/\mathfrak{v}-a/\mathfrak{v})$ results in the same slot in K, thanks to the identity $P_{+a,\times\mathfrak{v}}=P_{\times\mathfrak{v},+a/\mathfrak{v}}$.

Quasilinear slots. Note that $\operatorname{ndeg} P_{\times\mathfrak{m}}\geqslant 1$ by Lemma 3.2.21. We call $(P,\mathfrak{m},\widehat{a})$ **quasilinear** if $P_{\times\mathfrak{m}}$ is quasilinear, that is, $\operatorname{ndeg} P_{\times\mathfrak{m}}=1$. If $(P,\mathfrak{m},\widehat{a})$ is quasilinear, then so is any slot in K equivalent to $(P,\mathfrak{m},\widehat{a})$, any multiplicative conjugate of $(P,\mathfrak{m},\widehat{a})$, as well as any refinement of $(P,\mathfrak{m},\widehat{a})$, by Corollary 3.2.23. If $(P,\mathfrak{m},\widehat{a})$ is linear, then it is quasilinear by Lemma 3.2.21.

Let (a_ρ) be a divergent pc-sequence in K with $a_\rho\rightsquigarrow\widehat{a}$ and for each index ρ, let $\mathfrak{m}_\rho\in K^\times$ be such that $\mathfrak{m}_\rho\asymp\widehat{a}-a_\rho$. Take an index ρ_0 such that $\mathfrak{m}_\sigma\prec\mathfrak{m}_\rho\prec\mathfrak{m}$ for all $\sigma>\rho\geqslant\rho_0$, cf. [ADH, 2.2].

Lemma 3.2.24. *Let $\sigma\geqslant\rho\geqslant\rho_0$. Then*

- (i) $(P_{+a_{\rho+1}},\mathfrak{m}_\rho,\widehat{a}-a_{\rho+1})$ *is a refinement of* $(P,\mathfrak{m},\widehat{a})$*;*
- (ii) *if* $(P_{+a},\mathfrak{n},\widehat{a}-a)$ *is a refinement of* $(P,\mathfrak{m},\widehat{a})$*, then* $\mathfrak{m}_\rho\preccurlyeq\mathfrak{n}$ *for all sufficiently large* ρ*, and for such* ρ*,* $(P_{+a_{\rho+1}},\mathfrak{m}_\rho,\widehat{a}-a_{\rho+1})$ *refines* $(P_{+a},\mathfrak{n},\widehat{a}-a)$*;*
- (iii) $(P_{+a_{\sigma+1}},\mathfrak{m}_\sigma,\widehat{a}-a_{\sigma+1})$ *refines* $(P_{+a_{\rho+1}},\mathfrak{m}_\rho,\widehat{a}-a_{\rho+1})$*.*

Proof. Part (i) follows from $\widehat{a} - a_{\rho+1} \asymp \mathfrak{m}_{\rho+1} \prec \mathfrak{m}_\rho \preccurlyeq \mathfrak{m}$. For (ii) let $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ be a refinement of $(P, \mathfrak{m}, \widehat{a})$. Since $\widehat{a} - a \prec \mathfrak{n}$, we have $\mathfrak{m}_\rho \preccurlyeq \mathfrak{n}$ for all sufficiently large ρ. For such ρ, with $b := a_{\rho+1} - a$ we have

$$(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1}) \;=\; \big((P_{+a})_{+b}, \mathfrak{m}_\rho, (\widehat{a} - a) - b\big)$$

and

$$(\widehat{a} - a) - b \;=\; \widehat{a} - a_{\rho+1} \;\asymp\; \mathfrak{m}_{\rho+1} \;\prec\; \mathfrak{m}_\rho \;\preccurlyeq\; \mathfrak{n}.$$

Hence $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ refines $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$. Part (iii) follows from (i) and (ii). $\square$

Let $\boldsymbol{a} = c_K(a_\rho)$ be the cut defined by (a_ρ) in K and $\operatorname{ndeg}_{\boldsymbol{a}} P$ be the Newton degree of P in $\boldsymbol{a}$ as introduced in [ADH, 11.2]. Then $\operatorname{ndeg}_{\boldsymbol{a}} P$ is the eventual value of $\operatorname{ndeg} P_{+a_\rho, \times \mathfrak{m}_\rho}$. Increasing ρ_0 we arrange that additionally for all $\rho \geqslant \rho_0$ we have $\operatorname{ndeg} P_{+a_\rho, \times \mathfrak{m}_\rho} = \operatorname{ndeg}_{\boldsymbol{a}} P$.

Corollary 3.2.25. *$(P, \mathfrak{m}, \widehat{a})$ has a quasilinear refinement iff $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$.*

Proof. By Lemma 3.2.21 and [ADH, 11.2.8] we have

$$1 \leqslant \operatorname{ndeg} P_{+a_{\rho+1}, \times \mathfrak{m}_\rho} = \operatorname{ndeg} P_{+a_\rho, \times \mathfrak{m}_\rho}. \tag{3.2.1}$$

Thus if $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$, then for $\rho \geqslant \rho_0$, the refinement $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ of $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Conversely, if $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is a quasilinear refinement of $(P, \mathfrak{m}, \widehat{a})$, then Lemma 3.2.24(ii) yields a $\rho \geqslant \rho_0$ such that $\mathfrak{m}_\rho \preccurlyeq \mathfrak{n}$, and then $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ in K refines $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ and hence is also quasilinear, so $\operatorname{ndeg}_{\boldsymbol{a}} P = \operatorname{ndeg} P_{+a_\rho, \times \mathfrak{m}_\rho} = 1$ by (3.2.1). $\square$

Lemma 3.2.26. *Assume K is* d*-valued and ω-free, and Γ is divisible. Then every Z-minimal slot in K of positive order has a quasilinear refinement.*

Proof. Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Take a divergent pc-sequence (a_ρ) in K such that $a_\rho \rightsquigarrow \widehat{a}$. Then P is a minimal differential polynomial of (a_ρ) over K, by Corollary 3.2.13. Hence $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$ by [ADH, 14.5.1], where $\boldsymbol{a} := c_K(a_\rho)$. Now Corollary 3.2.25 gives a quasilinear refinement of $(P, \mathfrak{m}, \widehat{a})$. $\square$

Remark. Suppose K is a real closed H-field that is λ-free but not ω-free. (For example, the real closure of the H-field $\mathbb{R}\langle\omega\rangle$ from [ADH, 13.9.1] satisfies these conditions, by [ADH, 11.6.8, 11.7.23, 13.9.1].) Take $(P, \mathfrak{m}, \lambda)$ as in Lemma 3.2.10. Then by Corollary 3.2.25 and [ADH, 11.7.9], $(P, \mathfrak{m}, \lambda)$ has no quasilinear refinement. Thus Lemma 3.2.26 fails if "ω-free" is replaced by "λ-free".

Lemma 3.2.27. *Let L be an r-newtonian H-asymptotic extension of K such that $\Gamma^<$ is cofinal in $\Gamma_L^<$, and suppose $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Then $P(\widehat{b}) = 0$ and $\widehat{b} \prec \mathfrak{m}$ for some $\widehat{b} \in L$.*

Proof. Lemma 3.2.21 and $\operatorname{ndeg} P_{\times \mathfrak{m}} = 1$ gives $\mathfrak{n} \prec \mathfrak{m}$ with $\operatorname{ndeg}_{\times \mathfrak{n}} P = 1$. By [ADH, p. 480], $\operatorname{ndeg} P_{\times \mathfrak{n}}$ does not change in passing from K to L. As L is r-newtonian this yields $\widehat{b} \preccurlyeq \mathfrak{n}$ in L with $P(\widehat{b}) = 0$. $\square$

In the next two corollaries we assume that K is d-valued and ω-free, and that L is a newtonian H-asymptotic extension of K.

Corollary 3.2.28. *If $(P, \mathfrak{m}, \widehat{a})$ is quasilinear, then $P(\widehat{b}) = 0$, $\widehat{b} \prec \mathfrak{m}$ for some $\widehat{b} \in L$.*

Proof. By [159, Theorem B], K has a newtonization K^* inside L. Such K^* is d-algebraic over K by [ADH, remarks after 14.0.1], so $\Gamma^<$ is cofinal in $\Gamma^<_{K^*}$ by Theorem 1.4.1. Thus we can apply Lemma 3.2.27 to K^* in the role of L. $\square$

Here is a variant of Lemma 3.2.14:

Corollary 3.2.29. *Suppose Γ is divisible and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Then there exists $\widehat{b} \in L$ such that $K\langle\widehat{b}\rangle$ is an immediate extension of K and $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. (Thus if $(P, \mathfrak{m}, \widehat{a})$ is also a hole in K, then there is an embedding $K\langle\widehat{a}\rangle \to L$ of valued differential fields over K.)*

Proof. By Lemma 3.2.26 we may refine $(P, \mathfrak{m}, \widehat{a})$ to arrange that $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Then [ADH, 11.4.8] gives $\widehat{b}$ in an immediate H-asymptotic extension of K with $P(\widehat{b}) = 0$ and $v(\widehat{a} - a) = v(\widehat{b} - a)$ for all a. So $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. The immediate d-algebraic extension $K\langle\widehat{b}\rangle$ of K is ω-free by Theorem 1.4.1. Then [ADH, remarks following 14.0.1] gives a newtonian d-algebraic immediate extension M of $K\langle\widehat{b}\rangle$ and thus of K. Then M is a newtonization of K by [ADH, 14.5.4] and thus embeds over K into L. The rest follows from Corollary 3.2.15. $\square$

Remark. Lemma 3.2.26 and Corollary 3.2.29 go through with the hypothesis "Γ is divisible" replaced by "K is henselian". The proofs are the same, using [159, 3.3] in place of [ADH, 14.5.1] in the proof of Lemma 3.2.26, and [159, 3.5] in place of [ADH, 14.5.4] in the proof of Corollary 3.2.29.

For $r = 1$ we can weaken the hypothesis of ω-freeness in Corollary 3.2.29:

Corollary 3.2.30. *Suppose K is λ-free and Γ is divisible, and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal of order $r = 1$ with a quasilinear refinement. Let L be a newtonian H-asymptotic extension of K. Then there exists $\widehat{b} \in L$ such that $K\langle\widehat{b}\rangle$ is an immediate extension of K and $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. (So if $(P, \mathfrak{m}, \widehat{a})$ is also a hole in K, then we have an embedding $K\langle\widehat{a}\rangle \to L$ of valued differential fields over K.)*

Proof. Take a divergent pc-sequence (a_ρ) in K with $a_\rho \rightsquigarrow \widehat{a}$. Then $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$ for $\boldsymbol{a} := c_K(a_\rho)$, by Corollary 3.2.25, and P is a minimal differential polynomial of (a_ρ) over K, by [ADH, 11.4.13]. The equality $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$ remains valid when passing from K, $\boldsymbol{a}$ to L, $c_L(a_\rho)$, respectively, by Lemma 1.8.8. Hence [ADH, 14.1.10] yields $\widehat{b} \in L$ such that $P(\widehat{b}) = 0$ and $a_\rho \rightsquigarrow \widehat{b}$, so $v(\widehat{a} - a) = v(\widehat{b} - a)$ for all a. Then $K\langle\widehat{b}\rangle$ is an immediate extension of K by [ADH, 9.7.6], so $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. For the rest use Corollary 3.2.15. $\square$

The linear part of a slot. We define the **linear part** of $(P, \mathfrak{m}, \widehat{a})$ to be the linear part $L_{P_{\times\mathfrak{m}}} \in K[\partial]$ of $P_{\times\mathfrak{m}}$. By [ADH, p. 242] and Lemma 1.1.10 we have

$$L_{P_{\times\mathfrak{m}}} \;=\; L_P\,\mathfrak{m} \;=\; \sum_{n=0}^{r} \frac{\partial P_{\times\mathfrak{m}}}{\partial Y^{(n)}}(0)\,\partial^n \;=\; \mathfrak{m} S_P(0)\partial^r + \text{lower order terms in } \partial.$$

The slot $(P, \mathfrak{m}, \widehat{a})$ has the same linear part as each of its multiplicative conjugates. The linear part of a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is given by

$$\begin{aligned} L_{P_{+a,\times\mathfrak{n}}} \ &= \ L_{P_{+a}}\mathfrak{n} \ = \ \sum_{m=0}^{r}\left(\sum_{n=m}^{r}\binom{n}{m}\mathfrak{n}^{(n-m)}\frac{\partial P}{\partial Y^{(n)}}(a)\right)\partial^m \\ &= \ \mathfrak{n}\, S_P(a)\,\partial^r + \text{lower order terms in } \partial. \end{aligned}$$

(See [ADH, (5.1.1)].) By [ADH, 5.7.5] we have $(P^{\phi})_d = (P_d)^{\phi}$ for $d \in \mathbb{N}$; in particular $L_{P^{\phi}} = (L_P)^{\phi}$ and so $\operatorname{order}(L_{P^{\phi}}) = \operatorname{order}(L_P)$. A particularly favorable situation occurs when L_P splits over a given differential field extension E of K (which includes requiring $L_P \neq 0$). Typically, E is an algebraic closure of K. In any case, L_P splits over E iff $L_{P_{\times\mathfrak{n}}}$ splits over E, iff $L_{P^{\phi}}$ splits over E^{ϕ}. Thus:

Lemma 3.2.31. *Suppose* $\deg P = 1$ *and* L_P *splits over* E. *Then the linear part of any refinement of* $(P, \mathfrak{m}, \widehat{a})$ *and any multiplicative conjugate of* $(P, \mathfrak{m}, \widehat{a})$ *also splits over* E, *and any compositional conjugate of* $(P, \mathfrak{m}, \widehat{a})$ *by an active* ϕ *splits over* E^{ϕ}.

Let $\boldsymbol{i} = (i_0, \dots, i_r)$ range over $\mathbb{N}^{1+r}$. As in [ADH, 4.2] we set

$$P_{(\boldsymbol{i})} \ := \ \frac{P^{(\boldsymbol{i})}}{\boldsymbol{i}!} \qquad \text{where } P^{(\boldsymbol{i})} \ := \ \frac{\partial^{|\boldsymbol{i}|}P}{\partial^{i_0}Y \cdots \partial^{i_r}Y^{(r)}}.$$

If $|\boldsymbol{i}| = i_0 + \cdots + i_r \geqslant 1$, then $\operatorname{c}(P_{(\boldsymbol{i})}) < \operatorname{c}(P)$. Note that for $\boldsymbol{i} = (0, \dots, 0, 1)$ we have $P_{(\boldsymbol{i})} = S_P \neq 0$, since $\operatorname{order} P = r$. We now aim for Corollary 3.2.34.

Lemma 3.2.32. *Suppose that* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal. Then* $(P, \mathfrak{m}, \widehat{a})$ *has a refinement* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *such that for all* $\boldsymbol{i}$ *with* $|\boldsymbol{i}| \geqslant 1$ *and* $P_{(\boldsymbol{i})} \neq 0$,

$$\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} \ = \ 0.$$

Proof. Let $\boldsymbol{i}$ range over the (finitely many) elements of $\mathbb{N}^{1+r}$ satisfying $|\boldsymbol{i}| \geqslant 1$ and $P_{(\boldsymbol{i})} \neq 0$. Each $P_{(\boldsymbol{i})}$ has smaller complexity than P, so $P_{(\boldsymbol{i})} \notin Z(K, \widehat{a})$. Then $Q := \prod_{\boldsymbol{i}} P_{(\boldsymbol{i})} \notin Z(K, \widehat{a})$ by [ADH, 11.4.4], so Lemma 3.2.18 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}} = 0$. Then $\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} = 0$ for all $\boldsymbol{i}$, by [ADH, remarks before 11.2.6]. $\square$

From [ADH, (4.3.3)] we recall that $(P_{(\boldsymbol{i})})_{+a} = (P_{+a})_{(\boldsymbol{i})}$. Also recall that $(P_{+a})_{\boldsymbol{i}} = P_{(\boldsymbol{i})}(a)$ by Taylor expansion. In particular, if $P_{(\boldsymbol{i})} = 0$, then $(P_{+a})_{\boldsymbol{i}} = 0$.

Lemma 3.2.33. *Suppose* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *refines* $(P, \mathfrak{m}, \widehat{a})$ *and* $\boldsymbol{i}$ *is such that* $|\boldsymbol{i}| \geqslant 1$, $P_{(\boldsymbol{i})} \neq 0$, *and* $\operatorname{ndeg}(P_{(\boldsymbol{i})})_{\times\mathfrak{m}} = 0$. *Then*

$$\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} \ = \ 0, \qquad (P_{+a})_{\boldsymbol{i}} \ \sim \ P_{\boldsymbol{i}}.$$

Proof. Using [ADH, 11.2.4, 11.2.3(iii), 11.2.5] we get

$$\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} \ = \ \operatorname{ndeg}(P_{(\boldsymbol{i})})_{+\widehat{a},\times\mathfrak{n}} \ \leqslant \ \operatorname{ndeg}(P_{(\boldsymbol{i})})_{+\widehat{a},\times\mathfrak{m}} \ = \ \operatorname{ndeg}(P_{(\boldsymbol{i})})_{\times\mathfrak{m}} \ = \ 0,$$

so $\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} = 0$. Thus $P_{(\boldsymbol{i})} \notin Z(K, \widehat{a})$, hence $(P_{+a})_{\boldsymbol{i}} = P_{(\boldsymbol{i})}(a) \sim P_{(\boldsymbol{i})}(\widehat{a})$ by [ADH, 11.4.3]; applying this to $a = 0$, $\mathfrak{n} = \mathfrak{m}$ yields $P_{\boldsymbol{i}} = P_{(\boldsymbol{i})}(0) \sim P_{(\boldsymbol{i})}(\widehat{a})$. $\square$

Combining Lemmas 3.2.32 and 3.2.33 gives:

Corollary 3.2.34. *Every* Z*-minimal slot in* K *of order* r *has a refinement* $(P, \mathfrak{m}, \widehat{a})$ *such that for all refinements* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *of* $(P, \mathfrak{m}, \widehat{a})$ *and all* $\boldsymbol{i}$ *with* $|\boldsymbol{i}| \geqslant 1$ *and* $P_{(\boldsymbol{i})} \neq 0$ *we have* $(P_{+a})_{\boldsymbol{i}} \sim P_{\boldsymbol{i}}$ (*and thus* $\operatorname{order} L_{P_{+a}} = \operatorname{order} L_P = r$).

Here the condition "of order r" may seem irrelevant, but is forced on us because refinements preserve order and by our convention that P has order r.

Special slots. The slot $(P,\mathfrak{m},\widehat{a})$ in K is said to be **special** if $\widehat{a}/\mathfrak{m}$ is special over K in the sense of [ADH, p. 167]: some nontrivial convex subgroup Δ of Γ is cofinal in $v\big(\frac{\widehat{a}}{\mathfrak{m}}-K\big)$. If $(P,\mathfrak{m},\widehat{a})$ is special, then so are $(bP,\mathfrak{m},\widehat{a})$ for $b\neq 0$, any multiplicative conjugate of $(P,\mathfrak{m},\widehat{a})$, any compositional conjugate of $(P,\mathfrak{m},\widehat{a})$, and any slot in K equivalent to $(P,\mathfrak{m},\widehat{a})$. Also, by Lemma 1.6.1:

Lemma 3.2.35. *If $(P,\mathfrak{m},\widehat{a})$ is special, then so is any refinement.*

Here is our main source of special slots:

Lemma 3.2.36. *Let K be r-linearly newtonian, and $\upomega$-free if $r>1$. Suppose $(P,\mathfrak{m},\widehat{a})$ is quasilinear, and Z-minimal or a hole in K. Then $(P,\mathfrak{m},\widehat{a})$ is special.*

Proof. Use Lemma 3.2.14 to arrange $(P,\mathfrak{m},\widehat{a})$ is a hole in K. Next arrange $\mathfrak{m}=1$ by replacing $(P,\mathfrak{m},\widehat{a})$ with $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$. So $\operatorname{ndeg} P=1$, hence $\widehat{a}$ is special over K by Proposition 1.6.12 (if $r>1$) and 1.6.18 (if $r=1$). $\square$

Next an approximation result used in the proof of Corollary 6.5.19 in Part 6:

Lemma 3.2.37. *Suppose $\mathfrak{m}=1$, $(P,1,\widehat{a})$ is special and Z-minimal, and $\widehat{a}-a\preccurlyeq\mathfrak{n}\prec 1$ for some a. Then $\widehat{a}-b\prec\mathfrak{n}^{r+1}$ for some b, and $P(b)\prec\mathfrak{n}P$ for any such b.*

Proof. Using Lemma 3.2.14 we arrange $P(\widehat{a})=0$. The differential polynomial $Q(Y):=\sum_{|\boldsymbol{i}|\geqslant 1}P_{(\boldsymbol{i})}(\widehat{a})Y^{\boldsymbol{i}}\in\widehat{K}\{Y\}$ has order $\leqslant r$ and $\operatorname{mul}(Q)\geqslant 1$, and Taylor expansion yields, for all a:

$$P(a)\ =\ P(\widehat{a})+\sum_{|\boldsymbol{i}|\geqslant 1}P_{(\boldsymbol{i})}(\widehat{a})(a-\widehat{a})^{\boldsymbol{i}}\ =\ Q(a-\widehat{a}).$$

Since $\widehat{a}$ is special over K, we have b with $\widehat{a}-b\prec\mathfrak{n}^{r+1}$, and then by Lemma 1.1.13 we have $Q(b-\widehat{a})\prec\mathfrak{n}Q\preccurlyeq\mathfrak{n}P$. $\square$

3.3. The Normalization Theorem

Throughout this section K is an H-asymptotic field with small derivation and with rational asymptotic integration. We set $\Gamma:=v(K^\times)$. The notational conventions introduced in the last section remain in force: a, b, f, g range over K; ϕ, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{v}$, $\mathfrak{w}$ over $K^\times$. As at the end of Section 3.1 we shall frequently use for $\mathfrak{v}\prec 1$ the coarsening of v by the convex subgroup $\Delta(\mathfrak{v})=\big\{\gamma\in\Gamma:\ \gamma=o(v\mathfrak{v})\big\}$ of Γ.

We fix a slot $(P,\mathfrak{m},\widehat{a})$ in K of order $r\geqslant 1$, and set $w:=\operatorname{wt}(P)$ (so $w\geqslant r\geqslant 1$). In the next subsections we introduce various conditions on $(P,\mathfrak{m},\widehat{a})$. These conditions will be shown to be related as follows:

$$\begin{tikzcd}
\text{strictly normal} \arrow[r, Rightarrow] & \text{normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{steep} \\
& \text{quasilinear} & \text{deep} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}$$

Thus "deep + strictly normal" yields the rest. The main results of this section are Theorem 3.3.33 and its variants 3.3.34, 3.3.36, and 3.3.48.

Steep and deep slots. In this subsection, if $\operatorname{order}(L_{P_{\times\mathfrak{m}}}) = r$, then we set

$$\mathfrak{v} \ := \ \mathfrak{v}(L_{P_{\times\mathfrak{m}}}).$$

The slot $(P, \mathfrak{m}, \widehat{a})$ in K is said to be **steep** if $\operatorname{order}(L_{P_{\times\mathfrak{m}}}) = r$ and $\mathfrak{v} \prec^\flat 1$. Thus

$$(P, \mathfrak{m}, \widehat{a}) \text{ is steep} \iff (P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n}) \text{ is steep} \iff (bP, \mathfrak{m}, \widehat{a}) \text{ is steep}$$

for $b \neq 0$. If $(P, \mathfrak{m}, \widehat{a})$ is steep, then so is any slot in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is steep, then so is any slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ for active $\phi \preccurlyeq 1$, by Lemma 3.1.19, and thus $\operatorname{nwt}(L_{P_{\times\mathfrak{m}}}) < r$. Below we tacitly use that if $(P, \mathfrak{m}, \widehat{a})$ is steep, then

$$\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v} \implies [\mathfrak{n}] = [\mathfrak{v}], \qquad \mathfrak{n} \prec 1,\ [\mathfrak{n}] = [\mathfrak{v}] \implies \mathfrak{n} \prec^\flat 1.$$

Note also that if $(P, \mathfrak{m}, \widehat{a})$ is steep, then $\mathfrak{v}^\dagger \asymp_{\Delta(\mathfrak{v})} 1$ by [ADH, 9.2.10(iv)].

Lemma 3.3.1. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is steep,* $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$ *and* $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$*. Then*

$$\operatorname{order}(L_{P_{\times\mathfrak{n}}}) \ = \ r, \qquad \mathfrak{v}(L_{P_{\times\mathfrak{n}}}) \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v},$$

so $(P, \mathfrak{n}, \widehat{a})$ *is a steep refinement of* $(P, \mathfrak{m}, \widehat{a})$*.*

Proof. Replace $(P, \mathfrak{m}, \widehat{a})$ and $\mathfrak{n}$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ and $\mathfrak{n}/\mathfrak{m}$ to arrange $\mathfrak{m} = 1$. Set $L := L_P$ and $\widetilde{L} := L_{P_{\times\mathfrak{n}}}$. Then $\widetilde{L} = L\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} \mathfrak{n}L$ by [ADH, 6.1.3]. Hence

$$\widetilde{L}_r \ = \ \mathfrak{n}L_r \ \asymp \ \mathfrak{n}\mathfrak{v}L \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v}\widetilde{L}.$$

Since $\mathfrak{v}(\widetilde{L})\widetilde{L} \asymp \widetilde{L}_r$, this gives $\mathfrak{v}(\widetilde{L})\widetilde{L} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}\widetilde{L}$, and thus $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. $\square$

If $(P, \mathfrak{m}, \widehat{a})$ is steep and linear, then $L_{P_{+a,\times\mathfrak{m}}} = L_{P_{\times\mathfrak{m},+(a/\mathfrak{m})}} = L_{P_{\times\mathfrak{m}}}$, so any refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is also steep and linear.

Lemma 3.3.2. *Suppose* $\operatorname{order} L_{P_{\times\mathfrak{m}}} = r$*. Then* $(P, \mathfrak{m}, \widehat{a})$ *has a refinement* $(P, \mathfrak{n}, \widehat{a})$ *such that* $\operatorname{nwt} L_{P_{\times\mathfrak{n}}} = 0$*, and* $(P^\phi, \mathfrak{n}, \widehat{a})$ *is steep, eventually.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ we arrange $\mathfrak{m} = 1$. Take $\mathfrak{n}_1$ with $\widehat{a} \prec \mathfrak{n}_1 \prec 1$. Then $\operatorname{order}(P_1)_{\times\mathfrak{n}_1} = \operatorname{order} P_1 = \operatorname{order} L_P = r$, and thus $(P_1)_{\times\mathfrak{n}_1} \neq 0$. So [ADH, 11.3.6] applied to $(P_1)_{\times\mathfrak{n}_1}$ in place of P yields an $\mathfrak{n}$ with $\mathfrak{n}_1 \prec \mathfrak{n} \prec 1$ and $\operatorname{nwt}(P_1)_{\times\mathfrak{n}} = 0$, so $\operatorname{nwt} L_{P_{\times\mathfrak{n}}} = 0$. Hence by Lemma 3.1.20, $(P^\phi, \mathfrak{n}, \widehat{a})$ is steep, eventually. $\square$

Recall that the separant $S_P = \partial P/\partial Y^{(r)}$ of P has lower complexity than P. Below we sometimes use the identity $S_{P^\phi_{\times\mathfrak{m}}} = \phi^r (S_{P_{\times\mathfrak{m}}})^\phi$ from Lemma 1.1.10.

The slot $(P, \mathfrak{m}, \widehat{a})$ in K is said to be **deep** if it is steep and for all active $\phi \preccurlyeq 1$,

- (D1) $\operatorname{ddeg} S_{P^\phi_{\times\mathfrak{m}}} = 0$ (hence $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$), and
- (D2) $\operatorname{ddeg} P^\phi_{\times\mathfrak{m}} = 1$ (hence $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$).

If $\deg P = 1$, then (D1) is automatic, for all active $\phi \preccurlyeq 1$. If $(P, \mathfrak{m}, \widehat{a})$ is deep, then so are $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$, as well as every slot in K equivalent to $(P, \mathfrak{m}, \widehat{a})$ and the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ for active $\phi \preccurlyeq 1$. Every deep slot in K is quasilinear, by (D2). If $\deg P = 1$, then $(P, \mathfrak{m}, \widehat{a})$ is quasilinear iff $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep for some active $\phi \preccurlyeq 1$. Moreover, if $(P, \mathfrak{m}, \widehat{a})$ is a deep hole in K, then $\operatorname{dmul} P^\phi_{\times\mathfrak{m}} = 1$ for all active $\phi \preccurlyeq 1$, by (D2) and Lemma 3.2.9.

Example 3.3.3. Suppose $P = Y' + gY - u$ where $g, u \in K$ and $\mathfrak{m} = 1$, $r = 1$. Set $L := L_P = \partial + g$ and $\mathfrak{v} := \mathfrak{v}(L)$. Then $\mathfrak{v} = 1$ if $g \preccurlyeq 1$, and $\mathfrak{v} = 1/g$ if $g \succ 1$. Thus

$$(P, 1, \widehat{a}) \text{ is steep} \quad \Longleftrightarrow \quad g \succ^{\flat} 1 \quad \Longleftrightarrow \quad g \succ 1 \text{ and } g^{\dagger} \succcurlyeq 1.$$

Note that $(P, 1, \widehat{a})$ is steep iff L is steep as defined in Section 1.5. Also,

$$(P, 1, \widehat{a}) \text{ is deep} \quad \Longleftrightarrow \quad (P, 1, \widehat{a}) \text{ is steep and } g \succcurlyeq u.$$

Hence if $u = 0$, then $(P, 1, \widehat{a})$ is deep iff it is steep.

Lemma 3.3.4. *For steep $(P, \mathfrak{m}, \widehat{a})$, the following are equivalent:*

(i) $(P^{\phi}, \mathfrak{m}, \widehat{a})$ *is deep, eventually;*

(ii) $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$ *and* $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$.

Note that if $\operatorname{ddeg} S_{P_{\times\mathfrak{m}}} = 0$ or $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$, then $S_{P_{\times\mathfrak{m}}}(0) \neq 0$, so $\operatorname{order} L_{P_{\times\mathfrak{m}}} = r$.

Lemma 3.3.5. *Suppose $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ refines the hole $(P, \mathfrak{m}, \widehat{a})$ in K. Then:*

(i) $\operatorname{ddeg} S_{P_{\times\mathfrak{m}}} = 0 \implies \operatorname{ddeg} S_{P_{+a,\times\mathfrak{n}}} = 0$;

(ii) $\operatorname{ddeg} P_{\times\mathfrak{m}} = 1 \implies \operatorname{ddeg} P_{+a,\times\mathfrak{n}} = 1$;

(iii) $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0 \implies S_P(a) \sim S_P(0)$.

Thus if $(P, \mathfrak{m}, \widehat{a})$ is deep and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is steep, then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep.

Proof. Suppose $\operatorname{ddeg} S_{P_{\times\mathfrak{m}}} = 0$. Then $\operatorname{ddeg} S_{P_{+a,\times\mathfrak{n}}} = 0$ follows from

$$\operatorname{ddeg} S_{P_{+a,\times\mathfrak{n}}} \;=\; \operatorname{ddeg} (S_P)_{+a,\times\mathfrak{n}} \quad \text{and} \quad \operatorname{ddeg}(S_P)_{\times\mathfrak{m}} \;=\; \operatorname{ddeg} S_{P_{\times\mathfrak{m}}}$$

(consequences of Lemma 1.1.10), and

$$\operatorname{ddeg} (S_P)_{+a,\times\mathfrak{n}} \;=\; \operatorname{ddeg} (S_P)_{+\widehat{a},\times\mathfrak{n}} \;\leqslant\; \operatorname{ddeg} (S_P)_{+\widehat{a},\times\mathfrak{m}} \;=\; \operatorname{ddeg} (S_P)_{\times\mathfrak{m}}$$

which holds by [ADH, 6.6.7]. This proves (i). Corollary 3.2.20 yields (ii), and (iii) is contained in Lemma 3.2.33. □

Lemmas 3.2.14 and 3.3.5 give:

Corollary 3.3.6. *If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, then each steep refinement of $(P, \mathfrak{m}, \widehat{a})$ is deep.*

Here is another sufficient condition on refinements of deep holes to remain deep:

Lemma 3.3.7. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a deep hole in K, and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ refines $(P, \mathfrak{m}, \widehat{a})$ with $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$. Then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{n}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$.*

Proof. From $(P, \mathfrak{m}, \widehat{a})$ we pass to the hole $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ and then to $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$. We first show that $\operatorname{order} L_{P_{+a,\times\mathfrak{m}}} = r$ and $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \sim \mathfrak{v}$, from which it follows that $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is steep, hence deep by Lemma 3.3.5. By Corollary 3.2.20,

$$\operatorname{ddeg} P_{+a,\times\mathfrak{m}} \;=\; \operatorname{dmul} P_{+a,\times\mathfrak{m}} \;=\; 1,$$

so $(P_{+a,\times\mathfrak{m}})_1 \sim P_{+a,\times\mathfrak{m}}$. Also

$$(P_{\times\mathfrak{m}})_1 \sim P_{\times\mathfrak{m}} \sim P_{\times\mathfrak{m},+(a/\mathfrak{m})} = P_{+a,\times\mathfrak{m}},$$

by [ADH, 4.5.1(i)], and thus $(P_{+a,\times\mathfrak{m}})_1 \sim (P_{\times\mathfrak{m}})_1$. By Lemmas 1.1.10 and 3.3.5(iii),

$$S_{P_{+a,\times\mathfrak{m}}}(0) \;=\; \mathfrak{m} S_P(a) \;\sim\; \mathfrak{m} S_P(0) \;=\; S_{P_{\times\mathfrak{m}}}(0),$$

so $S_{P_{+a,\times\mathfrak{m}}}(0) \sim S_{P_{\times\mathfrak{m}}}(0)$. This gives $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \sim \mathfrak{v}$ as promised.

Next, Lemma 3.3.1 applied to $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ in the role of $(P, \mathfrak{m}, \widehat{a})$ gives that $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is steep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{n}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Now Lemma 3.3.5 applied to $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ in the role of $(P, \mathfrak{m}, \widehat{a})$ and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$, respectively, gives that $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep. $\square$

Lemmas 3.2.14 and 3.3.7 give a version for Z-minimal slots:

Corollary 3.3.8. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ refines $(P, \mathfrak{m}, \widehat{a})$ with $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$, where $\mathfrak{v} := \mathfrak{v}(L_{P_{\times \mathfrak{m}}})$. Then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep with $\mathfrak{v}(L_{P_{+a, \times \mathfrak{n}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$.*

Next we turn to the task of turning Z-minimal slots into deep ones.

Lemma 3.3.9. *Every quasilinear Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that:*

(i) $\operatorname{ndeg} (P_{(\boldsymbol{i})})_{\times \mathfrak{m}} = 0$ *for all $\boldsymbol{i}$ with $|\boldsymbol{i}| \geqslant 1$ and $P_{(\boldsymbol{i})} \neq 0$;*
(ii) $\operatorname{ndeg} P_{\times \mathfrak{m}} = \operatorname{nmul} P_{\times \mathfrak{m}} = 1$, *and*
(iii) $\operatorname{nwt} L_{P_{\times \mathfrak{m}}} = 0$.

Proof. By Corollary 3.2.22, any quasilinear $(P, \mathfrak{m}, \widehat{a})$ satisfies (ii). Any refinement of a quasilinear $(P, \mathfrak{m}, \widehat{a})$ remains quasilinear, by Corollary 3.2.23. By Lemma 3.2.32 and a subsequent remark any quasilinear Z-minimal slot in K of order r can be refined to a quasilinear $(P, \mathfrak{m}, \widehat{a})$ that satisfies (i), and by Lemma 3.2.33, any further refinement of such $(P, \mathfrak{m}, \widehat{a})$ continues to satisfy (i). Thus to prove the lemma, assume we are given a quasilinear $(P, \mathfrak{m}, \widehat{a})$ satisfying (i); it is enough to show that then $(P, \mathfrak{m}, \widehat{a})$ has a refinement $(P, \mathfrak{n}, \widehat{a})$ satisfying (iii) with $\mathfrak{n}$ instead of $\mathfrak{m}$ (and thus also (i) and (ii) with $\mathfrak{n}$ instead of $\mathfrak{m}$).

Take $\widetilde{\mathfrak{m}}$ with $\widehat{a} \prec \widetilde{\mathfrak{m}} \prec \mathfrak{m}$. Then $(P_{\times \widetilde{\mathfrak{m}}})_1 \neq 0$ by (ii), so [ADH, 11.3.6] applied to $(P_1)_{\times \widetilde{\mathfrak{m}}}$ in place of P yields an $\mathfrak{n}$ with $\widetilde{\mathfrak{m}} \prec \mathfrak{n} \prec \mathfrak{m}$ and $\operatorname{nwt} (P_1)_{\times \mathfrak{n}} = 0$. Hence the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ satisfies (iii) with $\mathfrak{n}$ instead of $\mathfrak{m}$. $\square$

Corollary 3.3.10. *Every quasilinear Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $\operatorname{nwt} L_{P_{\times \mathfrak{m}}} = 0$, and $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep, eventually.*

Proof. Given a quasilinear Z-minimal slot in K of order r, we take a refinement $(P, \mathfrak{m}, \widehat{a})$ as in Lemma 3.3.9. Then $\operatorname{ndeg} S_{P_{\times \mathfrak{m}}} = 0$ by (i) of that lemma, so $\operatorname{order} L_{P_{\times \mathfrak{m}}} = r$ by the remark that precedes Lemma 3.3.5. Then (iii) of Lemma 3.3.9 and Lemma 3.1.20 give that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is steep, eventually. Using now $\operatorname{ndeg} S_{P_{\times \mathfrak{m}}} = 0$ and (ii) of Lemma 3.3.9 we obtain from Lemma 3.3.4 that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep, eventually. $\square$

Lemma 3.2.26 and the previous lemma and its corollary now yield:

Lemma 3.3.11. *Suppose K is* d*-valued and $\boldsymbol{\omega}$-free, and Γ is divisible. Then every Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ satisfying* (i)–(iii) *in Lemma 3.3.9.*

Corollary 3.3.12. *Suppose K is* d*-valued and $\boldsymbol{\omega}$-free, and Γ is divisible. Then every Z-minimal slot in K of order r has a quasilinear refinement $(P, \mathfrak{m}, \widehat{a})$ such that $\operatorname{nwt} L_{P_{\times \mathfrak{m}}} = 0$, and $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep, eventually.*

Approximating Z-minimal slots. In this subsection we set, as before,

$$\mathfrak{v} \ := \ \mathfrak{v}(L_{P_{\times \mathfrak{m}}}),$$

provided $L_{P_{\times \mathfrak{m}}}$ has order r. The next lemma is a key approximation result.

Lemma 3.3.13. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and steep, and*

$$\operatorname{ddeg} P_{\times\mathfrak{m}} \ =\ \operatorname{ndeg} P_{\times\mathfrak{m}} \ =\ 1, \qquad \operatorname{ddeg} S_{P_{\times\mathfrak{m}}} \ =\ 0.$$

Then there exists an a such that $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$.

Proof. We can arrange $\mathfrak{m} = 1$ and $P \asymp 1$. Then $\operatorname{ddeg} P = 1$ gives $P_1 \asymp 1$, so $S_P(0) \asymp \mathfrak{v}$. Take $Q, R_1, \dots, R_n \in K\{Y\}$ $(n \geqslant 1)$ of order $< r$ such that

$$P \ =\ Q + R_1 Y^{(r)} + \cdots + R_n (Y^{(r)})^n, \qquad S_P \ =\ R_1 + \cdots + nR_n (Y^{(r)})^{n-1}.$$

Then $R_1(0) = S_P(0) \asymp \mathfrak{v}$. As $\operatorname{ddeg} S_P = 0$, this gives $S_P \sim R_1(0)$, hence

$$R \ :=\ P - Q \ \sim\ R_1(0) Y^{(r)} \ \asymp\ \mathfrak{v} \ \prec_{\Delta(\mathfrak{v})}\ 1 \ \asymp\ P,$$

so $P \sim_{\Delta(\mathfrak{v})} Q$. Thus $Q \neq 0$, and $Q \notin Z(K, \widehat{a})$ because $\operatorname{order} Q < r$. Now Lemma 3.2.18 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, 1, \widehat{a})$ such that $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}} = 0$ and $\mathfrak{n} \prec 1$. We claim that then $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} 1$. (Establishing this claim finishes the proof.) Suppose the claim is false. Then $\widehat{a} - a \asymp_{\Delta(\mathfrak{v})} 1$, so $\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} 1$, hence $Q_{+a,\times\mathfrak{n}} \asymp_{\Delta(\mathfrak{v})} Q_{+a} \asymp Q$ by [ADH, 4.5.1]. Likewise, $R_{+a,\times\mathfrak{n}} \asymp_{\Delta(\mathfrak{v})} R$. Using $P_{+a,\times\mathfrak{n}} = Q_{+a,\times\mathfrak{n}} + R_{+a,\times\mathfrak{n}}$ gives $Q_{+a,\times\mathfrak{n}} \sim_{\Delta(\mathfrak{v})} P_{+a,\times\mathfrak{n}}$, so $Q_{+a,\times\mathfrak{n}} \sim^\flat P_{+a,\times\mathfrak{n}}$. Then $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}} = \operatorname{ndeg} P_{+a,\times\mathfrak{n}} = 1$ by Lemma 1.8.2 and Corollary 3.2.23, a contradiction. $\square$

Lemmas 3.2.9 and 3.3.13, and a remark following the definition of *deep* give:

Corollary 3.3.14. *If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, steep, and linear, then there exists an a such that $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$.*

Corollary 3.3.15. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and special. Then for all $n \geqslant 1$ there is an a with $\widehat{a} - a \prec \mathfrak{v}^n \mathfrak{m}$.*

Proof. We arrange $\mathfrak{m} = 1$ in the usual way. Let Δ be the convex subgroup of Γ that is cofinal in $v(\widehat{a} - K)$. Lemma 3.3.13 gives an element $\gamma \in v(\widehat{a} - K)$ with $\gamma \geqslant \delta/m$ for some $m \geqslant 1$. Hence $v(\widehat{a} - K)$ contains for every $n \geqslant 1$ an element $> n\delta$. $\square$

Combining Lemma 3.2.36 with Corollary 3.3.15 yields:

Corollary 3.3.16. *If K is r-linearly newtonian, $\upomega$-free if $r > 1$, and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, then for all $n \geqslant 1$ there is an a such that $\widehat{a} - a \prec \mathfrak{v}^n \mathfrak{m}$.*

Normal slots. We say that our slot $(P, \mathfrak{m}, \widehat{a})$ in K, with linear part L, is **normal** if $\operatorname{order} L = r$ and, with $\mathfrak{v} := \mathfrak{v}(L)$ and $w := \operatorname{wt}(P)$,

(N1) $\mathfrak{v} \prec^\flat 1$;
(N2) $(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times\mathfrak{m}})_1$.

Note that then $\mathfrak{v} \prec 1$, $\operatorname{dwt}(L) < r$, $(P, \mathfrak{m}, \widehat{a})$ is steep, and

$$P_{\times\mathfrak{m}} \sim_{\Delta(\mathfrak{v})} P(0) + (P_{\times\mathfrak{m}})_1 \qquad (\text{so } \operatorname{ddeg} P_{\times\mathfrak{m}} \leqslant 1). \tag{3.3.1}$$

If $\operatorname{order} L = r$, $\mathfrak{v} := \mathfrak{v}(L)$, and L is monic, then $(P_{\times\mathfrak{m}})_1 \asymp \mathfrak{v}^{-1}$, so that (N2) is then equivalent to: $(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^w$. If $\deg P = 1$, then $\operatorname{order} L = r$ and (N2) automatically holds, hence $(P, \mathfrak{m}, \widehat{a})$ is normal iff it is steep. Thus by Lemma 3.1.20:

Lemma 3.3.17. *If $\deg P = 1$ and $\operatorname{nwt}(L) < r$, then $(P^\phi, \mathfrak{m}, \widehat{a})$ is normal, eventually.*

If $(P, \mathfrak{m}, \widehat{a})$ is normal, then so are $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$. In particular, $(P, \mathfrak{m}, \widehat{a})$ is normal iff $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ is normal. If $(P, \mathfrak{m}, \widehat{a})$ is normal, then so is any equivalent slot. Hence by (3.3.1) and Lemmas 3.2.9 and 3.2.14:

Lemma 3.3.18. *If* $(P, \mathfrak{m}, \widehat{a})$ *is normal, and* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal or is a hole in* K*, then* $\operatorname{ddeg} P_{\times\mathfrak{m}} = \operatorname{dmul} P_{\times\mathfrak{m}} = 1$.

Example. Let $K \supseteq \mathbb{R}(\mathrm{e}^x)$ be an H-subfield of $\mathbb{T}$, $\mathfrak{m} = 1$, $r = 2$. If $P = D + R$ where

$$D \ = \ \mathrm{e}^{-x}\, Y'' - Y, \qquad R \ = \ f + \mathrm{e}^{-4x}\, Y^5 \quad (f \in K),$$

then $\mathfrak{v} = -\,\mathrm{e}^{-x} \prec^{\flat} 1$, $P_1 = D \sim -Y$, $w = 2$, and $P_{>1} = \mathrm{e}^{-4x}\, Y^5 \prec_{\Delta(\mathfrak{v})} \mathrm{e}^{-3x}\, P_1$, so $(P, 1, \widehat{a})$ is normal. However, if $P = D + S$ with D as above and $S = f + \mathrm{e}^{-3x}\, Y^5$ $(f \in K)$, then $P_{>1} = \mathrm{e}^{-3x}\, Y^5 \succcurlyeq_{\Delta(\mathfrak{v})} \mathrm{e}^{-3x}\, P_1$, so $(P, 1, \widehat{a})$ is not normal.

Lemma 3.3.19. *Suppose* $\operatorname{order}(L) = r$ *and* $\mathfrak{v}$ *is such that* (N1) *and* (N2) *hold, and* $\mathfrak{v}(L) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$*. Then* $(P, \mathfrak{m}, \widehat{a})$ *is normal.*

Proof. Put $\mathfrak{w} := \mathfrak{v}(L)$. Then $[\mathfrak{w}] = [\mathfrak{v}]$, and so $\mathfrak{v} \prec^{\flat} 1$ gives $\mathfrak{w} \prec^{\flat} 1$. Also,

$$(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times\mathfrak{m}})_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}^{w+1} (P_{\times\mathfrak{m}})_1.$$

Hence (N1), (N2) hold with $\mathfrak{w}$ in place of $\mathfrak{v}$. □

Lemma 3.3.20. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal and* $\phi \preccurlyeq 1$ *is active. Then the slot* $(P^{\phi}, \mathfrak{m}, \widehat{a})$ *in* K^{ϕ} *is normal.*

Proof. We arrange $\mathfrak{m} = 1$ and put $\mathfrak{v} := \mathfrak{v}(L)$, $\mathfrak{w} := \mathfrak{v}(L_{P^{\phi}})$. Now $L_{P^{\phi}} = L^{\phi}$, so $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}$ and $\mathfrak{v} \prec^{\flat}_{\phi} 1$ by Lemma 3.1.19. By [ADH, 11.1.1], $[\phi] < [\mathfrak{v}]$, and (N2) we have

$$(P^{\phi})_{>1} \ = \ (P_{>1})^{\phi} \ \asymp_{\Delta(\mathfrak{v})} \ P_{>1} \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1} P_1 \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1} P_1^{\phi},$$

which by Lemma 3.3.19 applied to $(P^{\phi}, 1, \widehat{a})$ in the role of $(P, \mathfrak{m}, \widehat{a})$ gives that $(P^{\phi}, 1, \widehat{a})$ is normal. □

Corollary 3.3.21. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal. Then* $(P, \mathfrak{m}, \widehat{a})$ *is quasilinear.*

Proof. Lemma 3.2.21 gives $\operatorname{ndeg} P_{\times\mathfrak{m}} \geqslant 1$. The parenthetical remark after (3.3.1) above and Lemma 3.3.20 gives $\operatorname{ndeg} P_{\times\mathfrak{m}} \leqslant 1$. □

Combining Lemmas 3.3.18 and 3.3.20 yields:

Corollary 3.3.22. *If* $(P, \mathfrak{m}, \widehat{a})$ *is normal and linear, and* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal or a hole in* K*, then* $(P, \mathfrak{m}, \widehat{a})$ *is deep.*

There are a few occasions later where we need to change the "monomial" $\mathfrak{m}$ in $(P, \mathfrak{m}, \widehat{a})$ while preserving key properties of this slot. Here is what we need:

Lemma 3.3.23. *Let* $u \in K$, $u \asymp 1$*. Then* $(P, u\mathfrak{m}, \widehat{a})$ *refines* $(P, \mathfrak{m}, \widehat{a})$*, and if* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *refines* $(P, \mathfrak{m}, \widehat{a})$*, then so does* $(P_{+a}, u\mathfrak{n}, \widehat{a} - a)$*. If* $(P, \mathfrak{m}, \widehat{a})$ *is quasilinear, respectively deep, respectively normal, then so is* $(P, u\mathfrak{m}, \widehat{a})$*.*

Proof. The refinement claims are clearly true, and quasilinearity is preserved since $\operatorname{ndeg} P_{\times u\mathfrak{m}} = \operatorname{ndeg} P_{\times\mathfrak{m}}$ by [ADH, 11.2.3(iii)]. "Steep" is preserved by Lemma 3.3.1, and hence "deep" is preserved using Lemma 1.1.10 and [ADH, 6.6.5(ii)]. Normality is preserved because steepness is,

$$(P_{\times u\mathfrak{m}})_d \ = \ (P_d)_{\times u\mathfrak{m}} \ \asymp \ (P_d)_{\times\mathfrak{m}} \ = \ (P_{\times\mathfrak{m}})_d \quad \text{for all } d \in \mathbb{N}$$

by [ADH, 4.3, 4.5.1(ii)], and $\mathfrak{v}(L_{P_{\times u\mathfrak{m}}}) \asymp \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$ by Lemma 3.1.2. □

Here is a useful invariance property of normal slots:

Lemma 3.3.24. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is normal and $a \prec \mathfrak{m}$. Then L_P and $L_{P_{+a}}$ have order r. If in addition K is $\uplambda$-free or $r = 1$, then $\mathscr{E}^{\mathrm{e}}(L_P) = \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$.*

Proof. $L_{P_{\times\mathfrak{m}}} = L_P\mathfrak{m}$ (so L_P has order r), and $L_{P_{+a,\times\mathfrak{m}}} = L_{P_{\times\mathfrak{m},+a/\mathfrak{m}}} = L_{P_{+a}}\mathfrak{m}$. The slot $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ in K is normal and $a/\mathfrak{m} \prec 1$. Thus we can apply Lemma 3.1.27(i) to $\widehat{K}$, $P_{\times\mathfrak{m}}$, $a/\mathfrak{m}$ in place of K, P, a to give order $L_{P_{+a}} = r$. Next, applying likewise Lemma 3.1.28 with $L := L_{P_{\times\mathfrak{m}}}$, $\mathfrak{v} := \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$, $m = r$, $B = 0$, gives

$$L_P\mathfrak{m} - L_{P_{+a}}\mathfrak{m} \;=\; L_{P_{\times\mathfrak{m}}} - L_{P_{\times\mathfrak{m},+a/\mathfrak{m}}} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{r+1}L_P\mathfrak{m}.$$

Hence, if K is $\uplambda$-free, then $\mathscr{E}^{\mathrm{e}}(L_P\mathfrak{m}) = \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}\mathfrak{m})$ by Lemma 3.1.22, so

$$\mathscr{E}^{\mathrm{e}}(L_P) \;=\; \mathscr{E}^{\mathrm{e}}(L_P\mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}\mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}).$$

If $r = 1$ we obtain the same equality from Corollary 3.1.23. □

Normality under refinements. In this subsection we study how normality behaves under more general refinements. This is not needed to prove the main result of this section, Theorem 3.3.33, but is included to obtain useful variants of it.

Proposition 3.3.25. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is normal. Let a refinement $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ be given. Then this refinement is also normal.*

Proof. By the remarks following the definition of "multiplicative conjugate" in Section 3.2 and after replacing the slots $(P, \mathfrak{m}, \widehat{a})$ and $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ in K by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ and $\big(P_{\times\mathfrak{m},+a/\mathfrak{m}}, 1, (\widehat{a} - a)/\mathfrak{m}\big)$, respectively, we arrange that $\mathfrak{m} = 1$. Let $\mathfrak{v} := \mathfrak{v}(L_P)$. By Lemma 3.1.27 we have order$(L_{P_{+a}}) = r$, $\mathfrak{v}(L_{P_{+a}}) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}$, and $(P_{+a})_1 \sim_{\Delta(\mathfrak{v})} P_1$. Using [ADH, 4.5.1(i)] we have for $d > 1$ with $P_d \neq 0$,

$$(P_{+a})_d \;=\; \big((P_{\geqslant d})_{+a}\big)_d \;\preccurlyeq\; (P_{\geqslant d})_{+a} \;\sim\; P_{\geqslant d} \;\preccurlyeq\; P_{>1},$$

and using (N2), this yields

$$(P_{+a})_{>1} \;\preccurlyeq\; P_{>1} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1}P_1 \;\asymp\; \mathfrak{v}^{w+1}(P_{+a})_1.$$

Hence (N2) holds with $\mathfrak{m} = 1$ and with P replaced by P_{+a}. Thus $(P_{+a}, 1, \widehat{a} - a)$ is normal, by Lemma 3.3.19. □

Proposition 3.3.26. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a normal hole in K, $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}] \leqslant \big[\mathfrak{v}(L_{P_{\times\mathfrak{m}}})\big]$. Then the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ is also normal.*

Proof. As in the proof of Lemma 3.3.1 we arrange $\mathfrak{m} = 1$ and set $L := L_P$, $\mathfrak{v} := \mathfrak{v}(L)$, and $\widetilde{L} := L_{P_{\times\mathfrak{n}}}$, to obtain $[\mathfrak{n}] \leqslant [\mathfrak{v}]$ and $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Recall from [ADH, 4.3] that $(P_{\times\mathfrak{n}})_d = (P_d)_{\times\mathfrak{n}}$ for $d \in \mathbb{N}$. For such d we have by [ADH, 6.1.3],

$$(P_d)_{\times\mathfrak{n}} \;\asymp_{\Delta(\mathfrak{v})}\; \mathfrak{n}^d P_d \;\preccurlyeq\; \mathfrak{n}^d P_{\geqslant d}.$$

In particular, $(P_{\times\mathfrak{n}})_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{n}P_1$. By (N2) we also have, for $d > 1$:

$$P_{\geqslant d} \;\preccurlyeq\; P_{>1} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1}P_1.$$

By Lemma 3.3.18 we have $P \sim P_1$. For $d > 1$ we have by [ADH, 6.1.3],

$$\mathfrak{n}^d P \;\asymp\; \mathfrak{n}^d P_1 \;\asymp_{\Delta(\mathfrak{v})}\; \mathfrak{n}^{d-1}(P_1)_{\times\mathfrak{n}} \;\preccurlyeq\; (P_1)_{\times\mathfrak{n}} \;=\; (P_{\times\mathfrak{n}})_1 \;\preccurlyeq\; P_{\times\mathfrak{n}}$$

and thus

$$(P_{\times\mathfrak{n}})_d \;=\; (P_d)_{\times\mathfrak{n}} \;\preccurlyeq_{\Delta(\mathfrak{v})}\; \mathfrak{n}^d P_{\geqslant d} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1}\mathfrak{n}^d P_1 \;\preccurlyeq_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1}(P_{\times\mathfrak{n}})_1.$$

Hence (N2) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Thus $(P, \mathfrak{n}, \widehat{a})$ is normal, using $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ and Lemmas 3.3.1 and 3.3.19. □

From Lemma 3.2.14 and Proposition 3.3.26 we obtain:

Corollary 3.3.27. *Suppose $(P,\mathfrak{m},\widehat{a})$ is normal and Z-minimal, $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}]\leqslant\big[\mathfrak{v}(L_{P_{\times\mathfrak{m}}})\big]$. Then the refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ is also normal.*

In the rest of this subsection $\mathfrak{m}=1$, $\widehat{a}\prec\mathfrak{n}\prec 1$, $\operatorname{order}(L_P)=r$, *and* $[\mathfrak{v}]<[\mathfrak{n}]$ *where* $\mathfrak{v}:=\mathfrak{v}(L_P)$. So $(P,\mathfrak{n},\widehat{a})$ refines $(P,1,\widehat{a})$, $L_{P_{\times\mathfrak{n}}}=L_P\mathfrak{n}$, and $\operatorname{order}L_{P_{\times\mathfrak{n}}}=r$.

Lemma 3.3.28. *Suppose $(P,1,\widehat{a})$ is steep, $\mathfrak{v}(L_{P_{\times\mathfrak{n}}})\preccurlyeq\mathfrak{v}$, and $P_{>1}\preccurlyeq P_1$. Then $(P,\mathfrak{n},\widehat{a})$ is normal.*

Proof. Put $\mathfrak{w}:=\mathfrak{v}(L_{P_{\times\mathfrak{n}}})$. Then $[\mathfrak{w}]<[\mathfrak{n}]$ by Corollary 3.1.10, and $\mathfrak{w}\preccurlyeq\mathfrak{v}\prec^\flat 1$ gives $\mathfrak{w}\prec^\flat 1$. It remains to show that $(P_{\times\mathfrak{n}})_{>1}\prec_{\Delta(\mathfrak{w})}\mathfrak{w}^{w+1}(P_{\times\mathfrak{n}})_1$. Using $[\mathfrak{n}]>[\mathfrak{w}]$ it is enough that $(P_{\times\mathfrak{n}})_{>1}\prec_\Delta\mathfrak{w}^{w+1}(P_{\times\mathfrak{n}})_1$, where $\Delta:=\Delta(\mathfrak{n})$. Since $\mathfrak{w}\asymp_\Delta 1$, it is even enough that $(P_{\times\mathfrak{n}})_{>1}\prec_\Delta(P_{\times\mathfrak{n}})_1$, to be derived below. Let $d>1$. Then by [ADH, 6.1.3] and $P_d\preccurlyeq P_{>1}\preccurlyeq P_1$ we have

$$(P_{\times\mathfrak{n}})_d\ =\ (P_d)_{\times\mathfrak{n}}\ \asymp_\Delta\ P_d\mathfrak{n}^d\ \preccurlyeq\ P_1\,\mathfrak{n}^d.$$

In view of $\mathfrak{n}\prec_\Delta 1$ and $d>1$ we have

$$P_1\,\mathfrak{n}^d\ \prec_\Delta\ P_1\,\mathfrak{n}\ \asymp_\Delta\ (P_1)_{\times\mathfrak{n}}\ =\ (P_{\times\mathfrak{n}})_1,$$

using again [ADH, 6.1.3]. Thus $(P_{\times\mathfrak{n}})_d\prec_\Delta(P_{\times\mathfrak{n}})_1$, as promised. □

Corollary 3.3.29. *If $(P,1,\widehat{a})$ is normal and $\mathfrak{v}(L_{P_{\times\mathfrak{n}}})\preccurlyeq\mathfrak{v}$, then $(P,\mathfrak{n},\widehat{a})$ is normal.*

In the next lemma and its corollary K is d-valued and for every $q\in\mathbb{Q}^>$ there is given an element $\mathfrak{n}^q$ of $K^\times$ such that $(\mathfrak{n}^q)^\dagger=q\mathfrak{n}^\dagger$; the remark before Lemma 3.1.15 gives $v(\mathfrak{n}^q)=qv(\mathfrak{n})$ for $q\in\mathbb{Q}^>$. Hence for $0<q\leqslant 1$ in $\mathbb{Q}$ we have $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{n}^q\prec 1$, so $(P,\mathfrak{n}^q,\widehat{a})$ refines $(P,1,\widehat{a})$.

Lemma 3.3.30. *Suppose $(P,1,\widehat{a})$ is steep and $P_{>1}\preccurlyeq P_1$. Then $(P,\mathfrak{n}^q,\widehat{a})$ is normal, for all but finitely many $q\in\mathbb{Q}$ with $0<q\leqslant 1$.*

Proof. We have $\mathfrak{n}^\dagger\succcurlyeq 1$ by $\mathfrak{n}\prec\mathfrak{v}\prec 1$ and $\mathfrak{v}^\dagger\succcurlyeq 1$. Hence Lemma 3.1.16 gives $\mathfrak{v}(L_{P_{\times\mathfrak{n}^q}})\preccurlyeq\mathfrak{v}$ for all but finitely many $q\in\mathbb{Q}^>$. Suppose $\mathfrak{v}(L_{P_{\times\mathfrak{n}^q}})\preccurlyeq\mathfrak{v}$, $0<q\leqslant 1$ in $\mathbb{Q}$. Then $(P,\mathfrak{n}^q,\widehat{a})$ is normal by Lemma 3.3.28 applied with $\mathfrak{n}^q$ instead of $\mathfrak{n}$. □

Corollary 3.3.31. *If $(P,1,\widehat{a})$ is normal, then $(P,\mathfrak{n}^q,\widehat{a})$ is normal for all but finitely many $q\in\mathbb{Q}$ with $0<q\leqslant 1$.*

Normalizing. If in this subsection $\operatorname{order}(L_{P_{\times\mathfrak{m}}})=r$, then $\mathfrak{v}:=\mathfrak{v}(L_{P_{\times\mathfrak{m}}})$. Towards proving that normality can always be achieved we first show:

Lemma 3.3.32. *Suppose Γ is divisible, $(P,\mathfrak{m},\widehat{a})$ is a deep hole in K, and $\widehat{a}-a\prec\mathfrak{v}^{w+2}\mathfrak{m}$ for some a. Then $(P,\mathfrak{m},\widehat{a})$ has a refinement that is deep and normal.*

Proof. Replacing $(P,\mathfrak{m},\widehat{a})$ by $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$ and renaming we arrange $\mathfrak{m}=1$. Take a such that $\widehat{a}-a\prec\mathfrak{v}^{w+2}$. For $e:=w+\frac{3}{2}$, let $\mathfrak{v}^e$ be an element of $K^\times$ with $v(\mathfrak{v}^e)=e\,v(\mathfrak{v})$. *Claim*: the refinement $(P_{+a},\mathfrak{v}^e,\widehat{a}-a)$ of $(P,1,\widehat{a})$ is deep and normal. By Lemma 3.3.7, $(P_{+a},\mathfrak{v}^e,\widehat{a}-a)$ is deep, so we do have $\operatorname{order}(L_{P_{+a,\times\mathfrak{v}^e}})=r$ and $\mathfrak{v}(L_{P_{+a,\times\mathfrak{v}^e}})\prec^\flat 1$. Lemma 3.3.7 also yields $\mathfrak{v}(L_{P_{+a,\times\mathfrak{v}^e}})\asymp_{\Delta(\mathfrak{v})}\mathfrak{v}$. Since $\operatorname{ddeg}P=\operatorname{dmul}P=1$, we can use Corollary 3.2.20 for $\mathfrak{n}=\mathfrak{v}^e$ and for $\mathfrak{n}=1$ to obtain

$$\operatorname{ddeg}P_{+a,\times\mathfrak{v}^e}\ =\ \operatorname{dmul}P_{+a,\times\mathfrak{v}^e}\ =\ \operatorname{ddeg}P_{+a}\ =\ \operatorname{dmul}P_{+a}=1$$

and thus $(P_{+a,\times \mathfrak{v}^e})_1 \sim P_{+a,\times \mathfrak{v}^e}$; also $P_1 \sim P \sim P_{+a} \sim (P_{+a})_1$, where $P \sim P_{+a}$ follows from $a \prec 1$ and [ADH, 4.5.1(i)]. Now let $d > 1$. Then

$$\begin{aligned}(P_{+a,\times \mathfrak{v}^e})_d \ &\asymp_{\Delta(\mathfrak{v})} \ (\mathfrak{v}^e)^d (P_{+a})_d \ \preccurlyeq \ (\mathfrak{v}^e)^d P_{+a} \ \sim \ (\mathfrak{v}^e)^d (P_{+a})_1 \\ &\asymp_{\Delta(\mathfrak{v})} \ (\mathfrak{v}^e)^{d-1} (P_{+a,\times \mathfrak{v}^e})_1 \ \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{+a,\times \mathfrak{v}^e})_1,\end{aligned}$$

using [ADH, 6.1.3] for $\asymp_{\Delta(\mathfrak{v})}$. So $(P_{+a}, \mathfrak{v}^e, \widehat{a} - a)$ is normal by Lemma 3.3.19. $\square$

We can now finally show:

Theorem 3.3.33. *Suppose K is $\upomega$-free and r-linearly newtonian, and Γ is divisible. Then every Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

Proof. By Lemma 3.2.14 it is enough to show this for Z-minimal holes in K of order r. Given such hole in K, use Corollary 3.3.12 to refine it to a hole $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep, eventually. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P^\phi, \mathfrak{m}, \widehat{a})$ for a suitable active $\phi \preccurlyeq 1$ we arrange that $(P, \mathfrak{m}, \widehat{a})$ itself is deep. Then an appeal to Corollary 3.3.16 followed by an application of Lemma 3.3.32 yields a deep and normal refinement of $(P, \mathfrak{m}, \widehat{a})$. Now apply Lemma 3.3.20 to this refinement. $\square$

Next we indicate some variants of Theorem 3.3.33:

Corollary 3.3.34. *Suppose K is* d*-valued and $\upomega$-free, and Γ is divisible. Then every minimal hole in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

Proof. Given a minimal hole in K of order r, use Corollary 3.3.12 to refine it to a hole $(P, \mathfrak{m}, \widehat{a})$ in K such that $\operatorname{nwt} L_{P_{\times \mathfrak{m}}} = 0$ and $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep, eventually. If $\deg P = 1$, then $(P^\phi, \mathfrak{m}, \widehat{a})$ is normal, eventually, by Lemma 3.3.17. If $\deg P > 1$, then K is r-linearly newtonian by Corollary 3.2.6, so we can use Theorem 3.3.33. $\square$

For $r = 1$ we can follow the proof of Theorem 3.3.33, using Corollary 3.3.10 in place of Corollary 3.3.12, to obtain:

Corollary 3.3.35. *If K is 1-linearly newtonian and Γ is divisible, then every quasilinear Z-minimal slot in K of order 1 has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

Here is another variant of Theorem 3.3.33:

Proposition 3.3.36. *If K is* d*-valued and $\upomega$-free, and Γ is divisible, then every Z-minimal special slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

To establish this proposition we follow the proof of Theorem 3.3.33, using Lemma 3.2.35 to preserve specialness in the initial refining. Corollary 3.3.15 takes over the role of Corollary 3.3.16 in that proof.

For linear slots in K we can weaken the hypotheses of Theorem 3.3.33:

Corollary 3.3.37. *Suppose* $\deg P = 1$*. Then $(P, \mathfrak{m}, \widehat{a})$ has a refinement $(P, \mathfrak{n}, \widehat{a})$ such that $(P^\phi, \mathfrak{n}, \widehat{a})$ is deep and normal, eventually. Moreover, if K is $\uplambda$-free and $r > 1$, then $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

Proof. By the remarks before Lemma 3.3.17, $(P, \mathfrak{m}, \widehat{a})$ is normal iff it is steep. Moreover, if $(P, \mathfrak{m}, \widehat{a})$ is normal, then it is quasilinear by Corollary 3.3.21, and hence $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually, by the remarks before Example 3.3.3 and Lemma 3.3.20. By Lemma 3.3.2, $(P, \mathfrak{m}, \widehat{a})$ has a refinement $(P, \mathfrak{n}, \widehat{a})$ such that $(P^\phi, \mathfrak{n}, \widehat{a})$ is steep, eventually. This yields the first part. The second part follows from Corollary 3.1.21 and Lemma 3.3.17. □

Corollary 3.3.38. *Suppose K is λ-free, Γ is divisible, and $(P, \mathfrak{m}, \widehat{a})$ is a quasilinear minimal hole in K of order $r = 1$. Then $(P, \mathfrak{m}, \widehat{a})$ has a refinement $(Q, \mathfrak{n}, \widehat{b})$ such that $(Q^\phi, \mathfrak{n}, \widehat{b})$ is deep and normal, eventually.*

Proof. The case $\deg P = 1$ is part of Corollary 3.3.37. If $\deg P > 1$, then K is 1-linearly newtonian by Lemma 3.2.5, so we can use Corollary 3.3.35. □

Improving normality. *In this subsection $L := L_{P_{\times\mathfrak{m}}}$.* Note that if $(P, \mathfrak{m}, \widehat{a})$ is a normal hole in K, then $P_{\times\mathfrak{m}} \sim (P_{\times\mathfrak{m}})_1$ by Lemma 3.3.18. We call our slot $(P, \mathfrak{m}, \widehat{a})$ in K **strictly normal** if it is normal, but with the condition (N2) replaced by the stronger condition

(N2s) $(P_{\times\mathfrak{m}})_{\neq 1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

Thus for normal $(P, \mathfrak{m}, \widehat{a})$ and $\mathfrak{v} = \mathfrak{v}(L)$ we have:

$$(P, \mathfrak{m}, \widehat{a}) \text{ is strictly normal } \iff P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1.$$

So if $(P, \mathfrak{m}, \widehat{a})$ is normal and $P(0) = 0$, then $(P, \mathfrak{m}, \widehat{a})$ is strictly normal. Note that if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then

$$P_{\times\mathfrak{m}} \sim_{\Delta(\mathfrak{v})} (P_{\times\mathfrak{m}})_1 \qquad \text{(and hence } \operatorname{ddeg} P_{\times\mathfrak{m}} = 1\text{)}.$$

If $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then so are $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$. Thus $(P, \mathfrak{m}, \widehat{a})$ is strictly normal iff $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ is strictly normal. If $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then so is every equivalent slot in K. The proof of Lemma 3.3.23 shows that if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal and $u \in K$, $u \asymp 1$, then $(P, u\mathfrak{m}, \widehat{a})$ is also strictly normal. The analogue of Lemma 3.3.19 goes through, with $(P_{\times\mathfrak{m}})_{\neq 1}$ instead of $(P_{\times\mathfrak{m}})_{>1}$ in the proof:

Lemma 3.3.39. *Suppose* $\operatorname{order}(L) = r$ *and* $\mathfrak{v}$ *are such that* (N1) *and* (N2s) *hold, and* $\mathfrak{v}(L) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. *Then $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.*

Lemma 3.3.20 goes likewise through with "strictly normal" instead of "normal":

Lemma 3.3.40. *If $(P, \mathfrak{m}, \widehat{a})$ is strictly normal and $\phi \preccurlyeq 1$ is active, then the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ is strictly normal. (Hence if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then $(P, \mathfrak{m}, \widehat{a})$ is quasilinear, and if in addition $(P, \mathfrak{m}, \widehat{a})$ is linear, then it is deep.)*

As to Proposition 3.3.25, here is a weak version for strict normality:

Lemma 3.3.41. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a strictly normal hole in K and $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$ where $\mathfrak{v} := \mathfrak{v}(L)$. Then its refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is also strictly normal.*

Proof. As in the proof of Proposition 3.3.25 we arrange $\mathfrak{m} = 1$. We can also assume $P_1 \asymp 1$. From $P = P(0) + P_1 + P_{>1}$ we get

$$P(a) = P(0) + P_1(a) + P_{>1}(a),$$

where $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$ and $P_{>1}(a) \preccurlyeq P_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$ by (N2s) and $a \prec 1$; we show that also $P_1(a) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$. To see this note that

$$0 \ = \ P(\widehat{a}) \ = \ P(0) + P_1(\widehat{a}) + P_{>1}(\widehat{a}),$$

where as before $P(0), P_{>1}(\widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$, so $P_1(\widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$. Lemma 1.1.13 applied to $(\widehat{K}, \preccurlyeq_{\Delta(\mathfrak{v})}, P_1)$ in place of $(K, \preccurlyeq, P)$, with $m = w+1$, $y = a - \widehat{a}$, yields $P_1(a - \widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$, hence

$$P_1(a) \ = \ P_1(a - \widehat{a}) + P_1(\widehat{a}) \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1}$$

as claimed. It remains to use $\mathfrak{v}(L_{P_{+a}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ and the normality of $(P_{+a}, 1, \widehat{a} - a)$ obtained from Proposition 3.3.25 and its proof. □

We also have a version of Lemma 3.3.41 for Z-minimal slots, obtained from that lemma via Lemma 3.2.14:

Lemma 3.3.42. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and strictly normal. Set $\mathfrak{v} := \mathfrak{v}(L)$, and suppose $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$. Then the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.*

Next two versions of Proposition 3.3.26:

Lemma 3.3.43. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a strictly normal hole in K, $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}] < \big[\mathfrak{v}(L)\big]$. Then the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.*

Proof. As in the proof of Proposition 3.3.26 we arrange $\mathfrak{m} = 1$ and, setting $\mathfrak{v} := \mathfrak{v}(L)$, $\widetilde{L} := L_{P_{\times \mathfrak{n}}}$, show that $\operatorname{order}(\widetilde{L}) = r$, $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, and that (N2) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Now $[\mathfrak{n}] < [\mathfrak{v}]$ yields $\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} 1$; together with $(P_{\times \mathfrak{n}})_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{n} P_1$ this gives $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{n}})_1$. Hence (N2s) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Lemma 3.3.39 now yields that $(P, \mathfrak{n}, \widehat{a})$ is strictly normal. □

Lemma 3.3.44. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a strictly normal hole in K and $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ where $\mathfrak{v} := \mathfrak{v}(L)$. Assume also that for all $q \in \mathbb{Q}^>$ there is given an element $\mathfrak{v}^q$ of $K^\times$ with $v(\mathfrak{v}^q) = q\, v(\mathfrak{v})$. Then for all sufficiently small $q \in \mathbb{Q}^>$ and $\mathfrak{n}$ with $\mathfrak{n} \asymp \mathfrak{v}^q \mathfrak{m}$ we have: $\widehat{a} \prec \mathfrak{n}$ and the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.*

Proof. We arrange $\mathfrak{m} = 1$ as usual, and take $q_0 \in \mathbb{Q}^>$ such that $\widehat{a} \prec \mathfrak{v}^{q_0}$ and $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1+q_0} P_1$. Let $q \in \mathbb{Q}$, $0 < q \leqslant q_0$, and suppose $\mathfrak{n} \asymp \mathfrak{v}^q$. Then $(P, \mathfrak{n}, \widehat{a})$ is a refinement of $(P, 1, \widehat{a})$, and the proof of Proposition 3.3.26 gives: $\widetilde{L} := L_{P_{\times \mathfrak{n}}}$ has order r with $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, $\mathfrak{n} P_1 \asymp_{\Delta(\mathfrak{v})} (P_{\times \mathfrak{n}})_1$, and (N2) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Hence

$$P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1+q_0} P_1 \preccurlyeq \mathfrak{v}^{w+1} \mathfrak{n} P_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{n}})_1.$$

Now as in the proof of the previous lemma we conclude that $(P, \mathfrak{n}, \widehat{a})$ is strictly normal. □

Remark 3.3.45. In Lemmas 3.3.43 and 3.3.44 we assumed that $(P, \mathfrak{m}, \widehat{a})$ is a strictly normal hole in K. By Lemma 3.2.14 these lemmas go through if this hypothesis is replaced by "$(P, \mathfrak{m}, \widehat{a})$ is a strictly normal Z-minimal slot in K".

We now turn to refining a given normal hole to a strictly normal hole. We only do this under additional hypotheses, tailored so that we may employ Lemma 3.1.17. Therefore we assume in the rest of this subsection: *K is* d*-valued and for all $\mathfrak{v}$ and $q \in \mathbb{Q}^>$ we are given an element $\mathfrak{v}^q$ of $K^\times$ with $(\mathfrak{v}^q)^\dagger = q\mathfrak{v}^\dagger$.* Note that then $v(\mathfrak{v}^q) = q\, v(\mathfrak{v})$ for such q. (In particular, Γ is divisible.) We also adopt the convention that if $\operatorname{order} L = r$, then $\mathfrak{v} := \mathfrak{v}(L)$.

Lemma 3.3.46. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a normal hole in K and $\widehat{a} - a \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$. Then the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.*

Proof. As usual we arrange that $\mathfrak{m} = 1$. By Proposition 3.3.25, $(P_{+a}, 1, \widehat{a} - a)$ is normal; the proof of this proposition gives $\operatorname{order}(L_{P_{+a}}) = r$, $\mathfrak{v}(L_{P_{+a}}) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}$, $(P_{+a})_1 \sim_{\Delta(\mathfrak{v})} P_1$, and (N2) holds with $\mathfrak{m} = 1$ and P replaced by P_{+a}. It remains to show that $P_{+a}(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1$, equivalently, $P(a) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$.

Let $\widehat{L} := L_{P_{+\widehat{a}}} \in \widehat{K}[\partial]$ and $R := P_{>1} \in K\{Y\}$; note that $P_{(\boldsymbol{i})} = R_{(\boldsymbol{i})}$ for $|\boldsymbol{i}| > 1$ and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$. Hence Taylor expansion and $P(\widehat{a}) = 0$ give

$$\begin{aligned} P(a) &= P(\widehat{a}) + \widehat{L}(a - \widehat{a}) + \sum_{|\boldsymbol{i}|>1} P_{(\boldsymbol{i})}(\widehat{a}) \cdot (a - \widehat{a})^{\boldsymbol{i}} \\ &= \widehat{L}(a - \widehat{a}) + \sum_{|\boldsymbol{i}|>1} R_{(\boldsymbol{i})}(\widehat{a}) \cdot (a - \widehat{a})^{\boldsymbol{i}} \\ &\qquad\qquad \text{where } R_{(\boldsymbol{i})}(\widehat{a}) \cdot (a - \widehat{a})^{\boldsymbol{i}} \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1} P_1 \text{ for } |\boldsymbol{i}| > 1, \end{aligned}$$

so it is enough to show $\widehat{L}(a - \widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$. Lemma 3.1.27 applied to $(\widehat{K}, \widehat{a})$ in place of (K, a) gives $\operatorname{order} \widehat{L} = r$ and $\widehat{L} \sim_{\Delta(\mathfrak{v})} L$. Since $\widehat{K}$ is d-valued, Lemma 3.1.17 yields a $q \in \mathbb{Q}$ with $w + 1 < q \leqslant w + 2$ and a $\mathfrak{w}$ such that $\widehat{L}\mathfrak{v}^q \asymp \mathfrak{w}\, \mathfrak{v}^q\, \widehat{L}$ where $[\mathfrak{w}] \leqslant [\mathfrak{v}^\dagger]$ and hence $\mathfrak{w} \asymp_{\Delta(\mathfrak{v})} 1$ (see the remark before Lemma 3.3.1). With $\mathfrak{n} \asymp a - \widehat{a}$ we have $\mathfrak{n} \preccurlyeq \mathfrak{v}^{w+2} \preccurlyeq \mathfrak{v}^q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$ and therefore

$$\widehat{L}(a - \widehat{a}) \ \preccurlyeq \ \widehat{L}\mathfrak{n} \ \preccurlyeq \ \widehat{L}\mathfrak{v}^q \ \asymp \ \mathfrak{w}\, \mathfrak{v}^q\, \widehat{L} \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v}^q \widehat{L} \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1} \widehat{L}.$$

Hence $\widehat{L}(a - \widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$ as required. □

In particular, if $(P, \mathfrak{m}, \widehat{a})$ is a normal hole in K and $\widehat{a} \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$, then $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.

Corollary 3.3.47. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and normal. If $(P, \mathfrak{m}, \widehat{a})$ is special, then $(P, \mathfrak{m}, \widehat{a})$ has a deep and strictly normal refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ where $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ and $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. (Note that if K is r-linearly newtonian, and $\upomega$-free if $r > 1$, then $(P, \mathfrak{m}, \widehat{a})$ is special by Lemma 3.2.36.)*

Proof. By Lemma 3.2.14 we arrange that $(P, \mathfrak{m}, \widehat{a})$ is a hole in K. If $(P, \mathfrak{m}, \widehat{a})$ is special, Corollary 3.3.15 gives an a such that $\widehat{a} - a \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$, and then the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal by Lemma 3.3.46, and deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ by Lemma 3.3.7. □

This leads to a useful variant of the Normalization Theorem 3.3.33:

Corollary 3.3.48. *Suppose K is $\upomega$-free and r-linearly newtonian. Then every Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and strictly normal, eventually.*

Proof. Let a Z-minimal slot in K of order r be given. Use Theorem 3.3.33 to refine it to a slot $(P, \mathfrak{m}, \widehat{a})$ in K with an active ϕ_0 such that the slot $(P^{\phi_0}, \mathfrak{m}, \widehat{a})$ in K^{ϕ_0} is deep and normal. Corollary 3.3.47 gives a deep and strictly normal refinement $(P^{\phi_0}_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P^{\phi_0}, \mathfrak{m}, \widehat{a})$. By Lemma 3.3.40 the slot $(P^{\phi}_{+a}, \mathfrak{m}, \widehat{a} - a)$

in K^{ϕ} is deep and strictly normal, for all active $\phi \preccurlyeq \phi_0$ (in K). Thus $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ refines the original Z-minimal slot in K and has the desired property. $\square$

Corollaries 3.2.6 and 3.3.48 have the following consequence:

Corollary 3.3.49. *Suppose K is $\boldsymbol{\omega}$-free. Then every minimal hole in K of order r and degree >1 has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep and strictly normal, eventually.*

Corollary 3.3.47 also gives the following variant of Corollary 3.3.48, where the role of Theorem 3.3.33 in its proof is taken over by Proposition 3.3.36:

Corollary 3.3.50. *Suppose K is $\boldsymbol{\omega}$-free. Then every Z-minimal special slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep and strictly normal, eventually.*

3.4. Isolated Slots

In this short section we study the concept of isolation, which plays well together with normality. *Throughout this section K is an H-asymptotic field with small derivation and with rational asymptotic integration.* We let a, b range over K and ϕ, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{w}$ over $K^{\times}$. We also let $(P, \mathfrak{m}, \widehat{a})$ be a slot in K of order $r \geqslant 1$. Recall that $v(\widehat{a}-K)$ is a cut in Γ without largest element. Note that $v\big((\widehat{a}-a)-K\big) = v(\widehat{a}-K)$ and $v(\widehat{a}\mathfrak{n}-K) = v(\widehat{a}-K)+v\mathfrak{n}$.

Definition 3.4.1. We say that $(P, \mathfrak{m}, \widehat{a})$ is **isolated** if for all $a \prec \mathfrak{m}$,

$$\operatorname{order}(L_{P_{+a}}) = r \quad\text{and}\quad \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{a}-K) \ < \ v(\widehat{a}-a);$$

equivalently, for all $a \prec \mathfrak{m}$: $\operatorname{order}(L_{P_{+a}}) = r$ and whenever $\mathfrak{w} \preccurlyeq \widehat{a}-a$ is such that $v(\mathfrak{w}) \in \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$, then $\mathfrak{w} \prec \widehat{a}-b$ for all b.

In particular, if $(P, \mathfrak{m}, \widehat{a})$ is isolated, then $v(\widehat{a}) \notin \mathscr{E}^{\mathrm{e}}(L_P)$. If $(P, \mathfrak{m}, \widehat{a})$ is isolated, then so is every equivalent slot in K, as well as $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ and the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in K^{ϕ} for active ϕ in K. Moreover:

Lemma 3.4.2. *If $(P, \mathfrak{m}, \widehat{a})$ is isolated, then so is any refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of it.*

Proof. For the case $\mathfrak{n} = \mathfrak{m}$, use $v\big((\widehat{a}-a)-K\big) = v(\widehat{a}-K)$. The case $a=0$ is clear. The general case reduces to these two special cases. $\square$

Lemma 3.4.3. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is isolated. Then the multiplicative conjugate $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ of $(P, \mathfrak{m}, \widehat{a})$ by $\mathfrak{n}$ is isolated.*

Proof. Let $a \prec \mathfrak{m}/\mathfrak{n}$. Then $a\mathfrak{n} \prec \mathfrak{m}$, so $\operatorname{order}(L_{P_{\times\mathfrak{n},+a}}) = \operatorname{order}(L_{P_{+a\mathfrak{n},\times\mathfrak{n}}}) = \operatorname{order}(L_{P_{+a\mathfrak{n}}}) = r$. Suppose $\mathfrak{w} \preccurlyeq (\widehat{a}/\mathfrak{n})-a$ and $v(\mathfrak{w}) \in \mathscr{E}^{\mathrm{e}}\big(L_{P_{\times\mathfrak{n},+a}}\big)$. Now $L_{P_{\times\mathfrak{n},+a}} = L_{P_{+a\mathfrak{n},\times\mathfrak{n}}} = L_{P_{+a\mathfrak{n}}}\mathfrak{n}$ and thus $\mathfrak{w}\mathfrak{n} \preccurlyeq \widehat{a}-a\mathfrak{n}$, $v(\mathfrak{w}\mathfrak{n}) \in \mathscr{E}^{\mathrm{e}}\big(P_{+a\mathfrak{n}}\big)$. But $(P, \mathfrak{m}, \widehat{a})$ is isolated, so $v(\mathfrak{w}\mathfrak{n}) > v(\widehat{a}-K)$ and hence $v(\mathfrak{w}) > v\big((\widehat{a}/\mathfrak{n})-K\big)$. Thus $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ is isolated. $\square$

Lemma 3.4.4. *Suppose K is $\boldsymbol{\lambda}$-free or $r=1$, and $(P, \mathfrak{m}, \widehat{a})$ is normal. Then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is isolated} \quad\Longleftrightarrow\quad \mathscr{E}^{\mathrm{e}}(L_P) \cap v(\widehat{a}-K) \ \leqslant \ v\mathfrak{m}.$$

Proof. Use Lemma 3.3.24; for the direction $\Rightarrow$, use also that $\widehat{a}-a \prec \mathfrak{m}$ iff $a \prec \mathfrak{m}$. $\square$

Lemma 3.4.5. *Suppose* $\deg P = 1$. *Then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is isolated} \quad\Longleftrightarrow\quad \mathscr{E}^{\mathrm{e}}(L_P) \cap v(\widehat{a} - K) \ \leqslant\ v\mathfrak{m}.$$

Proof. Use that order $L_P = r$ and $L_{P_{+a}} = L_P$ for all a. $\square$

Proposition 3.4.6. *Suppose K is λ-free or $r = 1$, and $(P, \mathfrak{m}, \widehat{a})$ is normal. Then $(P, \mathfrak{m}, \widehat{a})$ has an isolated refinement.*

Proof. Suppose $(P, \mathfrak{m}, \widehat{a})$ is not already isolated. Then Lemma 3.4.4 gives γ with

$$\gamma \in \mathscr{E}^{\mathrm{e}}(L_P) \cap v(\widehat{a} - K), \quad \gamma > v\mathfrak{m}.$$

We have $|\mathscr{E}^{\mathrm{e}}(L_P)| \leqslant r$, by [ADH, p. 481] if $r = 1$, and Corollary 1.8.11 and λ-freeness of K if $r > 1$. Hence we can take $\gamma := \max \mathscr{E}^{\mathrm{e}}(L_P) \cap v(\widehat{a} - K)$, and then $\gamma > v\mathfrak{m}$. Take a and $\mathfrak{n}$ with $v(\widehat{a} - a) > \gamma = v(\mathfrak{n})$; then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$ and $a \prec \mathfrak{m}$. Let $b \prec \mathfrak{n}$; then $a + b \prec \mathfrak{m}$, so by Lemma 3.3.24,

$$\operatorname{order}(L_{(P_{+a})_{+b}}) \ =\ r, \qquad \mathscr{E}^{\mathrm{e}}(L_{(P_{+a})_{+b}}) \ =\ \mathscr{E}^{\mathrm{e}}(L_P).$$

Also $v\big((\widehat{a} - a) - b\big) > \gamma$, hence

$$\mathscr{E}^{\mathrm{e}}\big(L_{(P_{+a})_{+b}}\big) \cap v\big((\widehat{a} - a) - K\big) \ =\ \mathscr{E}^{\mathrm{e}}(L_P) \cap v(\widehat{a} - K) \ \leqslant\ \gamma \ <\ v\big((\widehat{a} - a) - b\big).$$

Thus $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is isolated. $\square$

Remark 3.4.7. Proposition 3.4.6 goes through if instead of assuming that $(P, \mathfrak{m}, \widehat{a})$ is normal, we assume that $(P, \mathfrak{m}, \widehat{a})$ is linear. (Same argument, using Lemma 3.4.5 in place of Lemma 3.4.4 and $L_{(P_{+a})_{+b}} = L_P$ in place of Lemma 3.3.24.)

Corollary 3.4.8. *Suppose $r = 1$, and $(P, \mathfrak{m}, \widehat{a})$ is normal or linear. If $\mathscr{E}^{\mathrm{e}}(L_P) = \emptyset$, then $(P, \mathfrak{m}, \widehat{a})$ is isolated. If $\mathscr{E}^{\mathrm{e}}(L_P) \neq \emptyset$, so $\mathscr{E}^{\mathrm{e}}(L_P) = \{v\mathfrak{g}\}$ where $\mathfrak{g} \in K^{\times}$, then $(P, \mathfrak{m}, \widehat{a})$ is isolated iff $\mathfrak{m} \preccurlyeq \mathfrak{g}$ or $\widehat{a} - K \succ \mathfrak{g}$.*

This follows immediately from Lemmas 3.4.4 and 3.4.5. The results in the rest of this subsection are the *raison d'être* of isolated holes:

Proposition 3.4.9. *Suppose K is ω-free and $(P, \mathfrak{m}, \widehat{a})$ is an isolated hole in K which is normal or linear. Let $\widehat{b}$ in an immediate asymptotic extension of K satisfy $P(\widehat{b}) = 0$ and $\widehat{b} \prec \mathfrak{m}$. Then $v(\widehat{a} - a) = v(\widehat{b} - a)$ for all a, so $\widehat{b} \notin K$.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{a})$, $\widehat{b}$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$, $\widehat{b}/\mathfrak{m}$, we arrange $\mathfrak{m} = 1$. Let a be given; we show $v(\widehat{a} - a) = v(\widehat{b} - a)$. This is clear if $a \succcurlyeq 1$, so assume $a \prec 1$. Corollary 3.3.21 (if $(P, \mathfrak{m}, \widehat{a})$ is normal) and Lemma 3.2.21 (if $(P, \mathfrak{m}, \widehat{a})$ is linear) give $\operatorname{ndeg} P = 1$. Thus P is in newton position at a by Corollary 3.2.23. Moreover $v(\widehat{a} - a) \notin \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$, hence $v(\widehat{a} - a) = v^{\mathrm{e}}(P, a)$ by Lemma 1.8.15. Likewise, if $v(\widehat{b} - a) \notin \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$, then $v(\widehat{b} - a) = v^{\mathrm{e}}(P, a)$ by Lemma 1.8.15, so $v(\widehat{a} - a) = v(\widehat{b} - a)$.

Thus to finish the proof it is enough to show that $\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{b} - K) \leqslant 0$. Now $|\mathscr{E}^{\mathrm{e}}(L_{P_{+a}})| \leqslant r$ by Corollary 1.5.5, so we have $b \prec 1$ such that

$$\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{b} - K) \ <\ v(\widehat{b} - b),$$

in particular, $v(\widehat{b} - b) \notin \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$. If $(P, \mathfrak{m}, \widehat{a})$ is normal, then Lemma 3.3.24 gives

$$\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \ =\ \mathscr{E}^{\mathrm{e}}(L_P) \ =\ \mathscr{E}^{\mathrm{e}}(L_{P_{+b}}),$$

so by the above with b instead of a we have $v(\widehat{a}-b)=v(\widehat{b}-b)$. If $(P,\mathfrak{m},\widehat{a})$ is linear, then $L_{P_{+a}}=L_P=L_{P_{+b}}$, and we obtain likewise $v(\widehat{a}-b)=v(\widehat{b}-b)$. Hence

$$\mathscr{E}^{\mathrm{e}}(L_{P_{+a}})\cap v(\widehat{b}-K)\ \subseteq\ \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})\cap\Gamma^{<v(\widehat{a}-b)}\ \subseteq\ \mathscr{E}^{\mathrm{e}}(L_P)\cap v(\widehat{a}-K)\ \leqslant\ 0.$$

using Lemmas 3.4.4 and 3.4.5 for the last step. □

Combining Proposition 3.4.9 with Corollary 3.2.15 yields:

Corollary 3.4.10. *Let K, $(P,\mathfrak{m},\widehat{a})$, $\widehat{b}$ be as in Proposition 3.4.9, and assume also that $(P,\mathfrak{m},\widehat{a})$ is Z-minimal. Then there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

Using the Normalization Theorem, we now obtain:

Corollary 3.4.11. *Suppose K is ω-free and Γ is divisible. Then every minimal hole in K of order r has an isolated refinement $(P,\mathfrak{m},\widehat{a})$ such that for any $\widehat{b}$ in an immediate asymptotic extension of K with $P(\widehat{b})=0$ and $\widehat{b}\prec\mathfrak{m}$ there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

Proof. Given a minimal linear hole in K of order r, use Remark 3.4.7 to refine it to an isolated minimal linear hole $(P,\mathfrak{m},\widehat{a})$ in K of order r, and use Corollary 3.4.10. Suppose we are given a minimal non-linear hole in K of order r. Then K is r-linearly newtonian by Corollary 3.2.6. Then Theorem 3.3.33 yields a refinement $(Q,\mathfrak{w},\widehat{d})$ of it and an active θ in K such that the minimal hole $(Q^\theta,\mathfrak{w},\widehat{d})$ in K^θ is normal. Proposition 3.4.6 gives an isolated refinement $(Q^\theta_{+d},\mathfrak{v},\widehat{d}-d)$ of $(Q^\theta,\mathfrak{w},\widehat{d})$. Suitably refining $(Q^\theta_{+d},\mathfrak{v},\widehat{d}-d)$ further followed by compositionally conjugating with a suitable active element of K^θ yields by Theorem 3.3.33 and Lemma 3.4.2 a refinement $(P,\mathfrak{m},\widehat{a})$ of $(Q,\mathfrak{w},\widehat{d})$ (and thus of the originally given hole) and an active ϕ in K such that $(P^\phi,\mathfrak{m},\widehat{a})$ is both normal and isolated. Then $(P,\mathfrak{m},\widehat{a})$ is isolated, and we can apply Corollary 3.4.10 to K^ϕ and $(P^\phi,\mathfrak{m},\widehat{a})$ in the role of K and $(P,\mathfrak{m},\widehat{a})$. □

For $r=1$ we can replace "ω-free" in Proposition 3.4.9 and Corollary 3.4.10 by the weaker "λ-free" (same proofs, using Lemma 1.8.20 instead of Lemma 1.8.15):

Proposition 3.4.12. *Suppose K is λ-free, $(P,\mathfrak{m},\widehat{a})$ is an isolated hole in K of order $r=1$, and suppose $(P,\mathfrak{m},\widehat{a})$ is normal or linear. Let $\widehat{b}$ in an immediate asymptotic extension of K satisfy $P(\widehat{b})=0$ and $\widehat{b}\prec\mathfrak{m}$. Then $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a. (Hence if $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.)*

This leads to an analogue of Corollary 3.4.11:

Corollary 3.4.13. *Suppose K is λ-free and Γ is divisible. Then every quasilinear minimal hole in K of order $r=1$ has an isolated refinement $(P,\mathfrak{m},\widehat{a})$ such that for any $\widehat{b}$ in an immediate asymptotic extension of K with $P(\widehat{b})=0$ and $\widehat{b}\prec\mathfrak{m}$ there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

Proof. Suppose we are given a quasilinear minimal hole in K of order $r=1$. Then Corollary 3.3.38 yields a refinement $(Q,\mathfrak{w},\widehat{d})$ of it and an active θ in K such that the quasilinear minimal hole $(Q^\theta,\mathfrak{w},\widehat{d})$ in K^θ of order 1 is normal. Proposition 3.4.6 gives an isolated refinement $(Q^\theta_{+d},\mathfrak{v},\widehat{d}-d)$ of $(Q^\theta,\mathfrak{w},\widehat{d})$, and then Corollary 3.3.38

yields a refinement $(P,\mathfrak{m},\widehat{a})$ of $(Q,\mathfrak{w},\widehat{d})$ and an active ϕ in K such that $(P^{\phi},\mathfrak{m},\widehat{a})$ is normal and isolated. Now apply Proposition 3.4.12 with K^{ϕ} and $(P^{\phi},\mathfrak{m},\widehat{a})$ in the role of K and $(P,\mathfrak{m},\widehat{a})$. $\square$

Next a variant of Lemma 3.2.1 for $r=1$ without assuming $\boldsymbol{\omega}$-freeness:

Corollary 3.4.14. *Suppose K is* 1*-newtonian and Γ is divisible. Then K has no quasilinear Z-minimal slot of order* 1*.*

Proof. By Proposition 1.8.28, K is $\boldsymbol{\lambda}$-free. Towards a contradiction, let $(P,\mathfrak{m},\widehat{a})$ be a quasilinear Z-minimal slot in K of order 1. By Lemma 3.2.14 we arrange that $(P,\mathfrak{m},\widehat{a})$ is a hole in H. Using Corollary 3.3.35, Lemma 3.4.2 and the remark before it, and Proposition 3.4.6, we can refine further so that $(P^{\phi},\mathfrak{m},\widehat{a})$ is normal and isolated for some active ϕ in K. Then there is no $y\in K$ with $P(y)=0$ and $y\prec\mathfrak{m}$, by Proposition 3.4.12, contradicting Lemma 3.2.27 for $L=K$. $\square$

Finally, for isolated linear holes, without additional hypotheses:

Lemma 3.4.15. *Suppose $(P,\mathfrak{m},\widehat{a})$ is an isolated linear hole in K, and $\widehat{a}-a\prec\mathfrak{m}$. Then $P(a)\neq 0$, and $\gamma=v(\widehat{a}-a)$ is the unique element of $\Gamma\setminus\mathscr{E}^{\mathrm{e}}(L_P)$ such that $v^{\mathrm{e}}_{L_P}(\gamma)=v\big(P(a)\big)$.*

Proof. By Lemma 3.4.5, $\gamma:=v(\widehat{a}-a)\in\Gamma\setminus\mathscr{E}^{\mathrm{e}}(L_P)$. Since $\deg P=1$,

$$L_P(\widehat{a}-a)\ =\ L_P(\widehat{a})-L_P(a)\ =\ -P(0)-L_P(a)\ =\ -P(a),$$

so $P(a)\neq 0$. By Lemma 1.5.6, $v^{\mathrm{e}}_{L_P}(\gamma)=v\big(L_P(\widehat{a}-a)\big)=v\big(P(a)\big)$. $\square$

In [15] we shall prove a version of Proposition 3.4.9 without the hypothesis that $\widehat{b}$ lies in an immediate extension of K. In Section 4.4 below we consider, in a more restricted setting, a variant of isolated slots, with ultimate exceptional values taking over the role played by exceptional values in Definition 3.4.1.

3.5. Holes of Order and Degree One

In this section K is a d-valued field of H-type with small derivation and rational asymptotic integration. (Later on we will impose additional restrictions on K.) We also let $\widehat{K}$ be an immediate asymptotic extension of K. We focus here on slots of complexity $(1,1,1)$ in K. As a byproduct we obtain in Corollary 3.5.18 a partial generalization of Corollary 3.3.49 to minimal holes in K of arbitrary degree. First we establish in the next subsection a useful formal identity. We let j, k range over $\mathbb{N}$ (in addition to m, n, as usual).

An integration identity. Let R be a differential ring, and let f, g, h, range over R. We use $\int f=g+\int h$ as a suggestive way to express that $f=g'+h$, and likewise, $\int f=g-\int h$ means that $f=g'-h$. For example,

$$\int f'g\ =\ fg-\int fg' \qquad \text{(integration by parts).}$$

Let $\mathrm{e},\xi\in R^{\times}$ satisfy $\mathrm{e}^{\dagger}=\xi$. We wish to expand $\int\mathrm{e}$ by iterated integration by parts. Now for $g=\mathrm{e}$ we have $g'\in R^{\times}$ with $\frac{g}{g'}=\frac{1}{\xi}$, so in view of $\mathrm{e}=g'\frac{\mathrm{e}}{g'}$:

$$\int\mathrm{e}\ =\ \int g'\frac{\mathrm{e}}{g'}\ =\ \frac{\mathrm{e}}{\xi}-\int g\left(\frac{\mathrm{e}}{g'}\right)',$$

and

$$\left(\frac{\mathrm{e}}{g'}\right)' = \left(\frac{1}{\xi}\right)' = \frac{-\xi'}{\xi^2} = \frac{-\xi^\dagger}{\xi},$$

and thus

$$\int \mathrm{e} \ = \ \frac{\mathrm{e}}{\xi} + \int \frac{\xi^\dagger}{\xi}\,\mathrm{e}\,.$$

More generally, using the above identities for $g = \mathrm{e}$,

$$\begin{aligned}
\int f\,\mathrm{e} \ &= \ \int g' f \frac{\mathrm{e}}{g'} \ = \ \frac{f}{\xi}\,\mathrm{e} - \int g\left(f\frac{\mathrm{e}}{g'}\right)' \\
&= \ \frac{f}{\xi}\,\mathrm{e} - \int g\left(f'\frac{\mathrm{e}}{g'} + f\left(\frac{\mathrm{e}}{g'}\right)'\right) \ = \ \frac{f}{\xi}\,\mathrm{e} - \int\left(\frac{f'}{\xi}\,\mathrm{e} + fg\left(\frac{-\xi^\dagger}{\xi}\right)\right) \\
&= \ \frac{f}{\xi}\,\mathrm{e} - \int\left(\frac{f'}{\xi}\,\mathrm{e} + \frac{-f\xi^\dagger}{\xi}\,\mathrm{e}\right) \ = \ \frac{f}{\xi}\,\mathrm{e} + \int\left(\frac{\xi^\dagger f - f'}{\xi}\right)\mathrm{e}\,.
\end{aligned}$$

Replacing f by f/ξ^k gives the following variant of this identity:

$$\int \frac{f}{\xi^k}\,\mathrm{e} \ = \ \frac{f}{\xi^{k+1}}\,\mathrm{e} + \int \frac{(k+1)\xi^\dagger f - f'}{\xi^{k+1}}\,\mathrm{e}\,.$$

Induction on m using the last identity yields:

Lemma 3.5.1. *Set $\zeta := \xi^\dagger$. Then*

$$\int f\,\mathrm{e} \ = \ \sum_{j=0}^{m} P_j(\zeta, f)\frac{\mathrm{e}}{\xi^{j+1}} + \int P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}},$$

where the $P_j \in \mathbb{Q}\{Z, V\} = \mathbb{Q}\{Z\}\{V\}$ are independent of R, e, ξ:

$$P_0 \ := \ V, \qquad P_{j+1} \ := \ (j+1)ZP_j - P_j'.$$

Thus $P_j = P_{j0}V + P_{j1}V' + \cdots + P_{jj}V^{(j)}$ with all $P_{jk} \in \mathbb{Q}\{Z\}$ and $P_{jj} = (-1)^j$.

For example,

$$P_0 = V, \quad P_1 = ZV - V', \quad P_2 = (2Z^2 - Z')V - 3ZV' + V''.$$

An asymptotic expansion. *In this subsection $\xi \in K$ and $\xi \succ^\flat 1$;* equivalently, $\xi \in K$ satisfies $\xi \succ 1$ and $\zeta := \xi^\dagger \succcurlyeq 1$. *We also assume that $\xi \notin \mathrm{I}(K) + K^\dagger$.* Since $\widehat{K}$ is d-valued of H-type with asymptotic integration, it has by [ADH, 10.2.7] an immediate asymptotic extension $\widehat{K}(\phi)$ with $\phi' = \xi$. Then the algebraic closure of $\widehat{K}(\phi)$ is still d-valued of H-type, by [ADH, 9.5], and so [ADH, 10.4.1] yields a d-valued H-asymptotic extension L of this algebraic closure with an element $\mathrm{e} \neq 0$ such that $\mathrm{e}^\dagger = \xi$. All we need about L below is that it is a d-valued H-asymptotic extension of $\widehat{K}$ with elements ϕ and e such that $\phi' = \xi$ and $\mathrm{e} \neq 0$, $\mathrm{e}^\dagger = \xi$. Note that then L has small derivation, and $\xi \succ^\flat 1$ in L. (The element ϕ will only play an auxiliary role later in this subsection.)

Lemma 3.5.2. $v(\mathrm{e}) \notin \Gamma$.

Proof. Suppose otherwise. Take $a \in K^\times$ with $a\,\mathrm{e} \asymp 1$. Then $a^\dagger + \xi = (a\,\mathrm{e})^\dagger \in \mathrm{I}(L) \cap K = \mathrm{I}(K)$ and thus $\xi \in \mathrm{I}(K) + K^\dagger$, a contradiction. □

By Lemma 3.5.2 there is for each $g \in L$ at most one $\widehat{f} \in \widehat{K}$ with $\big(\widehat{f}\frac{\mathrm{e}}{\xi}\big)' = g$. Let $f \in K^{\times}$ be given with $f \preccurlyeq 1$, and suppose $\widehat{f} \in \widehat{K}$ satisfies $\big(\widehat{f}\frac{\mathrm{e}}{\xi}\big)' = f\,\mathrm{e}$. Our aim is to show that with P_j as in Lemma 3.5.1, the series $\sum_{j=0}^{\infty} P_j(\zeta, f)\frac{1}{\xi^j}$ is a kind of asymptotic expansion of $\widehat{f}$. The partial sums

$$f_m := \sum_{j=0}^{m} P_j(\zeta, f)\frac{1}{\xi^j}$$

of this series lie in K, with $f_0 = f$ and $f_n - f_m \prec \xi^{-m}$ for $m < n$, by Lemma 1.1.16. More precisely, we show:

Proposition 3.5.3. *We have* $\widehat{f} - f_m \prec \xi^{-m}$ *for all* m. *(Thus:* $f \asymp 1 \Rightarrow \widehat{f} \sim f$.*)*

Towards the proof, note that by Lemma 3.5.1 with $R = L$,

$$\widehat{f}\,\frac{\mathrm{e}}{\xi} \;=\; \sum_{j=0}^{m} P_j(\zeta, f)\frac{\mathrm{e}}{\xi^{j+1}} + I_m, \quad I_m \in L, \text{ and thus}$$

$$\widehat{f} \;=\; f_m + \frac{\xi}{\mathrm{e}} I_m \tag{3.5.1}$$

where $I_m \in \widehat{K}\,\mathrm{e}$ satisfies $I_m' = P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}}$, a condition that determines I_m uniquely up to an additive constant from C_L. The proof of Proposition 3.5.3 now rests on the following lemmas:

Lemma 3.5.4. *In L we have* $(\mathrm{e}\,\xi^l)^{(k)} \;\sim\; \mathrm{e}\,\xi^{l+k}$, *for all* $l \in \mathbb{Z}$ *and all* k.

This is Corollary 1.1.17 with our L in the role of K there, and taking e^{ϕ} there as our $\mathrm{e} \in L$; note that here our $\phi \in L$ with $\phi' = \xi$ is needed.

Lemma 3.5.5. *Suppose* $\mathrm{e} \succ \xi^{m+1}$. *Then* $\frac{\xi}{\mathrm{e}} I_m \prec \xi^{-m}$.

Proof. This amounts to $I_m \prec \frac{\mathrm{e}}{\xi^{m+1}}$. Suppose $I_m \succcurlyeq \frac{\mathrm{e}}{\xi^{m+1}} \succ 1$. Then we have $I_m' \succcurlyeq \big(\frac{\mathrm{e}}{\xi^{m+1}}\big)' \sim \frac{\mathrm{e}}{\xi^m}$ by Lemma 3.5.4, so $P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}} \succcurlyeq \frac{\mathrm{e}}{\xi^m}$, and thus $P_{m+1}(\zeta, f) \succcurlyeq \xi$, contradicting Lemma 1.1.16. □

Lemma 3.5.6. *Suppose* $\mathrm{e} \preccurlyeq \xi^m$. *Then* $I_m \prec 1$ *and* $\frac{\xi}{\mathrm{e}} I_m \prec \xi^{-m}$.

Proof. Lemma 1.1.16 gives

$$P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}} \preccurlyeq \zeta^N \frac{\mathrm{e}}{\xi^{m+1}} \preccurlyeq \frac{\zeta^N}{\xi} \quad \text{for some } N \in \mathbb{N},$$

so $v(I_m') > \Psi_L$, and thus $I_m \preccurlyeq 1$. If $I_m \asymp 1$, then $v(\frac{\xi}{\mathrm{e}} I_m) = v(\xi) - v(\mathrm{e}) \notin \Gamma$, contradicting $\frac{\xi}{\mathrm{e}} I_m = \widehat{f} - f_m \in \widehat{K}$, by (3.5.1). Thus $I_m \prec 1$. Now assume towards a contradiction that $\frac{\xi}{\mathrm{e}} I_m \succcurlyeq \xi^{-m}$. Then $\frac{\mathrm{e}}{\xi^{m+1}} \preccurlyeq I_m \prec 1$, so $I_m' \succcurlyeq \big(\frac{\mathrm{e}}{\xi^{m+1}}\big)' \sim \frac{\mathrm{e}}{\xi^m}$ by Lemma 3.5.4, and this yields a contradiction as in the proof of Lemma 3.5.5. □

Proof of Proposition 3.5.3. Let m be given. If $\mathrm{e} \succ \xi^{m+1}$, then $\widehat{f} - f_m = \frac{\xi}{\mathrm{e}} I_m \prec \xi^{-m}$ by Lemma 3.5.5. Suppose $\mathrm{e} \preccurlyeq \xi^{m+1}$. Then Lemma 3.5.6 (with $m+1$ instead of m) gives $\widehat{f} - f_{m+1} \prec \xi^{-(m+1)}$, hence $\widehat{f} - f_m = (\widehat{f} - f_{m+1}) + (f_{m+1} - f_m) \prec \xi^{-m}$. □

Application to linear differential equations of order 1. Proposition 3.5.3 yields information about the asymptotics of solutions (in $\widehat{K}$) of certain linear differential equations of order 1 over K:

Corollary 3.5.7. *Let $f, \xi \in K$, $f \preccurlyeq 1$, $\xi \succ^{\flat} 1$, $\xi \notin \mathrm{I}(K) + K^{\dagger}$, and suppose $y \in \widehat{K}$ satisfies $y' + \xi y = f$. Then there is for every m an element $y_m \in K$ with $y - y_m \prec \xi^{-m}$. Also, $f \asymp 1 \ \Rightarrow\ y \sim f\xi^{-1}$.*

Proof. Take L and $\mathrm{e} \in L$ as at the beginning of the previous subsection, and set $\widehat{f} := y\xi \in \widehat{K}$. Then for $A := \partial + \xi$ we have $A(\widehat{f}/\xi) = f$, so

$$\left(\widehat{f}\,\frac{\mathrm{e}}{\xi}\right)' \ =\ (\widehat{f}/\xi)'\,\mathrm{e} + (\widehat{f}/\xi)\xi\,\mathrm{e} \ =\ A(\widehat{f}/\xi)\,\mathrm{e} \ =\ f\,\mathrm{e},$$

hence $\widehat{f}$ is as in the previous subsection. Now apply Proposition 3.5.3. $\square$

Corollary 3.5.8. *Let $g \in K$, $u \in K^{\times}$ be such that $g \notin \mathrm{I}(K) + K^{\dagger}$ and $\xi := g + u^{\dagger} \succ^{\flat} 1$. Suppose $z \in \widehat{K}$ satisfies $z' + gz = u$. Then $z \sim u/\xi$, and for every m there is a $z_m \in K$ such that $z - z_m \prec u\xi^{-m}$.*

Proof. Set $A := \partial + g$. Then $A_{\ltimes u} = \partial + \xi$, so $A(z) = u$ yields for $y := z/u$ that $y' + \xi y = 1$. Now observe that $\xi \notin \mathrm{I}(K) + K^{\dagger}$ and use the previous corollary. $\square$

Slots of order and degree 1. In the rest of this section we use the material above to analyze slots of order and degree 1 in K. *Below K is henselian and $(P, \mathfrak{m}, \widehat{f})$ is a slot in K with* $\operatorname{order} P = \deg P = 1$ *and $\widehat{f} \in \widehat{K} \setminus K$. We let f range over K, $\mathfrak{n}$ over $K^{\times}$, and ϕ over active elements of K.* Thus

$$\begin{aligned} P \ &=\ a(Y' + gY - u) \quad \text{where } a \in K^{\times},\ g, u \in K, \\ P_{\times \mathfrak{n}} \ &=\ a\mathfrak{n}\big(Y' + (g + \mathfrak{n}^{\dagger})Y - \mathfrak{n}^{-1}u\big). \end{aligned}$$

Since K is henselian, $(P, \mathfrak{m}, \widehat{f})$ is Z-minimal and thus equivalent to a hole in K, by Lemma 3.2.14. Also, $\operatorname{nmul} P_{\times \mathfrak{m}} = \operatorname{ndeg} P_{\times \mathfrak{m}} = 1$ by Lemma 3.2.21. We have $L_P = a(\partial + g)$, so

$$g \in K^{\dagger} \iff \ker L_P \neq \{0\}, \qquad\qquad g \in \mathrm{I}(K) + K^{\dagger} \iff \mathscr{E}^{\mathrm{e}}(L_P) \neq \emptyset,$$

using for the second equivalence the remark on $\mathscr{E}^{\mathrm{e}}(A)$ preceding Lemma 1.5.9. If $(P, \mathfrak{m}, \widehat{f})$ is isolated, then $P(f) \neq 0$ for $\widehat{f} - f \prec \mathfrak{m}$ by Lemmas 3.2.14 and 3.4.15, so, taking $f = 0$, we have $u \neq 0$.

Lemma 3.5.9. *Suppose $\partial K = K$ and $\mathrm{I}(K) \subseteq K^{\dagger}$. Then $\mathscr{E}^{\mathrm{e}}(L_P) = \emptyset$, so $(P, \mathfrak{m}, \widehat{f})$ is isolated by Lemma 3.4.5.*

Proof. Passing to an equivalent hole in K, arrange that $(P, \mathfrak{m}, \widehat{f})$ is a hole in K. Since $\partial K = K$ and $\widehat{f} \in \widehat{K} \setminus K$, the remark following Lemma 1.8.21 yields $g \notin K^{\dagger} = \mathrm{I}(K) + K^{\dagger}$, therefore $\mathscr{E}^{\mathrm{e}}(L_P) = \emptyset$. $\square$

Set $\mathfrak{v} := \mathfrak{v}(L_{P_{\times \mathfrak{m}}})$; thus $\mathfrak{v} = 1$ if $g + \mathfrak{m}^{\dagger} \preccurlyeq 1$ and $\mathfrak{v} = 1/(g + \mathfrak{m}^{\dagger})$ otherwise. Hence from Example 3.3.3 and the remarks before Lemma 3.3.17 we obtain:

$$\begin{aligned} (P, \mathfrak{m}, \widehat{f}) \text{ is normal} \quad &\iff \quad (P, \mathfrak{m}, \widehat{f}) \text{ is steep} \quad \iff \quad \mathfrak{v} \prec^{\flat} 1, \\ (P, \mathfrak{m}, \widehat{f}) \text{ is deep} \quad &\iff \quad \mathfrak{v} \prec^{\flat} 1 \text{ and } u \preccurlyeq \mathfrak{m}/\mathfrak{v}. \end{aligned}$$

We have $P(0) = -au$, and if $\mathfrak{v} \prec 1$, then $(P_{\times\mathfrak{m}})_1 \sim (a\mathfrak{m}/\mathfrak{v})Y$. Thus

$$(P, \mathfrak{m}, \widehat{f}) \text{ is strictly normal} \quad\Longleftrightarrow\quad \mathfrak{v} \prec^{\flat} 1 \text{ and } u \prec_{\Delta(\mathfrak{v})} \mathfrak{m}\mathfrak{v}.$$

We say that $(P, \mathfrak{m}, \widehat{f})$ is **balanced** if $(P, \mathfrak{m}, \widehat{f})$ is steep and $P(0) \preccurlyeq S_{P_{\times\mathfrak{m}}}(0)$, equivalently, $(P, \mathfrak{m}, \widehat{f})$ is steep and $u \preccurlyeq \mathfrak{m}$. Thus

$$(P, \mathfrak{m}, \widehat{f}) \text{ is strictly normal} \implies (P, \mathfrak{m}, \widehat{f}) \text{ is balanced} \implies (P, \mathfrak{m}, \widehat{f}) \text{ is deep},$$

and with $b \in K^{\times}$,

$$(P, \mathfrak{m}, \widehat{f}) \text{ is balanced} \iff (P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{f}/\mathfrak{n}) \text{ is balanced} \iff (bP, \mathfrak{m}, \widehat{f}) \text{ is balanced.}$$

If $(P, \mathfrak{m}, \widehat{f})$ is balanced, then so is any slot in K equivalent to $(P, \mathfrak{m}, \widehat{f})$. Moreover, if $(P, \mathfrak{m}, \widehat{f})$ is a hole in K, then $P(0) = -L_P(\widehat{f})$, so $(P, \mathfrak{m}, \widehat{f})$ is balanced iff it is steep and $L_P(\widehat{f}) \preccurlyeq S_{P_{\times\mathfrak{m}}}(0)$. By Corollary 3.3.14, if $(P, \mathfrak{m}, \widehat{f})$ is steep, then $\widehat{f} - f \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ for some f. For balanced $(P, \mathfrak{m}, \widehat{f})$ we have a variant of this fact:

Lemma 3.5.10. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is balanced and $g \notin \mathrm{I}(K) + K^{\dagger}$. Then there is for all n an f such that $\widehat{f} - f \prec \mathfrak{v}^n\mathfrak{m}$.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{f})$ by an equivalent hole in K, we arrange that $(P, \mathfrak{m}, \widehat{f})$ is a hole in K, and replacing $(P, \mathfrak{m}, \widehat{f})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$, that $\mathfrak{m} = 1$. Then $\widehat{f}' + g\widehat{f} = u$ with $g = 1/\mathfrak{v} \succ^{\flat} 1$, $g \notin \mathrm{I}(K) + K^{\dagger}$, and $u \preccurlyeq 1$. Hence the lemma follows from Corollary 3.5.7. $\square$

In the next corollary we assume that the subgroup $K^{\dagger}$ of K is divisible. (Since K is henselian and d-valued, this holds if the groups $C^{\times}$ and Γ are divisible.)

Corollary 3.5.11. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is balanced and $g \notin \mathrm{I}(K) + K^{\dagger}$. Then $(P, \mathfrak{m}, \widehat{f})$ has a strictly normal refinement $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$.*

Proof. First arrange that $(P, \mathfrak{m}, \widehat{f})$ is a hole in K. The previous lemma yields an f such that $\widehat{f} - f \preccurlyeq \mathfrak{v}^3\mathfrak{m}$. Then $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ is a strictly normal refinement of $(P, \mathfrak{m}, \widehat{f})$, by Lemma 3.3.46 (where the latter uses divisibility of $K^{\dagger}$). $\square$

Lemma 3.5.12. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is balanced with $v\widehat{f} \notin \mathscr{E}^{\mathrm{e}}(L_P)$ and $\widehat{f} - f \preccurlyeq \widehat{f}$. Then the refinement $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ is balanced.*

Proof. By Lemma 3.2.14 we arrange $(P, \mathfrak{m}, \widehat{f})$ is a hole. Replacing $(P, \mathfrak{m}, \widehat{f})$ and f by $(P_{\times\mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$ and $f/\mathfrak{m}$ we arrange next that $\mathfrak{m} = 1$. By the remark preceding Lemma 3.3.2, $(P_{+f}, 1, \widehat{f} - f)$ is steep. Take ϕ such that $v\widehat{f} \notin \mathscr{E}\big((L_P)^{\phi}\big)$, and set $\widehat{g} := \widehat{f} - f$, so $0 \neq \widehat{g} \preccurlyeq \widehat{f}$. Recall from [ADH, 5.7.5] that $L_{P^{\phi}} = (L_P)^{\phi}$ and hence $L_{P^{\phi}}(\widehat{f}) = L_P(\widehat{f})$ and $L_P(\widehat{g}) = L_{P^{\phi}}(\widehat{g})$. Thus

$$L_{P_{+f}}(\widehat{g}) \;=\; L_P(\widehat{g}) \;\preccurlyeq\; L_{P^{\phi}}\widehat{g} \;\preccurlyeq\; L_{P^{\phi}}\widehat{f} \;\asymp\; L_{P^{\phi}}(\widehat{f}) \;=\; L_P(\widehat{f}) \;\preccurlyeq\; S_P(0) \;=\; S_{P_{+f}}(0),$$

using [ADH, 4.5.1(iii)] to get the second $\preccurlyeq$ and $v\widehat{f} \notin \mathscr{E}(L_{P^{\phi}})$ to get $\asymp$; the last $\preccurlyeq$ uses $(P, 1, \widehat{f})$ being a hole. Therefore $(P_{+f}, 1, \widehat{g})$ is balanced. $\square$

Combining Lemmas 3.4.2 and 3.5.12 yields:

Corollary 3.5.13. *If $(P, \mathfrak{m}, \widehat{f})$ is balanced and isolated, and $\widehat{f} - f \preccurlyeq \widehat{f}$, then the refinement $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ is also balanced and isolated.*

We call $(P, \mathfrak{m}, \widehat{f})$ **proper** if the differential polynomial P is proper as defined in Section 1.8 (that is, $u \neq 0$ and $g + u^\dagger \succ^\flat 1$). If $(P, \mathfrak{m}, \widehat{f})$ is proper, then so are $(bP, \mathfrak{m}, \widehat{f})$ for $b \neq 0$ and $(P_{\times \mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{f}/\mathfrak{n})$, as well as each refinement $(P, \mathfrak{n}, \widehat{f})$ of $(P, \mathfrak{m}, \widehat{f})$ and each slot in K equivalent to $(P, \mathfrak{m}, \widehat{f})$. By Lemma 1.8.23, if $(P, \mathfrak{m}, \widehat{f})$ is proper, then so is $(P^\phi, \mathfrak{m}, \widehat{f})$ for $\phi \preccurlyeq 1$.

Lemma 3.5.14. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is proper and $\mathfrak{m} \asymp u$; then $(P, \mathfrak{m}, \widehat{f})$ is balanced.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{f})$ by $(P_{\times \mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$, we arrange $\mathfrak{m} = 1$. Then $u \asymp 1$ and thus $(P, 1, \widehat{f})$ is balanced. □

Proposition 3.5.15. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is proper and $v\widehat{f} \notin \mathscr{E}^{\mathrm{e}}(L_P)$. Then $(P, \mathfrak{m}, \widehat{f})$ has a balanced refinement.*

Proof. We arrange $\mathfrak{m} = 1$ as usual. By Lemmas 1.8.26 and 3.2.14 we have

$$\widehat{f} \ \sim\ u/(g + u^\dagger) \prec^\flat u.$$

Hence if $u \preccurlyeq 1$, then $(P, u, \widehat{f})$ refines $(P, 1, \widehat{f})$, and so $(P, u, \widehat{f})$ is balanced by Lemma 3.5.14. Assume now that $u \succ 1$. Then $1 \prec u \prec g$ by Lemma 1.8.25 and $\operatorname{nmul} P = 1$, and hence $u^\dagger \preccurlyeq g^\dagger \prec g$. So $g \sim g + u^\dagger \succ^\flat 1$, hence $(P, 1, \widehat{f})$ is steep, and $\widehat{f} \sim u/g$. Set $f := u/g \prec 1$; then $(P_{+f}, 1, \widehat{f} - f)$ is a steep refinement of $(P, 1, \widehat{f})$. Moreover

$$P_{+f}(0) = P(f) = af' \prec a = S_{P_{+f}}(0),$$

hence $(P_{+f}, 1, \widehat{f} - f)$ is balanced. □

Corollary 3.5.16. *Suppose K is $\uplambda$-free. Then there exists $\phi \preccurlyeq 1$ and a refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ such that $(P^\phi_{+f}, \mathfrak{n}, \widehat{f} - f)$ is balanced.*

Proof. Using Remark 3.4.7 we can replace $(P, \mathfrak{m}, \widehat{f})$ by a refinement to arrange that $(P, \mathfrak{m}, \widehat{f})$ is isolated. Then $u \neq 0$ by the remark before Lemma 3.5.9, so by Lemma 1.8.24 , P^ϕ is proper, eventually. Now apply Proposition 3.5.15 to a proper (and isolated) $(P^\phi, \mathfrak{m}, \widehat{f})$ with $\phi \preccurlyeq 1$. □

Corollary 3.5.17. *Suppose K is $\uplambda$-free, $\partial K = K$, $\mathrm{I}(K) \subseteq K^\dagger$, and $K^\dagger$ is divisible. Then $(P, \mathfrak{m}, \widehat{f})$ has a refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ such that $(P^\phi_{+f}, \mathfrak{n}, \widehat{f} - f)$ is strictly normal for some $\phi \preccurlyeq 1$.*

Proof. Corollary 3.5.16 yields a refinement $(P_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ of $(P, \mathfrak{m}, \widehat{f})$ and a $\phi \preccurlyeq 1$ such that $(P^\phi_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ is balanced. By Lemma 3.5.9 with K^ϕ in the role of K and $(P^\phi_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ in the role of $(P, \mathfrak{m}, \widehat{f})$ we can apply Corollary 3.5.11 to $(P^\phi_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ to give a strictly normal refinement $(P^\phi_{f_1+f_2}, \mathfrak{n}, \widehat{f} - f_1 - f_2)$ of it. Thus for $f := f_1 + f_2$ the refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ has the property that $(P^\phi_{+f}, \mathfrak{n}, \widehat{f} - f)$ is strictly normal. □

Combining this corollary with Corollaries 3.2.8, 3.3.49, and Lemma 3.3.40 yields:

Corollary 3.5.18. *If K is $\upomega$-free and algebraically closed with $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$, then every minimal hole in K of order $\geqslant 1$ has a refinement $(Q, \mathfrak{n}, \widehat{g})$ such that $(Q^\phi, \mathfrak{n}, \widehat{g})$ is deep and strictly normal, eventually.*

Remark. Suppose K is $\boldsymbol{\lambda}$-free, with $\partial K = K$, $\mathrm{I}(K) \subseteq K^\dagger$, and $K^\dagger$ is divisible. By Corollary 3.3.37 every linear slot in K of order $r \geqslant 1$ has a refinement $(Q, \mathfrak{n}, \widehat{g})$ such that $(Q^\phi, \mathfrak{n}, \widehat{g})$ is deep and normal, eventually. We don't know whether every linear minimal hole in K of order $r \geqslant 1$ has a refinement $(Q, \mathfrak{n}, \widehat{g})$ such that $(Q^\phi, \mathfrak{n}, \widehat{g})$ is deep and strictly normal, eventually. (For $r = 1$ this holds by Corollary 3.5.17.)

Part 4. Slots in H-Fields

Here we specialize to the case that K is the algebraic closure of a Liouville closed H-field H with small derivation. After the preliminary Sections 4.1 and 4.2 we come in Sections 4.3–4.5 to the technical heart of Part 4. Section 4.3 shows that every minimal hole in K gives rise to a *split-normal* slot $(Q, \mathfrak{n}, \widehat{b})$ in H: a normal slot in H whose linear part $L_{Q_{\times\mathfrak{n}}} \in H[\partial]$ "asymptotically" splits over K; see Definition 4.3.3 for the precise definition, and Theorem 4.3.9 for the detailed statement of the main result of this section. In the intended setting where H is a Hardy field, this asymptotic splitting will allow us to define in Part 6 a contractive operator on a space of real-valued functions; this operator then has a fixed point whose germ y satisfies $Q(y) = 0$, $y \prec \mathfrak{n}$. A main difficulty in that part will lie in guaranteeing that such germs y have similar asymptotic properties as $\widehat{b}$. Sections 4.4 and 4.5 prepare the ground for dealing with this: In Section 4.4 we strengthen the concept of isolated slot to *ultimate* slot (in H, or in K). This relies on the ultimate exceptional values of linear differential operators over K introduced in Part 2. In Section 4.5 we single out among split-normal slots in H those that are *repulsive-normal*, culminating in the proof of Theorem 4.5.28: an analogue of Theorem 4.3.9 producing repulsive-normal ultimate slots in H from minimal holes in K.

4.1. Some Valuation-Theoretic Lemmas

The present section contains preliminaries for the next section on approximating splittings of linear differential operators; these facts in turn will be used in Section 4.3 on split-normality. We shall often deal with real closed fields with extra structure, denoted usually by H, since the results in this section about such H will later be applied to H-fields and Hardy fields. We begin by summarizing some purely valuation-theoretic facts.

Completion and specialization of real closed valued fields. Let H be a real closed valued field whose valuation ring $\mathcal{O}$ is convex in H (with respect to the unique ordering on H making H an ordered field). Using [ADH, 3.5.15] we equip the algebraic closure $K = H[\mathrm{i}]$ ($\mathrm{i}^2 = -1$) of H with its unique valuation ring lying over $\mathcal{O}$, which is $\mathcal{O} + \mathcal{O}i$. We set $\Gamma := v(H^\times)$, so $\Gamma_K = \Gamma$.

Lemma 4.1.1. *The completion H^{c} of the valued field H is real closed, its valuation ring is convex in H^{c}, and there is a unique valued field embedding $H^{\mathrm{c}} \to K^{\mathrm{c}}$ over H. Identifying H^{c} with its image under this embedding we have $H^{\mathrm{c}}[\mathrm{i}] = K^{\mathrm{c}}$.*

Proof. For the first two claims, see [ADH, 3.5.20]. By [ADH, 3.2.20] we have a unique valued field embedding $H^{\mathrm{c}} \to K^{\mathrm{c}}$ over H, and viewing H^{c} as a valued subfield of K^{c} via this embedding we have $K^{\mathrm{c}} = H^{\mathrm{c}}K = H^{\mathrm{c}}[\mathrm{i}]$ by [ADH, 3.2.29]. $\square$

We identify H^{c} with its image in K^{c} as in the previous lemma. Fix a convex subgroup Δ of Γ. Let $\dot{\mathcal{O}}$ be the valuation ring of the coarsening of H by Δ, with maximal ideal $\dot{o}$. Then by [ADH, 3.5.11 and subsequent remarks] $\dot{\mathcal{O}}$ and $\dot{o}$ are convex in H, the specialization $\dot{H} = \dot{\mathcal{O}}/\dot{o}$ of H by Δ is naturally an ordered and valued field, and the valuation ring of $\dot{H}$ is convex in $\dot{H}$. Moreover, $\dot{H}$ is even real closed by [ADH, 3.5.16]. Likewise, the coarsening of K by Δ has valuation ring $\dot{\mathcal{O}}_K$ with maximal ideal $\dot{o}_K$ and valued residue field $\dot{K}$. Thus $\dot{\mathcal{O}}_K$ lies over $\dot{\mathcal{O}}$

by [ADH, 3.4, subsection *Coarsening and valued field extensions*], so $(K, \dot{\mathcal{O}}_K)$ is a valued field extension of $(H, \dot{\mathcal{O}})$. In addition:

Lemma 4.1.2. *$\dot{K}$ is a valued field extension of $\dot{H}$ and an algebraic closure of $\dot{H}$.*

Proof. The second part follows by general valuation theory from K being an algebraic closure of H. In fact, with the image of $\mathrm{i} \in \mathcal{O}_K \subseteq \dot{\mathcal{O}}_K$ in $\dot{K}$ denoted by the same symbol, we have $\dot{K} = \dot{H}[\mathrm{i}]$. $\square$

Next, let $\widehat{H}$ be an immediate valued field extension of H. We equip $\widehat{H}$ with the unique field ordering making it an ordered field extension of H in which $\mathcal{O}_{\widehat{H}}$ is convex; see [ADH, 3.5.12]. Choose i in a field extension of $\widehat{H}$ with $\mathrm{i}^2 = -1$. Equip $\widehat{H}[\mathrm{i}]$ with the unique valuation ring of $\widehat{H}[\mathrm{i}]$ that lies over $\mathcal{O}_{\widehat{H}}$, namely $\mathcal{O}_{\widehat{H}} + \mathcal{O}_{\widehat{H}}\mathrm{i}$ [ADH, 3.5.15]. Let $\widehat{a} = \widehat{b} + \widehat{c}\mathrm{i} \in \widehat{H}[\mathrm{i}] \setminus H[\mathrm{i}]$ with $\widehat{b}, \widehat{c} \in \widehat{H}$, and let b, c range over H. Then

$$v\big(\widehat{a} - (b + c\mathrm{i})\big) \;=\; \min\big\{v(\widehat{b} - b), v(\widehat{c} - c)\big\}$$

and thus $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{b} - H)$ and $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{c} - H)$.

Lemma 4.1.3. *We have $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$. Moreover, the following are equivalent:*

(i) *$v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$;*
(ii) *for all b there is a c with $v\big(\widehat{a} - (b + c\mathrm{i})\big) = v(\widehat{b} - b)$;*
(iii) *$v\big(\widehat{a} - H[\mathrm{i}]\big) = v(\widehat{b} - H)$.*

Proof. For the first assertion, use that $v(\widehat{b} - H), v(\widehat{c} - H) \subseteq \Gamma_\infty$ are downward closed. Suppose $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$, and let b be given. If $\widehat{c} \in H$, then for $c := \widehat{c}$ we have $v\big(\widehat{a} - (b + c\mathrm{i})\big) = v(\widehat{b} - b)$. Suppose $\widehat{c} \notin H$. Then $v(\widehat{c} - H) \subseteq \Gamma$ does not have a largest element and $v(\widehat{b} - b) \in v(\widehat{c} - H)$, so we have c with $v(\widehat{b} - b) < v(\widehat{c} - c)$; thus

$$v\big(\widehat{a} - (b + c\mathrm{i})\big) = \min\big\{v(\widehat{b} - b), v(\widehat{c} - c)\big\} = v(\widehat{b} - b).$$

This shows (i) $\Rightarrow$ (ii). Moreover, (ii) $\Rightarrow$ (iii) follows from $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{b} - H)$, and (iii) $\Rightarrow$ (i) from $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{c} - H)$. $\square$

So if $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$, then: $\widehat{a}$ is special over $H[\mathrm{i}]$ $\iff$ $\widehat{b}$ is special over H.

To apply Lemma 4.1.3 to H-fields we assume in the next lemma more generally that H is equipped with a derivation making it a d-valued field and that $\widehat{H}$ is equipped with a derivation ∂ making it an asymptotic field extension of H; then $\widehat{H}$ is also d-valued with the same constant field as H [ADH, 9.1.2].

Lemma 4.1.4. *Suppose H is closed under integration. Then we have:*

$$v(\widehat{b} - H) \subseteq v(\widehat{c} - H) \implies v(\partial\widehat{b} - H) \subseteq v(\partial\widehat{c} - H).$$

Proof. Assume $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. Let $b \in H$, and take $g \in H$ with $g' = b$; adding a suitable constant to g we arrange $\widehat{b} - g \not\asymp 1$. Next, take $h \in H$ with $\widehat{b} - g \asymp \widehat{c} - h$. Then

$$\partial\widehat{b} - b \;=\; \partial(\widehat{b} - g) \;\asymp\; \partial(\widehat{c} - h) \;=\; \partial\widehat{c} - h',$$

so $v(\partial\widehat{b} - b) \in v(\partial\widehat{c} - H)$. $\square$

Embedding into the completion. In this subsection K is an asymptotic field, $\Gamma := v(K^\times) \neq \{0\}$, and L is an asymptotic field extension of K such that Γ is cofinal in Γ_L.

Lemma 4.1.5. *Let $a \in L$ and let (a_ρ) be a c-sequence in K with $a_\rho \to a$ in L. Then for each n, $(a_\rho^{(n)})$ is a c-sequence in K with $a_\rho^{(n)} \to a^{(n)}$ in L.*

Proof. By induction on n it suffices to treat the case $n=1$. Let $\gamma \in \Gamma_L$; we need to show the existence of an index σ such that $v(a'-a'_\rho) > \gamma$ for all $\rho > \sigma$. By [ADH, 9.2.6] we have $f \in L^\times$ with $f \prec 1$ and $v(f') \geqslant \gamma$. Take σ such that $v(a-a_\rho) > vf$ for all $\rho > \sigma$. Then $v(a'-a'_\rho) > v(f') \geqslant \gamma$ for $\rho > \sigma$. $\square$

Let K^{c} be the completion of the valued differential field K; then K^{c} is asymptotic by [ADH, 9.1.6]. Lemma 4.1.5 and [ADH, 3.2.13 and 3.2.15] give:

Corollary 4.1.6. *Let $(a_i)_{i \in I}$ be a family of elements of L such that a_i is the limit in L of a c-sequence in K, for each $i \in I$. Then there is a unique embedding $K\langle (a_i)_{i\in I}\rangle \to K^{\mathrm{c}}$ of valued differential fields over K.*

Next suppose that H is a real closed asymptotic field whose valuation ring $\mathcal{O}$ is convex in H with $\mathcal{O} \neq H$, the asymptotic extension $\widehat{H}$ of H is immediate, and i is an element of an asymptotic extension of $\widehat{H}$ with $i^2 = -1$. Then $i \notin \widehat{H}$, and we identify H^{c} with a valued subfield of $H[i]^{\mathrm{c}}$ as in Lemma 4.1.1, so that $H^{\mathrm{c}}[i] = H[i]^{\mathrm{c}}$ as in that lemma. Using also Lemma 4.1.5 we see that H^{c} is actually a valued *differential* subfield of the asymptotic field $H[i]^{\mathrm{c}}$, and so $H^{\mathrm{c}}[i] = H[i]^{\mathrm{c}}$ also as *asymptotic* fields. Thus by Corollary 4.1.6 applied to $K := H$ and $L := \widehat{H}$:

Corollary 4.1.7. *Let $a \in \widehat{H}[i]$ be the limit in $\widehat{H}[i]$ of a c-sequence in $H[i]$. Then $\operatorname{Re} a$, $\operatorname{Im} a$ are limits in $\widehat{H}$ of c-sequences in H, hence there is a unique embedding $H[i]\langle \operatorname{Re} a, \operatorname{Im} a\rangle \to H^{\mathrm{c}}[i]$ of valued differential fields over $H[i]$.*

4.2. Approximating Linear Differential Operators

In this section K is a valued differential field with small derivation, $\Gamma := v(K^\times)$. For later use we prove here Corollaries 4.2.6 and 4.2.9 and consider *strong splitting*. Much of this section rests on the following basic estimate for linear differential operators which split over K:

Lemma 4.2.1. *Let $b_1,\dots,b_r \in K$ and n be given. Then there exists $\gamma_0 \in \Gamma^{\geqslant}$ such that for all $b_1^\bullet,\dots,b_r^\bullet \in K$ and $\gamma \in \Gamma$ with $\gamma > \gamma_0$ and $v(b_i - b_i^\bullet) \geqslant (n+r)\gamma$ for $i = 1,\dots,r$, we have $v(B - B^\bullet) \geqslant vB + n\gamma$, where*

$$B \ := \ (\partial - b_1)\cdots(\partial - b_r) \in K[\partial], \quad B^\bullet \ := \ (\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) \in K[\partial].$$

Proof. By induction on $r \in \mathbb{N}$. The case $r = 0$ is clear (any $\gamma_0 \in \Gamma^{\geqslant}$ works). Suppose the lemma holds for a certain r. Let $b_1,\dots,b_{r+1} \in K$ and n be given. Set $\beta_i := vb_i$ ($i = 1,\dots,r+1$). Take γ_0 as in the lemma applied to $b_1,\dots,b_r$ and $n+1$ in place of n, and let $\gamma_1 := \gamma_0$ if $b_{r+1} = 0$, $\gamma_1 := \max\{\gamma_0, |\beta_{r+1}|\}$ otherwise. Let $b_1^\bullet,\dots,b_{r+1}^\bullet \in K$ and $\gamma \in \Gamma$ with $\gamma > \gamma_1$ and $v(b_i - b_i^\bullet) \geqslant (n+r+1)\gamma$ for $i = 1,\dots,r+1$. Set

$$B \ := \ (\partial - b_1)\cdots(\partial - b_r), \qquad B^\bullet \ := \ (\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet), \qquad E \ := \ B - B^\bullet.$$

Then

$$B(\partial - b_{r+1}) \ = \ B^\bullet(\partial - b_{r+1}^\bullet) + B^\bullet(b_{r+1}^\bullet - b_{r+1}) + E(\partial - b_{r+1}).$$

Inductively we have $vE \geqslant vB + (n+1)\gamma$. Suppose $E \neq 0$ and $0 \neq b_{r+1} \not\asymp 1$. Then by [ADH, 6.1.5],

$$\begin{aligned} v_E(\beta_{r+1}) - v_B(\beta_{r+1}) &= vE - vB + o(\beta_{r+1}) \\ &\geqslant (n+1)\gamma + o(\beta_{r+1}) \\ &\geqslant n\gamma + |\beta_{r+1}| + o(\beta_{r+1}) > n\gamma. \end{aligned}$$

Hence, using $E(\partial - b_{r+1}) = E\partial - Eb_{r+1}$ and $v(E\partial) = v(E) \neq v_E(\beta_{r+1})$,

$$\begin{aligned} v\big(E(\partial - b_{r+1})\big) &= \min\big\{vE, v_E(\beta_{r+1})\big\} > \min\big\{vB, v_B(\beta_{r+1})\big\} + n\gamma \\ &= v\big(B(\partial - b_{r+1})\big) + n\gamma, \end{aligned}$$

where for the last equality we use $vB \neq v_B(\beta_{r+1})$. Also,

$$v\big(B^\bullet(b^\bullet_{r+1} - b_{r+1})\big) = v_{B^\bullet}\big(v(b^\bullet_{r+1} - b_{r+1})\big) \geqslant v_{B^\bullet}\big((n+r+1)\gamma\big) = v_B\big((n+r+1)\gamma\big)$$

where we use [ADH, 6.1.7] for the last equality. Moreover, by [ADH, 6.1.4],

$$v_B\big((n+r+1)\gamma\big) - n\gamma \geqslant vB + (r+1)\gamma + o(\gamma) > vB \geqslant v\big(B(\partial - b_{r+1})\big).$$

This yields the desired result for $E \neq 0$, $0 \neq b_{r+1} \not\asymp 1$. The cases $E \neq 0$, $b_{r+1} = 0$ and $E = 0$, $0 \neq b_{r+1} \not\asymp 1$ are simpler versions of the above, and so is the case $E \neq 0$, $b_{r+1} \asymp 1$ using [ADH, 5.6.1(i)]. The remaining cases, $E = 0$, $b_{r+1} = 0$ and $E = 0$, $b_{r+1} \asymp 1$, are even simpler to handle. $\square$

Corollary 4.2.2. *Let $a, b_1, \dots, b_r \in K$, $a \neq 0$. Then there exists $\gamma_0 \in \Gamma^{\geqslant}$ such that for all $a^\bullet, b^\bullet_1, \dots, b^\bullet_r \in K$ and $\gamma \in \Gamma$ with $\gamma > \gamma_0$, $v(a - a^\bullet) \geqslant va + \gamma$, and $v(b_i - b^\bullet_i) \geqslant (r+1)\gamma$ for $i = 1, \dots, r$, we have $v(A - A^\bullet) \geqslant vA + \gamma$, where*

$$A := a(\partial - b_1)\cdots(\partial - b_r) \in K[\partial], \quad A^\bullet := a^\bullet(\partial - b^\bullet_1)\cdots(\partial - b^\bullet_r) \in K[\partial].$$

Proof. Take γ_0 as in the previous lemma applied to $b_1, \dots, b_r$ and $n = 1$, and let $B = (\partial - b_1)\cdots(\partial - b_r)$, $A = aB$. Let $a^\bullet, b^\bullet_1, \dots, b^\bullet_r \in K$ and $\gamma \in \Gamma$ be such that $\gamma > \gamma_0$, $v(a - a^\bullet) \geqslant va + \gamma$, and $v(b_i - b^\bullet_i) \geqslant (r+1)\gamma$ for $i = 1, \dots, r$. Set $B^\bullet := (\partial - b^\bullet_1)\cdots(\partial - b^\bullet_r)$, $A^\bullet := a^\bullet B^\bullet$. Then

$$E := A - A^\bullet = a(B - B^\bullet) + (a - a^\bullet)B^\bullet.$$

Lemma 4.2.1 gives $vB^\bullet = vB$, and so

$$v\big(a(B - B^\bullet)\big) \geqslant va + vB + \gamma = vA + \gamma, \quad v\big((a - a^\bullet)B^\bullet\big) = v(a - a^\bullet) + vB \geqslant vA + \gamma,$$

so $vE \geqslant vA + \gamma$. $\square$

In the rest of this subsection we assume $P \in K\{Y\} \setminus K$, set $r := \operatorname{order} P$, and let $\boldsymbol{i}$, $\boldsymbol{j}$ range over $\mathbb{N}^{1+r}$.

Lemma 4.2.3. *For $\delta := v\big(P - P(0)\big)$ and all $h \in \mathcal{O}$ we have $v\big(P_{+h} - P\big) \geqslant \delta + \frac{1}{2}vh$.*

Proof. Note that $\delta \in \Gamma$ and $v(P_{\boldsymbol{j}}) \geqslant \delta$ for all $\boldsymbol{j}$ with $|\boldsymbol{j}| \geqslant 1$. Let $h \in \mathcal{O}^{\neq}$ and $\boldsymbol{i}$ be given; we claim that $v\big((P_{+h})_{\boldsymbol{i}} - P_{\boldsymbol{i}}\big) \geqslant \delta + \frac{1}{2}vh$. By [ADH, (4.3.1)] we have

$$(P_{+h})_{\boldsymbol{i}} = P_{\boldsymbol{i}} + Q(h) \quad \text{where } Q(Y) := \sum_{|\boldsymbol{j}| \geqslant 1} \binom{\boldsymbol{i}+\boldsymbol{j}}{\boldsymbol{i}} P_{\boldsymbol{i}+\boldsymbol{j}} Y^{\boldsymbol{j}} \in K\{Y\}.$$

From $Q(0) = 0$ and [ADH, 6.1.4] we obtain

$$v(Q_{\times h}) \geqslant v(Q) + vh + o(vh) \geqslant \delta + \tfrac{1}{2}vh.$$

Together with $v\big(Q(h)\big) \geqslant v(Q_{\times h})$ this yields the lemma. $\square$

Corollary 4.2.4. *Let $f \in K$. Then there exists $\delta \in \Gamma$ such that for all $f^\bullet \in K$ with $f - f^\bullet \prec 1$ we have $v\big(P_{+f^\bullet} - P_{+f}\big) \geqslant \delta + \frac{1}{2}v(f^\bullet - f)$.*

Proof. Take δ as in the preceding lemma with P_{+f} in place of P and $h = f^\bullet - f$. $\square$

Corollary 4.2.5. *Let $a, b_1, \dots, b_r, f \in K$ be such that*

$$A \;:=\; L_{P_{+f}} \;=\; a(\partial - b_1)\cdots(\partial - b_r), \qquad a \;\neq\; 0.$$

Then there exists $\gamma_1 \in \Gamma^{\geqslant}$ such that for all $a^\bullet, b_1^\bullet, \dots, b_r^\bullet, f^\bullet \in K$ and $\gamma \in \Gamma$, if

$$\gamma > \gamma_1,\ v(a - a^\bullet) \geqslant va + \gamma,\ v(b_i - b_i^\bullet) \geqslant (r+1)\gamma\ (i = 1, \dots, r),\ \textit{and}\ v(f - f^\bullet) \geqslant 4\gamma,$$

then

(i) $v\big(P_{+f^\bullet} - P_{+f}\big) \geqslant vA + \gamma$; *and*
(ii) $L_{P_{+f^\bullet}} = a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) + E$ *where* $vE \geqslant vA + \gamma$.

Proof. Take γ_0 as in Corollary 4.2.2 applied to $a, b_1, \dots, b_r$, and take δ as in Corollary 4.2.4. Then $\gamma_1 := \max\{\gamma_0, vA - \delta\}$ has the required property. $\square$

In the next result L is a valued differential field extension of K with small derivation such that Γ is cofinal in Γ_L. Then the natural inclusion $K \to L$ extends uniquely to an embedding $K^{\mathrm{c}} \to L^{\mathrm{c}}$ of valued fields by [ADH, 3.2.20]. It is easy to check that this is even an embedding of valued *differential* fields; we identify K^{c} with a valued differential subfield of L^{c} via this embedding.

Corollary 4.2.6. *Let $a, b_1, \dots, b_r \in L^{\mathrm{c}}$ and $f \in K^{\mathrm{c}}$ be such that in $L^{\mathrm{c}}[\partial]$,*

$$A \;:=\; L_{P_{+f}} \;=\; a(\partial - b_1)\cdots(\partial - b_r), \qquad a, f \neq 0, \quad \mathfrak{v} := \mathfrak{v}(A) \prec 1,$$

and let $w \in \mathbb{N}$. Then there are $a^\bullet, b_1^\bullet, \dots, b_r^\bullet \in L$ and $f^\bullet \in K$ such that

$$a^\bullet \;\sim\; a, \qquad f^\bullet \;\sim\; f, \qquad A^\bullet := L_{P_{+f^\bullet}} \;\sim\; A, \qquad \operatorname{order} A^\bullet \;=\; r, \qquad \mathfrak{v}(A^\bullet) \;\sim\; \mathfrak{v},$$

and such that for $\Delta := \big\{\alpha \in \Gamma_L : \alpha = o\big(v(\mathfrak{v})\big)\big\}$ we have in $L[\partial]$,

$$A^\bullet \;=\; a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) + E, \qquad E \prec_\Delta \mathfrak{v}^{w+1}A.$$

Proof. Let $\gamma_1 \in \Gamma_L^{\geqslant}$ be as in Corollary 4.2.5 applied to L^{c} in place of K, and take $\gamma_2 \in \Gamma$ such that $\gamma_2 \geqslant \max\{\gamma_1, \frac{1}{4}vf\} + vA$ and $\gamma_2 \geqslant v\big((P_{+f})_{\boldsymbol{i}}\big)$ for all $\boldsymbol{i}$ with $(P_{+f})_{\boldsymbol{i}} \neq 0$. Let $\gamma \in \Gamma$ and $\gamma > \gamma_2$. Then $\gamma - vA > \gamma_1$. By the density of K, L in K^{c}, L^{c}, respectively, we can take $a^\bullet, b_1^\bullet, \dots, b_r^\bullet \in L$ and $f^\bullet \in K$ such that

$$v(a - a^\bullet) \;\geqslant\; va + (\gamma - vA), \qquad v(b_i - b_i^\bullet) \geqslant (r+1)(\gamma - vA) \;\text{ for } i = 1, \dots, r,$$

and $v(f - f^\bullet) \geqslant 4(\gamma - vA) > vf$. Then $a^\bullet \sim a$, $f^\bullet \sim f$, and by Corollary 4.2.5,

$$v\big(P_{+f^\bullet} - P_{+f}\big) \;\geqslant\; \gamma, \quad A^\bullet := L_{P_{+f^\bullet}} \;=\; a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) + E, \quad vE \geqslant \gamma.$$

Hence $(P_{+f^\bullet})_{\boldsymbol{i}} \sim (P_{+f})_{\boldsymbol{i}}$ if $(P_{+f})_{\boldsymbol{i}} \neq 0$, and $v\big((P_{+f^\bullet})_{\boldsymbol{i}}\big) > \gamma_2 \geqslant vA$ if $(P_{+f})_{\boldsymbol{i}} = 0$, so $A^\bullet \sim A$, order $A^\bullet = r$, and $\mathfrak{v}(A^\bullet) \sim \mathfrak{v}$. Choosing γ so that also $\gamma > v(\mathfrak{v}^{w+1}A) + \Delta$ we achieve in addition that $E \prec_\Delta \mathfrak{v}^{w+1}A$. $\square$

Keeping it real. *In this subsection H is a real closed H-asymptotic field with small derivation whose valuation ring is convex, with $\Gamma := v(H^\times) \neq \{0\}$, and K is the asymptotic extension $H[\mathrm{i}]$ of H with $\mathrm{i}^2 = -1$.* Then H^{c} is real closed and $H^{\mathrm{c}}[\mathrm{i}] = K^{\mathrm{c}}$ as valued field extension of H according to Lemma 4.1.1, and as asymptotic field extension of H by the discussion after Corollary 4.1.6. Using the real splittings from Definition 1.1.5 we show here that we can "preserve the reality of A" in Corollary 4.2.6.

Lemma 4.2.7. *Let $A \in H^{\mathrm{c}}[\partial]$ be of order $r \geqslant 1$ and let $(g_1, \dots, g_r) \in H^{\mathrm{c}}[\mathrm{i}]^r$ be a real splitting of A over $H^{\mathrm{c}}[\mathrm{i}]$. Then for every $\gamma \in \Gamma$ there are $g_1^\bullet, \dots, g_r^\bullet$ in $H[\mathrm{i}]$ such that $v(g_i - g_i^\bullet) > \gamma$ for $i = 1, \dots, r$,*

$$A^\bullet \ := \ (\partial - g_1^\bullet) \cdots (\partial - g_r^\bullet) \in H[\partial],$$

and $(g_1^\bullet, \dots, g_r^\bullet)$ is a real splitting of $A^\bullet$ over $H[\mathrm{i}]$.

Proof. We can reduce to the case where $r = 1$ or $r = 2$. If $r = 1$, then the lemma holds trivially, so suppose $r = 2$. Then again the lemma holds trivially if $g_1, g_2 \in H^{\mathrm{c}}$, so we can assume instead that

$$g_1 \ = \ a - b\mathrm{i} + b^\dagger, \quad g_2 \ = \ a + b\mathrm{i}, \qquad a \in H^{\mathrm{c}},\ b \in (H^{\mathrm{c}})^\times.$$

Let $\gamma \in \Gamma$ be given. The density of H in H^{c} gives $a^\bullet \in H$ with $v(a - a^\bullet) \geqslant \gamma$. Next, choose $\gamma^\bullet \in \Gamma$ such that $\gamma^\bullet \geqslant \max\{\gamma, vb\}$ and $\alpha' > \gamma$ for all nonzero $\alpha > \gamma^\bullet - vb$ in Γ, and take $b^\bullet \in H$ with $v(b - b^\bullet) > \gamma^\bullet$. Then $v(b - b^\bullet) > \gamma$ and $b \sim b^\bullet$. In fact, $b = b^\bullet(1 + \varepsilon)$ where $v\varepsilon + vb = v(b - b^\bullet) > \gamma^\bullet$ and so $v\big((b/b^\bullet)^\dagger\big) = v(\varepsilon') > \gamma$. Set $g_1^\bullet := a^\bullet - b^\bullet\mathrm{i} + b^{\bullet\dagger}$ and $g_2^\bullet := a^\bullet + b^\bullet\mathrm{i}$. Then

$$v(g_1 - g_1^\bullet) \ = \ v\big(a - a^\bullet + (b/b^\bullet)^\dagger + (b^\bullet - b)\mathrm{i}\big) \ > \ \gamma, \quad v(g_2 - g_2^\bullet) \ > \ \gamma,$$
$$(\partial - g_1^\bullet) \cdot (\partial - g_2^\bullet) \ = \ \partial^2 - \big(2a^\bullet + b^{\bullet\dagger}\big)\partial + \big((-a^\bullet)' + a^{\bullet 2} + a^\bullet b^{\bullet\dagger} + b^{\bullet 2}\big) \ \in \ H[\partial].$$

Hence $(g_1^\bullet, g_2^\bullet)$ is a real splitting of $A^\bullet := (\partial - g_1^\bullet)(\partial - g_2^\bullet) \in H[\partial]$. □

In the next two corollaries $a \in (H^{\mathrm{c}})^\times$ and $b_1, \dots, b_r \in K^{\mathrm{c}}$ are such that

$$A \ := \ a(\partial - b_1) \cdots (\partial - b_r) \in H^{\mathrm{c}}[\partial],$$

$(b_1, \dots, b_r)$ is a real splitting of A over K^{c}, and $\mathfrak{v} := \mathfrak{v}(A) \prec 1$. We set $\Delta := \Delta(\mathfrak{v})$.

Corollary 4.2.8. *Suppose $A = L_{P_{+f}}$ with $P \in H\{Y\}$ of order $r \geqslant 1$ and f in $(H^{\mathrm{c}})^\times$. Let $\gamma \in \Gamma$ and $w \in \mathbb{N}$. Then there is $f^\bullet \in H^\times$ such that $v(f^\bullet - f) \geqslant \gamma$,*

$$f^\bullet \ \sim \ f, \quad A^\bullet \ := \ L_{P_{+f^\bullet}} \ \sim \ A, \quad \operatorname{order} A^\bullet \ = \ r, \quad \mathfrak{v}(A^\bullet) \ \sim \ \mathfrak{v}, \tag{4.2.1}$$

and we have $a^\bullet \in H^\times$, $b_1^\bullet, \dots, b_r^\bullet \in K$, and $B^\bullet, E^\bullet \in H[\partial]$ with $A^\bullet = B^\bullet + E^\bullet$, $E^\bullet \prec_\Delta \mathfrak{v}^{w+1} A$, such that

$$B^\bullet \ = \ a^\bullet(\partial - b_1^\bullet) \cdots (\partial - b_r^\bullet), \qquad v(a - a^\bullet),\ v(b_1 - b_1^\bullet),\ \dots\ , v(b_r - b_r^\bullet) \ \geqslant \ \gamma,$$

and $(b_1^\bullet, \dots, b_r^\bullet)$ is a real splitting of $B^\bullet$ over K.

Proof. We apply Corollary 4.2.6 with H, K in the role of K, L, and take γ_1, γ_2 as in the proof of that corollary. We can assume $\gamma > \gamma_2$, so that $\gamma - vA > 0$. The density of H in H^{c} gives $a^\bullet \in H$ such that $v(a - a^\bullet) \geqslant \max\big\{va + (\gamma - vA), \gamma\big\}$ (so $a^\bullet \sim a$), and Lemma 4.2.7 gives $b_1^\bullet, \dots, b_r^\bullet \in K$ such that $v(b_i - b_i^\bullet) \geqslant \max\big\{(r+1)(\gamma - vA), \gamma\big\}$ for $i = 1, \dots, r$, and $(b_1^\bullet, \dots, b_r^\bullet)$ is a real splitting of

$$B^\bullet \ := \ a^\bullet(\partial - b_1^\bullet) \cdots (\partial - b_r^\bullet) \in H[\partial]$$

over K. Take $f^\bullet \in H$ with $v(f - f^\bullet) \geqslant \max\{4(\gamma - vA), \gamma\}$. Then (4.2.1) follows from the proof of Corollary 4.2.6. We can increase γ so that $\gamma > v(\mathfrak{v}^{w+1}A) + \Delta$, and then we have $A^\bullet - B^\bullet \prec_\Delta \mathfrak{v}^{w+1}A$. $\square$

This result persists after multiplicative conjugation:

Corollary 4.2.9. *Suppose $A = L_{P_{+f,\times\mathfrak{m}}}$ with $P \in H\{Y\}$ of order $r \geqslant 1$, and f in $(H^c)^\times$, $\mathfrak{m} \in H^\times$. Let $\gamma \in \Gamma$, $w \in \mathbb{N}$. Then there is $f^\bullet \in H^\times$ such that*

$$v(f^\bullet - f) \geqslant \gamma, \quad f^\bullet \sim f, \quad A^\bullet := L_{P_{+f^\bullet,\times\mathfrak{m}}} \sim A, \quad \operatorname{order} A^\bullet = r, \quad \mathfrak{v}(A^\bullet) \sim \mathfrak{v},$$

and we have $a^\bullet \in H^\times$, $b_1^\bullet, \dots, b_r^\bullet \in K$, and $B^\bullet, E^\bullet \in H[\partial]$ with the properties stated in the previous corollary.

Proof. Put $Q := P_{\times\mathfrak{m}} \in H\{Y\}$, $g := f/\mathfrak{m} \in H^c$; then $Q_{+g} = P_{+f,\times\mathfrak{m}}$. Applying the previous corollary to Q, g in place of P, f yields $g^\bullet \in H^\times$, $a^\bullet \in H^\times$, and $b_1^\bullet, \dots, b_r^\bullet \in K$ such that $v(g^\bullet - g) \geqslant \gamma - v\mathfrak{m}$,

$$g^\bullet \sim g, \qquad A^\bullet := L_{Q_{+g^\bullet}} \sim A, \qquad \operatorname{order} A^\bullet = r, \qquad \mathfrak{v}(A^\bullet) \sim \mathfrak{v}$$

and $A^\bullet = B^\bullet + E^\bullet$, with $B^\bullet, E^\bullet \in H[\partial]$, $E^\bullet \prec_\Delta \mathfrak{v}^{w+1}A$, and

$$B^\bullet = a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet), \qquad v(a - a^\bullet),\ v(b_1 - b_1^\bullet),\ \dots\ , v(b_r - b_r^\bullet) \geqslant \gamma,$$

and $(b_1^\bullet, \dots, b_r^\bullet)$ is a real splitting of $B^\bullet$ over K. Therefore $f^\bullet := g^\bullet\mathfrak{m} \in H^\times$ and $a^\bullet, b_1^\bullet, \dots, b_r^\bullet$ have the required properties. $\square$

Strong splitting. *In this subsection H is a real closed H-field with small derivation and asymptotic integration.* Thus $K := H[\mathrm{i}]$ is a d-valued extension of H. Let $A \in K[\partial]^{\neq}$ have order $r \geqslant 1$ and set $\mathfrak{v} := \mathfrak{v}(A)$, and let f, g, h (possibly subscripted) range over K. Recall from Section 1.1 that a splitting of A over K is an r-tuple $(g_1, \dots, g_r)$ such that

$$A = f(\partial - g_1)\cdots(\partial - g_r) \quad \text{where } f \neq 0.$$

We call such a splitting $(g_1, \dots, g_r)$ of A over K **strong** if $\operatorname{Re} g_j \succcurlyeq \mathfrak{v}^\dagger$ for $j = 1, \dots, r$, and we say that A **splits strongly over** K if there is a strong splitting of A over K. This notion is mainly of interest for $\mathfrak{v} \prec 1$, since otherwise $\mathfrak{v} = 1$, and then any splitting of A over K is a strong splitting of A over K.

Lemma 4.2.10. *Let $(g_1, \dots, g_r)$ be a strong splitting of A over K. If $h \neq 0$, then $(g_1, \dots, g_r)$ is a strong splitting of hA over K. If $h \asymp 1$, then $(g_1 - h^\dagger, \dots, g_r - h^\dagger)$ is a strong splitting of Ah over K.*

Proof. Suppose $h \asymp 1$. Now use Lemma 1.1.1, and the fact that if $\mathfrak{v} \prec 1$, then $\operatorname{Re} h^\dagger \preccurlyeq h^\dagger \prec \mathfrak{v}^\dagger$. If $\mathfrak{v} = 1$, then use that $\mathfrak{v}(Ah) = 1$ by Corollary 3.1.3. $\square$

Lemma 4.2.11. *Suppose $g \asymp \operatorname{Re} g$. Then $A = \partial - g$ splits strongly over K.*

Proof. Assuming $\mathfrak{v} \prec 1$ gives $\mathfrak{v}' \prec 1$, so $\mathfrak{v}^\dagger \prec 1/\mathfrak{v} \asymp g \asymp \operatorname{Re} g$. $\square$

In particular, every $A \in H[\partial]^{\neq}$ of order 1 splits strongly over K.

Lemma 4.2.12. *Suppose $(g_1, \dots, g_r)$ is a strong splitting of A over K and $\mathfrak{v} \prec^\flat 1$. Let $\phi \preccurlyeq 1$ be active in H and set $h_j := \phi^{-1}\big(g_j - (r-j)\phi^\dagger\big)$ for $j = 1, \dots, r$. Then $(h_1, \dots, h_r)$ is a strong splitting of A^ϕ over $K^\phi = H^\phi[\mathrm{i}]$.*

Proof. By Lemma 1.1.2, $(h_1,\dots,h_r)$ is a splitting of A^ϕ over K^ϕ. We have $\phi^\dagger \prec 1 \preccurlyeq \mathfrak{v}^\dagger$, so $\operatorname{Re} h_j \sim \phi^{-1}\operatorname{Re} g_j \succcurlyeq \phi^{-1}\mathfrak{v}^\dagger$ for $j=1,\dots,r$. Set $\mathfrak{w} := \mathfrak{v}(A^\phi)$ and $\delta := \phi^{-1}\partial$. Lemma 3.1.19 gives $\mathfrak{v}^\dagger \asymp \mathfrak{w}^\dagger$, so $\phi^{-1}\mathfrak{v}^\dagger \asymp \delta(\mathfrak{w})/\mathfrak{w}$. $\square$

In the next two results we assume that for all $q \in \mathbb{Q}^>$ and $\mathfrak{n} \in H^\times$ there is given an element $\mathfrak{n}^q \in H^\times$ such that $(\mathfrak{n}^q)^\dagger = q\mathfrak{n}^\dagger$ (and thus $v(\mathfrak{n}^q) = q\,v(\mathfrak{n})$).

Lemma 4.2.13. *Suppose $(g_1,\dots,g_r)$ is a splitting of A over K, $\mathfrak{v} \prec 1$, $\mathfrak{n} \in H^\times$, and $[\mathfrak{v}] \leqslant [\mathfrak{n}]$. Then for all $q \in \mathbb{Q}^>$ with at most r exceptions, $(g_1 - q\mathfrak{n}^\dagger,\dots,g_r - q\mathfrak{n}^\dagger)$ is a strong splitting of $A\mathfrak{n}^q$ over K.*

Proof. Let $q \in \mathbb{Q}^>$. Then $(g_1 - q\mathfrak{n}^\dagger,\dots,g_r - q\mathfrak{n}^\dagger)$ is a splitting of $A\mathfrak{n}^q$ over K, by Lemma 1.1.1. Moreover, $\big[\mathfrak{v}(A\mathfrak{n}^q)\big] \leqslant [\mathfrak{n}]$, by Lemma 3.1.9, so $\mathfrak{v}(A\mathfrak{n}^q)^\dagger \preccurlyeq \mathfrak{n}^\dagger$. Thus if $\operatorname{Re} g_j \not\sim q\mathfrak{n}^\dagger$ for $j=1,\dots,r$, then $(g_1 - q\mathfrak{n}^\dagger,\dots,g_r - q\mathfrak{n}^\dagger)$ is a strong splitting of $A\mathfrak{n}^q$ over K. $\square$

Corollary 4.2.14. *Let $(P,\mathfrak{m},\widehat{a})$ be a steep slot in K of order $r \geqslant 1$ whose linear part $L := L_{P_{\times\mathfrak{m}}}$ splits over K and such that $\widehat{a} \prec_\Delta \mathfrak{m}$ for $\Delta := \Delta\big(\mathfrak{v}(L)\big)$. Then for all sufficiently small $q \in \mathbb{Q}^>$, any $\mathfrak{n} \asymp |\mathfrak{v}(L)|^q\mathfrak{m}$ in $K^\times$ gives a steep refinement $\big(P,\mathfrak{n},\widehat{a}\big)$ of $(P,\mathfrak{m},\widehat{a})$ whose linear part $L_{P_{\times\mathfrak{n}}}$ splits strongly over K.*

Proof. Note that $|f| \asymp f$ for all f. Lemma 3.3.1 gives $q_0 \in \mathbb{Q}^>$ such that for all $q \in \mathbb{Q}^>$ with $q \leqslant q_0$ and any $\mathfrak{n} \asymp |\mathfrak{v}(L)|^q\mathfrak{m}$, $(P,\mathfrak{n},\widehat{a})$ is a steep refinement of $(P,\mathfrak{m},\widehat{a})$. Now apply Lemma 4.2.13 with L, $\mathfrak{v}(L)$, $|\mathfrak{v}(L)|$ in the respective roles of A, $\mathfrak{v}$, $\mathfrak{n}$, and use Lemma 4.2.10 and the fact that for $\mathfrak{n} \asymp |\mathfrak{v}(L)|^q\mathfrak{m}$ we have $L_{P_{\times\mathfrak{n}}} = L\cdot\mathfrak{n}/\mathfrak{m} = L|\mathfrak{v}(L)|^q h$ with $h \asymp 1$. $\square$

We finish this section with a useful fact on slots in K. Given such a slot $(P,\mathfrak{m},\widehat{a})$, the element $\widehat{a}$ lies in an immediate asymptotic extension of K that might not be of the form $\widehat{H}[i]$ with $\widehat{H}$ an immediate H-field extension of H. By the next lemma we can nevertheless often reduce to this situation, and more:

Lemma 4.2.15. *Suppose H is ω-free. Then every Z-minimal slot in K of positive order is equivalent to a hole $(P,\mathfrak{m},\widehat{b})$ in K with $\widehat{b} \in \widehat{K} = \widehat{H}[i]$ for some immediate ω-free newtonian H-field extension $\widehat{H}$ of H.*

Proof. Let $(P,\mathfrak{m},\widehat{a})$ be a Z-minimal slot in K of order $\geqslant 1$. Take an immediate ω-free newtonian H-field extension $\widehat{H}$ of H; such $\widehat{H}$ exists by remarks following [ADH, 14.0.1]. Then $\widehat{K} = \widehat{H}[i]$ is also newtonian by [ADH, 14.5.7]. Now apply Corollary 3.2.29 with $L := \widehat{K}$ to obtain $\widehat{b} \in \widehat{K}$ such that $(P,\mathfrak{m},\widehat{b})$ is a hole in K equivalent to $(P,\mathfrak{m},\widehat{a})$. $\square$

4.3. Split-Normal Slots

In this section H is a real closed H-field with small derivation and asymptotic integration. We let $\mathcal{O} := \mathcal{O}_H$ be its valuation ring and $C := C_H$ its constant field. We fix an immediate asymptotic extension $\widehat{H}$ of H with valuation ring $\widehat{\mathcal{O}}$ and an element i of an asymptotic extension of $\widehat{H}$ with $i^2 = -1$. Then $\widehat{H}$ is also an H-field by [ADH, 10.5.8], $i \notin \widehat{H}$ and $K := H[i]$ is an algebraic closure of H. With $\widehat{K} := \widehat{H}[i]$

we have the inclusion diagram

$$\begin{array}{ccc} \widehat{H} & \text{———} & \widehat{K}=\widehat{H}[\mathrm{i}] \\ | & & | \\ H & \text{———} & K=H[\mathrm{i}] \end{array}$$

By [ADH, 3.5.15, 10.5.7], K and $\widehat{K}$ are d-valued with valuation rings $\mathcal{O}+\mathcal{O}\mathrm{i}$ and $\widehat{\mathcal{O}}+\widehat{\mathcal{O}}\mathrm{i}$ and with the same constant field $C[\mathrm{i}]$, and $\widehat{K}$ is an immediate extension of K. Thus H, K, $\widehat{H}$, $\widehat{K}$ have the same H-asymptotic couple (Γ,ψ).

Lemma 4.3.1. *Let $\widehat{a}\in\widehat{H}\setminus H$. Then $Z(H,\widehat{a})=Z\big(K,\widehat{a}\big)\cap H\{Y\}$.*

Proof. The inclusion "$\supseteq$" is obvious since the Newton degree of a differential polynomial $Q\in H\{Y\}^{\neq}$ does not change when H is replaced by its algebraic closure; see [ADH, 11.1]. Conversely, let $P\in Z(H,\widehat{a})$. Then for all $\mathfrak{v}\in H^{\times}$ and $a\in H$ such that $a-\widehat{a}\prec\mathfrak{v}$ we have $\operatorname{ndeg}_{\prec\mathfrak{v}}H_{+a}\geqslant 1$. Let $\mathfrak{v}\in H^{\times}$ and $z\in K$ be such that $z-\widehat{a}\prec\mathfrak{v}$. Take $a,b\in H$ such that $z=a+b\mathrm{i}$. Then $a-\widehat{a},b\mathrm{i}\prec\mathfrak{v}$ and hence $\operatorname{ndeg}_{\prec\mathfrak{v}}P_{+z}=\operatorname{ndeg}_{\prec\mathfrak{v}}P_{+a}\geqslant 1$, using [ADH, 11.2.7]. Thus $P\in Z\big(K,\widehat{a}\big)$. ☐

Corollary 4.3.2. *Let $(P,\mathfrak{m},\widehat{a})$ be a slot in H with $\widehat{a}\in\widehat{H}$. Then $(P,\mathfrak{m},\widehat{a})$ is also a slot in K, and if $(P,\mathfrak{m},\widehat{a})$ is Z-minimal as a slot in K, then $(P,\mathfrak{m},\widehat{a})$ is Z-minimal as a slot in H. Moreover, $(P,\mathfrak{m},\widehat{a})$ is a hole in H iff $(P,\mathfrak{m},\widehat{a})$ is a hole in K, and if $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in K, then $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in H.*

Proof. The first three claims are obvious from $\widehat{K}$ being an immediate extension of K and the previous lemma. Suppose $(P,\mathfrak{m},\widehat{a})$ is minimal as a hole in K. Let $(Q,\mathfrak{n},\widetilde{b})$ be a hole in H; thus $\widetilde{b}\in\widetilde{H}$ where $\widetilde{H}$ is an immediate asymptotic extension of H. By the first part of the corollary applied to $(Q,\mathfrak{n},\widetilde{b})$ and $\widetilde{H}$ in place of $(P,\mathfrak{m},\widehat{a})$ and $\widehat{H}$, respectively, $(Q,\mathfrak{n},\widetilde{b})$ is also a hole in K. Hence $\mathrm{c}(P)\leqslant\mathrm{c}(Q)$, proving the last claim. ☐

In the next subsection we define the notion of a *split-normal* slot in H. Later in this section we employ the results of Sections 3.3–4.2 to show, under suitable hypotheses on H, that minimal holes in K of order $\geqslant 1$ give rise to a split-normal Z-minimal slots in H. (Theorem 4.3.9.) We then investigate which kinds of refinements preserve split-normality, and also consider a strengthening of split-normality.

Defining split-normality. *In this subsection b ranges over H and $\mathfrak{m},\mathfrak{n}$ over $H^{\times}$. Also, $(P,\mathfrak{m},\widehat{a})$ is a slot in H of order $r\geqslant 1$ with $\widehat{a}\in\widehat{H}\setminus H$ and linear part $L:=L_{P_{\times\mathfrak{m}}}$. Set $w:=\operatorname{wt}(P)$, so $w\geqslant r$; if $\operatorname{order}L=r$, we set $\mathfrak{v}:=\mathfrak{v}(L)$.*

Definition 4.3.3. We say that $(P,\mathfrak{m},\widehat{a})$ is **split-normal** if $\operatorname{order}L=r$, and

- (SN1) $\mathfrak{v}\prec^{\flat}1$;
- (SN2) $(P_{\times\mathfrak{m}})_{\geqslant 1}=Q+R$ where $Q,R\in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits over K, and $R\prec_{\Delta(\mathfrak{v})}\mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

Note that in (SN2) we do not require that $Q=(P_{\times\mathfrak{m}})_1$.

Lemma 4.3.4. *Suppose $(P,\mathfrak{m},\widehat{a})$ is split-normal. Then $(P,\mathfrak{m},\widehat{a})$ is normal, and with Q, R as in* (SN2) *we have $(P_{\times\mathfrak{m}})_1-Q\prec_{\Delta(\mathfrak{v})}\mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, so $(P_{\times\mathfrak{m}})_1\sim Q$.*

Proof. We have $(P_{\times\mathfrak{m}})_1 = Q + R_1$ and $R_1 \preccurlyeq R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, and thus $(P_{\times\mathfrak{m}})_1 - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. Now $(P,\mathfrak{m},\widehat{a})$ is normal because $(P_{\times\mathfrak{m}})_{>1} = R_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. $\square$

If $(P,\mathfrak{m},\widehat{a})$ is normal and $(P_{\times\mathfrak{m}})_1 = Q+R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, then $(P,\mathfrak{m},\widehat{a})$ is split-normal. Thus if $(P,\mathfrak{m},\widehat{a})$ is normal and L splits over K, then $(P,\mathfrak{m},\widehat{a})$ is split-normal; in particular, if $(P,\mathfrak{m},\widehat{a})$ is normal of order $r=1$, then it is split-normal. If $(P,\mathfrak{m},\widehat{a})$ is split-normal, then so are $(bP,\mathfrak{m},\widehat{a})$ for $b \neq 0$ and $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$. Note also that if $(P,\mathfrak{m},\widehat{a})$ is split-normal, then with Q as in (SN2) we have $\mathfrak{v}(L) \sim \mathfrak{v}(L_Q)$, by Lemma 3.1.1. If $(P,\mathfrak{m},\widehat{a})$ is split-normal and H is $\uplambda$-free, then $\mathscr{E}^{\mathrm{e}}(L) = \mathscr{E}^{\mathrm{e}}(L_Q)$ with Q as in (SN2), by Lemmas 4.3.4 and 3.1.22.

Lemma 4.3.5. *Suppose $(P,\mathfrak{m},\widehat{a})$ is split-normal and $\phi \preccurlyeq 1$ is active in H and $\phi > 0$ (so H^ϕ is still an H-field). Then the slot $(P^\phi,\mathfrak{m},\widehat{a})$ in H^ϕ is split-normal.*

Proof. We first arrange $\mathfrak{m} = 1$. Note that $L_{P^\phi} = L^\phi$ has order r. Put $\mathfrak{w} := \mathfrak{v}(L_{P^\phi})$, and take Q, R as in (SN2). Then $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{w} \prec^\flat_\phi 1$ by Lemma 3.1.19. Moreover, $L_{Q^\phi} = L_Q^\phi$ splits over K^ϕ; see [ADH, p. 291] or Lemma 1.1.2. By [ADH, 11.1.4],

$$R^\phi \asymp_{\Delta(\mathfrak{v})} R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}P_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}^{w+1}P_1^\phi,$$

so $(P^\phi,\mathfrak{m},\widehat{a})$ is split-normal. $\square$

Since we need to preserve H being an H-field when compositionally conjugating, we say: $(P^\phi,\mathfrak{m},\widehat{a})$ *is eventually split-normal* if there exists an active ϕ_0 in H such that $(P^\phi,\mathfrak{m},\widehat{a})$ is split-normal for all active $\phi \preccurlyeq \phi_0$ in H with $\phi > 0$. We use this terminology in a similar way with "split-normal" replaced by other properties of slots of order $r \geqslant 1$ in real closed H-fields with small derivation and asymptotic integration, such as "deep" and "deep and split-normal".

Achieving split-normality. *Assume H is $\upomega$-free and $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in $K = H[\mathrm{i}]$ of order $r \geqslant 1$, with $\mathfrak{m} \in H^\times$ and $\widehat{a} \in \widehat{K} \setminus K$.* Note that then K is $\upomega$-free by [ADH, 11.7.23], K is $(r-1)$-newtonian by Corollary 3.2.3, and K is r-linearly closed by Corollary 3.2.4. In particular, the linear part of $(P,\mathfrak{m},\widehat{a})$ is 0 or splits over K. If $\deg P = 1$, then $r = 1$ by Corollary 3.2.8. If $\deg P > 1$, then K and H are r-linearly newtonian by Corollary 3.2.6 and Lemma 1.8.30. In particular, if H is 1-linearly newtonian, then H is r-linearly newtonian. *In this subsection we let a range over K, b, c over H, and $\mathfrak{n}$ over $H^\times$.*

Lemma 4.3.6. *Let $(Q,\mathfrak{n},\widehat{b})$ be a hole in H with $\mathrm{c}(Q) \leqslant \mathrm{c}(P)$ and $\widehat{b} \in \widehat{H}$. Then $\mathrm{c}(Q) = \mathrm{c}(P)$, $(Q,\mathfrak{n},\widehat{b})$ is minimal and remains a minimal hole in K. The linear part of $(Q,\mathfrak{n},\widehat{b})$ is 0 or splits over K, and $(Q,\mathfrak{n},\widehat{b})$ has a refinement $(Q_{+b},\mathfrak{p},\widehat{b}-b)$ (in H) such that $(Q^\phi_{+b},\mathfrak{p},\widehat{b}-b)$ is eventually deep and split-normal.*

Proof. By Corollary 4.3.2, $(Q,\mathfrak{n},\widehat{b})$ is a hole in K, and this hole in K is minimal with $\mathrm{c}(Q) = \mathrm{c}(P)$, since $(P,\mathfrak{m},\widehat{a})$ is minimal. By Corollary 4.3.2 again, $(Q,\mathfrak{n},\widehat{b})$ as a hole in H is also minimal. Since K is r-linearly closed, the linear part of $(Q,\mathfrak{n},\widehat{b})$ is 0 or splits over K. Corollary 3.3.34 gives a refinement $(Q_{+b},\mathfrak{p},\widehat{b}-b)$ of the minimal hole $(Q,\mathfrak{n},\widehat{b})$ in H such that $(Q^\phi_{+b},\mathfrak{p},\widehat{b}-b)$ is deep and normal, eventually. Thus the linear part of $(Q_{+b},\mathfrak{p},\widehat{b}-b)$ is not 0, and as $\mathrm{c}(Q_{+b}) = \mathrm{c}(P)$, this linear part splits over K. Hence for active ϕ in H the linear part of $(Q^{\phi}_{+b}, \mathfrak{p}, \widehat{b}-b)$ splits over $K^{\phi} = H^{\phi}[\mathrm{i}]$. Thus $(Q^{\phi}_{+b}, \mathfrak{p}, \widehat{b}-b)$ is eventually split-normal. $\square$

Now $\widehat{a} = \widehat{b} + \widehat{c}\,i$ with $\widehat{b}, \widehat{c} \in \widehat{H}$, and $\widehat{b}, \widehat{c} \prec \mathfrak{m}$. Moreover, $\widehat{b} \notin H$ or $\widehat{c} \notin H$. Since $\widehat{a}$ is differentially algebraic over H, so is its conjugate $\widehat{b} - \widehat{c}\,i$, and therefore its real and imaginary parts $\widehat{b}$ and $\widehat{c}$ are differentially algebraic over H; thus $Z(\widehat{b}, H) \neq \emptyset$ for $\widehat{b} \notin H$, and $Z(\widehat{c}, H) \neq \emptyset$ for $\widehat{c} \notin H$. More precisely:

Lemma 4.3.7. *We have* $\operatorname{trdeg}\big(H\langle\widehat{b}\rangle|H\big) \leqslant 2r$. *If* $\widehat{b} \notin H$, *then* $Z(H, \widehat{b}) \cap H[Y] = \emptyset$, *so* $1 \leqslant \operatorname{order} Q \leqslant 2r$ *for all* $Q \in Z(H, \widehat{b})$ *of minimal complexity. These statements also hold for* $\widehat{c}$ *instead of* $\widehat{b}$.

Proof. The first statement follows from $\widehat{b} \in H\langle \widehat{b} + \widehat{c}\,i, \widehat{b} - \widehat{c}\,i\rangle$. Suppose $\widehat{b} \notin H$. If $Q \in Z(H, \widehat{b})$ has minimal complexity, then [ADH, 11.4.8] yields an element f in a proper immediate asymptotic extension of H with $Q(f) = 0$, so $Q \notin H[Y]$. $\square$

Lemma 4.3.8. *Suppose* $\deg P = 1$ *and* $\widehat{b} \notin H$. *Let* $Q \in Z(H, \widehat{b})$ *be of minimal complexity; then either* $\operatorname{order} Q = 1$, *or* $\operatorname{order} Q = 2$, $\deg Q = 1$. *Let* $\widehat{Q} \in H\{Y\}$ *be a minimal annihilator of* $\widehat{b}$ *over* H; *then either* $\operatorname{order} \widehat{Q} = 1$, *or* $\operatorname{order} \widehat{Q} = 2$, $\deg \widehat{Q} = 1$, *and* $L_{\widehat{Q}} \in H[\partial]$ *splits over* K.

Proof. Recall that $r = 1$ by Corollary 3.2.8. Example 1.1.7 and Lemma 1.1.8 give a $\widetilde{Q} \in H\{Y\}$ of degree 1 and order 1 or 2 such that $\widetilde{Q}(\widehat{b}) = 0$ and $L_{\widetilde{Q}}$ splits over K. Then $\mathrm{c}(\widetilde{Q}) = (1,1,1)$ or $\mathrm{c}(\widetilde{Q}) = (2,1,1)$, which proves the claim about Q, using also Lemma 4.3.7. Also, $\widetilde{Q}, \widehat{Q} \in Z(H, \widehat{b})$, hence $\mathrm{c}(Q) \leqslant \mathrm{c}(\widehat{Q}) \leqslant \mathrm{c}(\widetilde{Q})$. If $\mathrm{c}(\widehat{Q}) = \mathrm{c}(\widetilde{Q})$, then $\widehat{Q} = a\widetilde{Q}$ for some $a \in H^{\times}$. The claim about $\widehat{Q}$ now follows easily. $\square$

By Corollary 3.3.34 and Lemma 3.3.23, our minimal hole $(P, \mathfrak{m}, \widehat{a})$ in K has a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ such that eventually $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep and normal. Moreover, as K is r-linearly closed, the linear part of $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ (for active ϕ in K) splits over $K^{\phi} = H^{\phi}[\mathrm{i}]$. Our main goal in this subsection is to prove analogues of these facts for suitable Z-minimal slots $(Q, \mathfrak{m}, \widehat{b})$ or $(R, \mathfrak{m}, \widehat{c})$ in H:

Theorem 4.3.9. *If H is* 1*-linearly newtonian, then one of the following holds:*

(i) $\widehat{b} \notin H$ *and there exists a Z-minimal slot* $(Q, \mathfrak{m}, \widehat{b})$ *in H with a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and split-normal;*

(ii) $\widehat{c} \notin H$ *and there exists a Z-minimal slot* $(R, \mathfrak{m}, \widehat{c})$ *in H with a refinement* $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ *such that* $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ *is eventually deep and split-normal.*

Lemmas 4.3.10, 4.3.11 and Corollaries 4.3.13–4.3.16 below are more precise (only Corollary 4.3.15 has H being 1-linearly newtonian as a hypothesis) and together give Theorem 4.3.9. We first deal with the case where $\widehat{b}$ or $\widehat{c}$ is in H:

Lemma 4.3.10. *Suppose* $\widehat{c} \in H$. *Then some hole* $(Q, \mathfrak{m}, \widehat{b})$ *in H has the same complexity as* $(P, \mathfrak{m}, \widehat{a})$. *Any such hole* $(Q, \mathfrak{m}, \widehat{b})$ *in H is minimal and has a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and split-normal.*

Proof. Let $A, B \in H\{Y\}$ be such that $P_{+\widehat{c}i}(Y) = A(Y) + B(Y)\,i$. Then $A(\widehat{b}) = B(\widehat{b}) = 0$. If $A \neq 0$, then $c(A) \leqslant c(P)$ gives that $Q := A$ has the desired property by Lemma 4.3.6. If $B \neq 0$, then likewise $Q := B$ has the desired property. The rest also follows from that lemma. $\square$

Thus if $\widehat{c} \in H$, we obtain a strong version of (i) in Theorem 4.3.9. Likewise, the next lemma gives a strong version of (ii) in Theorem 4.3.9 if $\widehat{b} \in H$.

Lemma 4.3.11. *Suppose $\widehat{b} \in H$. Then there is a hole $(R, \mathfrak{m}, \widehat{c})$ in H with the same complexity as $(P, \mathfrak{m}, \widehat{a})$. Every such hole in H is minimal and has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and split-normal.*

This follows by applying Lemma 4.3.10 with $(P, \mathfrak{m}, \widehat{a})$ replaced by the minimal hole $\big(P_{\times i}, \mathfrak{m}, -i\widehat{a}\big)$ in K, which has the same complexity as $(P, \mathfrak{m}, \widehat{a})$.

We assume in the rest of this subsection that $\widehat{b}, \widehat{c} \notin H$ and that $Q \in Z(H, \widehat{b})$ has minimal complexity. Hence $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H, and so is every refinement of $(Q, \mathfrak{m}, \widehat{b})$. If $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$ and $b = \operatorname{Re} a$, then $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ is a refinement of $(Q, \mathfrak{m}, \widehat{b})$. Conversely, if $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ is a refinement of $(Q, \mathfrak{m}, \widehat{b})$ and $v\big(\widehat{b} - H\big) \subseteq v(\widehat{c} - H)$, then Lemma 4.1.3 yields a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\operatorname{Re} a = b$. Recall from that lemma that $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ is equivalent to $v(\widehat{a} - K) = v(\widehat{b} - H)$; in this case, $(P, \mathfrak{m}, \widehat{a})$ is special iff $(Q, \mathfrak{m}, \widehat{b})$ is special. Recall also that if $(Q, \mathfrak{m}, \widehat{b})$ is deep, then so is each of its refinements $(Q_{+b}, \mathfrak{m}, \widehat{b} - b)$, by Corollary 3.3.8.

Here is a key technical fact underlying Theorem 4.3.9:

Proposition 4.3.12. *Suppose the hole $(P, \mathfrak{m}, \widehat{a})$ in K is special, the slot $(Q, \mathfrak{m}, \widehat{b})$ in H is normal, and $v\big(\widehat{b} - H\big) \subseteq v(\widehat{c} - H)$. Then some refinement $(Q_{+b}, \mathfrak{m}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$ has the property that $(Q^{\phi}_{+b}, \mathfrak{m}, \widehat{b} - b)$ is eventually split-normal.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{a})$, $(Q, \mathfrak{m}, \widehat{b})$ by $(P_{\times \mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$, $(Q_{\times \mathfrak{m}}, 1, \widehat{b}/\mathfrak{m})$, respectively, we reduce to the case $\mathfrak{m} = 1$; then $\widehat{a}, \widehat{b} \prec 1$. Since $\widehat{a}$ is special over $K = H[i]$,

$$\Delta \ := \ \big\{\delta \in \Gamma : \ |\delta| \in v(\widehat{a} - K)\big\}$$

is a convex subgroup of Γ which is cofinal in $v(\widehat{a} - K)$ and hence in $v(\widehat{b} - H)$, so $\widehat{b}$ is special over H. Compositionally conjugate H, $\widehat{H}$, K, $\widehat{K}$ by a suitable active $\phi \preccurlyeq 1$ in $H^{>}$, and replace P, Q by P^{ϕ}, Q^{ϕ}, to arrange $\Gamma^{\flat} \subseteq \Delta$; in particular, $\Psi \subseteq v(\widehat{b} - H)$ and $\psi(\Delta^{\neq}) \subseteq \Delta$. Multiplying P, Q by suitable elements of $H^{\times}$ we also arrange that $P, Q \asymp 1$. By Lemma 4.3.5 it suffices to show that then $(Q, 1, \widehat{b})$ has a split-normal refinement $(Q_{+b}, 1, \widehat{b} - b)$, and this is what we shall do.

Note that H, $\widehat{H}$, K, $\widehat{K}$ have small derivation, so the specializations $\dot{H}$, $\dot{\widehat{H}}$, $\dot{K}$, $\dot{\widehat{K}}$ of H, $\widehat{H}$, K, $\widehat{K}$, respectively, by Δ, are valued differential fields with small derivation. These specializations are asymptotic with asymptotic couple $(\Delta, \psi|\Delta^{\neq})$, and of H-type with asymptotic integration, by [ADH, 9.4.12]; in addition they are d-valued, by [ADH, 10.1.8]. The natural inclusions $\dot{\mathcal{O}} \to \dot{\mathcal{O}}_K$, $\dot{\mathcal{O}} \to \dot{\mathcal{O}}_{\widehat{H}}$, $\dot{\mathcal{O}}_{\widehat{H}} \to \dot{\mathcal{O}}_{\widehat{K}}$, and $\dot{\mathcal{O}}_K \to \dot{\mathcal{O}}_{\widehat{K}}$ induce valued differential field embeddings $\dot{H} \to \dot{K}$, $\dot{H} \to \dot{\widehat{H}}$, $\dot{\widehat{H}} \to \dot{\widehat{K}}$ and $\dot{K} \to \dot{\widehat{K}}$, which we make into inclusions by the usual identifications; see [ADH, pp. 405–406]. By Lemma 4.1.2 and the remarks preceding it, $\dot{H}$ is real closed with convex valuation ring and $\dot K$ is an algebraic closure of $\dot H$. Moreover, $\dot{\widehat{H}}$ is an immediate extension of $\dot H$ and $\dot{\widehat{K}}$ is an immediate extension of $\dot K$. Denoting the image of i under the residue morphism $\dot{\mathcal{O}}_{\widehat{K}} \to \dot{\widehat{K}}$ by the same symbol, we then have $\dot K = \dot H[\mathrm{i}]$, $\dot{\widehat{K}} = \dot{\widehat{H}}[\mathrm{i}]$, and $\mathrm{i} \notin \dot{\widehat{H}}$. This gives the following inclusion diagram:

$$\begin{tikzcd}
\dot{\widehat{H}} \arrow[r, dash] & \dot{\widehat{K}} = \dot{\widehat{H}}[\mathrm{i}] \\
\dot H \arrow[u, dash] \arrow[r, dash] & \dot K = \dot H[\mathrm{i}] \arrow[u, dash]
\end{tikzcd}$$

Now $\widehat a \in \mathcal{O}_{\widehat K} \subseteq \dot{\mathcal{O}}_{\widehat K}$ and $\widehat b, \widehat c \in \mathcal{O}_{\widehat H} \subseteq \dot{\mathcal{O}}_{\widehat H}$, and $\dot{\widehat a} = \dot{\widehat b} + \dot{\widehat c} i$, $\operatorname{Re} \dot{\widehat a} = \dot{\widehat b}$, $\operatorname{Im} \dot{\widehat a} = \dot{\widehat c}$. For all $a \in \dot{\mathcal{O}}_K$ we have $v(\dot{\widehat a} - \dot a) = v(\widehat a - a) \in \Delta$, hence $\dot{\widehat a} \notin \dot K$; likewise $v(\widehat b - b) \in \Delta$ for all $b \in \dot{\mathcal{O}}$, so $\dot{\widehat b} \notin \dot H$. Moreover, for all $\delta \in \Delta$ there is an $a \in \dot{\mathcal{O}}_K$ with $v(\dot{\widehat a} - \dot a) = \delta$; hence $\dot{\widehat a}$ is the limit of a c-sequence in $\dot K$. This leads us to consider the completions $\dot H^{\mathrm{c}}$ and $\dot K^{\mathrm{c}}$ of $\dot H$ and $\dot K$. By [ADH, 4.4.11] and Lemma 4.1.1, these yield an inclusion diagram of valued differential field extensions:

$$\begin{tikzcd}
\dot H^{\mathrm{c}} \arrow[r, dash] & \dot K^{\mathrm{c}} = \dot H^{\mathrm{c}}[\mathrm{i}] \\
\dot H \arrow[u, dash] \arrow[r, dash] & \dot K = \dot H[\mathrm{i}] \arrow[u, dash]
\end{tikzcd}$$

where $\dot H^{\mathrm{c}}$ is real closed with algebraic closure $\dot K^{\mathrm{c}} = \dot H^{\mathrm{c}}[\mathrm{i}]$. These completions are d-valued by [ADH, 9.1.6]. By Corollary 1.8.5, $\dot K$ and $\dot K^{\mathrm{c}}$ are ω-free and $(r-1)$-newtonian; thus $\dot K^{\mathrm{c}}$ is r-linearly closed by Corollary 1.8.42. We identify the valued differential subfield $\dot K\langle \operatorname{Re} \dot{\widehat a}, \operatorname{Im} \dot{\widehat a}\rangle$ of $\dot{\widehat K}$ with its image under the embedding into $\dot K^{\mathrm{c}}$ over $\dot K$ from Corollary 4.1.7; then $\dot{\widehat a} \in \dot K^{\mathrm{c}}$ and $\dot{\widehat b} = \operatorname{Re} \dot{\widehat a} \in \dot H^{\mathrm{c}}$. This leads to the next inclusion diagram:

$$\begin{tikzcd}
\dot H^{\mathrm{c}} \arrow[r, dash] & \dot K^{\mathrm{c}} \\
\dot H\langle \dot{\widehat b}\rangle \arrow[u, dash] & \dot K\langle \dot{\widehat a}\rangle \arrow[u, dash] \\
\dot H \arrow[u, dash] \arrow[r] & \dot K \arrow[u, dash]
\end{tikzcd}$$

By Corollary 1.6.21, $\dot P \in \dot K\{Y\}$ is a minimal annihilator of $\dot{\widehat a}$ over $\dot K$ and has the same complexity as P. Likewise, $\dot Q \in \dot H\{Y\}$ is a minimal annihilator of $\dot{\widehat b}$ over $\dot H$ and has the same complexity as Q. Let $s := \operatorname{order} Q = \operatorname{order} \dot Q$, so $1 \leqslant s \leqslant 2r$ by Lemma 4.3.7, and the linear part $A \in \dot H^{\mathrm{c}}[\partial]$ of $\dot Q_{+\dot{\widehat b}}$ has order s as well. By [ADH, 5.1.37] applied to $\dot H^{\mathrm{c}}$, $\dot H$, $\dot P$, $\dot Q$, $\dot{\widehat a}$ in the role of K, F, P, S, f, respectively, A splits over $\dot K^{\mathrm{c}} = \dot H^{\mathrm{c}}[\mathrm{i}]$, so Lemma 1.1.4 gives a real splitting $(g_1, \dots, g_s)$ of A over $\dot K^{\mathrm{c}}$:

$$A \;=\; f(\partial - g_1)\cdots(\partial - g_s), \qquad f, g_1, \dots, g_s \in \dot K^{\mathrm{c}},\ f \neq 0.$$

The slot $(Q,1,\widehat{b})$ in H is normal, so $\mathfrak{v}(L_{Q_{+\widehat{b}}}) \sim \mathfrak{v}(L_Q) \prec^\flat 1$ by Lemma 3.1.27, hence $\mathfrak{v}(A) \prec^\flat 1$ in $\dot{K}^{\mathrm{c}}$ by Lemma 3.1.7. Then Corollary 4.2.8 gives $a,b \in \dot{\mathcal{O}}$ and $b_1,\dots,b_s \in \dot{\mathcal{O}}_K$ with $\dot{a},\dot{b} \neq 0$ in $\dot{H}$ such that for the linear part $\widetilde{A} \in \dot{H}[\partial]$ of $\dot{Q}_{+\dot{b}}$,

$$\dot{b} \sim \dot{\widehat{b}}, \qquad \widetilde{A} \sim A, \qquad \operatorname{order} \widetilde{A} = s, \qquad \mathfrak{w} := \mathfrak{v}(\widetilde{A}) \sim \mathfrak{v}(A),$$

and such that for $w := \operatorname{wt}(Q)$ and with $\Delta(\mathfrak{w}) \subseteq \Delta$:

$$\widetilde{A} = \widetilde{B} + \widetilde{E},\ \widetilde{B} = \dot{a}(\partial - \dot{b}_1)\cdots(\partial - \dot{b}_s) \in \dot{H}[\partial], \quad \widetilde{E} \in \dot{H}[\partial], \quad \widetilde{E} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}\widetilde{A},$$

and $(\dot{b}_1,\dots,\dot{b}_s)$ is a real splitting of $\widetilde{B}$ over $\dot{K}$. Lemma 1.1.6 shows that we can change $b_1,\dots,b_s$ if necessary, without changing $\dot{b}_1,\dots,\dot{b}_s$, to arrange that $B := a(\partial - b_1)\cdots(\partial - b_s)$ lies in $\dot{\mathcal{O}}[\partial] \subseteq H[\partial]$ and $(b_1,\dots,b_s)$ is a real splitting of B over K. Now $\widehat{b} - b \prec \widehat{b} \prec 1$, so $(Q_{+b},1,\widehat{b}-b)$ is a refinement of the normal slot $(Q,1,\widehat{b})$. Hence $(Q_{+b},1,\widehat{b}-b)$ is normal by Proposition 3.3.25, so $\mathfrak{v} := \mathfrak{v}(L_{Q_{+b}}) \prec^\flat 1$. By Lemma 3.1.7 we have $\dot{\mathfrak{v}} = \mathfrak{w}$, so $\Delta(\mathfrak{v}) = \Delta(\mathfrak{w}) \subseteq \Delta$. Hence in $H[\partial]$:

$$L_{Q_{+b}} = B + E, \quad E \in \dot{\mathcal{O}}[\partial],\ E \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} L_{Q_{+b}}.$$

Thus $(Q_{+b},1,\widehat{b}-b)$ is split-normal. $\square$

Recall from the beginning of this subsection that if $\deg P > 1$, then $K = H[i]$ is r-linearly newtonian; this allows us to remove the assumptions that $(P,\mathfrak{m},\widehat{a})$ is special and $(Q,\mathfrak{m},\widehat{b})$ is normal in Proposition 4.3.12, by reducing to that case:

Corollary 4.3.13. *Suppose* $\deg P > 1$ *and* $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. *Then* $(Q,\mathfrak{m},\widehat{b})$ *has a special refinement* $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ *such that* $(Q^{\phi}_{+b},\mathfrak{n},\widehat{b}-b)$ *is eventually deep and split-normal.*

Proof. By Lemmas 3.2.26 and 3.3.23, the hole $(P,\mathfrak{m},\widehat{a})$ in K has a quasilinear refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$. (The use of Lemma 3.3.23 is because we require $\mathfrak{n} \in H^\times$.) Let $b = \operatorname{Re} a$. Then, using Lemma 4.1.3 for the second equality,

$$v\big((\widehat{a}-a)-K\big) = v(\widehat{a}-K) = v(\widehat{b}-H) = v\big((\widehat{b}-b)-H\big),$$

and $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ is a Z-minimal refinement of $(Q,\mathfrak{m},\widehat{b})$. We replace $(P,\mathfrak{m},\widehat{a})$ and $(Q,\mathfrak{m},\widehat{b})$ by $(P_{+a},\mathfrak{n},\widehat{a}-a)$ and $(Q_{+b},\mathfrak{n},\widehat{b}-b)$, respectively, to arrange that the hole $(P,\mathfrak{m},\widehat{a})$ in K is quasilinear. Then by Proposition 1.6.12 and K being r-linearly newtonian, $(P,\mathfrak{m},\widehat{a})$ is special. Hence $(Q,\mathfrak{m},\widehat{b})$ is also special, so Proposition 3.3.36 gives a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ of $(Q,\mathfrak{m},\widehat{b})$ and an active $\phi_0 \in H^>$ such that $(Q^{\phi_0}_{+b},\mathfrak{n},\widehat{b}-b)$ is deep and normal. Refinements of $(P,\mathfrak{m},\widehat{a})$ remain quasilinear, by Corollary 3.2.23. Since $v(\widehat{b}-H) \subseteq v(\widehat{c}-H)$ we have a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ with $\operatorname{Re} a = b$. Then by Lemma 3.2.35 the minimal hole $(P^{\phi_0}_{+a},\mathfrak{n},\widehat{a}-a)$ in $H^{\phi_0}[i]$ is special. Now apply Proposition 4.3.12 with H^{ϕ_0}, $(P^{\phi_0}_{+a},\mathfrak{n},\widehat{a}-a)$, $(Q^{\phi_0}_{+b},\mathfrak{n},\widehat{b}-b)$ in place of H, $(P,\mathfrak{m},\widehat{a})$, $(Q,\mathfrak{m},\widehat{b})$, respectively: it gives $b_0 \in H$ and a refinement

$$\big((Q^{\phi_0}_{+b})_{+b_0},\mathfrak{n},(\widehat{b}-b)-b_0\big) = \big(Q^{\phi_0}_{+(b+b_0)},\mathfrak{n},\widehat{b}-(b+b_0)\big)$$

of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$, and thus a refinement $\big(Q_{+(b+b_0)}, \mathfrak{n}, \widehat{b}-(b+b_0)\big)$ of $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$, such that $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}, \widehat{b}-(b+b_0)\big)$ is eventually split-normal. By the remark before Proposition 4.3.12, $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}, \widehat{b}-(b+b_0)\big)$ is also eventually deep. $\square$

Recall that $v(\widehat{b}-H) \subseteq v(\widehat{c}-H)$ or $v(\widehat{c}-H) \subseteq v(\widehat{b}-H)$. The following corollary concerns the second case:

Corollary 4.3.14. *If* $\deg P > 1$, $v(\widehat{c}-H) \subseteq v(\widehat{b}-H)$, *and* $R \in Z(H, \widehat{c})$ *has minimal complexity, then the* Z*-minimal slot* $(R, \mathfrak{m}, \widehat{c})$ *in* H *has a special refinement* $(R_{+c}, \mathfrak{n}, \widehat{c}-c)$ *such that* $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c}-c)$ *is eventually deep and split-normal.*

Proof. Apply Corollary 4.3.13 to the minimal hole $(P_{\times i}, \mathfrak{m}, -i\widehat{a})$ in $H[i]$. $\square$

In the next two corollaries we handle the case $\deg P = 1$. Recall from Lemma 4.3.8 that then $\operatorname{order} Q = 1$ or $\operatorname{order} Q = 2$, $\deg Q = 1$. Theorem 3.3.33 gives:

Corollary 4.3.15. *Suppose* H *is* 1*-linearly newtonian and* $\operatorname{order} Q = 1$. *Then the slot* $(Q, \mathfrak{m}, \widehat{b})$ *in* H *has a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b)$ *is eventually deep and split-normal.*

Corollary 4.3.16. *Suppose* $\deg P = 1$ *and* $\operatorname{order} Q = 2$, $\deg Q = 1$. *Let* $\widehat{Q} \in H\{Y\}$ *be a minimal annihilator of* $\widehat{b}$ *over* H. *Then* $\big(\widehat{Q}, \mathfrak{m}, \widehat{b}\big)$ *is a* Z*-minimal hole in* H *and has a refinement* $\big(\widehat{Q}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$ *such that* $\big(\widehat{Q}^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$ *is eventually deep and split-normal.*

Proof. By the proof of Lemma 4.3.8 we have $\mathrm{c}(Q) = \mathrm{c}(\widehat{Q})$ (hence $\big(\widehat{Q}, \mathfrak{m}, \widehat{b}\big)$ is a Z-minimal hole in H) and $L_{\widehat{Q}}$ splits over $H[i]$. Corollary 3.3.12 gives a refinement $\big(\widehat{Q}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$ of $\big(\widehat{Q}, \mathfrak{m}, \widehat{b}\big)$ whose linear part has Newton weight 0 and such that the slot $\big(\widehat{Q}^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$ in H^{ϕ} is deep, eventually. Moreover, by Lemmas 3.3.17 and 3.2.31, $\big(\widehat{Q}^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$ is normal and its linear part splits over $H^{\phi}[i]$, eventually. Thus $\big(\widehat{Q}^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$ is eventually deep and split-normal. $\square$

This concludes the proof of Theorem 4.3.9.

Split-normality and refinements. We now study the behavior of split-normality under refinements. In this subsection a ranges over H and $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{v}$ range over $H^{\times}$. Let $(P, \mathfrak{m}, \widehat{a})$ be a slot in H of order $r \geqslant 1$ with $\widehat{a} \in \widehat{H} \setminus H$, and $L := L_{P_{\times\mathfrak{m}}}$, $w := \mathrm{wt}(P)$. Here is the split-normal analogue of Lemma 3.3.19:

Lemma 4.3.17. *Suppose* $\operatorname{order}(L) = r$ *and* $\mathfrak{v}$ *is such that* (SN1) *and* (SN2) *hold, and* $\mathfrak{v}(L) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. *Then* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal.*

Proof. Same as that of 3.3.19, but with R as in (SN2) instead of $(P_{\times\mathfrak{m}})_{>1}$. $\square$

Now split-normal analogues of Propositions 3.3.25 and 3.3.26:

Lemma 4.3.18. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal. Let a refinement* $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ *of* $(P, \mathfrak{m}, \widehat{a})$ *be given. Then* $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ *is also split-normal.*

Proof. As in the proof of Proposition 3.3.25 we arrange $\mathfrak{m} = 1$ and show for $\mathfrak{v} := \mathfrak{v}(L_P)$, using Lemmas 3.1.27 and 4.3.4, that $\operatorname{order}(L_{P_{+a}}) = r$ and

$$(P_{+a})_1 \sim_{\Delta(\mathfrak{v})} P_1, \quad \mathfrak{v}(L_{P_{+a}}) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}, \quad (P_{+a})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1.$$

Now take Q, R as in (SN2) for $\mathfrak{m}=1$. Then $P_1 = Q + R_1$, and so by Lemma 3.1.28 for $A = L_Q$ we obtain $(P_{+a})_1 - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1$, and thus $(P_{+a})_{\geqslant 1} - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1$. Hence (SN2) holds with $\mathfrak{m}=1$ and P_{+a} instead of P. Thus the slot $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ in H is split-normal by Lemma 4.3.17. □

Lemma 4.3.19. *Suppose $(P,\mathfrak{m},\widehat{a})$ is split-normal, $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$, $\mathfrak{v} := \mathfrak{v}(L)$. Then the refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ is split-normal: if $\mathfrak{m}$, P, Q, $\mathfrak{v}$ are as in* (SN2)*, then* (SN2) *holds with $\mathfrak{n}$, $Q_{\times\mathfrak{n}/\mathfrak{m}}$, $R_{\times\mathfrak{n}/\mathfrak{m}}$, $\mathfrak{v}(L_{P_{\times\mathfrak{n}}})$ in place of $\mathfrak{m}$, Q, R, $\mathfrak{v}$.*

Proof. Set $\widetilde{L} := L_{P_{\times\mathfrak{n}}}$. Lemma 3.3.1 gives $\operatorname{order}(\widetilde{L}) = r$ and $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Thus $(P_{\times\mathfrak{n}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{n}})_1$ by Proposition 3.3.26. Now arrange $\mathfrak{m}=1$ in the usual way, and take Q, R as in (SN2) for $\mathfrak{m}=1$. Then

$$(P_{\times\mathfrak{n}})_1 \ = \ (P_1)_{\times\mathfrak{n}} \ = \ Q_{\times\mathfrak{n}} + (R_1)_{\times\mathfrak{n}}, \qquad (P_{\times\mathfrak{n}})_{>1} \ = \ (R_{\times\mathfrak{n}})_{>1} \ = \ (R_{>1})_{\times\mathfrak{n}}$$

by [ADH, 4.3], where $Q_{\times\mathfrak{n}}$ is homogeneous of degree 1 and order r, and $L_{Q_{\times\mathfrak{n}}} = L_Q\mathfrak{n}$ splits over K. Using [ADH, 4.3, 6.1.3] and $[\mathfrak{n}] \leqslant [\mathfrak{v}]$ we obtain

$$(R_1)_{\times\mathfrak{n}} \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{n}R_1 \ \preccurlyeq \ \mathfrak{n}R \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{n}\mathfrak{v}^{w+1}P_1 \ \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_1)_{\times\mathfrak{n}} \ = \ \mathfrak{v}^{w+1}(P_{\times\mathfrak{n}})_1.$$

Hence (SN2) holds for $\mathfrak{n}, Q_{\times\mathfrak{n}}, R_{\times\mathfrak{n}}, \mathfrak{v}(\widetilde{L})$ in place of $\mathfrak{m}, Q, R, \mathfrak{v}$. □

Recall our standing assumption in this section that H is a real closed H-field. Thus H is d-valued, and for all $\mathfrak{n}$ and $q \in \mathbb{Q}^>$ we have $\mathfrak{n}^q \in H^\times$ such that $(\mathfrak{n}^q)^\dagger = q\mathfrak{n}^\dagger$. *In the rest of this section we fix such an $\mathfrak{n}^q$ for all $\mathfrak{n}$ and $q \in \mathbb{Q}^>$.* Now we upgrade Corollary 3.3.31 with "split-normal" instead of "normal":

Lemma 4.3.20. *Suppose $\mathfrak{m}=1$, $(P,1,\widehat{a})$ is split-normal, $\widehat{a} \prec \mathfrak{n} \prec 1$, and for $\mathfrak{v} := \mathfrak{v}(L_P)$ we have $[\mathfrak{n}^\dagger] < [\mathfrak{v}] < [\mathfrak{n}]$. Then $(P,\mathfrak{n}^q,\widehat{a})$ is a split-normal refinement of $(P,1,\widehat{a})$ for all but finitely many $q \in \mathbb{Q}$ with $0 < q < 1$.*

Proof. Corollary 3.3.31 gives that $(P,\mathfrak{n}^q,\widehat{a})$ is a normal refinement of $(P,1,\widehat{a})$ for all but finitely many $q \in \mathbb{Q}$ with $0 < q < 1$. Take Q, R as in (SN2) for $\mathfrak{m}=1$. Then $L = L_Q + L_R$ where L_Q splits over $H[i]$ and $L_R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}L$, for $\mathfrak{v} := \mathfrak{v}(L)$. Applying Corollary 3.1.18 to $A := L$, $A_* := L_R$ we obtain: $L_R\mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}L\mathfrak{n}^q$, $\mathfrak{w} := \mathfrak{v}(L\mathfrak{n}^q)$, for all but finitely many $q \in \mathbb{Q}^>$.

Let $q \in \mathbb{Q}$ be such that $0 < q < 1$, $(P,\mathfrak{n}^q,\widehat{a})$ is a normal refinement of $(P,1,\widehat{a})$, and $L_R\mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}L\mathfrak{n}^q$, with $\mathfrak{w}$ as above. Then $(P_{\times\mathfrak{n}^q})_1 = Q_{\times\mathfrak{n}^q} + (R_1)_{\times\mathfrak{n}^q}$ where $Q_{\times\mathfrak{n}^q}$ is homogeneous of degree 1 and order r, $L_{Q_{\times\mathfrak{n}^q}} = L_Q\mathfrak{n}^q$ splits over $H[i]$, and $(R_1)_{\times\mathfrak{n}^q} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P_{\times\mathfrak{n}^q})_1$ for $\mathfrak{w} := \mathfrak{v}(L_{P_{\times\mathfrak{n}^q}})$. Since $(P,\mathfrak{n}^q,\widehat{a})$ is normal, we also have $(P_{\times\mathfrak{n}^q})_{>1} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P_{\times\mathfrak{n}^q})_1$. Thus $(P,\mathfrak{n}^q,\widehat{a})$ is split-normal. □

Remark. We do not know if in this last lemma we can drop the assumption $[\mathfrak{n}^\dagger] < [\mathfrak{v}]$.

Strengthening split-normality. *In this subsection a, b range over H and $\mathfrak{m}, \mathfrak{n}$ over $H^\times$, and $(P,\mathfrak{m},\widehat{a})$ is a slot in H of order $r \geqslant 1$ and weight $w := \operatorname{wt}(P)$, so $w \geqslant 1$, and $L := L_{P_{\times\mathfrak{m}}}$. If* order $L = r$*, we set $\mathfrak{v} := \mathfrak{v}(L)$.*

With an eye towards later use in connection with fixed point theorems over Hardy fields we strengthen here the concept of split-normality; in the next subsection we show how to improve Theorem 4.3.9 accordingly. See the last subsection of Section 4.2 for the notion of strong splitting.

Definition 4.3.21. Call $(P,\mathfrak m,\widehat a)$ **almost strongly split-normal** if $\operatorname{order} L = r$, $\mathfrak v \prec^\flat 1$, and the following strengthening of (SN2) holds:

(SN2as) $(P_{\times\mathfrak m})_{\geqslant 1} = Q+R$ where $Q,R\in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits strongly over K, and $R \prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_{\times\mathfrak m})_1$.

We say that $(P,\mathfrak m,\widehat a)$ is **strongly split-normal** if $\operatorname{order} L = r$, $\mathfrak v \prec^\flat 1$, and the following condition is satisfied:

(SN2s) $P_{\times\mathfrak m} = Q+R$ where $Q,R\in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits strongly over K, and $R \prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_{\times\mathfrak m})_1$.

To facilitate use of (SN2s) we observe:

Lemma 4.3.22. *Suppose $(P,\mathfrak m,\widehat a)$ is strongly split-normal and $P_{\times\mathfrak m} = Q+R$ as in* (SN2s)*. Then $Q\sim (P_{\times\mathfrak m})_1$, $\mathfrak v_Q := \mathfrak v(L_Q)\sim\mathfrak v$, so $R\prec_{\Delta(\mathfrak v)} \mathfrak v_Q^{w+1}Q$.*

Proof. We have $(P_{\times\mathfrak m})_1 = Q+R_1$, so $Q = (P_{\times\mathfrak m})_1 - R_1$ with $R_1 \prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_{\times\mathfrak m})_1$. Now apply Lemma 3.1.1 to $A := L$ and $B := -L_{R_1}$. $\square$

If $(P,\mathfrak m,\widehat a)$ is almost strongly split-normal, then $(P,\mathfrak m,\widehat a)$ is split-normal and hence normal by Lemma 4.3.4. If $(P,\mathfrak m,\widehat a)$ is normal and L splits strongly over K, then $(P,\mathfrak m,\widehat a)$ is almost strongly split-normal; in particular, if $(P,\mathfrak m,\widehat a)$ is normal of order $r=1$, then $(P,\mathfrak m,\widehat a)$ is almost strongly split-normal, by Lemma 4.2.11. Moreover:

Lemma 4.3.23. *The following are equivalent:*

(i) *$(P,\mathfrak m,\widehat a)$ is strongly split-normal;*
(ii) *$(P,\mathfrak m,\widehat a)$ is almost strongly split-normal and strictly normal;*
(iii) *$(P,\mathfrak m,\widehat a)$ is almost strongly split-normal and $P(0)\prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_1)_{\times\mathfrak m}$.*

Proof. Suppose $(P,\mathfrak m,\widehat a)$ is strongly split-normal, and let Q, R be as in (SN2s). Then $(P_{\times\mathfrak m})_{\geqslant 1} = Q+R_{\geqslant 1}$, L_Q splits strongly over K, and $R_{\geqslant 1}\prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_{\times\mathfrak m})_1$. Hence $(P,\mathfrak m,\widehat a)$ is almost strongly split-normal, and thus normal. Also $P(0) = R(0)\prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_{\times\mathfrak m})_1$, so $(P,\mathfrak m,\widehat a)$ is strictly normal. This shows (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is clear. For (iii) $\Rightarrow$ (i) suppose $(P,\mathfrak m,\widehat a)$ is almost strongly split-normal and $P(0)\prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_1)_{\times\mathfrak m}$. Take Q, R as in (SN2as). Then $P_{\times\mathfrak m} = Q+\widetilde R$ where $\widetilde R := P(0)+R\prec_{\Delta(\mathfrak v)} \mathfrak v^{w+1}(P_1)_{\times\mathfrak m}$. Thus $(P,\mathfrak m,\widehat a)$ is strongly split-normal. $\square$

Corollary 4.3.24. *If L splits strongly over K, then*

$$(P,\mathfrak m,\widehat a) \text{ is strongly split-normal} \iff (P,\mathfrak m,\widehat a) \text{ is strictly normal.}$$

The following diagram summarizes some implications between these variants of normality, for slots $(P,\mathfrak m,\widehat a)$ in H of order $r\geqslant 1$:

$$\begin{tikzcd}
\text{strongly split-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{almost strongly split-normal} \arrow[r, Rightarrow] & \text{split-normal} \arrow[d, Rightarrow] \\
\text{strictly normal} \arrow[rr, Rightarrow] & & \text{normal}
\end{tikzcd}$$

If $(P,\mathfrak m,\widehat a)$ is almost strongly split-normal, then so are $(bP,\mathfrak m,\widehat a)$ for $b\neq 0$ and $(P_{\times\mathfrak n},\mathfrak m/\mathfrak n,\widehat a/\mathfrak n)$, and likewise with “strongly” in place of “almost strongly”.

Here is a version of Lemma 4.3.18 for (almost) strong split-normality:

Lemma 4.3.25. *Suppose $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ refines $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal, then so is $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$. If $(P, \mathfrak{m}, \widehat{a})$ is strongly split-normal, Z-minimal, and $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$, then $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is strongly split-normal.*

Proof. The first part follows from Lemma 4.3.18 and its proof. In combination with Lemmas 3.3.42 and 4.3.23, this also yields the second part. $\square$

Lemma 4.3.26. *Suppose that $(P, \mathfrak{m}, \widehat{a})$ is split-normal and $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. Then for all sufficiently small $q \in \mathbb{Q}^{>}$, any $\mathfrak{n} \asymp \mathfrak{v}^q\mathfrak{m}$ yields an almost strongly split-normal refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$.*

Proof. We arrange $\mathfrak{m} = 1$, so $\widehat{a} \prec_{\Delta(\mathfrak{v})} 1$. Take Q, R as in (SN2) with $\mathfrak{m} = 1$, and take $q_0 \in \mathbb{Q}^{>}$ such that $\widehat{a} \prec \mathfrak{v}^{q_0} \prec 1$. By Lemma 4.2.13 we can decrease q_0 so that for all $q \in \mathbb{Q}$ with $0 < q \leqslant q_0$ and any $\mathfrak{n} \asymp \mathfrak{v}^q$, $L_{Q_{\times\mathfrak{n}}} = L_Q\mathfrak{n}$ splits strongly over K. Suppose $q \in \mathbb{Q}$, $0 < q \leqslant q_0$, and $\mathfrak{n} \asymp \mathfrak{v}^q$. Then $(P, \mathfrak{n}, \widehat{a})$ is an almost strongly split-normal refinement of $(P, 1, \widehat{a})$, by Lemma 4.3.19. $\square$

Corollary 4.3.27. *Suppose that $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and split-normal. Then $(P, \mathfrak{m}, \widehat{a})$ has a refinement which is deep and almost strongly split-normal.*

Proof. Lemma 3.3.13 gives a such that $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. By Corollary 3.3.8, the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, and by Lemma 4.3.18 it is also split-normal. Now apply Lemma 4.3.26 to $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ in place of $(P, \mathfrak{m}, \widehat{a})$ and again use Corollary 3.3.8. $\square$

We now turn to the behavior of these properties under compositional conjugation.

Lemma 4.3.28. *Let ϕ be active in H with $0 < \phi \preccurlyeq 1$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal, then so is the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ. Likewise with "strongly" in place of "almost strongly".*

Proof. We arrange $\mathfrak{m} = 1$, assume $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal, and take Q, R as in (SN2as). The proof of Lemma 4.3.5 shows that with $\mathfrak{w} := \mathfrak{v}(L_{P^\phi})$ we have $\mathfrak{w} \prec^\flat_\phi 1$ and $(P^\phi)_{\geqslant 1} = Q^\phi + R^\phi$ where $Q^\phi \in H^\phi\{Y\}$ is homogeneous of degree 1 and order r, L_{Q^ϕ} splits over $H^\phi[\mathrm{i}]$, and $R^\phi \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P^\phi)_1$. By Lemma 4.2.12, $L_{Q^\phi} = L_Q^\phi$ even splits strongly over $H[\mathrm{i}]$. Hence $(P^\phi, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal. The rest follows from Lemma 4.3.23 and the fact that if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then so is $(P^\phi, \mathfrak{m}, \widehat{a})$. $\square$

If H is $\upomega$-free and r-linearly newtonian, then by Corollary 3.3.48, every Z-minimal slot in H of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ is eventually deep and strictly normal. Corollary 4.3.30 of the next lemma is a variant of this fact for strong split-normality.

Lemma 4.3.29. *Assume H is $\upomega$-free and r-linearly newtonian, and every $A \in H[\partial]$ of order r splits over K. Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Then there is a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep and strictly normal, and its linear part splits strongly over K^ϕ (so $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ is strongly split-normal by Corollary 4.3.24).*

Proof. For any active ϕ in H with $0 < \phi \preccurlyeq 1$ we may replace H, $(P, \mathfrak{m}, \widehat{a})$ by H^ϕ, $(P^\phi, \mathfrak{m}, \widehat{a})$, respectively. We may also replace $(P, \mathfrak{m}, \widehat{a})$ by any of its refinements. Now Theorem 3.3.33 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H such that $0<\phi\preccurlyeq 1$ and $(P^{\phi}_{+a},\mathfrak{n},\widehat{a}-a)$ is deep and normal. Replacing H, $(P,\mathfrak{m},\widehat{a})$ by H^{ϕ}, $(P^{\phi}_{+a},\mathfrak{n},\widehat{a}-a)$, respectively, we thus arrange that $(P,\mathfrak{m},\widehat{a})$ itself is deep and normal. We show that then the lemma holds with $\phi=1$. For this we first replace $(P,\mathfrak{m},\widehat{a})$ by a suitable refinement $(P_{+a},\mathfrak{m},\widehat{a}-a)$ to arrange by Corollary 3.3.47 that $(P,\mathfrak{m},\widehat{a})$ is strictly normal and $\widehat{a}\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$. Now L splits over K, so by Corollary 4.2.14, for sufficiently small $q\in\mathbb{Q}^{>}$, any $\mathfrak{n}\asymp|\mathfrak{v}|^q\mathfrak{m}$ gives a refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ whose linear part $L_{P_{\times\mathfrak{n}}}$ has order r and splits strongly over K. For each such $\mathfrak{n}$, $(P,\mathfrak{n},\widehat{a})$ is deep by Corollary 3.3.8, and for some such $\mathfrak{n}$, $(P,\mathfrak{n},\widehat{a})$ is also strictly normal, by Remark 3.3.45. ☐

The previous lemma in combination with Lemma 4.3.28 yields:

Corollary 4.3.30. *With the same assumptions on H, K as in Lemma 4.3.29, every Z-minimal slot in H of order r has a refinement $(P,\mathfrak{m},\widehat{a})$ such that $(P^{\phi},\mathfrak{m},\widehat{a})$ is eventually deep and strongly split-normal.*

For $r=1$ the splitting assumption is automatically satisfied (and this is the case most relevant later). We do not know whether "every $A\in H[\partial]^{\neq}$ of order $\leqslant r$ splits over K" is strictly weaker than "K is r-linearly closed".

Achieving strong split-normality. We make the same assumptions as in the subsection *Achieving split-normality*: *H is ω-free and $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in $K=H[i]$ of order $r\geqslant 1$, with $\mathfrak{m}\in H^{\times}$ and $\widehat{a}\in\widehat{K}\setminus K$.* Recall that K is also ω-free [ADH, 11.7.23]. We have

$$\widehat{a} \;=\; \widehat{b}+\widehat{c}\,i,\qquad \widehat{b},\widehat{c}\in\widehat{H}.$$

We let a range over K, b, c over H, and $\mathfrak{n}$ over $H^{\times}$. In connection with the next two lemmas we note that given an active ϕ in H with $0<\phi\preccurlyeq 1$, if $(P,\mathfrak{m},\widehat{a})$ is normal (strictly normal, respectively), then so is $(P^{\phi},\mathfrak{m},\widehat{a})$, by Lemma 3.3.20 (Lemma 3.3.40, respectively); moreover, if the linear part of $(P,\mathfrak{m},\widehat{a})$ splits strongly over K, then the linear part of $(P^{\phi},\mathfrak{m},\widehat{a})$ splits strongly over $K^{\phi}=H^{\phi}[i]$, by Lemma 4.2.12. Here is a "complex" version of Lemma 4.3.29, with a similar proof:

Lemma 4.3.31. *For some refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ and active ϕ in H with $0<\phi\preccurlyeq 1$, the hole $(P^{\phi}_{+a},\mathfrak{n},\widehat{a}-a)$ in K^{ϕ} is deep and normal, its linear part splits strongly over K^{ϕ}, and it is moreover strictly normal if* $\deg P>1$.

Proof. For any active ϕ in H with $0<\phi\preccurlyeq 1$ we may replace H and $(P,\mathfrak{m},\widehat{a})$ by H^{ϕ} and the minimal hole $(P^{\phi},\mathfrak{m},\widehat{a})$ in K^{ϕ}. We may also replace $(P,\mathfrak{m},\widehat{a})$ by any of its refinements $(P_{+a},\mathfrak{n},\widehat{a}-a)$. As noted before Theorem 4.3.9, Corollary 3.3.34 and Lemma 3.3.23 give a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ and an active ϕ in H with $0<\phi\preccurlyeq 1$ such that $(P^{\phi}_{+a},\mathfrak{n},\widehat{a}-a)$ is deep and normal. Replacing H, $(P,\mathfrak{m},\widehat{a})$ by H^{ϕ}, $(P^{\phi}_{+a},\mathfrak{n},\widehat{a}-a)$, respectively, we thus arrange that $(P,\mathfrak{m},\widehat{a})$ itself is deep and normal. We show that then the lemma holds with $\phi=1$.

Set $L:=L_{P_{\times\mathfrak{m}}}$ and $\mathfrak{v}:=\mathfrak{v}(L)$. Lemma 3.3.13 gives a with $\widehat{a}-a\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$. If $\deg P>1$, then K is r-linearly newtonian and we use Corollary 3.3.16 to take a such that even $\widehat{a}-a\preccurlyeq\mathfrak{v}^{w+2}\mathfrak{m}$. Replacing $(P,\mathfrak{m},\widehat{a})$ by $(P_{+a},\mathfrak{m},\widehat{a}-a)$, we thus arrange by Lemma 3.3.7 and Proposition 3.3.25 that $\widehat{a}\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$, and also by Lemma 3.3.46 that $(P,\mathfrak{m},\widehat{a})$ is strictly normal if $\deg P>1$. Now L splits over K, since K is r-linearly closed by Corollary 3.2.4. Then by Corollary 4.2.14, for sufficiently small $q\in\mathbb{Q}^{>}$, any $\mathfrak{n}\asymp|\mathfrak{v}|^q\mathfrak{m}$ gives a refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ whose linear part $L_{P_{\times\mathfrak{n}}}$ splits strongly over K. For such $\mathfrak{n}$, $(P,\mathfrak{n},\widehat{a})$ is deep by Lemma 3.3.7 and normal by Proposition 3.3.26. If $(P,\mathfrak{m},\widehat{a})$ is strictly normal, then for some such $\mathfrak{n}$, $(P,\mathfrak{n},\widehat{a})$ is also strictly normal, thanks to Lemma 3.3.44. $\square$

The following version of Lemma 4.3.31 also encompasses linear $(P,\mathfrak{m},\widehat{a})$:

Lemma 4.3.32. *Suppose $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$. Then there is a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ and an active ϕ in H with $0<\phi\preccurlyeq 1$ such that the hole $(P^\phi_{+a},\mathfrak{n},\widehat{a}-a)$ in K^ϕ is deep and strictly normal, and its linear part splits strongly over K^ϕ.*

Proof. Thanks to Lemma 4.3.31 we need only consider the case $\deg P = 1$. Then we have $r=1$ by Corollary 3.2.8. (See now the remark following this proof.) As in the proof of Lemma 4.3.31 we may replace H and $(P,\mathfrak{m},\widehat{a})$ for any active $\phi\preccurlyeq 1$ in $H^>$ by H^ϕ and $(P^\phi,\mathfrak{m},\widehat{a})$, and also $(P,\mathfrak{m},\widehat{a})$ by any of its refinements $(P_{+a},\mathfrak{n},\widehat{a}-a)$. Recall here that $\mathfrak{n}\in H^\times$. Hence using a remark preceding Lemma 3.3.39 and Corollary 3.5.17 we arrange that $(P,\mathfrak{m},\widehat{a})$ is strictly normal, and thus balanced and deep. We show that then the lemma holds with $\phi=1$.

Set $L := L_{P_{\times\mathfrak{m}}}$, $\mathfrak{v} := \mathfrak{v}(L)$. Lemmas 3.5.9 and 3.5.10 yield an a with $\widehat{a}-a\preccurlyeq\mathfrak{v}^4\mathfrak{m}$. Replacing $(P,\mathfrak{m},\widehat{a})$ by $(P_{+a},\mathfrak{m},\widehat{a}-a)$ arranges that $\widehat{a}\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$, by Lemmas 3.3.7 and 3.3.41. As in the proof of Lemma 4.3.31, for sufficiently small $q\in\mathbb{Q}^>$, any $\mathfrak{n}\asymp|\mathfrak{v}|^q\mathfrak{m}$ now gives a strictly normal and deep refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ whose linear part splits strongly over K. $\square$

Remark. Suppose we replace our standing assumption that H is $\boldsymbol{\omega}$-free and $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in K by the assumption that H is $\boldsymbol{\lambda}$-free and $(P,\mathfrak{m},\widehat{a})$ is a slot in K of order and degree 1 (so K is $\boldsymbol{\lambda}$-free by [ADH, 11.6.8] and $(P,\mathfrak{m},\widehat{a})$ is Z-minimal). Then Lemma 4.3.32 goes through with "hole" replaced by "slot". Its proof also goes through with the references to Lemmas 3.3.7 and 3.3.41 replaced by references to Corollary 3.3.8 and Lemma 3.3.42. The end of that proof refers to the end of the proof of Lemma 4.3.31, and there one should replace Proposition 3.3.26 by Corollary 3.3.27, and Lemma 3.3.44 by Remark 3.3.45.

In the remainder of this subsection we prove the following variant of Theorem 4.3.9:

Theorem 4.3.33. *If H is 1-linearly newtonian, then one of the following holds:*

(i) *$\widehat{b}\notin H$ and there exists a Z-minimal slot $(Q,\mathfrak{m},\widehat{b})$ in H with a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ such that $(Q^\phi_{+b},\mathfrak{n},\widehat{b}-b)$ is eventually deep and almost strongly split-normal;*

(ii) *$\widehat{c}\notin H$ and there exists a Z-minimal slot $(R,\mathfrak{m},\widehat{c})$ in H with a refinement $(R_{+c},\mathfrak{n},\widehat{c}-c)$ such that $(R^\phi_{+c},\mathfrak{n},\widehat{c}-c)$ is eventually deep and almost strongly split-normal.*

Moreover, if H is 1-linearly newtonian and either $\deg P>1$*, or $\widehat{b}\notin H$ and $Z(H,\widehat{b})$ contains an element of order 1, or $\widehat{c}\notin H$ and $Z(H,\widehat{c})$ contains an element of order 1, then* (i) *holds with "almost" omitted, or* (ii) *holds with "almost" omitted.*

Towards the proof of this theorem we first show:

Lemma 4.3.34. *Suppose $\widehat{b}\notin H$ and $(Q,\mathfrak{m},\widehat{b})$ is a Z-minimal slot in H with a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ such that $(Q^\phi_{+b},\mathfrak{n},\widehat{b}-b)$ is eventually deep and split-normal. Then $(Q,\mathfrak{m},\widehat{b})$ has a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ such that $(Q^\phi_{+b},\mathfrak{n},\widehat{b}-b)$ is eventually deep and almost strongly split-normal.*

Proof. Let $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ be a refinement of $(Q, \mathfrak{m}, \widehat{b})$ and let ϕ_0 be active in H such that $0 < \phi_0 \preccurlyeq 1$ and $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$ is deep and split-normal. Then Corollary 4.3.27 yields a refinement $\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b}-b)-b_0\big)$ of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$ which is deep and almost strongly split-normal. Hence

$$\big((Q_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b}-b)-b_0\big) \ = \ \big(Q_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b}-(b+b_0)\big)$$

is a refinement of $(Q, \mathfrak{m}, \widehat{b})$, and $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b}-(b+b_0)\big)$ is eventually deep and almost strongly split-normal by Lemma 4.3.28. $\square$

Likewise:

Lemma 4.3.35. *Suppose $\widehat{c} \notin H$, and $(R, \mathfrak{m}, \widehat{c})$ is a Z-minimal slot in H with a refinement $(R_{+c}, \mathfrak{n}, \widehat{c}-c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c}-c)$ is eventually deep and split-normal. Then $(R, \mathfrak{m}, \widehat{c})$ has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c}-c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c}-c)$ is eventually deep and almost strongly split-normal.*

Theorem 4.3.9 and the two lemmas above give the first part of Theorem 4.3.33. We break up the proof of the "moreover" part into several cases, along the lines of the proof of Theorem 4.3.9. We begin with the case where $\widehat{b} \in H$ or $\widehat{c} \in H$.

Lemma 4.3.36. *Suppose H is 1-linearly newtonian, $\widehat{b} \notin H$, $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H of order r, and some refinement $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ of $(Q, \mathfrak{m}, \widehat{b})$ is such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b)$ is eventually deep and split-normal. Then $(Q, \mathfrak{m}, \widehat{b})$ has a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ with $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b)$ eventually deep and strongly split-normal.*

Proof. Lemma 4.3.34 gives a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ of $(Q, \mathfrak{m}, \widehat{b})$ with $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b)$ eventually deep and almost strongly split-normal. We upgrade this to "strongly split-normal" as follows: Take active ϕ_0 in H with $0 < \phi_0 \preccurlyeq 1$ such that the slot $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$ in H^{ϕ_0} is deep and almost strongly split-normal. Now H is 1-linearly newtonian, hence r-linearly newtonian. Therefore Corollary 3.3.47 yields a deep and strictly normal refinement $\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}, (\widehat{b}-b)-b_0\big)$ of $\big(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b\big)$. By Lemma 4.3.25, this refinement is still almost strongly split-normal, thus strongly split-normal by Lemma 4.3.23. Then by Lemma 4.3.28, $\big(Q_{+(b+b_0)}, \mathfrak{n}, \widehat{b}-(b+b_0)\big)$ is a refinement of $(Q, \mathfrak{m}, \widehat{b})$ such that $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}, \widehat{b}-(b+b_0)\big)$ is eventually deep and strongly split-normal. $\square$

Lemmas 4.3.10 and 4.3.36 give the following:

Corollary 4.3.37. *Suppose H is 1-linearly newtonian and $\widehat{c} \in H$. Then there is a hole $(Q, \mathfrak{m}, \widehat{b})$ in H of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. Every such hole $(Q, \mathfrak{m}, \widehat{b})$ in H is minimal and has a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b}-b)$ is eventually deep and strongly split-normal.*

Just as Lemma 4.3.10 gave rise to Lemma 4.3.11, Corollary 4.3.37 leads to:

Corollary 4.3.38. *Suppose H is 1-linearly newtonian and $\widehat{b} \in H$. Then there is a hole $(R, \mathfrak{m}, \widehat{c})$ in H of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. Every such hole in H is minimal and has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c}-c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c}-c)$ is eventually deep and strongly split-normal.*

In the following two lemmas we assume that $\widehat{b}, \widehat{c} \notin H$. Let $Q \in Z(H, \widehat{b})$ be of minimal complexity, so $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H, as is each of its refinements. The next lemma strengthens Corollary 4.3.13:

Lemma 4.3.39. *Suppose* $\deg P > 1$ *and* $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. *Then* $(Q, \mathfrak{m}, \widehat{b})$ *has a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and strongly split-normal.*

Proof. Corollary 4.3.13 and Lemma 4.3.34 give a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$ and an active ϕ_0 in H with $0 < \phi_0 \preccurlyeq 1$ such that the slot $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ in H^{ϕ_0} is deep and almost strongly split-normal. From $\deg P > 1$ we obtain that H is r-linearly newtonian. Now argue as in the proof of Lemma 4.3.36. $\square$

Similarly we obtain a strengthening of Corollary 4.3.14, using that corollary and Lemma 4.3.35 in place of Corollary 4.3.13 and Lemma 4.3.34 in the proof:

Lemma 4.3.40. *If* $\deg P > 1$, $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$, *and* $R \in Z(H, \widehat{c})$ *has minimal complexity, then the* Z*-minimal slot* $(R, \mathfrak{m}, \widehat{c})$ *in* H *has a refinement* $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ *such that* $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ *is eventually deep and strongly split-normal.*

We now prove the "moreover" part of Theorem 4.3.33. Thus, suppose H is 1-linearly newtonian. If $\widehat{b} \in H$, then $\widehat{c} \notin H$ and Corollary 4.3.38 yields a strong version of (ii) with "almost" omitted. Likewise, if $\widehat{c} \in H$, then $\widehat{b} \notin H$ and Corollary 4.3.37 yields a strong version of (i), with "almost" omitted. In the rest of the proof we assume $\widehat{b}, \widehat{c} \notin H$. By Lemma 4.1.3 we have $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$, and thus Lemmas 4.3.39 and 4.3.40 take care of the case $\deg P > 1$. If $Z(H, \widehat{b})$ contains an element of order 1, and $Q \in Z(H, \widehat{b})$ has minimal complexity, then $\operatorname{order} Q = 1$ by Lemma 4.3.7, so Corollary 4.3.30 and the remark following it yield (i) with "almost" omitted. Likewise, if $Z(H, \widehat{c})$ contains an element of order 1, then (ii) holds with "almost" omitted. $\square$

4.4. Ultimate Slots and Firm Slots

In this section H *is a Liouville closed* H*-field with small derivation,* $\widehat{H}$ *is an immediate asymptotic extension of* H, *and* i *be an element of an asymptotic extension of* $\widehat{H}$ *with* $\mathrm{i}^2 = -1$. Then $\widehat{H}$ is an H-field, $\mathrm{i} \notin \widehat{H}$, $K := H[\mathrm{i}]$ is an algebraic closure of H, and $\widehat{K} := \widehat{H}[\mathrm{i}]$ is an immediate d-valued extension of K. (See the beginning of Section 4.3.) Let C be the constant field of H, let $\mathcal{O}$ denote the valuation ring of H and Γ its value group. Accordingly, the constant field of K is $C_K = C[\mathrm{i}]$ and the valuation ring of K is $\mathcal{O}_K = \mathcal{O} + \mathcal{O}\mathrm{i}$. Let $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{w}$ range over $H^{\times}$ and ϕ over the elements of $H^{>}$ which are active in H (and hence in K).

In Section 1.2 we introduced

$$W \ := \ \big\{\operatorname{wr}(a,b) : \ a, b \in H, \ a^2 + b^2 = 1\big\}.$$

Note that W is a subspace of the $\mathbb{Q}$-linear space H, because $W i = S^{\dagger}$ where

$$S \ := \ \{a + bi : \ a, b \in H, \ a^2 + b^2 = 1\}$$

is a divisible subgroup of $K^{\times}$. We have $K^{\dagger} = H + Wi$ by Lemma 1.2.4. Thus there exists a complement Λ of the subspace $K^{\dagger}$ of K such that $\Lambda \subseteq Hi$, and in this section we fix such Λ and let λ range over Λ. Let $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K defined in Section 2.2.

For $A \in K[\partial]^{\neq}$ we have its set $\mathscr{E}^{\mathrm{u}}(A) \subseteq \Gamma$ of ultimate exceptional values, which a-priori might depend on our choice of Λ. We now make good on a promise from Section 2.6 by showing under the mild assumption $\mathrm{I}(K) \subseteq K^{\dagger}$ and with our restriction $\Lambda \subseteq Hi$ there is no such dependence:

Corollary 4.4.1. *Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$. Then for $A \in K[\partial]^{\neq}$, the status of A being terminal does not depend on the choice of Λ, and the set $\mathscr{E}^{\mathrm{u}}(A)$ of ultimate exceptional values of A also does not depend on this choice.*

Proof. Let $\Lambda^* \subseteq Hi$ also be a complement of $K^{\dagger}$. Let $\lambda \mapsto \lambda^*$ be the $\mathbb{Q}$-linear bijection $\Lambda \to \Lambda^*$ with $\lambda - \lambda^* \in Wi$ for all λ. Then by Lemmas 1.2.8 and 1.2.16,

$$\lambda - \lambda^* \in \mathrm{I}(H)\mathrm{i} \ \subseteq\ \mathrm{I}(K) \ \subseteq\ (\mathcal{O}_K^{\times})^{\dagger}$$

for all λ. Now use Lemma 2.6.8 and Corollary 2.6.9. $\square$

Corollary 4.4.2. *Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$. Let $A = \partial - g \in K[\partial]$ where $g \in K$ and let $\mathfrak{g} \in H^{\times}$ be such that $\mathfrak{g}^{\dagger} = \operatorname{Re} g$. Then*

$$\mathscr{E}^{\mathrm{u}}(A) \ =\ v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \ =\ \{v\mathfrak{g}\}.$$

In particular, if $\operatorname{Re} g \in \mathrm{I}(H)$, then $\mathscr{E}^{\mathrm{u}}(A) = \{0\}$.

Proof. Let $f \in K^{\times}$ and λ be such that $g = f^{\dagger} + \lambda$. Then

$$\mathscr{E}^{\mathrm{u}}(A) \ =\ v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \ =\ \{vf\}$$

by Lemma 2.6.14 and its proof. Recall that $K^{\dagger} = H + \mathrm{I}(H)\mathrm{i}$ by Lemma 1.2.16 and remarks preceding it, so $g \in K^{\dagger}$ iff $\operatorname{Im} g \in \mathrm{I}(H)$. Consider first the case $g \notin K^{\dagger}$. Then by Corollary 4.4.1 we can change Λ if necessary to arrange $\lambda := (\operatorname{Im} g)\mathrm{i} \in \Lambda$ so that we can take $f := \mathfrak{g}$ in the above. Now suppose $g \in K^{\dagger}$. Then $g = (\mathfrak{g}h)^{\dagger}$ where $h \in K^{\times}$, $h^{\dagger} = (\operatorname{Im} g)\mathrm{i}$. Then we can take $f := \mathfrak{g}h$, $\lambda := 0$, and we have $h \asymp 1$ since $h^{\dagger} \in \mathrm{I}(H)\mathrm{i} \subseteq \mathrm{I}(K)$. $\square$

Corollary 4.4.3. *Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$, and let F be a Liouville closed H-field extension of H, and $L := F[\mathrm{i}]$. Then the subspace $L^{\dagger}$ of the $\mathbb{Q}$-linear space L has a complement Λ_L with $\Lambda \subseteq \Lambda_L \subseteq Fi$. For any such Λ_L and $A \in K[\partial]^{\neq}$ we have $\mathscr{E}^{\mathrm{e}}(A_{\lambda}) = \mathscr{E}^{\mathrm{e}}_L(A_{\lambda}) \cap \Gamma$ for all λ, and thus $\mathscr{E}^{\mathrm{u}}(A) \subseteq \mathscr{E}^{\mathrm{u}}_L(A)$, where $\mathscr{E}^{\mathrm{u}}_L(A)$ is the set of ultimate exceptional values of $A \in L[\partial]^{\neq}$ with respect to Λ_L.*

Proof. By the remarks at the beginning of this subsection applied to F, L in place of H, K we have $L^{\dagger} = F + W_F i$ where W_F is a subspace of the $\mathbb{Q}$-linear space F. Also $K^{\dagger} = H + \mathrm{I}(H)i$ by Lemma 1.2.16, and $L^{\dagger} \cap K = K^{\dagger}$ by Lemma 2.6.24. This yields a complement Λ_L of $L^{\dagger}$ in L with $\Lambda \subseteq \Lambda_L \subseteq Fi$. Since H is Liouville closed and hence $\boldsymbol{\lambda}$-free by [ADH, 11.6.2], its algebraic closure K is $\boldsymbol{\lambda}$-free by [ADH, 11.6.8]. Now the rest follows from remarks preceding Lemma 2.6.12. $\square$

Given $A \in K[\partial]^{\neq}$, let $\mathscr{E}^{\mathrm{u}}(A^{\phi})$ be the set of ultimate exceptional values of the linear differential operator $A^{\phi} \in K^{\phi}[\delta]$, $\delta = \phi^{-1}\partial$, with respect to $\Lambda^{\phi} = \phi^{-1}\Lambda$. We summarize some properties of ultimate exceptional values used later in this section:

Lemma 4.4.4. *Let $A \in K[\partial]^{\neq}$ have order r. Then for all $b \in K^{\times}$ and all ϕ,*

$$\mathscr{E}^{\mathrm{u}}(bA) \ =\ \mathscr{E}^{\mathrm{u}}(A), \quad \mathscr{E}^{\mathrm{u}}(Ab) \ =\ \mathscr{E}^{\mathrm{u}}(A) - vb, \quad \mathscr{E}^{\mathrm{u}}(A^{\phi}) \ =\ \mathscr{E}^{\mathrm{u}}(A).$$

Moreover, if $\mathrm{I}(K) \subseteq K^{\dagger}$, then:

(i) *$|\mathscr{E}^{\mathrm{u}}(A)| \leqslant r$;*

(ii) $\dim_{C[i]} \ker_{\mathrm{U}} A = r \implies \mathscr{E}^{\mathrm{u}}(A) = v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)$;

(iii) *under the assumption that* $\mathfrak{v} := \mathfrak{v}(A) \prec^{\flat} 1$ *and* $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1} A$ *where* $B \in K[\partial]$ *has order* $\leqslant r$, *we have* $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$;

(iv) *for* $r = 1$ *we have* $|\mathscr{E}^{\mathrm{u}}(A)| = 1$ *and* $\mathscr{E}^{\mathrm{u}}(A) = v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)$.

Proof. For the displayed equalities, see Remark 2.6.10. Now assume $\mathrm{I}(K) \subseteq K^{\dagger}$. Then $K^{\dagger} = H + \mathrm{I}(H)i$, so (i) and (ii) follow from Proposition 2.6.26 and (iii) from Proposition 3.1.26. Corollary 4.4.2 yields (iv). □

Recall from Lemma 1.2.9 that if K is 1-linearly newtonian, then $\mathrm{I}(K) \subseteq K^{\dagger}$.

Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$. Then $K^{\dagger} = H + \mathrm{I}(H)i$, so our Λ has the form $\Lambda_H i$ with Λ_H a complement of $\mathrm{I}(H)$ in H. Conversely, any complement Λ_H of $\mathrm{I}(H)$ in H yields a complement $\Lambda = \Lambda_H i$ of $K^{\dagger}$ in K with $\Lambda \subseteq Hi$. Now $\mathrm{I}(H)$ is a C-linear subspace of H, so $\mathrm{I}(H)$ has a complement Λ_H in H that is a C-linear subspace of H, and then $\Lambda := \Lambda_H i$ is also a C-linear subspace of K.

Lemma 4.4.5. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and* $g \in K$, $g - \lambda \in K^{\dagger}$. *Then*

$$\operatorname{Im} g \in \mathrm{I}(H) \iff \lambda = 0, \qquad \operatorname{Im} g \notin \mathrm{I}(H) \implies \lambda \sim (\operatorname{Im} g)i.$$

Proof. Recall that $\Lambda = \Lambda_H i$ where Λ_H is a complement of $\mathrm{I}(H)$ in H, so $\lambda = \lambda_H i$ where $\lambda_H \in \Lambda_H$. Also, $K^{\dagger} = H \oplus \mathrm{I}(H)i$, hence $\operatorname{Im}(g) - \lambda_H \in \mathrm{I}(H)$; this proves the displayed equivalence. Suppose $\operatorname{Im} g \notin \mathrm{I}(H)$; since $\mathrm{I}(H)$ is an $\mathcal{O}_H$-submodule of H and $\lambda_H \notin \mathrm{I}(H)$, we then have $\operatorname{Im}(g) - \lambda_H \prec \lambda_H$, so $\lambda = \lambda_H i \sim \operatorname{Im}(g)i$. □

Corollary 4.4.6. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$, $A \in K[\partial]^{\neq}$ *has order* r, $\dim_{C[i]} \ker_{\mathrm{U}} A = r$, *and* λ *is an eigenvalue of* A *with respect to* Λ. *Then* $\lambda \preccurlyeq \mathfrak{v}(A)^{-1}$.

Proof. Take $f \neq 0$ and $g_1, \dots, g_r$ in K with $A = f(\partial - g_1)\cdots(\partial - g_r)$. By Corollary 3.1.6 we have $g_1, \dots, g_r \preccurlyeq \mathfrak{v}(A)^{-1}$, and so Corollary 2.5.6 gives $j \in \{1, \dots, r\}$ with $g_j - \lambda \in K^{\dagger}$. Now use Lemma 4.4.5. □

Ultimate slots in H**.** *In this subsection* a, b *range over* H. Also, $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$, where $\widehat{a} \in \widehat{H} \setminus H$. Recall that $L_{P_{\times \mathfrak{m}}} = L_P \mathfrak{m}$, so if $(P, \mathfrak{m}, \widehat{a})$ is normal, then L_P has order r.

Corollary 4.4.7. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and the slot* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal with linear part* $L := L_{P_{\times \mathfrak{m}}}$. *Then with* Q *and* R *as in* (SN2) *we have* $\mathscr{E}^{\mathrm{u}}(L) = \mathscr{E}^{\mathrm{u}}(L_Q)$.

This follows from Lemmas 4.3.4 and 4.4.4(iii). In a similar vein we have an analogue of Lemma 3.3.24:

Lemma 4.4.8. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal and* $a \prec \mathfrak{m}$. *Then* L_P *and* $L_{P_{+a}}$ *have order* r, *and if* $\mathrm{I}(K) \subseteq K^{\dagger}$, *then* $\mathscr{E}^{\mathrm{u}}(L_P) = \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$.

Proof. We have $L_{P_{\times \mathfrak{m}}} = L_P \mathfrak{m}$ and $L_{P_{+a, \times \mathfrak{m}}} = L_{P_{\times \mathfrak{m}, +a/\mathfrak{m}}} = L_{P_{+a}} \mathfrak{m}$. The slot $(P_{\times \mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ in H is normal and $a/\mathfrak{m} \prec 1$. Lemma 3.1.28 applied to $\widehat{H}$, $P_{\times \mathfrak{m}}$, $\widehat{a}/\mathfrak{m}$ in place of K, P, a, respectively, gives: L_P and $L_{P_{+a}}$ have order r, and

$$L_P \mathfrak{m} - L_{P_{+a}} \mathfrak{m} \;=\; L_{P_{\times \mathfrak{m}}} - L_{P_{\times \mathfrak{m}, +a/\mathfrak{m}}} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{r+1} L_P \mathfrak{m}$$

where $\mathfrak{v} := \mathfrak{v}(L_P \mathfrak{m}) \prec^{\flat} 1$ by (N1). Suppose now that $\mathrm{I}(K) \subseteq K^{\dagger}$. Then

$$\mathscr{E}^{\mathrm{u}}(L_P) \;=\; \mathscr{E}^{\mathrm{u}}(L_P \mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{u}}(L_{P_{+a}} \mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$$

by Lemma 4.4.4(iii). □

The notion introduced below is modeled on that of "isolated slot" (Definition 3.4.1):

Definition 4.4.9. Call $(P, \mathfrak{m}, \widehat{a})$ **ultimate** if for all $a \prec \mathfrak{m}$,

$$\operatorname{order}(L_{P_{+a}}) = r \quad \text{and} \quad \mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) \cap v(\widehat{a} - H) \ < \ v(\widehat{a} - a);$$

equivalently, for all $a \prec \mathfrak{m}$: $\operatorname{order}(L_{P_{+a}}) = r$ and whenever $\mathfrak{w} \preccurlyeq \widehat{a} - a$ is such that $v(\mathfrak{w}) \in \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$, then $\mathfrak{w} \prec \widehat{a} - b$ for all b. (Thus if $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then it is isolated.)

If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is every equivalent slot in H and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$, as well as the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ} (by Lemma 4.4.4). The proofs of the next two lemma are like those of their "isolated" versions, Lemmas 3.4.2 and 3.4.3:

Lemma 4.4.10. *If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is any of its refinements.*

Lemma 4.4.11. *If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is any of its multiplicative conjugates.*

The ultimate condition is most useful in combination with other properties:

Lemma 4.4.12. *If $\mathrm{I}(K) \subseteq K^{\dagger}$ and $(P, \mathfrak{m}, \widehat{a})$ is normal, then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad \Longleftrightarrow \quad \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H) \leqslant v\mathfrak{m}.$$

Proof. Use Lemma 4.4.8 and the equivalence $\widehat{a} - a \prec \mathfrak{m} \Leftrightarrow a \prec \mathfrak{m}$. □

The "ultimate" version of Lemma 3.4.5 has the same proof:

Lemma 4.4.13. *If $\deg P = 1$, then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad \Longleftrightarrow \quad \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H) \leqslant v\mathfrak{m}.$$

The next proposition is the "ultimate" version of Proposition 3.4.6:

Proposition 4.4.14. *Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$, and $(P, \mathfrak{m}, \widehat{a})$ is normal. Then $(P, \mathfrak{m}, \widehat{a})$ has an ultimate refinement.*

Proof. Suppose $(P, \mathfrak{m}, \widehat{a})$ is not already ultimate. Then Lemma 4.4.12 gives γ with

$$\gamma \in \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H), \quad \gamma > v\mathfrak{m}.$$

Lemma 4.4.4(i) gives $|\mathscr{E}^{\mathrm{u}}(L_P)| \leqslant r$, so we can take

$$\gamma \ := \ \max \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H),$$

and then $\gamma > v\mathfrak{m}$. Take a and $\mathfrak{n}$ with $v(\widehat{a} - a) > \gamma = v(\mathfrak{n})$; then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$ and $a \prec \mathfrak{m}$. Let $b \prec \mathfrak{n}$; then $a + b \prec \mathfrak{m}$, so by Lemma 4.4.8,

$$\operatorname{order}(L_{(P_{+a})_{+b}}) \ = \ r, \qquad \mathscr{E}^{\mathrm{u}}(L_{(P_{+a})_{+b}}) \ = \ \mathscr{E}^{\mathrm{u}}(L_P).$$

Also $v\big((\widehat{a} - a) - b\big) > \gamma$, hence

$$\mathscr{E}^{\mathrm{u}}\big(L_{(P_{+a})_{+b}}\big) \cap v\big((\widehat{a} - a) - H\big) \ = \ \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H) \ \leqslant \ \gamma \ < \ v\big((\widehat{a} - a) - b\big).$$

Thus $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is ultimate. □

Remark 4.4.15. Proposition 4.4.14 goes through if instead of assuming that $(P, \mathfrak{m}, \widehat{a})$ is normal, we assume that $(P, \mathfrak{m}, \widehat{a})$ is linear. (Same argument, using Lemma 4.4.13 in place of Lemma 4.4.12.)

Finally, here is a consequence of Corollaries 2.6.15, 4.4.2, and Lemma 4.4.12, where we recall that $\operatorname{order}(L_{P_{\times \mathfrak{m}}}) = \operatorname{order}(L_P \mathfrak{m}) = \operatorname{order}(L_P)$:

Corollary 4.4.16. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal of order* $r = 1$. *Then* $L_P = f(\partial - g)$ *with* $f \in H^{\times}$, $g \in H$, *and for* $\mathfrak{g} \in H^{\times}$ *with* $\mathfrak{g}^{\dagger} = g$ *we have:*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \iff (P, \mathfrak{m}, \widehat{a}) \text{ is isolated} \iff .\mathfrak{g} \succcurlyeq \mathfrak{m} \text{ or } \mathfrak{g} \prec \widehat{a} - H.$$

(*In particular, if* $g \in \mathrm{I}(H)$ *and* $\mathfrak{m} \preccurlyeq 1$, *then* $(P, \mathfrak{m}, \widehat{a})$ *is ultimate.*)

Ultimate slots in K**.** *In this subsection,* a, b *range over* $K = H[\mathrm{i}]$. Also $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order $r \geqslant 1$, where $\widehat{a} \in \widehat{K} \setminus K$. Lemma 4.4.8 goes through in this setting, with H in the proof replaced by K:

Lemma 4.4.17. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal, and* $a \prec \mathfrak{m}$. *Then* L_P *and* $L_{P_{+a}}$ *have order* r, *and if* $\mathrm{I}(K) \subseteq K^{\dagger}$, *then* $\mathscr{E}^{\mathrm{u}}(L_P) = \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$.

We adapt Definition 4.4.9 to slots in K: call $(P, \mathfrak{m}, \widehat{a})$ **ultimate** if for all $a \prec \mathfrak{m}$ we have $\operatorname{order}(L_{P_{+a}}) = r$ and $\mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) \cap v(\widehat{a} - K) < v(\widehat{a} - a)$. If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then it is isolated. Moreover, if $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ as well as the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in K^{ϕ}. Lemmas 4.4.10 and 4.4.11 go through in the present context, and so do Lemmas 4.4.12 and 4.4.13 with H replaced by K. The analogue of Proposition 4.4.14 follows likewise:

Proposition 4.4.18. *If* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal, then* $(P, \mathfrak{m}, \widehat{a})$ *has an ultimate refinement.*

Remark 4.4.19. Proposition 4.4.18 also holds if instead of assuming that $(P, \mathfrak{m}, \widehat{a})$ is normal, we assume that $(P, \mathfrak{m}, \widehat{a})$ is linear.

Corollary 4.4.2 and the K-versions of Lemmas 4.4.12 and 4.4.13 yield:

Corollary 4.4.20. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$, $r = 1$, *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal or linear. Then* $L_P = f(\partial - g)$ *with* $f \in K^{\times}, g \in K$, *and for* $\mathfrak{g} \in H^{\times}$ *with* $\mathfrak{g}^{\dagger} = \operatorname{Re} g$ *we have:*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \iff \mathfrak{g} \succcurlyeq \mathfrak{m} \text{ or } \mathfrak{g} \prec \widehat{a} - K.$$

(*In particular, if* $\operatorname{Re} g \in \mathrm{I}(H)$ *and* $\mathfrak{m} \preccurlyeq 1$, *then* $(P, \mathfrak{m}, \widehat{a})$ *is ultimate.*)

Using the norm to characterize being ultimate. We use here the "norm" $\|\cdot\|$ on U and the gaussian extension v_{g} of the valuation of K from Section 2.1.

Lemma 4.4.21. *For* $u \in \mathrm{U}^{\times}$ *we have* $\|u\|^{\dagger} = \operatorname{Re} u^{\dagger}$.

Proof. For $u = f\,\mathrm{e}(\lambda)$, $f \in K^{\times}$ we have $\|u\| = |f|$ and $u^{\dagger} = f^{\dagger} + \lambda$, so

$$\|u\|^{\dagger} = |f|^{\dagger} = \operatorname{Re} f^{\dagger} = \operatorname{Re} u^{\dagger},$$

using Corollary 1.2.5 for the second equality. □

Using Corollary 2.1.10, Lemma 4.4.21, and [ADH, 10.5.2(i)] we obtain:

Lemma 4.4.22. *Let* $\mathfrak{W} \subseteq H^{\times}$ *be* $\preccurlyeq$*-closed. Then for all* $u \in \mathrm{U}^{\times}$,

$$\|u\| \in \mathfrak{W} \iff v_{\mathrm{g}} u \in v(\mathfrak{W}) \iff \operatorname{Re} u^{\dagger} < \mathfrak{n}^{\dagger} \text{ for all } \mathfrak{n} \notin \mathfrak{W}.$$

Let $(P,\mathfrak{m},\widehat{a})$ be a slot in H of order $r\geqslant 1$. Applying Lemma 4.4.22 to the set $\mathfrak{W}=\{\mathfrak{w}:\ \mathfrak{w}\prec\widehat{a}-H\}$— so $v(\mathfrak{W})=\Gamma\setminus v(\widehat{a}-H)$—we obtain a reformulation of the condition "$(P,\mathfrak{m},\widehat{a})$ is ultimate" in terms of the "norm" $\|\cdot\|$ on U:

Corollary 4.4.23. *The following are equivalent (with a ranging over H):*

(i) *$(P,\mathfrak{m},\widehat{a})$ is ultimate;*

(ii) *for all $a\prec\mathfrak{m}$: $\operatorname{order}(L_{P_{+a}})=r$ and whenever $u\in\mathrm{U}^\times$, $v_{\mathrm{g}}u\in\mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$, and $\|u\|\preccurlyeq\widehat{a}-a$, then $\|u\|\prec\widehat{a}-H$;*

(iii) *for all $a\prec\mathfrak{m}$: $\operatorname{order}(L_{P_{+a}})=r$ and whenever $u\in\mathrm{U}^\times$, $v_{\mathrm{g}}u\in\mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$, and $\|u\|\preccurlyeq\widehat{a}-a$, then $\operatorname{Re}u^\dagger<\mathfrak{n}^\dagger$ for all $\mathfrak{n}$ with $v(\mathfrak{n})\in v(\widehat{a}-H)$.*

Firm slots and flabby slots in H (*). Let $(P,\mathfrak{m},\widehat{a})$ be a slot in H of order $r\geqslant 1$, where $\widehat{a}\in\widehat{H}\setminus H$. We let a, b range over H.

Definition 4.4.24. We call $(P,\mathfrak{m},\widehat{a})$ **firm** if for all $a\prec\mathfrak{m}$,

$$\operatorname{order}(L_{P_{+a}})=r\quad\text{and}\quad\mathscr{E}^{\mathrm{u}}(L_{P_{+a}})\subseteq v(\widehat{a}-H).$$

We call $(P,\mathfrak{m},\widehat{a})$ **flabby** if it is not firm, that is, if there is an $a\prec\mathfrak{m}$ such that $\operatorname{order}(L_{P_{+a}})<r$, or $\operatorname{order}(L_{P_{+a}})=r$ and $\gamma>v(\widehat{a}-H)$ for some $\gamma\in\mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$.

If $(P,\mathfrak{m},\widehat{a})$ is firm, then so are $(bP,\mathfrak{m},\widehat{a})$ for $b\neq 0$ and any slot $(P,\mathfrak{m},\widehat{b})$ in H that is equivalent to $(P,\mathfrak{m},\widehat{a})$. For any ϕ, the slot $(P,\mathfrak{m},\widehat{a})$ in H is firm iff the slot $(P^\phi,\mathfrak{m},\widehat{a})$ in H^ϕ is firm.

Lemma 4.4.25. *If $(P,\mathfrak{m},\widehat{a})$ is firm, then so is any of its refinements. If $(P,\mathfrak{m},\widehat{a})$ is flabby, then so is any refinement $(P_{+a},\mathfrak{m},\widehat{a}-a)$ of it.*

The proof is like that of Lemma 4.4.10.

Lemma 4.4.26. *Suppose $(P,\mathfrak{m},\widehat{a})$ is firm. Then $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$ is firm.*

Proof. Let $a\prec\mathfrak{m}/\mathfrak{n}$, so $a\mathfrak{n}\prec\mathfrak{m}$ with $L_{P_{\times\mathfrak{n},+a}}=L_{P_{+a\mathfrak{n}}}\mathfrak{n}$. Since $(P,\mathfrak{m},\widehat{a})$ is firm, this yields $\operatorname{order}(L_{P_{\times\mathfrak{n},+a}})=\operatorname{order}(L_{P_{+a\mathfrak{n}}})=r$ and

$$\mathscr{E}^{\mathrm{u}}(L_{P_{\times\mathfrak{n},+a}})=\mathscr{E}^{\mathrm{u}}(L_{P_{+a\mathfrak{n}}})-v\mathfrak{n}\subseteq v(\widehat{a}-H)-v\mathfrak{n}=v\big((\widehat{a}/\mathfrak{n})-H\big),$$

using Lemma 4.4.4 for the first equality. □

The proofs of the next two lemmas are clear, using Lemma 4.4.8 for the first one:

Lemma 4.4.27. *If $\mathrm{I}(K)\subseteq K^\dagger$ and $(P,\mathfrak{m},\widehat{a})$ is normal, then*

$$(P,\mathfrak{m},\widehat{a})\ \textit{is firm}\quad\Longleftrightarrow\quad\mathscr{E}^{\mathrm{u}}(L_P)\subseteq v(\widehat{a}-H).$$

Lemma 4.4.28. *If $\deg P=1$, then*

$$(P,\mathfrak{m},\widehat{a})\ \textit{is firm}\quad\Longleftrightarrow\quad\mathscr{E}^{\mathrm{u}}(L_P)\subseteq v(\widehat{a}-H).$$

Remark 4.4.29. If the hypothesis of Lemma 4.4.27 or Lemma 4.4.28 holds, then

$$(P,\mathfrak{m},\widehat{a})\ \text{is firm and ultimate}\quad\Longleftrightarrow\quad\mathscr{E}^{\mathrm{u}}(L_P)\leqslant v\mathfrak{m},$$

as a consequence of Lemmas 4.4.12 and 4.4.13.

Lemma 4.4.8 yields:

Corollary 4.4.30. *If the hypothesis of Lemma 4.4.27 or Lemma 4.4.28 holds and $(P,\mathfrak{m},\widehat{a})$ is flabby, then so is each refinement of $(P,\mathfrak{m},\widehat{a})$.*

Firm slots and flabby slots in K (*)**.** Let now $(P, \mathfrak{m}, \widehat{a})$ be a slot in K of order $r \geqslant 1$ with $\widehat{a} \in \widehat{K} \setminus K$, and let a, b range over K. We define $(P, \mathfrak{m}, \widehat{a})$ to be **firm** if for all $a \prec \mathfrak{m}$ we have $\operatorname{order}(L_{P_{+a}}) = r$ and $\mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) \subseteq v(\widehat{a} - H)$, and we say that $(P, \mathfrak{m}, \widehat{a})$ is **flabby** if it is not firm. The results in the subsection above about a slot $(P, \mathfrak{m}, \widehat{a})$ in H go through for the slot $(P, \mathfrak{m}, \widehat{a})$ in K, replacing H, $\widehat{H}$ by K, $\widehat{K}$ throughout.

Corollary 4.4.31. *Suppose* $\mathrm{I}(K) \subseteq K^\dagger$, $r = 1$, *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal or linear. Then* $L_P = f(\partial - g)$ *with* $f \in K^\times$, $g \in K$. *For* $\mathfrak{g} \in H^\times$ *with* $\mathfrak{g}^\dagger = \operatorname{Re} g$ *we have:*

(i) $(P, \mathfrak{m}, \widehat{a})$ *is flabby* $\iff$ $\mathfrak{g} \prec \widehat{a} - K$ $\implies$ $(P, \mathfrak{m}, \widehat{a})$ *is ultimate;*
(ii) $(P, \mathfrak{m}, \widehat{a})$ *is firm and ultimate* $\iff$ $\mathfrak{g} \succcurlyeq \mathfrak{m}$*;*
(iii) $\mathfrak{g} \succcurlyeq 1 \iff \operatorname{Re} g \in \mathrm{I}(H)$ *or* $\operatorname{Re} g > 0$.

Proof. The equivalence in (i) follows from Corollary 4.4.2 and the K-versions of Lemmas 4.4.27 and 4.4.28. Corollary 4.4.20 yields the last part of (i). For (ii), use Corollary 4.4.2 and the K-version of the equivalence in Remark 4.4.29. As to (iii), this is an elementary fact about the relation between $\mathfrak{g} \in H^\times$ and $\mathfrak{g}^\dagger$. □

For the significance of firm slots in the Hardy field setting, see Section 7.7 below.

Counterexamples (*). Suppose $\mathrm{I}(K) \subseteq K^\dagger$ and H is not ω-free. (In Example 7.5.40 we provide an H with these properties.) Let (λ_ρ) and (ω_ρ) be as in Lemma 3.2.10 with H in the role of K there. That lemma yields a minimal hole $(P, \mathfrak{m}, \lambda)$ in H with $P = 2Y' + Y^2 + \omega$ $(\omega \in H)$. This is a good source of counterexamples:

Lemma 4.4.32. *The minimal hole* $(P, \mathfrak{m}, \lambda)$ *in* H *is ultimate, and none of its refinements is quasilinear, normal, or firm.*

Proof. Let $a \in H$. Then $P_{+a} = 2Y' + 2aY + Y^2 + P(a)$ and thus $L_{P_{+a}} = 2(\partial + a)$, so for $b \in H^\times$ with $b^\dagger = -a$ we have $\mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) = \{vb\}$, by Corollary 4.4.2. Thus $(P, \mathfrak{m}, \lambda)$ is ultimate iff $\lambda - a \prec b$ for all $a \prec \mathfrak{m}$ in H and $b \in H^\times$ with $b^\dagger = -a$ and $vb \in v(\lambda - H)$; the latter holds by [ADH, 11.5.6] since $v(\lambda - H) = \Psi$. Hence $(P, \mathfrak{m}, \lambda)$ is ultimate. No refinement of $(P, \mathfrak{m}, \lambda)$ is quasilinear by Corollary 3.2.25 and [ADH, 11.7.9], and so by Corollary 3.3.21, no refinement of $(P, \mathfrak{m}, \lambda)$ is normal.

It remains to show that no refinement of $(P, \mathfrak{m}, \lambda)$ is firm. Let (ℓ_ρ), (γ_ρ), be the sequences from [ADH, 11.5] that give rise to $\lambda_\rho = -\gamma_\rho^\dagger$ with H in place of K. If $(P, \mathfrak{m}, \lambda)$ has a firm refinement, then it has a firm refinement $(P_{+\lambda_\rho}, \gamma_\rho, \lambda - \lambda_\rho)$, by Lemmas 3.2.24 and 4.4.25, so it suffices that $(P_{+\lambda_\rho}, \gamma_\rho, \lambda - \lambda_\rho)$ is flabby for all ρ. For $a \in H$ we have $L_{P_{+(\lambda_\rho + a)}} = 2(\partial + \lambda_\rho + a)$, so $\mathscr{E}^{\mathrm{u}}(L_{P_{+(\lambda_\rho + a)}}) = \{vb\}$ with $b \in H^\times$, $b^\dagger = -(\lambda_\rho + a)$. Also $v\big((\lambda - \lambda_\rho) - H\big) = v(\lambda - H) = \Psi$. Hence $(P_{+\lambda_\rho}, \gamma_\rho, \lambda - \lambda_\rho)$ is flabby if there is $a \prec \gamma_\rho$ in H and $b \in H^\times$, not active in H, such that $b^\dagger = -(\lambda_\rho + a)$. We take $a := 2\gamma_{\rho+1}$, $b := \gamma_\rho / \ell_{\rho+1}^2$. Then $b^\dagger = \gamma_\rho^\dagger - 2\ell_{\rho+1}^\dagger = -(\lambda_\rho + a)$ as required. Also, b is not active in H. To see this let $\sigma > \rho + 1$. Then $\gamma_\rho/\gamma_{\rho+1}, \gamma_{\rho+1}/\gamma_\sigma \succ 1$ and

$$(\gamma_\rho/\gamma_{\rho+1})^\dagger \;=\; \lambda_{\rho+1} - \lambda_\rho \;\sim\; \gamma_{\rho+1} \;\succ\; \gamma_{\rho+2} \;\sim\; \lambda_\sigma - \lambda_{\rho+1} \;=\; (\gamma_{\rho+1}/\gamma_\sigma)^\dagger$$

by [ADH, 11.5.2], hence $\gamma_\rho/\gamma_{\rho+1} \succ \gamma_{\rho+1}/\gamma_\sigma$. Also $\ell_{\rho+1} \asymp \gamma_\rho/\gamma_{\rho+1}$ by [ADH, proof of 11.5.2] and thus $b = \gamma_\rho/\ell_{\rho+1}^2 \asymp \gamma_{\rho+1}^2/\gamma_\rho \prec \gamma_\sigma$. □

4.5. Repulsive-Normal Slots

In this section H is a real closed H-field with small derivation and asymptotic integration, with $\Gamma := v(H^\times)$. Also $K := H[\mathrm{i}]$ with $\mathrm{i}^2 = -1$ is an algebraic closure of H. We study here the concept of a repulsive-normal slot in H, which strengthens that of a split-normal slot in H. Despite their name, repulsive-normal slots will turn out to have attractive analytic properties in the realm of Hardy fields.

Attraction and repulsion. In this subsection a, b range over H, $\mathfrak{m}$, $\mathfrak{n}$ over $H^\times$, f, g, h (possibly with subscripts) over K, and γ, δ over Γ. We say that f is **attractive** if $\operatorname{Re} f \succcurlyeq 1$ and $\operatorname{Re} f < 0$, and **repulsive** if $\operatorname{Re} f \succcurlyeq 1$ and $\operatorname{Re} f > 0$. If $\operatorname{Re} f \sim \operatorname{Re} g$, then f is attractive iff g is attractive, and likewise with "repulsive" in place of "attractive". Moreover, if $a > 0$, $a \succcurlyeq 1$, and f is attractive (repulsive), then af is attractive (repulsive, respectively).

Definition 4.5.1. Let $\gamma > 0$; we say f is **γ-repulsive** if $v(\operatorname{Re} f) < \gamma^\dagger$ or $\operatorname{Re} f > 0$. Given $S \subseteq \Gamma$, we say f is **S-repulsive** if f is γ-repulsive for all $\gamma \in S \cap \Gamma^>$, equivalently, $\operatorname{Re} f > 0$, or $v(\operatorname{Re} f) < \gamma^\dagger$ for all $\gamma \in S \cap \Gamma^>$.

Note the following implications for $\gamma > 0$:

$$\begin{aligned} f \text{ is } \gamma\text{-repulsive} \quad &\Longrightarrow \quad \operatorname{Re} f \neq 0, \\ f \text{ is } \gamma\text{-repulsive, } \operatorname{Re} g \sim \operatorname{Re} f \quad &\Longrightarrow \quad g \text{ is } \gamma\text{-repulsive.} \end{aligned}$$

The following is easy to show:

Lemma 4.5.2. *Suppose $\gamma > 0$ and $\operatorname{Re} f \succcurlyeq 1$. Then f is γ-repulsive iff $v(\operatorname{Re} f) < \gamma^\dagger$ or f is repulsive. Hence, if f is repulsive, then f is Γ-repulsive; the converse of this implication holds if Ψ is not bounded from below in Γ.*

Let $\gamma, \delta > 0$. If f is γ-repulsive and $a > 0$, $a \succcurlyeq 1$, then af is γ-repulsive. If f is γ-repulsive and δ-repulsive, then f is $(\gamma+\delta)$-repulsive. If f is γ-repulsive and $\gamma > \delta$, then f is $(\gamma - \delta)$-repulsive. Moreover:

Lemma 4.5.3. *Suppose $\gamma \geqslant \delta = v\mathfrak{n} > 0$. Set $g := f - \mathfrak{n}^\dagger$. Then:*

$$f \text{ is } \gamma\text{-repulsive} \quad \Longleftrightarrow \quad f \text{ is } \delta\text{-repulsive and } g \text{ is } \gamma\text{-repulsive.}$$

Proof. Note that $\gamma \geqslant \delta > 0$ gives $\gamma^\dagger \leqslant \delta^\dagger$. Suppose f is γ-repulsive; by our remark, f is δ-repulsive. Now if $v(\operatorname{Re} f) < \gamma^\dagger$, then $\operatorname{Re} g \sim \operatorname{Re} f$, whereas if $\operatorname{Re} f > 0$, then $\operatorname{Re}(g) = \operatorname{Re}(f) - \mathfrak{n}^\dagger > \operatorname{Re}(f) > 0$; in both cases, g is γ-repulsive. Conversely, suppose f is δ-repulsive and g is γ-repulsive. If $\operatorname{Re} f > 0$, then clearly f is γ-repulsive. Otherwise, $v(\operatorname{Re} f) < \delta^\dagger$, hence $\operatorname{Re} g \sim \operatorname{Re} f$, so f is also γ-repulsive. □

In a similar way we deduce a useful characterization of repulsiveness:

Lemma 4.5.4. *Suppose $\gamma = v\mathfrak{m} > 0$. Set $g := f - \mathfrak{m}^\dagger$. Then:*

$$f \text{ is repulsive} \quad \Longleftrightarrow \quad \operatorname{Re} f \succcurlyeq 1,\ f \text{ is } \gamma\text{-repulsive, and } g \text{ is repulsive.}$$

Proof. Suppose f is repulsive; then by Lemma 4.5.2, f is γ-repulsive. Moreover, $\operatorname{Re} g = \operatorname{Re}(f) - \mathfrak{m}^\dagger > \operatorname{Re} f > 0$, hence $\operatorname{Re} g \succcurlyeq 1$ and $\operatorname{Re} g > 0$, that is, g is repulsive. Conversely, suppose $\operatorname{Re} f \succcurlyeq 1$, f is γ-repulsive, and g is repulsive. If $v(\operatorname{Re} f) < \gamma^\dagger$, then $\operatorname{Re} f \sim \operatorname{Re} g$; otherwise $\operatorname{Re} f > 0$. In both cases, f is repulsive. □

Corollary 4.5.5. *Suppose f is γ-repulsive where $\gamma = v\mathfrak{m} > 0$, and $\operatorname{Re} f \succcurlyeq 1$. Then f is repulsive iff $f - \mathfrak{m}^\dagger$ is repulsive, and f is attractive iff $f - \mathfrak{m}^\dagger$ is attractive.*

Proof. The first equivalence is immediate from Lemma 4.5.4; this equivalence yields

$$\begin{aligned} f \text{ is attractive} \iff f \text{ is not repulsive} &\iff f - \mathfrak{m}^\dagger \text{ is not repulsive} \\ &\iff \operatorname{Re}(f) - \mathfrak{m}^\dagger \prec 1 \text{ or } f - \mathfrak{m}^\dagger \text{ is attractive.} \end{aligned}$$

Thus if $f - \mathfrak{m}^\dagger$ is attractive, so is f. Now assume towards a contradiction that f is attractive and $f - \mathfrak{m}^\dagger$ is not. Then $\operatorname{Re} f < 0$ and $\operatorname{Re}(f) - \mathfrak{m}^\dagger \prec 1$ by the above equivalence, so $\operatorname{Re} f \sim \mathfrak{m}^\dagger$ thanks to $\operatorname{Re} f \succcurlyeq 1$. But f is γ-repulsive, that is, $\operatorname{Re} f \succ \mathfrak{m}^\dagger$ or $\operatorname{Re} f > 0$, a contradiction. $\square$

Lemma 4.5.6. *Suppose $\gamma = v\mathfrak{m} > 0$ and $v(\operatorname{Re} g) \geqslant \gamma^\dagger$. Then for all sufficiently large $c \in C^>$ we have $\operatorname{Re}(g) - c\mathfrak{m}^\dagger > 0$ (and hence $g - c\mathfrak{m}^\dagger$ is Γ-repulsive).*

Proof. If $v(\operatorname{Re} g) > \gamma^\dagger$, then $\operatorname{Re}(g) - c\mathfrak{m}^\dagger \sim -c\mathfrak{m}^\dagger > 0$ for all $c \in C^>$. Suppose $v(\operatorname{Re} g) = \gamma^\dagger$. Take $c_0 \in C^\times$ with $\operatorname{Re} g \sim c_0\mathfrak{m}^\dagger$; then $\operatorname{Re}(g) - c\mathfrak{m}^\dagger > 0$ for $c > c_0$. $\square$

In the rest of this subsection we assume that $S \subseteq \Gamma$. If f is S-repulsive, then so is af for $a > 0$, $a \succcurlyeq 1$. If $S > 0$, $\delta > 0$, and f is S-repulsive and δ-repulsive, then f is $(S + \delta)$-repulsive.

Lemma 4.5.7. *Suppose f is S-repulsive and $0 < \delta = v\mathfrak{n} \in S$. Then*

(i) *f is $(S - \delta)$-repulsive;*
(ii) *$g := f - \mathfrak{n}^\dagger$ is S-repulsive.*

Proof. Let $\gamma \in (S - \delta)$, $\gamma > 0$. Then $\gamma + \delta \in S$, so f is $(\gamma + \delta)$-repulsive, hence γ-repulsive. This shows (i). For (ii), suppose $\gamma \in S$, $\gamma > 0$; we need to show that g is γ-repulsive. If $\gamma \geqslant \delta$, then g is γ-repulsive by Lemma 4.5.3. Taking $\gamma = \delta$ we see that g is δ-repulsive, hence if $\gamma < \delta$, then g is also γ-repulsive. $\square$

Let $A \in K[\partial]^{\neq}$ have order $r \geqslant 1$. An **S-repulsive splitting** of A over K is a splitting $(g_1, \dots, g_r)$ of A over K where $g_1, \dots, g_r$ are S-repulsive. An S-repulsive splitting of A over K remains an S-repulsive splitting of hA over K for $h \neq 0$. We say that A **splits S-repulsively** over K if there is an S-repulsive splitting of A over K. From Lemmas 1.1.1 and 4.5.7 we obtain:

Lemma 4.5.8. *Suppose $(g_1, \dots, g_r)$ is an S-repulsive splitting of A over K and $0 < \delta = v\mathfrak{n} \in S$. Then $(g_1, \dots, g_r)$ is an $(S - \delta)$-repulsive splitting of A over K, and $(h_1, \dots, h_r) := (g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is an S-repulsive splitting of $A\mathfrak{n}$ over K. (Hence $(h_1, \dots, h_r)$ is also an $(S - \delta)$-repulsive splitting of $A\mathfrak{n}$ over K.)*

Note that if ϕ is active in H with $0 < \phi \preccurlyeq 1$, and f is γ-repulsive (in K), then $\phi^{-1}f$ is γ-repulsive in $K^\phi = H^\phi[\mathrm{i}]$.

Lemma 4.5.9. *Suppose $(g_1, \dots, g_r)$ is an S-repulsive splitting of A over K and $S \cap \Gamma^> \not\subseteq \Gamma^\flat$. Let ϕ be active in H with $0 < \phi \prec 1$, and set $h_j := g_j - (r - j)\phi^\dagger$ for $j = 1, \dots, r$. Then $(\phi^{-1}h_1, \dots, \phi^{-1}h_r)$ is an S-repulsive splitting of A^ϕ over K^ϕ.*

Proof. By Lemma 1.1.2, $(\phi^{-1}h_1, \dots, \phi^{-1}h_r)$ is splitting of A^ϕ over K^ϕ. Let $j \in \{1, \dots, r\}$. If $\operatorname{Re} g_j > 0$, then $\phi^\dagger < 0$ yields $\operatorname{Re} h_j \geqslant \operatorname{Re} g_j > 0$. Otherwise, $v(\operatorname{Re} g_j) < \gamma^\dagger$ whenever $0 < \gamma \in S$; in particular, $\operatorname{Re} g_j \succ 1 \succ \phi^\dagger$, so $\operatorname{Re} h_j \sim \operatorname{Re} g_j$. In both cases h_j is S-repulsive, so $\phi^{-1}h_j$ is S-repulsive in K^ϕ. $\square$

Proposition 4.5.10. *Suppose $S \cap \Gamma^{>} \neq \emptyset$, $nS \subseteq S$ for all $n \geqslant 1$, the ordered constant field C of H is archimedean, and $(g_1, \dots, g_r)$ is a splitting of A over K. Then there exists $\gamma \in S \cap \Gamma^{>}$ such that for any $\mathfrak{m}$ with $\gamma = v\mathfrak{m}$: $(g_1 - n\mathfrak{m}^{\dagger}, \dots, g_r - n\mathfrak{m}^{\dagger})$ is an S-repulsive splitting of $A\mathfrak{m}^n$ over K, for all big enough n.*

Proof. Let J be the set of $j \in \{1, \dots, r\}$ such that g_j is not S-repulsive. If $\gamma > 0$ and g is not γ-repulsive, then g is not δ-repulsive, for all $\delta \geqslant \gamma$. Hence we can take $\gamma \in S \cap \Gamma^{>}$ such that g_j is not γ-repulsive, for all $j \in J$. Suppose $\gamma = v\mathfrak{m}$. Lemma 4.5.6 yields $m \geqslant 1$ such that for all $n \geqslant m$, setting $\mathfrak{n} := \mathfrak{m}^n$, $g_j - \mathfrak{n}^{\dagger}$ is Γ-repulsive for all $j \in J$. For such $\mathfrak{n}$ we have $v\mathfrak{n} \in S$, so by Lemma 4.5.7(ii), $g_j - \mathfrak{n}^{\dagger}$ is also S-repulsive for $j \notin J$. □

Corollary 4.5.11. *If C is archimedean and $(g_1, \dots, g_r)$ is a splitting of A over K, then there exists $\gamma > 0$ such that for all $\mathfrak{m}$ with $\gamma = v\mathfrak{m}$: $(g_1 - n\mathfrak{m}^{\dagger}, \dots, g_r - n\mathfrak{m}^{\dagger})$ is a Γ-repulsive splitting of $A\mathfrak{m}^n$ over K, for all big enough n. If $\Gamma \neq \Gamma^{\flat}$ then we can choose such $\gamma > \Gamma^{\flat}$.*

Proof. Taking $S = \Gamma$ this follows from Proposition 4.5.10 and its proof. □

In logical jargon, the condition that C is archimedean is not *first-order*. But it is satisfied when H is a Hardy field, the case where the results of this section will be applied. For other possible uses we indicate here a first-order variant of Proposition 4.5.10 with essentially the same proof:

Corollary 4.5.12. *Suppose $(g_1, \dots, g_r)$ is a splitting of A over K. Then there exists $\mathfrak{m} \prec 1$ such that for all sufficiently large $c \in C^{>}$ and all $\mathfrak{n}$, if $\mathfrak{n}^{\dagger} = c\mathfrak{m}^{\dagger}$, then $(g_1 - \mathfrak{n}^{\dagger}, \dots, g_r - \mathfrak{n}^{\dagger})$ is a Γ-repulsive splitting of $A\mathfrak{n}$ over K.*

In connection with this corollary we recall from [7, p. 105] that H is said to be *closed under powers* if for all $c \in C$ and $\mathfrak{m}$ there is an $\mathfrak{n}$ with $c\mathfrak{m}^{\dagger} = \mathfrak{n}^{\dagger}$.

In the rest of this section $\widehat{H}$ is an immediate asymptotic extension of H and i with $\mathrm{i}^2 = -1$ lies in an asymptotic extension of $\widehat{H}$. Also $K := H[\mathrm{i}]$ and $\widehat{K} := \widehat{H}[\mathrm{i}]$.

Let $\widehat{a} \in \widehat{H} \setminus H$, so $v(\widehat{a} - H)$ is a downward closed subset of Γ. We say that f is **$\widehat{a}$-repulsive** if f is $v(\widehat{a} - H)$-repulsive; that is, $\operatorname{Re} f > 0$, or $\operatorname{Re} f \succ \mathfrak{m}^{\dagger}$ for all a, $\mathfrak{m}$ with $\mathfrak{m} \asymp \widehat{a} - a \prec 1$. (Of course, this notion is only interesting if $v(\widehat{a} - H) \cap \Gamma^{>} \neq \emptyset$, since otherwise every f is $\widehat{a}$-repulsive.) Various earlier results give:

Lemma 4.5.13. *Suppose f is $\widehat{a}$-repulsive. Then*

(i) $b > 0,\ b \succcurlyeq 1 \implies bf$ *is $\widehat{a}$-repulsive;*
(ii) f *is $(\widehat{a} - a)$-repulsive;*
(iii) $\mathfrak{m} \asymp 1 \implies f$ *is $\widehat{a}\mathfrak{m}$-repulsive;*
(iv) $\mathfrak{n} \asymp \widehat{a} - a \prec 1 \implies f$ *is $\widehat{a}/\mathfrak{n}$ repulsive and $f - \mathfrak{n}^{\dagger}$ is $\widehat{a}$-repulsive.*

For (iv), use Lemma 4.5.7. An **$\widehat{a}$-repulsive splitting** of A over K is a $v(\widehat{a} - H)$-repulsive splitting $(g_1, \dots, g_r)$ of A over K:

$$A \ = \ f(\partial - g_1) \cdots (\partial - g_r) \qquad \text{where } f \neq 0 \text{ and } g_1, \dots, g_r \text{ are } \widehat{a}\text{-repulsive.}$$

We say that A **splits $\widehat{a}$-repulsively** over K if it splits $v(\widehat{a} - H)$-repulsively over K. Thus if A splits $\widehat{a}$-repulsively over K, then so does hA ($h \neq 0$), and A splits $(\widehat{a} - a)$-repulsively over K, and splits $\widehat{a}\mathfrak{m}$-repulsively over K for $\mathfrak{m} \asymp 1$. Moreover, from Lemma 4.5.8 we obtain:

Corollary 4.5.14. *Suppose $(g_1,\dots,g_r)$ is an $\widehat{a}$-repulsive splitting of A over K and $\mathfrak{n} \asymp \widehat{a}-a \prec 1$. Then $(g_1,\dots,g_r)$ is an $\widehat{a}/\mathfrak{n}$-repulsive splitting of A over K and $(g_1-\mathfrak{n}^\dagger,\dots,g_r-\mathfrak{n}^\dagger)$ is an $\widehat{a}$-repulsive splitting of $A\mathfrak{n}$ over K.*

Proposition 4.5.10 yields:

Corollary 4.5.15. *If $\widehat{a} \preccurlyeq 1$ is special over H, C is archimedean, and A splits over K, then $A\mathfrak{n}$ splits $\widehat{a}$-repulsively over K for some a and $\mathfrak{n} \asymp \widehat{a}-a \prec 1$.*

Recall that in Section 4.2 we defined a splitting $(g_1,\dots,g_r)$ of A over K to be *strong* if $\operatorname{Re} g_j \succcurlyeq \mathfrak{v}(A)^\dagger$ for $j=1,\dots,r$.

Lemma 4.5.16. *Suppose $\widehat{a}-a \prec^\flat 1$ for some a. Let $(g_1,\dots,g_r)$ be an $\widehat{a}$-repulsive splitting of A over K, let ϕ be active in H with $0<\phi\prec 1$, and set*

$$h_j \ := \ \phi^{-1}\big(g_j-(r-j)\phi^\dagger\big) \qquad (j=1,\dots,r).$$

Then $(h_1,\dots,h_r)$ is an $\widehat{a}$-repulsive splitting of A^ϕ over $K^\phi=H^\phi[\mathrm{i}]$. If $\mathfrak{v}(A)\prec^\flat 1$ and $(g_1,\dots,g_r)$ is strong, then $(h_1,\dots,h_r)$ is strong.

This follows from Lemmas 4.2.12 and 4.5.9.

Lemma 4.5.17. *Suppose $\mathfrak{v} := \mathfrak{v}(A) \prec 1$ and $\widehat{a} \prec_{\Delta(\mathfrak{v})} 1$. Let $(g_1,\dots,g_r)$ be an $\widehat{a}$-repulsive splitting of A over K. Then for all sufficiently small $q\in\mathbb{Q}^>$ and any $\mathfrak{n}\asymp|\mathfrak{v}|^q$, $(g_1-\mathfrak{n}^\dagger,\dots,g_r-\mathfrak{n}^\dagger)$ is a strong $\widehat{a}/\mathfrak{n}$-repulsive splitting of $A\mathfrak{n}$ over K.*

Proof. Take $q_0\in\mathbb{Q}^>$ with $\widehat{a}\prec|\mathfrak{v}|^{q_0}\prec 1$. Then for any $q\in\mathbb{Q}$ with $0<q\leqslant q_0$ and any $\mathfrak{n}\asymp|\mathfrak{v}|^q$, $(g_1-\mathfrak{n}^\dagger,\dots,g_r-\mathfrak{n}^\dagger)$ is an $\widehat{a}/\mathfrak{n}$-repulsive splitting of $A\mathfrak{n}$ over K, by Corollary 4.5.14. Using Lemmas 4.2.13 and 4.2.10 (in that order) we can decrease q_0 so that for all $q\in\mathbb{Q}$ with $0<q\leqslant q_0$ and $\mathfrak{n}\asymp|\mathfrak{v}|^q$, $(g_1-\mathfrak{n}^\dagger,\dots,g_r-\mathfrak{n}^\dagger)$ is also a strong splitting of $A\mathfrak{n}$ over K. ☐

In the rest of this subsection we assume that H is Liouville closed with $\mathrm{I}(K)\subseteq K^\dagger$. We choose a complement $\Lambda\subseteq H i$ of $K^\dagger$ in K as in Section 4.4 and set $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$. We then have the set $\mathscr{E}^{\mathrm{u}}(A)\subseteq\Gamma$ of ultimate exceptional values of A (which doesn't depend on Λ by Corollary 4.4.1). Recall from Corollary 1.2.31 that H is of Hardy type iff C is archimedean. *We now assume $r=1$ and $\widehat{a}\prec 1$ is special over H, and let Δ be the nontrivial convex subgroup of Γ that is cofinal in $v(\widehat{a}-H)$.*

Lemma 4.5.18. *Suppose C is archimedean and $\mathscr{E}^{\mathrm{u}}(A)\cap v(\widehat{a}-H)<0$. Then A splits $\widehat{a}$-repulsively over K.*

Proof. We may arrange $A=\partial-f$. Take $u\in\mathrm{U}^\times$ with $u^\dagger=f$, and set $b:=\|u\|\in H^>$. Then $\mathscr{E}^{\mathrm{u}}(A)=\{vb\}$ by Lemma 2.6.14 and its proof, hence

$$\mathscr{E}^{\mathrm{u}}(A)\cap v(\widehat{a}-H)<0 \quad\Longleftrightarrow\quad b\succ 1 \text{ or } vb>\Delta,$$

and $\operatorname{Re} f=b^\dagger$ by Lemma 4.4.21. If $b\succ 1$, then $\operatorname{Re} f>0$, and if $vb>\Delta$, then for all $\delta\in\Delta^{\neq}$ we have $\psi(vb)<\psi(\delta)$ by Lemma 1.2.27, so $\operatorname{Re} f\succ\mathfrak{m}^\dagger$ for all a, $\mathfrak{m}$ with $\widehat{a}-a\asymp\mathfrak{m}\prec 1$. In both cases A splits $\widehat{a}$-repulsively over K. ☐

Lemma 4.5.19. *Suppose $A\in H[\partial]$ and $\mathfrak{v}(A)\prec 1$. Then $0\notin\mathscr{E}^{\mathrm{u}}(A)$, and if A splits $\widehat{a}$-repulsively over K, then $\mathscr{E}^{\mathrm{u}}(A)\cap v(\widehat{a}-H)<0$.*

Proof. We again arrange $A=\partial-f$ and take u, b as in the proof of Lemma 4.5.18. Then $f\in H$ and $b^\dagger=f=-1/\mathfrak{v}(A)\succ 1$, so $b\not\asymp 1$, and thus $0\notin\{vb\}=\mathscr{E}^{\mathrm{u}}(A)$. Now suppose A splits $\widehat{a}$-repulsively over K, that is, $f>0$ or $f\succ\mathfrak{m}^\dagger$ for all a, $\mathfrak{m}$

with $\widehat{a} - a \asymp \mathfrak{m} \prec 1$. In the first case $f = b^\dagger$ and $b \not\asymp 1$ yield $b \succ 1$. In the second case $\psi(vb) = vf < \psi(\delta)$ for all $\delta \in \Delta^{\neq}$, hence $vb \notin \Delta$. $\square$

Combining Lemma 4.2.11 with the previous two lemmas yields:

Corollary 4.5.20. *Suppose $A \in H[\partial]$ and $\mathfrak{v}(A) \prec 1$, and H is of Hardy type. Then A splits strongly over K, and we have the equivalence*

$$A \text{ splits } \widehat{a}\text{-repulsively over } K \iff \mathscr{E}^{\mathrm{u}}(A) \cap v(\widehat{a} - H) \leqslant 0.$$

Defining repulsive-normality. *In this subsection $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$ with $\widehat{a} \in \widehat{H} \setminus H$ and linear part $L := L_{P_{\times \mathfrak{m}}}$.* Set $w := \mathrm{wt}(P)$; if order $L = r$, set $\mathfrak{v} := \mathfrak{v}(L)$. We let a, b range over H and $\mathfrak{n}$ over $H^\times$.

Definition 4.5.21. Call $(P, \mathfrak{m}, \widehat{a})$ **repulsive-normal** if order $L = r$, and

(RN1) $\mathfrak{v} \prec^\flat 1$;

(RN2) $(P_{\times \mathfrak{m}})_{\geqslant 1} = Q + R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits $\widehat{a}/\mathfrak{m}$-repulsively over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{m}})_1$.

Compare this with "split-normality" from Definition 4.3.3: clearly repulsive-normal implies split-normal, and hence normal. If $(P, \mathfrak{m}, \widehat{a})$ is normal and L splits $\widehat{a}/\mathfrak{m}$-repulsively over K, then $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal. If $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal, then so are $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ and $(P_{\times \mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$.

Lemma 4.5.22. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal and ϕ is active in H such that $0 < \phi \prec 1$, and $\widehat{a} - a \prec^\flat \mathfrak{m}$ for some a. Then the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ is repulsive-normal.*

Proof. First arrange $\mathfrak{m} = 1$, and let Q, R be as in (RN2) for $\mathfrak{m} = 1$. Now $(P^\phi, 1, \widehat{a})$ is split-normal by Lemma 4.3.5. In fact, $P^\phi_{\geqslant 1} = Q^\phi + R^\phi$, and the proof of this lemma shows that $R^\phi \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1} P^\phi_1$ where $\mathfrak{w} := \mathfrak{v}(L_{P^\phi})$. By Lemma 4.5.16, $L_{Q^\phi} = L^\phi_Q$ splits $\widehat{a}$-repulsively over K^ϕ. So $(P^\phi, 1, \widehat{a})$ is repulsive-normal. $\square$

If order $L = r$, $\mathfrak{v} \prec^\flat 1$, and $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$, then $\widehat{a} - a \prec^\flat \mathfrak{m}$. Thus we obtain from Lemmas 3.3.13 and 4.5.22 the following result:

Corollary 4.5.23. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and repulsive-normal. Let ϕ be active in H with $0 < \phi \prec 1$. Then the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ is repulsive-normal.*

Before we turn to the task of obtaining repulsive-normal slots, we deal with the preservation of repulsive-normality under refinements.

Lemma 4.5.24. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal, and let Q, R be as in* (RN2). *Let $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ be a steep refinement of $(P, \mathfrak{m}, \widehat{a})$ where $\mathfrak{n} \prec \mathfrak{m}$ or $\mathfrak{n} = \mathfrak{m}$. Suppose*

$$(P_{+a, \times \mathfrak{n}})_{\geqslant 1} - Q_{\times \mathfrak{n}/\mathfrak{m}} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1} (P_{+a, \times \mathfrak{n}})_1 \qquad \text{where } \mathfrak{w} := \mathfrak{v}(L_{P_{+a, \times \mathfrak{n}}}).$$

Then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is repulsive-normal.

Proof. By (RN2), L_Q splits $\widehat{a}/\mathfrak{m}$-repulsively over K, so L_Q also splits $(\widehat{a} - a)/\mathfrak{m}$-repulsively over K. We have $(\widehat{a} - a)/\mathfrak{m} \prec \mathfrak{n}/\mathfrak{m} \prec 1$ or $(\widehat{a} - a)/\mathfrak{m} \prec 1 = \mathfrak{n}/\mathfrak{m}$, so L_Q splits $(\widehat{a} - a)/\mathfrak{n}$-repulsively over K by the first part of Corollary 4.5.14, and hence $L_{Q_{\times \mathfrak{n}/\mathfrak{m}}} = L_Q \cdot (\mathfrak{n}/\mathfrak{m})$ splits $(\widehat{a} - a)/\mathfrak{n}$-repulsively over K by the second part of that Corollary 4.5.14. Thus $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is repulsive-normal. $\square$

The proofs of Lemmas 4.3.18, 4.3.19, 4.3.20 give the following repulsive-normal analogues of these lemmas, using also Lemma 4.5.24; for Lemma 4.5.27 below we adopt the notational conventions about $\mathfrak{n}^q$ ($q \in \mathbb{Q}^>$) stated before Lemma 4.3.20.

Lemma 4.5.25. *If $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal and $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$, then $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ is also repulsive-normal.*

Lemma 4.5.26. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal, $\widehat{a} \prec \mathfrak{n} \prec \mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$. Then the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal: if $\mathfrak{m}$, P, Q, $\mathfrak{v}$ are as in* (RN2)*, then* (RN2) *holds with $\mathfrak{n}$, $Q_{\times \mathfrak{n}/\mathfrak{m}}$, $R_{\times \mathfrak{n}/\mathfrak{m}}$, $\mathfrak{v}(L_{P_{\times \mathfrak{n}}})$ in place of $\mathfrak{m}$, Q, R, $\mathfrak{v}$.*

Lemma 4.5.27. *Suppose $\mathfrak{m} = 1$, $(P, 1, \widehat{a})$ is repulsive-normal, $\widehat{a} \prec \mathfrak{n} \prec 1$, and for $\mathfrak{v} := \mathfrak{v}(L_P)$ we have $[\mathfrak{n}^\dagger] < [\mathfrak{v}] < [\mathfrak{n}]$; then $(P, \mathfrak{n}^q, \widehat{a})$ is a repulsive-normal refinement of $(P, 1, \widehat{a})$ for all but finitely many $q \in \mathbb{Q}$ with $0 < q < 1$.*

Achieving repulsive-normality. In this subsection we adopt the setting of the subsection *Achieving split-normality* of Section 4.3: *H is $\upomega$-free and $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K of order $r \geqslant 1$, $\mathfrak{m} \in H^\times$, and $\widehat{a} \in \widehat{K} \setminus K$, with $\widehat{a} = \widehat{b} + \widehat{c}\mathrm{i}$, $\widehat{b}, \widehat{c} \in \widehat{H}$.* We let a range over K, b, c over H, and $\mathfrak{n}$ over $H^\times$. We prove here the following variant of Theorem 4.3.9:

Theorem 4.5.28. *Suppose the constant field C of H is archimedean and* $\deg P > 1$. *Then one of the following conditions is satisfied:*

- (i) *$\widehat{b} \notin H$ and some Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ such that $(Q^\phi_{+b}, \mathfrak{n}, \widehat{b}-b)$ is eventually deep and repulsive-normal;*
- (ii) *$\widehat{c} \notin H$ and some Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c}-c)$ such that $(R^\phi_{+c}, \mathfrak{n}, \widehat{c}-c)$ is eventually deep and repulsive-normal.*

To establish this theorem we need to take up the approximation arguments in the proof of Theorem 4.3.9 once again. While in that proof we treated the cases $\widehat{b} \in H$ and $\widehat{c} \in H$ separately to obtain stronger results in those cases (Lemmas 4.3.10, 4.3.11), here we proceed differently and first show a repulsive-normal version of Proposition 4.3.12 which also applies to those cases. *In the rest of this subsection we assume that C is archimedean.*

Proposition 4.5.29. *Suppose the hole $(P, \mathfrak{m}, \widehat{a})$ in K is special and $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ (so $\widehat{b} \notin H$). Let $(Q, \mathfrak{m}, \widehat{b})$ be a Z-minimal deep normal slot in H. Then $(Q, \mathfrak{m}, \widehat{b})$ has a repulsive-normal refinement.*

Proof. As in the proof of Proposition 4.3.12 we first arrange $\mathfrak{m} = 1$, and set

$$\Delta := \{\delta \in \Gamma : |\delta| \in v(\widehat{a} - K)\},$$

a convex subgroup of Γ which is cofinal in $v(\widehat{a} - K) = v(\widehat{b} - H)$, so $\widehat{b}$ is special over H. Lemma 3.3.13 applied to $(Q, 1, \widehat{b})$ and $\mathfrak{v}(L_Q) \prec^\flat 1$ gives that $\Gamma^\flat$ is strictly contained in Δ. To show that $(Q, 1, \widehat{b})$ has a repulsive-normal refinement, we follow the proof of Proposition 4.3.12, skipping the initial compositional conjugation, and arranging first that $P, Q \asymp 1$. Recall from that proof that $\dot{\widehat{a}} \in \dot{K}^{\mathrm{c}} = \dot{H}^{\mathrm{c}}[\mathrm{i}]$ and $\operatorname{Re} \dot{\widehat{a}} = \dot{\widehat{b}} \in \dot{H}^{\mathrm{c}} \setminus \dot{H}$, with $\dot{\widehat{b}} \prec 1$, $\dot{Q} \in \dot{H}\{Y\}$, and so $\dot{Q}_{+\dot{\widehat{b}}} \in \dot{H}^{\mathrm{c}}\{Y\}$. Let $A \in \dot{H}^{\mathrm{c}}[\partial]$ be the linear part of $\dot{Q}_{+\dot{\widehat{b}}}$. Recall from that proof that $1 \leqslant s := \operatorname{order} Q = \operatorname{order} A \leqslant 2r$ and that A splits over $\dot{K}^{\mathrm{c}}$. Then Lemma 1.1.4 gives a *real* splitting $(g_1, \dots, g_s)$ of A over $\dot{K}^{\mathrm{c}}$:

$$A \;=\; f(\partial - g_1)\cdots(\partial - g_s), \qquad 0 \neq f \in \dot{H}^{\mathrm{c}},\ g_1, \dots, g_s \in \dot{K}^{\mathrm{c}}.$$

It follows easily from [ADH, 10.1.8] that the real closed d-valued field $\dot{H}$ is an H-field, and so its completion $\dot{H}^{\mathrm{c}}$ is also a real closed H-field by [ADH, 10.5.9]. Recall also that $\Delta = v(\dot{H}^{\times})$ is the value group of $\dot{H}^{\mathrm{c}}$ and properly contains $\Gamma^{\flat}$. Thus we can apply Corollary 4.5.11 with $\dot{H}^{\mathrm{c}}$ in the role of H to get $\mathfrak{n} \in \dot{\mathcal{O}}$ with $0 \neq \dot{\mathfrak{n}} \prec^{\flat} 1$ and m such that for all $n > m$, $(h_1, \dots, h_s) := (g_1 - n\dot{\mathfrak{n}}^{\dagger}, \dots, g_s - n\dot{\mathfrak{n}}^{\dagger})$ is a Δ-repulsive splitting of $A\dot{\mathfrak{n}}^n$ over $\dot{K}^{\mathrm{c}}$, so $\operatorname{Re} h_1, \dots, \operatorname{Re} h_s \neq 0$. For any n, $A\dot{\mathfrak{n}}^n$ is the linear part of $\dot{Q}_{+\dot{\widehat{b}}, \times \dot{\mathfrak{n}}^n} \in \dot{H}^{\mathrm{c}}\{Y\}$, and $(h_1, \dots, h_s)$ is also a real splitting of $A\dot{\mathfrak{n}}^n$ over $\dot{K}^{\mathrm{c}}$:

$$A\dot{\mathfrak{n}}^n \;=\; \dot{\mathfrak{n}}^n f(\partial - h_1)\cdots(\partial - h_s).$$

By increasing m we arrange that for all $n > m$ we have $g_j \not\sim n\dot{\mathfrak{n}}^{\dagger}$ $(j = 1, \dots, s)$, and also $\mathfrak{v}(A\dot{\mathfrak{n}}^n) \preccurlyeq \mathfrak{v}(A)$ provided $\big[\mathfrak{v}(A)\big] < [\dot{\mathfrak{n}}]$; for the latter part use Lemma 3.1.16. Below we assume $n > m$. Then $\mathfrak{v}(A\dot{\mathfrak{n}}^n) \prec 1$: to see this use Corollary 3.1.4, $\mathfrak{v}(A) \prec 1$, and $g_j \preccurlyeq h_j$ $(j = 1, \dots, s)$. Note that $h_1, \dots, h_s \succcurlyeq 1$. We now apply Corollary 4.2.9 to $\dot{H}$, $\dot{K}$, $\dot{Q}$, s, $\dot{\mathfrak{n}}^n$, $\dot{\widehat{b}}$, $\dot{\mathfrak{n}}^n f$, $h_1, \dots, h_s$ in place of H, K, P, r, $\mathfrak{m}$, f, a, $b_1, \dots, b_r$, respectively, and any $\gamma \in \Delta$ with $\gamma > v(\dot{\mathfrak{n}}^n), v(\operatorname{Re} h_1), \dots, v(\operatorname{Re} h_s)$. This gives $a, b \in \dot{\mathcal{O}}$ and $b_1, \dots, b_s \in \dot{\mathcal{O}}_K$ such that $\dot{a}, \dot{b} \neq 0$ in $\dot{H}$ and such that for the linear part $\widetilde{A} \in \dot{H}[\partial]$ of $\dot{Q}_{+\dot{b}, \times \dot{\mathfrak{n}}^n}$ we have

$$\dot{b} - \dot{\widehat{b}} \;\prec\; \dot{\mathfrak{n}}^n, \qquad \widetilde{A} \;\sim\; A\dot{\mathfrak{n}}^n, \qquad \operatorname{order} \widetilde{A} \;=\; s, \qquad \mathfrak{w} \;:=\; \mathfrak{v}(\widetilde{A}) \;\sim\; \mathfrak{v}(A\dot{\mathfrak{n}}^n),$$

and such that for $w := \operatorname{wt}(Q)$ and with $\Delta(\mathfrak{w}) \subseteq \Delta$:

$$\begin{aligned} \widetilde{A} \;&=\; \widetilde{B} + \widetilde{E}, \quad \widetilde{B} \;=\; \dot{a}(\partial - \dot{b}_1)\cdots(\partial - \dot{b}_s) \in \dot{H}[\partial], \quad \widetilde{E} \in \dot{H}[\partial], \\ v(\dot{b}_1 - h_1), \dots, v(\dot{b}_s - h_s) \;&>\; \gamma, \quad \widetilde{E} \;\prec_{\Delta(\mathfrak{w})}\; \mathfrak{w}^{w+1}\widetilde{A}, \end{aligned}$$

and $(\dot{b}_1, \dots, \dot{b}_s)$ is a real splitting of $\widetilde{B}$ over $\dot{K}$. This real splitting over $\dot{K}$ has a consequence that will be crucial at the end of the proof: by changing $b_1, \dots, b_s$ if necessary, without changing $\dot{b}_1, \dots, \dot{b}_s$ we arrange that $B := a(\partial - b_1)\cdots(\partial - b_s)$ lies in $\dot{\mathcal{O}}[\partial] \subseteq H[\partial]$ and that $(b_1, \dots, b_s)$ is a real splitting of B over K. (Lemma 1.1.6.)

Since $\operatorname{Re} \dot{b}_1 \sim \operatorname{Re} h_1, \dots, \operatorname{Re} \dot{b}_s \sim \operatorname{Re} h_s$, the implication just before Lemma 4.5.2 gives that $(\dot{b}_1, \dots, \dot{b}_s)$ is a Δ-repulsive splitting of $\widetilde{B}$ over $\dot{K}$. Now $\widehat{b} - b \prec \mathfrak{n}^n \prec 1$, so $(Q_{+b}, 1, \widehat{b} - b)$ is a refinement of the normal slot $(Q, 1, \widehat{b})$ in H, hence $(Q_{+b}, 1, \widehat{b} - b)$ is normal by Proposition 3.3.25. We claim that the refinement $(Q_{+b}, \mathfrak{n}^n, \widehat{b} - b)$ of $(Q_{+b}, 1, \widehat{b} - b)$ is also normal. If $[\mathfrak{n}] \leqslant \big[\mathfrak{v}(L_{Q_{+b}})\big]$, this claim holds by Corollary 3.3.27. From Lemma 3.1.27 and 3.1.7 we obtain:

$$\begin{gathered} \operatorname{order} L_{Q_{+b}} \;=\; \operatorname{order} L_Q \;=\; \operatorname{order} L_{Q_{+\widehat{b}}} \;=\; s, \\ \mathfrak{v}(L_{Q_{+b}}) \;\sim\; \mathfrak{v}(L_Q) \;\sim\; \mathfrak{v}(L_{Q_{+\widehat{b}}}), \qquad v\big(\mathfrak{v}(L_{Q_{+\widehat{b}}})\big) \;=\; v\big(\mathfrak{v}(A)\big), \end{gathered}$$

so $v\big(\mathfrak{v}(L_{Q_{+b}})\big) = v\big(\mathfrak{v}(A)\big)$. Moreover, by Lemma 3.1.7 and the facts about $\widetilde{A}$,

$$v\big(\mathfrak{v}(L_{Q_{+b, \times \mathfrak{n}^n}})\big) \;=\; v\big(\mathfrak{v}(\widetilde{A})\big) \;=\; v\big(\mathfrak{v}(A\dot{\mathfrak{n}}^n)\big) \;=\; v(\mathfrak{w}).$$

Suppose $\big[\mathfrak{v}(L_{Q_{+b}})\big] < [\mathfrak{n}]$. Then $[\mathfrak{v}(A)] < [\dot{\mathfrak{n}}]$, so $\mathfrak{v}(A\dot{\mathfrak{n}}^n) \preccurlyeq \mathfrak{v}(A)$ using $n > m$. Now the asymptotic relations among the various $\mathfrak{v}(\dots)$ above give

$$\mathfrak{v}(L_{Q_{+b,\times\mathfrak{n}^n}}) \ \preccurlyeq\ \mathfrak{v}(L_{Q_{+b}}),$$

hence $(Q_{+b}, \mathfrak{n}^n, \widehat{b} - b)$ is normal by Corollary 3.3.29 applied to H and the normal slot $(Q_{+b}, 1, \widehat{b} - b)$ in H in the role of K and $(P, 1, \widehat{a})$, respectively. Put $\mathfrak{v} := \mathfrak{v}(L_{Q_{+b,\times\mathfrak{n}^n}})$, so $\mathfrak{v} \asymp \mathfrak{w}$. Note that $Q_{+b,\times\mathfrak{n}^n} \in \dot{\mathcal{O}}\{Y\}$, so the image of $L_{Q_{+b,\times\mathfrak{n}^n}} \in \dot{\mathcal{O}}[\partial]$ in $\dot{H}[\partial]$ is $\widetilde{A}$. Thus in $H[\partial]$ we have:

$$L_{Q_{+b,\times\mathfrak{n}^n}} \ =\ B + E \qquad \text{where } E \in \dot{\mathcal{O}}[\partial],\ \ E \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} L_{Q_{+b,\times\mathfrak{n}^n}}.$$

Now $\dot{b}_1, \dots, \dot{b}_s$ are Δ-repulsive, so $b_1, \dots, b_s$ are Δ-repulsive, hence

$$B \ =\ a(\partial - b_1)\cdots(\partial - b_s)$$

splits Δ-repulsively, and thus $(\widehat{b} - b)/\mathfrak{n}^n$-repulsively. Therefore $(Q_{+b}, \mathfrak{n}^n, \widehat{b} - b)$ is repulsive-normal. □

Instead of assuming in the above proposition that $(P, \mathfrak{m}, \widehat{a})$ is special and $(Q, \mathfrak{m}, \widehat{b})$ is deep and normal, we can assume, as with Corollary 4.3.13, that $\deg P > 1$:

Corollary 4.5.30. *Suppose* $\deg P > 1$ *and* $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. *Let* $Q \in Z(H, \widehat{b})$ *have minimal complexity. Then the Z-minimal slot* $(Q, \mathfrak{m}, \widehat{b})$ *in H has a special refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and repulsive-normal.*

Proof. The beginning of the subsection *Achieving split-normality* of Section 4.3 and $\deg P > 1$ give that K is r-linearly newtonian. Lemmas 3.2.26 and 3.3.23 yield a quasilinear refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of our hole $(P, \mathfrak{m}, \widehat{a})$ in K. Set $b := \operatorname{Re} a$. By Lemma 4.1.3 we have

$$v\big((\widehat{a} - a) - K\big) \ =\ v(\widehat{a} - K) \ =\ v\big(\widehat{b} - H\big) \ =\ v\big((\widehat{b} - b) - H\big).$$

Replacing $(P, \mathfrak{m}, \widehat{a})$ and $(Q, \mathfrak{m}, \widehat{b})$ by $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ and $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$, respectively, we arrange that $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Then by Proposition 1.6.12 and K being r-linearly newtonian, $(P, \mathfrak{m}, \widehat{a})$ is special; hence so is $(Q, \mathfrak{m}, \widehat{b})$. Proposition 3.3.36 gives a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$ and an active $\phi_0 \in H^{>}$ such that $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is deep and normal. Refinements of $(P, \mathfrak{m}, \widehat{a})$ remain quasilinear by Corollary 3.2.23. Since $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$, Lemma 4.1.3(ii) gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\operatorname{Re} a = b$. By Lemma 3.2.35 the minimal hole $(P^{\phi_0}_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K^{ϕ_0} is special. Proposition 4.5.29 applied to $(P^{\phi_0}_{+a}, \mathfrak{n}, \widehat{a} - a)$, $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ in place of $(P, \mathfrak{m}, \widehat{a})$, $(Q, \mathfrak{m}, \widehat{b})$, respectively, gives us $b_0 \in H$, $\mathfrak{n}_0 \in H^{\times}$ and a repulsive-normal refinement $\big(Q^{\phi_0}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$ of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$. This refinement is steep and hence deep by Corollary 3.3.6, since $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is deep. Thus by Corollary 4.5.23, $\big(Q_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$ is a refinement of $(Q, \mathfrak{m}, \widehat{b})$ such that that $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$ is eventually deep and repulsive-normal. As a refinement of $(Q, \mathfrak{m}, \widehat{b})$, it is special. □

In the same way that Corollary 4.3.13 gave rise to Corollary 4.3.14, Corollary 4.5.30 gives rise to the following:

Corollary 4.5.31. *If* $\deg P > 1$, $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$, *and* $R \in Z(H, \widehat{c})$ *has minimal complexity, then the* Z*-minimal slot* $(R, \mathfrak{m}, \widehat{c})$ *in* H *has a special refinement* $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ *such that* $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ *is eventually deep and repulsive-normal.*

By Lemma 4.1.3 we have $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$, hence the two corollaries above yield Theorem 4.5.28, completing its proof. $\square$

Strengthening repulsive-normality. In this subsection we adopt the setting of the subsection *Strengthening split-normality* of Section 4.3. Thus $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$ and weight $w := \operatorname{wt}(P)$, and $L := L_{P_{\times \mathfrak{m}}}$. If $\operatorname{order} L = r$, we set $\mathfrak{v} := \mathfrak{v}(L)$. We let a, b range over H and $\mathfrak{m}$, $\mathfrak{n}$ over $H^{\times}$.

Definition 4.5.32. We say that $(P, \mathfrak{m}, \widehat{a})$ is **almost strongly repulsive-normal** if $\operatorname{order} L = r$, $\mathfrak{v} \prec^{\flat} 1$, and there are $Q, R \in H\{Y\}$ such that

(RN2as) $(P_{\times \mathfrak{m}})_{\geqslant 1} = Q + R$, Q is homogeneous of degree 1 and order r, L_Q has a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{m}})_1$.

We say that $(P, \mathfrak{m}, \widehat{a})$ is **strongly repulsive-normal** if $\operatorname{order} L = r$, $\mathfrak{v} \prec^{\flat} 1$, and there are $Q, R \in H\{Y\}$ such that:

(RN2s) $P_{\times \mathfrak{m}} = Q + R$, Q is homogeneous of degree 1 and order r, L_Q has a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{m}})_1$.

If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal; likewise without "almost". Thus we can augment our diagram from Section 4.3 as follows, the implications holding for slots of order $\geqslant 1$ in real closed H-fields with small derivation and asymptotic integration:

$$
\begin{tikzcd}
\text{strongly repulsive-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{almost strongly repulsive-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{repulsive-normal} \arrow[d, Rightarrow] \\
\text{strongly split-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{almost strongly split-normal} \arrow[r, Rightarrow] & \text{split-normal} \arrow[d, Rightarrow] \\
\text{strictly normal} \arrow[rr, Rightarrow] & & \text{normal}
\end{tikzcd}
$$

Adapting the proof of Lemma 4.3.23 gives:

Lemma 4.5.33. *The following are equivalent:*

(i) $(P, \mathfrak{m}, \widehat{a})$ *is strongly repulsive-normal;*

(ii) $(P, \mathfrak{m}, \widehat{a})$ *is almost strongly repulsive-normal and strictly normal;*

(iii) $(P, \mathfrak{m}, \widehat{a})$ *is almost strongly repulsive-normal and* $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_1)_{\times \mathfrak{m}}$.

Corollary 4.5.34. *If* L *has a strong* $\widehat{a}/\mathfrak{m}$*-repulsive splitting over* K*, then:*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is almost strongly repulsive-normal} \iff (P, \mathfrak{m}, \widehat{a}) \text{ is normal},$$

$$(P, \mathfrak{m}, \widehat{a}) \text{ is strongly repulsive-normal} \iff (P, \mathfrak{m}, \widehat{a}) \text{ is strictly normal}.$$

If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so are $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ and $(P_{\times \mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$, and likewise with "strongly" in place of "almost strongly". The proof of the next lemma is like that of Lemma 4.3.25, using Lemmas 4.5.25 and 4.5.33 in place of Lemmas 4.3.18 and 4.3.23, respectively.

Lemma 4.5.35. *Suppose $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ refines $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so is $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$. If $(P, \mathfrak{m}, \widehat{a})$ is strongly repulsive-normal, Z-minimal, and $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$, then $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is strongly repulsive-normal.*

Here is the key to achieving almost strong repulsive-normality; its proof is similar to that of Lemma 4.3.26:

Lemma 4.5.36. *Suppose that $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal and $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. Then for all sufficiently small $q \in \mathbb{Q}^{>}$, any $\mathfrak{n} \asymp \mathfrak{v}^q \mathfrak{m}$ yields an almost strongly repulsive-normal refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$.*

Proof. First arrange $\mathfrak{m} = 1$. Take Q, R as in (RN2) for $\mathfrak{m} = 1$. Then Lemma 4.5.17 gives $q_0 \in \mathbb{Q}^{>}$ such that $\widehat{a} \prec \mathfrak{v}^{q_0}$ and for all $q \in \mathbb{Q}$ with $0 < q \leqslant q_0$ and $\mathfrak{n} \asymp \mathfrak{v}^q$, $L_{Q_{\times \mathfrak{n}}} = L_Q \mathfrak{n}$ has a strong $\widehat{a}/\mathfrak{n}$-repulsive splitting over K. Now Lemma 4.5.26 yields that $(P, \mathfrak{n}, \widehat{a})$ is almost strongly repulsive-normal for such $\mathfrak{n}$. □

Using this lemma we now adapt the proof of Corollary 4.3.27 to obtain:

Corollary 4.5.37. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and repulsive-normal. Then $(P, \mathfrak{m}, \widehat{a})$ has a deep and almost strongly repulsive-normal refinement.*

Proof. Lemma 3.3.13 gives a such that $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. By Corollary 3.3.8, the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is deep with $\mathfrak{v}(L_{P_{+a, \times \mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, and by Lemma 4.5.25 it is also repulsive-normal. Now apply Lemma 4.5.36 to $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ in place of $(P, \mathfrak{m}, \widehat{a})$ and again use Corollary 3.3.8 to preserve being deep. □

Next we adapt the proof of Lemma 4.3.28 to obtain a result about the behavior of (almost) repulsive-normality under compositional conjugation:

Lemma 4.5.38. *Suppose ϕ is active in H with $0 < \phi \prec 1$, and there exists a with $\widehat{a} - a \prec^{\flat} \mathfrak{m}$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so is the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ}. Likewise with "strongly" in place of "almost strongly".*

Proof. We arrange $\mathfrak{m} = 1$, assume $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, and take Q, R as in (RN2as). The proof of Lemma 4.3.5 shows that with $\mathfrak{w} := \mathfrak{v}(L_{P^{\phi}})$ we have $\mathfrak{w} \prec^{\flat}_{\phi} 1$ and $(P^{\phi})_{\geqslant 1} = Q^{\phi} + R^{\phi}$ where $Q^{\phi} \in H^{\phi}\{Y\}$ is homogeneous of degree 1 and order r, $L_{Q^{\phi}}$ splits over K^{ϕ}, and $R^{\phi} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P^{\phi})_1$. By Lemma 4.5.16, $L_{Q^{\phi}} = L_Q^{\phi}$ has even a strong $\widehat{a}$-repulsive splitting over K. Hence $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal. For the rest we use Lemma 4.5.33 and the fact that if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then so is $(P^{\phi}, \mathfrak{m}, \widehat{a})$. □

Lemma 3.3.13, the remark preceding Corollary 4.5.23, and Lemma 4.5.38 yield:

Corollary 4.5.39. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, and ϕ is active in H with $0 < \phi \prec 1$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so is the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ}. Likewise with "strongly" in place of "almost strongly".*

In the case $r = 1$, ultimateness yields almost strong repulsive-normality, under suitable assumptions; more precisely:

Lemma 4.5.40. *Suppose H is Liouville closed and of Hardy type, and $\mathrm{I}(K) \subseteq K^\dagger$. Assume also that $(P, \mathfrak{m}, \widehat{a})$ is normal and special, of order $r = 1$. Then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad \Longleftrightarrow \quad L \text{ has a strong } \widehat{a}/\mathfrak{m}\text{-repulsive splitting over } K,$$

in which case $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal.

Proof. By Lemma 4.4.12, $(P, \mathfrak{m}, \widehat{a})$ is ultimate iff $\mathscr{E}^{\mathrm{u}}(L) \cap v\big((\widehat{a}/\mathfrak{m}) - H\big) \leqslant 0$, and the latter is equivalent to L having a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, by Corollary 4.5.20. For the rest use Corollary 4.5.34. $\square$

Liouville closed H-fields are 1-linearly newtonian by Corollary 1.8.29, so in view of Lemma 3.2.36 and Corollary 3.3.21 we may replace the hypothesis "$(P, \mathfrak{m}, \widehat{a})$ is special" in the previous lemma by "$(P, \mathfrak{m}, \widehat{a})$ is Z-minimal or a hole in H". This leads to repulsive-normal analogues of Lemma 4.3.29 and Corollary 4.3.30 for $r = 1$:

Lemma 4.5.41. *Assume H is Liouville closed and of Hardy type, and $\mathrm{I}(K) \subseteq K^\dagger$. Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and quasilinear of order $r = 1$. Then there is a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep, strictly normal, and ultimate (so $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ is strongly repulsive-normal by Lemmas 4.5.40 and 4.5.33).*

Proof. For any active ϕ in H with $0 < \phi \preccurlyeq 1$ we may replace H, $(P, \mathfrak{m}, \widehat{a})$ by H^ϕ, $(P^\phi, \mathfrak{m}, \widehat{a})$. We may also replace $(P, \mathfrak{m}, \widehat{a})$ by any of its refinements. Since H is 1-linearly newtonian, Corollary 3.3.35 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H such that $0 < \phi \preccurlyeq 1$ and $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ is normal. Replacing H, $(P, \mathfrak{m}, \widehat{a})$ by H^ϕ, $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$, we arrange that $(P, \mathfrak{m}, \widehat{a})$ itself is normal. Then $(P, \mathfrak{m}, \widehat{a})$ has an ultimate refinement by Proposition 4.4.14, and applying Corollary 3.3.35 to this refinement and using Lemma 4.4.10, we obtain an ultimate refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the Z-minimal slot $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ in H^ϕ is deep, normal, and ultimate. Again replacing H, $(P, \mathfrak{m}, \widehat{a})$ by H^ϕ, $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$, we arrange that $(P, \mathfrak{m}, \widehat{a})$ is deep, normal, and ultimate. Corollary 3.3.47 yields a deep and strictly normal refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$; this refinement is still ultimate by Lemma 4.4.10. Hence $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$ as required, with $\phi = 1$. $\square$

Combining Lemmas 3.2.26 and 4.5.41 with Corollary 4.5.39 yields:

Corollary 4.5.42. *Assume H is Liouville closed, $\upomega$-free, and of Hardy type, and $\mathrm{I}(K) \subseteq K^\dagger$. Then every Z-minimal slot in H of order $r = 1$ has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is eventually deep, ultimate, and strongly repulsive-normal.*

In the next subsection we show how minimal holes of degree > 1 in K give rise to deep, ultimate, strongly repulsive-normal, Z-minimal slots in H.

Achieving strong repulsive-normality. Let H be an $\upomega$-free Liouville closed H-field with small derivation and constant field C, and $(P, \mathfrak{m}, \widehat{a})$ a minimal hole of order $r \geqslant 1$ in $K := H[\mathrm{i}]$. Other conventions are as in the subsection *Achieving repulsive-normality.* Our goal is to prove a version of Theorem 4.5.28 with "repulsive-normal" improved to "strongly repulsive-normal + ultimate":

Theorem 4.5.43. *Suppose C is archimedean, $\mathrm{I}(K) \subseteq K^\dagger$, and $\deg P > 1$. Then one of the following conditions is satisfied:*

(i) *$\widehat{b} \notin H$ and some Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^\phi_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep, strongly repulsive-normal, and ultimate;*

(ii) *$\widehat{c} \notin H$ and some Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^\phi_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep, strongly repulsive-normal, and ultimate.*

The proof of this theorem rests on the following two lemmas, where the standing assumption that H is Liouville closed can be dropped.

Lemma 4.5.44. *Suppose $\widehat{b} \notin H$ and $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H with a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^\phi_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep and repulsive-normal. Then $(Q, \mathfrak{m}, \widehat{b})$ has a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^\phi_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep and almost strongly repulsive-normal.*

Proof. We adapt the proof of Lemma 4.3.34. Let $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ be a refinement of $(Q, \mathfrak{m}, \widehat{b})$ and let ϕ_0 be active in H such that $0 < \phi_0 \preccurlyeq 1$ and $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is deep and repulsive-normal. Then Corollary 4.5.37 yields a refinement

$$\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b} - b) - b_0\big)$$

of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ which is deep and almost strongly repulsive-normal. Hence

$$\big((Q_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b} - b) - b_0\big) \ = \ \big(Q_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$$

is a refinement of $(Q, \mathfrak{m}, \widehat{b})$, and $\big(Q^\phi_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$ is eventually deep and almost strongly repulsive-normal by Corollary 4.5.39. $\square$

In the same way we obtain:

Lemma 4.5.45. *Suppose $\widehat{c} \notin H$ and $(R, \mathfrak{m}, \widehat{c})$ is a Z-minimal slot in H with a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^\phi_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and repulsive-normal. Then $(R, \mathfrak{m}, \widehat{c})$ has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^\phi_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and almost strongly repulsive-normal.*

Theorem 4.5.28 and the two lemmas above give Theorem 4.5.28 with "repulsive-normal" improved to "almost strongly repulsive-normal". We now upgrade this further to "strongly repulsive-normal + ultimate" (under an extra assumption).

Recall from Lemma 4.1.3 that $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$. Thus the next two lemmas finish the proof of Theorem 4.5.43.

Lemma 4.5.46. *Suppose C is archimedean, $\mathrm{I}(K) \subseteq K^\dagger$, $\deg P > 1$, and*

$$v(\widehat{b} - H) \ \subseteq \ v(\widehat{c} - H).$$

Let $Q \in Z(H, \widehat{b})$ have minimal complexity. Then the Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^\phi_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep, strongly repulsive-normal, and ultimate.

Proof. Here are two ways of modifying $(Q, \mathfrak{m}, \widehat{b})$. First, let $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ be a refinement of $(Q, \mathfrak{m}, \widehat{b})$. Lemma 4.1.3 gives $c \in H$ with $v(\widehat{a} - a) = v(\widehat{b} - b)$ with $a := b + ci$, and so the minimal hole $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K is a refinement of $(P, \mathfrak{m}, \widehat{a})$ that relates to $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ as $(P, \mathfrak{m}, \widehat{a})$ relates to $(Q, \mathfrak{m}, \widehat{b})$. So we can replace $(P, \mathfrak{m}, \widehat{a})$ and $(Q, \mathfrak{m}, \widehat{b})$ by $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ and $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$, whenever convenient. Second, let ϕ be active in H with $0 < \phi \preccurlyeq 1$. Then we can likewise replace H, K, $(P, \mathfrak{m}, \widehat{a})$, $(Q, \mathfrak{m}, \widehat{b})$ by H^ϕ, K^ϕ, $(P^\phi, \mathfrak{m}, \widehat{a})$, $(Q^\phi, \mathfrak{m}, \widehat{b})$.

In this way we first arrange as in the proof of Corollary 4.5.30 that $(Q, \mathfrak{m}, \widehat{b})$ is special. Next, we use Proposition 3.3.36 likewise to arrange that $(Q, \mathfrak{m}, \widehat{b})$ is also normal. By Propositions 4.4.14 (where the assumption $\mathrm{I}(K) \subseteq K^\dagger$ comes into play) and 3.3.25 we arrange that $(Q, \mathfrak{m}, \widehat{b})$ is ultimate as well. The properties "special" and "ultimate" persist under further refinements and compositional conjugations.

Now Corollary 4.5.30 and Lemma 4.5.44 give a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of the slot $(Q, \mathfrak{m}, \widehat{b})$ in H and an active ϕ_0 in H with $0 < \phi_0 \preccurlyeq 1$ such that the slot $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ in H^{ϕ_0} is deep and almost strongly repulsive-normal. Corollary 3.3.47 then yields a deep and strictly normal refinement

$$\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}, (\widehat{b} - b) - b_0\big)$$

of $\big(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b\big)$. This refinement is still almost strongly repulsive-normal by Lemma 4.5.35, and therefore strongly repulsive-normal by Lemma 4.5.33. Corollary 4.5.39 then gives that $\big(Q_{+(b+b_0)}, \mathfrak{n}, \widehat{b} - (b + b_0)\big)$ is a special refinement of our slot $(Q, \mathfrak{m}, \widehat{b})$ such that $\big(Q^\phi_{+(b+b_0)}, \mathfrak{n}, \widehat{b} - (b + b_0)\big)$ is eventually deep and strongly repulsive-normal. $\square$

Likewise:

Lemma 4.5.47. *Suppose C is archimedean,* $\mathrm{I}(K) \subseteq K^\dagger$, $\deg P > 1$, *and*

$$v(\widehat{c} - H) \ \subseteq\ v(\widehat{b} - H).$$

Let $R \in Z(H, \widehat{c})$ have minimal complexity. Then the Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^\phi_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep, strongly repulsive-normal, and ultimate.

Part 5. Hardy Fields and their Universal Exponential Extensions

In this part we turn to Hardy fields. Section 5.1 contains basic definitions and facts about germs of one-variable (real- or complex-valued) functions, and in Section 5.2 we collect the main facts we need about linear differential equations. In Section 5.3 we introduce Hardy fields and review some extension results due to Boshernitzan [32, 33, 34] and Rosenlicht [171]. In Section 5.4 we discuss upper and lower bounds on the growth of germs in Hardy fields from [34, 33, 170], and Section 5.5 contains a first study of second-order linear differential equations over Hardy fields (to be be completed in Section 7.5, with our main theorem available). Section 5.6 contains the proof of a significant result about maximal Hardy fields, Theorem 5.6.2: every such Hardy field is ω-free. (See the beginning of that section for a review of this important property of H-asymptotic fields, introduced in [ADH, 11.7].) The rest of Section 5.6 contains refinements and applications of this fact. In Section 5.7 we then prove a general fact about bounding the derivatives of solutions to linear differential equations, based on [67, 88, 121]. In Section 5.10 we give an analytic description of the universal exponential extension $\mathrm{U} = \mathrm{U}_K$, introduced in Part 2, of the algebraic closure K of a Liouville closed Hardy field extending $\mathbb{R}$. The elements of U are exponential sums with coefficients and exponents in K. To extract asymptotic information about the summands in such a sum we use results of Boshernitzan [36] about uniform distribution mod 1 over Hardy fields. We include proofs of these results in Section 5.9, preceded by a development of the required classical facts concerning almost periodic functions in Section 5.8. (None of the material in Sections 5.8 and 5.9 is original, we only aim for an efficient and self-contained exposition.)

5.1. Germs of Continuous Functions

Hardy fields consist of germs of one-variable differentiable real-valued functions. In this section we first consider the ring $\mathcal{C}$ of germs of *continuous* real-valued functions, and its complex counterpart $\mathcal{C}[i]$. With an eye towards applications to Hardy fields, we pay particular attention to extending subfields of $\mathcal{C}$.

Germs. As in [ADH, 9.1] we let $\mathcal{G}$ be the ring of germs at $+\infty$ of real-valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$; the domain may vary and the ring operations are defined as usual. If $g \in \mathcal{G}$ is the germ of a real-valued function on a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$, then we simplify notation by letting g also denote this function if the resulting ambiguity is harmless. With this convention, given a property P of real numbers and $g \in \mathcal{G}$ we say that $P\big(g(t)\big)$ *holds eventually* if $P\big(g(t)\big)$ holds for all sufficiently large real t. Thus for $g \in \mathcal{G}$ we have $g = 0$ iff $g(t) = 0$ eventually (and so $g \neq 0$ iff $g(t) \neq 0$ for arbitrarily large t). Note that the multiplicative group $\mathcal{G}^\times$ of units of $\mathcal{G}$ consists of the $f \in \mathcal{G}$ such that $f(t) \neq 0$, eventually. We identify each real number r with the germ at $+\infty$ of the function $\mathbb{R} \to \mathbb{R}$ that takes the constant value r. This makes the field $\mathbb{R}$ into a subring of $\mathcal{G}$. Given $g, h \in \mathcal{G}$, we set

$$g \leqslant h \quad :\Longleftrightarrow \quad g(t) \leqslant h(t), \text{ eventually.} \tag{5.1.1}$$

This defines a partial ordering $\leqslant$ on $\mathcal{G}$ which restricts to the usual ordering of $\mathbb{R}$.

Let $g, h \in \mathcal{G}$. Then $g, h \geqslant 0 \Rightarrow g + h,\ g \cdot h,\ g^2 \geqslant 0$, and $g \geqslant r \in \mathbb{R}^> \Rightarrow g \in \mathcal{G}^\times$. We define $g < h :\Leftrightarrow g \leqslant h$ and $g \neq h$. Thus if $g(t) < h(t)$, eventually, then $g < h$; the converse is not generally valid.

Continuous germs. We call a germ $g \in \mathcal{G}$ *continuous* if it is the germ of a continuous function $(a, +\infty) \to \mathbb{R}$ for some $a \in \mathbb{R}$, and we let $\mathcal{C} \supseteq \mathbb{R}$ be the subring of $\mathcal{G}$ consisting of the continuous germs $g \in \mathcal{G}$. We have $\mathcal{C}^\times = \mathcal{G}^\times \cap \mathcal{C}$; thus for $f \in \mathcal{C}^\times$, we have $f(t) \neq 0$, eventually, hence either $f(t) > 0$, eventually, or $f(t) < 0$, eventually, and so $f > 0$ or $f < 0$. More generally, if $g, h \in \mathcal{C}$ and $g(t) \neq h(t)$, eventually, then $g(t) < h(t)$, eventually, or $h(t) < g(t)$, eventually. We let x denote the germ at $+\infty$ of the identity function on $\mathbb{R}$, so $x \in \mathcal{C}^\times$.

The ring $\mathcal{C}[\mathrm{i}]$. In analogy with $\mathcal{C}$ we define its complexification $\mathcal{C}[\mathrm{i}]$ as the ring of germs at $+\infty$ of $\mathbb{C}$-valued continuous functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$. It has $\mathcal{C}$ as a subring. Identifying each complex number c with the germ at $+\infty$ of the function $\mathbb{R} \to \mathbb{C}$ that takes the constant value c makes $\mathbb{C}$ also a subring of $\mathcal{C}[\mathrm{i}]$ with $\mathcal{C}[\mathrm{i}] = \mathcal{C} + \mathcal{C}i$, justifying the notation $\mathcal{C}[\mathrm{i}]$. The "eventual" terminology for germs $f \in \mathcal{C}$ (like "$f(t) \neq 0$, eventually") is extended in the obvious way to germs $f \in \mathcal{C}[\mathrm{i}]$. Thus for $f \in \mathcal{C}[\mathrm{i}]$ we have: $f(t) \neq 0$, eventually, if and only if $f \in \mathcal{C}[\mathrm{i}]^\times$. In particular $\mathcal{C}^\times = \mathcal{C}[\mathrm{i}]^\times \cap \mathcal{C}$.

Let $\Phi \colon U \to \mathbb{C}$ be a continuous function where $U \subseteq \mathbb{C}$, and let $f \in \mathcal{C}[\mathrm{i}]$ be such that $f(t) \in U$, eventually; then $\Phi(f)$ denotes the germ in $\mathcal{C}[\mathrm{i}]$ with $\Phi(f)(t) = \Phi\big(f(t)\big)$, eventually. For example, taking $U = \mathbb{C}$, $\Phi(z) = \mathrm{e}^z$, we obtain for $f \in \mathcal{C}[\mathrm{i}]$ the germ $\exp f = \mathrm{e}^f \in \mathcal{C}[\mathrm{i}]$ with $(\mathrm{e}^f)(t) = \mathrm{e}^{f(t)}$, eventually. Likewise, for $f \in \mathcal{C}$ with $f(t) > 0$, eventually, we have the germ $\log f \in \mathcal{C}$. For $f \in \mathcal{C}[\mathrm{i}]$ we have $\overline{f} \in \mathcal{C}[\mathrm{i}]$ with $\overline{f}(t) = \overline{f(t)}$, eventually; the map $f \mapsto \overline{f}$ is an automorphism of the ring $\mathcal{C}[\mathrm{i}]$ with $\overline{\overline{f}} = f$ and $f \in \mathcal{C} \Leftrightarrow \overline{f} = f$. For $f \in \mathcal{C}[\mathrm{i}]$ we also have $\operatorname{Re} f, \operatorname{Im} f, |f| \in \mathcal{C}$ with $f(t) = (\operatorname{Re} f)(t) + (\operatorname{Im} f)(t)\mathrm{i}$ and $|f|(t) = |f(t)|$, eventually.

Asymptotic relations on $\mathcal{C}[\mathrm{i}]$. Although $\mathcal{C}[\mathrm{i}]$ is not a valued field, it will be convenient to equip $\mathcal{C}[\mathrm{i}]$ with the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ (which are defined on any valued field [ADH, 3.1]) as follows: for $f, g \in \mathcal{C}[\mathrm{i}]$,

$$\begin{aligned}
f \preccurlyeq g \quad &:\Longleftrightarrow \quad \text{there exists } c \in \mathbb{R}^> \text{ such that } |f| \leqslant c|g|, \\
f \prec g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[\mathrm{i}]^\times \text{ and } \lim_{t\to\infty} f(t)/g(t) = 0 \\
&\ \Longleftrightarrow \quad g \in \mathcal{C}[\mathrm{i}]^\times \text{ and } |f| \leqslant c|g| \text{ for all } c \in \mathbb{R}^>, \\
f \sim g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[\mathrm{i}]^\times \text{ and } \lim_{t\to\infty} f(t)/g(t) = 1 \\
&\ \Longleftrightarrow \quad f - g \prec g.
\end{aligned}$$

We also use these notations for continuous functions $[a, +\infty) \to \mathbb{C}$, $a \in \mathbb{R}$; for example, for continuous $f \colon [a, +\infty) \to \mathbb{C}$ and $g \colon [b, +\infty) \to \mathbb{C}$ $(a, b \in \mathbb{R})$, $f \preccurlyeq g$ means: (germ of f) $\preccurlyeq$ (germ of g). If $h \in \mathcal{C}[\mathrm{i}]$ and $1 \preccurlyeq h$, then $h \in \mathcal{C}[\mathrm{i}]^\times$. Also, for $f, g \in \mathcal{C}[\mathrm{i}]$ and $h \in \mathcal{C}[\mathrm{i}]^\times$ we have

$$f \preccurlyeq g \ \Leftrightarrow\ fh \preccurlyeq gh, \qquad f \prec g \ \Leftrightarrow\ fh \prec gh, \qquad f \sim g \ \Leftrightarrow\ fh \sim gh.$$

The binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ is reflexive and transitive, and $\sim$ is an equivalence relation on $\mathcal{C}[\mathrm{i}]^\times$. Moreover, for $f, g, h \in \mathcal{C}[\mathrm{i}]$ we have

$$f \prec g \ \Rightarrow\ f \preccurlyeq g, \qquad f \preccurlyeq g \prec h \ \Rightarrow\ f \prec h, \qquad f \prec g \preccurlyeq h \ \Rightarrow\ f \prec h.$$

Note that $\prec$ is a transitive binary relation on $\mathcal{C}[i]$. For $f, g \in \mathcal{C}[i]$ we also set

$$f \asymp g :\Leftrightarrow\ f \preccurlyeq g\ \&\ g \preccurlyeq f, \qquad f \succcurlyeq g :\Leftrightarrow\ g \preccurlyeq f, \qquad f \succ g :\Leftrightarrow\ g \prec f,$$

so $\asymp$ is an equivalence relation on $\mathcal{C}[i]$, and $f \sim g \Rightarrow f \asymp g$. Thus for $f, g, h \in \mathcal{C}[i]$,

$$f \preccurlyeq g\ \Rightarrow\ fh \preccurlyeq gh, \quad f \preccurlyeq h\ \&\ g \preccurlyeq h\ \Rightarrow\ f + g \preccurlyeq h, \quad f \preccurlyeq 1\ \&\ g \prec 1\ \Rightarrow\ fg \prec 1,$$

hence

$$\mathcal{C}[i]^{\preccurlyeq}\ :=\ \big\{f \in \mathcal{C}[i] : f \preccurlyeq 1\big\}\ =\ \big\{f \in \mathcal{C}[i] : |f| \leqslant n \text{ for some } n\big\}$$

is a subalgebra of the $\mathbb{C}$-algebra $\mathcal{C}[i]$ and

$$\mathcal{C}[i]^{\prec}\ :=\ \big\{f \in \mathcal{C}[i] : f \prec 1\big\}\ =\ \Big\{f \in \mathcal{C}[i] : \lim_{t\to\infty} f(t) = 0\Big\}$$

is an ideal of $\mathcal{C}[i]^{\preccurlyeq}$. The group of units of $\mathcal{C}[i]^{\preccurlyeq}$ is

$$\mathcal{C}[i]^{\asymp}\ :=\ \big\{f \in \mathcal{C}[i] : f \asymp 1\big\}\ =\ \big\{f \in \mathcal{C}[i] : 1/n \leqslant |f| \leqslant n \text{ for some } n \geqslant 1\big\}$$

and has the subgroup

$$\mathbb{C}^{\times}\big(1 + \mathcal{C}[i]^{\prec}\big)\ =\ \Big\{f \in \mathcal{C}[i] : \lim_{t\to\infty} f(t) \in \mathbb{C}^{\times}\Big\}.$$

We set $\mathcal{C}^{\preccurlyeq} := \mathcal{C}[i]^{\preccurlyeq} \cap \mathcal{C}$, and similarly with $\prec$, $\asymp$ in place of $\preccurlyeq$.

Lemma 5.1.1. *Let $f, g, f^*, g^* \in \mathcal{C}[i]^{\times}$ with $f \sim f^*$ and $g \sim g^*$. Then $1/f \sim 1/f^*$ and $fg \sim f^*g^*$. Moreover, $f \preccurlyeq g \Leftrightarrow f^* \preccurlyeq g^*$, and similarly with $\prec$, $\asymp$, or $\sim$ in place of $\preccurlyeq$.*

This follows easily from the observations above. For later reference we also note:

Lemma 5.1.2. *Let $f, g \in \mathcal{C}^{\times}$ be such that $1 \prec f \preccurlyeq g$; then $\log|f| \preccurlyeq \log|g|$.*

Proof. Clearly $\log|g| \succ 1$. Take $c \in \mathbb{R}^{>}$ such that $|f| \leqslant c|g|$. Then $\log|f| \leqslant \log c + \log|g|$ where $\log c + \log|g| \sim \log|g|$; hence $\log|f| \preccurlyeq \log|g|$. □

Lemma 5.1.3. *Let $f, g, h \in \mathcal{C}^{\times}$ be such that $f - g \prec h$ and $(f-h)(g-h) = 0$. Then $f \sim g$.*

Proof. Take $a \in \mathbb{R}$ and representatives $(a, +\infty) \to \mathbb{R}$ of f, g, h, denoted by the same symbols, such that for each $t > a$ we have $f(t), g(t), h(t) \neq 0$, and $f(t) = h(t)$ or $g(t) = h(t)$. Let $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leqslant 1$ be given, and choose $b \geqslant a$ such that $|f(t) - g(t)| \leqslant \frac{1}{2}\varepsilon|h(t)|$ for all $t > b$. Set $q := f/g$ and let $t > b$; we claim that then $|q(t) - 1| \leqslant \varepsilon$. This is clear if $g(t) = h(t)$, so suppose otherwise; then $f(t) = h(t)$, and $|1 - 1/q(t)| \leqslant \frac{1}{2}\varepsilon \leqslant \frac{1}{2}$. In particular, $0 < q(t) \leqslant 2$ and so $|1 - q(t)| = |1 - 1/q(t)| \cdot q(t) \leqslant \varepsilon$ as claimed. □

Subfields of $\mathcal{C}$. Let H be a *Hausdorff field*, that is, a subring of $\mathcal{C}$ that happens to be a field; see [12]. Then H has the subfield $H \cap \mathbb{R}$. If $f \in H^{\times}$, then $f(t) \neq 0$ eventually, hence either $f(t) < 0$ eventually or $f(t) > 0$ eventually. The partial ordering of $\mathcal{G}$ from (5.1.1) thus restricts to a total ordering on H making H an ordered field in the usual sense of that term. By [32, Propositions 3.4 and 3.6]:

Proposition 5.1.4. *Let H^{rc} consist of the $y \in \mathcal{C}$ with $P(y) = 0$ for some $P(Y)$ in $H[Y]^{\neq}$. Then H^{rc} is the unique real closed Hausdorff field that extends H and is algebraic over H. In particular, H^{rc} is a real closure of the ordered field H.*

Boshernitzan [32] assumes $H \supseteq \mathbb{R}$ for this result, but this is not really needed in the proof, much of which already occurs in Hausdorff [94]. For the reader's convenience we include a proof of Proposition 5.1.4, after some lemmas. Let

$$P(Y) \;=\; P_0 Y^n + P_1 Y^{n-1} + \cdots + P_n \in H[Y] \qquad (P_0, \dots, P_n \in H),$$

and take $a \in \mathbb{R}$ such that $P_0, \dots, P_n$ have representatives in $\mathcal{C}_a$, also denoted by $P_0, \dots, P_n$. This yields for $t \geqslant a$ the polynomial

$$P(t, Y) \;:=\; P_0(t) Y^n + P_1(t) Y^{n-1} + \cdots + P_n(t) \in \mathbb{R}[Y].$$

For any other choice of a and representatives of $P_0, \dots, P_n$ in $\mathcal{C}_a$ this gives for large enough t the same polynomial $P(t, Y) \in \mathbb{R}[Y]$, so the "eventual" terminology makes sense for properties mentioning $P(t, Y)$ with t ranging over $\mathbb{R}$. For example, for $y \in \mathcal{C}[\mathrm{i}]$, we have: $P(y) = 0 \Leftrightarrow y(t) \in \mathbb{C}$ is a zero of $P(t, Y)$, eventually.

Lemma 5.1.5. *Suppose P is irreducible (in $H[Y]$) of degree n, so $n \geqslant 1$. Then there are $y_1, \dots, y_m \in \mathcal{C}$ such that $y_1(t) < \cdots < y_m(t)$, eventually, and the distinct real zeros of the polynomial $P(t, Y) \in \mathbb{R}[Y]$ are exactly $y_1(t), \dots, y_m(t)$, eventually. Thus $P(y_1) = \cdots = P(y_m) = 0$, and if n is odd, then $m \geqslant 1$.*

Proof. Take $A, B \in H[Y]$ with $1 = AP + BP'$. Then

$$1 \;=\; A(t, Y) P(t, Y) + B(t, Y) P(t, Y)', \quad \text{ eventually.}$$

Hence $P(t, Y) \in \mathbb{R}[Y]$ has exactly n distinct complex zeros, eventually. Now use "continuity of roots" as used for example in [58, Chapter II, (2.4)]. ☐

Lemma 5.1.6. *Let $P, Q \in H[Y]$ be monic and irreducible with $P \neq Q$, and let $y, z \in \mathcal{C}[\mathrm{i}]$, $P(y) = Q(z) = 0$. Then $y(t) \neq z(t)$, eventually. In particular, if $y, z \in \mathcal{C}$, then either $y(t) < z(t)$ eventually, or $y(t) > z(t)$ eventually.*

Proof. Take $A, B \in H[Y]$ such that $1 = AP + BQ$. Then

$$1 \;=\; A(t, Y) P(t, Y) + B(t, Y) Q(t, Y), \quad \text{ eventually.}$$

Hence $Q(t, y(t)) \neq 0$, eventually, and thus $y(t) \neq z(t)$, eventually. ☐

Corollary 5.1.7. *Let $y \in \mathcal{C}$ and $P(y) = 0$, $P \in H[Y]^{\neq}$. Then $Q(y) = 0$ for some monic irreducible $Q \in H[Y]$.*

Proof. We have $P = h Q_1^{e_1} \cdots Q_n^{e_n}$ where $h \in H^{\neq}$, $e_1, \dots, e_n \in \mathbb{N}^{\geqslant 1}$, and $Q_1, \dots, Q_n$ in $H[Y]$ are distinct, and monic irreducible. Lemmas 5.1.5 and 5.1.6 now yield germs $y_1, \dots, y_m \in \mathcal{C}$ such that, eventually, $y_1(t) < \cdots < y_m(t)$ are the real zeros of the polynomials $Q_1(t, Y), \dots, Q_n(t, Y) \in \mathbb{R}[Y]$, and thus of $P(t, Y)$, and such that for each $i \in \{1, \dots, m\}$ there is a unique $j \in \{1, \dots, n\}$ with $Q_j\big(t, y_i(t)\big) = 0$, eventually. Continuity arguments and the connectedness of halflines $[a, +\infty)$ yields a single i with $y_i(t) = y(t)$, eventually, and thus $Q_j(y) = 0$ for some j. ☐

Proof of Proposition 5.1.4. Given $y \in H^{\mathrm{rc}}$ we have by Corollary 5.1.7 a monic irreducible $Q \in H[Y]$ with $Q(y) = 0$, so $H[y] \subseteq H^{\mathrm{rc}}$ is a Hausdorff field and algebraic over H. Since "algebraic over" is transitive, it follows that H^{rc} is a Hausdorff field and algebraic over H. Such transitivity also gives $(H^{\mathrm{rc}})^{\mathrm{rc}} = H^{\mathrm{rc}}$. Obviously, any algebraic Hausdorff field extension of H is contained in H^{rc}. So it only remains to show that the ordered field H^{rc} is real closed. First, if $y \in H^{\mathrm{rc}}$ and $y \geqslant 0$, then clearly $\sqrt{y} \in \mathcal{C}$ is algebraic over H^{rc}, and thus in it. Next, let $P(Y) \in H^{\mathrm{rc}}[Y]$ have odd degree. Then P has a zero in H^{rc}: this follows from Lemma 5.1.5 by considering an irreducible factor of P in $H^{\mathrm{rc}}[Y]$ of odd degree. ☐

We record the following useful consequence of Corollary 5.1.7 and its proof:

Corollary 5.1.8. *Let $P \in H[Y]^{\neq}$ and let $y_1, \dots, y_m$ be the distinct zeros of P in H^{rc}. Then $y_1(t), \dots y_m(t)$ are the distinct real zeros of $P(t, Y)$, eventually.*

Note that $H[\mathrm{i}]$ is a subfield of $\mathcal{C}[\mathrm{i}]$, and by Proposition 5.1.4 and [ADH, 3.5.4], the subfield $H^{\mathrm{rc}}[\mathrm{i}]$ of $\mathcal{C}[\mathrm{i}]$ is an algebraic closure of the field H. If $f \in \mathcal{C}[\mathrm{i}]$ is integral over H, then so is $\overline{f}$, and hence so are the elements $\operatorname{Re} f = \frac{1}{2}(f + \overline{f})$ and $\operatorname{Im} f = \frac{1}{2\mathrm{i}}(f - \overline{f})$ of $\mathcal{C}$ [ADH, 1.3.2]. Thus $H^{\mathrm{rc}}[\mathrm{i}]$ consists of the $y \in \mathcal{C}[\mathrm{i}]$ with $P(y) = 0$ for some $P(Y) \in H[Y]^{\neq}$.

The ordered field H has a convex subring

$$\mathcal{O} \ = \ \big\{f \in H : \ |f| \leqslant n \text{ for some } n\big\} \ = \ \mathcal{C}^{\preccurlyeq} \cap H,$$

which is a valuation ring of H, and we consider H accordingly as a valued ordered field. The maximal ideal of $\mathcal{O}$ is $\smallo = \mathcal{C}^{\prec} \cap H$. The residue morphism $\mathcal{O} \to \operatorname{res}(H)$ restricts to an ordered field embedding $H \cap \mathbb{R} \to \operatorname{res}(H)$, which is bijective if $\mathbb{R} \subseteq H$. Restricting the binary relations $\preccurlyeq$, $\prec$, $\sim$ from the previous subsection to H gives exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on H that it comes equipped with as a valued field. By [ADH, 3.5.15],

$$\mathcal{O} + \mathcal{O}\mathrm{i} \ = \ \big\{f \in H[\mathrm{i}] : |f| \leqslant n \text{ for some } n\big\} \ = \ \mathcal{C}[\mathrm{i}]^{\preccurlyeq} \cap H[\mathrm{i}]$$

is the unique valuation ring of $H[\mathrm{i}]$ whose intersection with H is $\mathcal{O}$. In this way we consider $H[\mathrm{i}]$ as a valued field extension of H. The maximal ideal of $\mathcal{O} + \mathcal{O}\mathrm{i}$ is $\smallo + \smallo\mathrm{i} = \mathcal{C}[\mathrm{i}]^{\prec} \cap H[\mathrm{i}]$. The asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $\mathcal{C}[\mathrm{i}]$ restricted to $H[\mathrm{i}]$ are exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $H[\mathrm{i}]$ that $H[\mathrm{i}]$ has as a valued field. Moreover, $f \asymp |f|$ in $\mathcal{C}[\mathrm{i}]$ for all $f \in H[\mathrm{i}]$.

Composition. Let $g \in \mathcal{C}$, and suppose that $\lim\limits_{t \to +\infty} g(t) = +\infty$; equivalently, $g \geqslant 0$ and $g \succ 1$. Then the composition operation

$$f \mapsto f \circ g \ : \ \mathcal{C}[\mathrm{i}] \to \mathcal{C}[\mathrm{i}], \qquad (f \circ g)(t) \ := \ f\big(g(t)\big) \ \text{ eventually},$$

is an injective endomorphism of the ring $\mathcal{C}[\mathrm{i}]$ that is the identity on the subring $\mathbb{C}$. For $f_1, f_2 \in \mathcal{C}[\mathrm{i}]$ we have: $f_1 \preccurlyeq f_2 \Leftrightarrow f_1 \circ g \preccurlyeq f_2 \circ g$, and likewise with $\prec$, $\sim$. This endomorphism of $\mathcal{C}[\mathrm{i}]$ commutes with the automorphism $f \mapsto \overline{f}$ of $\mathcal{C}[\mathrm{i}]$, and maps each subfield K of $\mathcal{C}[\mathrm{i}]$ isomorphically onto the subfield $K \circ g = \{f \circ g : f \in K\}$ of $\mathcal{C}[\mathrm{i}]$. Note that if the subfield K of $\mathcal{C}[\mathrm{i}]$ contains x, then $K \circ g$ contains g. Moreover, $f \mapsto f \circ g$ restricts to an endomorphism of the subring $\mathcal{C}$ of $\mathcal{C}[\mathrm{i}]$ such that if $f_1, f_2 \in \mathcal{C}$ and $f_1 \leqslant f_2$, then $f_1 \circ g \leqslant f_2 \circ g$. This endomorphism of $\mathcal{C}$ maps each Hausdorff field H isomorphically (as an ordered field) onto the Hausdorff field $H \circ g$.

Occasionally it is convenient to extend the composition operation on $\mathcal{C}$ to the ring $\mathcal{G}$ of all (not necessarily continuous) germs. Let $g \in \mathcal{G}$ with $\lim\limits_{t \to +\infty} g(t) = +\infty$. Then for $f \in \mathcal{G}$ we have the germ $f \circ g \in \mathcal{G}$ with

$$(f \circ g)(t) \ := \ f\big(g(t)\big) \ \text{ eventually}.$$

The map $f \mapsto f \circ g$ is an endomorphism of the $\mathbb{R}$-algebra $\mathcal{G}$. Let $f_1, f_2 \in \mathcal{G}$. Then $f_1 \leqslant f_2 \Rightarrow f_1 \circ g \leqslant f_2 \circ g$, and likewise with $\preccurlyeq$ and $\prec$ instead of $\leqslant$, where we extend the binary relations $\preccurlyeq$, $\prec$ from $\mathcal{C}$ to $\mathcal{G}$ in the natural way:

$$f_1 \preccurlyeq f_2 \quad :\Longleftrightarrow \quad \text{there exists } c \in \mathbb{R}^{>} \text{ such that } |f_1(t)| \leqslant c|f_2(t)|, \text{ eventually;}$$
$$f_1 \prec f_2 \quad :\Longleftrightarrow \quad f_2 \in \mathcal{G}^{\times} \text{ and } \lim_{t\to\infty} f_1(t)/f_2(t) = 0.$$

Compositional inversion. Suppose that $g \in \mathcal{C}$ is eventually strictly increasing such that $\lim_{t\to+\infty} g(t) = +\infty$. Then its compositional inverse $g^{\text{inv}} \in \mathcal{C}$ is given by $g^{\text{inv}}\big(g(t)\big) = t$, eventually, and g^{inv} is also eventually strictly increasing with $\lim_{t\to+\infty} g^{\text{inv}}(t) = +\infty$. Then $f \mapsto f \circ g$ is an automorphism of the ring $\mathcal{C}[i]$, with inverse $f \mapsto f \circ g^{\text{inv}}$. In particular, $g \circ g^{\text{inv}} = g^{\text{inv}} \circ g = x$. Moreover, $f \mapsto f \circ g$ restricts to an automorphism of $\mathcal{C}$, and if $h \in \mathcal{C}$ is eventually strictly increasing with $g \leqslant h$, then $h^{\text{inv}} \leqslant g^{\text{inv}}$.

Let now $f, g \in \mathcal{C}$ with $f, g \geqslant 0$, $f, g \succ 1$. It is not true in general that if f, g are eventually strictly increasing and $f \sim g$, then $f^{\text{inv}} \sim g^{\text{inv}}$. (Counterexample: $f = \log x$, $g = \log 2x$.) Corollary 5.1.10 below gives a useful condition on f, g under which this implication does hold. In addition, let $h \in \mathcal{C}^{\times}$ be eventually monotone and continuously differentiable with $h'/h \preccurlyeq 1/x$.

Lemma 5.1.9 (Entringer [65])**.** *Suppose $f \sim g$. Then $h \circ f \sim h \circ g$.*

Proof. Replacing h by $-h$ if necessary we arrange that $h \geqslant 0$, so $h(t) > 0$ eventually. Set $p := \min(f, g) \in \mathcal{C}$ and $q := \max(f, g) \in \mathcal{C}$. Then $0 \leqslant p \succ 1$ and $f - g \prec p$. The Mean Value Theorem gives $\xi \in \mathcal{G}$ such that $p \leqslant \xi \leqslant q$ (so $0 \leqslant \xi \succ 1$) and

$$h \circ f - h \circ g \ = \ (h' \circ \xi) \cdot (f - g).$$

From $h'/h \preccurlyeq 1/x$ we obtain $h' \circ \xi \preccurlyeq (h \circ \xi)/\xi \preccurlyeq (h \circ \xi)/p$, hence $h \circ f - h \circ g \prec h \circ \xi$. Set $u := \max(h \circ p, h \circ q)$. Then $0 \leqslant h \circ \xi \leqslant u$, hence $h \circ f - h \circ g \prec u$. Also $(u - h \circ f)(u - h \circ g) = 0$, so Lemma 5.1.3 yields $h \circ f \sim h \circ g$. □

Corollary 5.1.10. *Suppose f, g are eventually strictly increasing with $f \sim g$ and $f^{\text{inv}} \sim h$. Then $g^{\text{inv}} \sim h$.*

Proof. By the lemma above we have $h \circ f \sim h \circ g$, and from $f^{\text{inv}} \sim h$ we obtain $x = f^{\text{inv}} \circ f \sim h \circ f$. Therefore $g^{\text{inv}} \circ g = x \sim h \circ g$ and thus $g^{\text{inv}} \sim h$. □

Corollary 5.1.11. *If g, h are eventually strictly increasing, $0 \leqslant h \succ 1$, and $g \sim h^{\text{inv}}$, then $g^{\text{inv}} \sim h$.*

Proof. Take $f = h^{\text{inv}}$ in Corollary 5.1.10. □

Sometimes we prefer "big O" and "little o" notation instead of $\preccurlyeq$ and $\prec$: for $\phi, \xi, \theta \in \mathcal{C}[i]$, $\phi = \xi + O(\theta) :\Leftrightarrow \phi - \xi \preccurlyeq \theta$ and $\phi = \xi + o(\theta) :\Leftrightarrow \phi - \xi \prec \theta$. For use in Section 7.5 we note:

Corollary 5.1.12. *Suppose $g = x + cx^{-1} + o(x^{-1})$, $c \in \mathbb{R}$, and g is eventually strictly increasing. Then $g^{\text{inv}} = x - cx^{-1} + o(x^{-1})$.*

Proof. We have $g^{\text{inv}} \sim x$ by Corollary 5.1.11 (for $h = x$), so $g^{\text{inv}} = x(1 + \varepsilon)$ where $\varepsilon \in \mathcal{C}$, $\varepsilon \prec 1$. Now $(1 + \varepsilon)^{-1} = 1 + \delta$ with $\delta \in \mathcal{C}$, $\delta \prec 1$. Then

$$x \ = \ g \circ g^{\text{inv}} \ = \ x(1 + \varepsilon) + cx^{-1}(1 + \delta) + o(x^{-1})$$

and thus $\varepsilon = -cx^{-2}(1+\delta) + o(x^{-2}) = -cx^{-2} + o(x^{-2})$. This yields $g^{\text{inv}} = x(1+\varepsilon) = x - cx^{-1} + o(x^{-1})$, as claimed. □

Extending ordered fields inside an ambient partially ordered ring. Let R be a commutative ring with $1 \neq 0$, equipped with a translation-invariant partial ordering $\leqslant$ such that $r^2 \geqslant 0$ for all $r \in R$, and $rs \geqslant 0$ for all $r, s \in R$ with $r, s \geqslant 0$. It follows that for $a, b, r \in R$ we have:

(1) if $a \leqslant b$ and $r \geqslant 0$, then $ar \leqslant br$;
(2) if a is a unit and $a > 0$, then $a^{-1} = a \cdot (a^{-1})^2 > 0$;
(3) if a, b are units and $0 < a \leqslant b$, then $0 < b^{-1} \leqslant a^{-1}$.

Relevant cases: $R = \mathcal{G}$ and $R = \mathcal{C}$, with partial ordering given by (5.1.1).

An *ordered subring of* R is a subring of R that is totally ordered by the partial ordering of R. An *ordered subfield of* R is an ordered subring H of R which happens to be a field; then H equipped with the induced ordering is indeed an ordered field, in the usual sense of that term. (Thus any Hausdorff field is an ordered subfield of the partially ordered ring $\mathcal{C}$.) We identify $\mathbb{Z}$ with its image in R via the unique ring embedding $\mathbb{Z} \to R$, and this makes $\mathbb{Z}$ with its usual ordering into an ordered subring of R.

Lemma 5.1.13. *Assume D is an ordered subring of R and every nonzero element of D is a unit of R. Then D generates an ordered subfield* $\operatorname{Frac} D$ *of R.*

Proof. It is clear that D generates a subfield $\operatorname{Frac} D$ of R. For $a \in D$, $a > 0$, we have $a^{-1} > 0$. It follows that $\operatorname{Frac} D$ is totally ordered. □

Thus if every $n \geqslant 1$ is a unit of R, then we may identify $\mathbb{Q}$ with its image in R via the unique ring embedding $\mathbb{Q} \to R$, making $\mathbb{Q}$ into an ordered subfield of R.

Lemma 5.1.14. *Suppose H is an ordered subfield of R, all $g \in R$ with $g > H$ are units of R, and $H < f \in R$. Then we have an ordered subfield $H(f)$ of R.*

Proof. For $P(Y) \in H[Y]$ of degree $d \geqslant 1$ with leading coefficient $a > 0$ we have $P(f) = af^d(1+\varepsilon)$ with $-1/n < \varepsilon < 1/n$ for all $n \geqslant 1$, in particular, $P(f) > H$ is a unit of R. It remains to appeal to Lemma 5.1.13. □

Lemma 5.1.15. *Let H be a real closed ordered subfield of R. Let A be a nonempty downward closed subset of H such that A has no largest element and $B := H \setminus A$ is nonempty and has no least element. Let $f \in R$ be such that $A < f < B$. Then the subring $H[f]$ of R has the following properties:*

(i) *$H[f]$ is a domain;*
(ii) *$H[f]$ is an ordered subring of R;*
(iii) *H is cofinal in $H[f]$;*
(iv) *for all $g \in H[f] \setminus H$ and $a \in H$, if $a < g$, then $a < b < g$ for some $b \in H$, and if $g < a$, then $g < b < a$ for some $b \in H$.*

Proof. Let $P \in H[Y]^{\neq}$; to obtain (i) and (ii) it suffices to show that then $P(f) < 0$ or $P(f) > 0$. We have

$$P(Y) \ = \ c\,Q(Y)\,(Y - a_1) \cdots (Y - a_n)$$

where $c \in H^{\neq}$, $Q(Y)$ is a product of monic quadratic irreducibles in $H[Y]$, and $a_1, \dots, a_n \in H$. This gives $\delta \in H^{>}$ such that $Q(r) \geqslant \delta$ for all $r \in R$. Assume $c > 0$. (The case $c < 0$ is handled similarly.) We can arrange that $m \leqslant n$ is such that $a_i \in A$ for $1 \leqslant i \leqslant m$ and $a_j \in B$ for $m < j \leqslant n$. Take $\varepsilon > 0$ in H such that $a_i + \varepsilon \leqslant f$ for $1 \leqslant i \leqslant m$ and $f \leqslant a_j - \varepsilon$ for $m < j \leqslant n$. Then

$$P(f) \ = \ c\,Q(f)\,(f - a_1) \cdots (f - a_m)(f - a_{m+1}) \cdots (f - a_n),$$

and $(f-a_1)\cdots(f-a_m) \geqslant \varepsilon^m$. If $n-m$ is even, then $(f-a_{m+1})\cdots(f-a_n) \geqslant \varepsilon^{n-m}$, so $P(f) \geqslant c\delta\varepsilon^n > 0$. If $n-m$ is odd, then $(f-a_{m+1})\cdots(f-a_n) \leqslant -\varepsilon^{n-m}$, so $P(f) \leqslant -c\delta\varepsilon^n < 0$. These estimates also yield (iii) and (iv). $\square$

Lemma 5.1.16. *With H, A, f as in Lemma 5.1.15, suppose all $g \in R$ with $g \geqslant 1$ are units of R. Then we have an ordered subfield $H(f)$ of R such that* (iii) *and* (iv) *of Lemma 5.1.15 go through for $H(f)$ in place of $H[f]$.*

Proof. Note that if $g \in R$ and $g \geqslant \delta \in H^>$, then $g\delta^{-1} \geqslant 1$, so g is a unit of R and $0 < g^{-1} \leqslant \delta^{-1}$. For $Q \in H[Y]^{\neq}$ with $Q(f) > 0$ we can take $\delta \in H^>$ such that $Q(f) \geqslant \delta$, so $Q(f) \in R^\times$ and $0 < Q(f)^{-1} \leqslant \delta^{-1}$. Thus we have an ordered subfield $H(f)$ of R by Lemma 5.1.13, and the rest now follows easily. $\square$

Adjoining pseudolimits and increasing the value group. Let H be a real closed Hausdorff field and view H as an ordered valued field as before. Let (a_ρ) be a strictly increasing divergent pc-sequence in H. Set

$$A \ := \ \{a \in H :\ a < a_\rho \text{ for some } \rho\}, \qquad B \ := \ \{b \in H :\ b > a_\rho \text{ for all } \rho\},$$

so A is nonempty and downward closed without a largest element. Moreover, $B = H \setminus A$ is nonempty and has no least element, since a least element of B would be a limit and thus a pseudolimit of (a_ρ). Let $f \in \mathcal{C}$ satisfy $A < f < B$. Then by Lemma 5.1.16 for $R = \mathcal{C}$ we have an ordered subfield $H(f)$ of $\mathcal{C}$, and:

Lemma 5.1.17. *$H(f)$ is an immediate valued field extension of H with $a_\rho \rightsquigarrow f$.*

Proof. We can assume that $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$ for all indices $\tau > \sigma > \rho$. Set $d_\rho := a_{s(\rho)} - a_\rho$ ($s(\rho) :=$ successor of ρ). Then $a_\rho + 2d_\rho \in B$ for all indices ρ; see the discussion preceding [ADH, 2.4.2]. It then follows from that lemma that $a_\rho \rightsquigarrow f$. Now (a_ρ) is a divergent pc-sequence in the henselian valued field H, so it is of transcendental type over H, and thus $H(f)$ is an immediate extension of H. $\square$

Lemma 5.1.18. *Let H be a Hausdorff field with divisible value group $\Gamma := v(H^\times)$. Let P be a nonempty upward closed subset of Γ, and let $f \in \mathcal{C}$ be such that $a < f$ for all $a \in H^>$ with $va \in P$, and $f < b$ for all $b \in H^>$ with $vb < P$. Then f generates a Hausdorff field $H(f)$ with $P > vf > Q$, $Q := \Gamma \setminus P$.*

Proof. For any positive $a \in H^{\mathrm{rc}}$ there is $b \in H^>$ with $a \asymp b$ and $a < b$, and also an element $b \in H^>$ with $a \asymp b$ and $a > b$. Thus by Proposition 5.1.4 we can replace H by H^{rc} and arrange in this way that H is real closed. Set

$$A \ := \ \{a \in H :\ a \leqslant 0 \text{ or } va \in P\}, \qquad B := H \setminus A.$$

Then we are in the situation of Lemma 5.1.15 for $R = \mathcal{C}$, so by that lemma and Lemma 5.1.16 we have a Hausdorff field $H(f)$. Clearly then $P > vf > Q$. $\square$

Non-oscillation. A germ $f \in \mathcal{C}$ is said to **oscillate** if $f(t) = 0$ for arbitrarily large t and $f(t) \neq 0$ for arbitrarily large t. Thus for $f, g \in \mathcal{C}$,

$$f - g \text{ is non-oscillating} \quad \Longleftrightarrow \quad \begin{cases} \text{either } f(t) < g(t) \text{ eventually, or } f = g, \\ \text{or } f(t) > g(t) \text{ eventually.} \end{cases}$$

In particular, $f \in \mathcal{C}$ does not oscillate iff $f = 0$ or $f \in \mathcal{C}^\times$. If $g \in \mathcal{C}$ and $g(t) \to +\infty$ as $t \to +\infty$, then $f \in \mathcal{C}$ oscillates iff $f \circ g$ oscillates.

Lemma 5.1.19. *Let $f \in \mathcal{C}$ be such that for every $q \in \mathbb{Q}$ the germ $f - q$ is non-oscillating. Then $\lim\limits_{t\to\infty} f(t)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

Proof. Set $A := \{q \in \mathbb{Q} : f(t) > q \text{ eventually}\}$. If $A = \emptyset$, then $\lim_{t\to\infty} f(t) = -\infty$, whereas if $A = \mathbb{Q}$, then $\lim_{t\to\infty} f(t) = +\infty$. If $A \neq \emptyset, \mathbb{Q}$, then for $\ell := \sup A \in \mathbb{R}$ we have $\lim_{t\to\infty} f(t) = \ell$. $\square$

Lemma 5.1.20. *Let H be a real closed Hausdorff field and $f \in \mathcal{C}$. Then f lies in a Hausdorff field extension of H iff $f - h$ is non-oscillating for all $h \in H$.*

Proof. The forward direction is clear. For the converse, suppose $f - h$ is non-oscillating for all $h \in H$. We assume $f \notin H$, so $h < f$ or $h > f$ for all $h \in H$. Set $A := \{h \in H : h < f\}$, a downward closed subset of H. If $A = H$, then we are done by Lemma 5.1.14 applied to $R = \mathcal{C}$; if $A = \emptyset$ then we apply the same lemma to $R = \mathcal{C}$ and $-f$ in place of f. Suppose $A \neq \emptyset, H$. If A has a largest element a, then we replace f by $f - a$ to arrange $0 < f(t) < h(t)$ eventually, for all $h \in H^{>}$, and then Lemma 5.1.14 applied to $R = \mathcal{C}$, f^{-1} in place of f yields that f^{-1}, and hence also f, lies in a Hausdorff field extension of H. The case that $B := H \setminus A$ has a least element is handled in the same way. If A has no largest element and B has no least element, then we are done by Lemma 5.1.16. $\square$

5.2. Linear Differential Equations

In this section we fix notations and conventions concerning differentiable functions and summarize well-known results on linear differential equations as needed later, focusing on the case of order 2. We also discuss disconjugate linear differential equations, mainly following [52, Chapter 3], as well as work by Lyapunov and Perron on "bounded" matrix differential equations; this material is only used in Section 7.4 on applications and can be skipped upon first reading.

Differentiable functions. Let r range over $\mathbb{N} \cup \{\infty\}$, and let U be a nonempty open subset of $\mathbb{R}$. Then $\mathcal{C}^r(U)$ denotes the $\mathbb{R}$-algebra of r-times continuously differentiable functions $U \to \mathbb{R}$, with the usual pointwise defined algebra operations. (We use "$\mathcal{C}$" instead of "C" since C will often denote the constant field of a differential field.) For $r = 0$ this is the $\mathbb{R}$-algebra $\mathcal{C}(U)$ of continuous real-valued functions on U, so

$$\mathcal{C}(U) \;=\; \mathcal{C}^0(U) \;\supseteq\; \mathcal{C}^1(U) \;\supseteq\; \mathcal{C}^2(U) \;\supseteq\; \cdots \;\supseteq\; \mathcal{C}^\infty(U).$$

For $r \geqslant 1$ we have the derivation $f \mapsto f' \colon \mathcal{C}^r(U) \to \mathcal{C}^{r-1}(U)$ (with $\infty - 1 := \infty$). This makes $\mathcal{C}^\infty(U)$ a differential ring, with its subalgebra $\mathcal{C}^\omega(U)$ of real-analytic functions $U \to \mathbb{R}$ as a differential subring. The algebra operations on the algebras below are also defined pointwise. Note that

$$\mathcal{C}^r(U)^\times \;=\; \big\{f \in \mathcal{C}^r(U) : f(t) \neq 0 \text{ for all } t \in U\big\},$$

also for ω in place of r [57, (9.2), ex. 4].

Let a range over $\mathbb{R}$. Then $\mathcal{C}^r_a$ denotes the $\mathbb{R}$-algebra of functions $[a, +\infty) \to \mathbb{R}$ that extend to a function in $\mathcal{C}^r(U)$ for some open $U \supseteq [a, +\infty)$. Thus $\mathcal{C}^0_a$ (also denoted by $\mathcal{C}_a$) is the $\mathbb{R}$-algebra of real-valued continuous functions on $[a, +\infty)$, and

$$\mathcal{C}^0_a \;\supseteq\; \mathcal{C}^1_a \;\supseteq\; \mathcal{C}^2_a \;\supseteq\; \cdots \;\supseteq \mathcal{C}^\infty_a.$$

We have the subalgebra $\mathcal{C}^\omega_a$ of $\mathcal{C}^\infty_a$, consisting of the functions $[a, +\infty) \to \mathbb{R}$ that extend to a real-analytic function $U \to \mathbb{R}$ for some open $U \supseteq [a, +\infty)$. For $f \in \mathcal{C}^1_a$ and $g \in \mathcal{C}^1(U)$ extending f with open $U \subseteq \mathbb{R}$ containing $[a, +\infty)$, the restriction of g' to $[a,+\infty) \to \mathbb{R}$ depends only on f, not on g, so we may define $f' := g'|_{[a,+\infty)} \in \mathcal{C}_a$. For $r \geqslant 1$ this gives the derivation $f \mapsto f' \colon \mathcal{C}_a^r \to \mathcal{C}_a^{r-1}$. This makes $\mathcal{C}_a^\infty$ a differential ring with $\mathcal{C}_a^\omega$ as a differential subring.

For each of the algebras A above we also consider its complexification $A[\mathrm{i}]$ which consists by definition of the $\mathbb{C}$-valued functions $f = g+h\mathrm{i}$ with $g,h \in A$, so $g = \operatorname{Re} f$ and $h = \operatorname{Im} f$ for such f. We consider $A[\mathrm{i}]$ as a $\mathbb{C}$-algebra with respect to the natural pointwise defined algebra operations. We identify each complex number with the corresponding constant function to make $\mathbb{C}$ a subfield of $A[\mathrm{i}]$ and $\mathbb{R}$ a subfield of A. (This justifies the notation $A[\mathrm{i}]$.) We have $\mathcal{C}_a^r[\mathrm{i}]^\times = \mathcal{C}_a[\mathrm{i}]^\times \cap \mathcal{C}_a^r[\mathrm{i}]$ and $(\mathcal{C}_a^r)^\times = \mathcal{C}_a^\times \cap \mathcal{C}_a^r$, and likewise with r replaced by ω.

For $r \geqslant 1$ we extend $g \mapsto g' \colon \mathcal{C}_a^r \to \mathcal{C}_a^{r-1}$ to the derivation

$$g + h\mathrm{i} \mapsto g' + h'\mathrm{i} \;:\; \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a^{r-1}[\mathrm{i}] \qquad (g,h \in \mathcal{C}_a^r[\mathrm{i}]),$$

which for $r = \infty$ makes $\mathcal{C}_a^\infty$ a differential subring of $\mathcal{C}_a^\infty[\mathrm{i}]$. We shall use the map

$$f \mapsto f^\dagger := f'/f \;:\; \mathcal{C}_a^1[\mathrm{i}]^\times = \big(\mathcal{C}_a^1[\mathrm{i}]\big)^\times \to \mathcal{C}_a^0[\mathrm{i}],$$

with

$$(fg)^\dagger = f^\dagger + g^\dagger \qquad \text{for } f,g \in \mathcal{C}_a^1[\mathrm{i}]^\times,$$

in particular the fact that $f \in \mathcal{C}_a^1[\mathrm{i}]^\times$ and $f^\dagger \in \mathcal{C}_a^0[\mathrm{i}]$ are related by

$$f(t) \;=\; f(a)\exp\left[\int_a^t f^\dagger(s)\,ds\right] \qquad (t \geqslant a).$$

For $g \in \mathcal{C}_a^0[\mathrm{i}]$, let $\exp\int g$ denote the function $t \mapsto \exp\left[\int_a^t g(s)\,ds\right]$ in $\mathcal{C}_a^1[\mathrm{i}]^\times$. Then

$$(\exp\textstyle\int g)^\dagger = g \quad \text{ and } \quad \exp\int(g+h) = (\exp\int g)\cdot(\exp\int h) \qquad \text{for } g,h \in \mathcal{C}_a^0[\mathrm{i}].$$

Therefore $f \mapsto f^\dagger \colon \mathcal{C}_a^1[\mathrm{i}]^\times \to \mathcal{C}_a^0[\mathrm{i}]$ is surjective.

Notation. For $b \geqslant a$ and $f \in \mathcal{C}_a[\mathrm{i}]$ we set $f|_b := f|_{[b,+\infty)} \in \mathcal{C}_a[\mathrm{i}]$.

Differentiable germs. Let $r \in \mathbb{N} \cup \{\infty\}$ and let a range over $\mathbb{R}$. Let $\mathcal{C}^r$ be the partially ordered subring of $\mathcal{C}$ consisting of the germs at $+\infty$ of the functions in $\bigcup_a \mathcal{C}_a^r$; thus $\mathcal{C}^0 = \mathcal{C}$ consists of the germs at $+\infty$ of the continuous real valued functions on intervals $[a,+\infty)$, $a \in \mathbb{R}$. Note that $\mathcal{C}^r$ with its partial ordering satisfies the conditions on R from Section 5.1. Also, every $g \geqslant 1$ in $\mathcal{C}^r$ is a unit of $\mathcal{C}^r$, so Lemmas 5.1.14 and 5.1.16 apply to ordered subfields of $\mathcal{C}^r$. We have

$$\mathcal{C}^0 \;\supseteq\; \mathcal{C}^1 \;\supseteq\; \mathcal{C}^2 \;\supseteq\; \cdots \;\supseteq\; \mathcal{C}^\infty.$$

Each subring $\mathcal{C}^r$ of $\mathcal{C}$ yields the subring $\mathcal{C}^r[\mathrm{i}] = \mathcal{C}^r + \mathcal{C}^r\mathrm{i}$ of $\mathcal{C}^0[\mathrm{i}] = \mathcal{C}[\mathrm{i}]$, with

$$\mathcal{C}^0[\mathrm{i}] \;\supseteq\; \mathcal{C}^1[\mathrm{i}] \;\supseteq\; \mathcal{C}^2[\mathrm{i}] \;\supseteq\; \cdots \;\supseteq\; \mathcal{C}^\infty[\mathrm{i}].$$

Suppose $r \geqslant 1$; then for $f \in \mathcal{C}_a^r[\mathrm{i}]$ the germ of $f' \in \mathcal{C}_a^{r-1}[\mathrm{i}]$ only depends on the germ of f, and we thus obtain a derivation $g \mapsto g' \colon \mathcal{C}^r[\mathrm{i}] \to \mathcal{C}^{r-1}[\mathrm{i}]$ with (germ of $f)' =$ (germ of f') for $f \in \bigcup_a \mathcal{C}_a^r[\mathrm{i}]$. This derivation restricts to a derivation $\mathcal{C}^r \to \mathcal{C}^{r-1}$. Note that $\mathcal{C}[\mathrm{i}]^\times \cap \mathcal{C}^r[\mathrm{i}] = \mathcal{C}^r[\mathrm{i}]^\times$, and hence $\mathcal{C}^\times \cap \mathcal{C}^r = (\mathcal{C}^r)^\times$.

For open $U \subseteq \mathbb{C}$ and $\Phi \colon U \to \mathbb{C}$ of class $\mathcal{C}^r$ (that is, its real and imaginary parts are of class $\mathcal{C}^r$), if $f \in \mathcal{C}^r[\mathrm{i}]$ and $f(t) \in U$, eventually, then $\Phi(f) \in \mathcal{C}^r[\mathrm{i}]$. For example, if $f \in \mathcal{C}^r$, then $\exp f \in \mathcal{C}^r$, and if in addition $f(t) > 0$, eventually, then $\log f \in \mathcal{C}^r$.

We set

$$\mathcal{C}^{<\infty}[i] \ := \ \bigcap_n \mathcal{C}^n[i].$$

Thus $\mathcal{C}^{<\infty}[i]$ is naturally a differential ring with $\mathbb{C}$ as its ring of constants. We also have the differential subring

$$\mathcal{C}^{<\infty} \ := \ \bigcap_n \mathcal{C}^n$$

of $\mathcal{C}^{<\infty}[i]$, with $\mathbb{R}$ as its ring of constants and $\mathcal{C}^{<\infty}[i] = \mathcal{C}^{<\infty} + \mathcal{C}^{<\infty}i$. Note that $\mathcal{C}^{<\infty}[i]$ has $\mathcal{C}^{\infty}[i]$ as a differential subring. Similarly, $\mathcal{C}^{<\infty}$ has $\mathcal{C}^{\infty}$ as a differential subring, and the differential ring $\mathcal{C}^{\infty}$ has in turn the differential subring $\mathcal{C}^{\omega}$, whose elements are the germs at $+\infty$ of the functions in $\bigcup_a \mathcal{C}^{\omega}_a$. We have $\mathcal{C}[i]^\times \cap \mathcal{C}^{<\infty}[i] = (\mathcal{C}^{<\infty}[i])^\times$ and $\mathcal{C}^\times \cap \mathcal{C}^{<\infty} = (\mathcal{C}^{<\infty})^\times$, and likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{<\infty}$. If R is a subring of $\mathcal{C}^1$ such that $f' \in R$ for all $f \in R$, then $R \subseteq \mathcal{C}^{<\infty}$ is a differential subring of $\mathcal{C}^{<\infty}$.

Basic facts about linear differential equations. In this subsection we review the main analytic facts about linear differential equations used later. Let $a \in \mathbb{R}$, $r \in \mathbb{N}^{\geqslant 1}$, and $f_1, \dots, f_r \in \mathcal{C}_a[i]$. This gives the $\mathbb{C}$-linear map

$$y \mapsto A(y) \ := \ y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ : \ \mathcal{C}^r_a[i] \to \mathcal{C}_a[i].$$

We now have the classical existence and uniqueness theorem (see, e.g., [57, (10.6.3)] or [203, §19, I, II]):

Proposition 5.2.1. *Let $t \in \mathbb{R}^{\geqslant a}$ be given. Then for any $b \in \mathcal{C}_a[i]$ and $c \in \mathbb{C}^r$ there is a unique $y = y(b,c) \in \mathcal{C}^r_a[i]$ such that*

$$A(y) \ = \ b, \qquad \big(y(t), y'(t), \dots, y^{(r-1)}(t)\big) \ = \ c.$$

The map $c \mapsto y(0,c)\colon \mathbb{C}^r \to \ker A$ is an isomorphism of $\mathbb{C}$-linear spaces, and so in particular, $\dim_{\mathbb{C}} \ker A = r$.

Corollary 5.2.2. *Let $y \in \ker A$. If for some $t \in \mathbb{R}^{\geqslant a}$ we have $y^{(j)}(t) = 0$ for $j = 0, \dots, r-1$, then $y = 0$.*

Proposition 5.2.1 and $\operatorname{Re} A(y) = A(\operatorname{Re} y)$ for $f_1, \dots, f_r \in \mathcal{C}_a$ and $y \in \mathcal{C}^r_a[i]$ give:

Corollary 5.2.3. *Suppose $f_1, \dots, f_r \in \mathcal{C}_a$ and $t \in \mathbb{R}^{\geqslant a}$. Then for any $b \in \mathcal{C}_a$ and $c \in \mathbb{R}^r$ we have $y = y(b,c) \in \mathcal{C}^r_a$, and the map $c \mapsto y(0,c)\colon \mathbb{R}^r \to \mathcal{C}^r_a \cap \ker A$ is an isomorphism of $\mathbb{R}$-linear spaces.*

Let $b \in \mathcal{C}_a[i]$. Using $y^{(r)} = b - \sum_{i=1}^r f_i y^{(r-i)}$ for $y \in A^{-1}(b) \subseteq \mathcal{C}^r_a[i]$ gives

$$b, f_1, \dots, f_r \in \mathcal{C}^n_a[i] \implies A^{-1}(b) \subseteq \mathcal{C}^{n+r}_a[i]$$

by induction on n. Hence $b, f_1, \dots, f_r \in \mathcal{C}^\infty_a[i] \Rightarrow A^{-1}(b) \subseteq \mathcal{C}^\infty_a[i]$, in particular, $f_1, \dots, f_r \in \mathcal{C}^\infty_a[i] \Rightarrow \ker A \subseteq \mathcal{C}^\infty_a[i]$. Also $b, f_1, \dots, f_r \in \mathcal{C}^\omega_a[i] \Rightarrow A^{-1}(b) \subseteq \mathcal{C}^\omega_a[i]$ by Lemma 6.3.4 below (see also [57, (10.5.3)]), so $f_1, \dots, f_r \in \mathcal{C}^\omega_a[i] \Rightarrow \ker A \subseteq \mathcal{C}^\omega_a[i]$.

Let $y_1, \dots, y_r \in \mathcal{C}^r_a[i]$. The *Wronskian* $w = \operatorname{wr}(y_1, \dots, y_r)$ of $y_1, \dots, y_r$ is

$$\operatorname{wr}(y_1, \dots, y_r) \ := \ \det \begin{pmatrix} y_1 & \cdots & y_r \\ y_1' & \cdots & y_r' \\ \vdots & & \vdots \\ y_1^{(r-1)} & \cdots & y_r^{(r-1)} \end{pmatrix} \in \mathcal{C}^1_a[i].$$

Hence if $w \neq 0$ (that is, $w(t) \neq 0$ for some $t \geqslant a$), then $y_1, \dots, y_r$ are $\mathbb{C}$-linearly independent. The converse does not hold in general, even for $r = 2$ and $y_1, y_2 \in \mathcal{C}_a^\infty$, see [23], but we do have:

Lemma 5.2.4. *The following are equivalent:*

(i) $w \in \mathcal{C}_a[\mathrm{i}]^\times$ *(that is, $w(t) \neq 0$ for all $t \geqslant a$);*

(ii) $y_1, \dots, y_r$ *is a basis of* $\ker A$ *for some choice of* $f_1, \dots, f_r \in \mathcal{C}_a[\mathrm{i}]$.

Proof. For (i) $\Rightarrow$ (ii), assume $w \in \mathcal{C}_a[\mathrm{i}]^\times$, and use that $y_1, \dots, y_r \in \ker A$, where A is the $\mathbb{C}$-linear differential operator given by

$$y \mapsto A(y) \ := \ \mathrm{wr}(y_1, \dots, y_r, y)/w \ : \ \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}].$$

For (ii) $\Rightarrow$ (i), assume (ii) and suppose towards a contradiction that $t \geqslant a$ is such that $w(t) = 0$. This gives $c_1, \dots, c_r \in \mathbb{C}$, not all 0, such that for $y = \sum_{k=1}^r c_k y_k$ we have $y^{(j)}(t) = 0$ for $j = 0, \dots, r-1$. Hence $y = 0$ by Corollary 5.2.2. $\square$

Let now $y_1, \dots, y_r \in \ker A$. Then by the above

$$w \neq 0 \iff w \in \mathcal{C}_a^1[\mathrm{i}]^\times \iff y_1, \dots, y_r \text{ are } \mathbb{C}\text{-linearly independent.}$$

Moreover, $w' = -f_1 w$ (*Abel's Identity*, see [203, §19, p. 200]) and hence

$$w(t) \ = \ w(a) \exp\Big(-\int_a^t f_1(s)\, ds\Big) \qquad \text{for } t \geqslant a.$$

In particular, $w = w(a) \in \mathbb{C}$ if $f_1 = 0$.

In the next corollary we let $g_1, \dots, g_r \in \mathcal{C}_a[\mathrm{i}]$ and consider the $\mathbb{C}$-linear map

$$y \mapsto B(y) \ := \ y^{(r)} + g_1 y^{(r-1)} + \cdots + g_r y \ : \ \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}].$$

Corollary 5.2.5. $f_1 = g_1, \dots, f_{r-1} = g_{r-1} \iff A = B \iff \ker A = \ker B.$

Proof. Suppose $\ker A = \ker B$. Let $y_1, \dots, y_r$ be a basis of $\ker A$, and set $h_j := f_j - g_j$ ($j = 1, \dots, r-1$). Towards a contradiction suppose $h_j \neq 0$ for some j, and take j minimal with this property. Take a nonempty open interval $I \subseteq \mathbb{R}^{\geqslant a}$ with $h_j \in \mathcal{C}(I)[\mathrm{i}]^\times$. (Here and below we denote the restrictions of $h_1, \dots, h_{r-1}$ to functions $I \to \mathbb{C}$ by the same symbols.) Then $y_1, \dots, y_r$ restrict to $\mathbb{C}$-linearly independent functions in $\mathcal{C}^r(I)[\mathrm{i}]$ each satisfying the equation

$$y^{(r-j)} + (h_{j+1}/h_j) y^{(r-j-1)} + \cdots + (h_r/h_j) y = 0,$$

contradicting [203, §19, II]. $\square$

Next some basic properties of Wronskians:

Lemma 5.2.6. *Let* $u \in \mathcal{C}_a^r[\mathrm{i}]$. *Then* $\mathrm{wr}(uy_1, \dots, uy_r) = u^r \,\mathrm{wr}(y_1, \dots, y_r)$. *In particular, if* $y_1 \in \mathcal{C}_a^r[\mathrm{i}]^\times$, $r \geqslant 2$, *and* $z_j := (y_{j+1}/y_1)' \in \mathcal{C}_a^{r-1}[\mathrm{i}]$ $(j = 1, \dots, r-1)$, *then* $\mathrm{wr}(y_1, \dots, y_r) = y_1^r \,\mathrm{wr}(z_1, \dots, z_{r-1})$.

Proof. For the first identity, use that there are $u_{ij} \in \mathcal{C}_a[\mathrm{i}]$ $(0 \leqslant i \leqslant j < r)$ with $u_{0j} = u$ such that for all $y \in \mathcal{C}_a^r[\mathrm{i}]$ we have

$$(uy)^{(j)} \ = \ u_{0j} y^{(j)} + u_{1j} y^{(j-1)} + \cdots + u_{jj} y.$$

The first identity yields the second by taking $u := y_1^{-1}$. $\square$

Lemma 5.2.7. *Suppose* $v := \mathrm{wr}(y_1, \dots, y_{r-1})$ *and* $w := \mathrm{wr}(y_1, \dots, y_r)$ *lie in* $\mathcal{C}_a^1[\mathrm{i}]^\times$, *with* $v := 1$ *if* $r = 1$. *Then we have for all* $y \in \mathcal{C}_a^r[\mathrm{i}]$,

$$\big(\mathrm{wr}(y_1, \dots, y_{r-1}, y)/w\big)' \ = \ (v/w^2)\,\mathrm{wr}(y_1, \dots, y_r, y).$$

Proof. Expand the determinants $\operatorname{wr}(y_1, \dots, y_{r-1}, y)$ and $\operatorname{wr}(y_1, \dots, y_r, y)$ according to their last column to get functions $g_1, \dots, g_{r-1}, h_1, \dots, h_{r-1} \in \mathcal{C}_a[\mathrm{i}]$ such that

$$\begin{aligned}\big(\operatorname{wr}(y_1, \dots, y_{r-1}, y)/w\big)' &= (v/w)y^{(r)} + g_1 y^{(r-1)} + \cdots + g_r y, \\ (v/w^2)\operatorname{wr}(y_1, \dots, y_r, y) &= (v/w)y^{(r)} + h_1 y^{(r-1)} + \cdots + h_r y\end{aligned}$$

for all $y \in \mathcal{C}_a^r[\mathrm{i}]$. Both left hand sides have the $\mathbb{R}$-linearly independent $y_1, \dots, y_r$ among their zeros. Now use Corollary 5.2.5. $\square$

Let $y \in \mathcal{C}_a^r[\mathrm{i}]$ and $t \geqslant a$. The **multiplicity** of y at t is the largest $m \leqslant r$ such that $y(t) = y'(t) = \cdots = y^{(m-1)}(t) = 0$; notation: $\operatorname{mult}_t(y)$, or $\operatorname{mult}_t^r(y)$ if we need to indicate the dependence on r. So for $a \leqslant 0$ we have $\operatorname{mult}_0^2(x^3) = 2$, but $\operatorname{mult}_0^r(x^3) = 3$ for $r \geqslant 3$.) Thus t is a zero of y (that is, $y(t) = 0$) iff $\operatorname{mult}_t(y) \geqslant 1$. If $y \in \ker A$ has a zero of multiplicity r, then $y = 0$ by Corollary 5.2.2. Note that $\operatorname{mult}_t(y) = \min\big\{\operatorname{mult}_t(\operatorname{Re} y)\operatorname{mult}_t(\operatorname{Im} y)\big\}$. For $z \in \mathcal{C}_a^r[\mathrm{i}]$ we have

$$\operatorname{mult}_t(y+z) \geqslant \min\big\{\operatorname{mult}_t(y), \operatorname{mult}_t(z)\big\},$$

and using the Product Rule:

$$\operatorname{mult}_t(yz) = \min\big\{r, \operatorname{mult}_t(y) + \operatorname{mult}_t(z)\big\}.$$

If $r \geqslant 2$ and $y(t) = 0$, then $y' \in \mathcal{C}_a^{r-1}[\mathrm{i}]$ and $\operatorname{mult}_t^{r-1}(y') = \operatorname{mult}_t^r(y) - 1$. The following is obvious:

Lemma 5.2.8. *Let $y_1, \dots, y_r \in \mathcal{C}_a^r[\mathrm{i}]$, $w := \operatorname{wr}(y_1, \dots, y_r)$, and $t \geqslant a$. If $w(t) = 0$, then $\operatorname{mult}_t(y) = r$ for some $\mathbb{C}$-linear combination $y = c_1 y_1 + \cdots + c_r y_r$ of $y_1, \dots, y_r$, where $c_1, \dots, c_r \in \mathbb{C}$ are not all zero.*

We also call the sum

$$\operatorname{mult}(y) := \sum_{t \geqslant a} \operatorname{mult}_t(y) \in \mathbb{N} \cup \{\infty\}$$

of the multiplicities of all zeros of y the (total) **multiplicity** of y, and we denote it by $\operatorname{mult}^r(y)$ if we need to exhibit the dependence on r. Note that $\operatorname{mult}(y) < \infty$ iff y has finitely many zeros. If $z \in \mathcal{C}_a^r[\mathrm{i}]^\times$, then $\operatorname{mult}(yz) = \operatorname{mult}(y)$.

Lemma 5.2.9. *Suppose $y \in \mathcal{C}_a^r$, $r \geqslant 2$. Then (with $\infty - 1 := \infty$):*

$$\operatorname{mult}^{r-1}(y') \geqslant \operatorname{mult}^r(y) - 1.$$

Proof. Let $m \leqslant \operatorname{mult}^r(y)$; it is enough to show that then $m - 1 \leqslant \operatorname{mult}^{r-1}(y')$. Let $t_1 < \cdots < t_n$ be zeros of y such that $\sum_i \operatorname{mult}_{t_i}(y) \geqslant m$. For $i = 1, \dots, n-1$, Rolle's Theorem yields $s_i \in (t_i, t_{i+1})$ such that $y'(s_i) = 0$. Hence

$$\begin{aligned} m \leqslant \sum_{i=1}^n \operatorname{mult}_{t_i}^r(y) &= n + \sum_{i=1}^n \operatorname{mult}_{t_i}^{r-1}(y') \\ &\leqslant 1 + \sum_{i=1}^{n-1} \operatorname{mult}_{s_i}^{r-1}(y') + \sum_{i=1}^n \operatorname{mult}_{t_i}^{r-1}(y') \leqslant 1 + \operatorname{mult}^{r-1}(y')\end{aligned}$$

as required. $\square$

Oscillation. Let $y \in \mathcal{C}_a$. We say that y **oscillates** if its germ in $\mathcal{C}$ oscillates. So y does not oscillate iff $\operatorname{sign} y(t)$ is constant, eventually. If y oscillates, then so does cy for $c \in \mathbb{R}^\times$. If $y \in \mathcal{C}_a^1$ oscillates, then so does $y' \in \mathcal{C}_a$, by Rolle's Theorem.

Let now $r \in \mathbb{N}^{\geqslant 1}$ and $f_1, \dots, f_r \in \mathcal{C}_a$, and consider the $\mathbb{R}$-linear map

$$y \mapsto A(y) \ := \ y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ : \ \mathcal{C}_a^r \to \mathcal{C}_a.$$

By Corollary 5.2.3, the $\mathbb{R}$-linear subspace $\mathcal{C}_a^r \cap \ker A$ of $\mathcal{C}_a^r$ has dimension r.

Let $y \in \mathcal{C}_a^r \cap \ker A$, $y \neq 0$, and let $Z := y^{-1}(0)$ be the set of zeros of y, so $Z \subseteq [a, +\infty)$ is closed in $\mathbb{R}$. By a *limit point* of a set $S \subseteq \mathbb{R}$ we mean a point $b \in \mathbb{R}$ such that for every real $\varepsilon > 0$ we have $0 < |s - b| < \varepsilon$ for some $s \in S$.

Lemma 5.2.10. *Z has no limit points.*

Proof. For $j = 0, \dots, r$ let $Z_j := (y^{(j)})^{-1}(0)$ be the set of zeros of $y^{(j)}$, so $Z = Z_0$. Each Z_j is closed and hence contains its limit points. If $t_0 < t_1$ are in Z_j, $0 \leqslant j < r$, then $Z_{j+1} \cap (t_0, t_1) \neq \emptyset$, by Rolle, hence each limit point of Z_j is a limit point of Z_{j+1}. Thus if t is a limit point of Z, then $t \geqslant a$ and $y(t) = y'(t) = \cdots = y^{(r-1)}(t) = 0$, hence $y = 0$ by Corollary 5.2.2, a contradiction. □

By Lemma 5.2.10, $Z \cap [a, b]$ is finite for every $b \geqslant a$. Thus

$$y \text{ does not oscillate} \iff Z \text{ is finite} \iff Z \text{ is bounded}.$$

If $t_0 \in Z$ is not the largest element of Z, then $Z \cap (t_0, t_1) = \emptyset$ for some $t_1 > t_0$ in Z. We say that a pair of zeros $t_0 < t_1$ of y is **consecutive** if $Z \cap (t_0, t_1) = \emptyset$.

Next we consider the set $Z_1 := (y')^{-1}(0)$ of stationary points of y.

Lemma 5.2.11. *Suppose $f_r \in \mathcal{C}_a^\times$. Then Z_1 has no limit points.*

Proof. The proof of Lemma 5.2.10 shows that if t is a limit point of Z_1, then $t \geqslant a$ and $y'(t) = y''(t) = \cdots = y^{(r)}(t) = 0$, and as $y \in \ker A$, this gives $0 = A(y)(t) = f_r(t) y(t)$, so $y(t) = 0$, and thus $y = 0$, a contradiction. □

Thus if $f_r \in \mathcal{C}_a^\times$, then $Z_1 \cap [a, b]$ is finite for all $b \geqslant a$.

Second-order differential equations. Let $f \in \mathcal{C}_a$, that is, $f \colon [a, \infty) \to \mathbb{R}$ is continuous. We consider the differential equation

$$\text{(L)} \qquad Y'' + fY \ = \ 0.$$

The solutions $y \in \mathcal{C}_a^2$ of (L) form an $\mathbb{R}$-linear subspace $\operatorname{Sol}(f)$ of $\mathcal{C}_a^2$. The solutions $y \in \mathcal{C}_a^2[\mathrm{i}]$ of (L) are the $y_1 + y_2 i$ with $y_1, y_2 \in \operatorname{Sol}(f)$ and form a $\mathbb{C}$-linear subspace $\operatorname{Sol}_{\mathbb{C}}(f)$ of $\mathcal{C}_a^2[\mathrm{i}]$. For any complex numbers c, d there is a unique solution $y \in \mathcal{C}_a^2[\mathrm{i}]$ of (L) with $y(a) = c$ and $y'(a) = d$, and the map that assigns to (c, d) in $\mathbb{C}^2$ this unique solution is an isomorphism $\mathbb{C}^2 \to \operatorname{Sol}_{\mathbb{C}}(f)$ of $\mathbb{C}$-linear spaces; it restricts to an $\mathbb{R}$-linear bijection $\mathbb{R}^2 \to \operatorname{Sol}(f)$. We have $f \in \mathcal{C}_a^n \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}_a^{n+2}$ (hence $f \in \mathcal{C}_a^\infty \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}_a^\infty$) and $f \in \mathcal{C}_a^\omega \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}_a^\omega$. From [203, §27, XI]:

Lemma 5.2.12 (Sonin-Pólya)**.** *Suppose $f \in (\mathcal{C}_a^1)^\times$, $y \in \operatorname{Sol}(f)^{\neq}$, and $t_0 < t_1$ are stationary points of y. If f is increasing, then $|y(t_0)| \geqslant |y(t_1)|$. If f is decreasing, then $|y(t_0)| \leqslant |y(t_1)|$. If f is strictly increasing, respectively strictly decreasing, then these inequalities are strict.*

Proof. Put $u := y^2 + \big((y')^2/f\big) \in \mathcal{C}_a^1$. Then $u' = -f'(y'/f)^2$. Thus if f is increasing, then u is decreasing, and as $u(t_i) = y(t_i)^2$ for $i = 0, 1$, we get $|y(t_0)| \geqslant |y(t_1)|$. The other cases are similar, using also Lemma 5.2.11 for the strict inequalities. $\square$

Lemma 5.2.13. *Suppose $f \in (\mathcal{C}_a^1)^\times$, $y \in \operatorname{Sol}(f)$, and $t_0 < t_1$ are consecutive zeros of y. Then there is exactly one stationary point of y in the interval (t_0, t_1).*

Proof. If $s_0 < s_1$ were stationary points of y in the interval (t_0, t_1), then by Rolle y'' and thus y (in view of $y'' = -fy$) would have a zero in the interval (s_0, s_1). $\square$

Let $y_1, y_2 \in \operatorname{Sol}(f)$, with Wronskian $w = y_1 y_2' - y_1' y_2$. Then $w \in \mathbb{R}$, and

$$w \neq 0 \iff y_1,\ y_2 \text{ are } \mathbb{R}\text{-linearly independent.}$$

By [18, Chapter 6, Lemmas 2 and 3] we have:

Lemma 5.2.14. *Let $y_1, y_2 \in \operatorname{Sol}(f)$ be $\mathbb{R}$-linearly independent and $g \in \mathcal{C}_a$. Then*

$$t \mapsto y(t) \ := \ -y_1(t) \int_a^t \frac{y_2(s)}{w} g(s)\, ds + y_2(t) \int_a^t \frac{y_1(s)}{w} g(s)\, ds \ : \ [a, +\infty) \to \mathbb{R}$$

lies in $\mathcal{C}_a^2$ and satisfies $y'' + fy = g$, $y(a) = y'(a) = 0$.

Lemma 5.2.15. *Let $y_1 \in \operatorname{Sol}(f)$ with $y_1(t) \neq 0$ for $t \geqslant a$. Then the function*

$$t \mapsto y_2(t) := y_1(t) \int_a^t \frac{1}{y_1(s)^2}\, ds \colon [a, +\infty) \to \mathbb{R}$$

also lies in $\operatorname{Sol}(f)$, and y_1, y_2 are $\mathbb{R}$-linearly independent.

From [18, Chapter 2, Lemma 1] we also recall:

Lemma 5.2.16 (Gronwall's Lemma)**.** *Let $C \in \mathbb{R}^{\geqslant}$, $v, y \in \mathcal{C}_a$ satisfy $v(t), y(t) \geqslant 0$ for all $t \geqslant a$ and*

$$y(t) \ \leqslant \ C + \int_a^t v(s) y(s)\, ds \quad \text{for all } t \geqslant a.$$

Then

$$y(t) \ \leqslant \ C \exp\left[\int_a^t v(s)\, ds\right] \quad \text{for all } t \geqslant a.$$

Here is a simpler differential version:

Lemma 5.2.17. *Let $u \in \mathcal{C}_a$ and $y \in \mathcal{C}_a^1$ satisfy $y'(t) \leqslant u(t)y(t)$ for all $t \geqslant a$. Then $y(t) \leqslant y(a) \exp\Big(\int_a^t u(s)\, ds\Big)$ for all $t \geqslant a$.*

Proof. Put $z(t) := y(t) \exp\Big(-\int_a^t u(s)\, ds\Big)$ for $t \geqslant a$. Then $z \in \mathcal{C}_a^1$, and $z'(t) \leqslant 0$ for all $t \geqslant a$, so $z(t) \leqslant z(a) = y(a)$ for all $t \geqslant a$. This yields the desired result. $\square$

In the rest of this subsection we assume that $a \geqslant 1$ and that $c \in \mathbb{R}^{>}$ is such that $|f(t)| \leqslant c/t^2$ for all $t \geqslant a$. Under this hypothesis, Lemma 5.2.16 yields the following bound on the growth of the solutions $y \in \operatorname{Sol}(f)$; the proof we give is similar to that of [18, Chapter 6, Theorem 5].

Proposition 5.2.18. *Let $y \in \operatorname{Sol}(f)$. Then there is $C \in \mathbb{R}^{\geqslant}$ such that $|y(t)| \leqslant Ct^{c+1}$ and $|y'(t)| \leqslant Ct^c$ for all $t \geqslant a$.*

Proof. Let t range over $[a,+\infty)$. Integrating $y'' = -fy$ twice between a and t, we obtain constants c_1, c_2 such that for all t,

$$y(t) \;=\; c_1 + c_2 t - \int_a^t \int_a^{t_1} f(t_2)y(t_2)\,dt_2\,dt_1 \;=\; c_1 + c_2 t - \int_a^t (t-s)f(s)y(s)\,ds$$

and hence, with $C := |c_1| + |c_2|$,

$$|y(t)| \;\leqslant\; Ct + t\int_a^t |f(s)|\cdot|y(s)|\,ds, \quad \text{so} \quad \frac{|y(t)|}{t} \;\leqslant\; C + \int_a^t s|f(s)|\cdot\frac{|y(s)|}{s}\,ds.$$

Then by the lemma above,

$$\frac{|y(t)|}{t} \;\leqslant\; C\exp\left[\int_a^t s|f(s)|\,ds\right] \;\leqslant\; C\exp\left[\int_1^t c/s\,ds\right] \;=\; Ct^c$$

and thus $|y(t)| \leqslant Ct^{c+1}$. Now

$$\begin{aligned} y'(t) \;&=\; c_2 - \int_a^t f(s)y(s)\,ds, \text{ so} \\ |y'(t)| \;&\leqslant\; |c_2| + \int_a^t |f(s)y(s)|\,ds \;\leqslant\; C + Cc\int_1^t s^{c-1}\,ds \\ &=\; C + Cc\left[\frac{t^c}{c} - \frac{1}{c}\right] \;=\; Ct^c. \end{aligned}$$

□

Let $y_1, y_2 \in \mathrm{Sol}(f)$ be $\mathbb{R}$-linearly independent. Recall that $w = y_1y_2' - y_1'y_2 \in \mathbb{R}^\times$. It follows that y_1 and y_2 cannot be simultaneously very small:

Lemma 5.2.19. *There is a positive constant d such that*

$$\max\bigl(|y_1(t)|, |y_2(t)|\bigr) \;\geqslant\; dt^{-c} \qquad \textit{for all } t \geqslant a.$$

Proof. Proposition 5.2.18 yields $C \in \mathbb{R}^>$ such that $|y_i'(t)| \leqslant Ct^c$ for $i = 1,2$ and all $t \geqslant a$. Hence $|w| \leqslant 2\max\bigl(|y_1(t)|, |y_2(t)|\bigr)Ct^c$ for $t \geqslant a$, so

$$\max\bigl(|y_1(t)|, |y_2(t)|\bigr) \;\geqslant\; \frac{|w|}{2C}t^{-c} \qquad (t \geqslant a).$$

□

Corollary 5.2.20. *Set $y := y_1 + y_2 i$ and $z := y^\dagger$. Then for some $D \in \mathbb{R}^>$,*

$$|z(t)| \;\leqslant\; Dt^{2c} \qquad \textit{for all } t \geqslant a.$$

Proof. Take C as in the proof of Lemma 5.2.19, and d as in that lemma. Then

$$|z(t)| \;=\; \frac{|y_1'(t) + y_2'(t)i|}{|y_1(t) + y_2(t)i|} \;\leqslant\; \frac{|y_1'(t)| + |y_2'(t)|}{\max\bigl(|y_1(t)|, |y_2(t)|\bigr)} \;\leqslant\; \left(\frac{2C}{d}\right)t^{2c}$$

for $t \geqslant a$. □

More on oscillation. We continue with the study of (L). Sturm's Separation Theorem says that if $y, z \in \mathrm{Sol}(f)$ are $\mathbb{R}$-linearly independent and $t_0 < t_1$ are consecutive zeros of z, then (t_0, t_1) contains a unique zero of y [203, §27, VI]. Thus:

Lemma 5.2.21. *Some $y \in \mathrm{Sol}(f)^{\neq}$ oscillates $\iff$ every $y \in \mathrm{Sol}(f)^{\neq}$ oscillates.*

We say that f **generates oscillations** if some element of $\mathrm{Sol}(f)^{\neq}$ oscillates.

Lemma 5.2.22. *Let $b \in \mathbb{R}^{\geqslant a}$. Then*

$$f \textit{ generates oscillations} \quad \iff \quad f|_b \in \mathcal{C}_b \textit{ generates oscillations.}$$

Proof. The forward direction is obvious. For the backward direction, use that every $y \in \mathcal{C}_b^2$ with $y'' + gy = 0$ for $g := f|_b$ extends uniquely to a solution of (L). $\square$

By this lemma, whether f generates oscillations depends only on its germ in $\mathcal{C}$. So this induces the notion of an element of $\mathcal{C}$ generating oscillations. Here is another result of Sturm [203, loc. cit.] that we use below:

Theorem 5.2.23 (Sturm's Comparison Theorem)**.** *Let $g \in \mathcal{C}_a$ with $f(t) \geqslant g(t)$ for all $t \geqslant a$. Let $y \in \mathrm{Sol}(f)^{\neq}$ and $z \in \mathrm{Sol}(g)^{\neq}$, and let $t_0 < t_1$ be consecutive zeros of z. Then either (t_0, t_1) contains a zero of y, or on $[t_0, t_1]$ we have $f = g$ and $y = cz$ for some constant $c \in \mathbb{R}^{\times}$.*

Here is an immediate consequence:

Corollary 5.2.24. *If $g \in \mathcal{C}_a$ generates oscillations and $f(t) \geqslant g(t)$, eventually, then f also generates oscillations.*

Example. For $k \in \mathbb{R}^{\times}$ we have the differential equation of the harmonic oscillator,

$$y'' + k^2 y \;=\; 0.$$

A function $y \in \mathcal{C}_a^2$ is a solution iff for some real constants c, t_0 and all $t \geqslant a$,

$$y(t) \;=\; c \sin k(t - t_0).$$

For $c \neq 0$, any function $y \in \mathcal{C}_a^2$ as displayed oscillates. Thus if $f(t) \geqslant \varepsilon$, eventually, for some constant $\varepsilon > 0$, then f generates oscillations.

To (L) we associate the corresponding **Riccati equation**

$$z' + z^2 + f \;=\; 0. \tag{R}$$

Let $y \in \mathrm{Sol}(f)^{\neq}$ be a non-oscillating solution to (L), and take $b \geqslant a$ with $y(t) \neq 0$ for $t \geqslant b$. Then the function

$$t \mapsto z(t) \;:=\; y'(t)/y(t) \;:\; [b, +\infty) \to \mathbb{R}$$

in $\mathcal{C}_b^1$ satisfies (R). Conversely, if $z \in \mathcal{C}_b^1$ $(b \geqslant a)$ is a solution to (R), then

$$t \mapsto y(t) \;:=\; \exp\left(\int_b^t z(s)\, ds\right) \;:\; [b, +\infty) \to \mathbb{R}$$

is a non-oscillating solution to (L) with $y \in (\mathcal{C}_b^1)^{\times}$ and $z = y^{\dagger}$.

Let $g \in \mathcal{C}_a^1$, $h \in \mathcal{C}_a^0$ and consider the second-order linear differential equation

$$y'' + gy' + hy \;=\; 0. \tag{$\widetilde{\mathrm{L}}$}$$

Corollary 5.2.25. *Set $f := -\frac{1}{2}g' - \frac{1}{4}g^2 + h \in \mathcal{C}_a$. Then the following are equivalent:*

(i) *some nonzero solution of $(\widetilde{\mathrm{L}})$ oscillates;*
(ii) *all nonzero solutions of $(\widetilde{\mathrm{L}})$ oscillate;*
(iii) *f generates oscillations.*

Proof. Let $G \in (\mathcal{C}_a^2)^{\times}$ be given by $G(t) := \exp\Big(-\frac{1}{2}\int_a^t g(s)\, ds\Big)$. Then $y \in \mathcal{C}_a^2$ is a solution to (L) iff Gy is a solution to $(\widetilde{\mathrm{L}})$; cf. [ADH, 5.1.13]. $\square$

More on non-oscillation. We continue with (L). Let y_1, y_2 range over elements of $\mathrm{Sol}(f)$, and recall that its Wronskian $w = y_1 y_2' - y_1' y_2$ lies in $\mathbb{R}$.

Lemma 5.2.26. *Suppose $b \geqslant a$ is such that $y_2(t) \neq 0$ for $t \geqslant b$. Then for $q := y_1/y_2 \in \mathcal{C}_b^2$ we have $q'(t) = -w/y_2(t)^2$ for $t \geqslant b$, so q is monotone and $\lim_{t\to\infty} q(t)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

This leads to:

Corollary 5.2.27. *Suppose $b \geqslant a$ and $y_1(t), y_2(t) \neq 0$ for $t \geqslant b$. For $i = 1, 2$, set*

$$h_i(t) \ := \ \int_b^t \frac{1}{y_i(s)^2}\, ds \qquad \text{for } t \geqslant b, \text{ so } h_i \in \mathcal{C}_b^3.$$

Then: $y_1 \prec y_2 \iff h_1 \succ 1 \succcurlyeq h_2$.

Proof. Suppose $y_1 \prec y_2$. Then y_1, y_2 are $\mathbb{R}$-linearly independent, so $w \neq 0$. Moreover, $q \prec 1$ with q as in in Lemma 5.2.26, and $q' = -wh_2'$ by that lemma, so $q + wh_2$ is constant, and thus $h_2 \preccurlyeq 1$. Note that h_1 is strictly increasing. If $h_1(t) \to r \in \mathbb{R}$ as $t \to +\infty$, then $z := (r - h_1)y_1 \in \mathrm{Sol}(f)$ by Lemma 5.2.26 with y_1 and y_2 interchanged, and $z \prec y_1$, so $z = 0$, hence $h_1 = r$, a contradiction. Thus $h_1 \succ 1$.

For the converse, suppose $h_1 \succ 1 \succcurlyeq h_2$. Then y_1, y_2 are $\mathbb{R}$-linearly independent, so $w \neq 0$. From $h_2 \preccurlyeq 1$ and $q + wh_2$ being constant we obtain $q \preccurlyeq 1$. If $q(t) \to r \neq 0$ as $t \to +\infty$, then $y_1 = qy_2 \asymp y_2$, and thus $h_1 \asymp h_2$, a contradiction. Hence $q \prec 1$, and thus $y_1 \prec y_2$. □

The pair (y_1, y_2) is said to be a **principal system** of solutions of (L) if

(1) $y_1(t), y_2(t) > 0$ eventually, and
(2) $y_1 \prec y_2$.

Then y_1, y_2 form a basis of the $\mathbb{R}$-linear space $\mathrm{Sol}(f)$, and f does not generate oscillations, by Lemma 5.2.21. Moreover, for $y = c_1 y_1 + c_2 y_2$ with $c_1, c_2 \in \mathbb{R}$, $c_2 \neq 0$ we have $y \sim c_2 y_2$. Here are some facts about this notion:

Lemma 5.2.28. *If (y_1, y_2), (z_1, z_2) are principal systems of solutions of (L), then there are $c_1, d_1, d_2 \in \mathbb{R}$ such that $z_1 = c_1 y_1$, $z_2 = d_1 y_1 + d_2 y_2$, and $c_1, d_2 > 0$.*

Lemma 5.2.29. *Suppose f does not generate oscillations. Then (L) has a principal system of solutions.*

Proof. It suffices to find a basis y_1, y_2 of $\mathrm{Sol}(f)$ with $y_1 \prec y_2$. Suppose y_1, y_2 is any basis of $\mathrm{Sol}(f)$, and set $c := \lim_{t\to\infty} y_1(t)/y_2(t) \in \mathbb{R} \cup \{-\infty, +\infty\}$. If $c = \pm\infty$, then interchange y_1, y_2, otherwise replace y_1 by $y_1 - cy_2$. Then $c = 0$, so $y_1 \prec y_2$. □

One calls y_1 a **principal** solution of (L) if (y_1, y_2) is a principal system of solutions of (L) for some y_2. (See [91, Theorem XI.6.4] and [125, 127].) By the previous two lemmas, (L) has a principal solution iff f does not generate oscillations, and any two principal solutions differ by a multiplicative factor in $\mathbb{R}^>$. If $y_1 \in (\mathcal{C}_a)^\times$ and y_2 is given as in Lemma 5.2.15, then y_2 is a non-principal solution of (L) and $y_1 \notin \mathbb{R} y_2$.

Chebyshev systems and Markov systems (*). Let $r \in \mathbb{N}^{\geqslant 1}$ and $y_1, \dots, y_r \in \mathcal{C}_a^r$, and let V be the $\mathbb{R}$-linear subspace of $\mathcal{C}_a^r$ spanned by $y_1, \dots, y_r$. We call $y_1, \dots, y_r$ a **Chebyshev system** (on $\mathbb{R}^{\geqslant a}$) if for all $y = c_1 y_1 + \dots + c_r y_r$ with $c_1, \dots, c_r \in \mathbb{R}$ not all zero, we have $\mathrm{mult}^r(y) < r$. Note that if $y_1, \dots, y_r$ is a Chebyshev system, then $y_1, \dots, y_r$ are $\mathbb{R}$-linearly independent, and every basis of V is a Chebyshev system. Chebyshev systems can be used for interpolation:

Lemma 5.2.30. *Suppose $y_1, \dots, y_r$ are $\mathbb{R}$-linearly independent. Let $t_1, \dots, t_n \in \mathbb{R}^{\geqslant a}$ be pairwise distinct and let $r_1, \dots, r_n \in \mathbb{N}$ satisfy $r_1 + \cdots + r_n = r$. (So $n \geqslant 1$). Then the following are equivalent:*

(i) *the only $y \in V$ with $\operatorname{mult}^r_{t_i}(y) \geqslant r_i$ for $i = 1, \dots, n$ is $y = 0$;*
(ii) *for all $b_{ij} \in \mathbb{R}$ ($i = 1, \dots, n$, $j = 1, \dots, r_i$), there exists $y \in V$ with*

$$y^{(j-1)}(t_i) \ = \ b_{ij} \qquad (i = 1, \dots, n,\ j = 1, \dots, r_i).$$

Moreover, in this case, given any b_{ij} as in (ii)*, the element $y \in V$ in* (ii) *is unique.*

Proof. Each $y \in V$ equals $c_1 y_1 + \cdots + c_r y_r$ for a unique $(c_1, \dots, c_r) \in \mathbb{R}^r$. Let the b_{ij} in $\mathbb{R}$ be as in (ii), and set

$$M := \begin{pmatrix} y_1(t_1) & \cdots & y_r(t_1) \\ \vdots & & \vdots \\ y_1^{(r_1-1)}(t_1) & \cdots & y_r^{(r_1-1)}(t_1) \\ y_1(t_2) & \cdots & y_r(t_2) \\ \vdots & & \vdots \\ y_1^{(r_n-1)}(t_n) & \cdots & y_r^{(r_n-1)}(t_n) \end{pmatrix} \in \mathbb{R}^{r \times r}, \quad b := \begin{pmatrix} b_{11} \\ \vdots \\ b_{1r_1} \\ b_{21} \\ \vdots \\ b_{nr_n} \end{pmatrix} \in \mathbb{R}^r.$$

Then given $c = (c_1, \dots, c_r)^{\mathrm{t}} \in \mathbb{R}^r$, the element $y = c_1 y_1 + \cdots + c_r y_r$ of V satisfies the inequalities in (i) iff $Mc = 0$, and the displayed equations in (ii) iff $Mc = b$. Thus (i) means injectivity of $M : \mathbb{R}^r \to \mathbb{R}^r$, and (ii) its surjectivity. $\square$

In particular, if $y_1, \dots, y_r$ is a Chebyshev system, then for all t_i, r_i ($i = 1, \dots, n$) as in the previous lemma and for all $b_{ij} \in \mathbb{R}$ ($i = 1, \dots, n$, $j = 1, \dots, r_i$), there is a unique $y \in V$ with $y^{(j-1)}(t_i) = b_{ij}$ ($i = 1, \dots, n$, $j = 1, \dots, r_i$).

Remark. Suppose $y_1, \dots, y_r$ are $\mathbb{R}$-linearly independent. If $y_1, \dots, y_r$ is a Chebyshev system, then each $y \in V^{\neq}$ has $< r$ zeros. Remarkably, the converse of this implication also holds; this is due to Aramă [4] and (in greater generality) Hartman [89]; a simple proof, from [145], is in [52, Chapter 3, Proposition 3]. This links the notion of Chebyshev system considered here with the concept of the same name in approximation theory [46, Chapter 3, §4]. (These remarks are not used later.)

If $y_1, \dots, y_r$ is a Chebyshev system, then $\operatorname{wr}(y_1, \dots, y_r) \in \mathcal{C}_a^\times$ by Lemma 5.2.8. If $\operatorname{wr}(y_1, \dots, y_j) \in \mathcal{C}_a^\times$ for $j = 1, \dots, r$, then $y_1, \dots, y_r$ is called a **Markov system** (on $\mathbb{R}^{\geqslant a}$). Thus by Lemma 5.2.8, if $y_1, \dots, y_j$ is a Chebyshev system for $j = 1, \dots, r$, then $y_1, \dots, y_r$ is a Markov system. Here is a partial converse:

Lemma 5.2.31. *If $y_1, \dots, y_r$ is a Markov system, then it is a Chebyshev system.*

Proof. The case $r = 1$ is trivial, so let $r \geqslant 2$ and let $y_1, \dots, y_r$ be a Markov system; in particular, $y_1 \in \mathcal{C}_a^\times$. Set $z_j := (y_{j+1}/y_1)' \in \mathcal{C}_a^{r-1}$ for $j = 1, \dots, r-1$. Then $z_1, \dots, z_{r-1}$ is a Markov system by Lemma 5.2.6. Assume inductively that it is a Chebyshev system, and let $y = c_1 y_1 + \cdots + c_r y_r$, $c_1, \dots, c_r \in \mathbb{R}$ not all zero; we need to show $\operatorname{mult}(y) < r$. Towards a contradiction, assume $\operatorname{mult}(y) \geqslant r$. Then $z := (y/y_1)'$ satisfies $\operatorname{mult}(z) \geqslant r-1$, by Lemma 5.2.9 and the remarks before it. Moreover, $z = c_2 z_1 + \cdots + c_r z_{r-1}$, and so $c_2 = \cdots = c_r = 0$ and hence $y = c_1 y_1$, and thus $c_1 = 0$, a contradiction. $\square$

If $y_1, \dots, y_r$ is a Markov system and $b \geqslant a$, then $y_1|_b, \dots, y_r|_b$ is a Markov system on $\mathbb{R}^{\geqslant b}$, and likewise with "Chebyshev" in place of "Markov".

Disconjugacy (*). Let $r \in \mathbb{N}^{\geqslant 1}$ and $f_1, \dots, f_r \in \mathcal{C}_a$, and consider the linear differential equation

$$\text{(D)} \qquad y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ = \ 0$$

on $\mathbb{R}^{\geqslant a}$. Let $\operatorname{Sol}(\mathrm{D})$ be its set of solutions in $\mathcal{C}_a^r$, so $\operatorname{Sol}(\mathrm{D})$ is the kernel of the $\mathbb{R}$-linear map

$$y \mapsto A(y) := y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ : \ \mathcal{C}_a^r \to \mathcal{C}_a.$$

Recall that by Corollary 5.2.3 we have $\dim_{\mathbb{R}} \operatorname{Sol}(\mathrm{D}) = r$. The linear differential equation (D) is said to be **disconjugate** if $\operatorname{Sol}(\mathrm{D})$ contains a Chebyshev system; that is, every nonzero $y \in \operatorname{Sol}(D)$ has multiplicity $< r$. If (D) is disconjugate, then it has no oscillating solutions.

Example. The equation $y^{(r)} = 0$ is disconjugate, since its solutions in $\mathcal{C}_a^r$ are the polynomial functions $c_0 + c_1 x + \cdots + c_{r-1}x^{r-1}$ with $c_0, \dots, c_{r-1} \in \mathbb{R}$.

From Lemma 5.2.30 we obtain:

Corollary 5.2.32 (de la Vallée-Poussin [202]). *Suppose* (D) *is disconjugate. Then for all pairwise distinct $t_1, \dots, t_n \geqslant a$, all $r_1, \dots, r_n \in \mathbb{N}$ with $r_1 + \cdots + r_n = r$, and all $b_{ij} \in \mathbb{R}$ $(i = 1, \dots, n,\ j = 1, \dots, r_i)$, there is a unique $y \in \operatorname{Sol}(\mathrm{D})$ such that*

$$y^{(j-1)}(t_i) \ = \ b_{ij} \qquad (i = 1, \dots, n,\ j = 1, \dots, r_i).$$

Let $b \geqslant a$ and set $g_j := f_j|_b \in \mathcal{C}_b$ for $j = 1, \dots, r$. This yields the linear differential equation

$$(\mathrm{D}_b) \qquad y^{(r)} + g_1 y^{(r-1)} + \cdots + g_r y \ = \ 0$$

on $\mathbb{R}^{\geqslant b}$ with the $\mathbb{R}$-linear isomorphism $y \mapsto y|_b \colon \operatorname{Sol}(\mathrm{D}) \to \operatorname{Sol}(\mathrm{D}_b)$.

Corollary 5.2.33. *If* (D) *is disconjugate, then some basis $y_1, \dots, y_r$ of the $\mathbb{R}$-linear space* $\operatorname{Sol}(\mathrm{D})$ *yields for every $b > a$ a Markov system $y_1|_b, \dots, y_r|_b$ on $\mathbb{R}^{\geqslant b}$.*

Proof. Let $y_1, \dots, y_r \in \mathcal{C}_a^r$ be solutions of (D) such that

$$y_j(a) \ = \ y_j'(a) \ = \ \cdots \ = \ y_j^{(r-j-1)}(a) \ = \ 0,\ y_j^{(r-j)}(a) \neq 0 \qquad \text{for } j = 1, \dots, r.$$

Then $\operatorname{wr}(y_1, \dots, y_r)(a) \neq 0$, so $y_1, \dots, y_r$ are $\mathbb{R}$-linearly independent. Suppose (D) is disconjugate. Let $j \in \{1, \dots, r\}$, $t \in \mathbb{R}^{>a}$. Then $\operatorname{wr}(y_1, \dots, y_j)(t) \neq 0$: otherwise Lemma 5.2.8 yields an $\mathbb{R}$-linear combination $y \neq 0$ of $y_1, \dots, y_j$ with $\operatorname{mult}_t(y) \geqslant j$, but also $\operatorname{mult}_a(y) \geqslant r - j$ by choice of $y_1, \dots, y_r$, hence $\operatorname{mult}(y) \geqslant r$, contradicting disconjugacy of (D). Thus $y_1, \dots, y_r$ has the desired property. $\square$

With $n \geqslant 1$ understood from the context, let ∂ denote the $\mathbb{R}$-linear map

$$y \mapsto y' \ : \ \mathcal{C}_a^n \to \mathcal{C}_a^{n-1},$$

identify $f \in \mathcal{C}_a^{n-1}$ with the $\mathbb{R}$-linear operator $y \mapsto fy \colon \mathcal{C}_a^{n-1} \to \mathcal{C}_a^{n-1}$, and for maps $\alpha \colon \mathcal{C}_a^n \to \mathcal{C}_a^{n-1}$, $\beta \colon \mathcal{C}_a^{n+1} \to \mathcal{C}_a^n$, denote $\alpha \circ \beta \colon \mathcal{C}_a^{n+1} \to \mathcal{C}_a^{n-1}$ simply by $\alpha\beta$. With these conventions we can state an analytic version of Lemma 1.1.3:

Lemma 5.2.34. *If $g_j \in (\mathcal{C}_a^{r-j+1})^\times$ for $j = 1, \dots, r$ and we set*

$$(5.2.1) \qquad A \ = \ g_1 \cdots g_r (\partial g_r^{-1}) \cdots (\partial g_2^{-1})(\partial g_1^{-1}) \ : \ \mathcal{C}_a^r \to \mathcal{C}_a,$$

then $A = (\partial - h_r)\cdots(\partial - h_1)$ *with* $h_j := (g_1\cdots g_j)^\dagger \in \mathcal{C}_a^{r-j}$ *for* $j = 1,\dots,r$. *Conversely, if* $h_j \in \mathcal{C}_a^{r-j}$ *for* $j = 1,\dots,r$ *and* $A := (\partial - h_r)\cdots(\partial - h_1)\colon \mathcal{C}_a^r \to \mathcal{C}_a$ *and* $h_0 := 0$, *then* (5.2.1) *holds for* $g_j \in (\mathcal{C}_a^{r-j+1})^\times$ *given by*

$$g_j(t) \ := \ \exp\int_a^t \big(h_j(s) - h_{j-1}(s)\big)\,ds \qquad (j = 1,\dots,r),$$

and $h_j = (g_1\cdots g_j)^\dagger$ *for those* j.

We now link the notion of disconjugacy with factorization of the operator $A = \partial^r + f_1\partial^{r-1} + \cdots + f_r\colon \mathcal{C}_a^r \to \mathcal{C}_a$ considered earlier in connection with (D).

Proposition 5.2.35 (Frobenius [74], Libri [129]). *Suppose* $y_1,\dots,y_r \in \operatorname{Sol}(\mathrm{D})$ *is a Markov system. Set* $w_0 := 1$, $w_j := \operatorname{wr}(y_1,\dots,y_j) \in (\mathcal{C}_a^{r-j+1})^\times$ *for* $j = 1,\dots,r$, *and*

$$g_1 \ := \ w_1, \qquad g_j \ := \ w_j w_{j-2}/w_{j-1}^2 \quad (j = 2,\dots,r).$$

Then $g_j \in (\mathcal{C}_a^{r-j+1})^\times$ *for* $j = 1,\dots,r$ *and* $A = g_1\cdots g_r(\partial g_r^{-1})\cdots(\partial g_2^{-1})(\partial g_1^{-1})$.

Proof. It is clear that $g_j \in (\mathcal{C}_a^{r-j+1})^\times$ and easy to check that $w_j/w_{j-1} = g_1\cdots g_j$ for $j = 1,\dots,r$. We define for $j = 0,\dots,r$ the $\mathbb{R}$-linear map

$$y \mapsto A_j(y) := \operatorname{wr}(y_1,\dots,y_j,y)/w_j \ : \ \mathcal{C}_a^r \to \mathcal{C}_a^{r-j}.$$

We claim that $A_j = g_1\cdots g_j\partial g_j^{-1}\partial\cdots\partial g_1^{-1}$. The case $j = 0$ is trivial. Suppose the claim holds for a certain $j < r$. Then

$$g_1\cdots g_{j+1}\partial g_{j+1}^{-1}\partial\cdots\partial g_2^{-1}\partial g_1^{-1} \ = \ g_1\cdots g_{j+1}\partial(g_1\cdots g_{j+1})^{-1}A_j,$$

which sends $y \in \mathcal{C}_a^r$ to

$$\frac{w_{j+1}}{w_j}\left(\frac{w_j}{w_{j+1}}\frac{\operatorname{wr}(y_1,\dots,y_j,y)}{w_j}\right)' = \frac{w_{j+1}}{w_j}\left(\frac{\operatorname{wr}(y_1,\dots,y_j,y)}{w_{j+1}}\right)',$$

and this in turn equals $A_{j+1}(y) = \operatorname{wr}(y_1,\dots,y_j,y_{j+1},y)/w_{j+1}$ by Lemma 5.2.7. $\square$

Here is a converse, with A still the operator $\partial^r + f_1\partial^{r-1} + \cdots + f_r$ figuring in (D):

Theorem 5.2.36 (Pólya [156]). *Suppose*

$$A = g_1\cdots g_r(\partial g_r^{-1})\cdots(\partial g_2^{-1})(\partial g_1^{-1}) \qquad \textit{with } g_j \in (\mathcal{C}_a^{r-j+1})^\times \textit{ for } j = 1,\dots,r.$$

Then $\operatorname{Sol}(\mathrm{D})$ *contains a Markov system* $y_1,\dots,y_r$.

Proof. Let $t_1,\dots,t_r$ range over $\mathbb{R}^{\geqslant a}$ and define $y_1,\dots,y_r \in \mathcal{C}_a^r$ by

$$\begin{aligned}
y_1(t_1) \ &:= \ g_1(t_1),\\
y_2(t_1) \ &:= \ g_1(t_1)\int_a^{t_1} g_2(t_2)\,dt_2,\\
&\ \ \vdots\\
y_r(t_1) \ &:= \ g_1(t_1)\int_a^{t_1} g_2(t_2)\int_a^{t_2}\cdots\int_a^{t_{r-1}} g_r(t_r)\,dt_r\cdots dt_2.
\end{aligned}$$

For $j = 1,\dots,r$ we have $A(y_j) = 0$, and by an induction using Lemma 5.2.6, $\operatorname{wr}(y_1,\dots,y_j) = g_1^j g_2^{j-1}\cdots g_j$. So $y_1,\dots,y_r \in \operatorname{Sol}(\mathrm{D})$ is a Markov system. $\square$

Remark. Suppose Sol (D) contains a Markov system $y_1, \dots, y_r$. If $f_1, \dots, f_r \in \mathcal{C}_a^n$, then $y_1, \dots, y_r \in \mathcal{C}_a^{n+r}$, so $g_j \in (\mathcal{C}_a^{n+r-j+1})^\times$ for $j = 1, \dots, r$ where $g_1, \dots, g_r$ are as in Proposition 5.2.35. Likewise, if $f_1, \dots, f_r \in \mathcal{C}_a^\infty$, then those g_j lie in $(\mathcal{C}_a^\infty)^\times$, and the same with ω in place of ∞.

Corollary 5.2.37. *With* $A = \partial^r + f_1\partial^{r-1} + \dots + f_r : \mathcal{C}_a^r \to \mathcal{C}_a$, Sol (D) *contains a Markov system iff there are* $g_j \in (\mathcal{C}_a^{r-j+1})^\times$ $(j = 1, \dots, r)$ *such that*

$$A \;=\; g_1 \cdots g_r (\partial g_r^{-1}) \cdots (\partial g_2^{-1})(\partial g_1^{-1}).$$

Moreover, Sol (D_b) *contains a Markov system for all* $b > a$ *iff* (D_b) *is disconjugate for all* $b > a$.

Proof. The first equivalence follows from Proposition 5.2.35 and Theorem 5.2.36, and the second equivalence follows from Lemma 5.2.31 and Corollary 5.2.33, $\square$

We say that (D) is **eventually disconjugate** if (D_b) is disconjugate for some $b \geqslant a$. If (D) is disconjugate, then so is (D_b) for all $b \geqslant a$, and likewise with "eventually disconjugate" in place of "disconjugate". If (D) is eventually disconjugate, then no solution of (D) in $\mathcal{C}_a^r$ oscillates. If $r = 1$, then (D) is always disconjugate, since its solutions are the functions $t \mapsto c\exp\Big(-\int_a^t f_1(s)\,ds\Big)$ with $c \in \mathbb{R}$. Returning to the special case where $r = 2$ we have:

Corollary 5.2.38. *Suppose* (L) *has a non-oscillating solution* $y \neq 0$. *Then* (L) *is eventually disconjugate.*

Proof. Here $f_1 = 0$, $f_2 = f$, and f does not generate oscillations by Lemma 5.2.21. Let $y_1, y_2 \in \mathrm{Sol}(f)$ be non-oscillating and $\mathbb{R}$-linearly independent. Then $\mathrm{wr}(y_1, y_2) \in \mathbb{R}^\times$. Take $b \geqslant a$ such that $y_1|_b \in (\mathcal{C}_b)^\times$. Then $y_1|_b, y_2|_b$ is a Markov system. $\square$

Remark. By [81], there is for each $r > 2$ a linear differential equation (D) with only non-oscillating solutions in $\mathcal{C}_a^r$, but which is not eventually disconjugate. (This will not be used later but motivates Corollary 7.4.58 below.)

Passing to germs instead of functions, we now consider a monic operator

$$A \;=\; \partial^r + \phi_1\partial^{r-1} + \dots + \phi_r \in \mathcal{C}^{<\infty}[\partial] \qquad (\phi_1, \dots, \phi_r \in \mathcal{C}^{<\infty}).$$

It gives rise to the $\mathbb{R}$-linear map

$$y \mapsto A(y) = y^{(r)} + \phi_1 y^{(r-1)} + \dots + \phi_r y \;:\; \mathcal{C}^{<\infty} \to \mathcal{C}^{<\infty},$$

whose kernel we denote by $\ker A$.

Lemma 5.2.39. $\dim_{\mathbb{R}} \ker A = r$, *and if* $\theta_1, \dots, \theta_r \in \mathcal{C}^{<\infty}$ *and* $\ker A = \ker B$ *for* $B = \partial^r + \theta_1\partial^{r-1} + \dots + \theta_r \in \mathcal{C}^{<\infty}[\partial]$, *then* $A = B$, *that is* $\phi_i = \theta_i$ *for* $i = 1, \dots, r$.

Proof. Take $a \in \mathbb{R}$ and $f_1, \dots, f_r \in \mathcal{C}_a$ representing $\phi_1, \dots, \phi_r$. This gives an equation (D). Let $y_1, \dots, y_r$ be a basis of the $\mathbb{R}$-linear space Sol (D). Then the germs of $y_1, \dots, y_r$ lie in $\mathcal{C}^{<\infty}$, and denoting these germs also by $y_1, \dots, y_r$ one verifies easily that then $y_1, \dots, y_r$ is a basis of $\ker A$. The second part of the lemma follows in a similar way from Corollary 5.2.5. $\square$

We call A as above **disconjugate** if for some $a \in \mathbb{R}$ the germs $\phi_1, \dots, \phi_r$ have representatives $f_1, \dots, f_r$ in $\mathcal{C}_a$ such that the linear differential equation (D) on $\mathbb{R}^{\geqslant a}$ is disconjugate.

Lemma 5.2.40. *For A as above, the following are equivalent:*

(i) *A is disconjugate;*

(ii) *$A = g_1 \cdots g_r(\partial g_r^{-1}) \cdots (\partial g_2^{-1})(\partial g_1^{-1})$ for some $g_1, \dots, g_r \in (\mathcal{C}^{<\infty})^\times$;*

(iii) *$A = (\partial - h_r) \cdots (\partial - h_1)$ for some $h_1, \dots, h_r \in \mathcal{C}^{<\infty}$.*

Thus if monic $A_1, A_2 \in \mathcal{C}^{<\infty}[\partial]$ of order $\geqslant 1$ are disconjugate, then so is A_1A_2.

Proof. Assume (i). Then Corollary 5.2.33 yields $a \in \mathbb{R}$, representatives $f_1, \dots, f_r \in \mathcal{C}_a$ of $\phi_1, \dots, \phi_r$, and a Markov system $y_1, \dots, y_r \in \operatorname{Sol}(\mathrm{D})$. Let $g_1, \dots, g_r$ be as in Proposition 5.2.35. Then for $b \geqslant a$ with $f_1|_b, \dots, f_r|_b \in \mathcal{C}_a^n$ we have $g_j|_b \in (\mathcal{C}_b^{n+r-j+1})^\times$ for $j = 1, \dots, r$. So the germs of $g_1, \dots, g_r$ are in $(\mathcal{C}^{<\infty})^\times$, and denoting these germs also by $g_1, \dots, g_r$ gives $A = g_1 \cdots g_r(\partial g_r^{-1}) \cdots (\partial g_2^{-1})(\partial g_1^{-1})$ by Proposition 5.2.35 and Lemma 5.2.39. We have now shown (i) $\Rightarrow$ (ii). For the converse, we reverse the argument using Theorem 5.2.36. The equivalence (ii) $\Leftrightarrow$ (iii) is shown just like Lemma 1.1.3, using also that $f \mapsto f^\dagger \colon (\mathcal{C}^{<\infty})^\times \to \mathcal{C}^{<\infty}$ is surjective. $\square$

Remark. Lemma 5.2.40 goes through for monic $A \in \mathcal{C}^\infty[\partial]$ of order r, with $\mathcal{C}^\infty$ in place of $\mathcal{C}^{<\infty}$ everywhere. Likewise for monic $A \in \mathcal{C}^\omega[\partial]$ of order r, with $\mathcal{C}^\omega$ in place of $\mathcal{C}^{<\infty}$ everywhere.

A **principal system** of solutions of (D) is a tuple $y_1, \dots, y_r$ in $\operatorname{Sol}(\mathrm{D})$ such that

(1) $y_1(t), \dots, y_r(t) > 0$ eventually, and

(2) $y_1 \prec \cdots \prec y_r$ (in $\mathcal{C}$).

Note that then $y_1, \dots, y_r$ are $\mathbb{R}$-linearly independent, and $z_1, \dots, z_r \in \mathcal{C}_a^r$ is a principal system of solutions of (D) iff there are $c_{ij} \in \mathbb{R}$ ($1 \leqslant j \leqslant i \leqslant r$) such that

$$z_i \;=\; c_{ii}y_i + c_{i,i-1}y_{i-1} + \cdots + c_{i1}y_1 \quad \text{and} \quad c_{ii} > 0.$$

The next result generalizes Lemma 5.2.29. It seems slightly stronger than a similar result by Hartman [92] and Levin [128]:

Proposition 5.2.41. *Suppose* (D) *has no oscillating solutions. Then it has a principal system of solutions.*

Proof. Let $y, z \in \operatorname{Sol}(\mathrm{D})$, $y(t), z(t) > 0$ eventually. Claim: $\lim\limits_{t\to+\infty} y(t)/z(t)$ exists in $[0, +\infty]$. Suppose this limit doesn't exist. Then we have $c \in \mathbb{R}^>$ such that

$$\liminf_{t\to+\infty} y(t)/z(t) \;< c\; < \limsup_{t\to+\infty} y(t)/z(t),$$

so $y(t)/z(t) = c$ for arbitrarily large t, but then $y - cz = 0$, a contradiction. In particular, for such y, z we have either $y \prec z$, or $y \sim cz$ for some $c \in \mathbb{R}^>$, or $y \succ z$. If $y_1, \dots, y_n \in \operatorname{Sol}(\mathrm{D})^{\neq}$ and $y_1 \prec \cdots \prec y_n$, then $y_1, \dots, y_n$ are $\mathbb{R}$-linearly independent, so $n \leqslant r$, and for any nonzero $z \in \mathbb{R}y_1 + \cdots + \mathbb{R}y_n$ we have $z \sim cy_j$ for some $j \in \{1, \dots, n\}$ and $c \in \mathbb{R}^\times$. Now take such $y_1, \dots, y_n$ with maximal n, so $n \geqslant 1$. We claim that then $\operatorname{Sol}(\mathrm{D}) = \mathbb{R}y_1 + \cdots + \mathbb{R}y_n$ (so $n = r$). Let $z \in \operatorname{Sol}(\mathrm{D})^{\neq}$. We cannot have $z \prec y_1$, nor $y_j \prec z \prec y_{j+1}$ with $1 \leqslant j \leqslant n-1$, nor $z \succ y_n$; hence $z \sim cy_j$ where $1 \leqslant j \leqslant n$ and $c \in \mathbb{R}^\times$. Then $z - cy_j \prec y_j$. If $z \neq cy_j$, we take $z - cy_j$ as our new z and obtain likewise $z - cy_j \sim dy_i$ with $1 \leqslant i < j$ and $d \in \mathbb{R}^\times$. Continuing this way leads in a finite number of steps to $z \in \mathbb{R}y_1 + \cdots + \mathbb{R}y_n$. $\square$

The next result is due to Trench [200]. We do not give a proof, since we shall establish in Section 7.4 a version of it in the Hardy field context; see also Proposition 2.5.39.

Proposition 5.2.42. *Suppose* $\operatorname{Sol}(\mathrm{D})$ *contains a Markov system. Then there are* $g_j \in (\mathcal{C}_a^{r-j+1})^\times$ $(j = 1, \dots, r)$ *such that for* $A = \partial^r + f_1\partial^{r-1} + \cdots + f_r : \mathcal{C}_a^r \to \mathcal{C}_a$*,*

$$A \;=\; g_1 \cdots g_r(\partial g_r^{-1}) \cdots (\partial g_2^{-1})(\partial g_1^{-1}) \quad \textit{and} \quad \int_a^\infty |g_j(s)|\, ds = \infty \textit{ for } j = 2, \dots, r.$$

Moreover, such $g_1, \dots, g_r$ *are unique up to multiplication by nonzero constants.*

An application of l'Hôpital's Rule shows that for $g_1, \dots, g_r$ as in Proposition 5.2.42 the tuple $y_1, \dots, y_r$ in the proof of Theorem 5.2.36 is a principal system of solutions of (D).

Lyapunov exponents (*)**.** In this subsection f, g, h range over $\mathcal{C}[\mathrm{i}]$. Consider the downward closed subset

$$\Lambda = \Lambda(f) := \big\{\lambda \in \mathbb{R} : f\,\mathrm{e}^{\lambda x} \preccurlyeq 1\big\}$$

of $\mathbb{R}$. If $\lambda < \mu \in \Lambda$, then $f\,\mathrm{e}^{\lambda x} \prec 1$. Also

$$\Lambda(f) \;=\; \Lambda(\overline{f}) \;=\; \Lambda(|f|), \quad f \preccurlyeq g \;\Rightarrow\; \Lambda(f) \supseteq \Lambda(g).$$

Notation. Set $\mathbb{R}_{\pm\infty} := \mathbb{R} \cup \{-\infty, +\infty\}$. Then for $S \subseteq \mathbb{R}$ we have $\sup S \in \mathbb{R}_{\pm\infty}$ with $\sup \emptyset := -\infty$.

The **Lyapunov exponent** of f is $\lambda(f) := \sup \Lambda(f) \in \mathbb{R}_{\pm\infty}$. (See [45, §3.12].) Note:

$$\lambda(f) = +\infty \quad\Longleftrightarrow\quad \Lambda(f) = \mathbb{R} \quad\Longleftrightarrow\quad f \prec \mathrm{e}^{\lambda x} \text{ for all } \lambda \in \mathbb{R},$$

and

$$\lambda(f) \;=\; \lambda(\overline{f}) \;=\; \lambda(|f|), \quad f \preccurlyeq g \;\Rightarrow\; \lambda(f) \;\geqslant\; \lambda(g), \quad f \asymp g \;\Rightarrow\; \lambda(f) \;=\; \lambda(g).$$

If $\lambda = \lambda(f) \in \mathbb{R}$, then for each $\varepsilon \in \mathbb{R}^>$ we have $f\,\mathrm{e}^{(\lambda-\varepsilon)x} \prec 1$ and $f\,\mathrm{e}^{(\lambda+\varepsilon)x} \not\preccurlyeq 1$. One also verifies easily that for $f \in \mathcal{C}[\mathrm{i}]^\times$,

$$\lambda(f) \;=\; -\limsup_{t\to+\infty} \frac{\log|f(t)|}{t}. \tag{5.2.2}$$

If $f = \mathrm{e}^g$, then $\lambda(f) = -\limsup\limits_{t\to+\infty} \operatorname{Re} g(t)/t$. Thus $\lambda(c\,\mathrm{e}^{\mathrm{i}\phi}) = 0$ for $c \in \mathbb{C}^\times$, $\phi \in \mathcal{C}$.

Lemma 5.2.43. *Assume* $\lambda(f), \lambda(g) > -\infty$*. Then:*

(i) $\lambda(f+g) \geqslant \min\big\{\lambda(f), \lambda(g)\big\}$*, with equality if* $\lambda(f) \neq \lambda(g)$*;*
(ii) $\lambda(fg) \geqslant \lambda(f) + \lambda(g)$*;*
(iii) $\lambda(f^m) = m\lambda(f)$ *for all* m*.*

Proof. For (i) suppose $\lambda(f) \leqslant \lambda(g)$. Then for each $\lambda \in \Lambda(f)$ and $\varepsilon \in \mathbb{R}^>$ we have $(f+g)\,\mathrm{e}^{(\lambda-\varepsilon)x} \preccurlyeq 1$ and so $\lambda - \varepsilon \in \Lambda(f+g)$. This shows $\lambda(f+g) \geqslant \lambda(f)$, and $\lambda(f+g) = \lambda(f)$ if $\lambda(f) < \lambda(g)$ then follows using $f = (f+g) - g$. Parts (ii) and (iii) follow in a similar way. □

By Lemma 5.2.43(ii), if $f \in \mathcal{C}[\mathrm{i}]^\times$ and $\lambda(f), \lambda(f^{-1}) \in \mathbb{R}$, then $\lambda(f^{-1}) \leqslant -\lambda(f)$.

Example. If $f = \mathrm{e}^g$ and $\operatorname{Re} g - cx \prec x$, $c \in \mathbb{R}$, then $\lambda(f) = c$, $\lambda(f^{-1}) = -c$.

Set

$$\mathcal{C}[\mathrm{i}]^{\preceq} \;:=\; \big\{f : \lambda(f) > -\infty\big\} \;=\; \big\{f : f \preccurlyeq \mathrm{e}^{nx} \text{ for some } n\big\}.$$

Then $\mathcal{C}[\mathrm{i}]^{\preceq}$ is a subalgebra of the $\mathbb{C}$-algebra $\mathcal{C}[\mathrm{i}]$ and

$$\mathcal{C}[\mathrm{i}]^{\prec\!\!\prec} := \big\{f : \lambda(f) = +\infty\big\} = \big\{f : f \preccurlyeq \mathrm{e}^{-nx} \text{ for all } n\big\}$$

is an ideal of $\mathcal{C}[\mathrm{i}]^{\preceq}$. The group of units of $\mathcal{C}[\mathrm{i}]^{\preceq}$ is

$$\mathcal{C}[\mathrm{i}]^{\asymp} := \big\{f \in \mathcal{C}[\mathrm{i}]^{\times} : \lambda(f), \lambda(f^{-1}) \in \mathbb{R}\big\} = \big\{f : \mathrm{e}^{-nx} \preccurlyeq f \preccurlyeq \mathrm{e}^{nx} \text{ for some } n\big\}.$$

Lemma 5.2.44. *Suppose $f \in \mathcal{C}^1[\mathrm{i}]$. If $\lambda(f') \leqslant 0$, then $\lambda(f') \leqslant \lambda(f)$. If $\lambda(f') > 0$, then $c := \lim_{s\to\infty} f(s) \in \mathbb{C}$ exists and $\lambda(f') \leqslant \lambda(f-c)$.*

Proof. Let $\lambda \in \Lambda(f')$. Take $a \in \mathbb{R}$ and a representative of f in $\mathcal{C}^1_a[\mathrm{i}]$, also denoted by f, as well $C \in \mathbb{R}^>$, such that $|f'(t)| \leqslant C\,\mathrm{e}^{-\lambda t}$ for $t \geqslant a$. If $\lambda < 0$, then for $t \geqslant a$:

$$\begin{aligned}|f(t)| - |f(a)| \ \leqslant\ |f(t) - f(a)| \ =\ \left|\int_a^t f'(s)\,ds\right| \ \leqslant\ \int_a^t |f'(s)|\,ds \ \leqslant& \\ C\int_a^t \mathrm{e}^{-\lambda s}\,ds \ =\ -\frac{C}{\lambda}(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda a}),&\end{aligned}$$

hence $f \preccurlyeq \mathrm{e}^{-\lambda x}$. This yields $\lambda(f') \leqslant \lambda(f)$ if $\lambda(f') \leqslant 0$. Suppose $\lambda > 0$. Then for $a \leqslant s \leqslant t$:

$$|f(t) - f(s)| \ =\ \left|\int_s^t f'(u)\,du\right| \ \leqslant\ \int_s^t |f'(u)|\,du \ \leqslant\ -\frac{C}{\lambda}(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda s}).$$

Therefore $c := \lim_{s\to\infty} f(s)$ exists and $|c - f(s)| \leqslant \frac{C}{\lambda}\,\mathrm{e}^{-\lambda s}$ for $s \geqslant a$. Hence $f - c \preccurlyeq \mathrm{e}^{-\lambda x}$, so $\lambda \in \Lambda(f-c)$. This yields $\lambda(f') \leqslant \lambda(f-c)$. □

Let $y = (y_1, \dots, y_n) \in \mathcal{C}[\mathrm{i}]^n$, $n \geqslant 1$. Put $\lambda(y) := \min\big\{\lambda(y_1), \dots, \lambda(y_n)\big\}$. Then the function $\lambda\colon \mathcal{C}[\mathrm{i}]^n \to \mathbb{R}_{\pm\infty}$ on the product ring $\mathcal{C}[\mathrm{i}]^n$ also satisfies (i)–(iii) in Lemma 5.2.43 with f, g replaced by $y, z \in \mathcal{C}[\mathrm{i}]^n$ with $\lambda(y), \lambda(z) > -\infty$. Thus:

Corollary 5.2.45. *If $m \geqslant 2$, $y_1, \dots, y_m \in \mathcal{C}[\mathrm{i}]^n$ are $\mathbb{C}$-linearly dependent, and $\lambda(y_1), \dots, \lambda(y_m) > -\infty$, then $\lambda(y_i) = \lambda(y_j)$ for some $i \neq j$.*

We define $y \preccurlyeq g :\Leftrightarrow y_1, \dots, y_n \preccurlyeq g$ $\big(\Rightarrow \lambda(y) \geqslant \lambda(g)\big)$. Note that $\lambda(y) \in \mathbb{R}$ iff $y \preccurlyeq \mathrm{e}^{mx}$ and $y \not\preccurlyeq \mathrm{e}^{-mx}$ for some m.

Let $\|\cdot\|$ be a norm on the $\mathbb{C}$-linear space $\mathbb{C}^n$, and accordingly, let $\|y\|$ denote the germ of $t \mapsto \big\|\big(y_1(t), \dots, y_n(t)\big)\big\|$, so $\|y\| \in \mathcal{C}$.

Corollary 5.2.46. *$y \preccurlyeq \|y\|$, $y \preccurlyeq g \Leftrightarrow \|y\| \preccurlyeq g$, and $\lambda(\|y\|) = \lambda(y)$.*

Proof. Any two norms on $\mathbb{C}^n$ are equivalent, so we may arrange $\|\cdot\| = \|\cdot\|_1$. Then $y \preccurlyeq g \Rightarrow \|y\| = |y_1| + \dots + |y_n| \preccurlyeq g$. From $|y_j| \leqslant \|y\|$ we get $y_j \preccurlyeq \|y\|$ for $j = 1, \dots, n$ and thus $y \preccurlyeq \|y\|$. Thus $\|y\| \preccurlyeq g \Rightarrow y \preccurlyeq g$; also $\lambda(\|y\|) \leqslant \lambda(y)$. Finally, Lemma 5.2.43(i) and $\lambda(|f|) = \lambda(f)$ yield $\lambda(\|y\|) \geqslant \lambda(y)$. □

In particular, $\lambda(y) = \lambda(\|y\|) = \lambda(\|y\|_2) = \lambda\big((\operatorname{Re} y_1, \dots, \operatorname{Re} y_n, \operatorname{Im} y_1, \dots, \operatorname{Im} y_n)\big)$.

Remarks on matrix differential equations (*)**.** In this subsection N is an $n \times n$ matrix with entries in $\mathcal{C}[\mathrm{i}]$, $n \geqslant 1$. We consider tuples $y \in \mathcal{C}^1[\mathrm{i}]^n$ as column vectors $y = (y_1, \dots, y_n)^{\mathrm{t}}$ with entries y_j in $\mathcal{C}^1[\mathrm{i}]$. Later in this subsection and in Section 7.4 we shall tacitly use the following:

1. The $\mathbb{C}$-linear space of $y \in \mathcal{C}^1[\mathrm{i}]^n$ such that $y' = Ny$ has dimension n.
2. If all entries of N are in $\mathcal{C}^{<\infty}[\mathrm{i}]$ and $y \in \mathcal{C}^1[\mathrm{i}]^n$, $y' = Ny$, then $y \in \mathcal{C}^{<\infty}[\mathrm{i}]^n$.

Classical existence and uniqueness results on matrix linear differential equations give (1), and induction on the degree of smoothness of y yields (2).

Call a matrix (f_{ij}) over $\mathcal{C}[\mathrm{i}]$ **bounded** if $f_{ij} \preccurlyeq 1$ for all i, j. Similarly with $\mathcal{C}_a[\mathrm{i}]$ ($a \in \mathbb{R}$) in place of $\mathcal{C}[\mathrm{i}]$. In the proof of Corollary 7.4.28 we shall use the following (cf. [45, §3.13], [111, §A.3.11]):

Lemma 5.2.47 (Lyapunov [135], Perron [152])**.** *Suppose N is bounded. If $y \in \mathcal{C}^1[\mathrm{i}]^n$, $y' = Ny$, and $y \neq 0$, then $\lambda(y) \in \mathbb{R}$.*

Proof. We have $N = A + B\mathrm{i}$ where A, B are $n \times n$ matrices over $\mathcal{C}$. Consider the bounded $2n \times 2n$ matrix $M := \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$ over $\mathcal{C}$. For $y = (y_1, \dots, y_n)^{\mathrm{t}} \in \mathcal{C}^1[\mathrm{i}]^n$, set $v := (\operatorname{Re} y_1, \dots, \operatorname{Re} y_n, \operatorname{Im} y_1, \dots, \operatorname{Im} y_n)^{\mathrm{t}} \in (\mathcal{C}^1)^{2n}$; then $y' = Ny$ iff $v' = Mv$. Now assume $y' = Ny$ and $y \neq 0$. Then by the remark after Corollary 5.2.46 we may replace N, n, y by M, $2n$, v to arrange that the entries of N are in $\mathcal{C}$ and $y \in (\mathcal{C}^1)^n$.

Let $\lambda, \mu \in \mathbb{R}$ and consider $z := \mathrm{e}^{-\lambda x}\, y$. Then $z' = (N - \lambda I_n)z$ where I_n is the $n \times n$ identity matrix over $\mathcal{C}$, and thus

$$\langle z, z\rangle' \ =\ 2\langle z, z'\rangle \ =\ 2\big\langle z, \big(N - (\lambda - \tfrac{1}{2})I_n\big)z\big\rangle - \langle z, z\rangle.$$

The lemma below gives λ such that $\big\langle z, \big(N - (\lambda - \frac{1}{2})I_n\big)z\big\rangle \leqslant 0$, so $\langle z, z\rangle \in \mathcal{C}^1$ and $\langle z, z\rangle' \leqslant 0$, and thus $\langle z, z\rangle \preccurlyeq 1$. Corollary 5.2.46 yields $z \preccurlyeq 1$, so $y \preccurlyeq \mathrm{e}^{\lambda x}$. Likewise, set $w := \mathrm{e}^{\mu x}\, y$; then $w' = (N + \mu I_n)w$, and apply Lemma 5.2.48 to a representative F of $-N$ to get μ with $\langle w, w\rangle' \geqslant \langle w, w\rangle$, so $\langle w, w\rangle \succcurlyeq \mathrm{e}^x$ by Lemma 5.2.17, hence $w \not\preccurlyeq 1$, and thus $y \not\preccurlyeq \mathrm{e}^{-\mu x}$. So $\lambda(y) \in \mathbb{R}$. □

In the next lemma $F = (f_{ij})$ is an $n \times n$ matrix over $\mathcal{C}_a$, $a \in \mathbb{R}$. For $t \in \mathbb{R}^{\geqslant a}$ this yields the $n \times n$ matrix $F(t) := \big(f_{ij}(t)\big)$ over $\mathbb{R}$. Let I_n also be the $n \times n$ identity matrix over $\mathbb{R}$.

Lemma 5.2.48. *Suppose F is bounded. Then there exists $\mu \in \mathbb{R}^>$ such that for all real $\lambda \geqslant \mu$, $t \geqslant a$, and $z \in \mathbb{R}^n$: $\big\langle z, \big(F(t) - \lambda I_n\big)z\big\rangle \leqslant 0$.*

Proof. Put $G := \frac{1}{2}(F + F^{\mathrm{t}})$, a symmetric bounded $n \times n$ matrix over $\mathcal{C}_a$ such that $\big\langle z, F(t)z\big\rangle = \big\langle z, G(t)z\big\rangle$ for $t \geqslant a$ and $z \in \mathbb{R}^n$, and replace F by G to arrange that F is symmetric. Let

$$P(Y) \ :=\ \det(YI_n - F) \ =\ Y^n + P_1 Y^{n-1} + \dots + P_n \in \mathcal{C}_a[Y] \qquad (P_1, \dots, P_n \in \mathcal{C}_a),$$

and for $t \in \mathbb{R}^{\geqslant a}$ put

$$P(t, Y) \ :=\ Y^n + P_1(t) Y^{n-1} + \dots + P_n(t) \in \mathbb{R}[Y],$$

so for each $\lambda \in \mathbb{R}$, $P(t, Y + \lambda)$ is the characteristic polynomial of the symmetric $n \times n$ matrix $F(t) - \lambda I_n$ over $\mathbb{R}$. Now $P_1, \dots, P_n \preccurlyeq 1$ since F is bounded, so [ADH, 3.5.2] yields $\mu \in \mathbb{R}^>$ such that for all $t \in \mathbb{R}^{\geqslant a}$, all zeros of $P(t, Y)$ in $\mathbb{R}$ are in $[-\mu, \mu]$. Let $\lambda \geqslant \mu$. Then for $t \geqslant a$, no real zero of $P(t, Y + \lambda)$, and thus no eigenvalue of $F(t) - \lambda I_n$, is positive. Hence $\big\langle z, (F(t) - \lambda I_n)z\big\rangle \leqslant 0$ for all $z \in \mathbb{R}^n$. □

Let $V := \big\{y \in \mathcal{C}^1[\mathrm{i}]^n : y' = Ny\big\}$, an n-dimensional $\mathbb{C}$-linear subspace of $\mathcal{C}^1[\mathrm{i}]$. Suppose N is bounded. Then $S := \lambda(V^{\neq}) \subseteq \mathbb{R}$ by Lemma 5.2.47, and S, called the **Lyapunov spectrum** of $y' = Ny$, has at most n elements by Corollary 5.2.45. According to [ADH, 2.3] the surjective map

$$y \mapsto \lambda(y) \colon V \to S_\infty := S \cup \{\infty\}$$

makes V a valued vector space over $\mathbb{C}$. Thus by [ADH, remark before 2.3.10]:

Corollary 5.2.49 (Lyapunov [135]). *If N is bounded, then V has a basis $y_1,\dots,y_n$ such that for all $c_1,\dots,c_n\in\mathbb{C}$, not all zero, and $y=c_1y_1+\cdots+c_ny_n$,*

$$\lambda(y) \;=\; \min\bigl\{\lambda(y_j) : c_j\neq 0\bigr\}.$$

Whether or not N is bounded, a **Lyapunov fundamental system of solutions** of $y'=Ny$ is a basis $y_1,\dots,y_n$ of V as in the corollary above. (In [45, §3.14] this is called a *normal* fundamental system of solutions of $y'=Ny$.) A **Lyapunov fundamental matrix** for $y'=Ny$ is an $n\times n$ matrix with entries in $\mathcal{C}^1[\mathrm{i}]$ whose columns form a Lyapunov fundamental system of solutions of $y'=Ny$.

Lemma 5.2.47 also gives:

Corollary 5.2.50. *Let $f_1,\dots,f_n\in\mathcal{C}[\mathrm{i}]$ be such that $f_1,\dots,f_n\preccurlyeq 1$. Then there are $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ $(1\leqslant m\leqslant n)$ such that for all $y\in\mathcal{C}^n[\mathrm{i}]^{\neq}$ such that*

$$y^{(n)}+f_1y^{(n-1)}+\cdots+f_ny \;=\; 0$$

we have $\lambda(y,y',\dots,y^{(n-1)})\in\{\lambda_1,\dots,\lambda_m\}$.

5.3. Hardy Fields

Here we introduce Hardy fields and review some classical extension theorems for Hardy fields.

Hardy fields. A *Hardy field* is a subfield of $\mathcal{C}^{<\infty}$ that is closed under the derivation of $\mathcal{C}^{<\infty}$; see also [ADH, 9.1]. Let H be a Hardy field. Then H is considered as an ordered valued differential field in the obvious way; see Section 5.1 for the ordering and valuation on H. The field of constants of H is $\mathbb{R}\cap H$. Hardy fields are pre-H-fields, and H-fields if they contain $\mathbb{R}$; see [ADH, 9.1.9(i), (iii)]. As in Section 5.1 we equip the differential subfield $H[\mathrm{i}]$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ with the unique valuation ring lying over that of H. Then $H[\mathrm{i}]$ is a pre-d-valued field of H-type with small derivation and constant field $\mathbb{C}\cap H[\mathrm{i}]$; if $H\supseteq\mathbb{R}$, then $H[\mathrm{i}]$ is d-valued with constant field $\mathbb{C}$.

We also consider variants: a $\mathcal{C}^{\infty}$*-Hardy field* is a Hardy field $H\subseteq\mathcal{C}^{\infty}$, and a $\mathcal{C}^{\omega}$*-Hardy field* (also called an *analytic Hardy field*) is a Hardy field $H\subseteq\mathcal{C}^{\omega}$. Most Hardy fields arising in practice are actually $\mathcal{C}^{\omega}$-Hardy fields. Boshernitzan [32] (with details worked out in [77]) first suggested a Hardy field $H\not\subseteq\mathcal{C}^{\infty}$, and [79, Theorem 1] shows that each Hardy field with a largest comparability class extends to a Hardy field $H\not\subseteq\mathcal{C}^{\infty}$. Rolin, Speissegger, Wilkie [166] construct o-minimal expansions $\widetilde{\mathbb{R}}$ of the ordered field of real numbers such that $H\subseteq\mathcal{C}^{\infty}$ and $H\not\subseteq\mathcal{C}^{\omega}$ for the Hardy field H consisting of the germs of functions $\mathbb{R}\to\mathbb{R}$ that are definable in $\widetilde{\mathbb{R}}$. Le Gal and Rolin [124] construct such expansions such that $H\not\subseteq\mathcal{C}^{\infty}$ for the corresponding Hardy field H.

Hardian germs. Let $y\in\mathcal{G}$. Following [190] we call y **hardian** if it lies in a Hardy field (and thus $y\in\mathcal{C}^{<\infty}$). We also say that y is $\mathcal{C}^{\infty}$**-hardian** if y lies in a $\mathcal{C}^{\infty}$-Hardy field, equivalently, $y\in\mathcal{C}^{\infty}$ and y is hardian; likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$. Let H be a Hardy field. Call $y\in\mathcal{G}$ H**-hardian** (or **hardian over** H) if y lies in a Hardy field extension of H. (Thus y is hardian iff y is $\mathbb{Q}$-hardian.) If H is a $\mathcal{C}^{\infty}$-Hardy field and $y\in\mathcal{C}^{\infty}$ is hardian over H, then y generates a $\mathcal{C}^{\infty}$-Hardy field extension $H\langle y\rangle$ of H; likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.

Maximal and perfect Hardy fields. Let H be a Hardy field. Call H *maximal* if no Hardy field properly contains H. Following Boshernitzan [33] we denote by $\mathrm{E}(H)$ the intersection of all maximal Hardy fields containing H; thus $\mathrm{E}(H)$ is a Hardy field extension of H, and a maximal Hardy field contains H iff it contains $\mathrm{E}(H)$, so $\mathrm{E}(\mathrm{E}(H)) = \mathrm{E}(H)$. If H^* is a Hardy field extension of H, then $\mathrm{E}(H) \subseteq \mathrm{E}(H^*)$; hence if H^* is a Hardy field with $H \subseteq H^* \subseteq \mathrm{E}(H)$, then $\mathrm{E}(H^*) = \mathrm{E}(H)$. Note that $\mathrm{E}(H)$ consists of the $f \in \mathcal{G}$ that are hardian over each Hardy field $E \supseteq H$. Hence $\mathrm{E}(\mathbb{Q})$ consists of the germs in $\mathcal{G}$ that are hardian over each Hardy field. As in [33] we also say that H is **perfect** if $\mathrm{E}(H) = H$. (This terminology is slightly unfortunate, since Hardy fields, being of characteristic zero, are perfect as fields.) Thus $\mathrm{E}(H)$ is the smallest perfect Hardy field extension of H. Maximal Hardy fields are perfect.

Differentially maximal Hardy fields. Let H be a Hardy field. We now define differentially-algebraic variants of the above: call H **differentially maximal**, or d**-maximal** for short, if H has no proper d-algebraic Hardy field extension. Every maximal Hardy field is d-maximal, so each Hardy field is contained in a d-maximal one; in fact, by Zorn, each Hardy field H has a d-maximal Hardy field extension which is d-algebraic over H. Let $\mathrm{D}(H)$ be the intersection of all d-maximal Hardy fields containing H. Then $\mathrm{D}(H)$ is a d-algebraic Hardy field extension of H with $\mathrm{D}(H) \subseteq \mathrm{E}(H)$. By the next lemma, $\mathrm{D}(H) = \mathrm{E}(H)$ iff $\mathrm{E}(H)$ is d-algebraic over H:

Lemma 5.3.1. $\mathrm{D}(H) = \big\{ f \in \mathrm{E}(H) : f$ *is* d*-algebraic over* $H \big\}$.

Proof. We only need to show the inclusion "$\supseteq$". For this let $f \in \mathrm{E}(H)$ be d-algebraic over H, and let E be a d-maximal Hardy field extension of H; we need to show $f \in E$. To see this extend E to a maximal Hardy field M; then $f \in M$, hence f generates a Hardy field extension $E\langle f \rangle$ of E. Since f is d-algebraic over H and thus over E, this yields $f \in E$ by d-maximality of E, as required. $\square$

A d-maximal Hardy field contains H iff it contains $\mathrm{D}(H)$, hence $\mathrm{D}(\mathrm{D}(H)) = \mathrm{D}(H)$. If H^* is a Hardy field extension of H, then $\mathrm{D}(H) \subseteq \mathrm{D}(H^*)$; hence for each Hardy field H^* with $H \subseteq H^* \subseteq \mathrm{D}(H)$ we have $\mathrm{D}(H^*) = \mathrm{D}(H)$. We say that H is d**-perfect** if $\mathrm{D}(H) = H$. Thus $\mathrm{D}(H)$ is the smallest d-perfect Hardy field extension of H. Every perfect Hardy field is d-perfect, as is every d-maximal Hardy field. The following diagram summarizes the various implications among these properties of Hardy fields:

$$\begin{tikzcd} \text{maximal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{perfect} \arrow[d, Rightarrow] \\ \text{d-maximal} \arrow[r, Rightarrow] & \text{d-perfect} \end{tikzcd}$$

We call $\mathrm{D}(H)$ the d**-perfect hull** of H, and $\mathrm{E}(H)$ the **perfect hull** of H. It seems that the following question asked by Boshernitzan [33, p. 144] is still open:

Question. Is $\mathrm{E}(H)$ d-algebraic over H, in other words, is $\mathrm{D}(H) = \mathrm{E}(H)$?

Boshernitzan gave support for a positive answer: Lemma 5.4.1, Corollary 5.4.15, and Theorem 5.4.20 below. Our Theorems 5.6.11 and 7.5.32 (in combination with Theorem 1.4.1) can be seen as further support.

Variants of the perfect hull. Let H be a $\mathcal{C}^r$-Hardy field where $r \in \{\infty, \omega\}$. We say that H is $\mathcal{C}^r$**-maximal** if no $\mathcal{C}^r$-Hardy field properly contains it. By Zorn, H has a $\mathcal{C}^r$-maximal extension. In analogy with $\mathrm{E}(H)$, define the $\mathcal{C}^r$**-perfect hull** $\mathrm{E}^r(H)$ of H to be the intersection of all $\mathcal{C}^r$-maximal Hardy fields containing H. We say that H is $\mathcal{C}^r$**-perfect** if $\mathrm{E}^r(H) = H$. The penultimate subsection goes through with *Hardy field*, *maximal*, *hardian*, $\mathrm{E}(\,\cdot\,)$, and *perfect* replaced by $\mathcal{C}^r$*-Hardy field*, $\mathcal{C}^r$*-maximal*, $\mathcal{C}^r$*-hardian*, $\mathrm{E}^r(\,\cdot\,)$, and $\mathcal{C}^r$*-perfect*, respectively. (Corollary 7.2.13 shows that no analogue of $\mathrm{D}(H)$ is needed for the $\mathcal{C}^r$-category.)

Some basic extension theorems. We summarize some well-known extension results for Hardy fields:

Proposition 5.3.2. *Any Hardy field H has the following Hardy field extensions:*

(i) $H(\mathbb{R})$*, the subfield of $\mathcal{C}^{<\infty}$ generated by H and $\mathbb{R}$;*
(ii) H^{rc}*, the real closure of H as defined in Proposition 5.1.4;*
(iii) $H(\mathrm{e}^f)$ *for any $f \in H$;*
(iv) $H(f)$ *for any $f \in \mathcal{C}^1$ with $f' \in H$;*
(v) $H(\log f)$ *for any $f \in H^{>}$.*

If H is contained in $\mathcal{C}^{\infty}$, then so are the Hardy fields in (i), (ii), (iii), (iv), (v)*; likewise with $\mathcal{C}^{\omega}$ instead of $\mathcal{C}^{\infty}$.*

Note that (v) is a special case of (iv), since $(\log f)' = f^\dagger \in H$ for $f \in H^{>}$. Another special case of (iv) is that $H(x)$ is a Hardy field. A consequence of the proposition is that any Hardy field H has a smallest real closed Hardy field extension L with $\mathbb{R} \subseteq L$ such that for all $f \in L$ we have $\mathrm{e}^f \in L$ and $g' = f$ for some $g \in L$. Note that then L is a Liouville closed H-field as defined in [ADH, 10.6]. Let H be a Hardy field with $H \supseteq \mathbb{R}$. As in [6] and [ADH, p. 460] we then denote the above L by $\mathrm{Li}(H)$; so $\mathrm{Li}(H)$ is the smallest Liouville closed Hardy field containing H, called the *Hardy-Liouville closure* of H in [12]. We have $\mathrm{Li}(H) \subseteq \mathrm{D}(H)$, hence if H is d-perfect, then H is a Liouville closed H-field. Moreover, if $H \subseteq \mathcal{C}^{\infty}$ then $\mathrm{Li}(H) \subseteq \mathcal{C}^{\infty}$, and similarly with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.

The next more general result in Rosenlicht [171] is attributed there to M. Singer:

Proposition 5.3.3. *Let H be a Hardy field and $p(Y), q(Y) \in H[Y]$, $y \in \mathcal{C}^1$, such that $y'q(y) = p(y)$ with $q(y) \in \mathcal{C}^{\times}$. Then y generates a Hardy field $H(y)$ over H.*

Note that for H, p, q, y as in the proposition we have $y \in \mathrm{D}(H)$.

Compositional conjugation of differentiable germs. Let $\ell \in \mathcal{C}^1$, $\ell'(t) > 0$ eventually (so ℓ is eventually strictly increasing) and $\ell(t) \to +\infty$ as $t \to +\infty$. Then $\phi := \ell' \in \mathcal{C}^{\times}$, and the compositional inverse $\ell^{\mathrm{inv}} \in \mathcal{C}^1$ of ℓ satisfies

$$\ell^{\mathrm{inv}} > \mathbb{R}, \qquad (\ell^{\mathrm{inv}})' \;=\; (1/\phi) \circ \ell^{\mathrm{inv}} \in \mathcal{C}.$$

The $\mathbb{C}$-algebra automorphism $f \mapsto f^\circ := f \circ \ell^{\mathrm{inv}}$ of $\mathcal{C}[\mathrm{i}]$ (with inverse $g \mapsto g \circ \ell$) maps $\mathcal{C}^1[\mathrm{i}]$ onto itself and satisfies for $f \in \mathcal{C}^1[\mathrm{i}]$ a useful identity:

$$(f^\circ)' \;=\; (f \circ \ell^{\mathrm{inv}})' \;=\; (f' \circ \ell^{\mathrm{inv}}) \cdot (\ell^{\mathrm{inv}})' \;=\; (f'/\ell') \circ \ell^{\mathrm{inv}} \;=\; (\phi^{-1} f')^\circ.$$

Hence if $n \geqslant 1$ and $\ell \in \mathcal{C}^n$, then $\ell^{\mathrm{inv}} \in \mathcal{C}^n$ and $f \mapsto f^\circ$ maps $\mathcal{C}^n[\mathrm{i}]$ and $\mathcal{C}^n$ onto themselves, for each n. Therefore, if $\ell \in \mathcal{C}^{<\infty}$, then $\ell^{\mathrm{inv}} \in \mathcal{C}^{<\infty}$ and $f \mapsto f^\circ$ maps $\mathcal{C}^{<\infty}[\mathrm{i}]$ and $\mathcal{C}^{<\infty}$ onto themselves; likewise with $\mathcal{C}^{\infty}$ or $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{<\infty}$. *In the rest of this subsection we assume $\ell \in \mathcal{C}^{<\infty}$.* Denote the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$

by R, and as usual let R^ϕ be R with its derivation $f \mapsto \partial(f) = f'$ replaced by the derivation $f \mapsto \delta(f) = \phi^{-1}f'$ [ADH, 5.7]. Then $f \mapsto f^\circ \colon R^\phi \to R$ is an isomorphism of differential rings by the identity above. We extend it to the isomorphism

$$Q \mapsto Q^\circ \ : \ R^\phi\{Y\} \to R\{Y\}$$

of differential rings given by $Y^\circ = Y$. Let $y \in R$. Then

$$Q(y)^\circ \ = \ Q^\circ(y^\circ) \qquad \text{for } Q \in R^\phi\{Y\}$$

and thus

$$P(y)^\circ \ = \ P^\phi(y)^\circ \ = \ (P^\phi)^\circ(y^\circ) \qquad \text{for } P \in R\{Y\}.$$

This leads to a useful generalization of the identity for $(f^\circ)'$ above. For this, let $n \geqslant 1$ and let $G^n_k \in \mathbb{Q}\{X\}$ $(1 \leqslant k \leqslant n)$ be the differential polynomial introduced in [ADH, 5.7]; so G^n_k is homogeneous of degree n and isobaric of weight $n-k$. Viewing the G^n_k as elements of $R\{X\}$ and $\delta = \phi^{-1}\partial$ as an element of $R[\partial]$ we have

$$\delta^n \ = \ G^n_n(\phi^{-1})\partial^n + \cdots + G^n_1(\phi^{-1})\partial \qquad \text{in the ring } R[\partial].$$

Thus

$$\delta^2 \ = \ \phi^{-2}\partial^2 - \phi'\phi^{-3}\partial, \qquad \delta^3 \ = \ \phi^{-3}\partial^3 - 3\phi'\phi^{-4}\partial^2 + \big(3(\phi')^2 - \phi\phi''\big)\phi^{-5}\partial, \quad \dots$$

Set $\lambda := -\phi^\dagger$, and let

$$R^n_k \ := \ \operatorname{Ri}(G^n_k) \in \mathbb{Q}\{Z\}, \qquad \text{so} \qquad G^n_k(\phi^{-1}) = \phi^{-n}R^n_k(\lambda) \qquad (0 \leqslant k \leqslant n).$$

Thus

$$\delta^n \ = \ \phi^{-n}\big(R^n_n(\lambda)\partial^n + \cdots + R^n_1(\lambda)\partial\big).$$

For instance,

$$\begin{aligned} \delta^3 \ &= \ \phi^{-3}\big(R^3_3(\lambda)\partial^3 + R^3_2(\lambda)\partial^2 + R^3_1(\lambda)\partial\big) \\ &= \ \phi^{-3}\big(\partial^3 + 3\lambda\partial^2 + \big(2\lambda^2 + \lambda'\big)\partial\big). \end{aligned}$$

We now have:

Lemma 5.3.4. *Let $f \in R$ and $n \geqslant 1$. Then*

$$(f^\circ)^{(n)} \ = \ \Big(\phi^{-n}\big(R^n_n(\lambda)f^{(n)} + \cdots + R^n_1(\lambda)f'\big)\Big)^\circ.$$

Proof. Let $Q = Y^{(n)} \in R^\phi\{Y\}$, so $Q^\circ = Y^{(n)} \in R\{Y\}$. Then $(f^\circ)^{(n)} = Q^\circ(f^\circ) = Q(f)^\circ = \delta^n(f)^\circ$. Now use the above identity for δ^n. □

Note also: $(Q_{+f})^\circ = (Q^\circ)_{+f^\circ}$ and $(Q_{\times f})^\circ = (Q^\circ)_{\times f^\circ}$ for $Q \in R^\phi\{Y\}$, $f \in R$.

Compositional conjugation in Hardy fields. Let now H be a Hardy field, and let $\ell \in \mathcal{C}^1$ be such that $\ell > \mathbb{R}$ and $\ell' \in H$. Then $\ell \in \mathcal{C}^{<\infty}$, $\phi := \ell'$ is active in H, $\phi > 0$, and we have a Hardy field $H(\ell)$. The $\mathbb{C}$-algebra automorphism $f \mapsto f^\circ := f \circ \ell^{\text{inv}}$ of $\mathcal{C}[i]$ restricts to an ordered field isomorphism

$$h \mapsto h^\circ \ : \ H \to H^\circ := H \circ \ell^{\text{inv}}.$$

The identity $(f^\circ)' = (\phi^{-1}f')^\circ$, valid for each $f \in \mathcal{C}^1[i]$, shows that H° is again a Hardy field. Conversely, if E is a subfield of $\mathcal{C}^{<\infty}$ with $\phi \in E$ and $E^\circ := E \circ \ell^{\text{inv}}$ is a Hardy field, then E is a Hardy field. If $H \subseteq \mathcal{C}^\infty$ and $\ell \in \mathcal{C}^\infty$, then $H^\circ \subseteq \mathcal{C}^\infty$; likewise with $\mathcal{C}^\omega$ instead of $\mathcal{C}^\infty$. If E is a Hardy field extension of H, then E° is a Hardy field extension of H°, and E is d-algebraic over H iff E° is d-algebraic over H°. Hence H is maximal iff H° is maximal, and likewise with "d-maximal" in place of "maximal". So $\mathrm{E}(H^\circ) = \mathrm{E}(H)^\circ$ and $\mathrm{D}(H^\circ) = \mathrm{D}(H)^\circ$, and thus H is perfect iff H° is perfect, and likewise with "d-perfect" in place of "perfect". The next lemma is [32, Corollary 6.5]; see also [8, Theorem 1.7].

Lemma 5.3.5. *The germ ℓ^{inv} is hardian. Moreover, if ℓ is $\mathcal{C}^\infty$-hardian, then ℓ^{inv} is also $\mathcal{C}^\infty$-hardian, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Proof. By Proposition 5.3.2(iv) we can arrange that our Hardy field H contains both ℓ and x. Then $\ell^{\text{inv}} = x \circ \ell^{\text{inv}}$ is an element of the Hardy field $H \circ \ell^{\text{inv}}$. □

Next we consider the pre-d-valued field $K := H[i]$ of H-type, which gives rise to

$$K^\circ := K \circ \ell^{\text{inv}} = H^\circ[i],$$

also a pre-d-valued field of H-type, and we have the valued field isomorphism

$$h \mapsto h^\circ \ : \ K \to K^\circ.$$

Note: $h \mapsto h^\circ \colon H^\phi \to H^\circ$ is an isomorphism of pre-H-fields, and $h \mapsto h^\circ \colon K^\phi \to K^\circ$ is an isomorphism of valued differential fields. Recall that K and K^ϕ have the same underlying field. For $f, g \in K$ we have

$$f \preccurlyeq^\flat_\phi g \text{ (in } K) \quad \Longleftrightarrow \quad f \preccurlyeq^\flat g \text{ (in } K^\phi) \quad \Longleftrightarrow \quad f^\circ \preccurlyeq^\flat g^\circ \text{ (in } K^\circ),$$

and likewise with $\preccurlyeq^\flat_\phi$, $\preccurlyeq^\flat$ replaced by $\prec^\flat_\phi$, $\prec^\flat$.

Lemma 5.3.6. *From the isomorphisms $H^\phi \cong H^\circ$ and $K^\phi \cong K^\circ$ we obtain: If H is Liouville closed, then so is H°. If $\mathrm{I}(K) \subseteq K^\dagger$, then $\mathrm{I}(K^\circ) \subseteq (K^\circ)^\dagger$.*

So far we focused on pre-composition with ℓ^{inv}. As to pre-composition with ℓ, it seems not to be known whether $H \circ \ell \subseteq H$ whenever H is maximal. However, we have the following (cf. [33, Lemma 11.6(7)]):

Lemma 5.3.7. $\mathrm{E}(\mathbb{Q}) \circ \ell \subseteq \mathrm{E}(H)$.

Proof. $\mathrm{E}(H^\circ) = \mathrm{E}(H)^\circ$ gives $\mathrm{E}(H^\circ) \circ \ell = \mathrm{E}(H)$. Now use $\mathrm{E}(\mathbb{Q}) \subseteq \mathrm{E}(H^\circ)$. □

Lemma 5.3.7 gives $\mathrm{E}(\mathbb{Q}) \circ \mathrm{E}(\mathbb{Q})^{>\mathbb{R}} \subseteq \mathrm{E}(\mathbb{Q})$; cf. [32, Theorem 6.8]. Boshernitzan's conjecture [32, §10, Conjecture 3] that $\mathrm{E}(\mathbb{Q})^{>\mathbb{R}}$ is also closed under compositional inversion seems to be still open.

Differential algebraicity of compositional inverses (*)**.** In the next lemma we let $\ell \in \mathcal{C}^{<\infty}$ be hardian with $\ell > \mathbb{R}$. The argument in the proof of Lemma 5.3.5 shows that ℓ and ℓ^{inv} are both $\mathbb{R}(x)$-hardian; moreover (cf. [33, Lemma 14.10]):

Lemma 5.3.8. *We have*

$$\operatorname{trdeg}\big(\mathbb{R}\langle x, \ell^{\text{inv}}\rangle|\mathbb{R}\big) \ = \ \operatorname{trdeg}\big(\mathbb{R}\langle x, \ell\rangle|\mathbb{R}\big), \tag{5.3.1}$$

hence if ℓ is d*-algebraic over $\mathbb{R}$, then so is ℓ^{inv}, with*

$$\operatorname{trdeg}\big(\mathbb{R}\langle \ell^{\text{inv}}\rangle|\mathbb{R}\big) \ \leqslant \ \operatorname{trdeg}\big(\mathbb{R}\langle \ell\rangle|\mathbb{R}\big) + 1.$$

Proof. Set $H := \mathbb{R}\langle x, \ell\rangle = \mathbb{R}(x)\langle \ell\rangle$ and $\phi := \ell'$. With ∂ and $\delta = \phi^{-1}\partial$ denoting the derivations of H and H^ϕ, we have $\phi = 1/\delta(x)$ and for all $f \in H$ and $n \geqslant 1$,

$$\partial^n(f) \in \mathbb{Q}\big[\delta(f), \delta^2(f), \dots, \phi, \delta(\phi), \delta^2(\phi), \dots\big]$$

by [ADH, remarks before 5.7.3]. The differential fields H and H^ϕ have the same underlying field, and the former is generated as a field over $\mathbb{R}$ by x and the $\ell^{(n)}$, so applying the above to $f = \ell$ shows that H^ϕ is generated as a differential field over $\mathbb{R}$ by x and ℓ. We also have a differential field isomorphism $h \mapsto h^\circ \colon H^\phi \to H^\circ =$

$H\circ\ell^{\rm inv}$. This yields $H^\circ = \mathbb{R}\langle \ell^{\rm inv}, x\rangle$ and (5.3.1). Suppose now that ℓ is d-algebraic over $\mathbb{R}$; then by additivity of trdeg,

$$\operatorname{trdeg}\big(\mathbb{R}\langle x,\ell\rangle|\mathbb{R}\big) \ = \ \operatorname{trdeg}\big(\mathbb{R}\langle \ell,x\rangle|\mathbb{R}\langle\ell\rangle\big)+\operatorname{trdeg}\big(\mathbb{R}\langle\ell\rangle|\mathbb{R}\big) \ \leqslant \ 1+\operatorname{trdeg}\big(\mathbb{R}\langle\ell\rangle|\mathbb{R}\big),$$

and so by (5.3.1):

$$\operatorname{trdeg}\big(\mathbb{R}\langle\ell^{\rm inv}\rangle|\mathbb{R}\big) \ \leqslant \ \operatorname{trdeg}\big(\mathbb{R}\langle x,\ell^{\rm inv}\rangle|\mathbb{R}\big) \ \leqslant \ \operatorname{trdeg}\big(\mathbb{R}\langle\ell\rangle|\mathbb{R}\big)+1,$$

hence $\ell^{\rm inv}$ is d-algebraic over $\mathbb{R}$. □

In Corollary 5.3.12 below we prove a uniform version of Lemma 5.3.8. To prepare for this we prove the next two lemmas, where R is a differential ring and $x\in R$, $x'=1$. Also, $\phi\in R^\times$, and we take distinct differential indeterminates U, X, Y and let $G^n_k\in\mathbb{Q}\{U\}\subseteq R^\phi\{U\}$ $(k=1,\dots,n)$ be as in [ADH, p. 292], so with ∂ and $\delta=\phi^{-1}\partial$ denoting the derivations of R and R^ϕ, we have in $R^\phi[\delta]$ for $\partial=\phi\delta$:

$$\partial^n \ = \ G^n_n(\phi)\cdot\delta^n + G^n_{n-1}(\phi)\cdot\delta^{n-1}+\cdots+G^n_1(\phi)\cdot\delta.$$

Recall that the G^n_k do not depend on R, x, ϕ.

Lemma 5.3.9. *There are $H^n_k\in\mathbb{Q}\{X'\}\subseteq\mathbb{Q}\{X\}\subseteq R^\phi\{X\}$ $(k=1,\dots,n)$, independent of R, x, ϕ, such that $G^n_k(\phi)=\phi^{2n-1}H^n_k(x)$.*

Proof. By induction on $n\geqslant 1$. For $n=1$ we have $G^1_1=U$, so $H^1_1:=1$ does the job. Suppose for a certain $n\geqslant1$ we have H^n_k $(k=1,\dots,n)$ with the desired properties, and let $k\in\{1,\dots,n+1\}$. Now $G^{n+1}_k = U\cdot\big(\delta(G^n_k)+G^n_{k-1}\big)$ by [ADH, (5.7.2)] (with $G^n_0:=0$), so using $\delta(\phi)=-\phi^2\delta^2(x)$ and setting $H^n_0:=0$,

$$\begin{aligned} G^{n+1}_k(\phi) \ &= \ \phi\cdot\big((2n-1)\phi^{2n-2}\delta(\phi)H^n_k(x)+\phi^{2n-1}\delta(H^n_k(x))+\phi^{2n-1}H^n_{k-1}(x)\big)\\ &= \ \phi^{2n+1}\big((1-2n)\delta^2(x)H^n_k(x)+\delta(x)\delta(H^n_k(x))+\delta(x)H^n_{k-1}(x)\big).\end{aligned}$$

Thus we can take

$$H^{n+1}_k \ := \ (1-2n)X''H^n_k + X'(H^n_k)'+X'H^n_{k-1}. \qquad \square$$

Lemma 5.3.10. *Let C be a subfield of C_R and $P\in C\{X,Y\}\subseteq R\{X,Y\}$. Then there are $N\in\mathbb{N}$ and $Q\in C\{X,Y\}\subseteq R^\phi\{X,Y\}$ such that $P(x,Y)^\phi=\phi^N Q(x,Y)$ in $R^\phi\{Y\}$. Here we can take N, Q independent of x, ϕ.*

Note that $C\{X,Y\}$ as a differential subring of $R\{X,Y\}$ is the same as $C\{X,Y\}$ as a differential subring of $R^\phi\{X,Y\}$, but "$P\in C\{X,Y\}\subseteq R\{X,Y\}$" indicates that P is considered as an element of $R\{X,Y\}$ when substituting in P, while "$Q\in C\{X,Y\}\subseteq R^\phi\{X,Y\}$" indicates that Q is taken as an element of $R^\phi\{X,Y\}$ when substituting in Q.

Proof. For $i=1,2$, let $P_i\in C\{X,Y\}$, $N_i\in\mathbb{N}$, and $Q_i\in C\{X,Y\}\subseteq R^\phi\{X,Y\}$ be such that $P_i(x,Y)^\phi=\phi^{N_i}Q_i(x,Y)$. Then

$$(P_1\cdot P_2)(x,Y)^\phi \ = \ \phi^{N_1+N_2}(Q_1\cdot Q_2)(x,Y).$$

Moreover, $\delta(x)=\phi^{-1}$, hence if $N_1\leqslant N_2$, then

$$(P_1+P_2)(x,Y)^\phi \ = \ \phi^{N_2}Q(x,Y) \quad \text{for } Q:=(X')^{N_2-N_1}Q_1+Q_2.$$

For $P=X$ we have $P(x,Y)^\phi=x=\phi\cdot x\delta(x)$, so $N=1$ and $Q=XX'$ works. For $P=Y$ we can take $N=0$ and $Q=Y$. It is enough to prove the lemma for P

such that no monomial in P has any factor $X^{(m)}$ with $m \geqslant 1$. Thus it only remains to do the case $P = Y^{(n)}$ $(n \geqslant 1)$. With H^n_k as in Lemma 5.3.9 we have

$$(Y^{(n)})^\phi \;=\; G^n_n(\phi)Y^{(n)} + \cdots + G^n_1(\phi)Y' \;=\; \phi^N Q(x,Y)$$

for $N := 2n-1$ and $Q := H^n_n(X)Y^{(n)} + \cdots + H^n_1(X)Y'$. □

In the next lemma x has its usual meaning as the germ in $\mathcal{C}^{<\infty}$ of the identity function on $\mathbb{R}$, we take R as the differential ring $\mathcal{C}^{<\infty}[i]$ and $C\{X,Y\}$ as a differential subring of $R\{X,Y\}$ for any subfield C of $\mathbb{C} = C_R$.

Lemma 5.3.11. *Let $P \in C\{X,Y\}$ where C is a subfield of $\mathbb{C}$. Then there are $N \in \mathbb{N}$ and $P^\bullet \in C\{X,Y\}$ such that for all $y \in R$ and $\ell \in \mathcal{C}^{<\infty}$ with $\ell(t) \to +\infty$ as $t \to +\infty$ and $\ell'(t) > 0$, eventually, we have for $\phi := \ell'$:*

$$P(x,y)\circ \ell^{\text{inv}} \;=\; \big(\phi\circ\ell^{\text{inv}}\big)^N \cdot P^\bullet(\ell^{\text{inv}}, y\circ\ell^{\text{inv}}) \quad \textit{in } R.$$

Proof. Let $\ell \in \mathcal{C}^{<\infty}$ be such that $\ell(t) \to +\infty$ as $t \to +\infty$ and $\ell'(t) > 0$, eventually, and set $\phi := \ell'$. For $P_x := P(x,Y) \in R\{Y\}$ and $y \in R$ we have $P(x,y) = P_x(y) = P^\phi_x(y) = \phi^N Q(x,y)$, with $N \in \mathbb{N}$ and $Q \in R^\phi\{X,Y\}$ as in Lemma 5.3.10, so

$$P(x,y)\circ\ell^{\text{inv}} \;=\; \phi^N Q(x,y)\circ\ell^{\text{inv}} \;=\; \big(\phi\circ\ell^{\text{inv}}\big)^N \cdot Q(x,y)\circ\ell^{\text{inv}}.$$

Let $P^\bullet$ be the element of $C\{X,Y\}$ that is mapped to $Q \in R^\phi\{X,Y\}$ under the ring inclusion $C\{X,Y\} \to R^\phi\{X,Y\}$. The latter is not in general a differential ring morphism, but we have the differential ring isomorphism

$$y \mapsto y\circ\ell^{\text{inv}} \;:\; R^\phi \to R\circ\ell^{\text{inv}} = R,$$

which gives for $y \in R$ that

$$Q(x,y)\circ\ell^{\text{inv}} \;=\; P^\bullet(x\circ\ell^{\text{inv}}, y\circ\ell^{\text{inv}}) \;=\; P^\bullet(\ell^{\text{inv}}, y\circ\ell^{\text{inv}}). \qquad \square$$

Corollary 5.3.12. *For each $P \in \mathbb{R}\{X,Y\}$ there is a $P^\bullet \in \mathbb{R}\{X,Y\}$ such that for all $\ell \in \mathcal{C}^{<\infty}$ with $\ell(t) \to +\infty$ as $t \to +\infty$ and $\ell'(t) > 0$, eventually, we have*

$$P(x,\ell) = 0 \iff P^\bullet(\ell^{\text{inv}}, x) = 0.$$

We now indicate how Lemma 5.3.11 and Corollary 5.3.12 go through for transseries. Recall from [ADH, A.7] that there is a unique operation

$$(f,g) \mapsto f\circ g\colon \mathbb{T}\times\mathbb{T}^{>\mathbb{R}} \to \mathbb{T}$$

such that the following conditions hold for all $g \in \mathbb{T}^{>\mathbb{R}}$:

(1) $x\circ g = g$;
(2) $f \mapsto f\circ g\colon \mathbb{T}\to\mathbb{T}$ is an $\mathbb{R}$-linear embedding of ordered exponential fields;
(3) $f \mapsto f\circ g\colon \mathbb{T}\to\mathbb{T}$ is strongly additive.

By [60, Proposition 6.3] the Chain Rule holds:

$$(f\circ g)' = (f'\circ g)\cdot g' \qquad (f\in\mathbb{T},\ g\in\mathbb{T}^{>\mathbb{R}}).$$

Moreover, $(f,g)\mapsto f\circ g$ restricts to a binary operation on $\mathbb{T}^{>\mathbb{R}}$ which makes $\mathbb{T}^{>\mathbb{R}}$ a group with identity element x. For $f \in \mathbb{T}^{>\mathbb{R}}$ we denote the unique $g \in \mathbb{T}^{>\mathbb{R}}$ with $f\circ g = x$ by $g = f^{\text{inv}}$. We extend $\circ$ in a unique way to an operation

$$(f,g)\mapsto f\circ g\colon \mathbb{T}[i]\times\mathbb{T}^{>\mathbb{R}} \to \mathbb{T}[i]$$

by requiring that for all $g \in \mathbb{T}^{>\mathbb{R}}$, the operation $f\mapsto f\circ g\colon \mathbb{T}[i]\to\mathbb{T}[i]$ is $\mathbb{C}$-linear. It follows that for all $g \in \mathbb{T}^{>\mathbb{R}}$ the operation $f\mapsto f\circ g\colon \mathbb{T}[i]\to\mathbb{T}[i]$ is a field embedding. For $f \in \mathbb{T}[i]$, $g, h \in \mathbb{T}^{>\mathbb{R}}$ we have $(f \circ g) \circ h = f \circ (g \circ h)$ [ADH, A.7(vi)], so $\mathbb{T}[i] \circ h = \mathbb{T}[i]$. For $\ell \in \mathbb{T}^{>\mathbb{R}}$ and $\phi := \ell'$ we have a differential field isomorphism

$$y \mapsto y \circ \ell^{\mathrm{inv}} \colon \mathbb{T}[i]^{\phi} \to \mathbb{T}[i] \circ \ell^{\mathrm{inv}} = \mathbb{T}[i].$$

Let $P \in C\{X, Y\}$ where C is a subfield of $\mathbb{C}$. Let $N \in \mathbb{N}$ and $P^{\bullet} \in C\{X, Y\}$ be as obtained in the proof of Lemma 5.3.11. Then that proof gives for all $y \in \mathbb{T}[i]$, $\ell \in \mathbb{T}^{>\mathbb{R}}$, and $\phi := \ell'$:

$$P(x, y) \circ \ell^{\mathrm{inv}} \ = \ \left(\phi \circ \ell^{\mathrm{inv}}\right)^{N} \cdot P^{\bullet}(\ell^{\mathrm{inv}}, y \circ \ell^{\mathrm{inv}}) \ \text{ in } \mathbb{T}[i].$$

Hence for $C = \mathbb{R}$ we have $P^{\bullet} \in \mathbb{R}\{X, Y\}$ and for all $\ell \in \mathbb{T}^{>\mathbb{R}}$:

$$P(x, \ell) = 0 \iff P^{\bullet}(\ell^{\mathrm{inv}}, x) = 0.$$

5.4. Upper and Lower Bounds on the Growth of Hardian Germs (*)

This section elaborates on [33, 34, 170]. It is not used for proving our main theorem, but some of it is needed later, in the proofs of Corollary 5.5.40, Proposition 5.6.6, and Theorem 5.6.11.

Generalizing logarithmic decomposition. In this subsection K is a differential ring and $y \in K$. In [ADH, p. 213] we defined the nth iterated logarithmic derivative of $y^{\langle n \rangle}$ when K is a differential field. Generalizing this, set $y^{\langle 0 \rangle} := y$, and recursively, if $y^{\langle n \rangle} \in K$ is defined and a unit in K, then $y^{\langle n+1 \rangle} := (y^{\langle n \rangle})^{\dagger}$, while otherwise $y^{\langle n+1 \rangle}$ is not defined. (Thus if $y^{\langle n \rangle}$ is defined, then so are $y^{\langle 0 \rangle}, \dots, y^{\langle n-1 \rangle}$.) With L_n in $\mathbb{Z}[X_1, \dots, X_n]$ as in [ADH, p. 213], if $y^{\langle n \rangle}$ is defined, then

$$y^{(n)} \ = \ y^{\langle 0 \rangle} \cdot L_n(y^{\langle 1 \rangle}, \dots, y^{\langle n \rangle}).$$

If $y^{\langle n \rangle}$ is defined and $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{1+n}$, we set

$$y^{\langle \boldsymbol{i} \rangle} \ := \ (y^{\langle 0 \rangle})^{i_0} (y^{\langle 1 \rangle})^{i_1} \cdots (y^{\langle n \rangle})^{i_n} \in K.$$

Hence if H is a differential subfield of K, $P \in H\{Y\}$ has order at most n and logarithmic decomposition $P = \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i} \rangle} Y^{\langle \boldsymbol{i} \rangle}$ ($\boldsymbol{i}$ ranging over $\mathbb{N}^{1+n}$, all $P_{\langle \boldsymbol{i} \rangle} \in H$, and $P_{\langle \boldsymbol{i} \rangle} = 0$ for all but finitely many $\boldsymbol{i}$), and $y^{\langle n \rangle}$ is defined, then $P(y) = \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i} \rangle} y^{\langle \boldsymbol{i} \rangle}$. Below we apply these remarks to $K = \mathcal{C}^{<\infty}$, where for $y \in K^{\times}$ we have $y^{\dagger} = (\log |y|)'$, hence $y^{\langle n+1 \rangle} = (\log |y^{\langle n \rangle}|)'$ if $y^{\langle n+1 \rangle}$ is defined.

Transexponential germs. For $f \in \mathcal{C}$ we recursively define the germs $\exp_n f$ in $\mathcal{C}$ by $\exp_0 f := f$ and $\exp_{n+1} f := \exp(\exp_n f)$. Following [33] we say that a germ $y \in \mathcal{C}$ is **transexponential** if $y \geqslant \exp_n x$ for all n. *In the rest of this subsection H is a Hardy field.* By Corollary 1.3.9 and Proposition 5.3.2:

Lemma 5.4.1. *If the H-hardian germ y is* d*-algebraic over H, then $y \leqslant \exp_n h$ for some n and some $h \in H(x)$.*

Thus each transexponential hardian germ is d-transcendental (over $\mathbb{R}$). *In the rest of this subsection: $y \in \mathcal{C}^{<\infty}$ is transexponential and hardian, and $z \in \mathcal{C}^{<\infty}[i]$.* Then $y^{\langle n \rangle}$ is defined, and $y^{\langle n \rangle}$ is also transexponential and hardian, for all n. Next some variants of results from Section 1.3. For this, let n be given and let $f \in \mathcal{C}^{<\infty}$, not necessarily hardian, be such that $f \succ 1$, $f \geqslant 0$, and $y \succcurlyeq \exp_{n+1} f$.

Lemma 5.4.2. *We have $y^{\dagger} \succcurlyeq \exp_n f$ and $y^{\langle n \rangle} \succcurlyeq \exp f$.*

Proof. Since $y \succcurlyeq \exp_2 x$, we have $\log y \succcurlyeq \exp x$ by Lemma 5.1.2, and thus $y^\dagger = (\log y)' \succcurlyeq \log y$. Since $y \succcurlyeq \exp_{n+1} f$, the same lemma gives $\log y \succcurlyeq \exp_n f$. Thus $y^\dagger \succcurlyeq \exp_n f$. Now the second statement follows by an easy induction. $\square$

Corollary 5.4.3. *Let $\boldsymbol{i} \in \mathbb{Z}^{1+n}$ and suppose $\boldsymbol{i} > 0$ lexicographically. Then $y^{\langle \boldsymbol{i} \rangle} \succ f$.*

Proof. Let $m \in \{0, \dots, n\}$ be minimal such that $i_m \neq 0$; so $i_m \geqslant 1$. The remarks after Corollary 1.3.2 then give $y^{\langle \boldsymbol{i} \rangle} \succ 1$ and $[v(y^{\langle \boldsymbol{i} \rangle})] = [v(y^{\langle m \rangle})]$, so we have $k \in \mathbb{N}$, $k \geqslant 1$, such that $y^{\langle \boldsymbol{i} \rangle} \succcurlyeq (y^{\langle m \rangle})^{1/k}$. Then Lemma 5.4.2 gives $y^{\langle \boldsymbol{i} \rangle} \succcurlyeq (y^{\langle m \rangle})^{1/k} \succcurlyeq (\exp f)^{1/k} \succ f$ as required. $\square$

In the next proposition and lemma $P \in H\{Y\}^{\neq}$ has order at most n, and $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ range over $\mathbb{N}^{1+n}$. Let $\boldsymbol{j}$ be lexicographically maximal such that $P_{\langle \boldsymbol{j} \rangle} \neq 0$, and choose $\boldsymbol{k}$ so that $P_{\langle \boldsymbol{k} \rangle}$ has minimal valuation. If $P_{\langle \boldsymbol{k} \rangle}/P_{\langle \boldsymbol{j} \rangle} \succ x$, set $f := |P_{\langle \boldsymbol{k} \rangle}/P_{\langle \boldsymbol{j} \rangle}|$; otherwise set $f := x$. Then $f \in H(x)$, $f > 0$, $f \succ 1$, and $f \succcurlyeq P_{\langle \boldsymbol{i} \rangle}/P_{\langle \boldsymbol{j} \rangle}$ for all $\boldsymbol{i}$.

Proposition 5.4.4. *We have $P(y) \sim P_{\langle \boldsymbol{j} \rangle} y^{\langle \boldsymbol{j} \rangle}$ and thus*

$$P(y) \in \left(\mathcal{C}^{<\infty}\right)^\times, \qquad \operatorname{sign} P(y) \;=\; \operatorname{sign} P_{\langle \boldsymbol{j} \rangle} \neq 0.$$

Proof. For $\boldsymbol{i} < \boldsymbol{j}$ we have $y^{\langle \boldsymbol{j} - \boldsymbol{i} \rangle} \succ f \succcurlyeq P_{\langle \boldsymbol{i} \rangle}/P_{\langle \boldsymbol{j} \rangle}$ by Corollary 5.4.3, therefore $P_{\langle \boldsymbol{j} \rangle} y^{\langle \boldsymbol{j} \rangle} \succ P_{\langle \boldsymbol{i} \rangle} y^{\langle \boldsymbol{i} \rangle}$. Thus $P(y) \sim P_{\langle \boldsymbol{j} \rangle} y^{\langle \boldsymbol{j} \rangle}$. $\square$

Lemma 5.4.5. *Suppose that $z^{\langle n \rangle}$ is defined and $y^{\langle i \rangle} \sim z^{\langle i \rangle}$ for $i = 0, \dots, n$. Then $P(y) \sim P(z)$.*

Proof. For all $\boldsymbol{i}$ with $P_{\langle \boldsymbol{i} \rangle} \neq 0$ we have $P_{\langle \boldsymbol{i} \rangle} y^{\langle \boldsymbol{i} \rangle} \sim P_{\langle \boldsymbol{i} \rangle} z^{\langle \boldsymbol{i} \rangle}$, by Lemma 5.1.1. Now use that for $\boldsymbol{i} \neq \boldsymbol{j}$ we have $P_{\langle \boldsymbol{i} \rangle} y^{\langle \boldsymbol{i} \rangle} \prec P_{\langle \boldsymbol{j} \rangle} y^{\langle \boldsymbol{j} \rangle}$ by the proof of Proposition 5.4.4. $\square$

From here on n is no longer fixed.

Corollary 5.4.6 (Boshernitzan [33, Theorem 12.23])**.** *Suppose $y \geqslant \exp_n h$ for all $h \in H(x)$ and all n. Then y is H-hardian.*

This is an immediate consequence of Proposition 5.4.4. (In [33], the proof of this fact is only indicated.) From Lemma 5.4.5 we also obtain:

Corollary 5.4.7. *Suppose that y is as in Corollary 5.4.6 and $z \in \mathcal{C}^{<\infty}$, and $z^{\langle n \rangle}$ is defined and $y^{\langle n \rangle} \sim z^{\langle n \rangle}$, for all n. Then z is H-hardian, and there is a unique ordered differential field isomorphism $H\langle y \rangle \to H\langle z \rangle$ over H which sends y to z.*

Lemma 5.4.13 below contains another criterion for z to be H-hardian. This involves a certain binary relation $\sim_\infty$ on germs defined in the next subsection. Lemma 5.4.5 also yields a complex version of Corollary 5.4.7:

Corollary 5.4.8. *Suppose that y is as in Corollary 5.4.6 and that $z^{\langle n \rangle}$ is defined and $y^{\langle n \rangle} \sim z^{\langle n \rangle}$, for all n. Then z generates a differential subfield $H\langle z \rangle$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$, and there is a unique differential field isomorphism $H\langle y \rangle \to H\langle z \rangle$ over H which sends y to z. Moreover, the binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on $H\langle z \rangle$ which makes this an isomorphism of valued differential fields.*

A useful equivalence relation. We set

$$\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} \ := \ \{f \in \mathcal{C}^{<\infty}[\mathrm{i}] : f^{(n)} \preccurlyeq 1 \text{ for all } n\} \ \subseteq \ \mathcal{C}[\mathrm{i}]^{\preccurlyeq},$$

a differential $\mathbb{C}$-subalgebra of $\mathcal{C}^{<\infty}[\mathrm{i}]$, and

$$\mathcal{I} \ := \ \{f \in \mathcal{C}^{<\infty}[\mathrm{i}] : f^{(n)} \prec 1 \text{ for all } n\} \ \subseteq \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq},$$

a differential ideal of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$ (thanks to the Product Rule). Recall from the remarks preceding Lemma 5.1.1 that $(\mathcal{C}[\mathrm{i}]^{\preccurlyeq})^{\times} = \mathcal{C}[\mathrm{i}]^{\asymp}$.

Lemma 5.4.9. *The group of units of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$ is*

$$\mathcal{C}^{<\infty}[\mathrm{i}]^{\asymp} \ := \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} \cap \mathcal{C}[\mathrm{i}]^{\asymp} \ = \ \{f \in \mathcal{C}^{<\infty}[\mathrm{i}] : f \asymp 1,\ f^{(n)} \preccurlyeq 1 \textit{ for all } n\}.$$

Moreover, $1+\mathcal{I}$ is a subgroup of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\asymp}$.

Proof. It is clear that

$$(\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq})^{\times} \ \subseteq \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} \cap (\mathcal{C}[\mathrm{i}]^{\preccurlyeq})^{\times} \ = \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} \cap \mathcal{C}[\mathrm{i}]^{\asymp} \ = \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\asymp}.$$

Conversely, suppose $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ satisfies $f \asymp 1$ and $f^{(n)} \preccurlyeq 1$ for all n. For each n we have $Q_n \in \mathbb{Q}\{X\}$ such that $(1/f)^{(n)} = Q_n(f)/f^{n+1}$, hence $(1/f)^{(n)} \preccurlyeq 1$. Thus $f \in (\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq})^{\times}$. This shows the first statement. Clearly $1+\mathcal{I} \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]^{\asymp}$, and $1+\mathcal{I}$ is closed under multiplication. If $\delta \in \mathcal{I}$, then $1+\delta$ is a unit of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$ and $(1+\delta)^{-1} = 1+\varepsilon$ where $\varepsilon = -\delta(1+\delta)^{-1} \in \mathcal{I}$. $\square$

For $y, z \in \mathcal{C}[\mathrm{i}]^{\times}$ we define

$$y \sim_{\infty} z \quad :\Longleftrightarrow \quad y \in z \cdot (1+\mathcal{I});$$

hence $y \sim_{\infty} z \Rightarrow y \sim z$. Lemma 5.4.9 yields that $\sim_{\infty}$ is an equivalence relation on $\mathcal{C}[\mathrm{i}]^{\times}$, and for $y_i, z_i \in \mathcal{C}[\mathrm{i}]^{\times}$ $(i = 1, 2)$ we have

$$y_1 \sim_{\infty} y_2 \quad \& \quad z_1 \sim_{\infty} z_2 \qquad \Longrightarrow \qquad y_1 z_1 \sim_{\infty} y_2 z_2, \quad y_1^{-1} \sim_{\infty} y_2^{-1}.$$

Lemma 5.4.10. *Let $y, z \in \mathcal{C}^1[\mathrm{i}]^{\times}$ with $y \sim_{\infty} z$ and $z \in z'\,\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$. Then*

$$y', z' \in \mathcal{C}[\mathrm{i}]^{\times}, \qquad y' \sim_{\infty} z'.$$

Proof. Let $\delta \in \mathcal{I}$ and $f \in \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$ with $y = z(1+\delta)$ and $z = z'f$. Then $z' \in \mathcal{C}[\mathrm{i}]^{\times}$ and $y' = z'(1+\delta) + z\delta' = z'(1+\delta+f\delta')$ where $\delta + f\delta' \in \mathcal{I}$, so $y' \sim_{\infty} z'$. $\square$

If $\ell \in \mathcal{C}^n[\mathrm{i}]$ and $f \in \mathcal{C}^n$ with $f \geqslant 0$, $f \succ 1$, then $\ell \circ f \in \mathcal{C}^n[\mathrm{i}]$. In fact, for $n \geqslant 1$ and $1 \leqslant k \leqslant n$ we have a differential polynomial $Q^n_k \in \mathbb{Q}\{X'\} \subseteq \mathbb{Q}\{X\}$ of order $\leqslant n$, isobaric of weight n, and homogeneous of degree k, such that for all such ℓ, f,

$$(\ell \circ f)^{(n)} \ = \ (\ell^{(n)} \circ f)\, Q^n_n(f) + \cdots + (\ell' \circ f)\, Q^n_1(f).$$

For example,

$$Q^1_1 = X', \quad Q^2_2 = (X')^2,\ Q^2_1 = X'', \quad Q^3_3 = (X')^3,\ Q^3_2 = 3X'X'',\ Q^3_1 = X'''.$$

The following Lemma is only used in the proof of Theorem 5.6.11 below.

Lemma 5.4.11. *Let $f, g \in \mathcal{C}^{<\infty}$ be such that $f, g \geqslant 0$ and $f, g \succ 1$, and set $r := g - f$. Suppose $P(f) \cdot Q(r) \prec 1$ for all $P, Q \in \mathbb{Q}\{Y\}$ with $Q(0) = 0$, and let $\ell \in \mathcal{C}^{<\infty}[\mathrm{i}]$ be such that $\ell' \in \mathcal{I}$. Then $\ell \circ g - \ell \circ f \in \mathcal{I}$.*

Proof. Treating real and imaginary parts separately we arrange $\ell \in \mathcal{C}^{<\infty}$. Note that $r \prec 1$. Taylor expansion [ADH, 4.2] for $P \in \mathbb{Q}\{X\}$ of order $\leqslant n$ gives

$$P(g) - P(f) \;=\; \sum_{|\boldsymbol{i}|\geqslant 1} \frac{1}{\boldsymbol{i}!} P^{(\boldsymbol{i})}(f) \cdot r^{\boldsymbol{i}} \qquad (\boldsymbol{i} \in \mathbb{N}^{1+n}),$$

and thus $P(g) - P(f) \prec 1$ and $rP(g) \prec 1$. The Mean Value Theorem yields a germ $r_n \in \mathcal{G}$ such that

$$\ell^{(n)} \circ g - \ell^{(n)} \circ f \;=\; \big(\ell^{(n+1)} \circ (f + r_n)\big) \cdot r \quad \text{and} \quad |r_n| \leqslant |r|.$$

Now $r_0 \prec 1$, so $\ell' \circ (f + r_0) \prec 1$, hence $\ell \circ g - \ell \circ f \prec 1$. For $1 \leqslant k \leqslant n$,

$$\begin{aligned}(\ell^{(k)} \circ g)\, Q^n_k(g) - (\ell^{(k)} \circ f)\, Q^n_k(f) = \\ &(\ell^{(k)} \circ f)\,\big(Q^n_k(g) - Q^n_k(f)\big) + \big(\ell^{(k+1)} \circ (f + r_k)\big) \cdot r Q^n_k(g),\end{aligned}$$

so $(\ell^{(k)} \circ g)\, Q^n_k(g) - (\ell^{(k)} \circ f)\, Q^n_k(f) \prec 1$, and thus $\big(\ell \circ g - \ell \circ f\big)^{(n)} \prec 1$. $\square$

We consider next the differential $\mathbb{R}$-subalgebra

$$(\mathcal{C}^{<\infty})^{\preccurlyeq} \;:=\; \mathcal{C}^{<\infty}[i]^{\preccurlyeq} \cap \mathcal{C}^{<\infty} \;\subseteq\; \mathcal{C}^{\preccurlyeq}$$

of $\mathcal{C}^{<\infty}$. In the rest of this subsection H is a Hardy field and $y, z \in \mathcal{C}^{<\infty}$, $y, z \succ 1$. Note that $(\mathcal{C}^{<\infty})^{\preccurlyeq} \cap H = \mathcal{O}_H$ and $\mathcal{I} \cap H = o_H$. This yields:

Lemma 5.4.12. *Suppose $y - z \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$ and z is hardian. Then $y \sim_\infty z$.*

Proof. From $y = z + f$ with $f \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$ we obtain $y = z(1 + fz^{-1})$. Now $z^{-1} \in \mathcal{I}$, so $fz^{-1} \in \mathcal{I}$, and thus $y \sim_\infty z$. $\square$

We now formulate a sufficient condition involving $\sim_\infty$ for y to be H-hardian.

Lemma 5.4.13. *Suppose z is H-hardian with $z \geqslant \exp_n h$ for all $h \in H(x)$ and all n, and $y \sim_\infty z$. Then y is H-hardian, and there is a unique ordered differential field isomorphism $H\langle y\rangle \to H\langle z\rangle$ which is the identity on H and sends y to z.*

Proof. By Lemma 5.4.1 we may replace H by the Hardy subfield $\operatorname{Li}\big(H(\mathbb{R})\big)$ of $\operatorname{E}(H)$ to arrange that $H \supseteq \mathbb{R}$ is Liouville closed. By Corollary 5.4.7 (with the roles of y, z reversed) it is enough to show that for each n, $y^{\langle n\rangle}$ is defined, $y^{\langle n\rangle} \succ 1$, and $y^{\langle n\rangle} \sim_\infty z^{\langle n\rangle}$. This holds by hypothesis for $n = 0$. By Lemma 1.3.3, $z > H$ gives $z^\dagger > H$, so $z = z'f$ with $f \prec 1$ in the Hardy field $H\langle z\rangle$, hence $f^{(n)} \prec 1$ for all n. So by Lemma 5.4.10, $y^{\langle 1\rangle} = y^\dagger$ is defined, $y^{\langle 1\rangle} \in (\mathcal{C}^{<\infty})^\times$, $y^{\langle 1\rangle} \sim_\infty z^{\langle 1\rangle}$, and thus $y^{\langle 1\rangle} \succ 1$. Assume for a certain $n \geqslant 1$ that $y^{\langle n\rangle}$ is defined, $y^{\langle n\rangle} \succ 1$, and $y^{\langle n\rangle} \sim_\infty z^{\langle n\rangle}$. Then $z^{\langle n\rangle}$ is H-hardian and $H < z^{\langle n\rangle}$ by Lemma 1.3.5. Hence by the case $n = 1$ applied to $y^{\langle n\rangle}$, $z^{\langle n\rangle}$ in place of y, z, respectively, $y^{\langle n+1\rangle} = (y^{\langle n\rangle})^\dagger$ is defined, $y^{\langle n+1\rangle} \succ 1$, and $y^{\langle n+1\rangle} \sim_\infty z^{\langle n+1\rangle}$. $\square$

The next two corollaries are Theorems 13.6 and 13.10, respectively, in [33].

Corollary 5.4.14. *Suppose z is transexponential and hardian, and $y - z \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$. Then y is hardian, and there is a unique isomorphism $\mathbb{R}\langle y\rangle \to \mathbb{R}\langle z\rangle$ of ordered differential fields that is the identity on $\mathbb{R}$ and sends y to z.*

Proof. Take $H := \operatorname{Li}(\mathbb{R})$. Then z lies in a Hardy field extension of H, namely $\operatorname{Li}\big(\mathbb{R}\langle z\rangle\big)$, and $H < z$. So $y \sim_\infty z$ by Lemma 5.4.12. Now use Lemma 5.4.13. $\square$

Corollary 5.4.15. *If $z \in \mathrm{E}(H)^{>\mathbb{R}}$, then $z \leqslant \exp_n h$ for some $h \in H(x)$ and some n. (Thus if $x \in H$ and $\exp H \subseteq H$, then $H^{>\mathbb{R}}$ is cofinal in $\mathrm{E}(H)^{>\mathbb{R}}$.)*

Proof. Towards a contradiction, suppose $z \in \mathrm{E}(H)^{>\mathbb{R}}$ and $z > \exp_n h$ in $\mathrm{E}(H)$ for all $h \in H(x)$ and all n. Set $y := z + \sin x$. Then y is H-hardian by Lemmas 5.4.12 and 5.4.13, so y, z lie in a common Hardy field extension of H, a contradiction. □

The same proof shows that Corollary 5.4.15 remains true if H is assumed to be a $\mathcal{C}^{\infty}$-Hardy field and $\mathrm{E}(H)$ is replaced by $\mathrm{E}^{\infty}(H)$; likewise for ω in place of ∞.

Remarks on differential subfields of $\mathcal{C}^{<\infty}[\mathrm{i}]$. Let K be a subfield of $\mathcal{C}[\mathrm{i}]$. Then the following are equivalent:

(1) The binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on K;
(2) for all $f, g \in K$: $f \preccurlyeq g$ or $g \preccurlyeq f$;
(3) for all $f \in K$: $f \preccurlyeq 1$ or $1 \preccurlyeq f$.

If $K \subseteq H[\mathrm{i}]$ where H is a Hausdorff field, then $\preccurlyeq$ restricts to a dominance relation on K. (See Section 5.1.) Moreover, the following are equivalent:

(1) $K = H[\mathrm{i}]$ for some Hausdorff field H;
(2) $\mathrm{i} \in K$ and $\overline{f} \in K$ for each $f \in K$;
(3) $\mathrm{i} \in K$ and $\operatorname{Re} f, \operatorname{Im} f \in K$ for each $f \in K$.

Next a lemma similar to Lemma 5.4.13, but obtained using Corollary 5.4.8 instead of Corollary 5.4.7:

Lemma 5.4.16. *Let H be a Hardy field, let $z \in \mathcal{C}^{<\infty}$ be H-hardian with $z \geqslant \exp_n h$ for all $h \in H(x)$ and all n, and $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $y \sim_{\infty} z$. Then y generates a differential subfield $H\langle y\rangle$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$, and there is a unique differential field isomorphism $H\langle y\rangle \to H\langle z\rangle$ which is the identity on H and sends y to z. The binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on $H\langle y\rangle$ which makes this an isomorphism of valued differential fields.*

We use the above at the end of the next subsection to produce a differential subfield of $\mathcal{C}^{<\infty}[\mathrm{i}]$ that is not contained in $H[\mathrm{i}]$ for any Hardy field H.

Boundedness. Let $H \subseteq \mathcal{C}$. We say that $b \in \mathcal{C}$ **bounds** H if $h \leqslant b$ for each $h \in H$. We call H **bounded** if some $b \in \mathcal{C}$ bounds H, and we call H **unbounded** if H is not bounded. If $H_1, H_2 \subseteq \mathcal{C}$ and for each $h_2 \in H_2$ there is an $h_1 \in H_1$ with $h_2 \leqslant h_1$, then any $b \in \mathcal{C}$ bounding H_1 also bounds H_2. Every bounded subset of $\mathcal{C}$ is bounded by a germ in $\mathcal{C}^{\omega}$; this follows from [33, Lemma 14.3]:

Lemma 5.4.17. *For every $b \geqslant 0$ in $\mathcal{C}^{\times}$ there is a $\phi \geqslant 0$ in $(\mathcal{C}^{\omega})^{\times}$ such that $\phi^{(n)} \prec b$ for all n.*

Every countable subset of $\mathcal{C}$ is bounded, by du Bois-Reymond [30]; see also [87, Chapter II] or [39, Chapitre V, p. 53, ex. 8]. Thus $H \subseteq \mathcal{C}$ is bounded if it is totally ordered by the partial ordering $\leqslant$ of $\mathcal{C}$ and has countable cofinality. If H is a Hausdorff field and $b \in \mathcal{C}$ bounds H, then b also bounds the real closure $H^{\mathrm{rc}} \subseteq \mathcal{C}$ of H [ADH, 5.3.2]. *In the rest of this subsection H is a Hardy field.*

Lemma 5.4.18. *Let H^* be a* d*-algebraic Hardy field extension of H and suppose H is bounded. Then H^* is also bounded.*

Proof. By [ADH, 3.1.11] we have $f \in H(x)^{>}$ such that for all $g \in H(x)^{\times}$ there are $h \in H^{\times}$ and $q \in \mathbb{Q}$ with $g \asymp hf^{q}$. Hence $H(x)$ is bounded. Replacing H, H^{*} by $H(x)^{\mathrm{rc}}$, $\operatorname{Li}\big(H^{*}(\mathbb{R})\big)$, respectively, we arrange that H is real closed with $x \in H$, and $H^{*} \supseteq \mathbb{R}$ is Liouville closed. Let $b \in \mathcal{C}$ bound H. Then any $b^{*} \in \mathcal{C}$ such that $\exp_n b \leqslant b^{*}$ for all n bounds H^{*}, by Lemma 5.4.1. ☐

Lemma 5.4.19. *Suppose that H is bounded and $f \in \mathcal{C}^{<\infty}$ is hardian over H. Then $H\langle f\rangle$ is bounded.*

Proof. Lemma 5.4.18 gives that $\operatorname{Li}\big(H(\mathbb{R})\big)$ is bounded; also, f remains hardian over $\operatorname{Li}\big(H(\mathbb{R})\big)$. Using this we arrange that H is Liouville closed. The case that $H\langle f\rangle$ has no element $> H$ is trivial, so assume we have $y \in H\langle f\rangle$ with $y > H$. Then y is d-transcendental over H and the sequence $y, y^2, y^3, \dots$ is cofinal in $H\langle y\rangle$, by Corollary 1.3.8, so $H\langle y\rangle$ is bounded. Now use that f is d-algebraic over $H\langle y\rangle$. ☐

Theorem 5.4.20 (Boshernitzan [33, Theorem 14.4])**.** *Suppose H is bounded. Then the perfect hull $\mathrm{E}(H)$ of H is* d*-algebraic over H and hence bounded. If $H \subseteq \mathcal{C}^{\infty}$, then $\mathrm{E}^{\infty}(H)$ is* d*-algebraic over H; likewise with ω in place of ∞.*

Using the results above the proof is not difficult. It is omitted in [33], but we include it here for the sake of completeness. First, a lemma:

Lemma 5.4.21. *Let $b \in \mathcal{C}^{\times}$ bound H, let $\phi \geqslant 0$ in $\mathcal{C}^{<\infty}$ satisfy $\phi^{(n)} \prec b^{-1}$ for all n, and let $r \in \phi \cdot (\mathcal{C}^{<\infty})^{\preccurlyeq}$. Then $Q(r) \prec 1$ for all $Q \in H\{Y\}$ with $Q(0) = 0$.*

Proof. From $\phi \in \mathcal{I}$ we obtain $r \in \mathcal{I}$, so it is enough that $hr^{(n)} \prec 1$ for all $h \in H$ and all n. Now use the Product Rule and $h\phi^{(n)} \prec hb^{-1} \preccurlyeq 1$ for $h \in H^{\times}$. ☐

Proof of Theorem 5.4.20. Using Lemma 5.4.18, replace H by $\operatorname{Li}\big(H(\mathbb{R})\big)$ to arrange that $H \supseteq \mathbb{R}$ is Liouville closed. Let $b \in \mathcal{C}$ bound H. Then b also bounds $\mathrm{E}(H)$, by Corollary 5.4.15. Lemma 5.4.17 yields $\phi \geqslant 0$ in $(\mathcal{C}^{\omega})^{\times}$ such that $\phi^{(n)} \prec b^{-1}$ for all n; set $r := \phi \cdot \sin x \in \mathcal{C}^{\omega}$. Then $Q(r) \prec f$ for all $f \in \mathrm{E}(H)^{\times}$ and $Q \in \mathrm{E}(H)\{Z\}$ with $Q(0) = 0$, by Lemma 5.4.21.

Suppose towards a contradiction that $f \in \mathrm{E}(H)$ is d-transcendental over H, and set $g := f + r \in \mathcal{C}^{<\infty}$. Then f, g are not in a common Hardy field, so g is not hardian over H. On the other hand, let $P \in H\{Y\}^{\neq}$. Then $P(f) \in \mathrm{E}(H)^{\times}$, and by Taylor expansion,

$$P(f+Z) \;=\; P(f) + Q(Z) \quad \text{ where } Q \in \mathrm{E}(H)\{Z\} \text{ with } Q(0) = 0,$$

so $P(g) = P(f + r) \sim P(f)$. Hence g is hardian over H, a contradiction.

The proof in the case where $H \subseteq \mathcal{C}^{\infty}$ is similar, using the version of Corollary 5.4.15 for $\mathrm{E}^{\infty}(H)$; similarly for ω in place of ∞. ☐

As to the existence of transexponential hardian germs, we have:

Theorem 5.4.22. *For every $b \in \mathcal{C}$ there is a $\mathcal{C}^{\omega}$-hardian germ $y \geqslant b$.*

This is Boshernitzan [34, Theorem 1.2], and leads to [34, Theorem 1.1]:

Corollary 5.4.23. *No maximal Hardy field is bounded.*

Proof. Suppose $x \in H$, and $b \in \mathcal{C}$ bounds H. Take some $b^{*} \in \mathcal{C}$ such that $b^{*} \geqslant \exp_n b$ for each n. Now Theorem 5.4.22 yields a $\mathcal{C}^{\omega}$-hardian germ $y \geqslant b^{*}$. By Corollary 5.4.6, y is H-hardian, so $H\langle y\rangle$ is a proper Hardy field extension of H. ☐

The same proof shows also that no $\mathcal{C}^\infty$-maximal Hardy field and no $\mathcal{C}^\omega$-maximal Hardy field is bounded. In particular (Boshernitzan [34, Theorem 1.3]):

Corollary 5.4.24. *Every maximal Hardy field contains a transexponential germ. Likewise with "$\mathcal{C}^\infty$-maximal" or "$\mathcal{C}^\omega$-maximal" in place of "maximal".*

Remark. For $\mathcal{C}^\infty$-Hardy fields, some of the above is in Sjödin's [190], predating [33, 34]: if H is a bounded $\mathcal{C}^\infty$-Hardy field, then so is $\operatorname{Li}\big(H(\mathbb{R})\big)$ [190, Theorem 2]; no maximal $\mathcal{C}^\infty$-Hardy field is bounded [190, Theorem 6]; and $E := \mathrm{E}^\infty(\mathbb{Q})$ is bounded [190, Theorem 10] with $E \circ E^{>\mathbb{R}} \subseteq E$ [190, Theorem 11].

We can now produce a differential subfield K of $\mathcal{C}^\omega[i]$ containing i such that $\preccurlyeq$ restricts to a dominance relation on K making K a d-valued field of H-type with constant field $\mathbb{C}$, yet $K \not\subseteq H[i]$ for every Hardy field H:

Take a transexponential $\mathcal{C}^\omega$-hardian germ z, and $h \in \mathbb{R}(x)$ with $0 \neq h \prec 1$. Then $\varepsilon := h\,\mathrm{e}^{xi} \in \mathcal{I}$, so $y := z(1+\varepsilon) \in \mathcal{C}^\omega[i]$ with $y \sim_\infty z$. Lemma 5.4.16 applied with $H = \mathbb{R}$ shows that y generates a differential subfield $K_0 := \mathbb{R}\langle y\rangle$ of $\mathcal{C}^\omega[i]$, and $\preccurlyeq$ restricts to a dominance relation on K_0 making K_0 a d-valued field of H-type with constant field $\mathbb{R}$. Then $K := K_0[i]$ is a differential subfield of $\mathcal{C}^\omega[i]$ with constant field $\mathbb{C}$. Moreover, $\preccurlyeq$ also restricts to a dominance relation on K, and this dominance relation makes K a d-valued field of H-type [ADH, 10.5.15]. We cannot have $K \subseteq H[i]$ where H is a Hardy field, since $\operatorname{Im} y = zh\sin x \notin H$.

Lower bounds on d-algebraic hardian germs. In this subsection H is a Hardy field. Let $f \in \mathcal{C}$ and $f \succ 1$, $f \geqslant 0$. Then the germ $\log f \in \mathcal{C}$ also satisfies $\log f \succ 1$, $\log f \geqslant 0$. So we may inductively define the germs $\log_n f$ in $\mathcal{C}$ by $\log_0 f := f$, $\log_{n+1} f := \log\log_n f$. Lemma 5.4.1 gives exponential upper bounds on d-algebraic H-hardian germs. The next result leads to logarithmic lower bounds on such germs when H is grounded.

Theorem 5.4.25 (Rosenlicht [170, Theorem 3])**.** *Suppose H is grounded, and let E be a Hardy field extension of H such that $|\Psi_E \setminus \Psi_H| \leqslant n$ (so E is also grounded). Then there are $r, s \in \mathbb{N}$ with $r + s \leqslant n$ such that*

(i) *for any $h \in H^>$ with $h \succ 1$ and $\max\Psi_H = v(h^\dagger)$, there exists $g \in E^>$ such that $g \asymp \log_r h$ and $\max\Psi_E = v(g^\dagger)$;*
(ii) *for any $g \in E$ there exists $h \in H$ such that $g < \exp_s h$.*

This theorem is most useful in combination with the following lemma, which is [170, Proposition 5] (and also [7, Lemma 2.1] in the context of pre-H-fields).

Lemma 5.4.26. *Let E be a Hardy field extension of H such that $\operatorname{trdeg}(E|H) \leqslant n$. Then $|\Psi_E \setminus \Psi_H| \leqslant n$.*

From [ADH, 9.1.11] we recall that for $f, g \succ 1$ in a Hardy field we have $f^\dagger \preccurlyeq g^\dagger$ iff $|f| \leqslant |g|^n$ for some $n \geqslant 1$. (See also the discussion before Lemma 1.2.27.) Thus by Lemma 5.4.26 and Theorem 5.4.25:

Corollary 5.4.27. *Let E be a Hardy field extension of H with $\operatorname{trdeg}(E|H) \leqslant n$, and let $h \in H^>$ be such that $h \succ 1$ and $\max\Psi_H = v(h^\dagger)$. Then E is grounded, and for all $g \in E$ with $g \succ 1$ there is an $m \geqslant 1$ such that $\log_n h \preccurlyeq g^m$ (and hence $\log_{n+1} h \prec g$ for all $g \in E$ with $g \succ 1$).*

Applying Corollary 5.4.27 to $H = \mathbb{R}(x)$, $h = x$ yields:

Corollary 5.4.28 (Boshernitzan [33, Proposition 14.11]). *If $y \in \mathcal{C}$ is hardian and* d*-algebraic over* $\mathbb{R}$*, then the Hardy field $E = \mathbb{R}(x)\langle y\rangle$ is grounded, and there is an n such that $\log_n x \prec g$ for all $g \in E$ with $g \succ 1$.*

5.5. Second-Order Linear Differential Equations over Hardy Fields

In this section we review Boshernitzan's work [33, §16] on adjoining non-oscillating solutions of second-order linear differential equations to Hardy fields, deduce some consequences about complex exponentials over Hardy fields used later, and prove a conjecture from [33, §17]. *Throughout this section H is a Hardy field.*

Oscillation over Hardy fields. In this subsection we assume $f \in H$ and consider the linear differential equation

$$4Y'' + fY = 0 \tag{4L}$$

over H. The factor 4 is to simplify certain expressions, in conformity with [ADH, 5.2]. In [ADH, 5.2] we defined for any differential field K functions $\omega\colon K \to K$ and $\sigma\colon K^\times \to K$. We define likewise

$$\omega \ :\ \mathcal{C}^1[\mathrm{i}] \to \mathcal{C}^0[\mathrm{i}], \qquad \sigma\ :\ \mathcal{C}^2[\mathrm{i}]^\times \to \mathcal{C}^0[\mathrm{i}]$$

by

$$\omega(z) \ = \ -2z' - z^2 \quad \text{ and } \quad \sigma(y) \ = \ \omega(z) + y^2 \text{ for } z := -y^\dagger.$$

Note that $\omega(\mathcal{C}^1) \subseteq \mathcal{C}^0$ and $\sigma\big((\mathcal{C}^2)^\times\big) \subseteq \mathcal{C}^0$, and $\sigma(y) = \omega(z + y\mathrm{i})$ for $z := -y^\dagger$. To clarify the role of ω and σ in connection with second-order linear differential equations, suppose $y \in \mathcal{C}^2$ is a non-oscillating solution to (4L) with $y \neq 0$. Then $z := 2y^\dagger \in \mathcal{C}^1$ satisfies $-2z' - z^2 = f$, so z generates a Hardy field $H(z)$ with $\omega(z) = f$, by Proposition 5.3.3, which in turn yields a Hardy field $H(z, y)$ with $2y^\dagger = z$. Thus $y_1 := y$ lies in a Hardy field extension of H. From Lemma 5.2.15 and Proposition 5.3.2(iv) we also obtain a solution y_2 to (4L) in a Hardy field extension of $H\langle y_1\rangle = H(y, z)$ such that y_1, y_2 are $\mathbb{R}$-linearly independent; see also [171, Theorem 2, Corollary 2]. This shows:

Proposition 5.5.1. *If $f/4$ does not generate oscillations, then* $\mathrm{D}(H)$ *contains $\mathbb{R}$-linearly independent solutions y_1, y_2 to* (4L).

Indeed, if $f/4$ does not generate oscillations, then $\mathrm{D}(H)$ contains solutions y_1, y_2 to (4L) with $y_1, y_2 > 0$ and $y_1 \prec y_2$. Here y_1 is determined up to multiplication by a factor in $\mathbb{R}^>$; we call such y_1 a **principal solution** to (4L). (Lemmas 5.2.28, 5.2.29.) See Section 1.4 for the subsets $\Gamma(H)$, $\Lambda(H)$ of H.

Lemma 5.5.2. *Suppose H is* d*-perfect and $f/4$ does not generate oscillations, and let $y \in H$ be a principal solution to* (4L)*. Then $z := 2y^\dagger$ is the unique solution of the equation $\omega(z) = f$ in $\Lambda(H)$.*

Proof. We already know $\omega(z) = f$. The restriction of ω to $\Lambda(H)$ is strictly increasing [ADH, 11.8.20], so it remains to show that $z \in \Lambda(H)$. Let $h \in H$, $h' = 1/y^2$. Then $h \succ 1$ by Corollary 5.2.27, hence $1/y^2 \in \Gamma(H)$, so $z = -(1/y^2)^\dagger \in \Lambda(H)$. $\square$

By [ADH, p. 259], with $A = 4\partial^2 + f \in H[\partial]$ we have

$$4y'' + fy = 0 \text{ for some } y \in H^\times \ \Rightarrow\ A \text{ splits over } H \iff f \in \omega(H).$$

To simplify the discussion we now also introduce the subset

$$\overline{\omega}(H) := \big\{ f \in H : f/4 \text{ does not generate oscillations} \big\}$$

of H. If E is a Hardy field extension of H, then $\overline{\omega}(E) \cap H = \overline{\omega}(H)$. By Corollary 5.2.24, $\overline{\omega}(H)$ is downward closed, and $\omega(H) \subseteq \overline{\omega}(H)$ by the discussion following (R) in Section 5.2.

Corollary 5.5.3. *If H is* d*-perfect, then*

$$\omega(H) \;=\; \overline{\omega}(H) \;=\; \big\{f \in H :\ 4y'' + fy = 0 \text{ for some } y \in H^{\times}\big\},$$

and $\omega(H)$ is downward closed in H.

If H is d-perfect, then $H^{\dagger} = H$ by Proposition 5.3.2. The remarks after (R) show that a part of Corollary 5.5.3 holds under this weaker condition:

Corollary 5.5.4. *If $H^{\dagger} = H$, then*

$$\omega(H) \;=\; \big\{f \in H :\ 4y'' + fy = 0 \text{ for some } y \in H^{\times}\big\}.$$

Lemma 5.2.14 and Proposition 5.5.1 also yield:

Corollary 5.5.5. *If $f \in \overline{\omega}(H)$, then each $y \in \mathcal{C}^2$ such that $4y''+fy \in H$ is in* $\mathrm{D}(H)$.

For use in the proof of Corollary 5.5.32 we record the following property of $\overline{\omega}(H)$:

Lemma 5.5.6. $\Gamma(H) \cap \overline{\omega}(H) = \emptyset$.

Proof. We arrange that H is d-perfect. Hence $H \supseteq \mathbb{R}$ is Liouville closed and $\overline{\omega}(H) = \omega(H)$ by Corollary 5.5.3. From $x^{-1} = x^{\dagger} \in \Gamma(H)$ and $\sigma(x^{-1}) = 2x^{-2} \asymp (x^{-1})' \prec \ell^{\dagger}$ for all $\ell \succ 1$ in H we obtain $\Gamma(H) \subseteq \sigma\big(\Gamma(H)\big)^{\uparrow}$, so $\Gamma(H) \cap \omega(H) = \emptyset$ by [ADH, remark before 11.8.29]. □

Next some consequences of Proposition 5.5.1 for more general linear differential equations of order 2: Let $g, h \in H$, and consider the linear differential equation

$$(\widetilde{\mathrm{L}}) \qquad Y'' + gY' + hY \;=\; 0$$

over H. An easy induction on n shows that for a solution $y \in \mathcal{C}^2$ of $(\widetilde{\mathrm{L}})$ we have $y \in \mathcal{C}^n$ with $y^{(n)} \in Hy + Hy'$ for all n, so $y \in \mathcal{C}^{<\infty}$. To reduce $(\widetilde{\mathrm{L}})$ to an equation (4L) we take $f := \omega(g) + 4h = -2g' - g^2 + 4h \in H$, take $a \in \mathbb{R}$, and take a representative of g in $\mathcal{C}^1_a$, also denoted by g, and let $G \in (\mathcal{C}^2)^{\times}$ be the germ of

$$t \mapsto \exp\!\left(-\frac{1}{2}\int_a^t g(s)\,ds\right) \qquad (t \geqslant a).$$

This gives an isomorphism $y \mapsto Gy$ from the $\mathbb{R}$-linear space of solutions of (4L) in $\mathcal{C}^2$ onto the $\mathbb{R}$-linear space of solutions of $(\widetilde{\mathrm{L}})$ in $\mathcal{C}^2$, and $y \in \mathcal{C}^2$ oscillates iff Gy oscillates. By Proposition 5.3.2, $G \in \mathrm{D}(H)$. Using $\frac{f}{4} = -\frac{1}{2}g' - \frac{1}{4}g^2 + h$ we now obtain the following germ version of Corollary 5.2.25:

Corollary 5.5.7. *The following are equivalent:*

(i) *some solution in $\mathcal{C}^2$ of $(\widetilde{\mathrm{L}})$ oscillates;*
(ii) *all nonzero solutions in $\mathcal{C}^2$ of $(\widetilde{\mathrm{L}})$ oscillate;*
(iii) *$-\frac{1}{2}g' - \frac{1}{4}g^2 + h$ generates oscillations.*

Moreover, if $-\frac{1}{2}g' - \frac{1}{4}g^2 + h$ does not generate oscillations, then all solutions of $(\widetilde{\mathrm{L}})$ in $\mathcal{C}^2$ belong to $\mathrm{D}(H)$.

Set $A := \partial^2 + g\partial + h$, and let $f = \omega(g) + 4h$, G be as above. Then $A_{\ltimes G} = \partial^2 + \frac{f}{4}$. Thus by combining Corollary 5.5.5 and Corollary 5.5.7 we obtain:

Corollary 5.5.8. *If* $(\widetilde{\mathrm{L}})$ *has no oscillating solution in* $\mathcal{C}^2$*, and* $y \in \mathcal{C}^2$ *is such that* $y'' + gy' + hy \in H$*, then* $y \in \mathrm{D}(H)$.

The next corollary follows from Proposition 5.5.1 and [ADH, 5.1.21]:

Corollary 5.5.9. *The following are equivalent, for* $A \in H[\partial]$ *and* f *as above:*

(i) $f/4$ *does not generate oscillations;*
(ii) A *splits over some Hardy field extension of* H*;*
(iii) A *splits over* $\mathrm{D}(H)$.

For $A \in H[\partial]$ and f as before we have $A_{\ltimes G} = \partial^2 + \frac{f}{4}$ and $G^\dagger = -\frac{1}{2}g \in H$, so:

Corollary 5.5.10. A *splits over* $H[\mathrm{i}] \iff \partial^2 + \frac{f}{4}$ *splits over* $H[\mathrm{i}]$.

Proposition 5.5.1 and its corollaries 5.5.5–5.5.8 are from [33, Theorems 16.17, 16.18, 16.19], and Corollary 5.5.3 is essentially [33, Lemma 17.1].

Proposition 5.5.1 applies only when (4L) has a solution in $(\mathcal{C}^2)^\times$. Such a solution might not exist, but (4L) does have $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^2$, so $w := y_1y_2' - y_1'y_2 \in \mathbb{R}^\times$. Set $y := y_1 + y_2 i$. Then $4y'' + fy = 0$ and $y \in \mathcal{C}^2[\mathrm{i}]^\times$, and for $z = 2y^\dagger \in \mathcal{C}^1[\mathrm{i}]$ we have $-2z' - z^2 = f$. Now

$$z \;=\; \frac{2y_1' + 2iy_2'}{y_1 + iy_2} \;=\; \frac{2y_1'y_1 + 2y_2'y_2 - 2\mathrm{i}(y_1'y_2 - y_1y_2')}{y_1^2 + y_2^2} \;=\; \frac{2(y_1'y_1 + y_2'y_2) + 2\mathrm{i}w}{y_1^2 + y_2^2},$$

$$\text{so}\quad \operatorname{Re} z \;=\; \frac{2(y_1'y_1 + y_2'y_2)}{y_1^2 + y_2^2} \in \mathcal{C}^1, \qquad \operatorname{Im} z \;=\; \frac{2w}{y_1^2 + y_2^2} \in \mathcal{C}^2.$$

Thus $\operatorname{Im} z \in (\mathcal{C}^2)^\times$ and $(\operatorname{Im} z)^\dagger = -\operatorname{Re} z$, and so

$$\sigma(\operatorname{Im} z) \;=\; \omega\big(-(\operatorname{Im} z)^\dagger + (\operatorname{Im} z)\mathrm{i}\big) \;=\; \omega(z) \;=\; f \qquad \text{in } \mathcal{C}^1.$$

Replacing y_1 by $-y_1$ changes w to $-w$; this way we can arrange $w > 0$, so $\operatorname{Im} z > 0$.

Conversely, every $u \in (\mathcal{C}^2)^\times$ such that $u > 0$ and $\sigma(u) = f$ arises in this way. To see this, suppose we are given such u, take $\phi \in \mathcal{C}^3$ with $\phi' = \frac{1}{2}u$, and set

$$y_1 \;:=\; \frac{1}{\sqrt{u}}\cos\phi, \qquad y_2 \;:=\; \frac{1}{\sqrt{u}}\sin\phi \qquad (\text{elements of } \mathcal{C}^2).$$

Then $\mathrm{wr}(y_1, y_2) = 1/2$, and y_1, y_2 solve (4L). To see the latter, consider

$$y \;:=\; y_1 + y_2\mathrm{i} \;=\; \frac{1}{\sqrt{u}}\,\mathrm{e}^{\phi\mathrm{i}} \in \mathcal{C}^2[\mathrm{i}]^\times$$

and note that $z := 2y^\dagger$ satisfies

$$\omega(z) \;=\; \omega(-u^\dagger + u\mathrm{i}) \;=\; \sigma(u) \;=\; f,$$

hence $4y'' + fy = 0$. The computation above shows $\operatorname{Im} z = 1/(y_1^2 + y_2^2) = u$. We have $\phi' > 0$, so either $\phi > \mathbb{R}$ or $\phi - c \prec 1$ for some $c \in \mathbb{R}$, with $\phi > \mathbb{R}$ iff $f/4$ generates oscillations. As to uniqueness of the above pair (y_1, y_2), we have:

Lemma 5.5.11. *Suppose* $f \notin \overline{\omega}(H)$. *Let* $\widetilde{y}_1, \widetilde{y}_2 \in \mathcal{C}^2$ *be* $\mathbb{R}$*-linearly independent solutions of* (4L) *with* $\mathrm{wr}(\widetilde{y}_1, \widetilde{y}_2) = 1/2$. *Set* $\widetilde{y} := \widetilde{y}_1 + \widetilde{y}_2\mathrm{i}$, $\widetilde{z} := 2\widetilde{y}^\dagger$. *Then*

$$\operatorname{Im} \widetilde{z} = u \iff \widetilde{y} = \mathrm{e}^{\theta\mathrm{i}}\, y \text{ for some } \theta \in \mathbb{R}.$$

Proof. If $\widetilde{y} = \mathrm{e}^{\theta\mathrm{i}}\, y$ ($\theta \in \mathbb{R}$), then clearly $\widetilde{z} = 2\widetilde{y}^\dagger = 2y^\dagger = z$, hence $\operatorname{Im} z = \operatorname{Im} \widetilde{z}$. For the converse, let A be the invertible 2×2 matrix with real entries and $Ay = \widetilde{y}$; here $y = (y_1, y_2)^t$ and $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2)^t$, column vectors with entries in $\mathcal{C}^2$. As in the proof of [ADH, 4.1.18], $\operatorname{wr}(y_1, y_2) = \operatorname{wr}(\widetilde{y}_1, \widetilde{y}_2)$ yields $\det A = 1$.

Suppose $\operatorname{Im} \widetilde{z} = u$, so $y_1^2 + y_2^2 = \widetilde{y}_1^2 + \widetilde{y}_2^2$. Choose $a \in \mathbb{R}$ and representatives for u, y_1, y_2, $\widetilde{y}_1$, $\widetilde{y}_2$ in $\mathcal{C}_a$, denoted by the same symbols, such that in $\mathcal{C}_a$ we have $Ay = \widetilde{y}$ and $y_1^2 + y_2^2 = \widetilde{y}_1^2 + \widetilde{y}_2^2$, and $u(t) \cdot \big(y_1(t)^2 + y_2(t)^2\big) = 1$ for all $t \geqslant a$. With $\|\cdot\|$ the usual euclidean norm on $\mathbb{R}^2$, we then have $\|Ay(t)\| = \|y(t)\| = 1/\sqrt{u(t)}$ for $t \geqslant a$. Since $f/4$ generates oscillations, we have $\phi > \mathbb{R}$, and we conclude that $\|Av\| = 1$ for all $v \in \mathbb{R}^2$ with $\|v\| = 1$. It is well-known that then $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ with $\theta \in \mathbb{R}$ (see, e.g., [122, Chapter XV, Exercise 2]), so $\widetilde{y} = \mathrm{e}^{\theta\mathrm{i}}\, y$. $\square$

The observations above will be used in the proofs of Theorems 5.6.2 and 7.5.32 below. We finish with miscellaneous historical remarks (not used later):

Remarks. The connection between the second-order linear differential equation (4L) and the third-order non-linear differential equation $\sigma(y) = f$ was first investigated by Kummer [119] in 1834. Appell [3] noted that the linear differential equation

$$Y''' + fY' + (f'/2)Y \;=\; 0$$

has $\mathbb{R}$-linearly independent solutions $y_1^2, y_1y_2, y_2^2 \in \mathcal{C}^{<\infty}$, though some cases were known earlier [49, 131]; in particular, $1/u = y_1^2 + y_2^2$ is a solution. See also Lemma 2.4.23. Hartman [90, 93] investigates monotonicity properties of $y_1^2 + y_2^2$. Steen [197] in 1874, and independently Pinney [153], remarked that $r := 1/\sqrt{u} = \sqrt{y_1^2 + y_2^2} \in \mathcal{C}^{<\infty}$ satisfies $4r'' + fr = 1/r^3$. (See also [163].)

Complex exponentials over Hardy fields. We now use some of the above to prove an extension theorem for Hardy fields (cf. [33, Lemma 11.6(6)]):

Proposition 5.5.12. *If $\phi \in H$ and $\phi \preccurlyeq 1$, then $\cos\phi, \sin\phi \in \mathrm{D}(H)$.*

Proof. Replacing H by $\mathrm{D}(H)$ we arrange $\mathrm{D}(H) = H$. Then by Proposition 5.3.2, $H \supseteq \mathbb{R}$ is a Liouville closed H-field, and by Corollary 5.5.3, $\omega(H)$ is downward closed. Hence by Lemma 1.2.20, H is trigonometrically closed. Let now $\phi \in H$ and $\phi \preccurlyeq 1$. Then $(\mathrm{e}^{\phi\mathrm{i}})^\dagger = \phi'\mathrm{i} \in K^\dagger$, so $\cos\phi + \mathrm{i}\sin\phi = \mathrm{e}^{\phi\mathrm{i}} \in K$ using $K \supseteq \mathbb{C}$. Thus $\cos\phi, \sin\phi \in H$. $\square$

Corollary 5.5.13. *Let $\phi \in H$ and $\phi \preccurlyeq 1$. Then $\cos\phi$, $\sin\phi$ generate a* d*-algebraic Hardy field extension $E := H(\cos\phi, \sin\phi)$ of H. If H is a $\mathcal{C}^\infty$-Hardy field, then so is E, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Recall that for $\phi, \theta \in \mathbb{R}$ we have

$$\begin{aligned} \cos(\phi + \theta) \;&=\; \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta), \\ \cos(\phi - \theta) \;&=\; \cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta). \end{aligned}$$

Recall also the bijection $\arccos\colon [-1, 1] \to [0, \pi]$, the inverse of the cosine function on $[0, \pi]$. It follows that for any $a, b \in \mathbb{R}$ we have $d \in \mathbb{R}$ such that

$$a\cos(\phi) + b\sin(\phi) \;=\; \sqrt{a^2 + b^2} \cdot \cos(\phi + d) \text{ for all } \phi \in \mathbb{R}:$$

for a, b not both 0 this holds with $d = \arccos\big(a/\sqrt{a^2 + b^2}\big)$ when $b \leqslant 0$, and with $d = -\arccos\big(a/\sqrt{a^2 + b^2}\big)$ when $b \geqslant 0$. For later use we record some consequences:

Lemma 5.5.14 (Addition of sinusoids). *Let $y \in \mathcal{C}$. Then*

$$y = a\cos x + b\sin x \text{ for some } a, b \in \mathbb{R} \iff y = c\cos(x+d) \text{ for some } c, d \in \mathbb{R}.$$

Corollary 5.5.15. *Let $\phi \in \mathcal{C}$. Then $\mathbb{R}\cos\phi + \mathbb{R}\sin\phi = \{c\cos(\phi + d) : c, d \in \mathbb{R}\}$.*

Corollary 5.5.16. *Suppose $H \supseteq \mathbb{R}$ is real closed and closed under integration, and let $g, h \in H$. Then there is $u \in H$ such that $-\pi \leqslant u \leqslant \pi$ and $g\cos\phi + h\sin\phi = \sqrt{g^2+h^2}\cdot\cos(\phi+u)$ for all $\phi \in \mathcal{C}$: if $h < 0$ this holds for $u = \arccos\big(g/\sqrt{g^2+h^2}\big)$, and if $h > 0$ it holds for $u = -\arccos\big(g/\sqrt{g^2+h^2}\big)$.*

Proof. On the interval $(-1, 1)$ the function arccos is real analytic with derivative $t \mapsto -1/\sqrt{1-t^2}$. Thus $\arccos\big(g/\sqrt{g^2+h^2}\big) \in H$ for $h \neq 0$. $\square$

Corollary 5.5.17. *Let $a \in \mathbb{R}$ and let $g, \phi \in \mathcal{C}^1_a$ have germs in H such that $g(t) \neq 0$ eventually, and $\phi(t) \to +\infty$ as $t \to +\infty$. Then there is a real $b \geqslant a$ with the property that if $s_0, s_1 \in [b, +\infty)$ with $s_0 < s_1$ are any successive zeros of $y := g\cos\phi$, then y' has exactly one zero in the interval (s_0, s_1).*

Proof. By increasing a we arrange $g(t) \neq 0$ and $\phi'(t) > 0$ for all $t \geqslant a$. Replacing g by $-g$ if necessary we further arrange $g(t) > 0$ for all $t \geqslant a$. Let $s_0, s_1 \in [a, +\infty)$ with $s_0 < s_1$ be successive zeros of y. Later we impose a suitable lower bound $b \geqslant a$ on s_0. Then $\phi(s_1) = \phi(s_0) + \pi$, since s_1 is the next zero of $\cos\phi$ after s_0. Also

$$\begin{aligned} y' &= g'\cos\phi - g\phi'\sin\phi = \sqrt{g'^2 + (g\phi')^2}\cos(\phi + u), \quad \text{where} \\ u &= \arccos\big(g'/\sqrt{g'^2 + (g\phi')^2}\big), \quad \text{so } 0 < u(t) < \pi \text{ for all } t \geqslant a. \end{aligned}$$

By Rolle, y' has a zero in (s_0, s_1). Let $t \in (s_0, s_1)$ be a zero of y'. Then

$$\phi(s_0) < \phi(t) < \phi(s_0) + \pi, \quad \phi(t) + u(t) \in \phi(s_0) + \mathbb{Z}\pi,$$

so $\phi(t)+u(t) = \phi(s_0)+\pi$. Take $b \geqslant a$ in $\mathbb{R}$ so large that u is differentiable on $[b, +\infty)$ and $\phi'(t) + u'(t) > 0$ for all $t \geqslant b$; this is possible because $u \preccurlyeq 1$ is H-hardian by Corollary 5.5.16, and $\phi(t) + u(t) \to +\infty$ as $t \to +\infty$. Assuming now that $b \leqslant s_0$, we conclude that $t \in (s_0, s_1)$ is uniquely determined by $\phi(t) + u(t) = \phi(s_0) + \pi$. $\square$

The H-asymptotic field extension $K := H[\mathrm{i}]$ of H is a differential subring of $\mathcal{C}^{<\infty}[i]$. To handle ultimate dents in H in Section 4.4, we sometimes assumed $\mathrm{I}(K) \subseteq K^\dagger$, a condition that we consider more closely in the next proposition:

Proposition 5.5.18. *Suppose $H \supseteq \mathbb{R}$ is closed under integration. Then the following conditions are equivalent:*

(i) $\mathrm{I}(K) \subseteq K^\dagger$;
(ii) $\mathrm{e}^f \in K$ *for all* $f \in K$ *with* $f \prec 1$;
(iii) $\mathrm{e}^\phi, \cos\phi, \sin\phi \in H$ *for all* $\phi \in H$ *with* $\phi \prec 1$.

Proof. Assume (i), and let $f \in K$, $f \prec 1$. Then $f' \in \mathrm{I}(K)$, so we have $g \in K^\times$ with $f' = g^\dagger$ and thus $\mathrm{e}^f = cg$ for some $c \in \mathbb{C}^\times$. Therefore $\mathrm{e}^f \in K$. This shows (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is clear. Assume (iii), and let $f \in \mathrm{I}(K)$. Then $f = g + hi$, $g, h \in \mathrm{I}(H)$. Taking $\phi, \theta \prec 1$ in H with $\phi' = g$ and $\theta' = h$,

$$\exp(\phi + \theta\mathrm{i}) = \exp(\phi)\big(\cos(\theta) + \sin(\theta)\mathrm{i}\big) \in H[\mathrm{i}] = K$$

has the property that $f = \big(\exp(\phi + \theta i)\big)^\dagger \in K^\dagger$. This shows (iii) $\Rightarrow$ (i). $\square$

From Propositions 5.5.12 and 5.5.18 we obtain:

Corollary 5.5.19. *If H is* d*-perfect, then* $\mathrm{I}(K) \subseteq K^{\dagger}$.

Next we consider "polar coordinates" of nonzero elements of K:

Lemma 5.5.20. *Let $f \in \mathcal{C}[\mathrm{i}]^{\times}$. Then $|f| \in \mathcal{C}^{\times}$, and there exists $\phi \in \mathcal{C}$ with $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$; such ϕ is unique up to addition of an element of $2\pi\mathbb{Z}$. If also $f \in \mathcal{C}^r[\mathrm{i}]^{\times}$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, then $|f| \in \mathcal{C}^r$ and $\phi \in \mathcal{C}^r$ for such ϕ.*

Proof. The claims about $|f|$ are clearly true. To show existence of ϕ we may replace f by $f/|f|$ to arrange $|f| = 1$. Take $a \in \mathbb{R}$ and a representative of f in $\mathcal{C}_a[\mathrm{i}]$, also denoted by f, such that $|f(t)| = 1$ for all $t \geqslant a$. The proof of [57, (9.8.1)] shows that for $b \in (a, +\infty)$ and $\phi_a \in \mathbb{R}$ with $f(a) = \mathrm{e}^{\phi_a i}$ there is a unique continuous function $\phi \colon [a, b] \to \mathbb{R}$ such that $\phi(a) = \phi_a$ and $f(t) = \mathrm{e}^{\phi(t)\mathrm{i}}$ for all $t \in [a, b]$, and if also $f|_{[a,b]}$ is of class $\mathcal{C}^1$, then so is this ϕ with $\mathrm{i}\phi'(t) = f'(t)/f(t)$ for all $t \in [a, b]$. With $b \to +\infty$ this yields the desired result. $\square$

Lemma 5.5.21. *Suppose $H \supseteq \mathbb{R}$ is Liouville closed and $f \in \mathcal{C}^1[\mathrm{i}]^{\times}$. Then $f^{\dagger} \in K$ iff $|f| \in H^{>}$ and $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$ for some $\phi \in H$. If in addition $f \in K^{\times}$, then $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$ for some $\phi \preccurlyeq 1$ in H.*

Proof. Take $\phi \in \mathcal{C}$ as in Lemma 5.5.20. Then $\phi \in \mathcal{C}^1$ and $\operatorname{Re} f^{\dagger} = |f|^{\dagger}$, $\operatorname{Im} f^{\dagger} = \phi'$. If $f \in K^{\times}$, then the remarks preceding Lemma 1.2.16 give $\phi' \in \mathrm{I}(H)$, so $\phi \preccurlyeq 1$. $\square$

Corollary 5.5.22. *Suppose $H \supseteq \mathbb{R}$ is Liouville closed with $\mathrm{I}(K) \subseteq K^{\dagger}$. Let L be a differential subfield of $\mathcal{C}^{<\infty}[\mathrm{i}]$ containing K. Then $L^{\dagger} \cap K = K^{\dagger}$.*

Proof. Let $f \in L^{\times}$ satisfy $f^{\dagger} \in K$. Then $f = |f|\,\mathrm{e}^{\phi\mathrm{i}}$ with $|f| \in H^{>}$ and $\phi \in H$, by Lemma 5.5.21. Hence $\mathrm{e}^{\phi\mathrm{i}}, \mathrm{e}^{-\phi\mathrm{i}} \in L$ and so $\cos\phi = \frac{1}{2}(\mathrm{e}^{\phi\mathrm{i}} + \mathrm{e}^{-\phi\mathrm{i}}) \in L$. In particular, $\cos\phi$ does not oscillate, so $\phi \preccurlyeq 1$ and thus $f = |f|(\cos\phi + i\sin\phi) \in K$ by Proposition 5.5.18. $\square$

Corollary 5.5.23. *Let $\phi \in H$, and suppose $\mathrm{e}^{\phi\mathrm{i}} \sim f$ with $f \in E[\mathrm{i}]^{\times}$ for some Hardy field extension E of H. Then $\phi \preccurlyeq 1$.*

Proof. We can assume that $E = H$ is Liouville closed and contains $\mathbb{R}$. Towards a contradiction assume $\phi \succ 1$. Lemma 5.5.21 yields $\theta \preccurlyeq 1$ in H such that $f = |f|\,\mathrm{e}^{\theta\mathrm{i}}$. Then $\mathrm{e}^{(\phi-\theta)\mathrm{i}} \sim |f|$ and $\phi - \theta \sim \phi$. Thus replacing f, ϕ by $|f|$, $\phi - \theta$, respectively, we arrange $f \in H^{\times}$. Then $\mathrm{e}^{\phi\mathrm{i}} = \cos\phi + i\sin\phi \sim f$ in $\mathcal{C}^{<\infty}[\mathrm{i}]$ gives $\cos\phi \sim f$, contradicting that $\cos\phi$ has arbitrarily large zeros. $\square$

Corollary 5.5.24. *Let $f \in K^{\times}$, $\phi \in H$, so $y := f\,\mathrm{e}^{\phi\mathrm{i}} \in \mathcal{C}^{<\infty}[\mathrm{i}]^{\times}$. Then the following are equivalent:*

(i) $\phi \preccurlyeq 1$;
(ii) $y \in \mathrm{D}(H)[\mathrm{i}]$;
(iii) $y \in E[\mathrm{i}]$ *for some Hardy field extension E of H;*
(iv) $y \sim g$ *for some Hardy field extension E of H and $g \in E[\mathrm{i}]^{\times}$.*

Proof. Use Proposition 5.5.18 and Corollaries 5.5.19 and 5.5.23 to obtain the chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). $\square$

Finally, some observations about solutions to linear differential equations involving trigonometric functions.

Lemma 5.5.25. *Let $A \in K[\partial]^{\neq}$ and $\phi \in H$. Then $A(K\,\mathrm{e}^{\phi\mathrm{i}}) \subseteq K\,\mathrm{e}^{\phi\mathrm{i}}$. Moreover, if K is r-linearly surjective with $r := \operatorname{order} A$, or K is 1-linearly surjective and A splits over K, then $A(K\,\mathrm{e}^{\phi\mathrm{i}}) = K\,\mathrm{e}^{\phi\mathrm{i}}$.*

Proof. The differential operator $B := A_{\ltimes \mathrm{e}^{\phi i}} = \mathrm{e}^{-\phi i} A\, \mathrm{e}^{\phi i} \in (\mathcal{C}^{<\infty}[i])[\partial]$ of order r has coefficients in K. This follows from extending [ADH, 5.8.8] by allowing the element h there (which is $\mathrm{e}^{\phi i}$ here) to be a unit in a differential ring extension of K instead of a nonzero element in a differential field extension of K; the proof of [ADH, 5.8.8] goes through, *mutatis mutandis*, to give this extension. Thus if $y \in K$, then $A(y\, \mathrm{e}^{\phi i}) = B(y)\, \mathrm{e}^{\phi i}$. Also, if A splits over K, then so does B. Hence if K is r-linearly surjective, or K is 1-linearly surjective and A splits over K, then for each $b \in K$ we obtain $y \in K$ with $B(y) = b$, and so $A(y\, \mathrm{e}^{\phi i}) = b\, \mathrm{e}^{\phi i}$. $\square$

Lemma 5.5.26. *Let $A \in H[\partial]^{\neq}$, and suppose K is r-linearly surjective with $r :=$ order A, or K is 1-linearly surjective and A splits over K. Let also $h, \phi \in H$. Then there are $f, g \in H$ such that $A(f \cos\phi + g \sin\phi) = h \cos\phi$.*

Proof. Lemma 5.5.25 gives $y \in K$ such that

$$A(y\, \mathrm{e}^{\phi i}) \;=\; h\, \mathrm{e}^{\phi i} \;=\; (h \cos\phi) + (h \sin\phi)i.$$

Take $f, g \in H$ with $y = f - gi$. Then

$$y\, \mathrm{e}^{\phi i} \;=\; (f \cos\phi + g \sin\phi) + (-g \cos\phi + f \sin\phi)i$$

and hence $A(f \cos\phi + g \sin\phi) \;=\; h \cos\phi$. $\square$

Lemma 5.5.27. *Let $f, g \in K$, $\phi \in H$, $\phi \succ 1$, and $f \cos\phi + g \sin\phi \in \mathbb{C} \subseteq \mathcal{C}[i]$. Then $f = g = 0$.*

Proof. Take $c \in \mathbb{C}$ such that $f \cos\phi + g \sin\phi = c$. Since $\phi \in H$ and $\phi \succ 1$, there are arbitrarily large t with $\phi(t) \in 2\mathbb{Z}\pi$, so $f(t) = c$, and thus $f = c$. There are also arbitrarily large t with $\phi(t) \in (2\mathbb{Z}+1)\pi$, and this gives likewise $-f = c$, so $f = c = 0$. Hence $g \sin\phi = 0$, which easily gives $g = 0$. $\square$

Combining Lemmas 5.5.26 and 5.5.27 gives:

Corollary 5.5.28. *If K is 1-linearly surjective, and $h, \phi \in H$, $\phi \succ 1$, then there are unique $f, g \in H$ such that $(f \cos\phi + g \sin\phi)' = h \cos\phi$.*

Behavior of σ and ω under composition. In this subsection we fix $\ell \in \mathcal{C}^1$ with $\ell > \mathbb{R}$ and $\phi := \ell' \in H$, so $\phi > 0$. We use the superscript $\circ$ as in the subsection on compositional conjugation in Hardy fields of Section 5.3. We refer to [ADH, 11.8] (or Section 1.4) for the definition of the subsets $\Gamma(H)$, $\Lambda(H)$, and $\Delta(H)$ of H. The bijection

$$y \mapsto (y/\phi)^\circ \;:\; H \to H^\circ$$

restricts to bijections $\mathrm{I}(H) \to \mathrm{I}(H^\circ)$ and $\Gamma(H) \to \Gamma(H^\circ)$, and the bijection

$$z \mapsto \big((z + \phi^\dagger)/\phi\big)^\circ \;:\; H \to H^\circ$$

restricts to bijections $\Lambda(H) \to \Lambda(H^\circ)$ and $\Delta(H) \to \Delta(H^\circ)$. (See the transformation formulas in [ADH, p. 520].) Consider the bijection

$$f \mapsto \Phi(f) \;:=\; \big(\big(f - \omega(-\phi^\dagger)\big)/\phi^2\big)^\circ \;:\; H \to H^\circ.$$

Then for $y \in H^\times$, $z \in H$ we have

$$\sigma\big((y/\phi)^\circ\big) \;=\; \Phi\big(\sigma(y)\big), \qquad \omega\big(\big((z + \phi^\dagger)/\phi\big)^\circ\big) \;=\; \Phi\big(\omega(z)\big).$$

(See the formulas in [ADH, pp. 518–519].) Hence Φ restricts to bijections

$$\sigma(H^\times) \to \sigma\big((H^\circ)^\times\big), \quad \sigma\big(\mathrm{I}(H)^{\neq}\big) \to \sigma\big(\mathrm{I}(H^\circ)^{\neq}\big), \quad \sigma\big(\Gamma(H)\big) \to \sigma\big(\Gamma(H^\circ)\big),$$

and

$$\omega(H) \to \omega(H^\circ), \quad \omega\bigl(\Lambda(H)\bigr) \to \omega\bigl(\Lambda(H^\circ)\bigr), \quad \omega\bigl(\Delta(H)\bigr) \to \omega\bigl(\Delta(H^\circ)\bigr).$$

An example of compositional conjugation. Which "changes of variable" preserve the general form of the linear differential equation (4L)? The next lemma and Corollary 5.5.30 below give an answer.

Lemma 5.5.29. *Let K be a differential field, $f \in K$, and $P(Y) := 4Y'' + fY$. Then for $g \in K^\times$ and $\phi := g^{-2}$ we have*

$$g^3 P^\phi_{\times g}(Y) \;=\; 4Y'' + g^3 P(g)Y.$$

Proof. Let $g, \phi \in K^\times$. Then

$$\begin{aligned} P_{\times g}(Y) \;&=\; 4gY'' + 8g'Y' + (4g'' + fg)Y \;=\; 4gY'' + 8g'Y' + P(g)Y, \quad \text{so} \\ P^\phi_{\times g}(Y) \;&=\; 4g(\phi^2 Y'' + \phi' Y') + 8g'\phi Y' + P(g)Y \\ &=\; 4g\phi^2 Y'' + (4g\phi' + 8g'\phi)Y' + P(g)Y. \end{aligned}$$

Now $4g\phi' + 8g'\phi = 0$ is equivalent to $\phi^\dagger = -2g^\dagger$, which holds for $\phi = g^{-2}$. For this ϕ we get $P^\phi_{\times g}(Y) = g^{-3}\bigl(4Y'' + g^3P(g)Y\bigr)$, that is, $g^3 P^\phi_{\times g}(Y) = 4Y'' + g^3 P(g)Y$. □

Now let $\ell \in \mathcal{C}^1$ be such that $\ell > \mathbb{R}$ and $\phi := \ell' \in H$, and let $P := 4Y'' + fY$ where $f \in H$. Note that if $y \in \mathcal{C}^2[\mathrm{i}]$ and $4y'' + fy = 0$, then $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Towards using Lemma 5.5.29, suppose $\phi = g^{-2}$, $g \in H^\times$. Using notation from the previous subsection we set $h := (g^3 P(g))^\circ \in H^\circ$ to obtain the following reduction of solving the differential equation (4L) to solving a similar equation over H°:

Corollary 5.5.30. *Let $y \in \mathcal{C}^2[\mathrm{i}]$. Then $z := (y/g)^\circ \in \mathcal{C}^2[\mathrm{i}]$, and*

$$4y'' + fy \;=\; 0 \quad\Longleftrightarrow\quad 4z'' + hz \;=\; 0.$$

In particular, $f/4$ generates oscillations iff $h/4$ does. In connection with the formulas in the previous subsection, note that

$$g^3 P(g) \;=\; g^3(4g'' + fg) \;=\; \bigl(f - \omega(-\phi^\dagger)\bigr)/\phi^2,$$

so $h = \Phi(f)$.

Lemma 5.5.31. *The increasing bijection*

$$f \mapsto \Phi(f) \;=\; \Bigl(\bigl(f - \omega(-\phi^\dagger)\bigr)/\phi^2\Bigr)^\circ \;:\; H \to H^\circ$$

maps $\overline{\omega}(H)$ onto $\overline{\omega}(H^\circ)$.

Proof. First replace H by its real closure to arrange that H is real closed, then take $g \in H^\times$ with $g^{-2} = \phi$, and use the remarks following Corollary 5.5.30. □

We use the above to prove the Fite-Leighton-Wintner oscillation criterion for self-adjoint second-order linear differential equations over H [69, 126, 211]. (See also [99, §2] and [198, p. 45].) Let $A \in H[\partial]$ be self-adjoint of order 2. Then $A = f\partial^2 + f'\partial + g$ where $f, g \in H$, $f \neq 0$, by the example following Lemma 2.4.19. For $h \in \mathcal{C}$, let $\int h$ denote a germ in $\mathcal{C}^1$ with $(\int h)' = h$.

Corollary 5.5.32. *Suppose $\int f^{-1} > \mathbb{R}$ and $\int g > \mathbb{R}$. Then $A(y) = 0$ for some oscillating $y \in \mathcal{C}^{<\infty}$.*

Proof. We arrange that $H \supseteq \mathbb{R}$ is Liouville closed. Then $f^{-1}, g \in \Gamma(H)$ by [ADH, 11.8.19]. Note that $\phi := f^{-1}$ is active in H. Put $B := 4\phi A_{\ltimes \phi^{1/2}}$, so $B = 4\partial^2 + h$ with $h := \omega(-\phi^\dagger) + 4g\phi$. Then $A(y) = 0$ for some oscillating $y \in \mathcal{C}^{<\infty}$ iff $B(z) = 0$ for some oscillating $z \in \mathcal{C}^{<\infty}$ iff $h \notin \overline{\omega}(H)$, by Corollary 5.5.7. The latter is equivalent to $(4g/\phi)^\circ \notin \overline{\omega}(H^\circ)$, by Lemma 5.5.31 applied to h in place of f. Now $\Gamma(H^\circ) \cap \overline{\omega}(H^\circ) = \emptyset$ by Lemma 5.5.6, so it remains to note that $4g \in \Gamma(H)$ yields $(4g/\phi)^\circ \in \Gamma(H^\circ)$, by remarks in the previous subsection. $\square$

More about $\overline{\omega}(H)$ (*). For later use (in particular, in Section 7.5) we study here the downward closed subset $\overline{\omega}(H)$ of H in more detail. Recall that $\omega(H) \subseteq \overline{\omega}(H)$, with equality for d-perfect H. (Corollary 5.5.3.) In [ADH, 16.3] we introduced the concept of a $\Lambda\Omega$-*cut* in a pre-H-field; every pre-H-field has exactly one or exactly two $\Lambda\Omega$-cuts [ADH, remark before 16.3.19]. By [ADH, 16.3.14, 16.3.16]:

Lemma 5.5.33. *Suppose H is* d*-perfect. Then $\big(\mathrm{I}(H), \Lambda(H), \overline{\omega}(H)\big)$ is a $\Lambda\Omega$-cut in H, and this is the unique $\Lambda\Omega$-cut in H iff H is $\upomega$-free.*

Thus in general,

$$\big(\mathrm{I}\big(\mathrm{D}(H)\big) \cap H,\ \Lambda\big(\mathrm{D}(H)\big) \cap H,\ \overline{\omega}(H)\big)$$

is a $\Lambda\Omega$-cut in H, and hence (see [ADH, p. 692]):

$$\omega(H)^{\downarrow}\ \subseteq\ \overline{\omega}(H)\ \subseteq\ H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}.$$

The classification of $\Lambda\Omega$-cuts in H from [ADH, 16.3] can be used to narrow down the possibilities for $\overline{\omega}(H)$:

Lemma 5.5.34. *Let $\phi \in H^>$ be such that $v\phi \notin (\Gamma_H^{\neq})'$. Then*

$$\overline{\omega}(H)\ =\ \omega(-\phi^\dagger) + \phi^2 \mathcal{o}_H^{\downarrow} \quad \textit{or} \quad \overline{\omega}(H)\ =\ \omega(-\phi^\dagger) + \phi^2 \mathcal{O}_H^{\downarrow}.$$

The first alternative holds if H is grounded, and the second alternative holds if $v\phi$ is a gap in H with $\phi \asymp b'$ for some $b \asymp 1$ in H.

Proof. Either $v\phi = \max \Psi_H$ or $v\phi$ is a gap in H, by [ADH, 9.2]. The remark before the lemma yields an $\Lambda\Omega$-cut (I, Λ, Ω) in H where $\Omega = \overline{\omega}(H)$. Now use the proofs of [ADH, 16.3.11, 16.3.12, 16.3.13] together with the transformation formulas [ADH, (16.3.1)] for $\Lambda\Omega$-cuts. $\square$

By [ADH, 16.3.15] we have:

Lemma 5.5.35. *If H has asymptotic integration and the set $2\Psi_H$ does not have a supremum in Γ_H, then*

$$\overline{\omega}(H)\ =\ \omega\big(\Lambda(H)\big)^{\downarrow}\ =\ \omega(H)^{\downarrow} \quad \textit{or} \quad \overline{\omega}(H)\ =\ H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}.$$

Corollary 5.5.36. *Suppose H is $\upomega$-free. Then*

$$\overline{\omega}(H)\ =\ \omega\big(\Lambda(H)\big)^{\downarrow}\ =\ \omega(H)^{\downarrow}\ =\ H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}.$$

Proof. By [ADH, 11.8.30] we have $\omega\big(\Lambda(H)\big)^{\downarrow} = \omega(H)^{\downarrow} = H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$. It follows from [ADH, 9.2.19] that $2\Psi_H$ has no supremum in Γ_H. Now use Lemma 5.5.35. $\square$

In the next lemma $L \supseteq \mathbb{R}$ is a Liouville closed d-algebraic Hardy field extension of H such that $\omega(L) = \overline{\omega}(L)$. (By Corollary 5.5.3, this holds for $L = \mathrm{D}(H)$.) Note that then $\overline{\omega}(L) = \omega\big(\Lambda(L)\big)$ by [ADH, 11.8.20].

Lemma 5.5.37. *If H is not $\uplambda$-free or $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$, then L is $\upomega$-free.*

Proof. If H is $\upomega$-free or not $\uplambda$-free, then L is $\upomega$-free by Lemmas 1.4.18 and 1.4.20. Suppose H is $\uplambda$-free but not $\upomega$-free, and $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$. So [ADH, 11.8.30] gives $\upomega \in H$ with $\omega\big(\Lambda(H)\big) < \upomega < \sigma\big(\Gamma(H)\big)$. Then $\upomega \in \overline{\omega}(H) \subseteq \overline{\omega}(L) \subseteq \omega\big(\Lambda(L)\big)$. Thus L is $\upomega$-free by Corollary 1.4.21. $\square$

Theorem 7.5.32, which depends on much of what follows, shows that for $L = \mathrm{D}(H)$ the converse of Lemma 5.5.37 also holds.

Proof of a conjecture of Boshernitzan $(^*)$**.** In this last subsection we establish [33, Conjecture 17.11]: Corollary 5.5.40. For this, with $\ell_n := \log_n x$ we set $\upgamma_n := \ell_n^\dagger$, $\uplambda_n := -\upgamma_n^\dagger$, and $\upomega_n := \omega(\uplambda_n)$, as in [ADH, 11.5, 11.7], so

$$\upgamma_n \ = \ \frac{1}{\ell_0\ell_1\cdots\ell_n}, \qquad \upomega_n \ = \ \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2}.$$

(See also the beginning of Section 5.6 below.) For $c \in \mathbb{R}$, the germ

$$\frac{\upomega_n + c\upgamma_n^2}{4} \ = \ \frac{1}{4}\left(\frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \cdots + \frac{1}{(\ell_0\cdots\ell_{n-1})^2} + \frac{c+1}{(\ell_0\cdots\ell_n)^2}\right)$$

generates oscillations iff $c > 0$. (A. Kneser [112], Riemann-Weber [206, p. 63]; cf. [97].) This follows from the next corollary applied to $f = \upomega_n + c\upgamma_n^2$ and the grounded Hardy subfield $H := \mathbb{R}\langle \ell_n\rangle = \mathbb{R}(\ell_0,\dots,\ell_n)$ of $\operatorname{Li}(\mathbb{R})$:

Corollary 5.5.38. *Let H be a grounded Hardy field such that for some m we have $h \succ \ell_m$ for all $h \in H$ with $h \succ 1$. Then for $f \in H$, the following are equivalent:*

(i) $f \in \overline{\omega}(H)$;
(ii) $f < \upomega_n$ *for some* n;
(iii) *there exists* $c \in \mathbb{R}^>$ *such that for all* n *we have* $f < \upomega_n + c\upgamma_n^2$;
(iv) $f < \upomega_n + c\upgamma_n^2$ *for all* n *and all* $c \in \mathbb{R}^>$.

Proof. By [ADH, 10.3.2, 10.5.15] we may replace H by $H(\mathbb{R})$ to arrange $H \supseteq \mathbb{R}$. By Lemma 1.4.18, $L := \operatorname{Li}(H)$ is $\upomega$-free. With $H_\upomega$ as in the proof of that lemma, one verifies easily that for each $g \in H_\upomega$ with $g \succ 1$ there is an m such that $g \succ \ell_m$. Hence (ℓ_n) is a logarithmic sequence in L, in the sense of [ADH, p. 499]. Now the implication (i) $\Rightarrow$ (iv) follows from Corollary 5.5.36 and [ADH, 11.8.22], and (iv) $\Rightarrow$ (iii) is obvious. Since $0 < \upgamma_{n+1} \prec \upgamma_n$ we obtain for $c \in \mathbb{R}^>$:

$$\upomega_{n+1} + c\upgamma_{n+1}^2 \ = \ \upomega_n + \upgamma_{n+1}^2 + c\upgamma_{n+1}^2 \ < \ \upomega_n + \upgamma_n^2 \ = \ \sigma(\upgamma_n).$$

In view of [ADH, 11.8.30, 11.8.21] and Corollary 5.5.36, this yields (iii) $\Rightarrow$ (ii). Downward closedness of $\overline{\omega}(H)$ implies (ii) $\Rightarrow$ (i). $\square$

Using the above equivalence (i) $\Leftrightarrow$ (ii) we recover [33, Theorem 17.7]:

Corollary 5.5.39. *Suppose $f \in \mathcal{C}$ is hardian and* d*-algebraic over $\mathbb{R}$. Then*

$$f \text{ generates oscillations} \iff f > \upomega_n/4 \text{ for all } n.$$

Proof. By Corollary 5.4.28 the Hardy field $H := \mathbb{R}\langle f\rangle$ satisfies the hypotheses of Corollary 5.5.38. Also, f generates oscillations iff $4f \notin \overline{\omega}(H)$. $\square$

Using the above implication (iii) $\Rightarrow$ (i) we obtain in the same way:

Corollary 5.5.40. *Let $f \in \mathcal{C}$ be hardian and* d*-algebraic over $\mathbb{R}$, and suppose there is a $c \in \mathbb{R}^>$ with $f < \upomega_n + c\upgamma_n^2$ for all n. Then $f/4$ does not generate oscillations.*

In the beginning of this subsection we introduced the germs ℓ_n, and so this may be a good occasion to observe that the Hardy field $H = \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$ they generate over $\mathbb{R}$ is ω-free: since H is ungrounded and H is the union of the grounded Hardy subfields $\mathbb{R}(\ell_0, \dots, \ell_n)$, this follows from [ADH, 11.7.15]. Thus the Hardy field $\operatorname{Li}(\mathbb{R}) = \operatorname{Li}(H)$ is ω-free as well.

5.6. Maximal Hardy Fields are ω-Free

In this section we discuss the fundamental property of ω-freeness from [ADH] in the context of Hardy fields. The main result is Theorem 5.6.2, from which it follows that every maximal Hardy field is ω-free. As an application of this theorem, we answer a question from [34].

The property of ω-freeness for Hardy fields. Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field. Note that then $x \in H$ and $\log f \in H$ for all $f \in H^{>}$. To work with ω-freeness for H we introduce the "iterated logarithms" ℓ_ρ; more precisely, transfinite recursion yields a sequence (ℓ_ρ) in $H^{>\mathbb{R}}$ indexed by the ordinals ρ less than some infinite limit ordinal κ as follows: $\ell_0 = x$, and $\ell_{\rho+1} := \log \ell_\rho$; if λ is an infinite limit ordinal such that all ℓ_ρ with $\rho < \lambda$ have already been chosen, then we pick ℓ_λ to be any element in $H^{>\mathbb{R}}$ such that $\ell_\lambda \prec \ell_\rho$ for all $\rho < \lambda$, if there is such an ℓ_λ, while if there is no such ℓ_λ, we put $\kappa := \lambda$. From (ℓ_ρ) we obtain the sequences (γ_ρ) in $H^{>}$ and (λ_ρ) in H as follows:

$$\gamma_\rho \ := \ \ell_\rho^\dagger, \qquad \lambda_\rho \ := \ -\gamma_\rho^\dagger \ = \ -\ell_\rho^{\dagger\dagger} \ := \ -(\ell_\rho^{\dagger\dagger}).$$

Then $\lambda_{\rho+1} = \lambda_\rho + \gamma_{\rho+1}$ and we have

$$\begin{aligned} \gamma_0 \ &= \ \ell_0^{-1}, & \gamma_1 \ &= \ (\ell_0\ell_1)^{-1}, & \gamma_2 \ &= \ (\ell_0\ell_1\ell_2)^{-1}, \\ \lambda_0 \ &= \ \ell_0^{-1}, & \lambda_1 \ &= \ \ell_0^{-1} + (\ell_0\ell_1)^{-1}, & \lambda_2 \ &= \ \ell_0^{-1} + (\ell_0\ell_1)^{-1} + (\ell_0\ell_1\ell_2)^{-1}, \end{aligned}$$

and so on. Indeed, $v(\gamma_\rho)$ is strictly increasing as a function of ρ and is cofinal in $\Psi_H = \big\{v(f^\dagger) : f \in H,\ 0 \neq f \not\asymp 1\big\}$; we refer to [ADH, 11.5, 11.8] for this and some of what follows. Also, (λ_ρ) is a strictly increasing pc-sequence which is cofinal in $\Lambda(H) \subseteq H$. We recall here the relevant descriptions from [ADH, 11.8]:

$$\begin{aligned} \Gamma(H) \ &= \ \big\{a^\dagger :\ a \in H,\ a \succ 1\big\} \ &&= \ \{b \in H :\ b > \gamma_\rho \text{ for some } \rho\}, \\ \Lambda(H) \ &= \ -\Gamma(H)^\dagger \ &&= \ \big\{-a^{\dagger\dagger} :\ a \in H,\ a \succ 1\big\}. \end{aligned}$$

Here, $\Gamma(H) \subseteq H^{>}$ is upward closed and $\Lambda(H)$ is downward closed, since H is Liouville closed. The latter also gives that H is λ-free, that is, (λ_ρ) has no pseudolimit in H. The function $\omega\colon H \to H$ is strictly increasing on $\Lambda(H)$ and setting $\omega_\rho := \omega(\lambda_\rho)$ we obtain a strictly increasing pc-sequence (ω_ρ) which is cofinal in $\omega\big(\Lambda(H)\big) = \omega(H)$:

$$\omega_0 \ = \ \ell_0^{-2}, \qquad \omega_1 \ = \ \ell_0^{-2} + (\ell_0\ell_1)^{-2}, \qquad \omega_2 \ = \ \ell_0^{-2} + (\ell_0\ell_1)^{-2} + (\ell_0\ell_1\ell_2)^{-2},$$

and so on; see [ADH, 11.7, 11.8] for this and some of what follows. Now H being ω-free is equivalent to (ω_ρ) having no pseudolimit in H. By [ADH, 11.8.30] the pseudolimits of (ω_ρ) in H are exactly the $\omega \in H$ such that $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$. Also, σ is strictly increasing on $\Gamma(H)$. Thus H is not ω-free if and only if there exists an $\omega \in H$ such that $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$.

Lemma 5.6.1. *Let $\gamma \in (\mathcal{C}^1)^\times$, $\gamma > 0$, and $\lambda := -\gamma^\dagger$ with $\lambda_\rho < \lambda < \lambda_\rho + \gamma_\rho$ in $\mathcal{C}$, for all ρ. Then $\gamma_\rho > \gamma > \gamma_\rho/\ell_\rho = (-1/\ell_\rho)'$ in $\mathcal{C}$, for all ρ.*

Proof. Pick $a \in \mathbb{R}$ (independent of ρ) and functions in $\mathcal{C}_a$ whose germs at $+\infty$ are the elements ℓ_ρ, $\upgamma_\rho$, $\uplambda_\rho$ of H; denote these functions also by ℓ_ρ, $\uplambda_\rho$, $\upgamma_\rho$. From $\ell_\rho^\dagger = \upgamma_\rho$ and $\upgamma_\rho^\dagger = -\uplambda_\rho$ in H we obtain $c_\rho, d_\rho > 0$ such that for all sufficiently large $t \geqslant a$,

$$\ell_\rho(t) \;=\; c_\rho \exp\left[\int_a^t \upgamma_\rho(s)\,ds\right], \quad \upgamma_\rho(t) \;=\; d_\rho \exp\left[-\int_a^t \uplambda_\rho(s)\,ds\right].$$

(How large is "sufficiently large" depends on ρ.) Likewise we pick functions in $\mathcal{C}_a$ whose germ at $+\infty$ are $\upgamma$, $\uplambda$, and also denote these functions by $\upgamma$, $\uplambda$. From $\upgamma^\dagger = -\uplambda$ in H we obtain a real constant $d > 0$ such that for all sufficiently large $t \geqslant a$,

$$\upgamma(t) \;=\; d \exp\left[-\int_a^t \uplambda(s)\,ds\right].$$

Also, $\uplambda_\rho < \uplambda < \uplambda_\rho + \upgamma_\rho$ yields constants $a_\rho, b_\rho \in \mathbb{R}$ such that for all $t \geqslant a$

$$\int_a^t \uplambda_\rho(s)\,ds \;<\; a_\rho + \int_a^t \uplambda(s)\,ds \;<\; b_\rho + \int_a^t \uplambda_\rho(s)\,ds + \int_a^t \upgamma_\rho(s)\,ds,$$

which by applying $\exp(-*)$ yields that for all sufficiently large $t \geqslant a$,

$$\frac{1}{d_\rho}\upgamma_\rho(t) \;>\; \frac{1}{\mathrm{e}^{a_\rho}\,d}\upgamma(t) \;>\; \frac{c_\rho}{\mathrm{e}^{b_\rho}\,d_\rho}\upgamma_\rho(t)/\ell_\rho(t).$$

Here the positive constant factors don't matter, since the valuation of $\upgamma_\rho$ is strictly increasing and that of $\upgamma_\rho/\ell_\rho = (-1/\ell_\rho)'$ is strictly decreasing with ρ. Thus for all ρ we have $\upgamma_\rho > \upgamma > \upgamma_\rho/\ell_\rho = (-1/\ell_\rho)'$, in $\mathcal{C}$. □

We are now ready to prove a key result:

Theorem 5.6.2. *Every Hardy field has a* d*-algebraic* $\upomega$*-free Hardy field extension.*

Proof. It is enough to show that every d-maximal Hardy field is $\upomega$-free. That reduces to showing that every non-$\upomega$-free Liouville closed Hardy field containing $\mathbb{R}$ has a proper d-algebraic Hardy field extension. So assume $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and H is not $\upomega$-free. We shall construct a proper d-algebraic Hardy field extension of H. We have $\upomega \in H$ such that

$$\omega(H) \;<\; \upomega \;<\; \sigma\big(\Gamma(H)\big).$$

With $\upomega$ in the role of f in the discussion following Corollary 5.5.10, we have $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^2$ of the differential equation $4Y'' + \upomega Y = 0$; in fact, $y_1, y_2 \in \mathcal{C}^{<\infty}$. Then the complex solution $y = y_1 + y_2 i$ is a unit of $\mathcal{C}^{<\infty}[i]$, and so we have $z := 2y^\dagger \in \mathcal{C}^{<\infty}[i]$. We shall prove that the elements $\uplambda := \operatorname{Re} z$ and $\upgamma := \operatorname{Im} z$ of $\mathcal{C}^{<\infty}$ generate a Hardy field extension $K = H(\uplambda, \upgamma)$ of H with $\upomega = \sigma(\upgamma) \in \sigma(K^\times)$. We can assume that $w := y_1 y_2' - y_1' y_2 \in \mathbb{R}^>$, so $\upgamma = 2w/|y|^2 \in (\mathcal{C}^{<\infty})^\times$ and $\upgamma > 0$.

We have $\upomega_\rho \rightsquigarrow \upomega$, with $\upomega - \upomega_\rho \sim \upgamma_{\rho+1}^2$ by [ADH, 11.7.1]. Till further notice we fix ρ and set $g_\rho := \upgamma_\rho^{-1/2}$, so $2g_\rho^\dagger = \uplambda_\rho = -\upgamma_\rho^\dagger$. For $h \in H^\times$ we also have $\omega(2h^\dagger) = -4h''/h$, hence $P := 4Y'' + \upomega Y \in H\{Y\}$ gives

$$P(g_\rho) \;=\; g_\rho(\upomega - \upomega_\rho) \;\sim\; g_\rho \upgamma_{\rho+1}^2,$$

and so with an eye towards using Lemma 5.5.29:

$$g_\rho^3 P(g_\rho) \;\sim\; g_\rho^4 \upgamma_{\rho+1}^2 \;\sim\; \upgamma_{\rho+1}^2/\upgamma_\rho^2 \;\asymp\; 1/\ell_{\rho+1}^2.$$

Thus with $g := g_\rho = \upgamma_\rho^{-1/2}$, $\phi = g^{-2} = \upgamma_\rho$ we have $A_\rho \in \mathbb{R}^>$ such that

$$g^3 P^\phi_{\times g}(Y) \;=\; 4Y'' + g^3 P(g)Y, \quad |g^3 P(g)| \;\leqslant\; A_\rho/\ell^2_{\rho+1}. \tag{5.6.1}$$

From $P(y) = 0$ we get $P^\phi_{\times g}(y/g) = 0$, that is, $y/g \in \mathcal{C}^{<\infty}[\mathrm{i}]^\phi$ is a solution of $4Y'' + g^3P(g)Y = 0$, with $g^3P(g) \in H \subseteq \mathcal{C}^{<\infty}$. Set $\ell := \ell_{\rho+1}$, so $\ell' = \ell_\rho^\dagger = \phi$. The subsection on compositional conjugation in Section 5.3 yields the isomorphism $h \mapsto h^\circ = h \circ \ell^{\text{inv}} \colon H^\phi \to H^\circ$ of H-fields, where ℓ^{inv} is the compositional inverse of ℓ. Under this isomorphism the equation $4Y'' + g^3P(g)Y = 0$ corresponds to the equation

$$4Y'' + f_\rho Y \;=\; 0, \qquad f_\rho \;:=\; (g^3P(g))^\circ \in H^\circ \;\subseteq\; \mathcal{C}^{<\infty}.$$

By Corollary 5.5.30, the equation $4Y'' + f_\rho Y = 0$ has the "real" solutions

$$y_{j,\rho} \;:=\; (y_j/g)^\circ \in (\mathcal{C}^{<\infty})^\circ \;=\; \mathcal{C}^{<\infty} \qquad (j = 1, 2),$$

and the "complex" solution

$$y_\rho \;:=\; y_{1,\rho} + y_{2,\rho}\mathrm{i} \;=\; (y/g)^\circ,$$

which is a unit of the ring $\mathcal{C}^{<\infty}[\mathrm{i}]$. Set $z_\rho := 2y_\rho^\dagger \in \mathcal{C}^{<\infty}[\mathrm{i}]$. The bound in (5.6.1) gives $|f_\rho| \;\leqslant\; A_\rho/x^2$, which by Corollary 5.2.20 yields positive constants B_ρ, c_ρ such that $|z_\rho| \;\leqslant\; B_\rho x^{c_\rho}$. Using $(f^\circ)' = (\phi^{-1}f')^\circ$ for $f \in \mathcal{C}^{<\infty}[i]$ we obtain

$$z_\rho \;=\; 2\big((y/g)^\circ\big)^\dagger \;=\; 2\big(\phi^{-1}(y/g)^\dagger\big)^\circ \;=\; \big((z - 2g^\dagger)/\phi\big)^\circ$$

In combination with the bound on $|z_\rho|$ this yields

$$\begin{aligned} \left|\frac{z - 2g^\dagger}{\phi}\right| \;&\leqslant\; B_\rho\, \ell_{\rho+1}^{c_\rho}, \quad \text{hence} \\ |z - \uplambda_\rho| \;&\leqslant\; B_\rho\, \ell_{\rho+1}^{c_\rho}\, \phi \;=\; B_\rho\, \ell_{\rho+1}^{c_\rho}\, \upgamma_\rho, \quad \text{and so} \\ z \;&=\; \uplambda_\rho + R_\rho \quad \text{where} \quad |R_\rho| \leqslant B_\rho\, \ell_{\rho+1}^{c_\rho}\, \upgamma_\rho. \end{aligned}$$

We now use this last estimate with $\rho + 1$ instead of ρ, together with

$$\uplambda_{\rho+1} \;=\; \uplambda_\rho + \upgamma_{\rho+1}, \quad \ell_{\rho+1}\upgamma_{\rho+1} \;=\; \upgamma_\rho.$$

This yields

$$\begin{aligned} z \;&=\; \uplambda_\rho + \upgamma_{\rho+1} + R_{\rho+1} \\ &\quad \text{with } |R_{\rho+1}| \;\leqslant\; B_{\rho+1}\, \ell_{\rho+2}^{c_{\rho+1}}\, \upgamma_{\rho+1} \;=\; B_{\rho+1}\big(\ell_{\rho+2}^{c_{\rho+1}}/\ell_{\rho+1}\big)\,\upgamma_\rho, \\ \text{so} \quad z \;&=\; \uplambda_\rho + o(\upgamma_\rho) \;\; \text{that is, } z - \uplambda_\rho \prec \upgamma_\rho, \\ \text{and thus} \quad \uplambda \;&=\; \operatorname{Re} z \;=\; \uplambda_\rho + o(\upgamma_\rho), \quad \upgamma \;=\; \operatorname{Im} z \;\prec\; \upgamma_\rho. \end{aligned}$$

Now varying ρ again, $(\uplambda_\rho)$ is a strictly increasing divergent pc-sequence in H which is cofinal in $\Lambda(H)$. By the above, $\uplambda = \operatorname{Re} z$ satisfies $\Lambda(H) < \uplambda < \Delta(H)$. This yields an ordered subfield $H(\uplambda)$ of $\mathcal{C}^{<\infty}$, which by Lemma 5.1.17 is an immediate valued field extension of H with $\uplambda_\rho \rightsquigarrow \uplambda$. Now $\uplambda = -\upgamma^\dagger$ (see discussion before Lemma 5.5.11), so Lemma 5.6.1 gives $\upgamma_\rho > \upgamma > (-1/\ell_\rho)'$ in $\mathcal{C}^{<\infty}$, for all ρ. In view of Lemma 5.1.18 applied to $H(\uplambda)$, $\upgamma$ in the role of K, f this yields an ordered subfield $H(\uplambda, \upgamma)$ of $\mathcal{C}^{<\infty}$. Moreover, $\upgamma$ is transcendental over $H(\uplambda)$ and $\upgamma$ satisfies the second-order differential equation $2yy'' - 3(y')^2 + y^4 - \upomega y^2 = 0$ over H (obtained from the relation $\sigma(\upgamma) = \upomega$ by multiplication with $\upgamma^2$). It follows that $H(\uplambda, \upgamma)$ is closed under the derivation of $\mathcal{C}^{<\infty}$, and hence $H(\uplambda, \upgamma) = H\langle\upgamma\rangle$ is a Hardy field that is d-algebraic over H. □

The proof also shows that every $\mathcal{C}^{\infty}$-Hardy field has an ω-free d-algebraic $\mathcal{C}^{\infty}$-Hardy field extension, and the same with $\mathcal{C}^{\omega}$ instead of $\mathcal{C}^{<\infty}$. In Section 7.5 below we show that the perfect hull of an ω-free Hardy field remains ω-free (Lemma 7.5.39), but that not every perfect Hardy field is ω-free (Example 7.5.40).

Improving Theorem 5.6.2 (*)**.** *In this subsection $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and $\omega \in H$, $\gamma \in (\mathcal{C}^2)^{\times}$ satisfy $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$ and $\sigma(\gamma) = \omega$.* Lemma 1.4.18 leads to a more explicit version of Theorem 5.6.2:

Corollary 5.6.3. *The germ γ generates a Hardy field extension $H\langle\gamma\rangle$ of H with a gap $v\gamma$, and so $\operatorname{Li}\big(H\langle\gamma\rangle\big)$ is an ω-free Hardy field extension of H.*

Proof. Since $\sigma(-\gamma) = \sigma(\gamma)$, we may arrange $\gamma > 0$. The discussion before Lemma 5.5.11 with ω, γ in the roles of f, g, respectively, yields $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^{<\infty}$ of the differential equation $4Y'' + \omega Y = 0$ with Wronskian $1/2$ such that $\gamma = 1/(y_1^2 + y_2^2)$. The proof of Theorem 5.6.2 shows that γ generates a Hardy field extension $H\langle\gamma\rangle = H(\lambda, \gamma)$ of H. Recall that $v(\gamma_\rho)$ is strictly increasing as a function of ρ and cofinal in Ψ_H; as $\gamma \prec \gamma_\rho$ for all ρ, this gives $\Psi_H < v\gamma$. Also $\gamma > (-1/\ell_\rho)' > 0$ for all ρ and $v(1/\ell_\rho)'$ is strictly decreasing as a function of ρ and coinitial in $(\Gamma_H^{>})'$, and so $v\gamma < (\Gamma_H^{>})'$. Then by [ADH, 13.7.1 and subsequent remark (2) on p. 626], $v\gamma$ is a gap in $H\langle\gamma\rangle$, so $\operatorname{Li}\big(H\langle\gamma\rangle\big)$ is ω-free by Lemma 1.4.18. □

Corollary 5.6.4. *Suppose $\gamma > 0$. Then with $L := \operatorname{Li}\big(H\langle\gamma\rangle\big)$,*

$$\omega \notin \overline{\omega}(H) \iff \gamma \in \Gamma(L), \qquad \omega \in \overline{\omega}(H) \iff \gamma \in \mathrm{I}(L).$$

Proof. If $\gamma \notin \Gamma(L)$, then $\omega \in \omega(L)^{\downarrow}$ by [ADH, 11.8.31], hence $\omega \in \overline{\omega}(H)$. If $\gamma \in \Gamma(L)$, then we can use Corollary 5.5.36 for L to conclude $\omega \notin \overline{\omega}(H)$. The equivalence on the right now follows from that on the left and [ADH, 11.8.19]. □

We also note that if $\omega/4$ generates oscillations, then we have many choices for γ:

Corollary 5.6.5. *Suppose $\omega/4$ generates oscillations. Then there are continuum many $\widetilde{\gamma} \in (\mathcal{C}^{<\infty})^{\times}$ with $\widetilde{\gamma} > 0$ and $\sigma(\widetilde{\gamma}) = \omega$, and no Hardy field extension of H contains more than one such germ $\widetilde{\gamma}$.* (*In particular, H has continuum many maximal Hardy field extensions.*)

Proof. As before we arrange $\gamma > 0$ and set $L := \operatorname{Li}\big(H\langle\gamma\rangle\big)$. Take $\phi \in L$ with $\phi' = \frac{1}{2}\gamma$ and consider the germs

$$y_1 := \frac{1}{\sqrt{\gamma}}\cos\phi, \quad y_2 := \frac{1}{\sqrt{\gamma}}\sin\phi \qquad \text{in } \mathcal{C}^{<\infty}.$$

The remarks preceding Lemma 5.5.11 show: y_1, y_2 solve the differential equation $4Y'' + \omega Y = 0$, their Wronskian equals $1/2$, and $\phi \succ 1$ (since $\omega/4$ generates oscillations). We now dilate y_1, y_2: let $r \in \mathbb{R}^{>}$ be arbitrary and set

$$y_{1r} := ry_1, \qquad y_{2r} := r^{-1}y_2.$$

Then y_{1r}, y_{2r} still solve the equation $4Y'' + \omega Y = 0$, and their Wronskian is $1/2$. Put $\gamma_r := 1/(y_{1r}^2 + y_{2r}^2) \in \mathcal{C}^{<\infty}$. Then $\sigma(\gamma_r) = \omega$. Let $r, s \in \mathbb{R}^{>}$. Then

$$\gamma_r = \gamma_s \iff y_{1r}^2 + y_{2r}^2 = y_{1s}^2 + y_{2s}^2 \iff (r^2 - s^2)\cos^2\phi + \left(\tfrac{1}{r^2} - \tfrac{1}{s^2}\right)\sin^2\phi = 0,$$

and hence $\gamma_r = \gamma_s$ iff $r = s$. Next, suppose M is a d-perfect Hardy field extension of H containing both γ and $\widetilde{\gamma} \in (\mathcal{C}^{<\infty})^{\times}$ with $\widetilde{\gamma} > 0$ and $\sigma(\widetilde{\gamma}) = \omega$. Corollary 5.5.3

gives $\omega \notin \omega(M)$, hence $\gamma, \widetilde{\gamma} \in \Gamma(M)$ by [ADH, 11.8.31], and thus $\gamma = \widetilde{\gamma}$ by [ADH, 11.8.29]. $\square$

Answering a question of Boshernitzan $(^*)$**.** Following [34] we say that a germ y in $\mathcal{C}$ is **translogarithmic** if $r \leqslant y \leqslant \ell_n$ for all n and all $r \in \mathbb{R}$. Thus for eventually strictly increasing $y \succ 1$ in $\mathcal{C}$, y is translogarithmic iff its compositional inverse y^{inv} is transexponential. By Lemma 5.3.5 and Corollary 5.4.24 there exist $\mathcal{C}^{\omega}$-hardian translogarithmic germs; see also [ADH, 13.9]. Translogarithmic hardian germs are d-transcendental, by Corollary 5.4.28. In this subsection we use Theorem 5.6.2 to prove the following analogue of Corollary 5.4.24 for translogarithmic germs, thus giving a positive answer to Question 4 in [34, §7]:

Proposition 5.6.6. *Every maximal Hardy field contains a translogarithmic germ.*

Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field; then H has no translogarithmic element iff (ℓ_n) is a logarithmic sequence for H in the sense of [ADH, 11.5]. In this case, if H is also ω-free, then for each translogarithmic H-hardian germ y the isomorphism type of the ordered differential field $H\langle y\rangle$ over H is uniquely determined; more generally, by [ADH, 13.6.7, 13.6.8]:

Lemma 5.6.7. *Let H be an ω-free H-field, with asymptotic couple (Γ, ψ), and let $L = H\langle y\rangle$ be a pre-H-field extension of H with $\Gamma^{<} < vy < 0$. Then for all $P \in H\{Y\}^{\neq}$ we have*

$$v\big(P(y)\big) \ = \ \gamma + \operatorname{ndeg}(P)vy + \operatorname{nwt}(P)\psi_L(vy) \qquad \textit{where } \gamma = v^{\mathrm{e}}(P) \in \Gamma,$$

and thus

$$\Gamma_L \ = \ \Gamma \oplus \mathbb{Z}vy \oplus \mathbb{Z}\psi_L(vy) \qquad (\textit{internal direct sum}).$$

Moreover, if $L^* = H\langle y^*\rangle$ is a pre-H-field extension of H with $\Gamma^{<} < vy^* < 0$ and $\operatorname{sign} y = \operatorname{sign} y^*$, then there is a unique pre-H-field isomorphism $L \to L^*$ which is the identity on H and sends y to y^*.

This lemma suggests how to obtain Proposition 5.6.6: follow the arguments in the proof of [ADH, 13.6.7]. In the rest of this subsection we carry out this plan. For this, let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field and $y \in \mathcal{C}^{<\infty}$.

Lemma 5.6.8. *Suppose H is ω-free and for all $\ell \in H^{>\mathbb{R}}$ we have, in $\mathcal{C}$:*

(i) $1 \prec y \prec \ell$;
(ii) $\delta^n(y) \preccurlyeq 1$ *for all* $n \geqslant 1$, *where* $\delta := \phi^{-1}\partial$, $\phi := \ell'$;
(iii) $y' \in \mathcal{C}^{\times}$ *and* $(1/\ell)' \preccurlyeq y^{\dagger}$.

Let $P \in H\{Y\}^{\neq}$. Then in $\mathcal{C}$ we have

$$P(y) \sim a\, y^d\, (y^{\dagger})^w \qquad \textit{where } a \in H^{\times},\ d = \operatorname{ndeg}(P),\ w = \operatorname{nwt}(P).$$

(*Hence y is hardian over H and* d*-transcendental over H.*)

Proof. Since H is real closed, it has a monomial group, so the material of [ADH, 13.3] applies. Then [ADH, 13.3.3] gives a monic $D \in \mathbb{R}[Y]^{\neq}$, $b \in H^{\times}$, $w \in \mathbb{N}$, and an active element ϕ of H with $0 < \phi \prec 1$ such that:

$$P^{\phi} \ = \ b \cdot D \cdot (Y')^w + R, \qquad R \in H^{\phi}\{Y\},\ R \prec^{\flat}_{\phi} b.$$

Set $d := \operatorname{ndeg} P$, and note that by [ADH, 13.1.9] we have $d = \deg D + w$ and $w = \operatorname{nwt} P$. Replace P, b, R by $b^{-1}P, 1, b^{-1}R$, respectively, to arrange $b = 1$. Take $\ell \in H$ with $\ell' = \phi$, so $\ell > \mathbb{R}$; we use the superscript $\circ$ as in the subsection on compositional conjugation of Section 5.3; in particular, $y^\circ = y \circ \ell^{\text{inv}}$ with $(y^\circ)' = (\phi^{-1}y')^\circ$, so $(y^\circ)^\dagger \succcurlyeq 1/x^2$ by hypothesis (iii) of our lemma. In $H^\circ\{Y\}$ we now have

$$(P^\phi)^\circ \ = \ D \cdot (Y')^w + R^\circ \qquad \text{where} \qquad R^\circ \prec^\flat 1.$$

Evaluating at y° we have $D(y^\circ)\big((y^\circ)'\big)^w \sim (y^\circ)^d\big((y^\circ)^\dagger\big)^w$ and so $D(y^\circ)\big((y^\circ)'\big)^w \succcurlyeq x^{-2w} \asymp^\flat 1$. By (i) we have $(y^\circ)^m \prec x$ for $m \geqslant 1$, and by (ii) we have $(y^\circ)^{(n)} \preccurlyeq 1$ for $n \geqslant 1$. Hence $R^\circ(y^\circ) \preccurlyeq h^\circ$ for some $h \in H$ with $h^\circ \prec^\flat 1$. Thus in $\mathcal{C}$ we have

$$(P^\phi)^\circ(y^\circ) \sim (y^\circ)^d\big((y^\circ)^\dagger\big)^w.$$

Since $P(y)^\circ = (P^\phi)^\circ(y^\circ)$, this yields $P(y) \sim a \cdot y^d \cdot (y^\dagger)^w$ for $a = \phi^{-w}$. □

Corollary 5.6.9. *Suppose H is ω-free and $1 \prec y \prec \ell$ for all $\ell \in H^{>\mathbb{R}}$. Then the following are equivalent:*

(i) *y is hardian over H;*
(ii) *for all $h \in H^{>\mathbb{R}}$ there is an $\ell \in H^{>\mathbb{R}}$ such that $\ell \preccurlyeq h$ and y, ℓ lie in a common Hardy field;*
(iii) *for all $h \in H^{>\mathbb{R}}$ there is an $\ell \in H^{>\mathbb{R}}$ such that $\ell \preccurlyeq h$ and $y \circ \ell^{\text{inv}}$ is hardian.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are obvious. Let $\ell \in H^{>\mathbb{R}}$ be such that $y^\circ := y \circ \ell^{\text{inv}}$ lies in a Hardy field H_0; we arrange $x \in H_0$. For $\phi := \ell'$ we have $(\phi^{-1}y^\dagger)^\circ = (y^\circ)^\dagger \succ (1/x)' = -1/x^2$ and thus $y^\dagger \succ -\phi/\ell^2 = (1/\ell)'$. Also $y^\circ \prec x$, hence $z := (y^\circ)' \prec x' = 1$ and so $z^{(n)} \prec 1$ for all n. With $\delta := \phi^{-1}\partial$ and $n \geqslant 1$ we have $\delta^n(y)^\circ = z^{(n-1)}$ and thus $\delta^n(y) \prec 1$. Moreover, for $h \in H^{>\mathbb{R}}$ with $\ell \preccurlyeq h$ and $\theta := h'$ we have $\theta^{-1}\partial = f\delta$ where $f := \phi/\theta \in H$, $f \preccurlyeq 1$. Let $n \geqslant 1$. Then

$$(\theta^{-1}\partial)^n = (f\delta)^n = G^n_n(f)\delta^n + \cdots + G^n_1(f)\delta \quad \text{on } \mathcal{C}^{<\infty}$$
$$\text{where } G^n_j \in \mathbb{Q}\{X\} \subseteq H^\phi\{X\} \text{ for } j = 1, \dots, n.$$

As δ is small as a derivation on H, we have $G^n_j(f) \preccurlyeq 1$ for $j = 1, \dots, n$, and so $(\theta^{-1}\partial)^n(y) \prec 1$. Thus (iii) ⇒ (i) by Lemma 5.6.8. □

Proof of Proposition 5.6.6. Let $H \supseteq \mathbb{R}$ be any ω-free Liouville closed Hardy field not containing any translogarithmic element; in view of Theorem 5.6.2 it suffices to show that then some Hardy field extension of H contains a translogarithmic element. The remark before Proposition 5.6.6 yields a translogarithmic germ y in a $\mathcal{C}^\omega$-Hardy field $H_0 \supseteq \mathbb{R}$. Then for each n, the germs y, ℓ_n are contained in a common Hardy field, namely $\operatorname{Li}(H_0)$. Hence y generates a proper Hardy field extension of H by (ii) ⇒ (i) in Corollary 5.6.9. □

Proposition 5.6.6 goes through when "maximal" is replaced by "$\mathcal{C}^\infty$-maximal" or "$\mathcal{C}^\omega$-maximal". This follows from its proof, using also remarks after the proof of Theorem 5.6.2. Here is a conjecture that is much stronger than Proposition 5.6.6; it postulates an analogue of Corollary 5.4.23 for infinite "lower bounds":

Conjecture. If H is maximal, then there is no $y \in \mathcal{C}^1$ such that $1 \prec y \prec h$ for all $h \in H^{>\mathbb{R}}$, and $y' \in \mathcal{C}^\times$.

We observe that in this conjecture we may restrict attention to $\mathcal{C}^\omega$-hardian germs y:

Lemma 5.6.10. *Suppose there exists $y \in \mathcal{C}^1$ such that $1 \prec y \prec h$ for all $h \in H^{>\mathbb{R}}$ and $y' \in \mathcal{C}^\times$. Then there exists such a germ y which is $\mathcal{C}^\omega$-hardian.*

Proof. Take y as in the hypothesis. Replace y by $-y$ if necessary to arrange $y > \mathbb{R}$. Now Theorem 5.4.22 yields a $\mathcal{C}^\omega$-hardian germ $z \geqslant y^{\text{inv}}$. By Lemma 5.3.5, the germ z^{inv} is also $\mathcal{C}^\omega$-hardian, and $\mathbb{R} < z^{\text{inv}} \leqslant y \prec h$ for all $h \in H^{>\mathbb{R}}$. $\square$

Generalizing a theorem of Boshernitzan $(^*)$**.** *In this subsection H is a Hardy field.* Recall from Corollary 5.4.15 that for all $f \in \mathrm{E}(H)$ there are $h \in H(x)$ and n such that $f \leqslant \exp_n h$. In particular, the sequence $(\exp_n x)$ is cofinal in $\mathrm{E}(\mathbb{Q})$. By Theorem 5.4.20 and Corollary 5.4.27, (ℓ_n) is coinitial in $\mathrm{E}(\mathbb{Q})^{>\mathbb{R}}$; see also [33, Theorem 13.2]. In particular, for the Hardy field $H = \mathrm{Li}(\mathbb{R})$, the subset $H^{>\mathbb{R}}$ is coinitial in $\mathrm{E}(\mathbb{Q})^{>\mathbb{R}} = \mathrm{E}(H)^{>\mathbb{R}}$, equivalently, $\Gamma_H^<$ is cofinal in $\Gamma_{\mathrm{E}(H)}^<$. We now generalize this fact, recalling from the end of Section 5.5 that $\mathrm{Li}(\mathbb{R})$ is $\upomega$-free:

Theorem 5.6.11. *Suppose H is $\upomega$-free. Then $\Gamma_H^<$ is cofinal in $\Gamma_{\mathrm{E}(H)}^<$.*

Proof. Replacing H by $\mathrm{Li}\big(H(\mathbb{R})\big)$ and using Theorem 1.4.1 we arrange that H is Liouville closed and $H \supseteq \mathbb{R}$. Let $y \in \mathrm{E}(H)$ and suppose towards a contradiction that $\mathbb{R} < y < H^{>\mathbb{R}}$. Then $f := y^{\text{inv}}$ is transexponential and hardian (Lemma 5.3.5). Lemma 5.4.19 gives a bound $b \in \mathcal{C}$ for $\mathbb{R}\langle f\rangle$. Lemma 5.4.17 gives $\phi \in (\mathcal{C}^\omega)^\times$ such that $\phi^{(n)} \prec 1/b$ for all n; set $r := \phi \cdot \sin x \in \mathcal{C}^\omega$. Then by Lemma 5.4.21 (with $\mathbb{R}\langle f\rangle$ in place of H) we have $Q(r) \prec 1$ for all $Q \in \mathbb{R}\langle f\rangle\{Y\}$ with $Q(0) = 0$. Hence $g := f + r$ is eventually strictly increasing with $g \succ 1$, and $y = f^{\text{inv}}$ and $z := g^{\text{inv}} \in \mathcal{C}^{<\infty}$ do not lie in a common Hardy field. Thus in order to achieve the desired contradiction it suffices to show that z is H-hardian. For this we use Corollary 5.6.9. It is clear that $f \sim g$, so $y \sim z$ by Corollary 5.1.10, and thus $1 \prec z \prec \ell$ for all $\ell \in H^{>\mathbb{R}}$. Let $\ell \in H^{>\mathbb{R}}$ and $\ell \prec x$; we claim that $z \circ \ell^{\text{inv}}$ is hardian, equivalently, by Lemma 5.3.5, that $\ell \circ g = (z \circ \ell^{\text{inv}})^{\text{inv}}$ is hardian. Now $\ell \circ f = (y \circ \ell^{\text{inv}})^{\text{inv}}$ is hardian and $\ell \circ f \succ 1$, and Lemma 5.4.11 gives $\ell \circ f - \ell \circ g \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$. Hence $\ell \circ f \sim_\infty \ell \circ g$ by Lemma 5.4.12. For all n we have $\ell_n \circ \ell = \log_n \ell \in H^{>\mathbb{R}}$, so $y \leqslant \ell_n \circ \ell$, hence $y \circ \ell^{\text{inv}} \leqslant \ell_n$, which by compositional inversion gives $\ell \circ f \geqslant \exp_n x$. So $\ell \circ g$ is hardian by Corollary 5.4.14. Thus z is H-hardian by (iii) $\Rightarrow$ (i) of Corollary 5.6.9. $\square$

If $H \subseteq \mathcal{C}^\infty$ is $\upomega$-free, then $\Gamma_H^<$ is also cofinal in $\Gamma_{\mathrm{E}^\infty(H)}^<$, and similarly with ω in place of ∞. (Same proof as that of the previous theorem.) We also note that if $\mathrm{D}(H) = \mathrm{E}(H)$ (e.g., if H is bounded; cf. Theorem 5.4.20), then Theorem 5.6.11 already follows from Theorem 1.4.1.

5.7. Bounding Solutions of Linear Differential Equations

Let $r \in \mathbb{N}^{\geqslant 1}$, and with $\boldsymbol{i}$ ranging over $\mathbb{N}^r$, let

$$P = P(Y, Y', \dots, Y^{(r-1)}) = \sum_{\|\boldsymbol{i}\| < r} P_{\boldsymbol{i}} Y^{\boldsymbol{i}} \in \mathcal{C}[\mathrm{i}]\big[Y, Y', \dots, Y^{(r-1)}\big]$$

with $P_{\boldsymbol{i}} \in \mathcal{C}[\mathrm{i}]$ for all $\boldsymbol{i}$ with $\|\boldsymbol{i}\| < r$, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many such $\boldsymbol{i}$. Then P gives rise to an evaluation map

$$y \mapsto P\big(y, y', \dots, y^{(r-1)}\big) \ : \ \mathcal{C}^{r-1}[\mathrm{i}] \to \mathcal{C}[\mathrm{i}].$$

Let $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy the differential equation

$$y^{(r)} = P\big(y, y', \dots, y^{(r-1)}\big). \tag{5.7.1}$$

In addition, $\mathfrak{m}$ with $0 < \mathfrak{m} \preccurlyeq 1$ is a hardian germ, and $\eta \in \mathcal{C}$ is eventually increasing with $\eta(t) > 0$ eventually, and $n \geqslant r$.

Proposition 5.7.1. *Suppose $P_{\boldsymbol{i}} \preccurlyeq \eta$ for all $\boldsymbol{i}$, $P(0) \preccurlyeq \eta\,\mathfrak{m}^n$, and $y \preccurlyeq \mathfrak{m}^n$. Then*

$$y^{(j)} \ \preccurlyeq\ \eta^j \mathfrak{m}^{n-j(1+\varepsilon)} \qquad \text{for } j=0,\dots,r \text{ and all } \varepsilon \in \mathbb{R}^>,$$

with $\prec$ in place of $\preccurlyeq$ if $y \prec \mathfrak{m}^n$ and $P(0) \prec \eta\,\mathfrak{m}^n$.

The following immediate consequence is used in Section 5.10:

Corollary 5.7.2. *Suppose $f_1,\dots,f_r \in \mathcal{C}[\mathrm{i}]$ and $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy*

$$y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ =\ 0, \qquad f_1,\dots,f_r \preccurlyeq \eta, \quad y \ \preccurlyeq\ \mathfrak{m}^n.$$

Then $y^{(j)} \preccurlyeq \eta^j \mathfrak{m}^{n-j(1+\varepsilon)}$ for $j=0,\dots,r$ and all $\varepsilon \in \mathbb{R}^>$, with $\prec$ in place of $\preccurlyeq$ if $y \prec \mathfrak{m}^n$.

We obtain Proposition 5.7.1 from estimates due to Esclangon and Landau. To prepare for this we review an argument of Hardy-Littlewood which bounds the derivative f' of a twice continuously differentiable function f in terms of f, f''. (For another statement in the same spirit see Lemma 5.9.10.)

Bounding f' in terms of f, f''. Let $a \in \mathbb{R}$, let $\phi,\psi\colon [a,+\infty) \to (0,+\infty)$ be continuous and increasing, and $f \in \mathcal{C}^2_a[\mathrm{i}]$. If f and f'' are bounded, then so is f' by the next lemma:

Lemma 5.7.3 (Hardy-Littlewood [88])**.** *Suppose $|f| \leqslant \phi$, $|f''| \leqslant \psi$, and let $\varepsilon \in \mathbb{R}^>$. Then $|f'(t)| \leqslant (2+\varepsilon)\sqrt{\phi(t)\psi(t)}$, eventually.*

Proof (Mordell [140]). First arrange $a=0$ by translating the domain. Let $0<s<t$. Taylor expansion at t yields $\theta=\theta(s,t) \in [0,1]$ such that

$$f(t-s) \ =\ f(t) - sf'(t) + \tfrac{1}{2}s^2 f''(t-\theta s),$$

hence

$$|f(t-s) - f(t) + sf'(t)| \ \leqslant\ \tfrac{1}{2}s^2\psi(t-\theta s) \leqslant \tfrac{1}{2}s^2\psi(t).$$

Since $|f(t)| \leqslant \phi(t)$ and $|f(t-s)| \leqslant \phi(t-s) \leqslant \phi(t)$, this yields

$$|f'(t)| \ \leqslant\ (2/s)\phi(t) + (s/2)\psi(t).$$

Put $\rho(t) := \sqrt{\phi(t)/\psi(t)}$ for $t>0$. If $t > 2\rho(t)$, then $s := 2\rho(t)$ in the above gives $|f'(t)| \leqslant 2\rho(t)$. Hence if eventually $t > 2\rho(t)$, then we are done. Suppose otherwise; then ρ is unbounded, hence so is $\psi\rho = \sqrt{\phi\psi}$. Take $b>0$ such that $\sqrt{\phi(t)\psi(t)} \geqslant |f'(0)|/\varepsilon$ for all $t \geqslant b$. We claim that $|f'(t)| \leqslant (2+\varepsilon)\sqrt{\phi(t)\psi(t)}$ for $t \geqslant b$. If $t > 2\rho(t)$, then $|f'(t)| \leqslant 2\sqrt{\phi(t)\psi(t)} < (2+\varepsilon)\sqrt{\phi(t)\psi(t)}$, so suppose $t \leqslant 2\rho(t)$. Then

$$|f'(t) - f'(0)| \ =\ \left|\int_0^t f''(s)\,ds\right| \ \leqslant\ \int_0^t |f''(s)|\,ds \ \leqslant\ \int_0^t \psi(s)\,ds \ \leqslant\ t\psi(t)$$

and hence

$$|f'(t)| \ \leqslant\ |f'(0)| + t\psi(t) \leqslant |f'(0)| + 2\sqrt{\phi(t)\psi(t)} \ \leqslant\ (2+\varepsilon)\sqrt{\phi(t)\psi(t)}. \qquad \square$$

Corollary 5.7.4. *If $f \preccurlyeq \phi$, $f'' \preccurlyeq \psi$, then $f' \preccurlyeq \sqrt{\phi\psi}$, with $f' \prec \sqrt{\phi\psi}$ if also $f \prec \phi$ or $f'' \prec \psi$.*

In Corollary 5.7.4 we cannot drop the assumption that ϕ, ψ are increasing. (Take $a > 1$, $f(t) = t \log t$ for $t \geqslant a$, $\phi = f$, $\psi = f''$.) However, Mordell [140] also shows that if instead of assuming that ϕ, ψ are increasing, we assume that they are decreasing, then Lemma 5.7.3 holds in a stronger form: $|f| \leqslant \phi$ & $|f''| \leqslant \psi \Rightarrow |f'| \leqslant 2(\phi\psi)^{1/2}$. The next lemma (not used later) yields a variant of Corollary 5.7.4 where the germ of f lies in a complexified Hardy field; see also [88, §7].

Lemma 5.7.5. *Let H be a Hardy field, $K = H[\mathrm{i}]$, and $g \in K^\times$ such that $g \prec 1$ or $g \succ 1$, $g^\dagger \not\asymp x^{-1}$. Then $g' \preccurlyeq |gg''|^{1/2}$.*

Proof. Arranging that H is real closed and $x \in H$ and using $|h| \asymp h$ for $h \in K$ (see the remarks before Corollary 1.2.6), the lemma now follows from parts (1), (2), (4) of [7, Lemma 5.2] applied to the asymptotic couple of K. $\square$

We now generalize Corollary 5.7.4:

Lemma 5.7.6 (Hardy-Littlewood [88])**.** *Suppose $f \in \mathcal{C}^n_a[\mathrm{i}]$, $n \geqslant 1$, such that $f \preccurlyeq \phi$, $f^{(n)} \preccurlyeq \psi$. Then for $j = 0, \dots, n$ we have $f^{(j)} \preccurlyeq \phi^{1-j/n}\psi^{j/n}$. If additionally $f \prec \phi$ or $f^{(n)} \prec \psi$, then $f^{(j)} \prec \phi^{1-j/n}\psi^{j/n}$ for $j = 1, \dots, n-1$.*

Proof. The case $n = 1$ is trivial, so let $n \geqslant 2$. We may also assume $f \neq 0$, and by increasing a we arrange $f(a) \neq 0$. Let j range over $\{0, \dots, n\}$. Consider the continuous increasing functions

$$\chi_j : [a, +\infty) \to (0, +\infty), \qquad \chi_j(t) := \max_{a \leqslant s \leqslant t} |f^{(j)}(s)| \big/ \big(\phi(s)^{1-j/n}\psi(s)^{j/n}\big),$$

and set $\chi := \max\{\chi_0, \dots, \chi_n\}$. Then $\chi(t) \geqslant \chi_0(t) > 0$ for all $t \geqslant a$. We have

$$|f^{(j)}| / \big(\phi^{1-j/n}\psi^{j/n}\big) \leqslant \chi_j \leqslant \chi,$$

therefore

$$f^{(j)} \preccurlyeq \phi^{1-j/n}\psi^{j/n}\chi.$$

By induction on $j = 0, \dots, n-2$ we now show

$$f^{(j)} \preccurlyeq \phi^{1-j/n}\psi^{j/n}\chi^{1-1/2^j}. \tag{5.7.2}$$

The case $j = 0$ follows from $f \preccurlyeq \phi$. Suppose (5.7.2) holds for a certain $j < n-2$. Then Corollary 5.7.4 with $f^{(j)}$, $\phi^{1-j/n}\psi^{j/n}\chi^{1-1/2^j}$, $\phi^{1-(j+2)/n}\psi^{(j+2)/n}\chi$ in the role of f, ϕ, ψ, respectively, yields:

$$\begin{aligned} f^{(j+1)} &\preccurlyeq \big(\phi^{1-j/n}\psi^{j/n}\chi^{1-1/2^j} \cdot \phi^{1-(j+2)/n}\psi^{(j+2)/n}\chi\big)^{1/2} \\ &= \phi^{1-(j+1)/n}\psi^{(j+1)/n}\chi^{1-1/2^{j+1}}. \end{aligned}$$

This proves (5.7.2). We claim $\chi \preccurlyeq 1$. Suppose otherwise; so $\chi(t) \to +\infty$ as $t \to +\infty$, since χ is increasing, hence $f^{(n)} \preccurlyeq \psi \preccurlyeq \psi\chi^{1-1/2^n} \preccurlyeq \psi\chi$. Corollary 5.7.4 with $f^{(n-2)}$, $\phi^{2/n}\psi^{1-2/n}\chi^{1-1/2^{n-2}}$, $\psi\chi$ in the role of f, ϕ, ψ, respectively, yields

$$f^{(n-1)} \preccurlyeq \big(\phi^{2/n}\psi^{1-2/n}\chi^{1-1/2^{n-2}} \cdot \psi\chi\big)^{1/2} = \phi^{1/n}\psi^{1-1/n}\chi^{1-1/2^{n-1}}.$$

So (5.7.2) then also holds for $j = n-1$, and it clearly holds for $j = n$. But then $\chi_j \preccurlyeq \chi^{1-1/2^n}$ for all j and so $\chi \preccurlyeq \chi^{1-1/2^n}$, contradicting $\chi \succ 1$.

Now suppose $f \prec \phi$. By induction on $j = 0, \dots, n-1$ we show $f^{(j)} \prec \phi^{1-j/n}\psi^{j/n}$. The case $j = 0$ holds by assumption; suppose it holds for a certain $j \leqslant n-2$. Then $f^{(j+2)} \preccurlyeq \phi^{1-(j+2)/n}\psi^{(j+2)/n}$, so Corollary 5.7.4 with

$$f^{(j)}, \quad \phi^{1-j/n}\psi^{j/n}, \quad \phi^{1-(j+2)/n}\psi^{(j+2)/n}$$

in the role of f, ϕ, ψ, respectively, yields

$$f^{(j+1)} \prec \left(\phi^{1-j/n}\psi^{j/n} \cdot \phi^{1-(j+2)/n}\psi^{(j+2)/n}\right)^{1/2} = \phi^{1-(j+1)/n}\psi^{(j+1)/n}.$$

If $f^{(n)} \prec \psi$, then likewise $f^{(n-j)} \prec \phi^{j/n}\psi^{1-j/n}$ for $j = 0, \dots, n-1$. $\square$

Corollary 5.7.7. *Suppose $f \in \mathcal{C}^n_a[\mathrm{i}]$ and $f \preccurlyeq \phi$, $f^{(n)} \preccurlyeq \phi$. Then $f', \dots, f^{(n-1)} \preccurlyeq \phi$, and if in addition $f \prec \phi$ or $f^{(n)} \prec \phi$, then $f', \dots, f^{(n-1)} \prec \phi$.*

The theorem of Esclangon-Landau. In this subsection $n \geqslant r \geqslant 1$ and P is as at the beginning of this section, and $y \in \mathcal{C}^r[\mathrm{i}]$ satisfies (5.7.1). Also, $\eta \in \mathcal{C}$ is eventually increasing and positive, so $\eta \succcurlyeq 1$. The next theorem covers the case $\mathfrak{m} \asymp 1$ of Proposition 5.7.1:

Theorem 5.7.8 (Landau [121])**.** *Suppose $y \preccurlyeq 1$ and $P_{\boldsymbol{i}} \preccurlyeq \eta$ for all $\boldsymbol{i}$. Then $y^{(j)} \preccurlyeq \eta^j$ for $j = 0, \dots, r$. Moreover, if $y \prec 1$, then $y^{(j)} \prec \eta^j$ for $j = 0, \dots, r-1$, and if in addition $P(0) \prec \eta$, then also $y^{(r)} \prec \eta^r$.*

Proof. Take $a \in \mathbb{R}$ such that η is represented by an increasing continuous function $\eta\colon [a, +\infty) \to (0, +\infty)$, and y by a function $y \in \mathcal{C}^r_a[\mathrm{i}]$. Then

$$t \mapsto \psi(t) \ := \ \max\left(1, \max_{a \leqslant s \leqslant t} |y^{(r)}(s)|\right) \ : \ [a, +\infty) \to [1, +\infty)$$

is continuous and increasing with $|y^{(r)}| \leqslant \psi$. By Lemma 5.7.6 we have $y^{(j)} \preccurlyeq \psi^{j/r}$ for $j = 0, \dots, r-1$ and thus $P_{\boldsymbol{i}} y^{\boldsymbol{i}} \preccurlyeq \eta\psi^{\|\boldsymbol{i}\|/r} \preccurlyeq \eta\psi^{1-1/r}$ if $\|\boldsymbol{i}\| < r$. So $y^{(r)} = P\big(y, \dots, y^{(r-1)}\big) \preccurlyeq \eta\psi^{1-1/r}$. Take $C \in \mathbb{R}^>$ such that

$$|y^{(r)}(t)| \ \leqslant \ C\eta(t)\psi(t)^{1-1/r} \qquad \text{for all } t \geqslant a.$$

Increasing C we arrange $C\eta(a)\psi(a)^{1-1/r} \geqslant 1$. As $\eta\psi^{1-1/r}$ is increasing,

$$\psi(t) \ \leqslant \ \max\left(1, \max_{a \leqslant s \leqslant t} C\eta(s)\psi(s)^{1-1/r}\right) \ \leqslant \ C\eta(t)\psi(t)^{1-1/r} \qquad \text{for } t \geqslant a.$$

Hence $|y^{(r)}(t)| \leqslant \psi(t) \leqslant C^r\eta^r(t)$ for $t \geqslant a$, so $y^{(r)} \preccurlyeq \eta^r$. By Lemma 5.7.6 again this yields $y^{(j)} \preccurlyeq \eta^j$ for $j = 0, \dots, r$. Assume now that $y \prec 1$. Then by that same lemma, $y^{(j)} \prec \eta^j$ for $j < r$. We have $\eta \succcurlyeq 1$, so if $0 < \|\boldsymbol{i}\| < r$, then $y^{\boldsymbol{i}} \prec \eta^{\|\boldsymbol{i}\|} \preccurlyeq \eta^{r-1}$. Hence if additionally $P(0) \prec \eta$, then $y^{(r)} = P\big(y, \dots, y^{(r-1)}\big) \prec \eta^r$. $\square$

Corollary 5.7.9 (Esclangon [67])**.** *Suppose $f_1, \dots, f_r, g \in \mathcal{C}[\mathrm{i}]$ and $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy*

$$y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ = \ g, \qquad f_1, \dots, f_r, g, y \ \preccurlyeq \ 1.$$

Then $y', \dots, y^{(r)} \preccurlyeq 1$. If in addition $y \prec 1$ and $g \prec 1$, then $y', \dots, y^{(r)} \prec 1$.

Below H is a Hardy field and $\mathfrak{m} \in H$, $0 < \mathfrak{m} \prec 1$. Recall also that $n \geqslant r \geqslant 1$.

Lemma 5.7.10. *Let $z \in \mathcal{C}^r[\mathrm{i}]$. If $z^{(j)} \preccurlyeq \eta^j$ for $j = 0, \dots, r$, then $(z\mathfrak{m}^n)^{(j)} \preccurlyeq \eta^j\mathfrak{m}^{n-j}$ for $j = 0, \dots, r$, and likewise with $\prec$ instead of $\preccurlyeq$.*

Proof. Corollary 1.1.15 yields $(\mathfrak{m}^n)^{(m)} \preccurlyeq \mathfrak{m}^{n-m}$ for $m \leqslant n$. Thus if $z^{(j)} \preccurlyeq \eta^j$ for $j = 0, \dots, r$, then

$$z^{(k)}(\mathfrak{m}^n)^{(j-k)} \preccurlyeq \eta^k\mathfrak{m}^{n-(j-k)} \preccurlyeq \eta^j\mathfrak{m}^{n-j} \qquad (0 \leqslant k \leqslant j \leqslant r),$$

so $(z\mathfrak{m}^n)^{(j)} \preccurlyeq \eta^j\mathfrak{m}^{n-j}$ for $j = 0, \dots, r$, by the Product Rule. The argument with $\prec$ instead of $\preccurlyeq$ is similar. $\square$

We now return to the assumptions on P, y in the beginning of this section, so $y \in \mathcal{C}^r[\mathrm{i}]$ satisfies (5.7.1). Suppose also that $P_{\boldsymbol{i}} \preccurlyeq \eta$ for all $\boldsymbol{i}$, $P(0) \preccurlyeq \eta\,\mathfrak{m}^n$, and $y \preccurlyeq \mathfrak{m}^n$. Let $\varepsilon \in \mathbb{R}^>$ and set for $i = 0, \dots, r$,

$$Y_i \ := \ \sum_{j=0}^{i} \binom{i}{j} Y^{(i-j)}(\mathfrak{m}^n)^{(j)} \in H\big[Y, Y', \dots, Y^{(i)}\big] \ \subseteq \ \mathcal{C}[\mathrm{i}]\big[Y, Y', \dots, Y^{(r)}\big].$$

Then for $z := y\,\mathfrak{m}^{-n} \preccurlyeq 1$ in $\mathcal{C}^r[\mathrm{i}]$ the product rule gives

$$Y_i(z, z', \dots, z^{(i)}) \ = \ (z\,\mathfrak{m}^n)^{(i)} \ = \ y^{(i)} \qquad (i = 0, \dots, r),$$

so with

$$Q \ := \ Y^{(r)} - \mathfrak{m}^{-n}\big(Y_r - P(Y_0, \dots, Y_{r-1})\big) \in \mathcal{C}[\mathrm{i}]\big[Y, Y', \dots, Y^{(r-1)}\big]$$

we have by substitution of $z, \dots, z^{(r)}$ for $Y, Y', \dots, Y^{(r)}$,

$$\begin{aligned} z^{(r)} \ &= \ Q\big(z, z', \dots, z^{(r-1)}\big) + \mathfrak{m}^{-n}\big(y^{(r)} - P(y, y', \dots, y^{(r-1)})\big) \\ &= \ Q\big(z, z', \dots, z^{(r-1)}\big). \end{aligned}$$

For $Y_0, \dots, Y_r \in H\{Y\}$ we have $(Y^{\boldsymbol{i}})_{\times \mathfrak{m}^n} = Y_0^{i_0} \cdots Y_r^{i_r}$ for $\boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}$. Now $\mathfrak{m}^{-\varepsilon} \in \operatorname{Li}\big(H(\mathbb{R})\big)$. We equip $\operatorname{Li}\big(H(\mathbb{R})\big)\{Y\}$ with the gaussian extension of the valuation of $\operatorname{Li}\big(H(\mathbb{R})\big)$. Then by [ADH, 6.1.4],

$$\mathfrak{m}^{-n}(Y^{\boldsymbol{i}})_{\times \mathfrak{m}^n} \ \preccurlyeq \ \mathfrak{m}^{-\varepsilon} \qquad \text{for } \boldsymbol{i} \in \mathbb{N}^{1+r} \setminus \{0\}.$$

Let $\boldsymbol{i}$ range over $\mathbb{N}^r$ and take $Q_{\boldsymbol{i}} \in \mathcal{C}[\mathrm{i}]$ for $\|\boldsymbol{i}\| < r$ such that

$$Q \ = \ \sum_{\|\boldsymbol{i}\| < r} Q_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \qquad (Q_{\boldsymbol{i}} \neq 0 \text{ for only finitely many } \boldsymbol{i}).$$

Together with $P_{\boldsymbol{i}} \preccurlyeq \eta$ for all $\boldsymbol{i}$ and $P(0) \preccurlyeq \eta\,\mathfrak{m}^n$, the remarks above yield $Q_{\boldsymbol{i}} \preccurlyeq \eta\,\mathfrak{m}^{-\varepsilon}$ for all $\boldsymbol{i}$. By Theorem 5.7.8 applied to P, y, η replaced by Q, z, $\eta\,\mathfrak{m}^{-\varepsilon}$, respectively, we now obtain $z^{(j)} \preccurlyeq (\eta\,\mathfrak{m}^{-\varepsilon})^j$ $(j = 0, \dots, r)$, with $\prec$ in place of $\preccurlyeq$ if $y \prec \mathfrak{m}^n$ and $P(0) \prec \eta\,\mathfrak{m}^n$. Using Lemma 5.7.10 with $\eta\,\mathfrak{m}^{-\varepsilon}$ and $\operatorname{Li}\big(H(\mathbb{R})\big)$ in place of η and H finishes the proof of Proposition 5.7.1. ☐

5.8. Almost Periodic Functions

For later use we now discuss trigonometric polynomials, almost periodic functions, and their mean values; see [27, 53] for this material in the case $n = 1$. *In this section we assume $n \geqslant 1$, and for vectors $r = (r_1, \dots, r_n)$ and $s = (s_1, \dots, s_n)$ in $\mathbb{R}^n$ we let $r \cdot s := r_1 s_1 + \cdots + r_n s_n \in \mathbb{R}$ be the usual dot product of r and s. We also set $rs := (r_1 s_1, \dots, r_n s_n) \in \mathbb{R}^n$, not to be confused with $r \cdot s \in \mathbb{R}$. Moreover, we let $v, w \colon \mathbb{R}^n \to \mathbb{C}$ be complex-valued functions on $\mathbb{R}^n$, and let s range over $\mathbb{R}^n$, and T over $\mathbb{R}^>$; integrals are with respect to the usual Lebesgue measure of $\mathbb{R}^n$.* Set

$$\|w\| \ := \ \sup_s |w(s)| \ \in \ [0, +\infty].$$

We shall also have occasion to consider various functions $\mathbb{R}^n \to \mathbb{C}$ obtained from w: $\overline{w}$, $|w|$, as well as w_{+r} and $w_{\times r}$ (for $r \in \mathbb{R}^n$), defined by

$$\overline{w}(s) \ := \ \overline{w(s)}, \quad |w|(s) \ := \ |w(s)|, \quad w_{+r}(s) \ := \ w(r+s), \quad w_{\times r}(s) \ := \ w(rs).$$

We say that w is **1-periodic** if $w_{+k} = w$ for all $k \in \mathbb{Z}^n$.

Trigonometric polynomials. Let α range over $\mathbb{R}^n$. Call w a **trigonometric polynomial** if there are $w_\alpha \in \mathbb{C}$, with $w_\alpha = 0$ for all but finitely many α, such that for all s,

$$w(s) \;=\; \sum_\alpha w_\alpha\, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}\,. \tag{5.8.1}$$

The trigonometric polynomials form a subalgebra of the $\mathbb{C}$-algebra of uniformly continuous bounded functions $\mathbb{R}^n \to \mathbb{C}$. Let w be a trigonometric polynomial. Then $\overline{w}$ is a trigonometric polynomial, and for $r \in \mathbb{R}^n$, so are the functions w_{+r} and $w_{\times r}$. Note that w extends to a complex-analytic function $\mathbb{C}^n \to \mathbb{C}$, that $\operatorname{Re} w$ and $\operatorname{Im} w$ are real analytic, and that $\partial w/\partial x_j := (\partial \operatorname{Re} w/\partial x_j) + (\partial \operatorname{Im} w/\partial x_j)\mathrm{i}$ for $j = 1, \dots, n$ is also a trigonometric polynomial. The functions $s \mapsto \sin(\alpha\cdot s)$ and $s \mapsto \cos(\alpha\cdot s)$ on $\mathbb{R}^n$ are real valued trigonometric polynomials. By Corollary 5.8.18 below the coefficients w_α in (5.8.1) are uniquely determined by w.

If $w(s) = \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}$ for all s, then $w_{+r} = w$ for all $r \in \mathbb{R}^n$ with $\alpha\cdot r \in 2\pi\mathbb{Z}$. So if w is a trigonometric polynomial as in (5.8.1) with $w_\alpha = 0$ for all $\alpha \notin 2\pi\mathbb{Z}^n$, then w is 1-periodic. Next we state a well-known consequence of the Stone-Weierstrass Theorem; see [57, (7.4.2)] for the case $n = 1$.

Proposition 5.8.1. *If v is continuous and 1-periodic, then for every ε in $\mathbb{R}^{>}$ there is a 1-periodic trigonometric polynomial w with $\|v - w\| < \varepsilon$.*

Almost periodic functions. We call w **almost periodic** (in the sense of Bohr) if for every ε in $\mathbb{R}^{>}$ there is a trigonometric polynomial v such that $\|v - w\| \leqslant \varepsilon$. If w is almost periodic, then w is uniformly continuous and bounded (as the uniform limit of a sequence of uniformly continuous bounded functions $\mathbb{R}^n \to \mathbb{C}$). If w is almost periodic, then so are $\overline{w}$, and w_{+r}, $w_{\times r}$ for $r \in \mathbb{R}^n$.

Note that the $\mathbb{C}$-algebra of uniformly continuous bounded functions $\mathbb{R}^n \to \mathbb{C}$ is a Banach algebra with respect to $\|\cdot\|$: it is complete with respect to this norm. The closure of its subalgebra of trigonometric polynomials with respect to this norm is $\{w : w \text{ is almost perodic}\}$, which is therefore a Banach subalgebra. In particular, if $v, w\colon \mathbb{R}^n \to \mathbb{C}$ are almost periodic, so are $v + w$ and vw. Moreover:

Corollary 5.8.2. *Let $v_1, \dots, v_m\colon \mathbb{R}^n \to \mathbb{C}$ be almost periodic, let $X \subseteq \mathbb{C}^m$ be closed, and suppose $F\colon X \to \mathbb{C}$ is continuous with $\big(v_1(s), \dots, v_m(s)\big) \in X$ for all s. Then the function $F(v_1, \dots, v_m)\colon \mathbb{R}^n \to \mathbb{C}$ is almost periodic.*

Proof. Since $v_1, \dots, v_m$ are bounded we can arrange that X is compact. Let $\varepsilon \in \mathbb{R}^{>}$. Then Weierstrass Approximation [57, (7.4.1)] gives a polynomial

$$P(x_1, y_1, \dots, x_m, y_m) \;\in\; \mathbb{C}[x_1, y_1, \dots, x_m, y_m]$$

such that $|F(z_1, \dots, z_m) - P(z_1, \overline{z}_1, \dots, z_m, \overline{z}_m)| \leqslant \varepsilon$ for all $(z_1, \dots, z_m) \in X$. Hence $\|F(v_1, \dots, v_m) - P(v_1, \overline{v}_1, \dots, v_m \overline{v}_m)\| \leqslant \varepsilon$. It remains to note that the function $P(v_1, \overline{v}_1, \dots, v_m, \overline{v}_m)$ is almost periodic. $\square$

Call w **normal** if w is bounded and for every sequence (r_m) in $\mathbb{R}^n$ the sequence (w_{+r_m}) of functions $\mathbb{R}^n \to \mathbb{C}$ has a uniformly converging subsequence. One verifies easily that if v, w are normal, then so are the functions $v + w$ and cv ($c \in \mathbb{C}$); hence by the next lemma, each trigonometric polynomial is normal:

Lemma 5.8.3. *Suppose $w(s) = \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}$ for all s. Then w is normal.*

Proof. Let (r_m) be a sequence in $\mathbb{R}^n$. Passing to a subsequence of (r_m) we arrange that the sequence $\big(w(r_m)\big)$ of complex numbers of modulus 1 converges. Now use that for all l, m and all s we have $|w_{+r_l}(s) - w_{+r_m}(s)| = |w(r_l) - w(r_m)|$, and thus $\|w_{+r_l} - w_{+r_m}\| \leqslant |w(r_l) - w(r_m)|$. □

Lemma 5.8.4. *Let (w_m) be a sequence of normal functions with $\|w_m - w\| \to 0$ as $m \to \infty$. Then w is normal.*

Proof. Let $(r_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$. Using normality of the w_m we obtain inductively subsequences $(r_{k0}), (r_{k1}), \dots$ of (r_k) such that for all m, $\big((w_m)_{+r_{km}}\big)$ converges uniformly and $(r_{k,m+1})$ is a subsequence of (r_{km}). Then for every m, $(r_{m+l,m+l})_{l\geqslant 0}$ is a subsequence of (r_{km}); so $\big((w_m)_{+r_{kk}}\big)$ converges uniformly. Now let $\varepsilon \in \mathbb{R}^>$ be given. Take m so that $\|w_m - w\| \leqslant \varepsilon$, and then take k_0 so that $\|(w_m)_{+r_{kk}} - (w_m)_{+r_{ll}}\| \leqslant \varepsilon$ for all $k, l \geqslant k_0$. For such k, l we have

$$\begin{aligned}&\|w_{+r_{kk}} - w_{+r_{ll}}\| \leqslant \\ &\qquad \|w_{+r_{kk}} - (w_m)_{+r_{kk}}\| + \|(w_m)_{+r_{kk}} - (w_m)_{+r_{ll}}\| + \|(w_m)_{+r_{ll}} - w_{+r_{ll}}\| \leqslant 3\varepsilon.\end{aligned}$$

Thus $(w_{+r_{kk}})$ converges uniformly. □

Corollary 5.8.5 (Bochner)**.** *Every almost periodic function $\mathbb{R}^n \to \mathbb{C}$ is normal.*

For $\varepsilon \in \mathbb{R}^>$, we say that $r \in \mathbb{R}^n$ is an **ε-translation vector** for w if $\|w_{+r} - w\| < \varepsilon$. We define an **$n$-cube** of side length $\ell \in \mathbb{R}^>$ to be a subset of $\mathbb{R}^n$ of the form $I = I_1 \times \dots \times I_n$ where each $I_1, \dots, I_n$ is an open interval of length ℓ.

Proposition 5.8.6. *If w is normal, then for all $\varepsilon \in \mathbb{R}^>$ there is an $\ell = \ell(w,\varepsilon) \in \mathbb{R}^>$ such that every n-cube of side length ℓ contains an ε-translation vector for w.*

Proof. We assume that w is bounded and show the contrapositive. Let $\varepsilon \in \mathbb{R}^>$ be such that there are n-cubes of arbitrarily large sidelength that contain no ε-translation vector for w; to conclude that w is not normal it suffices to have a sequence $(r_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $r_j - r_i$ is not an ε-translation vector for w, for all $i < j$, since then $\|w_{+r_j} - w_{+r_i}\| = \|w_{+(r_j - r_i)} - w\| \geqslant \varepsilon$ for all $i < j$. Now suppose $r_0, \dots, r_m \in \mathbb{R}^n$ are such that $r_j - r_i$ is not an ε-translation vector for w, for all $i < j \leqslant m$. Then for $k = 1, \dots, n$ we take intervals $I_k = (a_k, b_k)$ ($a_k < b_k$ in $\mathbb{R}$) of equal length $b_k - a_k > 2\max\big\{|r_0|_\infty, \dots, |r_m|_\infty\big\}$ such that $I := I_1 \times \dots \times I_n$ does not contain an ε-translation vector for w. Set $r_{m+1} := \frac{1}{2}(a_1 + b_1, \dots, a_n + b_n)$; then for $i \leqslant m$ we have $r_{m+1} - r_i \in I$, hence $r_{m+1} - r_i$ is not an ε-translation vector for w. □

By Corollary 5.8.5, Proposition 5.8.6 applies to almost periodic w. Bohr [27] showed conversely that if w is continuous and satisfies the conclusion of Proposition 5.8.6, then w is almost periodic, but we do not use this elegant characterization of almost periodicity below. We now improve Proposition 5.8.6 for almost periodic w. *In the rest of this subsection we assume that w is almost periodic.*

Lemma 5.8.7. *Let $\varepsilon \in \mathbb{R}^>$; then there are $\delta, \ell \in \mathbb{R}^>$ such that every n-cube of side length ℓ contains an n-cube of side length δ consisting entirely of ε-translation vectors for w.*

Proof. Uniform continuity of w yields $\delta_1 \in \mathbb{R}^>$ such that all $d \in \mathbb{R}^n$ with $|d|_\infty < \delta_1$ are $(\varepsilon/3)$-translation vectors for w. Take $\ell_1 := \ell(w, \varepsilon/3)$ as in Proposition 5.8.6, and set $\delta := 2\delta_1$, $\ell := \ell_1 + \delta$. Let $J = a + (0,\ell)^n$ be a cube of side length ℓ, where $a \in \mathbb{R}^n$. Take an $(\varepsilon/3)$-translation vector $r \in a + (\delta_1, \ell_1 + \delta_1)^n$ for w. The cube $I := r + (-\delta_1, \delta_1)^n$ of side length δ is entirely contained in J. Let $p \in I$. Then for $d := p - r$ we have $|d|_\infty < \delta_1$, so for all s,

$$|w(s+p) - w(s)| \ \leqslant\ |w(s+d+r) - w(s+d)| + |w(s+d) - w(s)| \ <\ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ <\ \varepsilon,$$

hence p is an ε-translation vector for w. $\square$

Corollary 5.8.8. *Suppose $w(\mathbb{R}^n) \subseteq \mathbb{R}$, $s_0 \in \mathbb{R}^n$, and $w(s_0) > 0$. Then there are $\delta_1, \ell_1 \in \mathbb{R}^>$ such that every n-cube of side length ℓ_1 contains an n-cube I of side length δ_1 with $w(s) \geqslant w(s_0)/3$ for all $s \in I$.*

Proof. Let δ, ℓ be as in Lemma 5.8.7 for $\varepsilon := w(s_0)/3$. By decreasing δ we obtain from the uniform continuity of w that all $d \in \mathbb{R}^n$ with $|d|_\infty < \delta/2$ are ε-translation vectors for w. Set $\delta_1 := \delta$, $\ell_1 := \ell + \delta/2$. Let $J = a - (0, \ell_1)^n$ with $a \in \mathbb{R}^n$ be an n-cube of side length ℓ_1; we claim that J contains an n-cube I of side length δ with $w(s) \geqslant \varepsilon$ for all $s \in I$. To prove this claim, consider the n-cube $J_0 := (s_0 - a) + (\delta/2, \ell + \delta/2)^n$ of side length ℓ. Our choice of δ, ℓ gives an ε-translation vector $r \in J_0$ for w such that $r + (-\delta/2, \delta/2)^n \subseteq J_0$. Then

$$I \ :=\ (s_0 - r) + (-\delta/2, \delta/2)^n \ \subseteq\ s_0 - J_0 \ =\ a - (\delta/2, \ell + \delta/2)^n \ \subseteq\ J,$$

and for every $s \in I$, setting $d = s - s_0 + r$, we have $|d|_\infty < \delta/2$, so

$$w(s) \ =\ w(s_0) + \big(w(s_0 + d) - w(s_0)\big) - \big(w(s + r) - w(s)\big) \ \geqslant\ w(s_0) - \varepsilon - \varepsilon \ =\ \varepsilon$$

as required. $\square$

Lemma 5.8.9. *Suppose $w(\mathbb{R}^n) \subseteq \mathbb{R}$. Then with $|s| := |s|_\infty$,*

$$\liminf_{|s|\to\infty} w(s) \ =\ \inf_s w(s), \qquad \limsup_{|s|\to\infty} w(s) \ =\ \sup_s w(s).$$

Proof. It suffices to prove the second equality: applying it to $-w$ in place of w gives the first one. Set $\sigma := \sup_s w(s)$. Let $\varepsilon \in \mathbb{R}^>$, and take $s_0 \in \mathbb{R}^n$ with $w(s_0) > \sigma - \varepsilon$. By Corollary 5.8.8 applied to $s \mapsto v(s) := w(s) + \varepsilon - \sigma$ instead of w there are s with arbitrarily large $|s|$ and $v(s) \geqslant v(s_0)/3 > 0$, hence $w(s) > \sigma - \varepsilon$. Thus $\limsup_{|s|\to\infty} w(s) \geqslant \sigma$; the reverse inequality holds trivially. $\square$

The mean value. *In this subsection v and w are bounded and measurable.* If

$$\lim_{T\to\infty} \frac{1}{T^n} \int_{[0,T]^n} w(s)\, ds \tag{5.8.2}$$

exists (in $\mathbb{C}$), then we say that w has a mean value, and we call the quantity (5.8.2) the **mean value** of w and denote it by $M(w)$. One verifies easily that if v and w have a mean value, then so do the functions $v + w$, cw ($c \in \mathbb{C}$), and $\overline{w}$, with

$$M(v + w) \ =\ M(v) + M(w), \quad M(cw) \ =\ cM(w), \quad \text{and} \quad M(\overline{w}) \ =\ \overline{M(w)}.$$

If w has a mean value, then $|M(w)| \leqslant \|w\|$. If w and $|w|$ have a mean value, then $|M(w)| \leqslant M(|w|)$. If w has a mean value and $w(\mathbb{R}^n) \subseteq \mathbb{R}$, then $M(w) \in \mathbb{R}$, with $M(w) \geqslant 0$ if $w\big((\mathbb{R}^{\geqslant})^n\big) \subseteq \mathbb{R}^{\geqslant}$.

Lemma 5.8.10. *Let $d \in \mathbb{R}^n$. Then w has a mean value iff w_{+d} has a mean value, in which case $M(w) = M(w_{+d})$.*

Proof. It suffices to treat the case $d = (d_1, 0, \ldots, 0)$, $d_1 \in \mathbb{R}^>$. For $T > d_1$ we have

$$\left| \int_{[0,T]^n} w_{+d}(s)\, ds - \int_{[0,T]^n} w(s)\, ds \right| =$$

$$\left| \int_{[T,d_1+T]\times[0,T]^{n-1}} w(s)\, ds - \int_{[0,d_1]\times[0,T]^{n-1}} w(s)\, ds \right| \leqslant 2d_1 \|w\| T^{n-1},$$

and this yields the claim. □

Corollary 5.8.11. *Suppose w has a mean value, and let $T_0 \in \mathbb{R}^>$. If $w(s) = 0$ for all $s \in (\mathbb{R}^{\geqslant})^n$ with $|s| \geqslant T_0$, then $M(w) = 0$. If $w(\mathbb{R}^n) \subseteq \mathbb{R}$ and $w(s) \geqslant 0$ for all $s \in (\mathbb{R}^{\geqslant})^n$ with $|s| \geqslant T_0$, then $M(w) \geqslant 0$. (As before, $|s| := |s|_\infty$.)*

Lemma 5.8.12. *Suppose w has a mean value and $w(\mathbb{R}^n) \subseteq \mathbb{R}$; then*

$$\inf_s w(s) \leqslant \liminf_{|s|\to\infty} w(s) \leqslant M(w) \leqslant \limsup_{|s|\to\infty} w(s) \leqslant \sup_s w(s).$$

Proof. The first and last inequalities are clear. Towards a contradiction assume $L := \limsup_{|s|\to\infty} w(s) < M(w)$, and let $\varepsilon = \frac{1}{2}\big(M(w) - L\big)$. Take $T_0 \in \mathbb{R}^>$ such that $w(s) \leqslant M(w) - \varepsilon$ for all s with $|s| \geqslant T_0$. The previous corollary applied to $s \mapsto M(w) - \varepsilon - w(s)$ instead of w implies $M(w) \leqslant M(w) - \varepsilon$, a contradiction. This shows the third inequality; the second inequality is proved in a similar way. □

Note that if w has a mean value, then so does every v having the same restriction to $(\mathbb{R}^{\geqslant})^n$ as w, with $M(v) = M(w)$.

Lemma 5.8.13. *Let (v_m) be a sequence of bounded measurable functions $\mathbb{R}^n \to \mathbb{C}$ with a mean value, such that $\lim_{m\to\infty} \|v_m - w\| = 0$. Then w has a mean value, and $\lim\limits_{m\to\infty} M(v_m) = M(w)$.*

Proof. Let $\varepsilon \in \mathbb{R}^>$ be given, and take m with $||v_m - w|| \leqslant \varepsilon$. Since $v := v_m$ has a mean value, we have $T_0 \in \mathbb{R}^>$ such that for all $T_1, T_2 \geqslant T_0$,

$$\left| \frac{1}{T_1^n} \int_{[0,T_1]^n} v(s)\, ds - \frac{1}{T_2^n} \int_{[0,T_2]^n} v(s)\, ds \right| \leqslant \varepsilon.$$

Then for such T_1, T_2 we have

$$\left| \frac{1}{T_1^n} \int_{[0,T_1]^n} w(s)\, ds - \frac{1}{T_2^n} \int_{[0,T_2]^n} w(s)\, ds \right| \leqslant \frac{1}{T_1^n} \int_{[0,T_1]^n} \big|w(s) - v(s)\big|\, ds +$$

$$\left| \frac{1}{T_1^n} \int_{[0,T_1]^n} v(s)\, ds - \frac{1}{T_2^n} \int_{[0,T_2]^n} v(s)\, ds \right| + \frac{1}{T_2^n} \int_{[0,T_2]^n} \big|w(s) - v(s)\big|\, ds$$

where each term on the right of $\leqslant$ is $\leqslant \varepsilon$. Hence the limit (5.8.2) exists. To show $\lim\limits_{m\to\infty} M(v_m) = M(w)$, use $|M(v_m) - M(w)| = |M(v_m - w)| \leqslant \|v_m - w\|$. □

The mean value of an almost periodic function. *In this subsection v and w are almost periodic. As before, α ranges over $\mathbb{R}^n$.*

Lemma 5.8.14. *Suppose $w(s) = \mathrm{e}^{\mathrm{i}(\alpha\cdot s)}$ for all s. Then w has a mean value, with $M(w) = 1$ if $\alpha = 0$ and $M(w) = 0$ otherwise.*

Proof. This is obvious for $\alpha = 0$. Assume $\alpha \neq 0$. Then

$$\int_{[0,T]^n} \mathrm{e}^{\mathrm{i}(\alpha\cdot s)}\, ds \;=\; T^{|\{j:\alpha_j=0\}|} \cdot \prod_{j,\alpha_j\neq 0} \frac{\mathrm{e}^{\mathrm{i}\alpha_j T}-1}{\mathrm{i}\alpha_j}, \qquad \text{so}$$

$$\left|\frac{1}{T^n}\int_{[0,T]^n} \mathrm{e}^{i(\alpha\cdot s)}\, ds\right| \;\leqslant\; \frac{1}{T^{|\{j:\ \alpha_j\neq 0\}|}} \cdot \prod_{j,\alpha_j\neq 0} \frac{2}{|\alpha_j|},$$

and thus $\frac{1}{T^n}\int_{[0,T]^n} \mathrm{e}^{\mathrm{i}(\alpha\cdot s)}\, ds \to 0$ as $T \to \infty$. $\square$

It follows that every trigonometric polynomial w has a mean value. Using also Lemma 5.8.13, every almost periodic function $\mathbb{R}^n \to \mathbb{C}$ has a mean value.

Lemma 5.8.15. *Suppose $u\colon \mathbb{R}^n \to \mathbb{C}$ is continuous and 1-periodic. Then u is almost periodic with mean value $M(u) = \int_{[0,1]^n} u(s)\, ds$.*

Proof. By Proposition 5.8.1, u is almost periodic. Now use that for $T \in \mathbb{N}^{\geqslant 1}$,

$$\int_{[0,T]^n} u(s)\, ds \;=\; T^n \int_{[0,1]^n} u(s)\, ds. \qquad \square$$

Lemma 5.8.16. *Let $r \in (\mathbb{R}^\times)^n$. Then the almost periodic function $w_{\times r}$ has the same mean value as w.*

Proof. Choose a sequence (w_m) of trigonometric polynomials with $\|w_m - w\| \to 0$ as $m \to \infty$. Then $(w_m)_{\times r}$ is a trigonometric polynomial and $\|(w_m)_{\times r} - w_{\times r}\| \to 0$ as $m \to \infty$. Lemma 5.8.14 gives $M\big((w_m)_{\times r}\big) = M(w_m)$; now use Lemma 5.8.13. $\square$

Proposition 5.8.17 (Bohr). *Suppose $w(\mathbb{R}^n) \subseteq \mathbb{R}^{\geqslant}$. If $M(w) = 0$, then $w = 0$.*

Proof. Suppose $s_0 \in \mathbb{R}^n$ and $w(s_0) > 0$. We claim that then $M(w) > 0$. Take δ_1, ℓ_1 as in Corollary 5.8.8. Let k range over $\mathbb{N}^n$. Then $\int_{\ell_1 k+[0,\ell_1]^n} w(s)\, ds \geqslant \delta_1^n w(s_0)/3$ for all k, and hence for $m \geqslant 1$ and $T = \ell_1 m$:

$$\frac{1}{T^n}\int_{[0,T]^n} w(s)\, ds \;=\; \frac{1}{T^n}\sum_{|k|<m}\int_{\ell_1 k+[0,\ell_1]^n} w(s)\, ds \;\geqslant\; (\delta_1/\ell_1)^n w(s_0)/3.$$

Thus $M(w) \;\geqslant\; (\delta_1/\ell_1)^n w(s_0)/3 \;>\; 0$. $\square$

By Proposition 5.8.17, the map $(v,w) \mapsto \langle v,w\rangle := M(v\overline{w})$ is a positive definite hermitian form on the $\mathbb{C}$-linear space of almost periodic functions $\mathbb{R}^n \to \mathbb{C}$. Lemma 5.8.10 yields $\langle v_{+d}, w_{+d}\rangle = \langle v,w\rangle$ for $d \in \mathbb{R}^n$. By Lemma 5.8.14, the family $\big(s \mapsto \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}\big)_\alpha$ of trigonometric polynomials is orthonormal with respect to $\langle\ ,\ \rangle$. In particular, for a trigonometric polynomial w as in (5.8.1) we have $w_\alpha = \langle w, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}\rangle$, and thus:

Corollary 5.8.18. *If $w = 0$, then $w_\alpha = 0$ for all α.*

Corollary 5.8.19. *Suppose w is a trigonometric polynomial as in* (5.8.1). *Then*

$$w \text{ is 1-periodic} \iff w_\alpha = 0 \text{ for all } \alpha \notin 2\pi\mathbb{Z}^n.$$

Proof. If w is 1-periodic, then for $k \in \mathbb{Z}^n$ we have

$$w_\alpha \;=\; \langle w, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}\rangle \;=\; \langle w_{+k}, (\mathrm{e}^{(\alpha\cdot s)\mathrm{i}})_{+k}\rangle \;=\; \mathrm{e}^{-(\alpha\cdot k)\mathrm{i}}\langle w, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}\rangle \;=\; \mathrm{e}^{-(\alpha\cdot k)\mathrm{i}}\, w_\alpha,$$

which for $w_\alpha \neq 0$ gives $\alpha\cdot k \in 2\pi\mathbb{Z}$ for all $k \in \mathbb{Z}^n$, and thus $\alpha \in 2\pi\mathbb{Z}^n$. This yields the forward implication, and the backward direction is obvious. $\square$

In the next corollary we equip $\mathbb{R}^n$ with the lexicographic ordering.

Corollary 5.8.20. *Suppose w is a trigonometric polynomial. Then $w(\mathbb{R}^n) \subseteq \mathbb{R}$ iff there are $c \in \mathbb{R}$ and $u_\alpha, v_\alpha \in \mathbb{R}$ for $\alpha > 0$, with $u_\alpha = v_\alpha = 0$ for all but finitely many $\alpha > 0$, such that for all $s \in \mathbb{R}^n$,*

$$w(s) \;=\; c + \sum_{\alpha>0} \big(u_\alpha \cos(\alpha \cdot s) + v_\alpha \sin(\alpha \cdot s)\big). \tag{5.8.3}$$

Moreover, in this case c and the coefficients u_α, v_α are unique, and w is 1-periodic iff $u_\alpha = v_\alpha = 0$ for all $\alpha > 0$ with $\alpha \notin 2\pi\mathbb{Z}^n$.

Proof. Clearly if w has stated form, then $w(\mathbb{R}^n) \subseteq \mathbb{R}$. For the converse, suppose $w(\mathbb{R}^n) \subseteq \mathbb{R}$, and w is given as in (5.8.1). Then for $s \in \mathbb{R}^n$,

$$\sum_\alpha \overline{w_\alpha}\, \mathrm{e}^{-(\alpha\cdot s)\mathrm{i}} \;=\; \overline{w}(s) \;=\; w(s) \;=\; \sum_\alpha w_\alpha\, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}},$$

and hence $w_0 \in \mathbb{R}$ and $\overline{w_\alpha} = w_{-\alpha}$ for $\alpha > 0$, by Corollary 5.8.18, so

$$w(s) \;=\; w_0 + \sum_{\alpha>0} \big(w_\alpha\, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}} + \overline{w_\alpha}\, \mathrm{e}^{-(\alpha\cdot s)\mathrm{i}}\big) \qquad (s \in \mathbb{R}^n).$$

Put $c := w_0$ and $u_\alpha = \operatorname{Re}(2w_\alpha)$, $v_\alpha := \operatorname{Im}(2w_\alpha)$ for $\alpha > 0$. Then (5.8.3) holds for $s \in \mathbb{R}^n$. The rest follows from Corollaries 5.8.18 and 5.8.19. □

5.9. Uniform Distribution Modulo One

In this section we collect some basic facts about uniform distribution modulo 1 of functions as needed later. Our main references are [118, Chapter 1, §9] and [36].

Natural density. Below $\mathbb{R}$ is given its usual Lebesgue measure, *measurable* means *Lebesgue-measurable*, and μ denotes the Lebesgue measure on $\mathbb{R}$. By an "interval" we mean here a set $I = [a, b)$ where $a, b \in \mathbb{R}$, $a < b$, so $\mu(I) = b - a$. *In the rest of this subsection I is an interval and X, Y are measurable subsets of* $\mathbb{R}$. We let

$$\rho(I, X) \;:=\; \frac{\mu(I \cap X)}{\mu(I)} \in [0, 1]$$

be the **density of X in** I. So $\rho(I, X) = 0$ if $I \cap X = \emptyset$ and $\rho(I, X) = 1$ if $I \subseteq X$, and $\rho(I + d, X + d) = \rho(I, X)$ for $d \in \mathbb{R}$. Clearly $\rho(I, X) \leqslant \rho(I, Y)$ if $X \subseteq Y$, and if (X_n) is a family of pairwise disjoint measurable subsets of $\mathbb{R}$ and $X = \bigcup_n X_n$, then $\rho(I, X) = \sum_n \rho(I, X_n)$.

Let $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$ be the symmetric difference of X, Y. If $\mu(X) < \infty$ and $\mu(Y) < \infty$, then $\mu(X) - \mu(Y) \leqslant \mu(X \setminus Y)$ and $|\mu(X) - \mu(Y)| \leqslant \mu(X \triangle Y)$, so

Lemma 5.9.1. $|\rho(I, X) - \rho(I, Y)| \leqslant \rho(I, X \triangle Y)$.

Moreover:

Lemma 5.9.2. *Let $d \in \mathbb{R}$; then $\big|\rho(I, X) - \rho(I + d, X)\big| \leqslant |d|/\mu(I)$.*

Proof. We need to show $\big|\mu(I \cap X) - \mu\big((I + d) \cap X\big)\big| \leqslant |d|$. Replacing I and d by $I + d$ and $-d$, if necessary, we arrange $d \geqslant 0$. Then

$$-\mu(I) \;=\; -\mu(I + d) \;\leqslant\; \mu(I \cap X) - \mu\big((I + d) \cap X\big) \;\leqslant\; \mu(I),$$

hence we are done if $\mu(I) \leqslant d$. Suppose $\mu(I) > d$ and let $I = [a, b)$, $a, b \in \mathbb{R}$; so $\mu(I) = b - a$. Then $I \setminus (I + d) = [a, a + d)$ and $(I + d) \setminus I = [b, b + d)$, hence

$$-d \;=\; -\mu\big((I+d)\setminus I\big) \;\leqslant\; \mu(I \cap X) - \mu\big((I+d)\cap X\big) \;\leqslant\; \mu\big(I \setminus (I+d)\big) \;=\; d$$

as required. □

Let ρ range over $[0, 1]$ and T over $\mathbb{R}^{>}$. Lemma 5.9.2 gives:

Corollary 5.9.3. *The following conditions on X are equivalent:*

(i) $\lim_{T\to\infty} \rho\big([0, T), X\big) \;=\; \rho$;

(ii) *for all* $a \in \mathbb{R}$, $\lim_{T\to\infty} \rho\big([a, a + T), X\big) \;=\; \rho$;

(iii) *for some* $a \in \mathbb{R}$, $\lim_{T\to\infty} \rho\big([a, a + T), X\big) \;=\; \rho$.

We say that X has **natural density** ρ **at** $+\infty$ (short: X has density ρ) if one of the equivalent conditions in the corollary above holds, and in this case we set $\rho(X) := \rho$. If X has an upper bound in $\mathbb{R}$, then $\rho(X) = 0$, whereas if X contains a half-line $\mathbb{R}^{\geqslant a}$ ($a \in \mathbb{R}$), then $\rho(X) = 1$. By Lemma 5.9.1 we have:

Corollary 5.9.4. *If X has density ρ and $X \triangle Y$ has density 0, then Y has density ρ.*

In particular, the density of X only depends on the germ of X at $+\infty$, in the following sense: if $X \cap \mathbb{R}^{>a} = Y \cap \mathbb{R}^{>a}$ for some $a \in \mathbb{R}$, then X has density ρ iff Y has density ρ. The collection of measurable subsets of $\mathbb{R}$ that have a density is a boolean algebra of subsets of $\mathbb{R}$, and $X \mapsto \rho(X)$ is a finitely additive measure on this boolean algebra taking values in $[0, 1]$. If X has a density and $d \in \mathbb{R}$, then $X + d$ has the same density.

Uniform distribution mod 1**.** Let $f\colon \mathbb{R}^{\geqslant a} \to \mathbb{R}$ ($a \in \mathbb{R}$) be measurable. For $t \in \mathbb{R}$ we let $\{t\}$ be the fractional part of t: the element of $[0, 1)$ such that $t \in \mathbb{Z} + \{t\}$. Let $Y \subseteq [0, 1)$ be measurable; then $Y + \mathbb{Z}$ is measurable and hence so is

$$f^{-1}(Y + \mathbb{Z}) \;=\; \big\{t \in \mathbb{R}^{\geqslant a} : \{f(t)\} \in Y\big\}.$$

For $a \leqslant b < c$ we have

$$\mu\big([b, c) \cap f^{-1}(Y + \mathbb{Z})\big) \;=\; \int_b^c \chi_Y\big(\{f(t)\}\big)\, dt.$$

Let $\rho \in \mathbb{R}$; then $f^{-1}(Y + \mathbb{Z})$ has density ρ iff for some $b \geqslant a$ we have

$$\lim_{T\to\infty} \frac{1}{T} \int_b^{b+T} \chi_Y\big(\{f(t)\}\big) dt \;=\; \rho,$$

and in this case the displayed identity holds for all $b \geqslant a$. Hence if $f^{-1}(Y + \mathbb{Z})$ has density ρ and $g\colon \mathbb{R}^{\geqslant b} \to \mathbb{R}$ ($b \in \mathbb{R}$) is measurable with the same germ at $+\infty$ as f, then $g^{-1}(Y + \mathbb{Z})$ also has density ρ.

Definition 5.9.5. We say that f is **uniformly distributed** mod 1 (abbreviated: u.d. mod 1) if for every interval $I \subseteq [0, 1)$ the set $f^{-1}(I + \mathbb{Z})$ has density $\mu(I)$.

The function $f : \mathbb{R}^{\geqslant} \to \mathbb{R}$ with $f(t) = t$ for all $t \geqslant 0$ has $f^{-1}(I + \mathbb{Z}) = I + \mathbb{N}$ for I as above, so f is u.d mod 1. By the remarks above, if f is u.d. mod 1, then so is any measurable function $\mathbb{R}^{\geqslant b} \to \mathbb{R}$ ($b \in \mathbb{R}$) with the same germ at $+\infty$ as f. If f is u.d. mod 1 and eventually increasing or eventually decreasing, then $|f(t)| \to +\infty$ as $t \to +\infty$. If f is u.d. mod 1, then so are the functions $t \mapsto k \cdot f(t)\colon \mathbb{R}^{\geqslant a} \to \mathbb{R}$ for $k \in \mathbb{Z}^{\neq}$, and $t \mapsto f(d + t)\colon \mathbb{R}^{\geqslant (a-d)} \to \mathbb{R}$ with $d \in \mathbb{R}$.

The Weyl Criterion. In this subsection we fix a measurable function $f\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$. For a bounded measurable function $w\colon [0,1] \to \mathbb{R}$ we consider the relation

$$\text{(W)} \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T w\big(\{f(t)\}\big)dt \;=\; \int_0^1 w(s)\,ds.$$

Then f is u.d. mod 1 iff (W) holds whenever $w = \chi_I$ is the characteristic function of some interval $I \subseteq [0,1]$. It follows that if f is u.d. mod 1 and $w\colon [0,1] \to \mathbb{R}$ is a step function (that is, an $\mathbb{R}$-linear combination of characteristic functions χ_I of intervals $I \subseteq [0,1]$), then (W) holds.

Lemma 5.9.6. *Let $w\colon [0,1] \to \mathbb{R}$ be bounded and measurable, and suppose that for every $\varepsilon \in \mathbb{R}^{>}$ there are bounded measurable functions $w_1, w_2\colon [0,1] \to \mathbb{R}$ such that*

(i) *$w_1 \leqslant w \leqslant w_2$ on $[0,1)$,*
(ii) *$\int_0^1 \big(w_2(s) - w_1(s)\big)\,ds \leqslant \varepsilon$, and*
(iii) *for $i = 1, 2$, (W) holds for w_i instead of w.*

Then (W) *holds.*

Proof. Given $\varepsilon \in \mathbb{R}^{>}$ and $w_1, w_2\colon [0,1] \to \mathbb{R}$ satisfying (i), (ii), (iii), we have

$$\begin{aligned}
\int_0^1 w(s)\,ds - \varepsilon \;&\leqslant\; \int_0^1 w_1(s)\,ds \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T w_1\big(\{f(t)\}\big)dt \\
&\leqslant\; \liminf_{T\to\infty}\frac{1}{T}\int_0^T w\big(\{f(t)\}\big)dt \;\leqslant\; \limsup_{T\to\infty}\frac{1}{T}\int_0^T w\big(\{f(t)\}\big)dt \\
&\leqslant\; \lim_{T\to\infty}\frac{1}{T}\int_0^T w_2\big(\{f(t)\}\big)dt \;=\; \int_0^1 w_2(s)\,ds \\
&\leqslant\; \int_0^1 w(s)\,ds + \varepsilon.
\end{aligned}$$

□

Proposition 5.9.7. *f is u.d.* mod 1 *iff* (W) *holds for all continuous $w\colon [0,1] \to \mathbb{R}$.*

Proof. If $w\colon [0,1] \to \mathbb{R}$ is continuous, then partitioning $[0,1)$ into intervals as in Riemann integration we obtain for any $\varepsilon \in \mathbb{R}^{>}$ step functions $w_1, w_2\colon [0,1] \to \mathbb{R}$ such that $w_1 \leqslant w \leqslant w_2$ on $[0,1)$ and $\int_0^1 \big(w_2(s) - w_1(s)\big)\,ds \leqslant \varepsilon$. Moreover, if $I \subseteq [0,1)$ is an interval, then for any $\varepsilon \in \mathbb{R}^{>}$ there are continuous functions $w_1, w_2\colon [0,1] \to \mathbb{R}$ with $w_1 \leqslant \chi_I \leqslant w_2$ and $\int_0^1 \big(w_2(s) - w_1(s)\big)ds \leqslant \varepsilon$. The proposition follows from these facts and Lemma 5.9.6. □

It is convenient to extend the notion of mean value to bounded measurable functions $g\colon \mathbb{R}^{\geqslant} \to \mathbb{C}$: for such g, if $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T g(t)\,dt$ exists in $\mathbb{C}$, then we say that g has mean value $M(g) := \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T g(t)\,dt$.

Corollary 5.9.8. *The following conditions on f are equivalent:*

(i) *f is u.d.* mod 1*;*
(ii) $\lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T (w\circ f)(t)\,dt = \int_0^1 w(s)\,ds$ *for all continuous* 1*-periodic functions $w\colon \mathbb{R} \to \mathbb{C}$;*
(iii) *for every continuous* 1*-periodic $w\colon \mathbb{R} \to \mathbb{C}$, the function $w \circ f\colon \mathbb{R}^{\geqslant} \to \mathbb{C}$ has mean value $M(w \circ f) = M(w)$.*

Proof. We first show (i) ⇔ (ii). For the forward direction, apply Proposition 5.9.7 to the real and imaginary parts of w, using $w(\{t\}) = w(t)$ for $t \in \mathbb{R}$ and 1-periodic w. The converse follows from Lemma 5.9.6 and the observation that if $I \subseteq [0,1)$ is an interval, then for any $\varepsilon \in \mathbb{R}^>$ we can take continuous functions $w_1, w_2 \colon [0,1] \to \mathbb{R}$ with $w_1 \leqslant \chi_I \leqslant w_2$ and $\int_0^1 \big(w_2(s) - w_1(s)\big) ds \leqslant \varepsilon$ as in the proof of the proposition above, such that in addition $w_i(0) = w_i(1)$ for $i = 1, 2$, and then $v_i \colon \mathbb{R} \to \mathbb{R}$ given by $v_i(t) = w_i(\{t\})$ for $t \in \mathbb{R}$ $(i = 1, 2)$ is continuous and 1-periodic. The equivalence of (ii) and (iii) is immediate from Lemma 5.8.15. □

Theorem 5.9.9 (Weyl [207]). *The function f is u.d.* mod 1 *iff for all $n \geqslant 1$ we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{2\pi i n f(t)}\, dt \;=\; 0. \tag{5.9.1}$$

Proof. The forward direction follows from Corollary 5.9.8. Conversely, suppose that (5.9.1) holds for all $n \geqslant 1$. Note that then for all $k \in \mathbb{Z}^{\neq}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{2\pi i k f(t)}\, dt \;=\; 0.$$

Thus by Corollary 5.8.19, every 1-periodic trigonometric polynomial $v \colon \mathbb{R} \to \mathbb{C}$ gives a function $v \circ f$ with mean value $M(v \circ f) = M(v)$. Now let $w \colon \mathbb{R} \to \mathbb{C}$ be continuous and 1-periodic. Proposition 5.8.1 yields a sequence (v_m) of 1-periodic trigonometric polynomials $\mathbb{R} \to \mathbb{C}$ with $\|v_m - w\| \to 0$ as $m \to \infty$. So $M(v_m) \to M(w)$ as $m \to \infty$, by Lemma 5.8.13. Extend f to a measurable function $\mathbb{R} \to \mathbb{R}$, also denoted by f. Then $\|(v_m \circ f) - (w \circ f)\| \to 0$ as $m \to \infty$. Hence by Lemma 5.8.13 again, $w \circ f$ has a mean value and $M(v_m) = M(v_m \circ f) \to M(w \circ f)$ as $m \to \infty$. Therefore $M(w \circ f) = M(w)$. Hence f is u.d. mod 1 by Corollary 5.9.8. □

Remark. Let $w(s) = \mathrm{e}^{2\pi i s}$ $(s \in \mathbb{R})$ and let $g \colon \mathbb{R} \to \mathbb{R}$ be a continuous function whose restriction to $\mathbb{R}^{\geqslant}$ is u.d. mod 1. By Corollary 5.9.8, $w \circ g$ has a mean value. When is $w \circ g$ almost periodic? This happens only for very special g: if $w \circ g$ is almost periodic, then there are $r \in \mathbb{R}$ and an almost periodic $h \colon \mathbb{R} \to \mathbb{R}$ such that $g(t) = rt + h(t)$ for all $t \in \mathbb{R}$, by a theorem of Bohr [26]. Moreover, [143, Theorem 1] says that if $h \colon \mathbb{R} \to \mathbb{R}$ is almost periodic, then for all but countably many $r \in \mathbb{R}$ the function $t \mapsto rt + h(t) \colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ is u.d. mod 1. These facts are not used later.

Uniform distribution mod 1 **of differentiable functions.** Let $f \colon \mathbb{R}^{\geqslant a} \to \mathbb{R}$ $(a \in \mathbb{R})$ be continuously differentiable. We give here sufficient conditions for f to be u.d. mod 1 and for f not to be u.d. mod 1. First a lemma in the spirit of Corollary 5.7.4:

Lemma 5.9.10. *Let $F \colon \mathbb{R}^> \to \mathbb{R}$ be twice continuously differentiable such that $F(t)/t \to 0$ as $t \to +\infty$. Assume $t \mapsto tF''(t) \colon \mathbb{R}^> \to \mathbb{R}$ is bounded. Then $F'(t) \to 0$ as $t \to +\infty$.*

Proof. Let $t, \eta > 0$. Taylor's Theorem [16, Theorem 19.9] yields $\theta \in [0,1]$ such that

$$F(t+\eta) - F(t) \;=\; \eta F'(t) + \tfrac{1}{2}\eta^2 F''(t+\theta\eta),$$

and thus

$$F'(t) \;=\; \frac{F(t+\eta) - F(t)}{\eta} - \tfrac{1}{2}\eta F''(t+\theta\eta).$$

Take $M \in \mathbb{R}^>$ such that $|tF''(t)| \leqslant M$ for all $t \in \mathbb{R}^>$. Let $\varepsilon \in \mathbb{R}^>$, and set $\delta := \varepsilon/M$. Then for all $t > 0$, $\eta = \delta t$ yields $\theta = \theta_t \in [0,1]$ with

$$F'(t) \;=\; \left(\frac{F(t+\delta t)}{t+\delta t}\cdot\frac{1+\delta}{\delta} - \frac{F(t)}{t}\cdot\frac{1}{\delta}\right) - \frac{\delta}{2(1+\theta\delta)}(t+\theta\delta t)F''(t+\theta\delta t).$$

The difference in the parentheses tends to zero as $t \to \infty$ whereas the remaining term is $\leqslant \varepsilon/2$ in absolute value for all $t \in \mathbb{R}^>$. $\square$

Proposition 5.9.11 (Kuipers-Meulenbeld [117])**.** *Suppose the function*

$$t \mapsto f'(t)t \;:\; \mathbb{R}^{\geqslant a} \to \mathbb{R}$$

is bounded. Then f is not u.d. mod 1*.*

Proof. Replacing f by $t \mapsto f(a+t)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ we arrange $a = 0$. Assume towards a contradiction that (5.9.1) holds for $n = 1$, and consider $F\colon \mathbb{R}^> \to \mathbb{R}$ given by

$$F(t) \;:=\; \operatorname{Re}\left(\int_0^t e^{2\pi i f(s)}\,ds\right) \;=\; \int_0^t \cos\big(2\pi f(s)\big)\,ds.$$

Then F is twice continuously differentiable with

$$F'(t) \;=\; \cos\big(2\pi f(t)\big), \quad F''(t) \;=\; -2\pi f'(t)\sin\big(2\pi f(t)\big),$$

and $F(t)/t \to 0$ as $t \to \infty$ and $t \mapsto tF''(t)\colon \mathbb{R}^> \to \mathbb{R}$ is bounded. Hence by Lemma 5.9.10 we have $\cos\big(2\pi f(t)\big) \to 0$ as $t \to \infty$; likewise we show $\sin\big(2\pi f(t)\big) \to 0$ as $t \to \infty$. Hence $e^{2\pi i f(t)} \to 0$ as $t \to \infty$, a contradiction. $\square$

In the next proposition we assume $a = 0$ and consider the continuously differentiable function $t \mapsto g(t) := f(e^t)\colon \mathbb{R} \to \mathbb{R}$ (so $f(t) = g(\log t)$ for $t \in \mathbb{R}^>$).

Proposition 5.9.12 (Tsuji [201])**.** *Suppose g and g' are eventually strictly increasing with $g(t)/t \to +\infty$ as $t \to +\infty$. Then f is u.d.* mod 1*.*

Proof. Let $n \geqslant 1$; we claim that (5.9.1) holds. The continuous functions

$$\begin{aligned} t &\mapsto \varphi(t) := 2\pi n f(t) \;:\; \mathbb{R}^{\geqslant} \to \mathbb{R}, \\ t &\mapsto \gamma(t) := \varphi'(t)t = 2\pi n\, g'(\log t) \;:\; \mathbb{R}^> \to \mathbb{R} \end{aligned}$$

are eventually strictly increasing. We have $\varphi(t)/\log t \to +\infty$ as $t \to +\infty$. Therefore $\gamma(t) \to +\infty$ as $t \to +\infty$: otherwise $\varphi'(t) \leqslant M/t$ for all $t \geqslant b$, and some $b, M > 0$, and then integration gives $\varphi(t) = O(\log t)$ as $t \to +\infty$, a contradiction.

Take $a_0 \in \mathbb{R}^>$ such that φ and γ are strictly increasing on $\mathbb{R}^{\geqslant a_0}$ and $\gamma(a_0) > 0$. Set $\rho_0 = \varphi(a_0)$, and take $\eta\colon \mathbb{R}^{\geqslant \rho_0} \to \mathbb{R}^{\geqslant a_0}$ so that $(\eta \circ \varphi)(t) = t$ for $t \in \mathbb{R}^{\geqslant a_0}$. Then $\eta'\big(\varphi(t)\big) > 0$ and $\gamma(t) = \eta\big(\varphi(t)\big)/\eta'\big(\varphi(t)\big)$ for $t > a_0$. Hence the function

$$u \mapsto \eta^\dagger(u) \;:=\; \eta'(u)/\eta(u) \;:\; \mathbb{R}^{>\rho_0} \to \mathbb{R}^>$$

is strictly decreasing with $\lim\limits_{u \to +\infty} \eta^\dagger(u) = 0$. Let now $T > a_0$ and consider

$$I(T) \;:=\; \int_{a_0}^T \sin\varphi(t)\,dt.$$

Set $\rho_T = \varphi(T)$, and let $\tau \in (\rho_0, \rho_T)$. Substituting $u = \varphi(t)$ gives

$$I(T) \;=\; \int_{\rho_0}^{\rho_T} \eta'(u)\sin u\,du \;=\; \int_{\rho_0}^{\tau} \eta'(u)\sin u\,du + \int_{\tau}^{\rho_T} \eta(u)\eta^\dagger(u)\sin u\,du.$$

Two applications of the Second Mean Value Theorem for Integrals [16, Theorem 23.7] yield first τ_2 and then τ_1 such that $\tau \leqslant \tau_1 \leqslant \tau_2 \leqslant \rho_T$ and

$$\int_\tau^{\rho_T} \eta(u)\eta^\dagger(u)\sin u\,du \;=\; \eta^\dagger(\tau)\int_\tau^{\tau_2} \eta(u)\sin u\,du \;=\; \eta^\dagger(\tau)\eta(\tau_2)\int_{\tau_1}^{\tau_2} \sin u\,du,$$

hence

$$\int_\tau^{\rho_T} \eta(u)\eta^\dagger(u)\sin u\,du \;=\; \eta^\dagger(\tau)\,C \qquad \text{where } |C| \leqslant 2\eta(\rho_T) = 2T.$$

Let now $\varepsilon \in \mathbb{R}^>$ be given. Take $\tau > \rho_0$ so large that $\eta^\dagger(\tau) \leqslant \varepsilon/4$. Then for $T > a_0$ so large that $\varphi(T) > \tau$, and

$$\left|\int_{\rho_0}^{\tau} \eta'(u)\sin u\,du\right| \;\leqslant\; \varepsilon T/2$$

we have $|I(T)| \leqslant \varepsilon T$. Thus as $T \to \infty$ we have

$$\frac{1}{T}\int_0^T \sin\varphi(t)\,dt \;=\; \frac{1}{T}\int_0^{a_0} \sin\varphi(t)\,dt + \frac{I(T)}{T} \;\to\; 0.$$

Likewise, $\frac{1}{T}\int_0^T \cos\varphi(t)\,dt \to 0$ as $T \to \infty$. Thus (5.9.1) is satisfied. □

Theorem 5.9.13 (Boshernitzan [36])**.** *Suppose the germ of f, also denoted by f, lies in a Hardy field H. Then: f is u.d.* mod 1 $\Longleftrightarrow f \succ \log x$.

Proof. By increasing H we arrange $\log x \in H$. The claim is obvious if $f \preccurlyeq 1$, since then f is neither $\succ \log x$ nor u.d. mod 1. So suppose $f \succ 1$; then $f \succ \log x$ iff $f' \succ 1/x$. If $f' \preccurlyeq 1/x$, then f is not u.d. mod 1 by Proposition 5.9.11. Suppose $f' \succ 1/x$; to show that f is u.d. mod 1 we replace f by $t \mapsto f(a+t)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ and H by $H\circ(a+x)$ to arrange $a = 0$. Replacing f by $-f$ if necessary we also arrange $f > 0$ in H. Then $f(t) \to +\infty$ as $t \to +\infty$, hence the function $t \mapsto g(t) := f(\mathrm{e}^t)\colon \mathbb{R} \to \mathbb{R}$ is eventually strictly increasing and its germ, also denoted by g, lies in some Hardy field and satisfies $g \succ x$; thus $g' \succ 1$, hence $t \mapsto g'(t)\colon \mathbb{R} \to \mathbb{R}$ is also eventually strictly increasing. Thus f is u.d. mod 1 by Proposition 5.9.12. □

In particular, if f is u.d. mod 1 and its germ lies in a Hardy field, then αf is u.d. mod 1 for every $\alpha \in \mathbb{R}^\times$.

Uniform distribution mod 1 **in higher dimensions.** In this subsection $n \geqslant 1$, $\mathbb{R}^n$ is equipped with its usual Lebesgue measure μ_n, and *measurable* for a subset of $\mathbb{R}^n$ means measurable with respect to μ_n. Let $a \in \mathbb{R}$ and consider measurable functions $f_1, \dots, f_n\colon \mathbb{R}^{\geqslant a} \to \mathbb{R}$, which we combine into a single map

$$f \;:=\; (f_1, \dots, f_n) \;:\; \mathbb{R}^{\geqslant a} \to \mathbb{R}^n.$$

By a **box** (in $\mathbb{R}^n$) we mean a set $I = I_1 \times \cdots \times I_n$ where $I_1, \dots, I_n$ are intervals, so $\mu_n(I) = \mu(I_1)\cdots\mu(I_n)$ and $f^{-1}(I + \mathbb{Z}^n) = \bigcap_{j=1}^n f_j^{-1}(I_j + \mathbb{Z})$ is measurable.

Definition 5.9.14. We say that f is **uniformly distributed** mod 1 (abbreviated: u.d. mod 1) if for every box $I \subseteq [0,1)^n$ the set $f^{-1}(I + \mathbb{Z}^n)$ has density $\mu_n(I)$.

For $s = (s_1, \dots, s_n) \in \mathbb{R}^n$, set $\{s\} := \big(\{s_1\}, \dots, \{s_n\}\big) \in [0,1)^n$. With this notation, f is u.d. mod 1 iff for every box $I \subseteq [0,1)^n$ we have

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} \chi_I\big(\{f(t)\}\big)dt \;=\; \mu_n(I).$$

Let $b \in \mathbb{R}^{\geqslant a}$, $d \in \mathbb{R}$. Then f is u.d. mod 1 iff the restriction of f to $\mathbb{R}^{\geqslant b}$ is u.d. mod 1, and if f is u.d. mod 1, then so is $t \mapsto f(d+t) \colon \mathbb{R}^{\geqslant (a-d)} \to \mathbb{R}^n$.

In the rest of this subsection we assume $a = 0$. Proposition 5.9.7 and its Corollary 5.9.8 generalize to this setting:

Proposition 5.9.15. *The map f is u.d.* mod 1 *if and only if*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T w\big(\{f(t)\}\big)dt \ = \ \int_{[0,1]^n} w(s)\, ds$$

for every continuous function $w \colon [0,1]^n \to \mathbb{R}$.

Proof. First we use the proof of Lemma 5.9.6 to obtain the analogue of that lemma for bounded measurable functions $w \colon [0,1]^n \to \mathbb{R}$. Now, given a continuous function $w \colon [0,1]^n \to \mathbb{R}$ and $\varepsilon \in \mathbb{R}^{>}$, there are $\mathbb{R}$-linear combinations $w_1, w_2 \colon [0,1]^n \to \mathbb{R}$ of characteristic functions of pairwise disjoint boxes contained in $[0,1]^n$ such that $w_1 \leqslant w \leqslant w_2$ on $[0,1)^n$ and $\int_{[0,1]^n} \big(w_2(s) - w_1(s)\big)\, ds \leqslant \varepsilon$. This gives one direction.

Next, let $I = I_1 \times \cdots \times I_n \subseteq [0,1)^n$ be a box and $\varepsilon \in \mathbb{R}^{>}$. For $j = 1, \dots, n$ we have continuous functions $w_{1j}, w_{2j} \colon [0,1] \to \mathbb{R}^{\geqslant}$ such that

$$0 \ \leqslant \ w_{1j} \ \leqslant \ \chi_{I_j} \ \leqslant \ w_{2j} \ \leqslant \ 1 \quad \text{and} \quad \int_0^1 \big(w_{2j}(t) - w_{1j}(t)\big)dt \leqslant \varepsilon/2^n.$$

For $s = (s_1, \dots, s_n) \in [0,1]^n$ set $w_i(s) := w_{i1}(s_1) \cdots w_{in}(s_n)$. Then the functions $w_1, w_2 \colon [0,1]^n \to \mathbb{R}$ are continuous with

$$w_1 \leqslant \chi_I \leqslant w_2 \quad \text{and} \quad \int_{[0,1]^n} \big(w_2(s) - w_1(s)\big)ds \leqslant \varepsilon.$$

The proposition follows from these facts just as in the proof of Proposition 5.9.7. $\square$

As Proposition 5.9.7 led to Corollary 5.9.8, so does Proposition 5.9.15 give:

Corollary 5.9.16. *The following conditions on f are equivalent:*

(i) *the map f is u.d.* mod 1*;*

(ii) $\displaystyle\lim_{T\to\infty} \frac{1}{T} \int_0^T (w \circ f)(t)\, dt \ = \ \int_{[0,1]^n} w(s)\, ds$ *for every continuous* 1*-periodic function $w \colon \mathbb{R}^n \to \mathbb{C}$;*

(iii) *for every continuous* 1*-periodic $w \colon \mathbb{R}^n \to \mathbb{C}$, the function $w \circ f \colon \mathbb{R}^{\geqslant} \to \mathbb{C}$ has mean value $M(w \circ f) = M(w)$.*

Corollary 5.9.17. *Let $w \colon \mathbb{R}^n \to \mathbb{R}^{\geqslant}$ be* 1*-periodic and continuous, and suppose f is u.d.* mod 1*. Then*

$$\limsup_{t\to\infty} w\big(f(t)\big) = 0 \iff \limsup_{|s|\to\infty} w(s) = 0 \iff w = 0.$$

Proof. Corollary 5.9.16 gives $M(w \circ f) = M(w)$, and $\|w\| = \limsup_{|s|\to\infty} w(s)$ by Lemma 5.8.9. One verifies easily that $M(w \circ f) \leqslant \limsup_{t\to\infty} w\big(f(t)\big)$. The equivalences now follow from these facts and Proposition 5.8.17. $\square$

Corollary 5.9.18. *Let $w \colon \mathbb{R}^n \to \mathbb{R}$ be* 1*-periodic and continuous, and suppose f is u.d.* mod 1*. Then $\limsup\limits_{t\to\infty} w\big(f(t)\big) = \sup_s w(s)$ and $\liminf\limits_{t\to\infty} w\big(f(t)\big) = \inf_s w(s)$.*

Proof. Let $a \in \mathbb{R}$, $a < \sup_s w(s)$. Then $w = u + v$ where $u, v\colon \mathbb{R}^n \to \mathbb{R}$ are given by $u(s) := \min\bigl(a, w(s)\bigr)$ for all s. Then u, and thus v, is 1-periodic and continuous. Now $v \geqslant 0$, but $v \neq 0$, so $\limsup_{t\to\infty} v\bigl(f(t)\bigr) > 0$ by Corollary 5.9.17. This gives $\varepsilon > 0$ with $v(f(t)) > \varepsilon$ for arbitrarily large t. For such t we have $u\bigl(f(t)\bigr) = a$: $u\bigl(f(t)\bigr) < a$ would give $u\bigl(f(t)\bigr) = w\bigl(f(t)\bigr)$, so $v\bigl(f(t)\bigr) = 0$. Hence $w\bigl(f(t)\bigr) = u\bigl(f(t)\bigr) + v\bigl(f(t)\bigr) > a + \varepsilon$ for such t. The other equality follows likewise. ☐

Weyl's Theorem 5.9.9 also generalizes:

Theorem 5.9.19. *The map f is u.d.* mod 1 *if and only if for all $k \in (\mathbb{Z}^n)^{\neq}$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{e}^{2\pi\mathrm{i}(k\cdot f(t))}\,dt \;=\; 0.$$

Proof. Like that of Theorem 5.9.9, using 5.9.16 instead of 5.9.8. ☐

Theorems 5.9.9 and 5.9.19 yield:

Corollary 5.9.20. *The map f is u.d.* mod 1 *if and only if for all $k \in (\mathbb{Z}^n)^{\neq}$, the function $t \mapsto k \cdot f(t)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ is u.d.* mod 1.

Strengthening uniform distribution. *In this subsection $n \geqslant 1$, the functions $f_1, \dots, f_n\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ are measurable, $f = (f_1, \dots, f_n)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}^n$ is the resulting map, and for $\alpha \in \mathbb{R}^n$ we set $\alpha f := (\alpha_1 f_1, \dots, \alpha_n f_n)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}^n$.*

Lemma 5.9.21. *The following conditions on f are equivalent:*

(i) *αf is u.d.* mod 1 *for all $\alpha \in (\mathbb{R}^{\times})^n$;*

(ii) $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{e}^{2\pi\mathrm{i}(\beta\cdot f(t))}\,dt \;=\; 0$ *for all $\beta \in (\mathbb{R}^n)^{\neq}$;*

(iii) *for every almost periodic $w\colon \mathbb{R}^n \to \mathbb{C}$, the function $w \circ f\colon \mathbb{R}^{\geqslant} \to \mathbb{C}$ has mean value $M(w \circ f) = M(w)$.*

Proof. Assume (i); let $\beta \in (\mathbb{R}^n)^{\neq}$. For $i = 1, \dots, n$ set $\alpha_i := 1$, $k_i := 0$ if $\beta_i = 0$ and $\alpha_i := \beta_i$, $k_i := 1$ if $\beta_i \neq 0$. Then $k = (k_1, \dots, k_n) \in (\mathbb{Z}^n)^{\neq}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is in $(\mathbb{R}^{\times})^n$, and $\beta \cdot f(t) = k \cdot (\alpha f)(t)$ for all $t \in \mathbb{R}$. Now (ii) follows from Theorem 5.9.19 applied to αf in place of f. The implication (ii) $\Rightarrow$ (iii) follows as in the proof of Theorem 5.9.9, using the definition of almost periodicity instead of Proposition 5.8.1. Finally, assume (iii), and let $\alpha \in (\mathbb{R}^{\times})^n$; to show that αf is u.d. mod 1 we verify that condition (iii) in Corollary 5.9.16 holds for αf in place of f. Thus let $w\colon \mathbb{R}^n \to \mathbb{C}$ be continuous and 1-periodic. By (iii) applied to the almost periodic function

$$s \mapsto w_{\times\alpha}(s) = w(\alpha s) \;:\; \mathbb{R}^n \to \mathbb{C}$$

in place of w, the function $w_{\times\alpha} \circ f = w \circ (\alpha f)\colon \mathbb{R}^{\geqslant} \to \mathbb{C}$ has a mean value and $M(w_{\times\alpha} \circ f) = M(w_{\times\alpha})$; now use that $M(w_{\times\alpha}) = M(w)$ by Lemma 5.8.16. ☐

We say that f is **uniformly distributed** (abbreviated: u.d.) if it satisfies one of the equivalent conditions in Lemma 5.9.21. This lemma also yields:

Corollary 5.9.22. *The map f is u.d. if and only if for all $\beta \in (\mathbb{R}^n)^{\neq}$, the function $t \mapsto \beta \cdot f(t)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ is u.d.* mod 1.

The proof of the next result is like that of Corollary 5.9.17, using Lemma 5.9.21 instead of Corollary 5.9.16:

Corollary 5.9.23. *Suppose $w\colon \mathbb{R}^n \to \mathbb{R}^{\geqslant}$ is almost periodic and f is u.d. Then*

$$\limsup_{t\to\infty} w\big(f(t)\big) = 0 \iff \limsup_{|s|\to\infty} w(s) = 0 \iff w = 0.$$

Application to Hardy fields. *In this subsection $f_1, \dots, f_n\colon \mathbb{R}^{\geqslant} \to \mathbb{R}$ with $n \geqslant 1$ are continuous, their germs, denoted also by $f_1, \dots, f_n$, lie in a common Hardy field, and $f := (f_1, \dots, f_n)\colon \mathbb{R}^{\geqslant} \to \mathbb{R}^n$.* Theorem 5.9.13 with Corollary 5.9.20 gives:

Corollary 5.9.24 (Boshernitzan)**.** *We have the following equivalence:*

$$f \text{ is u.d. mod } 1 \iff k_1 f_1 + \cdots + k_n f_n \succ \log x \text{ for all } (k_1, \dots, k_n) \in (\mathbb{Z}^n)^{\neq}.$$

Combining Theorem 5.9.13 with Corollary 5.9.22 yields likewise:

Corollary 5.9.25. *We have the following equivalence:*

$$f \text{ is u.d.} \iff \alpha_1 f_1 + \cdots + \alpha_n f_n \succ \log x \text{ for all } (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}^n)^{\neq}.$$

In particular, if $\log x \prec f_1 \prec \cdots \prec f_n$, then f is u.d.

Here is an immediate application of Corollary 5.9.24:

Corollary 5.9.26 (Weyl)**.** *Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Define $g\colon \mathbb{R}^{\geqslant} \to \mathbb{R}^n$ by $g(t) = (\lambda_1 t, \dots, \lambda_n t)$. Then g is u.d.* mod 1 *iff $\lambda_1, \dots, \lambda_n$ are $\mathbb{Q}$-linearly independent.*

We now get to the result that we actually need in Section 5.10:

Proposition 5.9.27. *Suppose $w\colon \mathbb{R}^n \to \mathbb{R}^{\geqslant}$ is almost periodic, $1 \prec f_1 \prec \cdots \prec f_n$, and $\limsup\limits_{t\to+\infty} w\big(f(t)\big) = 0$. Then $w = 0$.*

Proof. We first arrange $f_1 > \mathbb{R}$, replacing $f_1, \dots, f_n$ and w by $-f_1, \dots, -f_n$ and the function $s \mapsto w(-s)\colon \mathbb{R}^n \to \mathbb{R}^{\geqslant}$, if $f_1 < \mathbb{R}$. Pick $a \geqslant 0$ such that the restriction of f_1 to $\mathbb{R}^{\geqslant a}$ is strictly increasing, set $b := f_1(a)$, and let $f_1^{\mathrm{inv}}\colon \mathbb{R}^{\geqslant b} \to \mathbb{R}$ be the compositional inverse of this restriction. Set $g_j(t) := (f_j \circ f_1^{\mathrm{inv}})(t)$ for $t \geqslant b$ and $j = 1, \dots, n$ and consider the map

$$g \;=\; (g_1, \dots, g_n) \;=\; f \circ f_1^{\mathrm{inv}} \;:\; \mathbb{R}^{\geqslant b} \to \mathbb{R}^n.$$

The germs of $g_1, \dots, g_n$, denoted by the same symbols, lie in a common Hardy field and satisfy $x = g_1 \prec g_2 \prec \cdots \prec g_n$. Now f_1^{inv} is strictly increasing and moreover $f_1^{\mathrm{inv}}(t) \to +\infty$ as $t \to +\infty$, so

$$\limsup_{t\to\infty} w\big(f(t)\big) \;=\; \limsup_{t\to\infty} w\big(f\big(f_1^{\mathrm{inv}}(t)\big)\big) \;=\; \limsup_{t\to\infty} w\big(g(t)\big) \;=\; 0.$$

Thus replacing $f_1, \dots, f_n$ by continuous functions $\mathbb{R}^{\geqslant} \to \mathbb{R}$ with the same germs as $g_1, \dots, g_n$, we arrange $x = f_1 \prec f_2 \prec \cdots \prec f_n$. Then f is u.d. by Corollary 5.9.25. Now use Corollary 5.9.23. □

The next three results are not used later but included for their independent interest.

Corollary 5.9.28. *Assume $w\colon \mathbb{R}^n \to \mathbb{R}$ is almost periodic and $1 \prec f_1 \prec \cdots \prec f_n$. Then $\limsup\limits_{t\to\infty} w\big(f(t)\big) = \sup_s w(s)$ and $\liminf\limits_{t\to\infty} w\big(f(t)\big) = \inf_s w(s)$.*

Proof. Let $a \in \mathbb{R}$, $a < \sup_s w(s)$. Then $w = u + v$ where $u, v\colon \mathbb{R}^n \to \mathbb{R}$ are given by $u(s) := \min\big(a, w(s)\big)$ for all s. Then u, and thus v, is almost periodic by Corollary 5.8.2. Now argue as in the proof of Corollary 5.9.18, using Proposition 5.9.27 instead of Corollary 5.9.17. □

Corollary 5.9.29. *If $w\colon \mathbb{R}^n \to \mathbb{C}$ is almost periodic and $1 \prec f_1 \prec \cdots \prec f_n$, then*

$$\lim_{t\to\infty} w\big(f(t)\big) \text{ exists in } \mathbb{C} \quad \Longleftrightarrow \quad w \text{ is constant.}$$

Proof. Apply the previous corollary to the real and imaginary part of w. $\square$

Finally, we use these results to reprove [101, Theorem 8]. Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ we put $\mathrm{e}^\alpha := (\mathrm{e}^{\alpha_1}, \dots, \mathrm{e}^{\alpha_n}) \in \mathbb{C}^n$. Let $m \geqslant 1$ and set

$$S \;:=\; \big\{(z_1, \dots, z_m) \in \mathbb{C}^m : \ |z_1| = \cdots = |z_m| = 1\big\}.$$

Corollary 5.9.30. *Suppose $1 \prec f_1 \prec \cdots \prec f_n$. Let $\varphi\colon S \to \mathbb{R}$ be continuous and let $k_1, \dots, k_n \in \mathbb{N}^{\geqslant 1}$ with $k_1 + \cdots + k_n = m$ and $\lambda_j = (\lambda_{j1}, \dots, \lambda_{jk_j}) \in \mathbb{R}^{k_j}$ for $j = 1, \dots, n$ be such that $\lambda_{j1}, \dots, \lambda_{jk_j}$ are $\mathbb{Q}$-linearly independent. Then*

$$\limsup_{t\to\infty} \varphi\big(\,\mathrm{e}^{\mathrm{i}f_1(t)\lambda_1}, \dots, \mathrm{e}^{\mathrm{i}f_n(t)\lambda_n}\,\big) \;=\; \max \varphi(S).$$

Proof. By Corollary 5.8.2, the function

$$s = (s_1, \dots, s_n) \mapsto w(s) := \varphi\big(\,\mathrm{e}^{\mathrm{i}s_1\lambda_1}, \dots, \mathrm{e}^{\mathrm{i}s_n\lambda_n}\,\big) \;:\; \mathbb{R}^n \to \mathbb{R}$$

is almost periodic. We have

$$w\big(f(t)\big) \;=\; \varphi\big(\,\mathrm{e}^{\mathrm{i}f_1(t)\lambda_1}, \dots, \mathrm{e}^{\mathrm{i}f_n(t)\lambda_n}\,\big) \quad \text{for } t \geqslant 0,$$

so by Corollary 5.9.28,

$$\limsup_{t\to\infty} \varphi\big(\,\mathrm{e}^{\mathrm{i}f_1(t)\lambda_1}, \dots, \mathrm{e}^{\mathrm{i}f_n(t)\lambda_n}\,\big) \;=\; \limsup_{t\to\infty} w\big(f(t)\big) \;=\; \sup_s w(s).$$

For $j = 1, \dots, n$ it follows from Corollary 5.9.26 that the image of the map

$$t \mapsto \mathrm{e}^{\mathrm{i}t\lambda_j} \;:\; \mathbb{R}^{\geqslant} \to \big\{(z_1, \dots, z_{k_j}) \in \mathbb{C}^{k_j} : |z_1| = \cdots = |z_{k_j}| = 1\big\}$$

is dense in its codomain, so the image of the map

$$(s_1, \dots, s_n) \mapsto \big(\,\mathrm{e}^{\mathrm{i}s_1\lambda_1}, \dots, \mathrm{e}^{\mathrm{i}s_n\lambda_n}\,\big) \;:\; \mathbb{R}^n \to S$$

is dense in S. Hence $\sup_s w(s) = \max \varphi(S)$. $\square$

Examples involving real-valued trigonometric polynomials (*)**.** The material in this subsection is only used later to justify a remark after Corollary 5.10.11.

Example 5.9.31. Let $a, b \in \mathbb{R}^\times$, and consider the 1-periodic trigonometric polynomial $w\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$w(s) \;=\; a\cos(2\pi s_1) + b\cos(2\pi s_2) \qquad \text{for } s = (s_1, s_2) \in \mathbb{R}^2.$$

Let $\lambda, \mu \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Then by Corollaries 5.9.18 and 5.9.26:

$$\limsup_{t\to-\infty} w(\lambda t, \mu t) \;=\; \limsup_{t\to+\infty} w(\lambda t, \mu t) \;=\; |a| + |b|.$$

Note also that $|a| + |b| > w(\lambda t, \mu t)$ for all $t \in \mathbb{R}^{\neq}$.

Now let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and consider

$$v\colon \mathbb{R} \to \mathbb{R}, \qquad v(t) := 2 - \cos(t) - \cos(\alpha t).$$

Then $v(t) > 0$ for all $t \in \mathbb{R}^{\neq}$. Moreover, $\liminf\limits_{t\to+\infty} v(t) = 0$, that is, for each $\varepsilon > 0$ there are arbitrarily large $t \in \mathbb{R}$ with $v(t) < \varepsilon$. With a suitable choice of α we can replace here ε by any prescribed function $\varepsilon\colon \mathbb{R} \to \mathbb{R}^>$ with $\varepsilon(t) \to 0$ as $t \to +\infty$:

Theorem 5.9.32 (Basu-Bose-Vijayaraghavan [17]). *Let $\phi\colon \mathbb{R} \to \mathbb{R}^{>}$ be such that $\phi(t) \to +\infty$ as $t \to +\infty$. Then there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that*

$$2 - \cos(t) - \cos(\alpha t) < 1/\phi(t) \quad \textit{for arbitrarily large } t \in \mathbb{R}.$$

Proof. We first arrange $\phi \geqslant 1$. We then choose a sequence $(d_n)_{n\geqslant 1}$ of positive integers such that with $q_n := d_1 d_2 \cdots d_n$ (so $q_0 = 1$):

$$d_n \geqslant (2\pi + 1)\,\phi(2\pi q_{n-1}) \qquad \text{for } n \geqslant 1,$$

and set

$$\alpha \ := \ \sum_{n=1}^{\infty} \frac{1}{q_n}.$$

We have $q_{m+1} = q_m d_{m+1} \geqslant (2\pi+1) q_m\, \phi(2\pi q_m)$, so if $q_{m+n} \geqslant (2\pi+1)^n q_m\, \phi(2\pi q_m)$, then

$$q_{m+n+1} \geqslant (2\pi + 1) q_{m+n}\, \phi(2\pi q_{m+n}) \geqslant (2\pi + 1) q_{m+n} \geqslant (2\pi + 1)^{n+1} q_m\, \phi(2\pi q_m).$$

Thus by induction on n we obtain

$$q_{m+n} \geqslant (2\pi + 1)^n q_m\, \phi(2\pi q_m) \qquad \text{for } n \geqslant 1.$$

This yields

$$\sum_{n=1}^{\infty} \frac{1}{q_{m+n}} \leqslant \frac{1}{2\pi q_m\, \phi(2\pi q_m)} \qquad \text{for all } m \geqslant 1. \tag{5.9.2}$$

Take $p_m \in \mathbb{N}$ $(m \geqslant 1)$ such that

$$\sum_{n=1}^{m} \frac{1}{q_n} = \frac{p_m}{q_m}.$$

Then

$$0 < \alpha - \frac{p_m}{q_m} = \sum_{n=1}^{\infty} \frac{1}{q_{m+n}} \leqslant \frac{1}{2\pi q_m\, \phi(2\pi q_m)} \qquad \text{for all } m \geqslant 1.$$

Suppose $\alpha = p/q$ where $p, q \in \mathbb{N}^{\geqslant 1}$; then for all $m \geqslant 1$ we have

$$\frac{pq_m - qp_m}{qq_m} = \alpha - \frac{p_m}{q_m} \leqslant \frac{1}{2\pi q_m\, \phi(2\pi q_m)}$$

and so

$$1 \leqslant pq_m - qp_m \leqslant \frac{q}{2\pi\, \phi(2\pi q_m)},$$

contradicting $\phi(2\pi q_m) \to +\infty$ as $m \to +\infty$. Hence $\alpha \notin \mathbb{Q}$. Next note that q_m and $\alpha q_m - q_m \sum_{n\geqslant 1} \frac{1}{q_{m+n}}$ are integers and so

$$2 - \cos(2\pi q_m) - \cos(2\pi \alpha q_m) = 1 - \cos\left(2\pi q_m \sum_{n=1}^{\infty} \frac{1}{q_{m+n}}\right) < \frac{1}{\phi(2\pi q_m)}$$

using (5.9.2). This yields the theorem. $\square$

In this section $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field. Then $\mathcal{C}^{<\infty}[\mathrm{i}]$ is a differential ring extension of the d-valued field $K := H[\mathrm{i}]$ with the same ring of constants as K, namely $\mathbb{C}$. Note that for any $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ we have a $g \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $g' = f$, and then $u = \mathrm{e}^g \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times$ satisfies $u^\dagger = f$.

Lemma 5.10.1. *Suppose $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ is purely imaginary, that is, $f \in \mathrm{i}\mathcal{C}^{<\infty}$. Then there is a $u \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times$ such that $u^\dagger = f$ and $|u| = 1$.*

Proof. Taking $g \in \mathrm{i}\mathcal{C}^{<\infty}$ with $g' = f$, the resulting $u = \mathrm{e}^g$ works. $\square$

We define the subgroup $\mathrm{e}^{H\mathrm{i}}$ of $\mathcal{C}^{<\infty}[\mathrm{i}]^\times$ by

$$\mathrm{e}^{H\mathrm{i}} \ :=\ \big\{\mathrm{e}^{h\mathrm{i}} :\ h \in H\big\} \ =\ \big\{u \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times :\ |u| = 1,\ u^\dagger \in H\mathrm{i}\big\}.$$

Then $(\mathrm{e}^{H\mathrm{i}})^\dagger = H\mathrm{i}$ by Lemma 5.10.1, so $(H^\times \cdot \mathrm{e}^{H\mathrm{i}})^\dagger = K$ and thus $K[\mathrm{e}^{H\mathrm{i}}]$ is an exponential extension of K (in the sense of Section 2.2) with the same ring of constants $\mathbb{C}$ as K.

As in the beginning of Section 4.4 we fix a complement Λ of $K^\dagger$ with $\Lambda \subseteq H\mathrm{i}$, set $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ as usual, and let λ range over Λ. The differential K-algebras U and $K[\mathrm{e}^{H\mathrm{i}}]$ are isomorphic by Corollary 2.2.10, but we need something better:

Lemma 5.10.2. *There is an isomorphism $\mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$ of differential K-algebras that maps $\mathrm{e}(\Lambda)$ into $\mathrm{e}^{H\mathrm{i}}$.*

Proof. We have a short exact sequence of commutative groups

$$1 \to S \xrightarrow{\subseteq} \mathrm{e}^{H\mathrm{i}} \xrightarrow{\ell} H\mathrm{i} \to 0,$$

where $S = \big\{z \in \mathbb{C}^\times :\ |z| = 1\big\}$ and $\ell(u) := u^\dagger$ for $u \in \mathrm{e}^{H\mathrm{i}}$. Since the subgroup S of $\mathbb{C}^\times$ is divisible, this sequence splits: we have a group embedding $e\colon H\mathrm{i} \to \mathrm{e}^{H\mathrm{i}}$ such that $e(b)^\dagger = b$ for all $b \in H\mathrm{i}$. Then the group embedding

$$\mathrm{e}(\lambda) \mapsto e(\lambda)\ :\ \mathrm{e}(\Lambda) \to \mathrm{e}^{H\mathrm{i}}$$

extends uniquely to a K-algebra morphism $\mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$. Since $\mathrm{e}(\lambda)^\dagger = \lambda = e(\lambda)^\dagger$ for all λ, this is a differential K-algebra morphism, and even an isomorphism by Lemma 2.2.9 applied to $R = K[\mathrm{e}^{H\mathrm{i}}]$. $\square$

Complex conjugation $f + g\mathrm{i} \mapsto \overline{f + g\mathrm{i}} = f - g\mathrm{i}$ $(f, g \in \mathcal{C}^{<\infty})$ is an automorphism of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$ over H and maps $K[\mathrm{e}^{H\mathrm{i}}]$ onto itself, sending each $u \in \mathrm{e}^{H\mathrm{i}}$ to u^{-1}. Thus any isomorphism $\iota\colon \mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$ of differential K-algebras with $\iota\big(\mathrm{e}(\Lambda)\big) \subseteq \mathrm{e}^{H\mathrm{i}}$—such ι exists by Lemma 5.10.2—also satisfies

$$\iota(\overline{f}) \ =\ \overline{\iota(f)} \qquad (f \in \mathrm{U}).$$

(See Section 2.2 for the definition of $\overline{f}$ for $f \in \mathrm{U}$. Given such an isomorphism ι, any differential K-algebra isomorphism $\mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$ mapping $\mathrm{e}(\Lambda)$ into $\mathrm{e}^{H\mathrm{i}}$ equals $\iota \circ \sigma_\chi$ for a unique character $\chi\colon \Lambda \to \mathbb{C}^\times$ with $|\chi(\lambda)| = 1$ for all λ, by Lemma 2.2.17.) Fix such an isomorphism ι and identify U with its image $K[\mathrm{e}^{H\mathrm{i}}]$ via ι. We have the asymptotic relations $\preccurlyeq_{\mathrm{g}}$ and $\prec_{\mathrm{g}}$ on U coming from the gaussian extension v_{g} of the valuation on K. But we also have the asymptotic relations induced on $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}]$ by the relations $\preccurlyeq$ and $\prec$ defined on $\mathcal{C}[\mathrm{i}]$ in Section 5.1. It is clear that for $f \in \mathrm{U}$:

$$f \preccurlyeq_{\mathrm{g}} 1 \quad\Longrightarrow\quad f \preccurlyeq 1 \quad\Longleftrightarrow\quad \text{for some } n \text{ we have } |f(t)| \leqslant n \text{ eventually},$$

$$f \prec_{\mathrm{g}} 1 \quad\Longrightarrow\quad f \prec 1 \quad\Longleftrightarrow\quad \lim_{t\to+\infty} f(t) = 0.$$

As a tool for later use we derive a converse of the implication $f \prec_{\mathrm{g}} 1 \Rightarrow f \prec 1$: Lemma 5.10.8 below, where we assume in addition that $\mathrm{I}(K) \subseteq K^{\dagger}$ and Λ is an $\mathbb{R}$-linear subspace of K. This requires the material from Section 5.9 and some considerations about exponential sums treated in the next subsection.

Exponential sums over Hardy fields. In this subsection $n \geqslant 1$. In the next lemma, $f = (f_1, \dots, f_m) \in H^m$ where $m \geqslant 1$ and $1 \prec f_1 \prec \cdots \prec f_m$. (In that lemma it doesn't matter which functions we use to represent the germs $f_1, \dots, f_m$.) For $r = (r_1, \dots, r_m) \in \mathbb{R}^m$ we set $r \cdot f := r_1 f_1 + \cdots + r_m f_m \in H$.

Lemma 5.10.3. *Let $r^1, \dots, r^n \in \mathbb{R}^m$ be distinct and $c_1, \dots, c_n \in \mathbb{C}^{\times}$. Then*

$$\limsup_{t \to \infty} \left| c_1 \,\mathrm{e}^{(r^1 \cdot f)(t)i} + \cdots + c_n \,\mathrm{e}^{(r^n \cdot f)(t)i} \right| \ > \ 0.$$

Proof. Consider the trigonometric polynomial $w\colon \mathbb{R}^m \to \mathbb{R}^{\geqslant}$ given by

$$w(s) \ := \ \left| c_1 \,\mathrm{e}^{(r^1 \cdot s)i} + \cdots + c_n \,\mathrm{e}^{(r^n \cdot s)i} \right|^2.$$

By Corollary 5.8.18 we have $w(s) > 0$ for some $s \in \mathbb{R}^m$. Taking continuous representatives $\mathbb{R}^{\geqslant} \to \mathbb{R}$ of $f_1, \dots, f_m$, to be denoted also by $f_1, \dots, f_m$, the lemma now follows from Proposition 5.9.27. $\square$

Next, let $h_1, \dots, h_n \in H$ be distinct such that $(\mathbb{R}h_1 + \cdots + \mathbb{R}h_n) \cap \mathrm{I}(H) = \{0\}$. Since H is Liouville closed we have $\phi_1, \dots, \phi_n \in H$ such that $\phi_1' = h_1, \dots, \phi_n' = h_n$.

Lemma 5.10.4. *Let $c_1, \dots, c_n \in \mathbb{C}^{\times}$. Then for $\phi_1, \dots, \phi_n$ as above,*

$$\limsup_{t \to \infty} \left| c_1 \,\mathrm{e}^{\phi_1(t)i} + \cdots + c_n \,\mathrm{e}^{\phi_n(t)i} \right| \ > \ 0.$$

Proof. The case $n = 1$ is trivial, so let $n \geqslant 2$. Then $\phi_1, \dots, \phi_n$ are not all in $\mathbb{R}$. Set $V := \mathbb{R} + \mathbb{R}\phi_1 + \cdots + \mathbb{R}\phi_n \subseteq H$, so $\partial V = \mathbb{R}h_1 + \cdots + \mathbb{R}h_n$. We claim that $V \cap \mathcal{o}_H = \{0\}$. To see this, let $\phi \in V \cap \mathcal{o}_H$; then $\phi' \in \partial(V) \cap \mathrm{I}(H) = \{0\}$ and hence $\phi \in \mathbb{R} \cap \mathcal{o}_H = \{0\}$, proving the claim. Now H is a Hahn space over $\mathbb{R}$ by [ADH, p. 109], so by [ADH, 2.3.13] we have $f_1, \dots, f_m \in V$ $(1 \leqslant m \leqslant n)$ such that $V = \mathbb{R} + \mathbb{R}f_1 + \cdots + \mathbb{R}f_m$ and $1 \prec f_1 \prec \cdots \prec f_m$. For $j = 1, \dots, n$, $k = 1, \dots, m$, take $t_j, r_{jk} \in \mathbb{R}$ such that $\phi_j = t_j + \sum_{k=1}^m r_{jk} f_k$ and set $r^j := (r_{j1}, \dots, r_{jm}) \in \mathbb{R}^m$. Since $\phi_{j_1} - \phi_{j_2} \notin \mathbb{R}$ for $j_1 \neq j_2$, we have $r^{j_1} \neq r^{j_2}$ for $j_1 \neq j_2$. It remains to apply Lemma 5.10.3 to $c_1 \,\mathrm{e}^{t_1 i}, \dots, c_n \,\mathrm{e}^{t_n i}$ in place of $c_1, \dots, c_n$. $\square$

Corollary 5.10.5. *Let $f_1, \dots, f_n \in K$ and set $f := f_1 \,\mathrm{e}^{\phi_1 i} + \cdots + f_n \,\mathrm{e}^{\phi_n i} \in \mathcal{C}^{<\infty}[i]$, and suppose $f \prec 1$. Then $f_1, \dots, f_n \prec 1$.*

Proof. We may assume $0 \neq f_1 \preccurlyeq \cdots \preccurlyeq f_n$. Towards a contradiction, suppose that $f_n \succcurlyeq 1$, and take $m \leqslant n$ minimal such that $f_m \asymp f_n$. Then with $g_j := f_j / f_n \in K^{\times}$ and $g := g_1 \,\mathrm{e}^{\phi_1 i} + \cdots + g_n \,\mathrm{e}^{\phi_n i}$ we have $g \prec 1$ and $g_1, \dots, g_n \preccurlyeq 1$, with $g_j \prec 1$ iff $j < m$. Replacing $f_1, \dots, f_n$ by $g_m, \dots, g_n$ and $\phi_1, \dots, \phi_n$ by $\phi_m, \dots, \phi_n$ we arrange $f_1 \asymp \cdots \asymp f_n \asymp 1$. So

$$f_1 = c_1 + \varepsilon_1, \dots, f_n = c_n + \varepsilon_n \quad \text{ with } c_1, \dots, c_n \in \mathbb{C}^{\times} \text{ and } \varepsilon_1, \dots, \varepsilon_n \in \mathcal{o}.$$

Then $\varepsilon_1 \,\mathrm{e}^{\phi_1 i} + \cdots + \varepsilon_n \,\mathrm{e}^{\phi_n i} \prec 1$, hence

$$c_1 \,\mathrm{e}^{\phi_1 i} + \cdots + c_n \,\mathrm{e}^{\phi_n i} \ = \ f - \left(\varepsilon_1 \,\mathrm{e}^{\phi_1 i} + \cdots + \mathrm{e}^{\phi_n i} \right) \ \prec \ 1.$$

Now Lemma 5.10.4 yields the desired contradiction. $\square$

Here is an application of Corollary 5.10.5:

Lemma 5.10.6. *Let $f_1, \dots, f_n, g_1, \dots, g_n \in K$ be such that in $\mathcal{C}[\mathrm{i}]$ we have*

$$f_1 \,\mathrm{e}^{\phi_1 \mathrm{i}} + \dots + f_n \,\mathrm{e}^{\phi_n \mathrm{i}} \;\sim\; g_1 \,\mathrm{e}^{\phi_1 \mathrm{i}} + \dots + g_n \,\mathrm{e}^{\phi_n \mathrm{i}}.$$

Let $j \in \{1, \dots, n\}$ be such that $0 \neq f_j \succcurlyeq f_k$ for all $k \in \{1, \dots, n\}$. Then $f_j \sim g_j$, and $f_k - g_k \prec f_j$ for all $k \neq j$.

Proof. We arrange $j = 1$ and $f_1 = 1$. Then

$$\mathrm{e}^{\phi_1 \mathrm{i}} + f_2 \,\mathrm{e}^{\phi_2 \mathrm{i}} + \dots + f_n \,\mathrm{e}^{\phi_n \mathrm{i}} \;\sim\; g_1 \,\mathrm{e}^{\phi_1 \mathrm{i}} + \dots + g_n \,\mathrm{e}^{\phi_n \mathrm{i}}, \qquad f_2, \dots, f_n \preccurlyeq 1.$$

Hence

$$(1 - g_1)\,\mathrm{e}^{\phi_1 \mathrm{i}} + (f_2 - g_2)\,\mathrm{e}^{\phi_2 \mathrm{i}} + \dots + (f_n - g_n)\,\mathrm{e}^{\phi_n \mathrm{i}} \prec \mathrm{e}^{\phi_1 \mathrm{i}} + f_2 \,\mathrm{e}^{\phi_2 \mathrm{i}} + \dots + f_n \,\mathrm{e}^{\phi_n \mathrm{i}} \preccurlyeq 1,$$

so $1 - g_1 \prec 1$ and $f_k - g_k \prec 1$ for all $k \neq j$, by Corollary 5.10.5. □

This leads to a partial generalization of Corollary 5.5.23, included for use in [15]:

Corollary 5.10.7. *Let $f \in K^\times$, $g_1, \dots, g_n \in K$, and $j \in \{1, \dots, n\}$ such that in $\mathcal{C}[\mathrm{i}]$,*

$$f \,\mathrm{e}^{\phi_j \mathrm{i}} \sim g_1 \,\mathrm{e}^{\phi_1 \mathrm{i}} + \dots + g_n \,\mathrm{e}^{\phi_n \mathrm{i}}.$$

Then $f \sim g_j$, and $g_j \succ g_k$ for all $k \neq j$.

Proof. Use Lemma 5.10.6 with $f_j := f$ and $f_k := 0$ for $k \neq j$. □

In the rest of this subsection we assume that $\mathrm{I}(K) \subseteq K^\dagger$. As noted in Section 4.4 we can then take $\Lambda = \Lambda_H i$ where Λ_H is an $\mathbb{R}$-linear complement of $\mathrm{I}(H)$ in H. *We assume Λ has this form,* giving rise to the valuation v_{g} on $\mathrm{U} = K[\mathrm{e}^{Hi}]$ as explained in the beginning of this section.

Lemma 5.10.8. *Let $f \in \mathrm{U}$ be such that $f \prec 1$. Then $f \prec_{\mathrm{g}} 1$.*

Proof. We have $f = f_1 \,\mathrm{e}(h_1 i) + \dots + f_n \,\mathrm{e}(h_n i)$ with $f_1, \dots, f_n \in K$ and distinct $h_1, \dots, h_n \in \Lambda_H$, so $(\mathbb{R}h_1 + \dots + \mathbb{R}h_n) \cap \mathrm{I}(H) = \{0\}$. For $h \in \Lambda_H$ we have $\mathrm{e}(hi) = \mathrm{e}^{\phi i}$ with $\phi \in H$ and $\phi' = h$. Hence $f = f_1 \,\mathrm{e}^{\phi_1 i} + \dots + f_n \,\mathrm{e}^{\phi_n i}$ with $\phi_1, \dots, \phi_n \in H$ such that $\phi_1' = h_1, \dots, \phi_n' = h_n$. Now Corollary 5.10.5 yields $f \prec_{\mathrm{g}} 1$. □

Corollary 5.10.9. *Let $f \in \mathrm{U}$ and $\mathfrak{m} \in H^\times$. Then $f \prec \mathfrak{m}$ iff $f \prec_{\mathrm{g}} \mathfrak{m}$.*

Lemma 5.10.10. *Let $f \in \mathrm{U}$ and $\mathfrak{m} \in H^\times$. Then $f \preccurlyeq \mathfrak{m}$ iff $f \preccurlyeq_{\mathrm{g}} \mathfrak{m}$.*

Proof. Replace $f, \mathfrak{m}$ by $f/\mathfrak{m}, 1$, respectively, to arrange $\mathfrak{m} = 1$. The backward direction was observed earlier in this section. For the forward direction suppose $f \preccurlyeq 1$. Then $f \prec \mathfrak{n}$ for all $\mathfrak{n} \in H^\times$ with $1 \prec \mathfrak{n}$, hence $f \prec_{\mathrm{g}} \mathfrak{n}$ for all $\mathfrak{n} \in H^\times$ with $1 \prec_{\mathrm{g}} \mathfrak{n}$, by two applications of Corollary 5.10.9, and thus $f \preccurlyeq_{\mathrm{g}} 1$. □

Corollary 5.10.11. *Let $f, g \in \mathrm{U}$. Then*

$$f \preccurlyeq g \implies f \preccurlyeq_{\mathrm{g}} g, \tag{5.10.1}$$

and likewise with $(\preccurlyeq, \preccurlyeq_{\mathrm{g}})$ replaced by $(\asymp, \asymp_{\mathrm{g}})$, $(\prec, \prec_{\mathrm{g}})$, or $(\sim, \sim_{\mathrm{g}})$. In particular, $\mathrm{e}^{\phi i} \asymp_{\mathrm{g}} 1$ for all $\phi \in H$.

Proof. The case $g = 0$ is trivial, so let $g \neq 0$. Then $g \asymp_{\mathrm{g}} \mathfrak{n}$ with $\mathfrak{n} \in H^\times$, so $g \preccurlyeq_{\mathrm{g}} \mathfrak{n}$ and $\mathfrak{n} \preccurlyeq_{\mathrm{g}} g$, and thus $g \preccurlyeq \mathfrak{n}$ by Lemma 5.10.10. If $f \preccurlyeq g$, then $f \preccurlyeq \mathfrak{n}$, hence $f \preccurlyeq_{\mathrm{g}} \mathfrak{n}$ by Lemma 5.10.10, so $f \preccurlyeq_{\mathrm{g}} g$. Likewise, if $f \prec g$, then $f \prec \mathfrak{n}$, so $f \prec_{\mathrm{g}} \mathfrak{n}$ by Corollary 5.10.9, hence $f \prec_{\mathrm{g}} g$. The rest is now clear. □

Remark. The converse of (5.10.1) doesn't hold in general, even when we restrict to $f = 1$ and $g \in \mathrm{U} \cap \mathcal{C}^\times$: let $\lambda, \mu \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent and set

$$g \ := \ 2 - \cos(\lambda x) - \cos(\mu x) \in \mathrm{U};$$

then $1 \asymp_{\mathrm{g}} g$, and by Example 5.9.31 we have $g \in \mathcal{C}^\times$ and $1 \not\preccurlyeq g$. Next, take $\phi \in H$ with $\phi > \mathbb{R}$, choose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ as in Theorem 5.9.32 applied to a representative of the germ ϕ, and set

$$h \ := \ \phi \cdot \big(2 - \cos(x) - \cos(\alpha x)\big) \in \mathrm{U}.$$

Then $h \in \mathcal{C}^\times$ and $h \asymp_{\mathrm{g}} \phi$, so $1 \prec_{\mathrm{g}} h$. By choice of α we also have $1 \not\prec h$. Hence the converse of (5.10.1) for $(\prec, \prec_{\mathrm{g}})$ in place of $(\preccurlyeq, \preccurlyeq_{\mathrm{g}})$ fails for $f := 1$, $g := h$.

An application to slots in H. *In this subsection we assume* $\mathrm{I}(K) \subseteq K^\dagger$. *We take* $\Lambda = \Lambda_H \mathrm{i}$ *where* Λ_H *is an* $\mathbb{R}$*-linear complement of* $\mathrm{I}(H)$ *in* H*, and accordingly identify* U *with* $K[\mathrm{e}^{H\mathrm{i}}]$ *as explained in the beginning of this section.* Until further notice we let $(P, 1, \widehat{h})$ be a slot in H of order $r \geqslant 1$. We also let $A \in K[\partial]$ have order r, and we let $\mathfrak{m}$ range over the elements of $H^\times$ such that $v\mathfrak{m} \in v(\widehat{h} - H)$. We begin with an important consequence of the material in Section 5.7:

Lemma 5.10.12. *Suppose $(P, 1, \widehat{h})$ is Z-minimal, deep, and special, and $\mathfrak{v}(L_P) \asymp \mathfrak{v} := \mathfrak{v}(A)$. Let $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy $A(y) = 0$ and $y \prec \mathfrak{m}$ for all $\mathfrak{m}$. Then $y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. Corollary 3.3.15 gives an $\mathfrak{m} \preccurlyeq \mathfrak{v}$, so it is enough to show $y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m} \preccurlyeq \mathfrak{v}$. Accordingly we assume $0 < \mathfrak{m} \preccurlyeq \mathfrak{v}$ below. As $\widehat{h}$ is special over H, we have $2(r+1)v\mathfrak{m} \in v(\widehat{h} - H)$, so $y \prec \mathfrak{m}^{2(r+1)}$. Then Corollary 5.7.2 with $n = 2(r+1)$, $\eta = |\mathfrak{v}|^{-1}$, $\varepsilon = 1/r$ gives for $j = 0, \dots, r$:

$$y^{(j)} \ \prec \ \mathfrak{v}^{-j}\mathfrak{m}^{n-j(1+\varepsilon)} \ \preccurlyeq \ \mathfrak{m}^{n-j(2+\varepsilon)} \ \preccurlyeq \ \mathfrak{m}^{n-r(2+\varepsilon)} \ = \ \mathfrak{m}. \qquad \square$$

Note that by Proposition 5.2.1, if $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$ and $A(y) = 0$, $y \in \mathcal{C}^r[\mathrm{i}]$, then $y \in \mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$. Corollary 5.10.9 is typically used in combination with the ultimate condition. Here is a first easy application:

Lemma 5.10.13. *Suppose $(P, 1, \widehat{h})$ is linear and ultimate,* $\dim_{\mathbb{C}} \ker_{\mathrm{U}} L_P = r$*, and $y \in \mathcal{C}^r[\mathrm{i}]$ satisfies $L_P(y) = 0$ and $y \prec 1$. Then $y \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. We have $y \in \mathrm{U}$, so $y \prec_{\mathrm{g}} 1$ by Lemma 5.10.8. If $y = 0$ we are done, so assume $y \neq 0$. Lemma 4.4.4(ii) gives $0 < v_{\mathrm{g}} y \in v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} L_P) = \mathscr{E}^{\mathrm{u}}(L_P)$, hence $v_{\mathrm{g}} y > v(\widehat{h} - H)$ by Lemma 4.4.13, so $y \prec_{\mathrm{g}} \mathfrak{m}$ for all $\mathfrak{m}$. Now Corollary 5.10.9 yields the desired conclusion. $\square$

Corollary 5.10.14. *Suppose that $(P, 1, \widehat{h})$ is Z-minimal, deep, special, linear, and ultimate, and that* $\dim_{\mathbb{C}} \ker_{\mathrm{U}} L_P = r$*. Let $f, g \in \mathcal{C}^r[\mathrm{i}]$ be such that $P(f) = P(g) = 0$ and $f, g \prec 1$. Then $(f - g)^{(j)} \prec \mathfrak{m}$ for $j = 0, \dots, r$ and all $\mathfrak{m}$.*

Proof. Use Lemmas 5.10.12 and 5.10.13 for $A = L_P$ and $y = f - g$. $\square$

In the rest of this subsection we assume that $(P, 1, \widehat{h})$ is ultimate and normal, $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$*, and $L_P = A + B$ where*

$$B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1} A, \qquad \mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1.$$

Then Lemma 3.1.1 gives $\mathfrak{v}(L_P) \sim \mathfrak{v}$, and by Lemma 4.4.4,

$$v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \;=\; \mathscr{E}^{\mathrm{u}}(A) \;=\; \mathscr{E}^{\mathrm{u}}(L_P).$$

This yields a variant of Lemma 5.10.13:

Proposition 5.10.15. *If $y \in \mathcal{C}^r[\mathrm{i}]$ and $A(y)=0$, $y \prec 1$, then $y \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. Like that of Lemma 5.10.13, using Lemma 4.4.12 instead of 4.4.13. □

The following result will be used in establishing a crucial non-linear version of Corollary 5.10.14, namely Proposition 6.5.14.

Corollary 5.10.16. *If $(P,1,\widehat{h})$ is Z-minimal, deep, and special, and $y \in \mathcal{C}^r[\mathrm{i}]$ is such that $A(y)=0$ and $y \prec 1$, then $y, y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. Use first Proposition 5.10.15 and then Lemma 5.10.12. □

So far we didn't have to name an immediate asymptotic extension of H where $\widehat{h}$ is located, but for the "complex" version of the above we need to be more specific.

As in the beginning of Section 4.4, let $\widehat{H}$ be an immediate asymptotic extension of H and $\widehat{K} = \widehat{H}[\mathrm{i}] \supseteq \widehat{H}$ a corresponding immediate d-valued extension of K. The results in this subsection then go through if instead of $(P,1,\widehat{h})$ being a slot in H of order $r \geqslant 1$ we assume that $(P,1,\widehat{h})$ is a slot in K of order $r \geqslant 1$ with $\widehat{h} \in \widehat{K} \setminus K$, with $\mathfrak{m}$ now ranging over the elements of $K^\times$ such that $v\mathfrak{m} \in v(\widehat{h} - K)$.

Solution spaces of linear differential operators. Recall that $\Lambda \subseteq H\mathrm{i}$, $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big] = K[\mathrm{e}^{H\mathrm{i}}]$ where $\mathrm{e}(\Lambda) \subseteq \mathrm{e}^{H\mathrm{i}} \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]^\times$. Hence for each λ we have an element $\phi(\lambda)$ of H (unique up to addition of an element of $2\pi\mathbb{Z}$) such that $\mathrm{e}(\lambda) = \mathrm{e}^{\phi(\lambda)\mathrm{i}}$; we take $\phi(0) := 0$. Then $\mathrm{e}(\lambda)^\dagger = \lambda$ gives $\phi(\lambda)'\mathrm{i} = \lambda$, and

$$\phi(\lambda_1+\lambda_2) \;\equiv\; \phi(\lambda_1)+\phi(\lambda_2) \bmod 2\pi\mathbb{Z} \quad \text{for } \lambda_1,\lambda_2 \in \Lambda.$$

If $\mathrm{I}(K) \subseteq K^\dagger$, then $\Lambda \cap \mathrm{I}(H)\mathrm{i} = \{0\}$ (see Lemma 1.2.16), so $\phi(\lambda) \succ 1$ for $\lambda \neq 0$, hence for $\mu \in \Lambda$: $\lambda = \mu \Leftrightarrow \phi(\lambda) = \phi(\mu) \Leftrightarrow \phi(\lambda) - \phi(\mu) \preccurlyeq 1$.

Lemma 5.10.17. *Let $\phi \in H$. Then there exists λ such that $\phi - \phi(\lambda) \preccurlyeq 1$. If $\phi \succ 1$, then for any such λ we have $\operatorname{sign}\phi = \operatorname{sign}\operatorname{Im}\lambda$.*

Proof. From $\mathrm{e}^{\phi\mathrm{i}} \in \mathrm{e}^{H\mathrm{i}} \subseteq \mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$ we get $f \in K^\times$ and λ with $\mathrm{e}^{\phi\mathrm{i}} = f\,\mathrm{e}(\lambda) = f\,\mathrm{e}^{\phi(\lambda)\mathrm{i}}$. Note that $|f|=1$, so $f = \mathrm{e}^{\theta\mathrm{i}}$ where $\theta \in H$ with $\theta \preccurlyeq 1$, by Lemma 5.5.21. This yields $\phi - \phi(\lambda) - \theta \in 2\pi\mathbb{Z}$ and so $\phi - \phi(\lambda) \preccurlyeq 1$. This proves the first statement. Now suppose we have any λ with $\phi - \phi(\lambda) \preccurlyeq 1$. Then $\phi \sim \phi(\lambda)$ if $\phi \succ 1$. So if $\phi > \mathbb{R}$, then $\phi(\lambda) > \mathbb{R}$ and thus $\operatorname{Im}\lambda = \phi(\lambda)' > 0$; likewise, $\phi < \mathbb{R}$ implies $\operatorname{Im}\lambda < 0$. □

Corollary 5.10.18. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$. Let $f = f_1\,\mathrm{e}^{\phi_1\mathrm{i}} + \cdots + f_m\,\mathrm{e}^{\phi_m\mathrm{i}} \in \mathrm{U}$ where $f_1,\dots,f_m \in K$ and $\phi_1,\dots,\phi_m \in H$ are such that $\phi_j = \phi_k$ or $\phi_j - \phi_k \succ 1$ for $j,k = 1,\dots,m$. Then*

$$f = 0 \quad\Longleftrightarrow\quad \sum_{1\leqslant k\leqslant m,\ \phi_k=\phi_j} f_k \;=\; 0 \ \text{ for } j=1,\dots,m,$$

and for $\mathfrak{m} \in H^\times$:

$$f \prec \mathfrak{m} \quad\Longleftrightarrow\quad \sum_{1\leqslant k\leqslant m,\ \phi_k=\phi_j} f_k \;\prec\; \mathfrak{m} \text{ for } j=1,\dots,m,$$

and likewise with $\preccurlyeq$ in place of $\prec$.

Proof. We first arrange that $\phi_1, \dots, \phi_m$ are distinct, and we then need to show: $f = 0 \Leftrightarrow f_1 = \cdots = f_m = 0$, and $f \prec \mathfrak{m} \Leftrightarrow f_1, \dots, f_m \prec \mathfrak{m}$, and likewise with $\preccurlyeq$ in place of $\prec$. To make Corollary 5.10.9 applicable we also arrange that $\Lambda = \Lambda_H \mathrm{i}$ with Λ_H an $\mathbb{R}$-linear complement of $\mathrm{I}(H)$ in H. Lemma 5.10.17 yields $\lambda_j \in \Lambda$ with $\phi_j - \phi(\lambda_j) \preccurlyeq 1$ for $j = 1, \dots, m$; then $\lambda_1, \dots, \lambda_m$ are distinct. For $j = 1, \dots, m$, put $g_j := f_j \, \mathrm{e}^{(\phi_j - \phi(\lambda_j))\mathrm{i}} \in K$, so $f_j \, \mathrm{e}^{\phi_j \mathrm{i}} = g_j \, \mathrm{e}(\lambda_j)$ and $g_j \asymp |g_j| = |f_j| \asymp f_j$. Now the claim follows from the K-linear independence of $\mathrm{e}(\lambda_1), \dots, \mathrm{e}(\lambda_m)$, Corollary 5.10.9, and Lemma 5.10.10. $\square$

Let $A \in K[\partial]^{\neq}$, $r := \operatorname{order} A$, and set $V := \ker_{\mathrm{U}} A$, a $\mathbb{C}$-linear subspace of U of dimension at most r, with $\dim_{\mathbb{C}} V = r$ iff $V = \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$. We describe in our present setting some consequences of the results obtained in Sections 2.3 and 2.5 about zeros of linear differential operators in the universal exponential extension.

Lemma 5.10.19. *The $\mathbb{C}$-linear space V has a basis*

$$f_1 \, \mathrm{e}(\lambda_1), \dots, f_d \, \mathrm{e}(\lambda_d) \quad \textit{where } f_j \in K^\times,\ \lambda_j \in \Lambda\ (j = 1, \dots, d).$$

For any such basis the set of eigenvalues of A with respect to Λ is $\{\lambda_1, \dots, \lambda_d\}$, and

$$\operatorname{mult}_\lambda(A) \ = \ |\{j \in \{1, \dots, d\} : \lambda_j = \lambda\}| \quad \textit{for every } \lambda.$$

This follows from Lemma 2.5.1 and the considerations preceding it.

Call $\phi_1, \dots, \phi_m \in H$ **apart** if $\phi_j = 0$ or $\phi_j \succ 1$ for $j = 1, \dots, m$, and $\phi_j = \phi_k$ or $\phi_j - \phi_k \succ 1$ for $j, k = 1, \dots, m$. (This holds in particular if $\phi_1 = \cdots = \phi_m = 0$.) If $\mathrm{I}(K) \subseteq K^\dagger$, then $\phi(\lambda_1), \dots, \phi(\lambda_m)$ are apart for any $\lambda_1, \dots, \lambda_m \in \Lambda$.

Corollary 5.10.20. *The $\mathbb{C}$-linear space V has a basis*

$$f_1 \, \mathrm{e}^{\phi_1 \mathrm{i}}, \dots, f_d \, \mathrm{e}^{\phi_d \mathrm{i}} \quad \textit{where } f_j \in K^\times,\ \phi_j \in H\ (j = 1, \dots, d).$$

If $\mathrm{I}(K) \subseteq K^\dagger$, then for any such basis the $f_j \, \mathrm{e}^{\phi_j \mathrm{i}}$ with $\phi_j \preccurlyeq 1$ form a basis of the $\mathbb{C}$-linear space $V \cap K = \ker_K A$, and we can choose the f_j, ϕ_j such that additionally $\phi_1, \dots \phi_d$ are apart and $(\phi_1, vf_1), \dots, (\phi_d, vf_d)$ are distinct.

Proof. The first claim holds by Lemma 5.10.19. Suppose $\mathrm{I}(K) \subseteq K^\dagger$, and let a basis of V as in the corollary be given. Then by Lemma 5.10.17 we obtain $\lambda_j \in \Lambda$ such that $\phi_j - \phi(\lambda_j) \preccurlyeq 1$, and so $f_j \, \mathrm{e}^{\phi_j \mathrm{i}} = g_j \, \mathrm{e}(\lambda_j)$ where $g_j := f_j \, \mathrm{e}^{(\phi_j - \phi(\lambda_j))\mathrm{i}} \in K^\times$ by Proposition 5.5.18. Now $\lambda_j = 0 \Leftrightarrow \phi_j \preccurlyeq 1$, by the remarks preceding Lemma 5.10.17, hence the $f_j \, \mathrm{e}^{\phi_j \mathrm{i}}$ with $\phi_j \preccurlyeq 1$ form a basis of $V \cap K = \ker_K A$. Moreover, $g_1 \, \mathrm{e}^{\phi(\lambda_1)\mathrm{i}}, \dots, g_d \, \mathrm{e}^{\phi(\lambda_d)\mathrm{i}}$ is a basis of V and $\phi(\lambda_1), \dots, \phi(\lambda_d)$ are apart.

We have $V = \bigoplus_\lambda V_\lambda$ (internal direct sum of $\mathbb{C}$-linear subspaces) where $V_\lambda = (\ker_K A_\lambda) \, \mathrm{e}(\lambda)$, by the remarks before (2.5.1). For each λ, the subspace $\ker_K A_\lambda$ of the $\mathbb{C}$-linear space K is generated by the g_j with $\lambda_j = \lambda$. Applying [ADH, 5.6.6] to each A_λ we obtain $h_j \in K^\times$ such that $h_1 \, \mathrm{e}^{\phi(\lambda_1)\mathrm{i}}, \dots, h_d \, \mathrm{e}^{\phi(\lambda_d)\mathrm{i}}$ is a basis of V where for all $j \neq k$ with $\phi(\lambda_j) = \phi(\lambda_k)$ we have $h_j \not\asymp h_k$. $\square$

A **Hahn basis** of V is a basis of V as in Corollary 5.10.20 such that $\phi_1, \dots, \phi_d$ are apart and $(\phi_1, vf_1), \dots, (\phi_d, vf_d)$ are distinct. (It should really be "Hahn basis with respect to $\phi_1, \dots, \phi_d$" but in the few cases we use this notion we shall rely on the context as to what tuple $(\phi_1, \dots, \phi_d) \in H^d$ we are dealing with.) If $\mathrm{I}(K) \subseteq K^\dagger$, then for such a Hahn basis the f_j with $\phi_j = 0$ form a valuation basis of the subspace $V \cap K$ of the valued $\mathbb{C}$-linear space K [ADH, 2.3].

In the next lemma we assume $\mathrm{I}(K) \subseteq K^\dagger$, and we recall that then

$$d \ \leqslant\ \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \ \leqslant\ r$$

by Lemma 2.6.16 and Proposition 2.6.26, and so by Lemma 2.6.16,

$$\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \ =\ d \implies \mathscr{E}^{\mathrm{e}}(A_\lambda) \ =\ v(\ker^{\neq} A_\lambda) \ \text{ for all } \lambda.$$

Lemma 5.10.21. *Suppose* $\mathrm{I}(K) \subseteq K^\dagger$, *and let* $f_1 \mathrm{e}^{\phi_1 \mathrm{i}}, \dots, f_d \mathrm{e}^{\phi_d \mathrm{i}}$ *be a Hahn basis of* V *as in Corollary 5.10.20. Then for all* λ,

$$\mathscr{E}^{\mathrm{e}}(A_\lambda) \ \supseteq\ v(\ker^{\neq} A_\lambda) \ =\ \big\{ v f_j :\ 1 \leqslant j \leqslant d,\ \phi_j - \phi(\lambda) \preccurlyeq 1 \big\}.$$

and so $\mathscr{E}^{\mathrm{u}}(A) \supseteq \{v f_1, \dots, v f_d\}$, *with* $\mathscr{E}^{\mathrm{u}}(A) = \{v f_1, \dots, v f_d\}$ *if* $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = d$.

Proof. Take g_j, λ_j as in the proof of Corollary 5.10.20. Then $g_j \asymp |g_j| = |f_j| \asymp f_j$, and $\lambda_j = \lambda \Leftrightarrow \phi_j - \phi(\lambda) \preccurlyeq 1$, for all λ. So we can replace f_j, ϕ_j by g_j, $\phi(\lambda_j)$ to arrange $\phi_j = \phi(\lambda_j)$ for $j = 1, \dots, d$. Then for all λ the $\mathbb{C}$-linear space $\ker A_\lambda \subseteq K$ is generated by the f_j with $\lambda_j = \lambda$, so

$$\mathscr{E}^{\mathrm{e}}(A_\lambda) \ \supseteq\ v(\ker^{\neq} A_\lambda) \ =\ \{ v f_j :\ 1 \leqslant j \leqslant d,\ \lambda_j = \lambda \}.$$

For the rest use $\mathscr{E}^{\mathrm{u}}(A) = \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda)$ and the remarks preceding the lemma. $\square$

Corollaries 2.5.8 and 2.5.23 yield conditions on A, K that guarantee $\dim_{\mathbb{C}} V = r$:

Lemma 5.10.22. *Suppose* A *splits over* K. *If* $r \leqslant 1$, *or* $r = 2$, $A \in H[\partial]$, *or* K *is* 1*-linearly surjective, then*

$$\dim_{\mathbb{C}} V \ =\ \sum_\lambda \operatorname{mult}_\lambda(A) \ =\ r.$$

Next a complement to Lemma 5.5.25:

Corollary 5.10.23. *Suppose* K *is* r*-linearly surjective, or* K *is* 1*-linearly surjective and* A *splits over* K. *Let* $\phi \in H$ *be such that* $\phi' \mathrm{i} + K^\dagger$ *is not an eigenvalue of* A. *Then* A *maps* $K \mathrm{e}^{\phi \mathrm{i}}$ *bijectively onto* $K \mathrm{e}^{\phi \mathrm{i}}$.

Proof. Let $y \in K$, $A(y \mathrm{e}^{\phi \mathrm{i}}) = 0$. By Lemma 5.5.25 it is enough to show that $y = 0$. Suppose towards a contradiction that $y \neq 0$. Then $y \mathrm{e}^{\phi \mathrm{i}} \in \mathrm{U}^\times = K^\times \mathrm{e}(\Lambda)$, so $y \mathrm{e}^{\phi \mathrm{i}} = z \mathrm{e}(\lambda)$, $z \in K^\times$. Then $\phi' \mathrm{i} - \lambda \in K^\dagger$, so λ is not an eigenvalue of A with respect to Λ. Since $y \in V$, this contradicts Lemma 5.10.19. $\square$

Let N be an $n \times n$ matrix over K, $n \geqslant 1$. We end this subsection with a variant of Lemma 5.10.19 for the matrix differential equation $y' = Ny$. Set $S := \mathrm{sol}_{\mathrm{U}}(N)$, so S is a $\mathbb{C}$-linear subspace of U^n of dimension $\leqslant n$.

Lemma 5.10.24. *Suppose* S *has a basis*

$$\mathrm{e}^{\phi_1 \mathrm{i}} f_1, \dots, \mathrm{e}^{\phi_d \mathrm{i}} f_d \quad \text{where } \phi_1, \dots, \phi_d \in H \text{ and } f_1, \dots, f_d \in K^n \subseteq \mathrm{U}^n.$$

Set $\alpha_j := \phi_j' \mathrm{i} + K^\dagger \in K/K^\dagger$ *for* $j = 1, \dots, d$. *Then*

$$\operatorname{mult}_\alpha(N) \ =\ |\{ j \in \{1, \dots, d\} :\ \alpha_j = \alpha \}| \quad \text{for all } \alpha \in K/K^\dagger.$$

Proof. We have $f_j = (f_{1j}, \dots, f_{nj})^{\mathrm{t}} \in \mathrm{U}^n$ for $j = 1, \dots, d$. We first consider the case that N is the companion matrix of a monic $B \in K[\partial]$ of order n. Then we have the $\mathbb{C}$-linear isomorphism $z \mapsto (z, z', \dots, z^{(n-1)})^{\mathrm{t}} \colon \ker_{\mathrm{U}} B \to S$; its inverse maps the given basis to a basis $\mathrm{e}^{\phi_1 \mathrm{i}} f_{11}, \dots, \mathrm{e}^{\phi_d \mathrm{i}} f_{1d}$ of $\ker_{\mathrm{U}} B$. For $j = 1, \dots, d$ we have $\phi_j - \phi(\lambda_j) \preccurlyeq 1$ with $\lambda_j \in \Lambda$, and so this basis has the form $g_1\,\mathrm{e}(\lambda_1), \dots, g_d\,\mathrm{e}(\lambda_d)$ with $g_1, \dots, g_d \in K^\times$. Now use Lemmas 2.4.35 and 5.10.19, and the fact that $\alpha_j = \lambda_j + K^\dagger$ for $j = 1, \dots, d$.

For the general case, [ADH, 5.5.9] gives the companion matrix M of a monic $B \in K[\partial]$ of order n such that $y' = Ny$ is equivalent to $y' = My$. This yields $P \in \mathrm{GL}_n(K)$ such that $f \mapsto Pf \colon S \to \mathrm{sol}_{\mathrm{U}}(M)$ is a $\mathbb{C}$-linear isomorphism, and so $PS = \mathrm{sol}_{\mathrm{U}}(M)$. Since $P\,\mathrm{e}^{\phi_j \mathrm{i}} f_j = \mathrm{e}^{\phi_j \mathrm{i}} g_j$ with $g_j \in K^n$ for $j = 1, \dots, d$, we obtain a basis $\mathrm{e}^{\phi_1 \mathrm{i}} g_1, \dots, \mathrm{e}^{\phi_d \mathrm{i}} g_d$ of the $\mathbb{C}$-linear subspace $\mathrm{sol}_{\mathrm{U}}(M)$ of U^n, so we are in the special case treated earlier. $\square$

A relative version of Corollary 5.10.20 (*)**.** In this subsection $\mathrm{I}(K) \subseteq K^\dagger$. We use an isomorphism as in Lemma 5.10.2 to identify $\mathrm{U} = K[\mathrm{e}(\lambda)]$ with $K[\mathrm{e}^{H\mathrm{i}}]$.

Let F be a Liouville closed Hardy field extension of H; set $L := F[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$. We show here how various results about H, K extend in a coherent way to F, L. First, Corollary 4.4.3 yields a complement Λ_L of the $\mathbb{Q}$-linear subspace $L^\dagger$ of L with $\Lambda \subseteq \Lambda_L \subseteq F\mathrm{i}$. Let $\mathrm{U}_L = L\big[\mathrm{e}(\Lambda_L)\big]$ be the universal exponential extension of L containing $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ as a differential subring described in the remarks following Corollary 2.2.13. We also have the differential subring $L[\mathrm{e}^{F\mathrm{i}}]$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ with $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}] \subseteq L[\mathrm{e}^{F\mathrm{i}}]$.

Lemma 5.10.25. *There is an isomorphism $\iota \colon \mathrm{U}_L \to L[\mathrm{e}^{F\mathrm{i}}]$ of differential L-algebras with $\iota\big(\mathrm{e}(\Lambda_L)\big) \subseteq \mathrm{e}^{F\mathrm{i}}$ that is the identity on* U*. Thus the diagram below commutes:*

$$\begin{tikzcd} \mathrm{U}_L \arrow[r, dashed, "\iota", "\cong"'] & L[\mathrm{e}^{F\mathrm{i}}] \\ \mathrm{U} \arrow[u, "\subseteq"] \arrow[r, equal] & K[\mathrm{e}^{H\mathrm{i}}] \arrow[u, "\subseteq"'] \end{tikzcd}$$

Proof. Lemma 5.10.2 yields an isomorphism $\iota_L \colon \mathrm{U}_L \to L[\mathrm{e}^{F\mathrm{i}}]$ of differential L-algebras with $\iota_L\big(\mathrm{e}(\Lambda_L)\big) \subseteq \mathrm{e}^{F\mathrm{i}}$. By Lemma 2.2.12 we have $\iota_L^{-1}\big(K[\mathrm{e}^{H\mathrm{i}}]\big) = K[E]$ where $E = \{u \in \mathrm{U}_L^\times : u^\dagger \in K\}$. From $\mathrm{U}_L^\times = L^\times\,\mathrm{e}(\Lambda_L)$ we get $E = K^\times\,\mathrm{e}(\Lambda)$, so $K[E] = \mathrm{U}$. Hence ι_L^{-1} restricts to an automorphism of the differential K-algebra U. So this restriction equals σ_χ where $\chi \in \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$. (Lemma 2.2.14.) Extending χ to $\chi_L \in \mathrm{Hom}(\Lambda_L, \mathbb{C}^\times)$ yields an isomorphism

$$\iota := \iota_L \circ \sigma_{\chi_L} \ :\ \mathrm{U}_L \to L[\mathrm{e}^{F\mathrm{i}}]$$

of differential L-algebras with the desired property. $\square$

Fix an isomorphism $\iota \colon \mathrm{U}_L \to L[\mathrm{e}^{F\mathrm{i}}]$ as in the previous lemma and identify U_L with its image via ι; thus $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}] \subseteq L[\mathrm{e}^{F\mathrm{i}}] = \mathrm{U}_L \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$. For each $\mu \in \Lambda_L$ we have an element $\phi(\mu)$ of F (unique up to addition of an element of $2\pi\mathbb{Z}$) such that $\mathrm{e}(\mu) = \mathrm{e}^{\phi(\mu)\mathrm{i}}$; we take $\phi(0) := 0$. The $\phi(\lambda) \in F$ are actually in H and agree with the $\phi(\lambda)$ defined earlier, up to addition of elements of $2\pi\mathbb{Z}$. *In the rest of this subsection we assume* $\mathrm{I}(L) \subseteq L^\dagger$. So for $\mu_1, \mu_2 \in \Lambda_L$: $\mu_1 = \mu_2 \Leftrightarrow \phi(\mu_1) - \phi(\mu_2) \preccurlyeq 1$.

Lemma 5.10.26. *Let $\mu \in \Lambda_L$. Then*

$$\phi(\mu) \in H \iff \phi(\mu) \in H + \mathcal{O}_F \iff \mu \in \Lambda.$$

Proof. We have $\mu = \phi(\mu)'\mathrm{i}$. So if $\phi(\mu) \in H + \mathcal{O}_F$, then $\mu \in H\mathrm{i} + \mathrm{I}(L) \subseteq K + L^\dagger = \Lambda + L^\dagger$, and hence $\mu \in \Lambda$. Conversely, if $\mu \in \Lambda$, then $\phi(\mu)'\mathrm{i} \in \Lambda \subseteq H\mathrm{i}$, so $\phi(\mu)' \in H$, and thus $\phi(\mu) \in H$. $\square$

Lemma 5.10.17 with F, L, Λ_L in place of H, K, Λ, and Lemma 5.10.26 yield:

Corollary 5.10.27. $H + \mathcal{O}_F = \big\{\phi \in F : \phi - \phi(\lambda) \preccurlyeq 1$ *for some* $\lambda\big\}$.

Let $A \in K[\partial]^{\neq}$, $r :=$ order A, $V := \ker_{\mathrm{U}} A$, $V_L := \ker_{\mathrm{U}_L} A$, so $V = V_L \cap \mathrm{U}$. Corollary 5.10.20 applied to F, L in place of H, K then gives a Hahn basis

$$f_1 \,\mathrm{e}^{\phi_1 \mathrm{i}},\ \ldots,\ f_d \,\mathrm{e}^{\phi_d \mathrm{i}} \qquad (f_j \in L^\times, \phi_j \in F)$$

of V_L. Recall from Corollary 4.4.3 that $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma$ for all λ. Applying Lemma 5.10.21 to such a Hahn basis of V_L and F, L, Λ_L, V_L in place of H, K, Λ, V, and using Corollary 5.10.27 we obtain:

$$\mathscr{E}^{\mathrm{u}}(A) \ \supseteq\ \{vf_j : j = 1, \ldots, d,\ \phi_j \in H + \mathcal{O}_F\} \cap \Gamma.$$

Recall from Corollary 2.6.27 that if A is terminal, then $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for all λ, and $\mathscr{E}^{\mathrm{u}}(A) = \mathscr{E}^{\mathrm{u}}_L(A)$. We have $d = \dim_{\mathbb{C}} V_L \leqslant r$, and Lemma 5.10.22 gives conditions on A, F, L which guarantee $d = r$. The next corollary shows that if A is terminal and $d = r$, then the “frequencies” ϕ_j of the elements of our Hahn basis of V_L above can be taken in H:

Corollary 5.10.28. *Suppose A is terminal and $d = r$. Then V_L has a Hahn basis*

$$f_1 \,\mathrm{e}^{\phi_1 \mathrm{i}},\ \ldots,\ f_r \,\mathrm{e}^{\phi_r \mathrm{i}} \qquad (f_j \in L^\times,\ \phi_j \in H).$$

For any such basis and all λ we have

$$\mathscr{E}^{\mathrm{e}}(A_\lambda) \ =\ \mathscr{E}^{\mathrm{e}}_L(A_\lambda) \ =\ v(\ker^{\neq}_L A_\lambda) \ =\ \big\{vf_j : j = 1, \ldots, r,\ \phi_j - \phi(\lambda) \preccurlyeq 1\big\}$$

and $\mathscr{E}^{\mathrm{u}}(A) = \mathscr{E}^{\mathrm{u}}_L(A) = \{vf_1, \ldots, vf_r\} \subseteq \Gamma$, *and the eigenvalues of A viewed as element of $L[\partial]$ are* $\phi_1'\mathrm{i} + L^\dagger, \ldots, \phi_r'\mathrm{i} + L^\dagger$.

Proof. Lemma 2.6.16 and Corollary 2.6.27 give $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda) = v(\ker^{\neq}_L A_\lambda)$ for all λ, and $\ker^{\neq}_L A_\mu = \emptyset$ for $\mu \in \Lambda_L \setminus \Lambda$. Take any Hahn basis of V_L as described before the corollary. Lemma 5.10.17 yields $\lambda_j \in \Lambda_L$ with $\phi_j - \phi(\lambda_j) \preccurlyeq 1$. We have $g_j := f_j \,\mathrm{e}^{(\phi_j - \phi(\lambda_j))\mathrm{i}} \in L^\times$ by Proposition 5.5.18 and $f_j \,\mathrm{e}^{\phi_j \mathrm{i}} = g_j \,\mathrm{e}(\lambda_j)$, so $g_j \in \ker^{\neq}_L A_{\lambda_j}$. This yields $\lambda_j \in \Lambda$ for $j = 1, \ldots, r$. Replacing each pair f_j, ϕ_j by $g_j, \phi(\lambda_j)$ we obtain a Hahn basis of V_L as claimed. The rest follows from Lemmas 5.10.21 and 5.10.19. $\square$

Duality considerations $(^*)$**.** As before, $A \in K[\partial]^{\neq}$ has order r, and $V := \ker_{\mathrm{U}} A$. Recall from Section 2.4 the bilinear form $[\ ,\ ]_A$ on the $\mathbb{C}$-linear space $\Omega = \operatorname{Frac}(\mathrm{U})$. As in the previous subsection we take $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}]$ and fix values $\mathrm{e}(\lambda)$.

Corollary 5.10.29. *Suppose A splits over K,* $\mathrm{I}(K) \subseteq K^\dagger$, *and $r \leqslant 1$ or K is 1-linearly surjective. Let f_j, ϕ_j be as in Corollary 5.10.20. Then the $\mathbb{C}$-linear space* $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A^*$ *equals* $W := \ker_{\mathrm{U}} A^*$ *and has a basis*

$$f_1^* \,\mathrm{e}^{-\phi_1 \mathrm{i}},\ \ldots,\ f_r^* \,\mathrm{e}^{-\phi_r \mathrm{i}} \qquad \text{where } f_k^* \in K^\times\ (k = 1, \ldots, r)$$

such that $\big[f_j \,\mathrm{e}^{\phi_j \mathrm{i}}, f_k^* \,\mathrm{e}^{-\phi_k \mathrm{i}}\big]_A = \delta_{jk}$ *for* $j, k = 1, \ldots, r$.

Proof. By Lemma 5.10.22 we have $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W = r$. As in the proof of Corollary 5.10.20 we obtain $g_j \in K^\times$, $\lambda_j \in \Lambda$ with $\phi_j - \phi(\lambda_j) \preccurlyeq 1$, and

$$y_j \ :=\ f_j\, \mathrm{e}^{\phi_j \mathrm{i}} = g_j\, \mathrm{e}(\lambda_j) \in \mathrm{U}^\times, \quad j = 1,\dots,r.$$

The basis $y_1,\dots,y_r$ of V yields by Corollary 2.5.5 that $A = a(\partial - a_r)\cdots(\partial - a_1)$ with $a \in K^\times$ and $(a_1,\dots,a_r) = \operatorname{split}(y_1,\dots,y_r)$. It is easy to reduce to the case $a = 1$. Then Corollary 2.5.16 provides a basis $y_1^*,\dots,y_r^*$ of W with $[y_j, y_k^*]_A = \delta_{jk}$ for all j, k, $\operatorname{split}(y_r^*,\dots,y_1^*) = (-a_r,\dots,-a_1)$, and $y_k^* = h_k\, \mathrm{e}(-\lambda_k)$, $h_k \in K^\times$, so $y_k^* = f_k^*\, \mathrm{e}^{-\phi_k \mathrm{i}}$, where $f_k^* := h_k\, \mathrm{e}^{(\phi_k - \phi(\lambda_k))\mathrm{i}} \in K^\times$, for $k = 1,\dots,r$. □

Corollary 5.10.30. *Suppose $\dim_{\mathbb{C}} V = r \geqslant 1$ and $\mathrm{I}(K) \subseteq K^\dagger$, and let f_j, ϕ_j be as in Corollary 5.10.20. Let $A = \partial^r + a_{r-1}\partial^{r-1} + \cdots + a_0$ $(a_0,\dots,a_{r-1} \in K)$. Then*

$$\phi_1 + \cdots + \phi_r \equiv b \bmod \mathcal{O}_H \qquad \text{for any } b \in H \text{ with } b' = -\operatorname{Im} a_{r-1},$$

and hence $\phi_1 + \cdots + \phi_r \preccurlyeq 1 \iff a_{r-1} \in K^\dagger$. In particular, if $A^* = (-1)^r A_{\ltimes a}$ $(a \in K^\times)$ or $a_{r-1} \in H$, then $\phi_1 + \cdots + \phi_r \preccurlyeq 1$.

Proof. Take g_j, λ_j as in the proof of Corollary 5.10.20. Then

$$\lambda_1 + \cdots + \lambda_r \ \equiv\ -a_{r-1} \bmod K^\dagger$$

by Corollary 2.5.2 and Lemma 5.10.19. Now $K^\dagger \cap H\mathrm{i} = \mathrm{I}(H)\mathrm{i}$ by Lemma 1.2.16 and the remarks preceding it. Also $\phi(\lambda_j)'\mathrm{i} = \lambda_j$ for all j, and this yields the first claim. For the rest note that if $A^* = (-1)^r A_{\ltimes a}$ $(a \in K^\times)$, then $a_{r-1} \in K^\dagger$ by the remarks after the proof of Proposition 2.4.11. □

Corollary 5.10.31. *Suppose A is self-dual and $\mathrm{I}(K) \subseteq K^\dagger$. Also assume K is 1-linearly surjective and $\dim_{\mathbb{C}} V = r$, or $r \geqslant 1$ and K is $(r-1)$-linearly surjective. Then with the f_j, ϕ_j as in Corollary 5.10.20 we have $\phi_1 + \cdots + \phi_d \preccurlyeq 1$, and there is for each $i \in \{1,\dots,d\}$ a $j \in \{1,\dots,d\}$ with $\phi_i + \phi_j \preccurlyeq 1$.*

Proof. By Corollary 2.4.9 and (2.5.1) we have $\operatorname{mult}_\lambda A = \operatorname{mult}_{-\lambda} A$ for all λ. With $\lambda_1,\dots,\lambda_d$ as in the proof of Corollary 5.10.20 this gives $\lambda_1 + \cdots + \lambda_d = 0$, by Lemma 5.10.19, and thus $\phi_1 + \cdots + \phi_d \preccurlyeq 1$. For $i = 1,\dots,d$ we have $\operatorname{mult}_{\lambda_i} A = \operatorname{mult}_{-\lambda_i} A > 0$ by Lemma 5.10.19, so that same lemma gives $j \in \{1,\dots,d\}$ such that $\lambda_i + \lambda_j = 0$, hence $\phi_i + \phi_j \preccurlyeq 1$. □

Corollary 5.10.32. *Let A, K be as in Corollary 5.10.31. Then V has a basis*

$$f_1\, \mathrm{e}^{\phi_1 \mathrm{i}}, g_1\, \mathrm{e}^{-\phi_1 \mathrm{i}}, \dots, f_m\, \mathrm{e}^{\phi_m \mathrm{i}}, g_m\, \mathrm{e}^{-\phi_m \mathrm{i}},\ h_1,\dots,h_n \qquad (2m+n=d)$$

where $f_1,\dots,f_m, g_1,\dots,g_m, h_1,\dots,h_n \in K^\times$, and $\phi_1,\dots,\phi_m \in H^{>\mathbb{R}}$ are apart.

Proof. By the proof of Corollary 5.10.31, if λ is an eigenvalue of A, then so is $-\lambda$, with the same multiplicity. Hence Lemma 5.10.19 yields a basis

$$f_1\, \mathrm{e}(\lambda_1), g_1\, \mathrm{e}(-\lambda_1), \dots, f_m\, \mathrm{e}(\lambda_m), g_m\, \mathrm{e}(-\lambda_m),\ h_1,\dots,h_n \qquad (2m+n=d)$$

of V where $f_j, g_j, h_k \in K$ for $j = 1,\dots,m$, $k = 1,\dots,n$ and $\lambda_j \in \Lambda$ with $\operatorname{Im}\lambda_j > 0$ for $j = 1,\dots,m$. Note $\mathrm{e}(-\lambda) = \mathrm{e}(\lambda)^{-1} = \mathrm{e}^{-\phi(\lambda)\mathrm{i}}$. Setting $\phi_j := \phi(\lambda_j)$ for $j = 1,\dots,m$ thus yields a basis of V as claimed. □

In Section 2.1 we defined a "positive definite hermitian form" on the K-linear space $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$, which via our isomorphism $\iota\colon \mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$ transfers to a "positive definite hermitian form" $\langle\ ,\ \rangle$ on the K-linear space $K[\mathrm{e}^{H\mathrm{i}}]$. Note that $\langle\ ,\ \rangle$ does not depend on the initial choice of isomorphism ι as in Lemma 5.10.2 at the beginning of this section, by the remarks following that lemma and Corollary 2.2.18. Suppose

$$y_1 = f_1\,\mathrm{e}^{\phi_1 \mathrm{i}},\ \dots,\ y_d = f_d\,\mathrm{e}^{\phi_d \mathrm{i}}$$

is a basis of the $\mathbb{C}$-linear space V as in Corollary 5.10.20 such that for $j,k = 1,\dots,d$ we have $\phi_j = \phi_k$ or $\phi_j - \phi_k \succ 1$. Then by Lemma 2.1.4 and Corollary 5.5.23,

$$\langle y_j, y_k\rangle \ =\ 0 \ \text{ if } \phi_j \neq \phi_k, \qquad \langle y_j, y_k\rangle \ =\ f_j\overline{f_k} \neq 0 \ \text{ if } \phi_j = \phi_k.$$

The case that $A \in H[\partial]$. *In this subsection we assume $A \in H[\partial]^{\neq}$ has order r.* Then $V := \ker_{\mathrm{U}} A$ is closed under the complex conjugation automorphism of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$. We have $\mathrm{U}_{\mathrm{r}} = \mathrm{U} \cap \mathcal{C}^{<\infty}$ and by Corollary 2.2.20 a decomposition of U_{r} as an internal direct sum of H-linear subspaces:

$$\mathrm{U}_{\mathrm{r}} \ =\ H \oplus \bigoplus_{\operatorname{Im}\lambda > 0} \big(H\cos\phi(\lambda) \oplus H\sin\phi(\lambda)\big).$$

Set $V_{\mathrm{r}} := V \cap \mathcal{C}^{<\infty}$, an $\mathbb{R}$-linear subspace of V with $V = V_{\mathrm{r}} \oplus V_{\mathrm{r}}\mathrm{i}$ (internal direct sum of $\mathbb{R}$-linear subspaces of V). Each basis of the $\mathbb{R}$-linear space V_{r} is a basis of the $\mathbb{C}$-linear space V; in particular, $\dim_{\mathbb{C}} V = \dim_{\mathbb{R}} V_{\mathrm{r}}$. If $\dim_{\mathbb{C}} V = r$, then $V_{\mathrm{r}} = \ker_{\mathcal{C}^{<\infty}} A$. If λ is an eigenvalue of A, then so is $-\lambda$, with $\operatorname{mult}_{\lambda}(A) = \operatorname{mult}_{-\lambda}(A)$.

Lemma 5.10.33. *The $\mathbb{C}$-linear space $V = \ker_{\mathrm{U}} A$ has a basis*

$$g_1\,\mathrm{e}^{\phi_1\mathrm{i}},\, g_1\,\mathrm{e}^{-\phi_1\mathrm{i}},\ \dots,\ g_m\,\mathrm{e}^{\phi_m\mathrm{i}},\, g_m\,\mathrm{e}^{-\phi_m\mathrm{i}},\ h_1,\ \dots,\ h_n \qquad (2m+n \leqslant r),$$

where $g_1,\dots,g_m \in H^{>}$, $\phi_1,\dots,\phi_m \in H$ with $\phi_j - \phi(\lambda_j) \preccurlyeq 1$ and $\operatorname{Im}\lambda_j > 0$ for some $\lambda_j \in \Lambda$ for $j = 1,\dots,m$, and $h_1,\dots,h_n \in H^{\times}$. For any such basis of V,

$$g_1\cos\phi_1,\, g_1\sin\phi_1,\ \dots,\ g_m\cos\phi_m,\, g_m\sin\phi_m,\ h_1,\ \dots,\ h_n$$

is a basis of the $\mathbb{R}$-linear space V_{r}, and $h_1,\dots,h_n$ is a basis of the $\mathbb{R}$-linear subspace $\ker_H A = V \cap H$ of H.

Proof. By Corollary 2.5.18 the $\mathbb{C}$-linear space V has a basis

$$f_1\,\mathrm{e}(\lambda_1),\ \overline{f_1}\,\mathrm{e}(-\lambda_1),\ \dots,\ f_m\,\mathrm{e}(\lambda_m),\ \overline{f_m}\,\mathrm{e}(-\lambda_m),\ h_1,\ \dots,\ h_n$$

with $f_1,\dots,f_m \in K^{\times}$, $\lambda_1,\dots,\lambda_m \in \Lambda$ with $\operatorname{Im}\lambda_1,\dots,\operatorname{Im}\lambda_m > 0$ and $h_1,\dots,h_n$ in $H^{\times}$. Moreover, for each such basis,

$$\operatorname{Re}\big(f_1\,\mathrm{e}(\lambda_1)\big),\ \operatorname{Im}\big(f_1\,\mathrm{e}(\lambda_1)\big),\ \dots,\ \operatorname{Re}\big(f_m\,\mathrm{e}(\lambda_m)\big),\ \operatorname{Im}\big(f_m\,\mathrm{e}(\lambda_1)\big),\ h_1,\ \dots,\ h_n$$

is a basis of the $\mathbb{R}$-linear space V_{r}, and $h_1,\dots,h_n$ is a basis of its $\mathbb{R}$-linear subspace $\ker_H A = V \cap H$. Set $g_j := |f_j| = |\overline{f_j}| \in H^{>}$ $(j = 1,\dots,m)$. Lemma 5.5.21 gives $\phi_j \in H$ such that $\phi_j - \phi(\lambda_j) \preccurlyeq 1$ and $f_j = g_j\,\mathrm{e}^{(\phi_j - \phi(\lambda_j))\mathrm{i}}$, and thus $f_j\,\mathrm{e}(\lambda_j) = g_j\,\mathrm{e}^{\phi_j\mathrm{i}}$, for $j = 1,\dots,m$. Then $g_1,\dots,g_m,\phi_1,\dots,\phi_m,h_1,\dots,h_n$ have the desired properties. $\square$

Corollary 5.10.34. *Suppose K is 1-linearly surjective when $r \geqslant 2$, $\mathrm{I}(K) \subseteq K^{\dagger}$, and A splits over K. Then $V = \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$ and the $\mathbb{C}$-linear space V has a basis*

$$g_1\,\mathrm{e}^{\phi_1\mathrm{i}},\, g_1\,\mathrm{e}^{-\phi_1\mathrm{i}},\ \dots,\ g_m\,\mathrm{e}^{\phi_m\mathrm{i}},\, g_m\,\mathrm{e}^{-\phi_m\mathrm{i}},\ h_1,\ \dots,\ h_n \qquad (2m+n = r),$$

where $g_j,\phi_j \in H^{>}$ with $\phi_j \succ 1$ $(j = 1,\dots,m)$ and $h_k \in H^{\times}$ $(k = 1,\dots,n)$. For any such basis of V, the $\mathbb{R}$-linear space $\ker_{\mathcal{C}^{<\infty}} A$ has basis

$$g_1\cos\phi_1,\, g_1\sin\phi_1,\ \dots,\ g_m\cos\phi_m,\, g_m\sin\phi_m,\ h_1,\ \dots,\ h_n,$$

and the $\mathbb{R}$-linear subspace $\ker_H A = H \cap \ker_{\mathcal{C}^{<\infty}} A$ of H has basis $h_1,\dots,h_n$.

Proof. By Corollary 2.5.8 we have $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$, hence $V = \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$ and $V_{\mathrm{r}} = \ker_{\mathcal{C}^{<\infty}} A$. Now use Lemmas 5.10.17 and 5.10.33. □

From Lemma 5.10.33 we obtain likewise, using Lemma 2.5.22 and Corollary 5.5.15:

Corollary 5.10.35. *Suppose $r = 2$ and A splits over K but not over H. Then there are $g, \phi \in H^{>}$ such that*

$$\ker_{\mathcal{C}^{<\infty}} A \;=\; \mathbb{R}g\cos\phi + \mathbb{R}g\sin\phi \;=\; \big\{cg\cos(\phi+d) :\ c, d \in \mathbb{R}\big\}.$$

Moreover, if $\mathrm{I}(K) \subseteq K^{\dagger}$, then we can choose here in addition $\phi \succ 1$.

Remark. Let A, g, ϕ, r be as in Corollary 5.10.35. If $\phi \succ 1$, then all $y \in \ker^{\neq}_{\mathcal{C}^{<\infty}} A$ oscillate. If $\phi \preccurlyeq 1$, then no $y \in \ker_{\mathcal{C}^{<\infty}} A$ oscillates.

The following generalizes Corollary 5.5.28:

Corollary 5.10.36. *Suppose K is 1-linearly surjective and A splits over K. Let ϕ be an element of H with $\phi > \mathbb{R}$ such that $\phi'\mathrm{i} + K^{\dagger}$ is not an eigenvalue of A. Then for every $h \in H$ there are unique $f, g \in H$ such that $A(f\cos\phi + g\sin\phi) = h\cos\phi$.*

Proof. Let $f, g \in H$ and $A(f\cos\phi + g\sin\phi) = 0$. By Lemma 5.5.26 it is enough to show $f = g = 0$. Set $y := \frac{1}{2}(f - g\mathrm{i}) \in K$, so $y\,\mathrm{e}^{\phi\mathrm{i}} + \overline{y}\,\mathrm{e}^{-\phi\mathrm{i}} = f\cos\phi + g\sin\phi$. The hypothesis and Corollary 2.5.8 give $V = \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$, so $f\cos\phi + g\sin\phi \in V$. Suppose towards a contradiction that $y \neq 0$. As in the proof of Corollary 5.10.23 we obtain $y\,\mathrm{e}^{\phi\mathrm{i}} = z\,\mathrm{e}(\lambda)$, $z \in K^{\times}$, where λ is not an eigenvalue with respect to Λ. Also $\lambda \neq 0$ in view of $K^{\dagger} \subseteq H + \mathrm{I}(H)\mathrm{i}$. Hence

$$0 \;=\; A(y\,\mathrm{e}^{\phi\mathrm{i}} + \overline{y}\,\mathrm{e}^{-\phi\mathrm{i}}) \;=\; A\big(z\,\mathrm{e}(\lambda) + \overline{z}\,\mathrm{e}(-\lambda)\big) \;=\; A_{\lambda}(z)\,\mathrm{e}(\lambda) + A_{-\lambda}(\overline{z})\,\mathrm{e}(-\lambda),$$

so $A_{\lambda}(z) = 0$, contradicting that λ is not an eigenvalue of A with respect to Λ. □

Next a version of Lemma 5.10.8 for U_{r}. *In the rest of this subsection* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and $\Lambda = \Lambda_H\mathrm{i}$ where Λ_H is an $\mathbb{R}$-linear complement of $\mathrm{I}(H)$ in H.*

Lemma 5.10.37. *Let $\lambda_1, \dots, \lambda_n \in \Lambda$ be distinct,* $\operatorname{Im}\lambda_j > 0$ *for $j = 1, \dots, n$, and*

$$y \;=\; f_1\cos\phi_1 + g_1\sin\phi_1 + \cdots + f_n\cos\phi_n + g_n\sin\phi_n + h$$

where $f_1, \dots, f_n, g_1, \dots, g_n, h \in H$ and $\phi_j \in \phi(\lambda_j) + \mathcal{O}_H$ for $j = 1, \dots, n$. Then

$$y \prec 1 \quad\Longrightarrow\quad f_1, \dots, f_n, g_1, \dots, f_n, h \prec 1.$$

Proof. Let j range over $\{1, \dots, n\}$. Setting $a_j := \frac{1}{2}(f_j - g_j\mathrm{i}) \in K$ we have

$$y \;=\; a_1\,\mathrm{e}^{\phi_1\mathrm{i}} + \overline{a_1}\,\mathrm{e}^{-\phi_1\mathrm{i}} + \cdots + a_n\,\mathrm{e}^{\phi_n\mathrm{i}} + \overline{a_n}\,\mathrm{e}^{-\phi_n\mathrm{i}} + h,$$

and so with $b_j := a_j\,\mathrm{e}^{\phi_j - \phi(\lambda_j)} \in K$ we have $a_j \asymp b_j$ and

$$y \;=\; b_1\,\mathrm{e}^{\phi(\lambda_1)\mathrm{i}} + \overline{b_1}\,\mathrm{e}^{-\phi(\lambda_1)\mathrm{i}} + \cdots + b_n\,\mathrm{e}^{\phi(\lambda_n)\mathrm{i}} + \overline{b_n}\,\mathrm{e}^{-\phi(\lambda_n)\mathrm{i}} + h,$$

Set $h_j := \phi(\lambda_j)' \in H$. Then $h_j\mathrm{i} = \lambda_j$, so the elements

$$h_1,\, \dots,\, h_n,\, -h_1,\, \dots,\, -h_n,\, 0$$

of H are distinct, and $(\mathbb{R}h_1 + \cdots + \mathbb{R}h_n) \cap \mathrm{I}(H) = \{0\}$ in view of $\Lambda \cap \mathrm{I}(H)\mathrm{i} = \{0\}$. Assuming $y \prec 1$, Corollary 5.10.5 then yields $b_1, \overline{b_1}, \dots, b_n, \overline{b_n}, h \prec 1$, and thus $f_1, \dots, f_n, g_1, \dots, f_n, h \prec 1$. □

Lemma 5.10.38. *Recalling that* $\mathrm{U}_r = \mathrm{U} \cap \mathcal{C}^{<\infty}$ *we have:*

$$\begin{aligned} H &= \big\{y \in \mathrm{U_r} : y - h \text{ is non-oscillating for all } h \in H\big\} \\ &= \big\{y \in \mathrm{U_r} : y \text{ lies in a Hausdorff field extension of } H\big\}. \end{aligned}$$

Proof. Let j range over $\{1,\dots,n\}$. Suppose $y \in \mathrm{U_r}$ and $y-h$ is non-oscillating for all $h \in H$. Take distinct $\lambda_1,\dots,\lambda_n \in \Lambda$ with $\operatorname{Im}\lambda_1,\dots,\operatorname{Im}\lambda_n > 0$, and take $f_1,\dots,f_n,g_1,\dots,g_n,h \in H$ such that

$$y = f_1\cos\phi(\lambda_1) + g_1\sin\phi(\lambda_1) + \cdots + f_n\cos\phi(\lambda_n) + g_n\sin\phi(\lambda_n) + h.$$

We claim that $y = h$. To prove this claim, replace y by $y-h$ to arrange $h = 0$. Towards a contradiction, assume $y \neq 0$. Then $f_j \neq 0$ or $g_j \neq 0$ for some j. Divide y and $f_1,\dots,f_n,g_1,\dots,g_n$ by a suitable element of $H^\times$ to arrange $f_j, g_j \preccurlyeq 1$ for all j and $f_j \asymp 1$ or $g_j \asymp 1$ for some j. Then $y \preccurlyeq 1$ and $y-s$ is non-oscillating for all $s \in \mathbb{R}$, and so Lemma 5.1.19 yields $\ell \in \mathbb{R}$ such that $y - \ell \prec 1$. Then Lemma 5.10.37 gives $f_j, g_j \prec 1$ for all j, a contradiction. This proves the first equality. The second equality follows from Lemma 5.1.20. $\square$

In combination with Corollary 5.10.34 this yields:

Corollary 5.10.39. *Recalling that* $V_r = \ker_{\mathrm{U}} A \cap \mathcal{C}^{<\infty}$ *we have:*

$$\begin{aligned} \ker_H A &= \big\{y \in V_{\mathrm{r}} : y - h \text{ is non-oscillating for all } h \in H\big\} \\ &= \big\{y \in V_{\mathrm{r}} : y \text{ lies in a Hausdorff field extension of } H\big\}. \end{aligned}$$

Hence if K is 1*-linearly surjective in case* $r \geqslant 2$*, and A splits over K, then every y in* $\ker_{\mathcal{C}^{<\infty}} A$ *such that $y-h$ is non-oscillating for all $h \in H$ lies in H.*

Connection to Lyapunov exponents (*). *In this subsection* $\mathrm{I}(K) \subseteq K^\dagger$*, and we take* $\Lambda = \Lambda_H \mathrm{i}$ *where* Λ_H *is an* $\mathbb{R}$*-linear complement of* $\mathrm{I}(H)$ *in* H*. Accordingly,* $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}]$*. Let also* $n \geqslant 1$. In Section 5.2 we introduced the Lyapunov exponent $\lambda(f) \in \mathbb{R}_{\pm\infty}$ of $f \in \mathcal{C}[\mathrm{i}]^n$. For use in Section 7.4 we collect here some properties of these exponents $\lambda(f)$ for $f \in \mathrm{U}^n \subseteq \mathcal{C}[\mathrm{i}]^n$. Recall: $f, g \in \mathcal{C}[\mathrm{i}],\ f \preccurlyeq g \Rightarrow \lambda(f) \geqslant \lambda(g)$.

Lemma 5.10.40. *Let* $f, g \in \mathrm{U}$. *Then*

$$f \preccurlyeq_{\mathrm{g}} g \Rightarrow \lambda(f) \geqslant \lambda(g), \qquad f \asymp_{\mathrm{g}} g \Rightarrow \lambda(f) = \lambda(g).$$

Proof. We first treat the special case $g = \mathfrak{m} \in H^\times$. Then the first statement follows from the remark before the lemma and Lemma 5.10.10. Suppose $f \asymp_{\mathrm{g}} \mathfrak{m}$; thanks to the first statement it suffices to show $\Lambda(f) \subseteq \Lambda(\mathfrak{m})$. Towards a contradiction, suppose $\Lambda(f) \not\subseteq \Lambda(\mathfrak{m})$. Then we have $a \in \mathbb{R}$ with $f \preccurlyeq \mathrm{e}^{-ax}$ and $\mathfrak{m} \not\preccurlyeq \mathrm{e}^{-ax}$, so $\mathrm{e}^{-ax} \prec \mathfrak{m}$ (since $\mathfrak{m}, \mathrm{e}^{-ax} \in H$), hence $f \prec \mathfrak{m}$ and thus $f \prec_{\mathrm{g}} \mathfrak{m}$ by Corollary 5.10.9, contradicting $f \asymp_{\mathrm{g}} \mathfrak{m}$.

The case $g = 0$ being trivial, we now assume $g \neq 0$ for the general case and take $\mathfrak{m} \in H^\times$ with $g \asymp_{\mathrm{g}} \mathfrak{m}$; then $\lambda(g) = \lambda(\mathfrak{m})$ by the special case (with $f = g$), so we may replace g by $\mathfrak{m}$ to reduce the lemma to the special case. $\square$

We turn U^n into a valued $\mathbb{C}$-linear space with valuation $v_{\mathrm{g}} \colon \mathrm{U}^n \to \Gamma_\infty$ given by

$$v_{\mathrm{g}}(f) := \min\big\{v_{\mathrm{g}}(f_1),\dots,v_{\mathrm{g}}(f_n)\big\} \quad \text{for } f = (f_1,\dots,f_n) \in \mathrm{U}^n,$$

and denote by $\preccurlyeq_{\mathrm{g}}$ the associated dominance relation on U^n. In the next four corollaries, f, g range over U^n.

Corollary 5.10.41. $f \preccurlyeq_{\mathrm{g}} g \Rightarrow \lambda(f) \geqslant \lambda(g)$ *and* $f \asymp_{\mathrm{g}} g \Rightarrow \lambda(f) = \lambda(g)$.

Proof. Suppose $f=(f_1,\dots,f_n)\preccurlyeq_{\mathrm{g}} g=(g_1,\dots,g_n)$. Take k with $v_{\mathrm{g}}g=v_{\mathrm{g}}g_k$. Then $f_j\preccurlyeq_{\mathrm{g}} g_k$ and so $\lambda(f_j)\geqslant\lambda(g_k)\geqslant\lambda(g)$, for all j, by Lemma 5.10.40, and thus $\lambda(f)\geqslant\lambda(g)$. $\square$

Corollary 5.10.42. *Let $m\geqslant 1$, $g_1,\dots,g_m\in \mathrm{U}^n$, and $g=g_1+\cdots+g_m$ be such that $v_{\mathrm{g}}(g)=\min\big\{v_{\mathrm{g}}(g_1),\dots,v_{\mathrm{g}}(g_m)\big\}$. Then $\lambda(g)=\min\big\{\lambda(g_1),\dots,\lambda(g_m)\big\}$.*

Proof. We may arrange $g_i\preccurlyeq_{\mathrm{g}} g_1$ for all i, so $v_{\mathrm{g}}g=v_{\mathrm{g}}g_1$. Then $\lambda(g_1)\leqslant\lambda(g_i)$ for all i and $\lambda(g)=\lambda(g_1)$, by Corollary 5.10.41. $\square$

Here is a special case of Corollary 5.10.42:

Corollary 5.10.43. *Suppose $m\geqslant 1$, $f=\mathrm{e}(h_1\mathrm{i})f_1+\cdots+\mathrm{e}(h_m\mathrm{i})f_m$ with $f_1,\dots,f_m$ in K^n and distinct $h_1,\dots,h_m\in\Lambda_H$. Then $\lambda(f)=\min\big\{\lambda(f_1),\dots,\lambda(f_m)\big\}$.*

For the notion of valuation-independence, see [ADH, p. 92].

Corollary 5.10.44. *Suppose $m\geqslant 1$, $f=\mathrm{e}^{\phi_1\mathrm{i}}f_1+\cdots+\mathrm{e}^{\phi_m\mathrm{i}}f_m$, $f_1,\dots,f_m\in K^n$ and $\phi_1,\dots,\phi_m\in H$. Suppose also $\phi_j=\phi_k$ or $\phi_j-\phi_k\succ 1$ for $j,k=1,\dots,m$, and for $k=1,\dots,m$ the f_j with $1\leqslant j\leqslant m$ and $\phi_j=\phi_k$ are valuation-independent. Then $v_{\mathrm{g}}(f)=\min\{v(f_1),\dots,v(f_m)\}$, and thus $\lambda(f)=\min\big\{\lambda(f_1),\dots,\lambda(f_m)\big\}$.*

Proof. First arrange that $l\in\{1,\dots,m\}$ is such that $\phi_1,\dots,\phi_l$ are distinct and each ϕ_j with $l<j\leqslant m$ equals one of $\phi_1,\dots,\phi_l$. For $k=1,\dots,l$, take $\lambda_k\in\Lambda$ with $\phi_k-\phi(\lambda_k)\preccurlyeq 1$ and put $g_k:=\sum_{1\leqslant j\leqslant m,\ \phi_j=\phi_k}f_j$ and $h_k:=\mathrm{e}^{(\phi_k-\phi(\lambda_k))\mathrm{i}}\,g_k\in K^n$. Then $v(g_k)=v(h_k)$, $\mathrm{e}^{\phi_k\mathrm{i}}\,g_k=\mathrm{e}(\lambda_k)h_k$, and

$$f\ =\ \mathrm{e}^{\phi_1\mathrm{i}}\,g_1+\cdots+\mathrm{e}^{\phi_l\mathrm{i}}\,g_l\ =\ \mathrm{e}(\lambda_1)h_1+\cdots+\mathrm{e}(\lambda_l)h_l$$

with distinct $\lambda_1,\dots,\lambda_l$. Hence

$$v_{\mathrm{g}}(f)\ =\ \min\{v(h_1),\dots,v(h_l)\}\ =\ \min\{v(g_1),\dots,v(g_l)\}.$$

Now use $v(g_k)=\min\{v(f_j):\ 1\leqslant j\leqslant m,\ \phi_j=\phi_k\}$ for $k=1,\dots,l$. $\square$

For what we say below about Δ and $\Gamma^\flat$, see [ADH, 9.1.11]. Set

$$\Delta\ :=\ \big\{\gamma\in\Gamma:\psi(\gamma)\geqslant 0\big\}\ =\ \big\{\gamma\in\Gamma:\gamma=O\big(v(\mathrm{e}^x)\big)\big\},$$

the smallest convex subgroup of $\Gamma=v(K^\times)$ containing $v(\mathrm{e}^x)\in\Gamma^<$. Then Δ has the convex subgroup

$$\Gamma^\flat\ =\ \big\{\gamma\in\Gamma:\psi(\gamma)>0\big\}\ =\ \big\{\gamma\in\Gamma:\gamma=o\big(v(\mathrm{e}^x)\big)\big\},$$

and we have an ordered group isomorphism $r\mapsto v(\mathrm{e}^{-rx})+\Gamma^\flat\colon\mathbb{R}\to\Delta/\Gamma^\flat$. Note also that for $f\in K$ we have: $v(f)\in\Gamma^\flat\Leftrightarrow\lambda(f)=0$.

Lemma 5.10.45. *Let $f\in\mathrm{U}$. Then*

$$\lambda(f)\ =\ +\infty\ \Leftrightarrow\ v_{\mathrm{g}}(f)\ >\ \Delta,\qquad \lambda(f)\ =\ -\infty\ \Leftrightarrow\ v_{\mathrm{g}}(f)\ <\ \Delta,$$

and if $\lambda(f)\in\mathbb{R}$, then $v_{\mathrm{g}}(f)\in\Delta$ and $v_{\mathrm{g}}(f)\equiv v(\mathrm{e}^{-\lambda(f)x})\bmod\Gamma^\flat$.

Proof. We assume $f\neq 0$, and use Lemma 5.10.40 to replace f by $\mathfrak{m}\in H$ with $f\asymp_{\mathrm{g}}\mathfrak{m}$ so as to arrange $f\in H^\times$. The displayed claims then follow. Suppose $\lambda(f)\in\mathbb{R}$, and let $a\in\mathbb{R}^>$. Then $f\,\mathrm{e}^{(\lambda(f)-\frac{1}{2}a)x}\preccurlyeq 1$, so $f\,\mathrm{e}^{\lambda(f)x}\prec\mathrm{e}^{\frac{1}{2}ax}\prec\mathrm{e}^{ax}$. Also $f\,\mathrm{e}^{\lambda(f)x}\not\preccurlyeq\mathrm{e}^{-ax}$, thus $\mathrm{e}^{-ax}\prec f\,\mathrm{e}^{\lambda(f)x}\prec\mathrm{e}^{ax}$. This holds for all $a\in\mathbb{R}^>$, so $v(f\,\mathrm{e}^{\lambda(f)x})\in\Gamma^\flat$. $\square$

Lemma 5.10.45 yields $\lambda(fg)=\lambda(f)+\lambda(g)$ for all $f,g\in\mathrm{U}\cap\mathcal{C}[\mathrm{i}]^{\preceq}$.

Corollary 5.10.46. *Assume $f, g \in K$, $g \preccurlyeq f$, and $\lambda(f) \in \mathbb{R}$. Then $g' + \lambda(f)g \prec f$.*

Proof. Lemma 5.10.45 gives $v(f\,\mathrm{e}^{\lambda(f)x}) \in \Gamma^{\flat}$, so we can replace f, g by $f\,\mathrm{e}^{\lambda(f)x}$, $g\,\mathrm{e}^{\lambda(f)x}$, to arrange $f' \prec f$ and $\lambda(f) = 0$. Now if $f \asymp 1$, then $g \preccurlyeq f \asymp 1$ and so $g' \prec 1 \asymp f$, and if $f \not\asymp 1$, then $g' \preccurlyeq f' \prec f$ using [ADH, 9.1.3(iii) and 9.1.4(i)]. $\square$

From Corollary 5.10.46 we easily obtain:

Corollary 5.10.47. *Suppose $f \in K^n$ is such that $\lambda(f) \in \mathbb{R}$. Then $f' + \lambda(f)f \prec f$.*

Note that $K \cap \mathcal{C}[\mathrm{i}]^{\preceq} = \mathcal{O}_\Delta$ is by Lemma 5.10.45 the valuation ring of the coarsening v_Δ of the valuation of K by Δ, with maximal ideal $K \cap \mathcal{C}[\mathrm{i}]^{\prec} = \mathcal{o}_\Delta$, cf. [ADH, 3.4]. By Corollary 5.10.43, the $\mathbb{C}$-subalgebra $\mathrm{U} \cap \mathcal{C}[\mathrm{i}]^{\preceq}$ of U satisfies

$$\mathrm{U}\cap\mathcal{C}[\mathrm{i}]^{\preceq} = \bigoplus_{h\in\Lambda_H} \mathcal{O}_\Delta\,\mathrm{e}(h\mathrm{i}) \quad \text{(internal direct sum of } \mathcal{O}_\Delta\text{-submodules of } \mathrm{U}\cap\mathcal{C}[\mathrm{i}]^{\preceq}).$$

We put

$$\mathrm{U}^{\preceq} := \bigoplus_{h\in\Lambda_H\cap\mathcal{O}_H} \mathcal{O}_\Delta\,\mathrm{e}(h\mathrm{i}),$$

a $\mathbb{C}$-subalgebra of $\mathrm{U}\cap\mathcal{C}[\mathrm{i}]^{\preceq}$. Then

$$\begin{aligned}(\mathrm{U}^{\preceq})^\times &= \big\{g\,\mathrm{e}(h\mathrm{i}) : g \in K^\times,\ h \in \Lambda_H,\ g^\dagger, h \preccurlyeq 1\big\}\\ &= \big\{g\,\mathrm{e}(h\mathrm{i}) : g \in K^\times,\ h \in \Lambda_H,\ \lambda(g) \in \mathbb{R},\ h \preccurlyeq 1\big\}.\end{aligned}$$

In the next lemma $f = g\,\mathrm{e}^{\phi\mathrm{i}} \in \mathrm{U}^\times$ where $g \in K^\times$, $\phi \in H$. Then $|f| = |g| \in H$ and so $-\lambda(f) = -\lambda(g) = \lim\limits_{t\to\infty} \frac{1}{t}\log|g(t)|$.

Lemma 5.10.48. $f \in (\mathrm{U}^{\preceq})^\times \Leftrightarrow g^\dagger, \phi' \preccurlyeq 1 \Leftrightarrow f^\dagger \preccurlyeq 1$. *If $f^\dagger \preccurlyeq 1$, then*

$$-\lambda(f) = \lim_{t\to\infty} \operatorname{Re} f^\dagger(t) = \lim_{t\to\infty} \operatorname{Re} g^\dagger(t), \qquad \lim_{t\to\infty} \operatorname{Im} f^\dagger(t) = \lim_{t\to\infty} \phi'(t)$$

and these limits are in $\mathbb{R}$.

Proof. Take $h \in \Lambda_H$ with $\phi - \phi(h\mathrm{i}) \preccurlyeq 1$ and put $g_1 := g\,\mathrm{e}^{(\phi-\phi(h\mathrm{i}))\mathrm{i}} \in K^\times$, so $f = g_1\,\mathrm{e}^{\phi(h\mathrm{i})\mathrm{i}} = g_1\,\mathrm{e}(h\mathrm{i})$. Now $g^\dagger - |g|^\dagger \prec 1$, since $g \asymp |g|$. Also $\mathrm{e}(h\mathrm{i}) = \mathrm{e}^{\phi(h\mathrm{i})\mathrm{i}}$ gives $h = \phi(h\mathrm{i})'$ by differentiation. Hence $g_1^\dagger - g^\dagger = (\phi' - h)\mathrm{i} = (\phi - \phi(h\mathrm{i}))'\mathrm{i} \prec 1$, so

$$|g|^\dagger \preccurlyeq 1 \Leftrightarrow g^\dagger \preccurlyeq 1 \Leftrightarrow g_1^\dagger \preccurlyeq 1, \qquad \phi' \preccurlyeq 1 \Leftrightarrow h \preccurlyeq 1.$$

This yields the equivalences of the Lemma, using for $f^\dagger \preccurlyeq 1 \Rightarrow g^\dagger, \phi' \preccurlyeq 1$ that $f^\dagger = g^\dagger + \phi'\mathrm{i}$, and $\operatorname{Im}(g^\dagger) \in \mathrm{I}(H)\mathrm{i} \subseteq K^{\prec 1}$, the latter a consequence of Lemma 1.2.16 and the remarks preceding it. Now assume $f^\dagger \preccurlyeq 1$. Then $g^\dagger, \phi' \preccurlyeq 1$, so $vg \in \Delta$, hence by Lemma 5.10.45, $\lambda(f) = \lambda(g) \in \mathbb{R}$ and $v\big(g\,\mathrm{e}^{\lambda(g)x}\big) \in \Gamma^\flat$, that is, $g^\dagger + \lambda(g) \prec 1$, so $\operatorname{Re}(g^\dagger) + \lambda(f) \prec 1$, and thus $-\lambda(f) = \lim\limits_{t\to\infty} \operatorname{Re} g^\dagger(t)$. Now use $\operatorname{Re} f^\dagger = \operatorname{Re} g^\dagger$ and $\operatorname{Im} f^\dagger = \operatorname{Im} g^\dagger + \phi'$ and $\operatorname{Im} g^\dagger \prec 1$. $\square$

Lemma 5.10.49. *Let $f \in \mathrm{U}^{\preceq}$. Then $f' \in \mathrm{U}^{\preceq}$ and $\lambda(f) \leqslant \lambda(f')$. Moreover, if $\lambda(f) \in \mathbb{R}$, then $f' \preccurlyeq_{\mathrm{g}} f$, and if $\lambda(f) \in \mathbb{R}^\times$, then $f' \asymp_{\mathrm{g}} f$.*

Proof. Suppose first that $f = g\,\mathrm{e}(h\mathrm{i})$ where $g \in \mathcal{O}_\Delta^{\neq}$, $h \in \Lambda_H \cap \mathcal{O}_H$, so $\lambda(f) = \lambda(g)$ and $f' = (g' + gh\mathrm{i})\,\mathrm{e}(h\mathrm{i})$. Then by [ADH, 9.2.24, 9.2.26] we have $g' \in \mathcal{O}_\Delta$, with $g' \in \mathcal{o}_\Delta$ if $g \in \mathcal{o}_\Delta$. So $f' \in \mathcal{O}_\Delta\,\mathrm{e}(h\mathrm{i})$, with $f' \in \mathcal{o}_\Delta\,\mathrm{e}(h\mathrm{i})$ if $g \in \mathcal{o}_\Delta$. This yields $f' \in \mathrm{U}^{\preceq}$ as well as $\lambda(f') = +\infty$ if $\lambda(f) = +\infty$, by Lemma 5.10.45. Now suppose $\lambda(f) \in \mathbb{R}$. Then $v(g\,\mathrm{e}^{\lambda(g)x}) \in \Gamma^\flat$ by Lemma 5.10.45, hence $g^\dagger + \lambda(g) \prec 1$, so $g^\dagger \preccurlyeq 1$, and thus $f' = g(g^\dagger + hi)\,\mathrm{e}(hi) \preccurlyeq_{\mathrm{g}} f$, and this yields $\lambda(f') \geqslant \lambda(f)$ by Lemma 5.10.40. If $\lambda(f) \neq 0$, then $g^\dagger \sim -\lambda(g)$ and so $g^\dagger + hi \sim -\lambda(g) + hi \asymp 1$, and thus $f' \asymp_{\mathrm{g}} f$.

The case $f = 0$ is trivial, so we can assume next that $f = f_1 + \cdots + f_m$, $f_j = g_j\,\mathrm{e}(h_j i)$, $g_j \in \mathcal{O}_\Delta^{\neq}$, $h_j \in \Lambda_H \cap \mathcal{O}_H$ for $j = 1, \dots, m$, $m \geqslant 1$, with distinct $h_1, \dots, h_m$. We arrange $f_1 \succcurlyeq_{\mathrm{g}} \cdots \succcurlyeq_{\mathrm{g}} f_m$, so $f \asymp_{\mathrm{g}} f_1$ and $\lambda(f_1) \leqslant \cdots \leqslant \lambda(f_m)$, and $\lambda(f) = \min\{\lambda(f_1), \dots, \lambda(f_m)\} = \lambda(f_1)$ by Corollary 5.10.43. The special case gives $f_j' \in \mathrm{U}^{\not\preccurlyeq}$ and $\lambda(f_j) \leqslant \lambda(f_j')$ for $j = 1, \dots, m$, so $f' \in \mathrm{U}^{\not\preccurlyeq}$ and $\lambda(f) \leqslant \lambda(f')$. Suppose $\lambda(f) \in \mathbb{R}$. Then $v_{\mathrm{g}}(f_1) \in \Delta$ by Lemma 5.10.45. If $\lambda(f_j) = +\infty$, then $\lambda(f_j') = +\infty$ by the special case, so $v_{\mathrm{g}}(f_j') > \Delta$ and thus $f_j' \prec_{\mathrm{g}} f_1 \asymp_{\mathrm{g}} f$. If $\lambda(f_j) \in \mathbb{R}$, then $f_j' \preccurlyeq_{\mathrm{g}} f_j \preccurlyeq_{\mathrm{g}} f$, again by the special case. This yields $f' \preccurlyeq_{\mathrm{g}} f$. Likewise, if $\lambda(f) \in \mathbb{R}^\times$, then $f_1' \asymp_{\mathrm{g}} f_1 \asymp_{\mathrm{g}} f$ and thus $f' \asymp_{\mathrm{g}} f$. $\square$

Corollary 5.10.50. *If $f \in \mathrm{U}^{\not\preccurlyeq}$ and $\lambda(f) \in \mathbb{R}^\times$, then for all n,*

$$f^{(n)} \asymp_{\mathrm{g}} f, \qquad \lambda(f^{(n)}) = \lambda(f).$$

For use in the next lemma and then in Section 7.4 we also define for $f \in \mathcal{C}^1[i]^\times$,

$$\mu(f) := \limsup_{t\to\infty} \operatorname{Im}\big(f'(t)/f(t)\big) \in \mathbb{R}_{\pm\infty}.$$

If $f \in (\mathrm{U}^{\not\preccurlyeq})^\times$, then $\lambda(f), \mu(f) \in \mathbb{R}$ by Lemma 5.10.48, and $f^\dagger = \operatorname{Re}(f^\dagger) + \operatorname{Im}(f^\dagger)i$ then yields $f^\dagger - \big(-\lambda(f) + \mu(f)i\big) \prec 1$.

In the next lemma, suppose $f_1, \dots, f_n \in (\mathrm{U}^{\not\preccurlyeq})^\times$ are such that

$$c_1 := -\lambda(f_1) + \mu(f_1)i, \ \dots, \ c_n := -\lambda(f_n) + \mu(f_n)i \in \mathbb{C}$$

are distinct. Also, let $c \in \mathbb{C}$ and suppose $f := f_1 + \cdots + f_n \in \mathcal{C}[i]^\times$ and $c - f^\dagger \prec 1$.

Lemma 5.10.51. *Let $i \in \{1, \dots, n\}$ be such that $f_i \succcurlyeq f_k$ for all $k \in \{1, \dots, n\}$. Then $c_i = c$ and $\operatorname{Re} c_k \leqslant \operatorname{Re} c$ for all k.*

Proof. We let j, k, l range over $\{1, \dots, n\}$. Take $g_k \in \mathcal{O}_\Delta^{\neq}$ and $h_k \in \Lambda_H \cap \mathcal{O}_H$ such that $f_k = g_k\,\mathrm{e}(h_k i)$. Then $f_k^\dagger = g_k^\dagger + h_k i \in \mathcal{O}$, $c_k - f_k^\dagger \prec 1$, and

$$f' = f_1^\dagger g_1\,\mathrm{e}(h_1 i) + \cdots + f_n^\dagger g_n\,\mathrm{e}(h_n i).$$

Suppose $h_j = h_k$ and $g_j \asymp g_k$; then $f_j^\dagger - f_k^\dagger = (g_j/g_k)^\dagger \in \mathrm{I}(K) \subseteq \smallo$ and so $c_j - c_k \prec 1$, hence $j = k$. We arrange $l \geqslant i$ so that $h_1, \dots, h_l$ are distinct and the h_k with $k > l$ are in $\{h_1, \dots, h_l\}$. For $j \leqslant l$, set $g_j^* := \sum_{h_k = h_j} g_k$ and $g_j^\partial := \sum_{h_k = h_j} f_k^\dagger g_k$, so

$$f = \sum_{j \leqslant l} g_j^*\,\mathrm{e}(h_j i), \qquad f' = \sum_{j \leqslant l} g_j^\partial\,\mathrm{e}(h_j i).$$

For $j \leqslant l$ we have a unique $k = k(j)$ with $g_j^* \sim g_k$. Now $g_i \asymp f_i \succcurlyeq f_k \asymp g_k$ for all k, so $i = k(i)$, hence $0 \neq g_i^* \sim g_i \succcurlyeq g_{k(j)} \asymp g_j^*$ for $j \leqslant l$. In particular, $f \asymp_{\mathrm{g}} g_i$.

Suppose $c \neq 0$. Then $c - f^\dagger \prec 1$ gives $cf \sim f'$. Hence by Lemma 5.10.6 we have $cg_i^* \sim g_i^\partial$ and $\sum_{h_k = h_j} (c - f_k^\dagger) g_k = cg_j^* - g_j^\partial \prec cg_i^*$ for $j \neq i$, $j \leqslant l$. Then $cg_i \sim g_i^\partial = \sum_{h_k = h_i} f_k^\dagger g_k$. But if $k \neq i$ and $h_k = h_i$, then $f_k^\dagger g_k \preccurlyeq g_k \prec g_i$, hence $cg_i \sim f_i^\dagger g_i$, so $c \sim f_i^\dagger$. This proves $c = c_i$. Also, if $k \neq i$ and $h_k = h_i$, then $g_k \prec g_i$, so $\operatorname{Re}(f_k^\dagger) = \operatorname{Re}(g_k^\dagger) < \operatorname{Re}(g_i^\dagger) = \operatorname{Re}(f_i^\dagger)$ by Corollary 1.2.6, hence $\operatorname{Re} c_k \leqslant \operatorname{Re} c_i = \operatorname{Re} c$. If $j \leqslant l$, $j \neq i$ and $h_k = h_j$, then $c \neq c_k$ gives $g_k \asymp (c - f_k^\dagger) g_k \preccurlyeq cg_j^* - g_j^\partial \prec cg_i^* \asymp g_i$, and as before this yields $\operatorname{Re} c_k \leqslant \operatorname{Re} c$. Hence $\operatorname{Re} c_k \leqslant \operatorname{Re} c$ for all k.

Next suppose $c = 0$. Then $f' \prec f$ and so $f' \prec_{\mathrm{g}} f \asymp_{\mathrm{g}} g_i$ by Corollary 5.10.11, hence $g_j^{\partial} \prec g_i$ for $j \leqslant l$, and this yields $f_k^{\dagger} g_k \prec g_i$ for all k. Taking $k = i$ now gives $f_i^{\dagger} \prec 1$ and so $c_i = 0$, and if $k \neq i$, then $c_k \neq 0$ and thus $f_k^{\dagger} \asymp 1$, so $g_k \asymp f_k^{\dagger} g_k \prec g_i$, and as in the case $c \neq 0$ this gives $\operatorname{Re} c_k \leqslant \operatorname{Re} c_i = 0$. $\square$

Part 6. Filling Holes in Hardy Fields

This part contains in Section 6.7 the proof of our main theorem. Important tools for this are the normalization and approximation theorems for holes and slots established in Parts 3 and 4. On the analytic side we need a suitable fixed point theorem proved in Section 6.2: Theorem 6.2.3. The definition of the operator used there is based on the right-inverses for linear differential operators over Hardy fields constructed in Section 6.1. Section 6.3 complements Section 6.2 by showing how to recover suitable smoothness for the fixed points obtained this way.

Let $(P, \mathfrak{m}, \widehat{f})$ be a hole in a Liouville closed Hardy field $H \supseteq \mathbb{R}$ and recall that $\widehat{f}$ lies in an immediate H-field extension of H and satisfies $P(\widehat{f}) = 0$, $\widehat{f} \prec \mathfrak{m}$. (This extension is not assumed to be a Hardy field.) Under suitable hypotheses on H and $(P, \mathfrak{m}, \widehat{f})$, our fixed point theorem (or rather its "real" variant, Corollary 6.2.8) produces a germ f of a one-variable real-valued function such that $P(f) = 0$, $f \prec \mathfrak{m}$; see Section 6.4. The challenge in the proof of our main result is to show that such an f generates a Hardy field extension $H\langle f\rangle$ of H isomorphic to $H\langle \widehat{f}\rangle$ over H (as ordered differential fields). In particular, we need to demonstrate that this zero f of P has the same asymptotic properties (relative to H) as its formal counterpart $\widehat{f}$, and the notion of *asymptotic similarity* established in Section 6.6 provides a suitable general framework for doing so. In order to show that f is indeed asymptotically similar to $\widehat{f}$ over H, we are naturally led to the following task: *given another germ g satisfying $P(g) = 0$, $g \prec \mathfrak{m}$, bound the growth of $h, h', \dots, h^{(r)}$ where $h := (f - g)/\mathfrak{m}$ and $r := \operatorname{order} P$.* Assuming (among other things) that $(P, \mathfrak{m}, \widehat{f})$ is repulsive-normal in the sense of Part 4, this is accomplished in Section 6.5, after revisiting parts of the material from Sections 6.1, 6.2, and 6.4 for certain weighted function spaces. (See Proposition 6.5.14.)

6.1. Inverting Linear Differential Operators over Hardy Fields

Given a Hardy field H and $A \in H[\partial]$ we shall construe A as a $\mathbb{C}$-linear operator on various spaces of functions. We wish to construct right-inverses to such operators. A key assumption here is that A splits over $H[\mathrm{i}]$. This reduces the construction of such inverses mainly to the case of order 1, and this case is handled in the first two subsections using suitable twisted integration operators. In the third subsection we put things together and also show how to "preserve reality" by taking real parts. In the fourth subsection we introduce damping factors. Throughout we pay attention to the continuity of various operators with respect to various norms, for use in Section 6.2.

We let a range over $\mathbb{R}$ and r over $\mathbb{N} \cup \{\infty, \omega\}$. If $r \in \mathbb{N}$, then $r-1$ and $r+1$ have the usual meaning, while for $r \in \{\infty, \omega\}$ we set $r - 1 = r + 1 := r$. (This convention is just to avoid case distinctions.) We have the usual absolute value on $\mathbb{C}$ given by $|a + bi| = \sqrt{a^2 + b^2} \in \mathbb{R}^{\geqslant}$ for $a, b \in \mathbb{R}$, so for $f \in \mathcal{C}_a[\mathrm{i}]$ we have $|f| \in \mathcal{C}_a$.

Integration and some useful norms. For $f \in \mathcal{C}_a[\mathrm{i}]$ we define $\partial_a^{-1} f \in \mathcal{C}_a^1[\mathrm{i}]$ by

$$\partial_a^{-1} f(t) \ := \ \int_a^t f(s)\,ds \ := \ \int_a^t \operatorname{Re} f(s)\,ds + \mathrm{i} \int_a^t \operatorname{Im} f(s)\,ds,$$

so $\partial_a^{-1} f$ is the unique $g \in \mathcal{C}_a^1[\mathrm{i}]$ such that $g' = f$ and $g(a) = 0$. The integration operator $\partial_a^{-1} \colon \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}_a^1[\mathrm{i}]$ is $\mathbb{C}$-linear and maps $\mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a^{r+1}[\mathrm{i}]$. For $f \in \mathcal{C}_a[\mathrm{i}]$

we have

$$\left|\partial_a^{-1} f(t)\right| \leqslant \left(\partial_a^{-1}|f|\right)(t) \qquad \text{for all } t \geqslant a.$$

Let $f \in \mathcal{C}_a[\mathrm{i}]$. Call f **integrable at** ∞ if $\lim_{t\to\infty} \int_a^t f(s)\,ds$ exists in $\mathbb{C}$. In that case we denote this limit by $\int_a^\infty f(s)\,ds$ and put

$$\int_\infty^a f(s)\,ds \ := \ -\int_a^\infty f(s)\,ds,$$

and define $\partial_\infty^{-1} f \in \mathcal{C}_a^1[\mathrm{i}]$ by

$$\partial_\infty^{-1} f(t) \ := \ \int_\infty^t f(s)\,ds \ = \ \int_\infty^a f(s)\,ds + \int_a^t f(s)\,ds \ = \ \int_\infty^a f(s)\,ds + \partial_a^{-1} f(t),$$

so $\partial_\infty^{-1} f$ is the unique $g \in \mathcal{C}_a^1[\mathrm{i}]$ such that $g' = f$ and $\lim_{t\to\infty} g(t) = 0$. Note that

(6.1.1) $$\mathcal{C}_a[\mathrm{i}]^{\mathrm{int}} \ := \ \left\{f \in \mathcal{C}_a[\mathrm{i}] :\ f \text{ is integrable at } \infty\right\}$$

is a $\mathbb{C}$-linear subspace of $\mathcal{C}_a[\mathrm{i}]$ and that ∂_∞^{-1} defines a $\mathbb{C}$-linear operator from this subspace into $\mathcal{C}_a^1[\mathrm{i}]$ which maps $\mathcal{C}_a^r[\mathrm{i}] \cap \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$ into $\mathcal{C}_a^{r+1}[\mathrm{i}]$. If $f \in \mathcal{C}_a[\mathrm{i}]$ and $g \in \mathcal{C}_a^{\mathrm{int}} := \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}} \cap \mathcal{C}_a$ with $|f| \leqslant g$ as germs in $\mathcal{C}$, then $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$; in particular, if $f \in \mathcal{C}_a[\mathrm{i}]$ and $|f| \in \mathcal{C}_a^{\mathrm{int}}$, then $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$. Moreover:

Lemma 6.1.1. *Let $f \in \mathcal{C}_a[\mathrm{i}]$ and $g \in \mathcal{C}_a^{\mathrm{int}}$ be such that $|f(t)| \leqslant g(t)$ for all $t \geqslant a$. Then $|\partial_\infty^{-1} f(t)| \leqslant |\partial_\infty^{-1} g(t)|$ for all $t \geqslant a$.*

Proof. Let $t \geqslant a$. We have $g \geqslant 0$ on $[a, \infty)$, hence $\partial_\infty^{-1} g(t) \leqslant 0$. Also $\left|\int_t^\infty f(s)\,ds\right| \leqslant \int_t^\infty |f(s)|\,ds \leqslant \int_t^\infty g(s)\,ds$. Thus

$$|\partial_\infty^{-1} f(t)| \ = \ \left|\int_t^\infty f(s)\,ds\right| \ \leqslant \ \int_t^\infty g(s)\,ds \ = \ -\partial_\infty^{-1} g(t) \ = \ |\partial_\infty^{-1} g(t)|$$

as claimed. □

For $f \in \mathcal{C}_a[\mathrm{i}]$ we set

$$\|f\|_a \ := \ \sup_{t \geqslant a} |f(t)| \ \in \ [0, \infty],$$

so (with b for "bounded"):

$$\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \ := \ \left\{f \in \mathcal{C}_a[\mathrm{i}] :\ \|f\|_a < \infty\right\}$$

is a $\mathbb{C}$-linear subspace of $\mathcal{C}_a[\mathrm{i}]$, and $f \mapsto \|f\|_a$ is a norm on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ making it a Banach space over $\mathbb{C}$. It is also convenient to define for $t \geqslant a$ the seminorm

$$\|f\|_{[a,t]} \ := \ \max_{a \leqslant s \leqslant t} |f(s)|$$

on $\mathcal{C}_a[\mathrm{i}]$. More generally, let $r \in \mathbb{N}$. Then for $f \in \mathcal{C}_a^r[\mathrm{i}]$ we set

$$\|f\|_{a;r} \ := \ \max\left\{\|f\|_a, \dots, \|f^{(r)}\|_a\right\} \ \in \ [0, \infty],$$

so

$$\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} := \left\{f \in \mathcal{C}_a^r[\mathrm{i}] :\ \|f\|_{a;r} < \infty\right\}$$

is a $\mathbb{C}$-linear subspace of $\mathcal{C}_a^r[\mathrm{i}]$, and $f \mapsto \|f\|_{a;r}$ makes $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ a normed vector space over $\mathbb{C}$. Note that by Corollary 5.7.7,

$$\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \ = \ \left\{f \in \mathcal{C}_a^r[\mathrm{i}] : \|f\|_a < \infty \text{ and } \|f^{(r)}\|_a < \infty\right\},$$

although we do not use this later. Note that for $f, g \in \mathcal{C}_a^r[\mathrm{i}]$ we have

$$\|fg\|_{a;r} \ \leqslant \ 2^r \|f\|_{a;r} \|g\|_{a;r},$$

so $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ is a subalgebra of the $\mathbb{C}$-algebra $\mathcal{C}^r_a[\mathrm{i}]$. If $f \in \mathcal{C}^{r+1}_a[\mathrm{i}]$, then $f' \in \mathcal{C}^r_a[\mathrm{i}]$ with $\|f'\|_{a;r} \leqslant \|f\|_{a;r+1}$.

With $\boldsymbol{i} = (i_0, \dots, i_r)$ ranging over $\mathbb{N}^{1+r}$, let $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ (all $P_{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]$) be a polynomial in $\mathcal{C}_a[\mathrm{i}]\big[Y, Y', \dots, Y^{(r)}\big]$. For $f \in \mathcal{C}^r_a[\mathrm{i}]$ we set

$$P(f) \ :=\ \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}] \qquad \text{where } f^{\boldsymbol{i}} := f^{i_0}(f')^{i_1}\cdots(f^{(r)})^{i_r} \in \mathcal{C}_a[\mathrm{i}].$$

We also let

$$\|P\|_a \ :=\ \max_{\boldsymbol{i}} \|P_{\boldsymbol{i}}\|_a \in [0,\infty].$$

Then $\|P\|_a < \infty$ iff $P \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\big[Y, \dots, Y^{(r)}\big]$, and $\|\cdot\|_a$ is a norm on the $\mathbb{C}$-linear space $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\big[Y, \dots, Y^{(r)}\big]$. In the following assume $\|P\|_a < \infty$. Then for $j = 0, \dots, r$ such that $\partial P/\partial Y^{(j)} \neq 0$ we have

$$\|\partial P/\partial Y^{(j)}\|_a \ \leqslant\ (\deg_{Y^{(j)}} P)\cdot\|P\|_a.$$

Moreover:

Lemma 6.1.2. *If P is homogeneous of degree $d \in \mathbb{N}$ and $f \in \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$, then*

$$\|P(f)\|_a \ \leqslant\ \binom{d+r}{r}\cdot\|P\|_a\cdot\|f\|^d_{a;r}.$$

Corollary 6.1.3. *Let $d \leqslant e$ in $\mathbb{N}$ be such that $P_{\boldsymbol{i}} = 0$ whenever $|\boldsymbol{i}| < d$ or $|\boldsymbol{i}| > e$. Then for $f \in \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ we have*

$$\|P(f)\|_a \ \leqslant\ D\cdot\|P\|_a\cdot\big(\|f\|^d_{a;r} + \cdots + \|f\|^e_{a;r}\big)$$

where $D = D(d,e,r) := \binom{e+r+1}{r+1} - \binom{d+r}{r+1} \in \mathbb{N}^{\geqslant 1}$.

Let $B\colon V \to \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ be a $\mathbb{C}$-linear map from a normed vector space V over $\mathbb{C}$ into $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. Then we set

$$\|B\|_{a;r} \ :=\ \sup\big\{\|B(f)\|_{a;r} :\ f \in V,\ \|f\| \leqslant 1\big\} \ \in\ [0,\infty],$$

the **operator norm of** B. Hence with the convention $\infty\cdot b := b\cdot\infty := \infty$ for $b \in [0,\infty]$ we have

$$\|B(f)\|_{a;r} \ \leqslant\ \|B\|_{a;r}\cdot\|f\| \qquad \text{for } f \in V.$$

Note that B is continuous iff $\|B\|_{a;r} < \infty$. If the map $D\colon \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}^s_a[\mathrm{i}]^{\mathrm{b}}$ ($s \in \mathbb{N}$) is also $\mathbb{C}$-linear, then

$$\|D\circ B\|_{a;s} \ \leqslant\ \|D\|_{a;s}\cdot\|B\|_{a;r}.$$

For $r = 0$ we drop the subscript: $\|B\|_a := \|B\|_{a;0}$.

Lemma 6.1.4. *Let $r \in \mathbb{N}^{\geqslant 1}$ and $\phi \in \mathcal{C}^{r-1}_a[\mathrm{i}]^{\mathrm{b}}$. Then the $\mathbb{C}$-linear operator*

$$\partial - \phi \ :\ \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}^{r-1}_a[\mathrm{i}], \quad f \mapsto f' - \phi f$$

maps $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}^{r-1}_a[\mathrm{i}]^{\mathrm{b}}$, and its restriction $\partial - \phi\colon \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}^{r-1}_a[\mathrm{i}]^{\mathrm{b}}$ is continuous with operator norm $\|\partial - \phi\|_{a;r-1} \leqslant 1 + 2^{r-1}\|\phi\|_{a;r-1}$.

Let $r \in \mathbb{N}$, $a_0 \in \mathbb{R}$, and let a range over $[a_0,\infty)$. The $\mathbb{C}$-linear map

$$f \mapsto f|_{[a,+\infty)} \ :\ \mathcal{C}^r_{a_0}[\mathrm{i}] \to \mathcal{C}^r_a[\mathrm{i}]$$

satisfies $\|f|_{[a,+\infty)}\|_{a;r} \leqslant \|f\|_{a_0;r}$ for $f \in \mathcal{C}^r_{a_0}[\mathrm{i}]$, so it maps $\mathcal{C}^r_{a_0}[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. For $f \in \mathcal{C}^0_{a_0}[\mathrm{i}]$ also denoting its germ at $+\infty$ and its restriction $f|_{[a,+\infty)}$, we have:

$$f \preccurlyeq 1 \iff \|f\|_a < \infty \text{ for some } a \iff \|f\|_a < \infty \text{ for all } a,$$
$$f \prec 1 \iff \|f\|_a \to 0 \text{ as } a \to \infty.$$

Twisted integration. For $f \in \mathcal{C}_a[\mathrm{i}]$ we have the $\mathbb{C}$-linear operator

$$g \mapsto fg \ : \ \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}],$$

which we also denote by f. We now fix an element $\phi \in \mathcal{C}_a[\mathrm{i}]$, and set $\Phi := \partial_a^{-1}\phi$, so $\Phi \in \mathcal{C}^1_a[\mathrm{i}]$, $\Phi(t) = \int_a^t \phi(s)\,ds$ for $t \geqslant a$, and $\Phi' = \phi$. Thus $\mathrm{e}^{\Phi}, \mathrm{e}^{-\Phi} \in \mathcal{C}^1_a[\mathrm{i}]$ with $(\mathrm{e}^{\Phi})^\dagger = \phi$. Consider the $\mathbb{C}$-linear operator

$$B := \mathrm{e}^{\Phi} \circ \partial_a^{-1} \circ \mathrm{e}^{-\Phi} \ : \ \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}^1_a[\mathrm{i}],$$

so

$$Bf(t) \ = \ \mathrm{e}^{\Phi(t)} \int_a^t \mathrm{e}^{-\Phi(s)}\, f(s)\, ds \qquad \text{for } f \in \mathcal{C}_a[\mathrm{i}].$$

It is easy to check that B is a right inverse to $\partial - \phi \colon \mathcal{C}^1_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ in the sense that $(\partial - \phi) \circ B$ is the identity on $\mathcal{C}_a[\mathrm{i}]$. Note that for $f \in \mathcal{C}_a[\mathrm{i}]$ we have $Bf(a) = 0$, and thus $(Bf)'(a) = f(a)$, using $(Bf)' = f + \phi B(f)$. Set $R := \operatorname{Re}\Phi$ and $S := \operatorname{Im}\Phi$, so $R, S \in \mathcal{C}^1_a$, $R' = \operatorname{Re}\phi$, $S' = \operatorname{Im}\phi$, and $R(a) = S(a) = 0$. Note also that if $\phi \in \mathcal{C}^r_a[\mathrm{i}]$, then $\mathrm{e}^{\Phi} \in \mathcal{C}^{r+1}_a[\mathrm{i}]$, so B maps $\mathcal{C}^r_a[\mathrm{i}]$ into $\mathcal{C}^{r+1}_a[\mathrm{i}]$.

Suppose $\varepsilon > 0$ and $\operatorname{Re}\phi(t) \leqslant -\varepsilon$ for all $t \geqslant a$. Then $-R$ has derivative $-R'(t) \geqslant \varepsilon$ for all $t \geqslant a$, so $-R$ is strictly increasing with image $[-R(a), \infty) = [0, \infty)$ and compositional inverse $(-R)^{\mathrm{inv}} \in \mathcal{C}^1_0$. Making the change of variables $-R(s) = u$ for $s \geqslant a$, we obtain for $t \geqslant a$ and $f \in \mathcal{C}_a[\mathrm{i}]$, and with $s := (-R)^{\mathrm{inv}}(u)$,

$$\begin{aligned}
\int_a^t \mathrm{e}^{-\Phi(s)}\, f(s)\, ds \ &= \ \int_0^{-R(t)} \mathrm{e}^{-\Phi(s)}\, f(s) \frac{1}{-R'(s)}\, du, \quad \text{and thus} \\
|Bf(t)| \ &\leqslant \ \mathrm{e}^{R(t)} \cdot \left(\int_0^{-R(t)} \mathrm{e}^{u}\ du \cdot \|f\|_{[a,t]} \right) \cdot \left\| \frac{1}{\operatorname{Re}\phi} \right\|_{[a,t]} \\
&= \ \big[1 - \mathrm{e}^{R(t)}\,\big] \cdot \|f\|_{[a,t]} \cdot \left\| \frac{1}{\operatorname{Re}\phi} \right\|_{[a,t]} \\
&\leqslant \ \|f\|_{[a,t]} \cdot \left\| \frac{1}{\operatorname{Re}\phi} \right\|_{[a,t]} \ \leqslant \ \|f\|_a \cdot \left\| \frac{1}{\operatorname{Re}\phi} \right\|_a.
\end{aligned}$$

Thus B maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}^1_a[\mathrm{i}]$ and $B\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ is continuous with operator norm $\|B\|_a \leqslant \left\|\frac{1}{\operatorname{Re}\phi}\right\|_a$.

Next, suppose $\varepsilon > 0$ and $\operatorname{Re}\phi(t) \geqslant \varepsilon$ for all $t \geqslant a$. Then $R'(t) \geqslant \varepsilon$ for all $t \geqslant a$, so $R(t) \geqslant \varepsilon \cdot (t - a)$ for such t. Hence if $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, then $\mathrm{e}^{-\Phi} f$ is integrable at ∞. Recall from (6.1.1) that $\mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$ is the $\mathbb{C}$-linear subspace of $\mathcal{C}_a[\mathrm{i}]$ consisting of the $g \in \mathcal{C}_a[\mathrm{i}]$ that are integrable at ∞. We have the $\mathbb{C}$-linear maps

$$f \mapsto \mathrm{e}^{-\Phi} f \colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}, \qquad \partial_\infty^{-1} \colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}} \to \mathcal{C}^1_a[\mathrm{i}], \qquad f \mapsto \mathrm{e}^{\Phi} f \colon \mathcal{C}^1_a[\mathrm{i}] \to \mathcal{C}^1_a[\mathrm{i}].$$

Composition yields the $\mathbb{C}$-linear operator $B\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}^1_a[\mathrm{i}]$,

$$Bf(t) \ := \ \mathrm{e}^{\Phi(t)} \int_\infty^t \mathrm{e}^{-\Phi(s)}\, f(s)\, ds \qquad (f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}).$$

It is a right inverse to $\partial-\phi$ in the sense that $(\partial-\phi)\circ B$ is the identity on $\mathcal{C}_a[i]^{\mathrm{b}}$. Note that R is strictly increasing with image $[0,\infty)$ and compositional inverse $R^{\mathrm{inv}}\in\mathcal{C}_0^1$. Making the change of variables $R(s)=u$ for $s\geqslant a$, we obtain for $t\geqslant a$ and $f\in\mathcal{C}_a[i]^{\mathrm{b}}$ with $s:=R^{\mathrm{inv}}(u)$,

$$\begin{aligned}\int_\infty^t \mathrm{e}^{-\Phi(s)}\,f(s)\,ds\ &=\ -\int_{R(t)}^\infty \mathrm{e}^{-\Phi(s)}\,f(s)\frac{1}{R'(s)}\,du,\quad\text{and thus}\\ |Bf(t)|\ &\leqslant\ \mathrm{e}^{R(t)}\cdot\left(\int_{R(t)}^\infty \mathrm{e}^{-u}\ du\right)\cdot\|f\|_t\cdot\left\|\frac{1}{\operatorname{Re}\phi}\right\|_t\\ &\leqslant\ \|f\|_t\cdot\left\|\frac{1}{\operatorname{Re}\phi}\right\|_t\ \leqslant\ \|f\|_a\cdot\left\|\frac{1}{\operatorname{Re}\phi}\right\|_a\end{aligned}$$

Hence B maps $\mathcal{C}_a[i]^{\mathrm{b}}$ into $\mathcal{C}_a[i]^{\mathrm{b}}\cap\mathcal{C}_a^1[i]$, and as a $\mathbb{C}$-linear operator $\mathcal{C}_a[i]^{\mathrm{b}}\to\mathcal{C}_a[i]^{\mathrm{b}}$, B is continuous with operator norm $\|B\|_a\leqslant\left\|\frac{1}{\operatorname{Re}\phi}\right\|_a$. If $\phi\in\mathcal{C}_a^r[i]$, then B maps $\mathcal{C}_a[i]^{\mathrm{b}}\cap\mathcal{C}_a^r[i]$ into $\mathcal{C}_a[i]^{\mathrm{b}}\cap\mathcal{C}_a^{r+1}[i]$.

The case that for some $\varepsilon>0$ we have $\operatorname{Re}\phi(t)\leqslant-\varepsilon$ for all $t\geqslant a$ is called the *attractive case*, and the case that for some $\varepsilon>0$ we have $\operatorname{Re}\phi(t)\geqslant\varepsilon$ for all $t\geqslant a$ is called the *repulsive case*. In both cases the above yields a continuous operator $B\colon\mathcal{C}_a[i]^{\mathrm{b}}\to\mathcal{C}_a[i]^{\mathrm{b}}$ with operator norm $\leqslant\left\|\frac{1}{\operatorname{Re}\phi}\right\|_a$ which is right-inverse to the operator $\partial-\phi\colon\mathcal{C}_a^1[i]\to\mathcal{C}_a[i]$. We denote this operator B by B_ϕ if we need to indicate its dependence on ϕ. Note also its dependence on a. In both the attractive and the repulsive case, B maps $\mathcal{C}_a[i]^{\mathrm{b}}$ into $\mathcal{C}_a[i]^{\mathrm{b}}\cap\mathcal{C}_a^1[i]$, and if $\phi\in\mathcal{C}_a^r[i]$ then B maps $\mathcal{C}_a[i]^{\mathrm{b}}\cap\mathcal{C}_a^r[i]$ into $\mathcal{C}_a[i]^{\mathrm{b}}\cap\mathcal{C}_a^{r+1}[i]$.

Given a Hardy field H and $f\in H[i]$ with $\operatorname{Re}f\succcurlyeq 1$ we can choose a and a representative of f in $\mathcal{C}_a[i]$, to be denoted also by f, such that $\operatorname{Re}f(t)\neq 0$ for all $t\geqslant a$, and then $f\in\mathcal{C}_a[i]$ falls either under the attractive case or under the repulsive case. The original germ $f\in H[i]$ as well as the function $f\in\mathcal{C}_a[i]$ is accordingly said to be attractive, respectively repulsive. (This agrees with the terminology introduced at the beginning of Section 4.5.)

Twists and right-inverses of linear operators over Hardy fields. Let H be a Hardy field, $K:=H[i]$, and let $A\in K[\partial]$ be a monic operator of order $r\geqslant 1$,

$$A\ =\ \partial^r+f_1\partial^{r-1}+\cdots+f_r,\qquad f_1,\dots,f_r\in K.$$

Take a real number a_0 and functions in $\mathcal{C}_{a_0}[i]$ that represent the germs $f_1,\dots,f_r$ and to be denoted also by $f_1,\dots,f_r$. Whenever we increase below the value of a_0, it is understood that we also update the functions $f_1,\dots,f_r$ accordingly, by restriction; the same holds for any function on $[a_0,\infty)$ that gets named. Throughout, a ranges over $[a_0,\infty)$, and $f_1,\dots,f_r$ denote also the restrictions of these functions to $[a,\infty)$, and likewise for any function on $[a_0,\infty)$ that we name. Thus for any a we have the $\mathbb{C}$-linear operator

$$A_a\ :\ \mathcal{C}_a^r[i]\to\mathcal{C}_a[i],\quad y\mapsto y^{(r)}+f_1y^{(r-1)}+\cdots+f_ry.$$

Next, let $\mathfrak{m}\in H^\times$ be given. It gives rise to the twist $A_{\ltimes\mathfrak{m}}\in K[\partial]$,

$$A_{\ltimes\mathfrak{m}}\ :=\ \mathfrak{m}^{-1}A\mathfrak{m}\ =\ \partial^r+g_1\partial^{r-1}+\cdots+g_r,\qquad g_1,\dots,g_r\in K.$$

Now [ADH, (5.1.1), (5.1.2), (5.1.3)] gives universal expressions for $g_1,\dots,g_r$ in terms of $f_1,\dots,f_r,\mathfrak{m},\mathfrak{m}^{-1}$; for example, $g_1=f_1+r\mathfrak{m}^\dagger$. Suppose the germ $\mathfrak{m}$

is represented by a function in $\mathcal{C}^r_{a_0}[\mathrm{i}]^\times$, also denoted by $\mathfrak{m}$. Let $\mathfrak{m}^{-1}$ likewise do double duty as the multiplicative inverse of $\mathfrak{m}$ in $\mathcal{C}^r_{a_0}[\mathrm{i}]$. The expressions above can be used to show that the germs $g_1,\dots,g_r$ are represented by functions in $\mathcal{C}_{a_0}[\mathrm{i}]$, to be denoted also by $g_1,\dots,g_r$, such that for all a and all $y\in\mathcal{C}^r_a[\mathrm{i}]$ we have

$$\mathfrak{m}^{-1}A_a(\mathfrak{m}y) \ = \ (A_{\ltimes\mathfrak{m}})_a(y), \text{ where } (A_{\ltimes\mathfrak{m}})_a(y) \ := \ y^{(r)}+g_1y^{(r-1)}+\cdots+g_ry.$$

The operator $A_a\colon \mathcal{C}^r_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}]$ is surjective (Proposition 5.2.1); we aim to construct a right-inverse of A_a on the subspace $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ of $\mathcal{C}_a[\mathrm{i}]$. For this, we assume given a splitting of A over K,

$$A \ = \ (\partial-\phi_1)\cdots(\partial-\phi_r), \qquad \phi_1,\dots,\phi_r\in K.$$

Take functions in $\mathcal{C}_{a_0}[\mathrm{i}]$, to be denoted also by $\phi_1,\dots,\phi_r$, that represent the germs $\phi_1,\dots,\phi_r$. We increase a_0 to arrange $\phi_1,\dots,\phi_r \in \mathcal{C}^{r-1}_{a_0}[\mathrm{i}]$. Note that for $j=1,\dots,r$ the $\mathbb{C}$-linear map $\partial-\phi_j\colon \mathcal{C}^1_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}]$ restricts to a $\mathbb{C}$-linear map $A_j\colon \mathcal{C}^j_a[\mathrm{i}]\to\mathcal{C}^{j-1}_a[\mathrm{i}]$, so that we obtain a map $A_1\circ\cdots\circ A_r\colon \mathcal{C}^r_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}]$. It is routine to verify that for all sufficiently large a we have

$$A_a \ = \ A_1\circ\cdots\circ A_r \ : \ \mathcal{C}^r_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}].$$

We increase a_0 so that $A_a=A_1\circ\cdots\circ A_r$ for all a. Note that $A_1,\dots,A_r$ depend on a, but we prefer not to indicate this dependence notationally.

Now $\mathfrak{m}\in H^\times$ gives over K the splitting

$$A_{\ltimes\mathfrak{m}} \ = \ (\partial-\phi_1+\mathfrak{m}^\dagger)\cdots(\partial-\phi_r+\mathfrak{m}^\dagger).$$

Suppose as before that the germ $\mathfrak{m}$ is represented by a function $\mathfrak{m}\in\mathcal{C}^r_{a_0}[\mathrm{i}]^\times$. With the usual notational conventions we have $\phi_j-\mathfrak{m}^\dagger\in\mathcal{C}^{r-1}_{a_0}[\mathrm{i}]$, giving the $\mathbb{C}$-linear map $\widetilde{A}_j := \partial-(\phi_j-\mathfrak{m}^\dagger)\colon \mathcal{C}^j_a[\mathrm{i}]\to\mathcal{C}^{j-1}_a[\mathrm{i}]$ for $j=1,\dots,r$, which for all sufficiently large a gives, just as for A_a, a factorization

$$(A_{\ltimes\mathfrak{m}})_a \ = \ \widetilde{A}_1\circ\cdots\circ\widetilde{A}_r.$$

To construct a right-inverse of A_a we now assume $\operatorname{Re}\phi_1,\dots,\operatorname{Re}\phi_r\succcurlyeq 1$. Then we increase a_0 once more so that for all $t\geqslant a_0$,

$$\operatorname{Re}\phi_1(t),\dots,\operatorname{Re}\phi_r(t)\neq 0.$$

Recall that for $j=1,\dots,r$ we have the continuous $\mathbb{C}$-linear operator

$$B_j \ := \ B_{\phi_j} \ : \ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\to\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

from the previous subsection. The subsection on twisted integration now yields:

Lemma 6.1.5. *The continuous $\mathbb{C}$-linear operator*

$$A_a^{-1} \ := \ B_r\circ\cdots\circ B_1 \ : \ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\to\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

is a right-inverse of A_a: it maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\cap\mathcal{C}^r_a[\mathrm{i}]$, and $A_a\circ A_a^{-1}$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. For its operator norm we have $\|A_a^{-1}\|_a \ \leqslant \ \prod_{j=1}^r \left\|\frac{1}{\operatorname{Re}\phi_j}\right\|_a$.

Suppose A is real in the sense that $A\in H[\partial]$. Then by increasing a_0 we arrange that $f_1,\dots,f_r\in\mathcal{C}_{a_0}$. Next, set

$$\mathcal{C}^{\mathrm{b}}_a \ := \ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\cap\mathcal{C}_a \ = \ \big\{f\in\mathcal{C}_a:\ \|f\|_a<\infty\big\},$$

an $\mathbb{R}$-linear subspace of $\mathcal{C}_a$. Then the real part

$$\operatorname{Re}A_a^{-1} \ : \ \mathcal{C}^{\mathrm{b}}_a\to\mathcal{C}^{\mathrm{b}}_a, \qquad (\operatorname{Re}A_a^{-1})(f) \ := \ \operatorname{Re}\big(A_a^{-1}(f)\big)$$

is $\mathbb{R}$-linear and maps $\mathcal{C}_a^{\mathrm{b}}$ into $\mathcal{C}_a^r$. Moreover, it is right-inverse to A_a on $\mathcal{C}_a^{\mathrm{b}}$ in the sense that $A_a \circ \operatorname{Re} A_a^{-1}$ is the identity on $\mathcal{C}_a^{\mathrm{b}}$, and for $f \in \mathcal{C}_a^{\mathrm{b}}$,

$$\|(\operatorname{Re} A_a^{-1})(f)\|_a \;\leqslant\; \|A_a^{-1}(f)\|_a.$$

Damping factors. Here H, K, A, $f_1, \dots, f_r$, $\phi_1, \dots, \phi_r$, a_0 are as in Lemma 6.1.5. In particular, $r \in \mathbb{N}^{\geqslant 1}$, $\operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \succcurlyeq 1$, and a ranges over $[a_0, \infty)$. For later use we choose damping factors u to make the operator uA_a^{-1} more manageable than A_a^{-1}. For $j = 0, \dots, r$ we set

$$A_j^\circ \;:=\; A_1 \circ \dots \circ A_j \;:\; \mathcal{C}_a^j[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}], \tag{6.1.2}$$

with A_0° the identity on $\mathcal{C}_a[\mathrm{i}]$ and $A_r^\circ = A_a$, and

$$B_j^\circ \;:=\; B_j \circ \dots \circ B_1 \;:\; \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}, \tag{6.1.3}$$

where B_0° is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and $B_r^\circ = A_a^{-1}$. Then B_j° maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^j[\mathrm{i}]$ and $A_j^\circ \circ B_j^\circ$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ by Lemma 6.1.5.

Lemma 6.1.6. *Let $u \in \mathcal{C}_a^r[\mathrm{i}]^\times$. Then for $i = 0, \dots, r$ and $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$,*

$$\big[u \cdot A_a^{-1}(f)\big]^{(i)} \;=\; \sum_{j=r-i}^{r} u_{i,j} \cdot u \cdot B_j^\circ(f) \quad \text{in } \mathcal{C}_a^{r-i}[\mathrm{i}] \tag{6.1.4}$$

with coefficient functions $u_{i,j} \in \mathcal{C}_a^{r-i}[\mathrm{i}]$ given by $u_{i,r-i} = 1$, and for $0 \leqslant i < r$,

$$u_{i+1,j} \;=\; \begin{cases} u_{i,r}' + u_{i,r}(u^\dagger + \phi_r) & \text{if } j = r, \\ u_{i,j}' + u_{i,j}(u^\dagger + \phi_j) + u_{i,j+1} & \text{if } r - i \leqslant j < r. \end{cases}$$

Proof. Recall that for $j = 1, \dots, r$ and $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have $B_j(f)' = f + \phi_j B_j(f)$. It is obvious that (6.1.4) holds for $i = 0$. Assuming (6.1.4) for a certain $i < r$ we get

$$\big[uA_a^{-1}(f)\big]^{(i+1)} \;=\; \sum_{j=r-i}^{r} u_{i,j}' \cdot uB_j^\circ(f) + \sum_{j=r-i}^{r} u_{i,j} \cdot \big[uB_j^\circ(f)\big]',$$

and for $j = r - i, \dots, r$,

$$\big[uB_j^\circ(f)\big]' \;=\; u'B_j^\circ(f) + u \cdot \big[B_j^\circ(f)\big]' \;=\; u^\dagger \cdot uB_j^\circ(f) + uB_{j-1}^\circ(f) + \phi_j uB_j^\circ(f),$$

which gives the desired result. □

Let $\mathfrak{v} \in \mathcal{C}_{a_0}^r$ be such that $\mathfrak{v}(t) > 0$ for all $t \geqslant a_0$, $\mathfrak{v} \in H$, $\mathfrak{v} \prec 1$. Then we have the convex subgroup

$$\Delta \;:=\; \big\{\gamma \in v(H^\times) :\; \gamma = o(v\mathfrak{v})\big\}$$

of $v(H^\times)$. *We assume that $\phi_1, \dots, \phi_r \preccurlyeq_\Delta \mathfrak{v}^{-1}$ in the asymptotic field K, where ϕ_j and $\mathfrak{v}$ also denote their germs.* For real $\nu > 0$ we have $\mathfrak{v}^\nu \in (\mathcal{C}_{a_0}^r)^\times$, so

$$u \;:=\; \mathfrak{v}^\nu|_{[a,\infty)} \in (\mathcal{C}_a^r)^\times, \qquad \|u\|_a < \infty.$$

In the next proposition u has this meaning, a meaning which accordingly varies with a. Recall that A_a^{-1} maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r[\mathrm{i}]$ with $\|A_a^{-1}\|_a < \infty$.

Proposition 6.1.7. *Assume H is real closed and $\nu \in \mathbb{Q}$, $\nu > r$. Then:*

(i) *the $\mathbb{C}$-linear operator $uA_a^{-1} \colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$;*
(ii) *$uA_a^{-1} \colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ is continuous;*
(iii) *there is a real constant $c \geqslant 0$ such that $\|uA_a^{-1}\|_{a;r} \leqslant c$ for all a;*
(iv) *for all $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have $uA_a^{-1}(f) \preccurlyeq \mathfrak{v}^\nu \prec 1$;*
(v) *$\|uA_a^{-1}\|_{a;r} \to 0$ as $a \to \infty$.*

Proof. Note that $\mathfrak{v}^\dagger \preccurlyeq_\Delta 1$ by [ADH, 9.2.10(iv)]. Denoting the germ of u also by u we have $u \in H$ and $u^\dagger = \nu \mathfrak{v}^\dagger \preccurlyeq_\Delta 1$, in particular, $u^\dagger \preccurlyeq \mathfrak{v}^{-1/2}$. Note that the $u_{i,j}$ from Lemma 6.1.6—that is, their germs—lie in K. Induction on i gives $u_{i,j} \preccurlyeq_\Delta \mathfrak{v}^{-i}$ for $r-i \leqslant j \leqslant r$. Hence $uu_{i,j} \prec_\Delta \mathfrak{v}^{\nu-i} \prec_\Delta 1$ for $r-i \leqslant j \leqslant r$. Thus for $i = 0, \dots, r$ we have a real constant

$$c_{i,a} \ := \ \sum_{j=r-i}^{r} \|u\, u_{i,j}\|_a \cdot \|B_j\|_a \cdots \|B_1\|_a \in [0,\infty)$$

with $\big\|\big[uA_a^{-1}(f)\big]^{(i)}\big\|_a \leqslant c_{i,a}\|f\|_a$ for all $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Therefore uA_a^{-1} maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$, and the operator $uA_a^{-1}\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ is continuous with

$$\|uA_a^{-1}\|_{a;r} \ \leqslant \ c_a := \max\{c_{0,a}, \dots, c_{r,a}\}.$$

As to (iii), this is because for all i, j, $\|u\, u_{ij}\|_a$ is decreasing as a function of a, and $\|B_j\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\phi_j}\big\|_a$ for all j. For $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have $A_a^{-1}(f) \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, so (iv) holds. As to (v), $u\, u_{i,j} \prec 1$ gives $\|uu_{ij}\|_a \to 0$ as $a \to \infty$, for all i, j. In view of $\|B_j\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\phi_j}\big\|_a$ for all j, this gives $c_{i,a} \to 0$ as $a \to \infty$ for $i = 0, \dots, r$, so $c_a \to 0$ as $a \to \infty$. $\square$

6.2. Solving Split-Normal Equations over Hardy Fields

We construct here solutions of suitable algebraic differential equations over Hardy fields. These solutions lie in rings $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ ($r \in \mathbb{N}^{\geqslant 1}$) and are obtained as fixed points of certain contractive maps, as is common in solving differential equations. Here we use that $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ is a Banach space with respect to the norm $\|\cdot\|_{a;r}$. It will take some effort to define the right contractions using the operators from Section 6.1.

In this section H, K, A, $f_1, \dots, f_r$, $\phi_1, \dots, \phi_r$, a_0 are as in Lemma 6.1.5. In particular, H is a Hardy field, $K = H[\mathrm{i}]$, and

$$A = (\partial - \phi_1)\cdots(\partial - \phi_r) \qquad \text{where } r \in \mathbb{N}^{\geqslant 1},\ \phi_1, \dots, \phi_r \in K,\ \operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \succcurlyeq 1.$$

Here a_0 is chosen so that we have representatives for $\phi_1, \dots, \phi_r$ in $\mathcal{C}_{a_0}^{r-1}[\mathrm{i}]$, denoted also by $\phi_1, \dots, \phi_r$. We let a range over $[a_0, \infty)$. In addition we assume that H is real closed, and that we are given a germ $\mathfrak{v} \in H^>$ such that $\mathfrak{v} \prec 1$ and $\phi_1, \dots, \phi_r \preccurlyeq_\Delta \mathfrak{v}^{-1}$ for the convex subgroup

$$\Delta \ := \ \big\{\gamma \in v(H^\times) :\ \gamma = o(v\mathfrak{v})\big\}$$

of $v(H^\times)$. We increase a_0 so that $\mathfrak{v}$ is represented by a function in $\mathcal{C}_{a_0}^r$, also denoted by $\mathfrak{v}$, with $\mathfrak{v}(t) > 0$ for all $t \geqslant a_0$.

Constructing fixed points over H. Consider a differential equation

$$(*) \qquad\qquad A(y) \ = \ R(y), \qquad y \prec 1,$$

where $R \in K\{Y\}$ has order $\leqslant r$, degree $\leqslant d \in \mathbb{N}^{\geqslant 1}$ and weight $\leqslant w \in \mathbb{N}^{\geqslant r}$, with $R \prec_\Delta \mathfrak{v}^w$. Now $R = \sum_{\boldsymbol{j}} R_{\boldsymbol{j}} Y^{\boldsymbol{j}}$ with $\boldsymbol{j}$ ranging here and below over the tuples $(j_0, \dots, j_r) \in \mathbb{N}^{1+r}$ with $|\boldsymbol{j}| \leqslant d$ and $\|\boldsymbol{j}\| \leqslant w$; likewise for $\boldsymbol{i}$. For each $\boldsymbol{j}$ we take a function in $\mathcal{C}_{a_0}[\mathrm{i}]$ that represents the germ $R_{\boldsymbol{j}} \in K$ and let $R_{\boldsymbol{j}}$ denote this function as well as its restriction to any $[a, \infty)$. Thus R is represented on $[a, \infty)$ by a polynomial $\sum_{\boldsymbol{j}} R_{\boldsymbol{j}} Y^{\boldsymbol{j}} \in \mathcal{C}_a[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$, to be denoted also by R for simplicity. This yields for each a an evaluation map

$$f \mapsto R(f) := \sum_{\boldsymbol{j}} R_{\boldsymbol{j}} f^{\boldsymbol{j}} \ : \ \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}].$$

As in [ADH, 4.2] we also have for every $\boldsymbol{i}$ the formal partial derivative

$$R^{(\boldsymbol{i})} \ := \ \frac{\partial^{|\boldsymbol{i}|} R}{\partial^{i_0} Y \cdots \partial^{i_r} Y^{(r)}} \ \in \ \mathcal{C}_a[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$$

with $R^{(\boldsymbol{i})} = \sum_{\boldsymbol{j}} R^{(\boldsymbol{i})}_{\boldsymbol{j}} Y^{\boldsymbol{j}}$, all $R^{(\boldsymbol{i})}_{\boldsymbol{j}} \in \mathcal{C}_a[\mathrm{i}]$ having their germs in K.

A *solution of* $(*)$ *on* $[a,\infty)$ is a function $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ such that $A_a(f) = R(f)$ and $f \prec 1$. One might try to obtain a solution as a fixed point of the operator $f \mapsto A_a^{-1}\big(R(f)\big)$, but this operator might fail to be contractive on a useful space of functions. Therefore we twist A and arrange things so that we can use Proposition 6.1.7. In the rest of this section we fix $\nu \in \mathbb{Q}$ with $\nu > w$ (so $\nu > r$) such that $R \prec_{\Delta} \mathfrak{v}^{\nu}$ and $\nu\mathfrak{v}^{\dagger} \not\sim \operatorname{Re}\phi_j$ in H for $j = 1,\dots,r$. (Note that such ν exists.) Then the twist $\widetilde{A} := A_{\ltimes \mathfrak{v}^{\nu}} = \mathfrak{v}^{-\nu} A \mathfrak{v}^{\nu} \in K[\partial]$ splits over K as follows:

$$\widetilde{A} \ = \ (\partial - \phi_1 + \nu\mathfrak{v}^{\dagger}) \cdots (\partial - \phi_r + \nu\mathfrak{v}^{\dagger}), \qquad \text{with}$$
$$\phi_j - \nu\mathfrak{v}^{\dagger} \ \preccurlyeq_{\Delta} \ \mathfrak{v}^{-1}, \quad \operatorname{Re}\phi_j - \nu\mathfrak{v}^{\dagger} \ \succcurlyeq \ 1 \qquad (j = 1,\dots,r).$$

We also increase a_0 so that $\operatorname{Re}\phi_j(t) - \nu\mathfrak{v}^{\dagger}(t) \neq 0$ for all $t \geqslant a_0$ and such that for all a and $u := \mathfrak{v}^{\nu}|_{[a,\infty)} \in (\mathcal{C}_a^r)^{\times}$ the operator $\widetilde{A}_a \colon \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ satisfies

$$\widetilde{A}_a(y) \ = \ u^{-1} A_a(uy) \qquad (y \in \mathcal{C}_a^r[\mathrm{i}]).$$

(See the explanations before Lemma 6.1.5 for definitions of A_a and $\widetilde{A}_a$.) We now increase a_0 once more, fixing it for the rest of the section except in the subsection "Preserving reality", so as to obtain as in Lemma 6.1.5, with $\widetilde{A}$ in the role of A, a right-inverse $\widetilde{A}_a^{-1} \colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ for such $\widetilde{A}_a$.

Lemma 6.2.1. *We have a continuous operator* (*not necessarily* $\mathbb{C}$*-linear*)

$$\Xi_a \ : \ \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}, \quad f \mapsto u\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big).$$

It has the property that $\Xi_a(f) \preccurlyeq \mathfrak{v}^{\nu} \prec 1$ *and* $A_a\big(\Xi_a(f)\big) = R(f)$ *for all* $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$.

Proof. We have $\|u^{-1}R_{\boldsymbol{i}}\|_a < \infty$ for all $\boldsymbol{i}$, so $u^{-1}R(f) = \sum_{\boldsymbol{i}} u^{-1}R_{\boldsymbol{i}} f^{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ for all $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$, and thus $u\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ for such f, by Proposition 6.1.7(i). Continuity of Ξ_a follows from Proposition 6.1.7(ii) and continuity of $f \mapsto u^{-1}R(f) \colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. For $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ we have $\Xi_a(f) \preccurlyeq \mathfrak{v}^{\nu} \prec 1$ by Proposition 6.1.7(iv), and

$$u^{-1}A_a\big(\Xi_a(f)\big) \ = \ u^{-1}A_a\big[u\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big)\big] \ = \ \widetilde{A}_a\big[\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big)\big] \ = \ u^{-1}R(f),$$

so $A_a\big(\Xi_a(f)\big) = R(f)$. ∎

By Lemma 6.2.1, each $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ with $\Xi_a(f) = f$ is a solution of $(*)$ on $[a,\infty)$.

Lemma 6.2.2. *There is a constant* $C_a \in \mathbb{R}^{\geqslant}$ *such that for all* $f, g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$,

$$\|\Xi_a(f+g) - \Xi_a(f)\|_{a;r} \ \leqslant \ C_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(\|g\|_{a;r} + \cdots + \|g\|_{a;r}^d\big).$$

We can take these C_a *such that* $C_a \to 0$ *as* $a \to \infty$*, and we do so below.*

Proof. Let $f, g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$. We have the Taylor expansion

$$R(f+g) \;=\; \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!} R^{(\boldsymbol{i})}(f) g^{\boldsymbol{i}} \;=\; \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!}\Bigg[\sum_{\boldsymbol{j}} R_{\boldsymbol{j}}^{(\boldsymbol{i})} f^{\boldsymbol{j}}\Bigg] g^{\boldsymbol{i}}.$$

Now for all $\boldsymbol{i}$, $\boldsymbol{j}$ we have $R_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec_{\Delta} \mathfrak{v}^{\nu}$ in K, so $u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec 1$. Hence

$$D_a \;:=\; \sum_{\boldsymbol{i},\boldsymbol{j}} \big\|u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})}\big\|_a \;\in\; [0,\infty)$$

has the property that $D_a \to 0$ as $a \to \infty$, and

$$\big\|u^{-1}\big(R(f+g)-R(f)\big)\big\|_a \;\leqslant\; D_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(\|g\|_{a;r} + \cdots + \|g\|_{a;r}^d\big).$$

So $h := u^{-1}\big(R(f+g)-R(f)\big) \in \mathcal{C}_a^0[\mathrm{i}]^{\mathrm{b}}$ gives $\Xi_a(f+g) - \Xi_a(f) = u\widetilde{A}_a^{-1}(h)$, and

$$\|\Xi_a(f+g) - \Xi_a(f)\|_{a;r} = \|u\widetilde{A}_a^{-1}(h)\|_{a;r} \leqslant \|u\widetilde{A}_a^{-1}\|_{a;r} \cdot \|h\|_a.$$

Thus the lemma holds for $C_a := \|u\widetilde{A}_a^{-1}\|_{a;r} \cdot D_a$. □

In the proof of the next theorem we use the well-known fact that the normed vector space $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ over $\mathbb{C}$ is actually a Banach space. Hence if $S \subseteq \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ is nonempty and closed and $\Phi\colon S \to S$ is contractive (that is, there is a real number $\lambda \in [0,1)$ such that $\|\Phi(f) - \Phi(g)\|_{a;r} \leqslant \lambda\|f-g\|_{a;r}$ for all $f, g \in S$), then Φ has a unique fixed point f_0, and $\Phi^n(f) \to f_0$ as $n \to \infty$, for every $f \in S$. (See for example [203, Chapter II, §5, IX].)

Theorem 6.2.3. *For all sufficiently large a the operator Ξ_a maps the closed ball*

$$\big\{f \in \mathcal{C}_a^r[\mathrm{i}] : \; \|f\|_{a;r} \leqslant 1/2\big\}$$

of the Banach space $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ into itself and has a unique fixed point on this ball.

Proof. We have $\Xi_a(0) = u\widetilde{A}_a^{-1}(u^{-1}R_0)$, so $\|\Xi_a(0)\|_{a;r} \leqslant \|u\widetilde{A}_a^{-1}\|_{a;r}\|u^{-1}R_0\|_a$. Now $\|u^{-1}R_0\|_a \to 0$ as $a \to \infty$, so by Proposition 6.1.7(iii) we can take a so large that $\|u\widetilde{A}_a^{-1}\|_{a;r}\|u^{-1}R_0\|_a \leqslant \frac{1}{4}$. For f, g in the closed ball above we have by Lemma 6.2.2,

$$\|\Xi_a(f) - \Xi_a(g)\|_{a;r} \;=\; \|\Xi_a(f+(g-f)) - \Xi_a(f)\|_{a;r} \;\leqslant\; C_a \cdot d\|f-g\|_{a;r}.$$

Take a so large that also $C_a d \leqslant \frac{1}{2}$. Then $\|\Xi_a(f) - \Xi_a(g)\|_{a;r} \leqslant \frac{1}{2}\|f-g\|_{a;r}$. Applying this to $g=0$ we see that Ξ_a maps the closed ball above to itself. Thus Ξ_a has a unique fixed point on this ball. □

Note that if $\deg R \leqslant 0$ (so $R = R_0$), then $\Xi_a(f) = u\widetilde{A}_a^{-1}(u^{-1}R_0)$ is independent of $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$, so for sufficiently large a, the fixed point $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ of Ξ_a with $\|f\|_{a;r} \leqslant 1/2$ is $f = \Xi_a(0) = u\widetilde{A}_a^{-1}(u^{-1}R_0)$. Here is a variant of Theorem 6.2.3:

Lemma 6.2.4. *Let $h \in \mathcal{C}_{a_0}^r[\mathrm{i}]$ be such that $\|h\|_{a_0;r} \leqslant 1/8$. Then for sufficiently large a there is a unique $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ such that $\|f\|_{a;r} \leqslant 1/2$ and $\Xi_a(f) = f + h$.*

Proof. Consider the operator $\Theta_a = \Xi_a - h\colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ given by

$$\Theta_a(y) \;:=\; \Xi_a(y) - h.$$

Arguing as in the proof of Theorem 6.2.3 we take a so large that $\|\Xi_a(0)\|_{a;r} \leqslant \frac{1}{8}$. Then $\|\Theta_a(0)\|_{a;r} \leqslant \|\Xi_a(0)\|_{a;r} + \|h\|_{a;r} \leqslant \frac{1}{4}$. Also, take a so large that $C_a d \leqslant \frac{1}{2}$. Then for $f, g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ with $\|f\|_{a;r}, \|g\|_{a;r} \leqslant 1/2$ we have

$$\|\Theta_a(f) - \Theta_a(g)\|_{a;r} \ = \ \|\Xi_a(f) - \Xi_a(g)\|_{a;r} \ \leqslant \ \tfrac{1}{2}\|f-g\|_{a;r}.$$

Now finish as in the proof of Theorem 6.2.3. □

Next we investigate the difference between solutions of $(*)$ on $[a_0, \infty)$:

Lemma 6.2.5. *Suppose $f, g \in \mathcal{C}_{a_0}^r[\mathrm{i}]^{\mathrm{b}}$ and $A_{a_0}(f) = R(f)$, $A_{a_0}(g) = R(g)$. Then there are positive reals E, ε such that for all a there exists an $h_a \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ with the property that for $\theta_a := (f-g)|_{[a,\infty)}$,*

$$A_a(h_a) = 0, \quad \theta_a - h_a \prec \mathfrak{v}^w, \quad \|\theta_a - h_a\|_{a;r} \ \leqslant \ E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \big(\|\theta_a\|_{a;r} + \cdots + \|\theta_a\|_{a;r}^d\big),$$

and thus $h_a \prec 1$ in case $f - g \prec 1$.

Proof. Set $\eta_a := A_a(\theta_a) = R(f) - R(g)$, where f and g stand for their restrictions to $[a, \infty)$. From $R \prec \mathfrak{v}^\nu$ we get $u^{-1}R(f) \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and $u^{-1}R(g) \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, so $u^{-1}\eta_a \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. By Proposition 6.1.7(i),(iv) we have

$$\xi_a \ := \ u\widetilde{A}_a^{-1}(u^{-1}\eta_a) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}, \qquad \xi_a \prec \mathfrak{v}^w.$$

Now $\widetilde{A}_a(u^{-1}\xi_a) = u^{-1}\eta_a$, that is, $A_a(\xi_a) = \eta_a$. Note that then $h_a := \theta_a - \xi_a$ satisfies $A_a(h_a) = 0$. Now $\xi_a = \theta_a - h_a$ and $\xi_a = \Xi_a(g + \theta_a) - \Xi_a(g)$, hence by Lemma 6.2.2 and its proof,

$$\|\theta_a - h_a\|_{a;r} \ = \ \|\xi_a\|_{a;r} \ \leqslant \ C_a \cdot \max\big\{1, \|g\|_{a;r}^d\big\} \cdot \big(\|\theta\|_{a;r} + \cdots + \|\theta\|_{a;r}^d\big), \text{ with}$$
$$C_a \ := \ \big\|u\widetilde{A}_a^{-1}\big\|_{a;r} \cdot \sum_{\boldsymbol{i},\boldsymbol{j}} \big\|u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})}\big\|_a.$$

Take a real $\varepsilon > 0$ such that $R \prec \mathfrak{v}^{\nu+\varepsilon}$. This gives a real $e > 0$ such that $\sum_{\boldsymbol{i},\boldsymbol{j}} \big\|u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})}\big\|_a \leqslant e\|\mathfrak{v}^\varepsilon\|_a$ for all a. Now use Proposition 6.1.7(iii). □

The situation we have in mind in the lemma above is that f and g are close at infinity, in the sense that $\|f - g\|_{a;r} \to 0$ as $a \to \infty$. Then the lemma yields solutions of $A(y) = 0$ that are very close to $f - g$ at infinity. However, being very close at infinity as stated in Lemma 6.2.5, namely $\theta_a - h_a \prec \mathfrak{v}^w$ and the rest, is too weak for later use. We take up this issue again in Section 6.5 below. (In Corollary 6.2.15 later in the present section we already show: if $f \neq g$ as germs, then $h_a \neq 0$ for sufficiently large a.)

Preserving reality. We now assume in addition that A and R are real, that is, $A \in H[\partial]$ and $R \in H\{Y\}$. It is not clear that the fixed points constructed in the proof of Theorem 6.2.3 are then also real. Therefore we slightly modify this construction using real parts. We first apply the discussion following Lemma 6.1.5 to $\widetilde{A}$ as well as to A, increasing a_0 so that for all a the $\mathbb{R}$-linear real part

$$\operatorname{Re} \widetilde{A}_a^{-1} \ : \ \mathcal{C}_a^{\mathrm{b}} \to \mathcal{C}_a^{\mathrm{b}}$$

maps $\mathcal{C}_a^{\mathrm{b}}$ into $\mathcal{C}_a^r$ and is right-inverse to $\widetilde{A}_a$ on $(\mathcal{C}_a^0)^{\mathrm{b}}$, with

$$\big\|(\operatorname{Re} \widetilde{A}_a^{-1})(f)\big\|_a \leqslant \big\|\widetilde{A}_a^{-1}(f)\big\|_a \qquad \text{for all } f \in \mathcal{C}_a^{\mathrm{b}}.$$

Next we set

$$(\mathcal{C}_a^r)^{\mathrm{b}} \ := \ \big\{f \in \mathcal{C}_a^r : \ \|f\|_{a;r} < \infty\big\} \ = \ \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r,$$

which is a real Banach space with respect to $\|\cdot\|_{a;r}$. Finally, this increasing of a_0 is done so that the original $R_{\boldsymbol{j}} \in \mathcal{C}_{a_0}[i]$ restrict to updated functions $R_{\boldsymbol{j}} \in \mathcal{C}_{a_0}$. For all a, take u, Ξ_a as in Lemma 6.2.1. This lemma has the following real analogue as a consequence:

Lemma 6.2.6. *The operator*

$$\operatorname{Re}\Xi_a \ : \ (\mathcal{C}^r_a)^{\mathrm{b}} \to (\mathcal{C}^r_a)^{\mathrm{b}}, \quad f \mapsto \operatorname{Re}\big(\Xi_a(f)\big)$$

satisfies $(\operatorname{Re}\Xi_a)(f) \preccurlyeq \mathfrak{v}^{\nu}$ *for* $f \in (\mathcal{C}^r_a)^{\mathrm{b}}$, *and any fixed point of* $\operatorname{Re}\Xi_a$ *is a solution of* $(*)$ *on* $[a,\infty)$.

Below the constants C_a are as in Lemma 6.2.2.

Lemma 6.2.7. *For* $f, g \in (\mathcal{C}^r_a)^{\mathrm{b}}$,

$$\big\|(\operatorname{Re}\Xi_a)(f+g) - (\operatorname{Re}\Xi_a)(f)\big\|_{a;r} \ \leqslant \ C_a \cdot \max\big\{1, \|f\|^d_{a;r}\big\} \cdot \big(\|g\|_{a;r} + \cdots + \|g\|^d_{a;r}\big).$$

The next corollary is derived from Lemma 6.2.7 in the same way as Theorem 6.2.3 from Lemma 6.2.2:

Corollary 6.2.8. *For all sufficiently large a the operator* $\operatorname{Re}\Xi_a$ *maps the closed ball*

$$\big\{f \in \mathcal{C}^r_a : \ \|f\|_{a;r} \leqslant 1/2\big\}$$

of the Banach space $(\mathcal{C}^r_a)^{\mathrm{b}}$ *into itself and has a unique fixed point on this ball.*

Here is the real analogue of Lemma 6.2.4, with a similar proof:

Corollary 6.2.9. *Let* $h \in \mathcal{C}^r_{a_0}$ *be such that* $\|h\|_{a_0;r} \leqslant 1/8$. *Then for sufficiently large a there is a unique* $f \in \mathcal{C}^r_a$ *such that* $\|f\|_{a;r} \leqslant 1/2$ *and* $(\operatorname{Re}\Xi_a)(f) = f + h$.

We also have a real analogue of Lemma 6.2.5:

Corollary 6.2.10. *Suppose* $f, g \in (\mathcal{C}^r_{a_0})^{\mathrm{b}}$ *and* $A_{a_0}(f) = R(f)$, $A_{a_0}(g) = R(g)$. *Then there are positive reals* E, ε *such that for all a there exists an* $h_a \in (\mathcal{C}^r_a)^{\mathrm{b}}$ *with the property that for* $\theta_a := (f-g)|_{[a,\infty)}$,

$$A_a(h_a) = 0, \quad \theta_a - h_a \prec \mathfrak{v}^w, \quad \|\theta_a - h_a\|_{a;r} \ \leqslant \ E \cdot \|\mathfrak{v}^{\varepsilon}\|_a \cdot \big(\|\theta_a\|_{a;r} + \cdots + \|\theta_a\|^d_{a;r}\big).$$

Proof. Take h_a to be the real part of an h_a as in Lemma 6.2.5. $\square$

Some useful bounds. To prepare for Section 6.5 we derive in this subsection some bounds from Lemmas 6.2.2 and 6.2.5. Throughout we assume $d, r \in \mathbb{N}^{\geqslant 1}$. We begin with an easy inequality:

Lemma 6.2.11. *Let* $(V, \|\cdot\|)$ *be a normed* $\mathbb{C}$*-linear space, and* $f, g \in V$. *Then*

$$\|f+g\|^d \ \leqslant \ 2^d \cdot \max\big\{1, \|f\|^d\big\} \cdot \max\big\{1, \|g\|^d\big\}.$$

Proof. Use that $\|f+g\| \leqslant \|f\| + \|g\| \leqslant 2\max\big\{1, \|f\|\big\} \cdot \max\big\{1, \|g\|\big\}$. $\square$

Now let u, Ξ_a be as in Lemma 6.2.1. By that lemma, the operator

$$\Phi_a \ : \ \mathcal{C}^r_a[i]^{\mathrm{b}} \times \mathcal{C}^r_a[i]^{\mathrm{b}} \to \mathcal{C}^r_a[i]^{\mathrm{b}}, \quad (f,y) \mapsto \Xi_a(f+y) - \Xi_a(f)$$

is continuous. Furthermore $\Phi_a(f,0) = 0$ for $f \in \mathcal{C}^r_a[i]^{\mathrm{b}}$ and

$$\Phi_a(f, g+y) - \Phi_a(f,g) \ = \ \Phi_a(f+g, y) \qquad \text{for } f,g,y \in \mathcal{C}^r_a[i]^{\mathrm{b}}. \tag{6.2.1}$$

Lemma 6.2.12. *There are $C_a, C_a^+ \in \mathbb{R}^{\geqslant}$ such that for all $f, g, y \in \mathcal{C}_a^r[i]^b$,*

$$\|\Phi_a(f,y)\|_{a;r} \ \leqslant\ C_a \cdot \max\{1, \|f\|_{a;r}^d\} \cdot \big(\|y\|_{a;r} + \cdots + \|y\|_{a;r}^d\big), \tag{6.2.2}$$

$$\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r} \ \leqslant \ C_a^+ \cdot \max\{1, \|f\|_{a;r}^d\} \cdot \max\{1, \|g\|_{a;r}^d\} \cdot \big(\|y\|_{a;r} + \cdots + \|y\|_{a;r}^d\big). \tag{6.2.3}$$

We can take these C_a, C_a^+ such that $C_a, C_a^+ \to 0$ as $a \to \infty$, and do so below.

Proof. The C_a as in Lemma 6.2.2 satisfy the requirements on the C_a here. Now let $f, g, y \in \mathcal{C}_a^r[i]^b$. Then by (6.2.1) and (6.2.2) we have

$$\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r} \ \leqslant\ C_a \cdot \max\{1, \|f+g\|_{a;r}^d\} \cdot \big(\|y\|_{a;r} + \cdots + \|y\|_{a;r}^d\big).$$

Thus by Lemma 6.2.11, $C_a^+ := 2^d \cdot C_a$ has the required property. $\square$

Next, let f, g be as in the hypothesis of Lemma 6.2.5 and take E, ε, and h_a (for each a) as in its conclusion. Thus for all a and $\theta_a := (f-g)|_{[a,\infty)}$,

$$\|\theta_a - h_a\|_{a;r} \ \leqslant\ E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \big(\|\theta_a\|_{a;r} + \cdots + \|\theta_a\|_{a;r}^d\big),$$

and if $f - g \prec 1$, then $h_a \prec 1$. So

$$\|\theta_a - h_a\|_{a;r} \ \leqslant\ E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \big(\rho + \cdots + \rho^d\big), \quad \rho := \|f-g\|_{a_0;r}.$$

We let

$$B_a \ :=\ \big\{y \in \mathcal{C}_a^r[i]^b :\ \|y - h_a\|_{a;r} \leqslant 1/2\big\}$$

be the closed ball of radius $1/2$ around h_a in $\mathcal{C}_a^r[i]^b$. Using $\mathfrak{v}^\varepsilon \prec 1$ we take $a_1 \geqslant a_0$ so that $\theta_a \in B_a$ for all $a \geqslant a_1$. Then for $a \geqslant a_1$ we have

$$\|h_a\|_{a;r} \ \leqslant\ \|h_a - \theta_a\|_{a;r} + \|\theta_a\|_{a;r} \ \leqslant\ \tfrac{1}{2} + \rho,$$

and hence for $y \in B_a$,

$$\|y\|_{a;r} \ \leqslant\ \|y - h_a\|_{a;r} + \|h_a\|_{a;r} \ \leqslant\ \tfrac{1}{2} + \big(\tfrac{1}{2} + \rho\big) \ =\ 1 + \rho. \tag{6.2.4}$$

Consider now the continuous operators

$$\Phi_a, \Psi_a \ :\ \mathcal{C}_a^r[i]^b \to \mathcal{C}_a^r[i]^b, \qquad \Phi_a(y) := \Xi_a(g+y) - \Xi_a(g), \quad \Psi_a(y) := \Phi_a(y) + h_a.$$

In the notation introduced above, $\Phi_a(y) = \Phi_a(g,y)$ for $y \in \mathcal{C}_a^r[i]^b$. With ξ_a as in the proof of Lemma 6.2.5 we also have $\Phi_a(\theta_a) = \xi_a$ and $\Psi_a(\theta_a) = \xi_a + h_a = \theta_a$. Below we reconstruct the fixed point θ_a of Ψ_a from h_a, for sufficiently large a.

Lemma 6.2.13. *There exists $a_2 \geqslant a_1$ such that for all $a \geqslant a_2$ we have $\Psi_a(B_a) \subseteq B_a$, and $\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \leqslant \frac{1}{2}\|y - z\|_{a;r}$ for all $y, z \in B_a$.*

Proof. Take C_a as in Lemma 6.2.12, and let $y \in B_a$. Then by (6.2.2),

$$\|\Phi_a(y)\|_{a;r} \ \leqslant\ C_a \cdot \max\{1, \|g\|_{a;r}^d\} \cdot \big(\|y\|_{a;r} + \cdots + \|y\|_{a;r}^d\big), \quad \theta_a \in B_a, \text{ so}$$

$$\|\Psi_a(y) - h_a\|_{a;r} \ \leqslant\ C_a M, \quad M := \max\{1, \|g\|_{a_0;r}^d\} \cdot \big((1+\rho) + \cdots + (1+\rho)^d\big).$$

Recall that $C_a \to 0$ as $a \to \infty$. Suppose $a \geqslant a_1$ is so large that $C_a M \leqslant 1/2$. Then $\Psi_a(B_a) \subseteq B_a$. With C_a^+ as in Lemma 6.2.12, (6.2.3) gives for $y, z \in \mathcal{C}_a^r[i]^b$,

$$\|\Phi_a(y) - \Phi_a(z)\|_{a;r} \leqslant \ C_a^+ \cdot \max\{1, \|g\|_{a;r}^d\} \cdot \max\{1, \|z\|_{a;r}^d\} \cdot \big(\|y-z\|_{a;r} + \cdots + \|y-z\|_{a;r}^d\big).$$

Hence with $N := \max\{1, \|g\|^d_{a_0;r}\} \cdot (1+\rho)^d \cdot d$ we obtain for $y, z \in B_a$ that

$$\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \ \leqslant \ C_a^+ N \|y - z\|_{a;r},$$

so $\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \leqslant \frac{1}{2}\|y - z\|_{a;r}$ if $C_a^+ N \leqslant 1/2$. ☐

Below a_2 is as in Lemma 6.2.13.

Corollary 6.2.14. *If $a \geqslant a_2$, then $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$ in $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$.*

Proof. Let $a \geqslant a_2$. Then Ψ_a has a unique fixed point on B_a. As $\Psi_a(\theta_a) = \theta_a \in B_a$, this fixed point is θ_a. ☐

Corollary 6.2.15. *If $f \neq g$ as germs, then $h_a \neq 0$ for $a \geqslant a_2$.*

Proof. Let $a \geqslant a_2$. Then $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$. If $h_a = 0$, then $\Psi_a = \Phi_a$, and hence $\theta_a = 0$, since $\Phi_a(0) = 0$. ☐

6.3. Smoothness Considerations

We assume $r \in \mathbb{N}$ in this section. We prove here as much smoothness of solutions of algebraic differential equations over Hardy fields as could be hoped for. In particular, the solutions in $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ of the equation $(*)$ in Section 6.2 actually have their germs in $\mathcal{C}^{<\infty}[\mathrm{i}]$. To make this precise, consider a "differential" polynomial

$$P \ = \ P(Y, \dots, Y^{(r)}) \ \in \ \mathcal{C}^n[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big].$$

We put *differential* in quotes since $\mathcal{C}^n[\mathrm{i}]$ is not naturally a differential ring. Nevertheless, P defines an obvious evaluation map

$$f \mapsto P\big(f, \dots, f^{(r)}\big) \ : \ \mathcal{C}^r[\mathrm{i}] \to \mathcal{C}[\mathrm{i}].$$

We also have the "separant"

$$S_P \ := \ \frac{\partial P}{\partial Y^{(r)}} \ \in \ \mathcal{C}^n[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$$

of P.

Proposition 6.3.1. *Assume $n \geqslant 1$. Let $f \in \mathcal{C}^r[\mathrm{i}]$ be such that*

$$P\big(f, \dots, f^{(r)}\big) = 0 \in \mathcal{C}[\mathrm{i}] \quad \text{ and } \quad S_P\big(f, \dots, f^{(r)}\big) \in \mathcal{C}[\mathrm{i}]^\times.$$

Then $f \in \mathcal{C}^{n+r}[\mathrm{i}]$. Thus if $P \in \mathcal{C}^{<\infty}[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$, then $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Moreover, if $P \in \mathcal{C}^{\infty}[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$, then $f \in \mathcal{C}^{\infty}[\mathrm{i}]$, and likewise with $\mathcal{C}^{\omega}[\mathrm{i}]$ in place of $\mathcal{C}^{\infty}[\mathrm{i}]$.

We deduce this from the lemma below, which has a complex-analytic hypothesis. Let $U \subseteq \mathbb{R} \times \mathbb{C}^{1+r}$ be open. Let t range over $\mathbb{R}$ and $z_0, \dots, z_r$ over $\mathbb{C}$, and set $x_j := \operatorname{Re} z_j$, $y_j := \operatorname{Im} z_j$ for $j = 0, \dots, r$. We also set

$$U(t, z_0, \dots, z_{r-1}) \ := \ \big\{z_r : (t, z_0, \dots, z_{r-1}, z_r) \in U\big\},$$

an open subset of $\mathbb{C}$, and we assume $\Phi\colon U \to \mathbb{C}$ and $n \geqslant 1$ are such that

$$\operatorname{Re}\Phi, \ \operatorname{Im}\Phi \ : \ U \to \mathbb{R}$$

are $\mathcal{C}^n$-functions of $(t, x_0, y_0, \dots, x_r, y_r)$, and for all $t, z_0, \dots, z_{r-1}$ the function

$$z_r \mapsto \Phi(t, z_0, \dots, z_{r-1}, z_r) \ : \ U(t, z_0, \dots, z_{r-1}) \to \mathbb{C}$$

is holomorphic (the complex-analytic hypothesis alluded to).

Lemma 6.3.2. *Let $I \subseteq \mathbb{R}$ be a nonempty open interval and suppose $f \in \mathcal{C}^r(I)[\mathrm{i}]$ is such that for all $t \in I$,*

- $\big(t, f(t), \dots, f^{(r)}(t)\big) \in U$;
- $\Phi\big(t, f(t), \dots, f^{(r)}(t)\big) = 0$; *and*
- $(\partial\Phi/\partial z_r)\big(t, f(t), \dots, f^{(r)}(t)\big) \neq 0$.

Then $f \in \mathcal{C}^{n+r}(I)[\mathrm{i}]$.

Proof. Set $A := \operatorname{Re}\Phi$, $B := \operatorname{Im}\Phi$ and $g := \operatorname{Re} f$, $h := \operatorname{Im} f$. Then for all $t \in I$,

$$\begin{aligned} A\big(t, g(t), h(t), g'(t), h'(t) \dots, g^{(r)}(t), h^{(r)}(t)\big) &= 0 \\ B\big(t, g(t), h(t), g'(t), h'(t) \dots, g^{(r)}(t), h^{(r)}(t)\big) &= 0. \end{aligned}$$

Consider the $\mathcal{C}^n$-map $(A,B)\colon U \to \mathbb{R}^2$, with U identified in the usual way with an open subset of $\mathbb{R}^{1+2(1+r)}$. The Cauchy-Riemann equations give

$$\frac{\partial\Phi}{\partial z_r} = \frac{\partial A}{\partial x_r} + \mathrm{i}\frac{\partial B}{\partial x_r}, \qquad \frac{\partial A}{\partial x_r} = \frac{\partial B}{\partial y_r}, \qquad \frac{\partial B}{\partial x_r} = -\frac{\partial A}{\partial y_r}.$$

Thus the Jacobian matrix of the map (A,B) with respect to its last two variables x_r and y_r has determinant

$$D = \left(\frac{\partial A}{\partial x_r}\right)^2 + \left(\frac{\partial B}{\partial x_r}\right)^2 = \left|\frac{\partial \Phi}{\partial z_r}\right|^2 : U \to \mathbb{R}.$$

Let $t_0 \in I$. Then

$$D\big(t_0, g(t_0), h(t_0), \dots, g^{(r)}(t_0), h^{(r)}(t_0)\big) \neq 0,$$

so by the Implicit Mapping Theorem [57, (10.2.2), (10.2.3)] we have a connected open neighborhood V of the point

$$\big(t_0, g(t_0), h(t_0), \dots, g^{(r-1)}(t_0), h^{(r-1)}(t_0)\big) \in \mathbb{R}^{1+2r},$$

open intervals $J, K \subseteq \mathbb{R}$ containing $g^{(r)}(t_0)$, $h^{(r)}(t_0)$, respectively, and a $\mathcal{C}^n$-map

$$(G,H)\colon V \to J \times K$$

such that $V \times J \times K \subseteq U$ and the zero set of Φ on $V \times J \times K$ equals the graph of (G,H). Take an open subinterval I_0 of I with $t_0 \in I_0$ such that for all $t \in I_0$,

$$\big(t, g(t), h(t), g'(t), h'(t), \dots, g^{(r-1)}(t), h^{(r-1)}(t), g^{(r)}(t), h^{(r)}(t)\big) \in V \times J \times K.$$

Then the above gives that for all $t \in I_0$ we have

$$\begin{aligned} g^{(r)}(t) &= G\big(t, g(t), h(t), g'(t), h'(t), \dots, g^{(r-1)}(t), h^{(r-1)}(t)\big), \\ h^{(r)}(t) &= H\big(t, g(t), h(t), g'(t), h'(t), \dots, g^{(r-1)}(t), h^{(r-1)}(t)\big). \end{aligned}$$

It follows easily from these two equalities that g, h are of class $\mathcal{C}^{n+r}$ on I_0. $\square$

Let f continue to be as in Lemma 6.3.2. If $\operatorname{Re}\Phi$, $\operatorname{Im}\Phi$ are $\mathcal{C}^\infty$, then by taking n arbitrarily high we conclude that $f \in \mathcal{C}^\infty(I)[\mathrm{i}]$. Moreover:

Corollary 6.3.3. *If $\operatorname{Re}\Phi$, $\operatorname{Im}\Phi$ are real-analytic, then $f \in \mathcal{C}^\omega(I)[\mathrm{i}]$.*

Proof. Same as that of Lemma 6.3.2, with the reference to [57, (10.2.3)] replaced by [57, (10.2.4)] to obtain that G, H are real-analytic, and noting that then the last displayed relations for $t \in I_0$ force g, h to be real-analytic on I_0 by [57, (10.5.3)]. $\square$

Lemma 6.3.4. *Let $I \subseteq \mathbb{R}$ be a nonempty open interval, $n \geqslant 1$, and*

$$P = P\big(Y, \dots, Y^{(r)}\big) \in \mathcal{C}^n(I)[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big].$$

Let $f \in \mathcal{C}^r(I)[\mathrm{i}]$ be such that

$$P\big(f, \dots, f^{(r)}\big) = 0 \in \mathcal{C}(I)[\mathrm{i}] \quad \text{and} \quad (\partial P/\partial Y^{(r)})\big(f, \dots, f^{(r)}\big) \in \mathcal{C}(I)[\mathrm{i}]^{\times}.$$

Then $f \in \mathcal{C}^{n+r}(I)[\mathrm{i}]$. Moreover, if $P \in \mathcal{C}^{\infty}(I)[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$, then $f \in \mathcal{C}^{\infty}(I)[\mathrm{i}]$, and likewise with $\mathcal{C}^{\omega}(I)[\mathrm{i}]$ in place of $\mathcal{C}^{\infty}(I)[\mathrm{i}]$.

Proof. Let $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ where all $P_{\boldsymbol{i}} \in \mathcal{C}^n(I)[\mathrm{i}]$. Set $U := I \times \mathbb{C}^{1+r}$, and consider the map $\Phi \colon U \to \mathbb{C}$ given by

$$\Phi(t, z_0, \dots, z_r) := \sum_{\boldsymbol{i}} P_{\boldsymbol{i}}(t) z^{\boldsymbol{i}} \quad \text{where } z^{\boldsymbol{i}} := z_0^{i_0} \cdots z_r^{i_r} \text{ for } \boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}.$$

From Lemma 6.3.2 we obtain $f \in \mathcal{C}^{n+r}(I)[\mathrm{i}]$. In view of Corollary 6.3.3 and the remark preceding it, and replacing n by ∞ respectively ω, this argument also gives the second part of the lemma. □

Proposition 6.3.1 follows from Lemma 6.3.4 by taking suitable representatives of the germs involved. Let now H be a Hardy field and $P \in H[\mathrm{i}]\{Y\}$ of order r. Then $P \in \mathcal{C}^{<\infty}[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$, and so $P(f) := P\big(f, \dots, f^{(r)}\big) \in \mathcal{C}[\mathrm{i}]$ for $f \in \mathcal{C}^r[\mathrm{i}]$ as explained in the beginning of this section.

For notational convenience we set

$$\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq} := \big\{f \in \mathcal{C}^n[\mathrm{i}] : f, f', \dots, f^{(n)} \preccurlyeq 1\big\}, \qquad (\mathcal{C}^n)^{\preccurlyeq} := \mathcal{C}^n[\mathrm{i}]^{\preccurlyeq} \cap \mathcal{C}^n,$$

and likewise with $\prec$ instead of $\preccurlyeq$. Then $\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq}$ is a $\mathbb{C}$-subalgebra of $\mathcal{C}^n[\mathrm{i}]$ and $(\mathcal{C}^n)^{\preccurlyeq}$ is an $\mathbb{R}$-subalgebra of $\mathcal{C}^n$. Also, $\mathcal{C}^n[\mathrm{i}]^{\prec}$ is an ideal of $\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq}$, and likewise with $\mathcal{C}^n$ instead of $\mathcal{C}^n[\mathrm{i}]$. We have $\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq} \supseteq \mathcal{C}^{n+1}[\mathrm{i}]^{\preccurlyeq}$ and $(\mathcal{C}^n)^{\preccurlyeq} \supseteq (\mathcal{C}^{n+1})^{\preccurlyeq}$, and likewise with $\prec$ instead of $\preccurlyeq$. Thus in the notation from Section 5.4:

$$\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} = \bigcap_n \mathcal{C}^n[\mathrm{i}]^{\preccurlyeq}, \quad \mathcal{I} = \bigcap_n \mathcal{C}^n[\mathrm{i}]^{\prec}, \quad (\mathcal{C}^{<\infty})^{\preccurlyeq} = \bigcap_n (\mathcal{C}^n)^{\preccurlyeq}.$$

Corollary 6.3.5. *Suppose*

$$P = Y^{(r)} + f_1 Y^{(r-1)} + \dots + f_r Y - R \quad \text{with } f_1, \dots, f_r \text{ in } H[\mathrm{i}] \text{ and } R_{\geqslant 1} \prec 1.$$

Let $f \in \mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}$ be such that $P(f) = 0$. Then $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Moreover, if $H \subseteq \mathcal{C}^{\infty}$, then $f \in \mathcal{C}^{\infty}[\mathrm{i}]$, and if $H \subseteq \mathcal{C}^{\omega}$, then $f \in \mathcal{C}^{\omega}[\mathrm{i}]$.

Proof. We have $S_P = \frac{\partial P}{\partial Y^{(r)}} = 1 - S$ with $S := \frac{\partial R_{\geqslant 1}}{\partial Y^{(r)}} \prec 1$ and thus

$$S_P(f, \dots, f^{(r)}) = 1 - S(f, \dots, f^{(r)}), \qquad S(f, \dots, f^{(r)}) \prec 1,$$

so $S_P(f, \dots, f^{(r)}) \in \mathcal{C}[\mathrm{i}]^{\times}$. Now appeal to Proposition 6.3.1. □

Thus the germ of any solution on $[a, \infty)$ of the asymptotic equation $(*)$ of Section 6.2 lies in $\mathcal{C}^{<\infty}[\mathrm{i}]$, and even in $\mathcal{C}^{\infty}[\mathrm{i}]$ (respectively $\mathcal{C}^{\omega}[\mathrm{i}]$) if H is in addition a $\mathcal{C}^{\infty}$-Hardy field (respectively, a $\mathcal{C}^{\omega}$-Hardy field).

Corollary 6.3.6. *Suppose $(P, 1, \widehat{a})$ is a normal slot in $H[\mathrm{i}]$ of order $r \geqslant 1$, and $f \in \mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}$, $P(f) = 0$. Then $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$. If $H \subseteq \mathcal{C}^{\infty}$, then $f \in \mathcal{C}^{\infty}[\mathrm{i}]$. If $H \subseteq \mathcal{C}^{\omega}$, then $f \in \mathcal{C}^{\omega}[\mathrm{i}]$.*

Proof. Multiplying P by an element of $H[i]^\times$ we arrange

$$P_1 \;=\; Y^{(r)} + f_1 Y^{(r-1)} + \cdots + f_r Y, \quad f_1,\dots,f_r \in H[i],$$

and then the hypothesis of Corollary 6.3.5 is satisfied. $\square$

For the differential subfield $K := H[i]$ of the differential ring $\mathcal{C}^{<\infty}[i]$ we have:

Corollary 6.3.7. *Suppose $f \in \mathcal{C}^{<\infty}[i]$ is such that $P(f) = 0$ and f generates a differential subfield $K\langle f\rangle$ of $\mathcal{C}^{<\infty}[i]$ over K. If H is a $\mathcal{C}^\infty$-Hardy field, then $K\langle f\rangle \subseteq \mathcal{C}^\infty[i]$, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Proof. Suppose H is a $\mathcal{C}^\infty$-Hardy field; it suffices to show $f \in \mathcal{C}^\infty[i]$. We may assume that P is a minimal annihilator of f over K; then $S_P(f) \neq 0$ in $K\langle f\rangle$ and so $S_P(f) \in \mathcal{C}[i]^\times$. Hence the claim follows from Proposition 6.3.1. $\square$

With H replacing K in this proof we obtain the "real" version:

Corollary 6.3.8. *Suppose $f \in \mathcal{C}^{<\infty}$ is hardian over H and $P(f) = 0$ for some $P \in H\{Y\}^{\neq}$. Then $H \subseteq \mathcal{C}^\infty \;\Rightarrow\; f \in \mathcal{C}^\infty$, and $H \subseteq \mathcal{C}^\omega \;\Rightarrow\; f \in \mathcal{C}^\omega$.*

This leads to:

Corollary 6.3.9. *Suppose H is a $\mathcal{C}^\infty$-Hardy field. Then every* d*-algebraic Hardy field extension of H is a $\mathcal{C}^\infty$-Hardy field; in particular, $\mathrm{D}(H) \subseteq \mathcal{C}^\infty$. Likewise with $\mathcal{C}^\infty$ replaced by $\mathcal{C}^\omega$.*

In particular, $\mathrm{D}(\mathbb{Q}) \subseteq \mathcal{C}^\omega$ [33, Theorems 14.3, 14.9].

Let H be a $\mathcal{C}^\infty$-Hardy field. Then by Corollary 6.3.9, H is d-maximal iff H has no proper d-algebraic $\mathcal{C}^\infty$-Hardy field extension; thus every $\mathcal{C}^\infty$-maximal Hardy field is d-maximal (so $\mathrm{D}(H) \subseteq \mathrm{E}^\infty(H)$), and H has a d-maximal d-algebraic $\mathcal{C}^\infty$-Hardy field extension. The same remarks apply with ω in place of ∞.

Existence and uniqueness theorems $(^*)$**.** We finish this section with some existence and uniqueness results for algebraic differential equations. From this subsection, only Corollary 6.3.13 is used later (in the proofs of Lemmas 7.7.42 and 7.7.51, which are not needed for the proof of our main theorem). First, let U, Φ be as in Lemma 6.3.2 for $n = 1$; the argument in the proof of that lemma combined with the existence and uniqueness theorem for scalar differential equations [203, §11] yields:

Lemma 6.3.10. *Let $(t_0, c_0, \dots, c_r) \in U$ be such that*

$$\Phi(t_0, c_0, \dots, c_r) = 0 \quad \textit{and} \quad (\partial\Phi/\partial z_r)(t_0, c_0, \dots, c_r) \neq 0.$$

Then for some open interval $I \subseteq \mathbb{R}$ containing t_0 there is a unique $f \in \mathcal{C}^r(I)[i]$ such that $\big(f(t_0), f'(t_0), \dots, f^{(r)}(t_0)\big) = (c_0, \dots, c_r)$ and for all $t \in I$,

$$\big(t, f(t), \dots, f^{(r)}(t)\big) \in U \quad \textit{and} \quad \Phi\big(t, f(t), \dots, f^{(r)}(t)\big) = 0. \tag{6.3.1}$$

Proof. Set $A := \operatorname{Re}\Phi$, $B := \operatorname{Im}\Phi$, $a_j := \operatorname{Re} c_j$, $b_j := \operatorname{Im} c_j$ $(j = 0, \dots, r)$. As in the proof of Lemma 6.3.2 we identify U with an open subset of $\mathbb{R}^{1+2(1+r)}$ and consider the $\mathcal{C}^1$-map $(A, B)\colon U \to \mathbb{R}^2$. The Jacobian matrix of the map (A, B) with respect to its last two variables x_r and y_r has determinant

$$D \;=\; \left(\frac{\partial A}{\partial x_r}\right)^2 + \left(\frac{\partial B}{\partial x_r}\right)^2 \;=\; \left|\frac{\partial \Phi}{\partial z_r}\right|^2 \;:\; U \to \mathbb{R},$$

with $D(t_0, a_0, b_0, \dots, a_r, b_r) \neq 0$, hence the Implicit Mapping Theorem [57, (10.2.2)] yields a connected open neighborhood V in $\mathbb{R}^{1+2r}$ of the point

$$u_0 := (t_0, a_0, b_0, \dots, a_{r-1}, b_{r-1}),$$

open intervals $J, K \subseteq \mathbb{R}$ containing a_r, b_r, respectively, such that $V \times J \times K \subseteq U$, as well as a $\mathcal{C}^1$-map $F = (G, H) \colon V \to J \times K$ whose graph is $\Phi^{-1}(0) \cap (V \times J \times K)$. Now by [203, §11, II] we have an open interval $I \subseteq \mathbb{R}$ containing t_0 as well as a $\mathcal{C}^r$-map $u \colon I \to \mathbb{R}^2$ such that $\big(t_0, u(t_0), u'(t_0), \dots, u^{(r-1)}(t_0)\big) = u_0$ and for all $t \in I$:

$$\big(t, u(t), \dots, u^{(r-1)}(t)\big) \in V \quad \text{and} \quad u^{(r)}(t) = F\big(t, u(t), \dots, u^{(r-1)}(t)\big).$$

Then the function $f \colon I \to \mathbb{C}$ with $(\operatorname{Re} f, \operatorname{Im} f) = u$ is an element of $\mathcal{C}^r(I)[\mathrm{i}]$ such that $\big(f(t_0), f'(t_0), \dots, f^{(r)}(t_0)\big) = (c_0, \dots, c_r)$ and (6.3.1) holds for all $t \in I$.

Let any $f_1 \in \mathcal{C}^r(I)[\mathrm{i}]$ be given with $\big(f_1(t_0), f_1'(t_0), \dots, f_1^{(r)}(t_0)\big) = (c_0, \dots, c_r)$ such that (6.3.1) holds for all $t \in I$ with f_1 in place of f. The closed subset

$$S \ := \ \big\{t \in I : \ f_1^{(j)}(t) = f^{(j)}(t) \text{ for } j = 0, \dots, r\big\}$$

of I contains t_0; it is enough to show that S is open. Towards this, let $t_1 \in S$. The map $u_1 := (\operatorname{Re} f_1, \operatorname{Im} f_1) \colon I \to \mathbb{R}^2$ is of class $\mathcal{C}^r$ and

$$\big(t_1, u_1(t_1), \dots, u_1^{(r)}(t_1)\big) \ = \ \big(t_1, u(t_1), \dots, u^{(r)}(t_1)\big) \in V \times J \times K,$$

which gives an open interval $I_1 \subseteq I$ containing t_1 such that $\big(t, u_1(t), \dots, u_1^{(r)}(t)\big) \in V \times J \times K$ for all $t \in I_1$. Since $\Phi\big(t, f_1(t), \dots, f_1^{(r)}(t)\big) = 0$ for $t \in I_1$, this yields $u_1^{(r)}(t) = F\big(t, u_1(t), \dots, u_1^{(r-1)}(t)\big)$ for $t \in I_1$. So $u_1 = u$ on I_1 by the uniqueness part of [203, §11, III], hence $f_1 = f$ on I_1, and thus $I_1 \subseteq S$. $\square$

The second part of the proof gives a bit more: Suppose $I, J \subseteq \mathbb{R}$ are open intervals with $t_0 \in I \cap J$ and the functions $f \in \mathcal{C}^r(I)[\mathrm{i}]$, $g \in \mathcal{C}^r(J)[\mathrm{i}]$ are such that

$$\big(f(t_0), f'(t_0), \dots, f^{(r)}(t_0)\big) \ = \ (c_0, \dots, c_r) \ = \ \big(g(t_0), g'(t_0), \dots, g^{(r)}(t_0)\big),$$

(6.3.1) holds for all $t \in I$, and (6.3.1) holds for all $t \in J$ with g instead of f. Assume also that $(\partial \Phi / \partial z_r)\big(t, f(t), \dots, f^{(r)}(t)\big) \neq 0$ for all $t \in I$. Then

$$f(t) \ = \ g(t) \ \text{ for all } t \in I \cap J.$$

Next, let $I \subseteq \mathbb{R}$ be a nonempty open interval and

$$P \ = \ P\big(Y, \dots, Y^{(r)}\big) \ \in \ \mathcal{C}^1(I)[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big].$$

Applying Lemma 6.3.10 to the map $\Phi \colon U := I \times \mathbb{C}^{1+r} \to \mathbb{C}$ introduced in the proof of Lemma 6.3.4, we obtain:

Lemma 6.3.11. *Let $t_0 \in I$ and $c_0, \dots, c_r \in \mathbb{C}$ be such that*

$$\Phi(t_0, c_0, \dots, c_r) = 0 \quad \textit{and} \quad (\partial \Phi / \partial z_r)(t_0, c_0, \dots, c_r) \neq 0.$$

Then there is an open interval $J \subseteq I$ contaning t_0 with a unique $f \in \mathcal{C}^r(J)[\mathrm{i}]$ such that $\big(f(t_0), f'(t_0), \dots, f^{(r)}(t_0)\big) = (c_0, \dots, c_r)$ and $P\big(f, \dots, f^{(r)}\big) = 0 \in \mathcal{C}(J)[\mathrm{i}]$.

This lemma and the remark following the proof of Lemma 6.3.10 yield:

Corollary 6.3.12. *Given $t_0 \in I$ and $c_0, \dots, c_r \in \mathbb{C}$, there is at most one function $f \in \mathcal{C}^r(I)[\mathrm{i}]$ such that $\big(f(t_0), f'(t_0), \dots, f^{(r)}(t_0)\big) = (c_0, \dots, c_r)$ as well as*

$$P\big(f, \dots, f^{(r)}\big) = 0 \in \mathcal{C}(I)[\mathrm{i}] \quad \textit{and} \quad (\partial P / \partial Y^{(r)})\big(f, \dots, f^{(r)}\big) \in \mathcal{C}(I)[\mathrm{i}]^{\times}.$$

Now let a range over $\mathbb{R}$, $\boldsymbol{i}$ over $\mathbb{N}^{1+r}$, and

$$P \;=\; P\big(Y,\dots,Y^{(r)}\big) \;=\; \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}} \qquad (\text{all } P_{\boldsymbol{i}} \in \mathcal{C}^1_a[\mathrm{i}])$$

over polynomials in $\mathcal{C}^1_a[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$ of degree at most $d \in \mathbb{N}^{\geqslant 1}$, and set $P_{\geqslant 1} := \sum_{|\boldsymbol{i}|\geqslant 1} P_{\boldsymbol{i}} Y^{\boldsymbol{i}} = P - P_0$. Recall that in Section 6.1 we defined

$$P(f) \;:=\; \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}] \qquad (f \in \mathcal{C}^r_a[\mathrm{i}]).$$

Corollary 6.3.13. *There is an $E = E(d,r) \in \mathbb{N}^{\geqslant 1}$ with the following property: if*

$$P = Y^{(r)} + f_1 Y^{(r-1)} + \cdots + f_r Y - R, \quad f_1,\dots,f_r \in \mathcal{C}^1_a[\mathrm{i}],\ \|R_{\geqslant 1}\|_a \leqslant 1/E,$$

then for any $t_0 \in \mathbb{R}^{>a}$ and $c_0,\dots,c_r \in \mathbb{C}$ there is at most one $f \in \mathcal{C}^r_a[\mathrm{i}]$ such that

$$P(f) = 0, \quad \big(f(t_0), f'(t_0),\dots,f^{(r)}(t_0)\big) = (c_0,\dots,c_r), \quad \text{and} \quad \|f\|_{a;r} \leqslant 1.$$

Proof. Set $E := 2d(d+1)D$ with $D = D(0,d,r)$ as in Corollary 6.1.3. Let P be as in the hypothesis and $f \in \mathcal{C}^r_a[\mathrm{i}]$, $\|f\|_{a;r} \leqslant 1$. Then $\partial P/\partial Y^{(r)} = 1 - S$ where $S := \partial R_{\geqslant 1}/\partial Y^{(r)}$ and therefore $(\partial P/\partial Y^{(r)})(f,\dots,f^{(r)}) = 1 - S(f,\dots,f^{(r)})$. We have $\|S\|_a \leqslant d/E = 1/\big(2(d+1)D\big)$ and hence by Corollary 6.1.3:

$$\|S(f,\dots,f^{(r)})\|_a \;\leqslant\; D \cdot \|S\|_a \cdot \big(1 + \|f\|^1_{a;r} + \cdots + \|f\|^d_{a;r}\big) \;\leqslant\; 1/2.$$

Thus $(\partial P/\partial Y^{(r)})(f,\dots,f^{(r)}) \in \mathcal{C}_a[\mathrm{i}]^\times$. Now use Corollary 6.3.12. ∎

Thus in the context of Section 6.2, if a is so large that the functions $f_1,\dots,f_r$ and the $R_{\boldsymbol{j}}$ there are $\mathcal{C}^1$ on $[a,\infty)$ with $\|R_{\boldsymbol{j}}\|_a \leqslant 1/E(d,r)$, then for all $t_0 \in \mathbb{R}^{>a}$ and $c_0,\dots,c_r \in \mathbb{C}$, there is at most one $f \in \mathcal{C}^r_a[\mathrm{i}]$ with $\|f\|_{a;r} \leqslant 1$ such that $A_a(f) = R(f)$ and $\big(f(t_0), f'(t_0),\dots,f^{(r)}(t_0)\big) = (c_0,\dots,c_r)$.

A theorem of Boshernitzan (*)**.** Here we supply a proof of the following result stated in [33, Theorem 11.8], and to be used in Section 7.7. (The proof in loc. cit. is only indicated there.) Below, Y and Z are distinct indeterminates.

Theorem 6.3.14. *Let H be a Hardy field, $P \in H[Y,Z]^{\neq}$, and suppose $P(y,y') = 0$ with $y \in \mathcal{C}^1$ lying in a Hausdorff field extension of H. Then $y \in \mathrm{D}(H)$.*

In particular, if H is a d-perfect Hardy field and F is a Hardy field properly extending H, then $\operatorname{trdeg}(F|H) \geqslant 2$.

For the proof of Theorem 6.3.14 we first observe:

Corollary 6.3.15. *Let H be a Hausdorff field. Then $H \subseteq \mathcal{C}^n \Rightarrow H^{\mathrm{rc}} \subseteq \mathcal{C}^n$, and likewise with $<\infty$, ∞, and ω in place of n.*

Proof. If $y \in H^{\mathrm{rc}}$ has minimum polynomial $P \in H[Y]$ over H, then $P(y) = 0 \in \mathcal{C}$ and $S_P(y) = P'(y) \in \mathcal{C}^\times$. Now use Proposition 6.3.1. ∎

Lemma 6.3.16. *Suppose $f \in \mathcal{C}^1$ oscillates. Then we are in case* (i) *or case* (ii)*:*

(i) *there are arbitrarily large s with $f'(s) = 0$ and $f(s) > 0$,*
(ii) *there are arbitrarily large s with $f'(s) = 0$ and $f(s) < 0$.*

In case (i) *there are also arbitrarily large s with $f'(s) = 0$ and $f(s) \leqslant 0$, and in case* (ii) *there are also arbitrarily large s with $f'(s) = 0$ and $f(s) \geqslant 0$.*

Proof. Let f be represented by a $\mathcal{C}^1$-function on an interval $(a, +\infty)$, also denoted by f. Take $b > a$ such that $f(b) = 0$, and then $c > b$ with $f(c) = 0$ such that $f(t) \neq 0$ for some $t \in (b, c)$. Next, take $s \in [b, c]$ such that $|f(s)| = \max_{b \leqslant t \leqslant c} |f(t)|$. Then $f(s) \neq 0$ and $f'(s) = 0$. Since b can be taken arbitrarily large, we are in case (i) or in case (ii) above. (Of course, this is not an exclusive *or.*) For the remaining part of the lemma, use that in case (i) there are arbitrarily large $s > a$ where f has a local minimum $f(s) \leqslant 0$, and that in case (ii) there are arbitrarily large $s > a$ where f has a local maximum $f(s) \geqslant 0$. □

In the next two lemmas H, P, y are as in Theorem 6.3.14.

Lemma 6.3.17. *The germ y generates a Hardy field extension $H\langle y\rangle$ of H. If $H \subseteq \mathcal{C}^\infty$, then $H\langle y\rangle \subseteq \mathcal{C}^\infty$, and likewise with ω in place of ∞.*

Proof. We are done if $y \in H^{\mathrm{rc}}$, since H^{rc} is a Hardy field with $H \subseteq \mathcal{C}^\infty \Rightarrow H^{\mathrm{rc}} \subseteq \mathcal{C}^\infty$ and $H \subseteq \mathcal{C}^\omega \Rightarrow H^{\mathrm{rc}} \subseteq \mathcal{C}^\omega$, by Proposition 5.3.2.

Suppose $y \notin H^{\mathrm{rc}}$. We have the Hausdorff field $F := H(y) \subseteq \mathcal{C}^1$, and its real closure is by Proposition 5.1.4 the Hausdorff field

$$F^{\mathrm{rc}} \ = \ \big\{z \in \mathcal{C} :\ Q(z) = 0 \text{ for some } Q \in F[Z]^{\neq}\big\}.$$

By Corollary 6.3.15 we have $F^{\mathrm{rc}} \subseteq \mathcal{C}^1$. Set $Q(Z) := P(y, Z) \in F[Z]^{\neq}$. We have $Q(y') = 0$, so $y' \in F^{\mathrm{rc}}$, and thus $\partial F \subseteq F^{\mathrm{rc}}$. Let now $z \in F^{\mathrm{rc}}$, and let $A(Z)$ be the minimum polynomial of z over F, say $A = Z^n + A_1 Z^{n-1} + \cdots + A_n$ ($A_1, \dots, A_n \in F$, $n \geqslant 1$). With $A^\partial := A_1' Z^{n-1} + \cdots + A_n' \in F^{\mathrm{rc}}[Z]$ we have

$$0 \ = \ A(z)' \ = \ A^\partial(z) + A'(z) \cdot z'$$

with $0 \neq A'(z) \in F^{\mathrm{rc}}$ and so $z' = -A^\partial(z)/A'(z) \in F^{\mathrm{rc}}$. Hence F^{rc} is a Hardy field, and so y generates a Hardy field extension $H\langle y\rangle \subseteq F^{\mathrm{rc}}$ of H. For the rest use Corollary 6.3.9. □

In the proof of the next lemma we encounter an ordered field isomorphism

$$f \mapsto \widetilde{f} \ :\ E \to \widetilde{E}$$

between Hausdorff fields E and $\widetilde{E}$. It extends uniquely to an ordered field isomorphism $E^{\mathrm{rc}} \to \widetilde{E}^{\mathrm{rc}}$, also denoted by $f \mapsto \widetilde{f}$, and to a ring isomorphism

$$Q \mapsto \widetilde{Q} \ :\ E[Y] \to \widetilde{E}[Y], \qquad \text{with } \widetilde{Y} = Y.$$

Let $Q \in E[Y]^{\neq}$, and let $y_1 < \cdots < y_m$ be the distinct zeros of Q in E^{rc}. Then by Corollary 5.1.8, $y_1(t) < \cdots < y_m(t)$ are the distinct real zeros of $Q(t, Y)$, eventually. By the isomorphism, $\widetilde{y}_1 < \cdots < \widetilde{y}_m$ are the distinct zeros of $\widetilde{Q}$ in $\widetilde{E}^{\mathrm{rc}}$, and so $\widetilde{y}_1(t) < \cdots < \widetilde{y}_m(t)$ are the distinct real zeros of $\widetilde{Q}(t, Y)$, eventually. This has the trivial but useful consequence that, eventually, that is, for all sufficiently large t,

$$Q(t, Y) = \widetilde{Q}(t, Y) \text{ in } \mathbb{R}[Y] \implies y_1(t) = \widetilde{y}_1(t), \dots, y_m(t) = \widetilde{y}_m(t).$$

Lemma 6.3.18. *Let u be an H-hardian germ. Then $y - u$ is non-oscillating.*

Proof. By Lemma 6.3.17, y is H-hardian. Replacing H by H^{rc} we arrange that H is real closed. Suppose towards a contradiction that $w := y - u$ oscillates. Then $w' = y' - u'$ oscillates. But y' and u' are H-hardian, so $y', u' \notin H$ and for all $h \in H$: $y' > h \Leftrightarrow u' > h$. This yields an ordered field isomorphism $H(y') \to H(u')$ over H mapping y' to u', which extends uniquely to an ordered field isomorphism

$$f \mapsto \widetilde{f} \ :\ H(y')^{\mathrm{rc}} \to H(u')^{\mathrm{rc}}.$$

Now $P(y, y') = 0$ gives $y \in H(y')^{\mathrm{rc}}$, so $\widetilde{y} \in H(u')^{\mathrm{rc}} \subseteq H\langle u \rangle^{\mathrm{rc}}$. The remarks preceding the lemma applied to $E = H(y')$, $\widetilde{E} = H(u')$ and $Q(Y) := P(Y, y')$ in $E[Y]$ give that for all sufficiently large t with $y'(t) = u'(t)$ (that is, $w'(t) = 0$) we have $y(t) = \widetilde{y}(t)$. Now $u, \widetilde{y} \in H\langle u \rangle^{\mathrm{rc}}$, so $\widetilde{y} < u$ or $\widetilde{y} = u$ or $u < \widetilde{y}$. Suppose $\widetilde{y} < u$. (The other two cases lead to a contradiction in a similar way.) Then for all sufficiently large t with $w'(t) = 0$ we have $y(t) = \widetilde{y}(t) < u(t)$, so $w(t) < 0$, contradicting Lemma 6.3.16 for $f := w$. $\square$

With these lemmas in place, Theorem 6.3.14 now follows quickly:

Proof of Theorem 6.3.14. Let E be a d-maximal Hardy field extension of H; we show that then $y \in E$. Now E is real closed by the remarks after Proposition 5.3.2, and $y - u$ is non-oscillating for all $u \in E$ by Lemma 6.3.18, so y lies in a Hausdorff field extension of E by Lemma 5.1.20, hence y is E-hardian by Lemma 6.3.17 with E in place of H, and thus $y \in E$ by d-maximality of E. $\square$

As an application of Theorem 6.3.14 we record [32, Theorem 8.1]:

Corollary 6.3.19. *Let $\ell \in \mathrm{D}(\mathbb{Q})$ be such that $\ell > \mathbb{R}$ and $\operatorname{trdeg}\big(\mathbb{R}\langle x, \ell\rangle | \mathbb{R}\big) \leqslant 2$. Then $\ell^{\mathrm{inv}} \in \mathrm{D}(\mathbb{Q})$.*

Proof. By Lemma 5.3.8 and the remark preceding it, ℓ^{inv} is $\mathbb{R}(x)$-hardian with

$$\operatorname{trdeg}\big(\mathbb{R}\langle x, \ell^{\mathrm{inv}}\rangle | \mathbb{R}(x)\big) = \operatorname{trdeg}\big(\mathbb{R}\langle x, \ell^{\mathrm{inv}}\rangle | \mathbb{R}\big) - 1 = \operatorname{trdeg}\big(\mathbb{R}\langle x, \ell\rangle | \mathbb{R}\big) - 1 \leqslant 1.$$

Now Theorem 6.3.14 with $H := \mathbb{R}(x)$ and $y := \ell^{\mathrm{inv}}$ yields $y \in \mathrm{D}(H) = \mathrm{D}(\mathbb{Q})$. $\square$

6.4. Application to Filling Holes in Hardy Fields

This section combines the analytic material above with the normalization results of Parts 3 and 4. *Throughout H is a Hardy field with $H \not\subseteq \mathbb{R}$, and $r \in \mathbb{N}^{\geqslant 1}$.* Thus $K := H[i] \subseteq \mathcal{C}^{<\infty}[i]$ is an H-asymptotic extension of H. (Later we impose extra assumptions on H like being real closed with asymptotic integration.) Note that $v(H^\times) \neq \{0\}$: take $f \in H \setminus \mathbb{R}$; then $f' \neq 0$, and if $f \asymp 1$, then $f' \prec 1$.

Evaluating differential polynomials at germs. Any $Q \in K\{Y\}$ of order $\leqslant r$ can be evaluated at any germ $y \in \mathcal{C}^r[i]$ to give a germ $Q(y) \in \mathcal{C}[i]$, with $Q(y) \in \mathcal{C}$ for $Q \in H\{Y\}$ of order $\leqslant r$ and $y \in \mathcal{C}^r$. (See the beginning of Section 6.3.) Here is a variant that we shall need. Let $\phi \in H^\times$; with ∂ denoting the derivation of K, the derivation of the differential field K^ϕ is then $\delta := \phi^{-1}\partial$. We also let δ denote its extension $f \mapsto \phi^{-1}f' \colon \mathcal{C}^1[i] \to \mathcal{C}[i]$, which maps $\mathcal{C}^{n+1}[i]$ into $\mathcal{C}^n[i]$ and $\mathcal{C}^{n+1}$ into $\mathcal{C}^n$, for all n. Thus for $j \leqslant r$ we have the maps

$$\mathcal{C}^r[i] \xrightarrow{\ \delta\ } \mathcal{C}^{r-1}[i] \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } \mathcal{C}^{r-j+1}[i] \xrightarrow{\ \delta\ } \mathcal{C}^{r-j}[i],$$

which by composition yield $\delta^j \colon \mathcal{C}^r[i] \to \mathcal{C}^{r-j}[i]$, mapping $\mathcal{C}^r$ into $\mathcal{C}^{r-j}$. This allows us to define for $Q \in K^\phi\{Y\}$ of order $\leqslant r$ and $y \in \mathcal{C}^r[i]$ the germ $Q(y) \in \mathcal{C}[i]$ by

$$Q(y) := q\big(y, \delta(y), \dots, \delta^r(y)\big) \quad \text{where } Q = q\big(Y, \dots, Y^{(r)}\big) \in K^\phi\big[Y, \dots, Y^{(r)}\big].$$

Note that H^ϕ is a differential subfield of K^ϕ, and if $Q \in H^\phi\{Y\}$ is of order $\leqslant r$ and $y \in \mathcal{C}^r$, then $Q(y) \in \mathcal{C}$.

Lemma 6.4.1. *Let $y \in \mathcal{C}^r[i]$ and $\mathfrak{m} \in K^\times$. Each of the following conditions implies $y \in \mathcal{C}^r[i]^{\preccurlyeq}$:*

(i) *$\phi \preccurlyeq 1$ and $\delta^0(y), \dots, \delta^r(y) \preccurlyeq 1$;*

(ii) $\mathfrak{m} \preccurlyeq 1$ *and* $y \in \mathfrak{m}\,\mathcal{C}^r[i]^{\preccurlyeq}$.

Moreover, if $\mathfrak{m} \preccurlyeq 1$ *and* $(y/\mathfrak{m})^{(0)}, \dots, (y/\mathfrak{m})^{(r)} \prec 1$, *then* $y^{(0)}, \dots, y^{(r)} \prec 1$.

Proof. For (i), use the smallness of the derivation of H and the transformation formulas in [ADH, 5.7] expressing the iterates of ∂ in terms of iterates of δ. For (ii) and the "moreover" part, set $y = \mathfrak{m}z$ with $z = y/\mathfrak{m}$ and use the Product Rule and the smallness of the derivation of K. $\square$

Equations over Hardy fields and over their complexifications. Let $\phi > 0$ be active in H. We recall here from Section 5.3 how the the asymptotic field $K^\phi = H[i]^\phi$ (with derivation $\delta = \phi^{-1}\partial$) is isomorphic to the asymptotic field $K^\circ := H^\circ[i]$ for a certain Hardy field H°: Let $\ell \in \mathcal{C}^1$ be such that $\ell' = \phi$; then $\ell > \mathbb{R}$, $\ell \in \mathcal{C}^{<\infty}$, and $\ell^{\text{inv}} \in \mathcal{C}^{<\infty}$ for the compositional inverse ℓ^{inv} of ℓ. The $\mathbb{C}$-algebra automorphism $f \mapsto f^\circ := f \circ \ell^{\text{inv}}$ of $\mathcal{C}[i]$ (with inverse $g \mapsto g \circ \ell$) maps $\mathcal{C}^n[i]$ and $\mathcal{C}^n$ onto themselves, and hence restricts to a $\mathbb{C}$-algebra automorphism of $\mathcal{C}^{<\infty}[i]$ and $\mathcal{C}^{<\infty}$ mapping $\mathcal{C}^{<\infty}$ onto itself. Moreover,

$$(\partial, \circ, \delta) \qquad (f^\circ)' \;=\; (\phi^\circ)^{-1}(f')^\circ \;=\; \delta(f)^\circ \qquad \text{for } f \in \mathcal{C}^1[i].$$

Thus we have an isomorphism $f \mapsto f^\circ : (\mathcal{C}^{<\infty}[i])^\phi \to \mathcal{C}^{<\infty}[i]$ of differential rings, and likewise with $\mathcal{C}^{<\infty}$ in place of $\mathcal{C}^{<\infty}[i]$. As already pointed out in Section 5.3,

$$H^\circ \;:=\; \{h^\circ : h \in H\} \;\subseteq\; \mathcal{C}^{<\infty}$$

is a Hardy field, and $f \mapsto f^\circ$ restricts to an isomorphism $H^\phi \to H^\circ$ of pre-H-fields, and to an isomorphism $K^\phi \to K^\circ$ of asymptotic fields. We extend the latter to the isomorphism

$$Q \mapsto Q^\circ \;:\; K^\phi\{Y\} \to K^\circ\{Y\}$$

of differential rings given by $Y^\circ = Y$, which restricts to a differential ring isomorphism $H^\phi\{Y\} \to H^\circ\{Y\}$. Using the identity $(\partial, \circ, \delta)$ it is routine to check that for $Q \in K^\phi\{Y\}$ of order $\leqslant r$ and $y \in \mathcal{C}^r[i]$,

$$Q(y)^\circ \;=\; (Q^\circ)(y^\circ).$$

This allows us to translate algebraic differential equations over K into algebraic differential equations over K°: Let $P \in K\{Y\}$ have order $\leqslant r$ and let $y \in \mathcal{C}^r[i]$.

Lemma 6.4.2. $P(y)^\circ = P^\phi(y)^\circ = P^{\phi\circ}(y^\circ)$ *where* $P^{\phi\circ} := (P^\phi)^\circ \in K^\circ\{Y\}$, *hence*

$$P(y) = 0 \iff P^{\phi\circ}(y^\circ) = 0.$$

Moreover, $y \prec \mathfrak{m} \iff y^\circ \prec \mathfrak{m}^\circ$, for $\mathfrak{m} \in K^\times$, so asymptotic side conditions are automatically taken care of under this "translation". Also, if $\phi \preccurlyeq 1$ and $y^\circ \in \mathcal{C}^r[i]^{\preccurlyeq}$, then $y \in \mathcal{C}^r[i]^{\preccurlyeq}$, by Lemma 6.4.1(i) and $(\partial, \circ, \delta)$.

In the rest of this section $H \supseteq \mathbb{R}$ is real closed with asymptotic integration. Then H is an H-field, and $K = H[i]$ is the algebraic closure of H, a d-valued field with small derivation extending H, constant field $\mathbb{C}$, and value group $\Gamma := v(K^\times) = v(H^\times)$.

Slots in Hardy fields and compositional conjugation. *In this subsection we let $\phi > 0$ be active in H; as in the previous subsection we take $\ell \in \mathcal{C}^1$ such that $\ell' = \phi$ and use the superscript $\circ$ accordingly: $f^\circ := f \circ \ell^{\text{inv}}$ for $f \in \mathcal{C}[i]$.*

Let $(P, \mathfrak{m}, \widehat{a})$ be a slot in K of order r, so $\widehat{a} \notin K$ is an element of an immediate asymptotic extension $\widehat{K}$ of K with $P \in Z(K, \widehat{a})$ and $\widehat{a} \prec \mathfrak{m}$. We associate to $(P, \mathfrak{m}, \widehat{a})$ a slot in K° as follows: choose an immediate asymptotic extension $\widehat{K}^\circ$

of K° and an isomorphism $\widehat{f} \mapsto \widehat{f}^\circ\colon \widehat{K}^\phi \to \widehat{K}^\circ$ of asymptotic fields extending the isomorphism $f \mapsto f^\circ\colon K^\phi \to K^\circ$. Then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a slot in K° of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. The equivalence class of $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ does not depend on the choice of $\widehat{K}^\circ$ and the isomorphism $\widehat{K}^\phi \to \widehat{K}^\circ$. If $(P, \mathfrak{m}, \widehat{a})$ is a hole in K, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a hole in K°, and likewise with "minimal hole" in place of "hole". Moreover, by Lemmas 3.1.19, 3.3.20, and 3.3.40:

Lemma 6.4.3. *If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, then so is $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$, and likewise with "quasi-linear" and "special" in place of "Z-minimal". If $(P, \mathfrak{m}, \widehat{a})$ is steep and $\phi \preccurlyeq 1$, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is steep, and likewise with "deep", "normal", and "strictly normal" in place of "steep".*

Next, let $(P, \mathfrak{m}, \widehat{a})$ be a slot in H of order r, so $\widehat{a} \notin H$ is an element of an immediate asymptotic extension $\widehat{H}$ of H with $P \in Z(H, \widehat{a})$ and $\widehat{a} \prec \mathfrak{m}$. We associate to $(P, \mathfrak{m}, \widehat{a})$ a slot in H° as follows: choose an immediate asymptotic extension $\widehat{H}^\circ$ of H° and an isomorphism $\widehat{f} \mapsto \widehat{f}^\circ\colon \widehat{H}^\phi \to \widehat{H}^\circ$ of asymptotic fields extending the isomorphism $f \mapsto f^\circ\colon H^\phi \to H^\circ$. Then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a slot in H° of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. The equivalence class of $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ does not depend on the choice of $\widehat{H}^\circ$ and the isomorphism $\widehat{H}^\phi \to \widehat{H}^\circ$. If $(P, \mathfrak{m}, \widehat{a})$ is a hole in H, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a hole in H°, and likewise with "minimal hole" in place of "hole". Lemma 6.4.3 goes through in this setting. Also, recalling Lemma 5.3.6, if H is Liouville closed and $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is ultimate.

Moreover, by Lemmas 4.3.5 and 4.3.28, and Corollaries 4.5.23 and 4.5.39:

Lemma 6.4.4.

(i) *If $\phi \preccurlyeq 1$ and $(P, \mathfrak{m}, \widehat{a})$ is split-normal, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is split-normal; likewise with "split-normal" replaced by "(almost) strongly split-normal".*

(ii) *If $\phi \prec 1$ and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and repulsive-normal, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is repulsive-normal; likewise with "repulsive-normal" replaced by "(almost) strongly repulsive-normal".*

Reformulations. We reformulate here some results of Sections 6.2 and 6.3 to facilitate their use. As in Section 3.1 we set for $\mathfrak{v} \in K^\times$, $\mathfrak{v} \prec 1$:

$$\Delta(\mathfrak{v}) \ := \ \big\{\gamma \in \Gamma : \gamma = o(v\mathfrak{v})\big\},$$

a proper convex subgroup of Γ. In the next lemma, $P \in K\{Y\}$ has order r and $P = Q - R$, where $Q, R \in K\{Y\}$ and Q is homogeneous of degree 1 and order r. We set $w := \operatorname{wt}(P)$, so $w \geqslant r \geqslant 1$.

Lemma 6.4.5. *Suppose that L_Q splits strongly over K, $\mathfrak{v}(L_Q) \prec^\flat 1$, and*

$$R \prec_\Delta \mathfrak{v}(L_Q)^{w+1} Q, \quad \Delta := \Delta\big(\mathfrak{v}(L_Q)\big).$$

Then $P(y) = 0$ and $y', \dots, y^{(r)} \preccurlyeq 1$ for some $y \prec \mathfrak{v}(L_Q)^w$ in $\mathcal{C}^{<\infty}[i]$. Moreover:

(i) *if $P, Q \in H\{Y\}$, then there is such y in $\mathcal{C}^{<\infty}$;*

(ii) *if $H \subseteq \mathcal{C}^\infty$, then for any $y \in \mathcal{C}^r[i]^\preccurlyeq$ with $P(y) = 0$ we have $y \in \mathcal{C}^\infty[i]$; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Proof. Set $\mathfrak{v} := |\mathfrak{v}(L_Q)| \in H^>$, so $\mathfrak{v} \asymp \mathfrak{v}(L_Q)$. Take $f \in K^\times$ such that $A := f^{-1} L_Q$ is monic; then $\mathfrak{v}(A) = \mathfrak{v}(L_Q) \asymp \mathfrak{v}$ and $f^{-1} R \prec_\Delta f^{-1} \mathfrak{v}^{w+1} Q \asymp \mathfrak{v}^w$. We have $A = (\partial - \phi_1) \cdots (\partial - \phi_r)$ where $\phi_j \in K$ and $\operatorname{Re} \phi_j \succcurlyeq \mathfrak{v}^\dagger \succcurlyeq 1$ for $j = 1, \dots, r$ by the strong splitting assumption. Also $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$ by Corollary 3.1.6. The claims now follow from various results in Section 6.2 applied to the equation $A(y) = f^{-1}R(y)$, $y \prec 1$ in the role of $(*)$, using also Corollary 6.3.5. $\square$

Lemma 6.4.6. *Let $(P, \mathfrak{n}, \widehat{h})$ be a slot in H of order r and let ϕ be active in H, $0 < \phi \preccurlyeq 1$, such that $(P^{\phi}, \mathfrak{n}, \widehat{h})$ is strongly split-normal. Then for some y in $\mathcal{C}^{<\infty}$,*

$$P(y) \;=\; 0, \quad y \;\prec\; \mathfrak{n}, \quad y \in \mathfrak{n}\,(\mathcal{C}^{r})^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^{\infty}$, then there exists such y in $\mathcal{C}^{\infty}$, and likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.

Proof. First we consider the case $\phi = 1$. Replace $(P, \mathfrak{n}, \widehat{h})$ by $(P_{\times\mathfrak{n}}, 1, \widehat{h}/\mathfrak{n})$ to arrange $\mathfrak{n} = 1$. Then L_P has order r, $\mathfrak{v} := \mathfrak{v}(L_P) \prec^{\flat} 1$, and $P = Q - R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, $L_Q \in H[\partial]$ splits strongly over K, and $R \prec_{\Delta} \mathfrak{v}^{w+1}P_1$, where $\Delta := \Delta(\mathfrak{v})$ and $w := \operatorname{wt}(P)$. Now $P_1 = Q - R_1$, so $\mathfrak{v} \sim \mathfrak{v}(L_Q)$ by Lemma 3.1.1(ii), and thus $\Delta = \Delta\big(\mathfrak{v}(L_Q)\big)$. Lemma 6.4.5 gives y in $\mathcal{C}^{<\infty}$ such that $y \prec \mathfrak{v}^{w} \prec 1$, $P(y) = 0$, and $y^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. Then y has for $\mathfrak{n} = 1$ the properties displayed in the lemma.

Now suppose ϕ is arbitrary. Employing $(\ \ )^{\circ}$ as explained earlier in this section, the slot $(P^{\phi\circ}, \mathfrak{n}^{\circ}, \widehat{h}^{\circ})$ in the Hardy field H° is strongly split-normal, hence by the case $\phi = 1$ we have $z \in \mathcal{C}^{<\infty}$ with $P^{\phi\circ}(z) = 0$, $z \prec \mathfrak{n}^{\circ}$, and $(z/\mathfrak{n}^{\circ})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. Take $y \in \mathcal{C}^{<\infty}$ with $y^{\circ} = z$. Then $P(y) = 0$, $y \prec \mathfrak{n}$, and $y \in \mathfrak{n}\,(\mathcal{C}^{r})^{\preccurlyeq}$ by Lemma 6.4.2 and a subsequent remark. Moreover, if $\phi, z \in \mathcal{C}^{\infty}$, then $y \in \mathcal{C}^{\infty}$, and likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$. $\square$

In the next "complex" version, $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order r with $\mathfrak{m} \in H^{\times}$.

Lemma 6.4.7. *Let ϕ be active in H, $0 < \phi \preccurlyeq 1$, such that the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in K^{ϕ} is strictly normal, and its linear part splits strongly over K^{ϕ}. Then for some $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ we have*

$$P(y) \;=\; 0, \quad y \;\prec\; \mathfrak{m}, \quad y \in \mathfrak{m}\,\mathcal{C}^{r}[\mathrm{i}]^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^{\infty}$, then there is such y in $\mathcal{C}^{\infty}[\mathrm{i}]$. If $H \subseteq \mathcal{C}^{\omega}$, then there is such y in $\mathcal{C}^{\omega}[\mathrm{i}]$.

Proof. Consider first the case $\phi = 1$. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ we arrange $\mathfrak{m} = 1$. Set $L := L_P \in K[\partial]$, $Q := P_1$, and $R := P - Q$. Since $(P, 1, \widehat{a})$ is strictly normal, we have $\operatorname{order}(L) = r$, $\mathfrak{v} := \mathfrak{v}(L) \prec^{\flat} 1$, and $R \prec_{\Delta} \mathfrak{v}^{w+1}Q$ where $\Delta := \Delta(\mathfrak{v})$, $w := \operatorname{wt}(P)$. As L splits strongly over K, Lemma 6.4.5 gives y in $\mathcal{C}^{<\infty}[\mathrm{i}]$ such that $P(y) = 0$, $y \prec \mathfrak{v}^{w} \prec 1$, and $y^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. For the last part of the lemma, use the last part of Lemma 6.4.5. The general case reduces to this special case as in the proof of Lemma 6.4.6. $\square$

Finding germs in holes. *In this subsection $\widehat{H}$ is an immediate asymptotic extension of H.* This fits into the setting of Section 4.3 on split-normal slots: $K = H[\mathrm{i}]$ and $\widehat{H}$ have H as a common asymptotic subfield and $\widehat{K} := \widehat{H}[\mathrm{i}]$ as a common asymptotic extension, $\widehat{H}$ is an H-field, and $\widehat{K}$ is d-valued. *Assume also that H is ω-free.* Thus K is ω-free by [ADH, 11.7.23]. *Let $(P, \mathfrak{m}, \widehat{a})$ with $\mathfrak{m} \in H^{\times}$ and $\widehat{a} \in \widehat{K} \setminus K$ be a minimal hole in K of order $r \geqslant 1$. Take $\widehat{b}, \widehat{c} \in \widehat{H}$ so that $\widehat{a} = \widehat{b} + \widehat{c}\,\mathrm{i}$.*

Proposition 6.4.8. *Suppose* $\deg P > 1$*. Then for some $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ we have*

$$P(y) \;=\; 0, \quad y \;\prec\; \mathfrak{m}, \quad y \in \mathfrak{m}\,\mathcal{C}^{r}[\mathrm{i}]^{\preccurlyeq}.$$

If $\mathfrak{m} \preccurlyeq 1$, then $y \prec \mathfrak{m}$ and $y \in \mathcal{C}^{r}[\mathrm{i}]^{\preccurlyeq}$ for such y. Moreover, if $H \subseteq \mathcal{C}^{\infty}$, then we can take such y in $\mathcal{C}^{\infty}[\mathrm{i}]$, and if $H \subseteq \mathcal{C}^{\omega}$, then we can take such y in $\mathcal{C}^{\omega}[\mathrm{i}]$.

Proof. Lemma 4.3.31 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\mathfrak{n} \in H^\times$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the hole $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K^ϕ is strictly normal and its linear part splits strongly over K^ϕ. Lemma 6.4.7 applied to $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ in place of $(P, \mathfrak{m}, \widehat{a})$ yields $z \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $P_{+a}(z) = 0$, $z \prec \mathfrak{n}$ and $(z/\mathfrak{n})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. Lemma 6.4.1(ii) applied to $z/\mathfrak{m}$, $\mathfrak{n}/\mathfrak{m}$ in place of y, $\mathfrak{m}$, respectively, yields $(z/\mathfrak{m})^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r$. Also, $a \prec \mathfrak{m}$ (in K), hence $(a/\mathfrak{m})^{(j)} \prec 1$ for $j = 0, \dots, r$. Set $y := a + z$; then $P(y) = 0$, $y \prec \mathfrak{m}$, and $(y/\mathfrak{m})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. For the rest use Lemma 6.4.1(ii) and the last statement in Lemma 6.4.7. □

Next we treat the linear case:

Proposition 6.4.9. *Suppose* $\deg P = 1$. *Then for some* $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ *we have*

$$P(y) \ = \ 0, \quad y \ \prec \ \mathfrak{m}, \quad (y/\mathfrak{m})' \ \preccurlyeq \ 1.$$

If $\mathfrak{m} \preccurlyeq 1$, *then* $y \prec 1$ *and* $y' \preccurlyeq 1$ *for each such* y. *Moreover, if* $H \subseteq \mathcal{C}^\infty$, *then we can take such* y *in* $\mathcal{C}^\infty[\mathrm{i}]$, *and if* $H \subseteq \mathcal{C}^\omega$, *then we can take such* y *in* $\mathcal{C}^\omega[\mathrm{i}]$.

Proof. We have $r = 1$ by Corollary 3.2.8. If $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$, then Lemma 4.3.32 applies, and we can argue as in the proof of Proposition 6.4.8, using this lemma instead of Lemma 4.3.31. We reduce the general case to this special case as follows: Set $H_1 := \mathrm{D}(H)$; then H_1 is an $\upomega$-free Hardy field by Theorem 1.4.1, and $K_1 := H_1[\mathrm{i}]$ satisfies $\partial K_1 = K_1$ and $\mathrm{I}(K_1) \subseteq K_1^\dagger$, by Corollary 5.5.19. Moreover, by Corollary 6.3.9, if $H \subseteq \mathcal{C}^\infty$, then $H_1 \subseteq \mathcal{C}^\infty$, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. The newtonization $\widehat{H}_1$ of H_1 is an immediate asymptotic extension of H_1, and $\widehat{K}_1 := \widehat{H}_1[\mathrm{i}]$ is newtonian [ADH, 14.5.7]. Corollary 3.2.29 gives an embedding $K\langle \widehat{a} \rangle \to \widehat{K}_1$ over K; let $\widehat{a}_1$ be the image of $\widehat{a}$ under this embedding. If $\widehat{a}_1 \in K_1$, then we are done by taking $y := \widehat{a}_1$, so we may assume $\widehat{a}_1 \notin K_1$. Then $(P, \mathfrak{m}, \widehat{a}_1)$ is a minimal hole in K_1, and the above applies with H, K, $\widehat{a}$ replaced by H_1, K_1, $\widehat{a}_1$, respectively. □

We can improve on these results in a useful way:

Corollary 6.4.10. *Suppose* $\widehat{a} \sim a \in K$. *Then for some* $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ *we have*

$$P(y) \ = \ 0, \quad y \ \sim \ a, \quad (y/a)^{(j)} \ \prec \ 1 \ \ \textit{for } j = 1, \dots, r.$$

If $H \subseteq \mathcal{C}^\infty$, *then there is such* y *in* $\mathcal{C}^\infty[\mathrm{i}]$. *If* $H \subseteq \mathcal{C}^\omega$, *then there is such* y *in* $\mathcal{C}^\omega[\mathrm{i}]$.

Proof. Take $a_1 \in K$ and $\mathfrak{n} \in H^\times$ with $\mathfrak{n} \asymp \widehat{a} - a \sim a_1$, and set $b := a + a_1$. Then $(P_{+b}, \mathfrak{n}, \widehat{a} - b)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$. Propositions 6.4.8 and 6.4.9 give $z \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $P(b + z) = 0$, $z \prec \mathfrak{n}$ and $(z/\mathfrak{n})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. We have $(a_1/a)^{(j)} \prec 1$ for $j = 0, \dots, r$, since K has small derivation. Likewise, $(\mathfrak{n}/a)^{(j)} \prec 1$ for $j = 0, \dots, r$, and hence $(z/a)^{(j)} \prec 1$ for $j = 0, \dots, r$, by $z/a = (z/\mathfrak{n}) \cdot (\mathfrak{n}/a)$ and the Product Rule. So $y := b + z$ has the desired property. The rest follows from the "moreover" parts of these propositions. □

Remark 6.4.11. Suppose we replace our standing assumption that H is $\upomega$-free and $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K by the assumption that H is $\uplambda$-free, $\partial K = K$, $\mathrm{I}(K) \subseteq K^\dagger$, and $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order and degree 1. Then Proposition 6.4.9 and Corollary 6.4.10 go through by the remark following Lemma 4.3.32.

Now also drawing upon Theorem 4.3.33, we arrive at the main result of this section:

Corollary 6.4.12. *Suppose H is 1-linearly newtonian. Then one of the following two conditions is satisfied:*

(i) *$\widehat{b} \notin H$, and there are $Q \in Z(H, \widehat{b})$ of minimal complexity and $y \in \mathcal{C}^{<\infty}$ such that $Q(y) = 0$ and $y \prec \mathfrak{m}$;*

(ii) *$\widehat{c} \notin H$, and there are $R \in Z(H, \widehat{c})$ of minimal complexity and $y \in \mathcal{C}^{<\infty}$ such that $R(y) = 0$ and $y \prec \mathfrak{m}$.*

If $H \subseteq \mathcal{C}^{\infty}$, then there is such y in $\mathcal{C}^{\infty}$, and likewise with $\mathcal{C}^{\infty}$ replaced by $\mathcal{C}^{\omega}$.

Proof. Suppose $\deg P > 1$, or $\widehat{b} \notin H$ and $Z(H, \widehat{b})$ has an element of order 1, or $\widehat{c} \notin H$ and $Z(H, \widehat{c})$ has an element of order 1. Let ϕ range over active elements of H with $0 < \phi \preccurlyeq 1$. By the "moreover" part of Theorem 4.3.33, one of the following holds:

(1) $\widehat{b} \notin H$ and there exist ϕ and a Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H with a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is strongly split-normal;

(2) $\widehat{c} \notin H$ and there exist ϕ and a Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H with a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is strongly split-normal.

Suppose $\widehat{b} \notin H$ and ϕ, Q, b are as in (1); then Lemma 6.4.6 applied to $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ in place of $(P, \mathfrak{n}, \widehat{h})$ yields $z \in \mathcal{C}^{<\infty}$ with $Q_{+b}(z) = 0$, $z \prec \mathfrak{n}$; hence $Q(y) = 0$, $y \prec \mathfrak{m}$ for $y := b + z$, so (i) holds. Similarly, (2) implies (ii).

Suppose now that $\deg P = 1$, that if $\widehat{b} \notin H$, then $Z(H, \widehat{b})$ has no element of order 1, and that if $\widehat{c} \notin H$, then $Z(H, \widehat{c})$ has no element of order 1. Since $\deg P = 1$, Proposition 6.4.9 gives $z \in \mathcal{C}^{<\infty}[i]$ such that $P(z) = 0$ and $z \prec \mathfrak{m}$. Recall also that P has order 1 by Corollary 3.2.8. Consider now the case $\widehat{b} \notin H$. In view of $P(\widehat{a}) = 0$ and $P(z) = 0$ we obtain from Example 1.1.7 and Remark 1.1.9 a $Q \in H\{Y\}$ of degree 1 and order 1 or 2 such that $Q(\widehat{b}) = 0$ and $Q(y) = 0$ for $y := \operatorname{Re} z \prec \mathfrak{m}$. But $Z(H, \widehat{b})$ has no element of order 1, so $\operatorname{order} Q = 2$ and $Q \in Z(H, \widehat{b})$ has minimal complexity. Thus (i) holds. If $\widehat{c} \notin H$, then the same reasoning shows that (ii) holds. □

Is y as in (i) or (ii) of Corollary 6.4.12 hardian over H? At this stage we cannot claim this. In the next section we introduce weights and their corresponding norms as a more refined tool. This will allow us to obtain Corollary 6.5.20 as a key approximation result for later use.

6.5. Weights

In this section we prove Proposition 6.5.14 to strengthen Lemma 6.2.5. This uses the material on repulsive-normal slots from Section 4.5, but we also need more refined norms for differentiable functions, to which we turn now.

Weighted spaces of differentiable functions. *In this subsection we fix $r \in \mathbb{N}$ and a* weight *function $\tau \in \mathcal{C}_a[i]^{\times}$.* For $f \in \mathcal{C}^r_a[i]$ we set

$$\|f\|^{\tau}_{a;r} \;:=\; \max\bigl\{\|\tau^{-1} f\|_a, \|\tau^{-1} f'\|_a, \dots, \|\tau^{-1} f^{(r)}\|_a\bigr\} \;\in\; [0, +\infty],$$

and $\|f\|^{\tau}_a := \|f\|^{\tau}_{a;0}$ for $f \in \mathcal{C}_a[i]$. Then

$$\mathcal{C}^r_a[i]^{\tau} \;:=\; \bigl\{f \in \mathcal{C}^r_a[i] : \|f\|^{\tau}_{a;r} < +\infty\bigr\}$$

is a $\mathbb{C}$-linear subspace of

$$\mathcal{C}_a[i]^{\tau} \;:=\; \mathcal{C}^0_a[i]^{\tau} \;=\; \tau\, \mathcal{C}_a[i]^{\mathrm{b}} \;=\; \bigl\{f \in \mathcal{C}_a[i] : f \preccurlyeq \tau\bigr\}.$$

Below we consider the $\mathbb{C}$-linear space $\mathcal{C}^r_a[\mathrm{i}]^\tau$ to be equipped with the norm

$$f \mapsto \|f\|^\tau_{a;r}.$$

Recall from Section 6.1 the convention $b \cdot \infty = \infty \cdot b = \infty$ for $b \in [0, \infty]$. Note that

$$\|fg\|^\tau_{a;r} \leqslant 2^r \|f\|_{a;r} \|g\|^\tau_{a;r} \quad \text{for } f, g \in \mathcal{C}^r_a[\mathrm{i}], \tag{6.5.1}$$

so $\mathcal{C}^r_a[\mathrm{i}]^\tau$ is a $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$-submodule of $\mathcal{C}^r_a[\mathrm{i}]$. Note also that $\|1\|^\tau_{a;r} = \|\tau^{-1}\|_a$, hence

$$\|f\|^\tau_{a;r} \leqslant 2^r \|f\|_{a;r} \|\tau^{-1}\|_a \quad \text{for } f \in \mathcal{C}^r_a[\mathrm{i}]$$

and

$$\tau^{-1} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \iff 1 \in \mathcal{C}^r_a[\mathrm{i}]^\tau \iff \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} \subseteq \mathcal{C}^r_a[\mathrm{i}]^\tau.$$

We have

$$\|f\|_{a;r} \leqslant \|f\|^\tau_{a;r} \|\tau\|_a \quad \text{for } f \in \mathcal{C}^r_a[\mathrm{i}], \tag{6.5.2}$$

and thus

$$\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \iff \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\tau^{-1}} \implies \mathcal{C}^r_a[\mathrm{i}]^\tau \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}. \tag{6.5.3}$$

Hence if $\tau, \tau^{-1} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, then $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} = \mathcal{C}^r_a[\mathrm{i}]^\tau$, and the norms $\|\cdot\|^\tau_{a;r}$ and $\|\cdot\|_{a;r}$ on this $\mathbb{C}$-linear space are equivalent. (In later use, $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, $\tau^{-1} \notin \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$.) If $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, then $\mathcal{C}^r_a[\mathrm{i}]^\tau$ is an ideal of the commutative ring $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. From (6.5.1) and (6.5.2) we obtain

$$\|fg\|^\tau_{a;r} \leqslant 2^r \|\tau\|_a \|f\|^\tau_{a;r} \|g\|^\tau_{a;r} \quad \text{for } f, g \in \mathcal{C}^r_a[\mathrm{i}].$$

For $f \in \mathcal{C}^{r+1}_a[\mathrm{i}]^\tau$ we have $\|f\|^\tau_{a;r}, \|f'\|^\tau_{a;r} \leqslant \|f\|^\tau_{a;r+1}$. From (6.5.2) and (6.5.3):

Lemma 6.5.1. *Suppose $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ (so $\mathcal{C}^r_a[\mathrm{i}]^\tau \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$) and $f \in \mathcal{C}^r_a[\mathrm{i}]^\tau$. If (f_n) is a sequence in $\mathcal{C}^r_a[\mathrm{i}]^\tau$ and $f_n \to f$ in $\mathcal{C}^r_a[\mathrm{i}]^\tau$, then also $f_n \to f$ in $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$.*

This is used to show:

Lemma 6.5.2. *Suppose $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Then the $\mathbb{C}$-linear space $\mathcal{C}^r_a[\mathrm{i}]^\tau$ equipped with the norm $\|\cdot\|^\tau_{a;r}$ is complete.*

Proof. We proceed by induction on r. Let (f_n) be a cauchy sequence in the normed space $\mathcal{C}_a[\mathrm{i}]^\tau$. Then the sequence $(\tau^{-1}f_n)$ in the Banach space $\mathcal{C}^0_a[\mathrm{i}]^{\mathrm{b}}$ is cauchy, hence has a limit $g \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, so with $f := \tau g \in \mathcal{C}_a[\mathrm{i}]^\tau$ we have $\tau^{-1}f_n \to \tau^{-1}f$ in $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and hence $f_n \to f$ in $\mathcal{C}_a[\mathrm{i}]^\tau$. Thus the lemma holds for $r = 0$. Suppose the lemma holds for a certain value of r, and let (f_n) be a cauchy sequence in $\mathcal{C}^{r+1}_a[\mathrm{i}]^\tau$. Then (f'_n) is a cauchy sequence in $\mathcal{C}^r_a[\mathrm{i}]^\tau$ and hence has a limit $g \in \mathcal{C}^r_a[\mathrm{i}]^\tau$, by inductive hypothesis. By Lemma 6.5.1, $f'_n \to g$ in $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Now (f_n) is also a cauchy sequence in $\mathcal{C}_a[\mathrm{i}]^\tau$, hence has a limit $f \in \mathcal{C}_a[\mathrm{i}]^\tau$ (by the case $r = 0$), and by Lemma 6.5.1 again, $f_n \to f$ in $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Thus f is differentiable and $f' = g$ by [57, (8.6.4)]. This yields $f_n \to f$ in $\mathcal{C}^{r+1}_a[\mathrm{i}]^\tau$. $\square$

Lemma 6.5.3. *Suppose $\tau \in \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. If $f \in \mathcal{C}^r_a[\mathrm{i}]$ and $f^{(k)} \preccurlyeq \tau^{r-k+1}$ for $k = 0, \dots, r$, then $f\tau^{-1} \in \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. (Thus $\mathcal{C}^r_a[\mathrm{i}]^{\tau^{r+1}} \subseteq \tau\,\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$.)*

Proof. Let $Q^n_k \in \mathbb{Q}\{X\}$ $(0 \leqslant k \leqslant n)$ be as in Lemma 1.1.11. Although $\mathcal{C}^r_a[\mathrm{i}]$ is not closed under taking derivatives, the proof of that lemma and the computation leading to Corollary 1.1.12 does give for $f \in \mathcal{C}^r_a[\mathrm{i}]$ and $n \leqslant r$:

$$(f\tau^{-1})^{(n)} = \sum_{k=0}^{n} Q^n_k(\tau) f^{(k)} \tau^{k-n-1}.$$

Now use that $Q^n_k(\tau) \preccurlyeq 1$ for $n \leqslant r$ and $k = 0, \dots, n$. $\square$

Next we generalize the inequality (6.5.1):

Lemma 6.5.4. *Let $f_1, \dots, f_{m-1}, g \in \mathcal{C}^r_a[\mathrm{i}]$, $m \geqslant 1$; then*

$$\|f_1 \cdots f_{m-1} g\|^{\tau}_{a;r} \ \leqslant\ m^r \, \|f_1\|_{a;r} \cdots \|f_{m-1}\|_{a;r} \, \|g\|^{\tau}_{a;r}.$$

Proof. Use the generalized Product Rule [ADH, p. 199] and the well-known identity $\sum \frac{n!}{i_1! \cdots i_m!} = m^n$ with the sum over all $(i_1, \dots, i_m) \in \mathbb{N}^m$ with $i_1 + \cdots + i_m = n$. $\square$

With $\boldsymbol{i}$ ranging over $\mathbb{N}^{1+r}$, let $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ (all $P_{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]$) be a polynomial in $\mathcal{C}_a[\mathrm{i}]\big[Y, Y', \dots, Y^{(r)}\big]$; for $f \in \mathcal{C}^r_a[\mathrm{i}]$ we have $P(f) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]$. (See also the beginning of Section 6.2.) We set

$$\|P\|_a \ :=\ \max_{\boldsymbol{i}} \|P_{\boldsymbol{i}}\|_a \in [0, \infty].$$

In the rest of this subsection we assume $\|P\|_a < \infty$, that is, $P \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\big[Y, \dots, Y^{(r)}\big]$. Hence if $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, $P(0) \in \mathcal{C}_a[\mathrm{i}]^{\tau}$, and $f \in \mathcal{C}^r_a[\mathrm{i}]^{\tau}$, then $P(f) \in \mathcal{C}_a[\mathrm{i}]^{\tau}$. Here are weighted versions of Lemma 6.1.2 and 6.1.3:

Lemma 6.5.5. *Suppose P is homogeneous of degree $d \geqslant 1$, and let $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and $f \in \mathcal{C}^r_a[\mathrm{i}]^{\tau}$. Then*

$$\|P(f)\|^{\tau}_a \ \leqslant\ \binom{d+r}{r} \cdot \|P\|_a \cdot \|f\|^{d-1}_{a;r} \cdot \|f\|^{\tau}_{a;r}.$$

Proof. For $j = 0, \dots, r$ we have $\|f^{(j)}\|_a \leqslant \|f\|_{a;r}$ and $\|f^{(j)}\|^{\tau}_a \leqslant \|f\|^{\tau}_{a;r}$. Now $f^{\boldsymbol{i}}$, where $\boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{r+1}$ and $i_0 + \cdots + i_r = d$, is a product of d such factors $f^{(j)}$, so Lemma 6.5.4 with $m := d$, $r := 0$, gives

$$\|f^{\boldsymbol{i}}\|^{\tau}_a \ \leqslant\ \|f\|^{d-1}_{a;r} \cdot \|f\|^{\tau}_{a;r}.$$

It remains to note that by (6.5.1) we have $\|P_{\boldsymbol{i}} f^{\boldsymbol{i}}\|^{\tau}_a \leqslant \|P_{\boldsymbol{i}}\|_a \cdot \|f^{\boldsymbol{i}}\|^{\tau}_a$. $\square$

Corollary 6.5.6. *Let $1 \leqslant d \leqslant e$ in $\mathbb{N}$ be such that $P_{\boldsymbol{i}} = 0$ if $|\boldsymbol{i}| < d$ or $|\boldsymbol{i}| > e$. Then for $f \in \mathcal{C}^r_a[\mathrm{i}]^{\tau}$ and $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have*

$$\|P(f)\|^{\tau}_a \ \leqslant\ D \cdot \|P\|_a \cdot \big(\|f\|^{d-1}_{a;r} + \cdots + \|f\|^{e-1}_{a;r}\big) \cdot \|f\|^{\tau}_{a;r}$$

where $D = D(d, e, r) := \binom{e+r+1}{r+1} - \binom{d+r}{r+1} \in \mathbb{N}^{\geqslant 1}$.

Doubly-twisted integration. In this subsection we adopt the setting in *Twisted integration* of Section 6.1. Thus $\phi \in \mathcal{C}_a[\mathrm{i}]$ and $\Phi = \partial_a^{-1}\phi$. Let $\tau \in \mathcal{C}^1_a$ satisfy $\tau(s) > 0$ for $s \geqslant a$, and set $\widetilde{\phi} := \phi - \tau^{\dagger} \in \mathcal{C}_a[\mathrm{i}]$ and $\widetilde{\Phi} := \partial_a^{-1}\widetilde{\phi}$. Thus

$$\widetilde{\Phi}(t) \ =\ \int_a^t (\phi - \tau^{\dagger})(s)\, ds \ =\ \Phi(t) - \log \tau(t) + \log \tau(a) \qquad \text{for } t \geqslant a.$$

Consider the right inverses $B, \widetilde{B} \colon \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}^1_a[\mathrm{i}]$ to, respectively, $\partial - \phi \colon \mathcal{C}^1_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ and $\partial - \widetilde{\phi} \colon \mathcal{C}^1_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$, given by

$$B \ :=\ \mathrm{e}^{\Phi} \circ \partial_a^{-1} \circ \mathrm{e}^{-\Phi}, \quad \widetilde{B} \ :=\ \mathrm{e}^{\widetilde{\Phi}} \circ \partial_a^{-1} \circ \mathrm{e}^{-\widetilde{\Phi}}.$$

For $f \in \mathcal{C}_a[i]$ and $t \geqslant a$ we have

$$\begin{aligned}
\widetilde{B}f(t) &= e^{\widetilde{\Phi}(t)} \int_a^t e^{-\widetilde{\Phi}(s)} f(s)\, ds \\
&= \mathfrak{r}(t)^{-1}\mathfrak{r}(a)\, e^{\Phi(t)} \int_a^t e^{-\Phi(s)}\, \mathfrak{r}(s)\mathfrak{r}(a)^{-1} f(s)\, ds \\
&= \mathfrak{r}(t)^{-1}\, e^{\Phi(t)} \int_a^t e^{-\Phi(s)}\, \mathfrak{r}(s) f(s)\, ds \ = \ \mathfrak{r}^{-1}(t)\big(B(\mathfrak{r}f)\big)(t)
\end{aligned}$$

and so $\widetilde{B} = \mathfrak{r}^{-1} \circ B \circ \mathfrak{r}$. Hence if $\widetilde{\phi}$ is attractive, then $B_{\ltimes \mathfrak{r}} := \mathfrak{r}^{-1} \circ B \circ \mathfrak{r}$ maps $\mathcal{C}_a[i]^{b}$ into $\mathcal{C}_a[i]^{b} \cap \mathcal{C}_a^1[i]$, and the operator $B_{\ltimes\mathfrak{r}} \colon \mathcal{C}_a[i]^{b} \to \mathcal{C}_a[i]^{b}$ is continuous with $\|B_{\ltimes\mathfrak{r}}\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\widetilde{\phi}}\big\|_a$; if in addition $\widetilde{\phi} \in \mathcal{C}_a^r[i]$, then $B_{\ltimes\mathfrak{r}}$ maps $\mathcal{C}_a[i]^{b} \cap \mathcal{C}_a^r[i]$ into $\mathcal{C}_a[i]^{b} \cap \mathcal{C}_a^{r+1}[i]$. Note that if $\phi \in \mathcal{C}_a^r[i]$ and $\mathfrak{r} \in \mathcal{C}_a^{r+1}$, then $\widetilde{\phi} \in \mathcal{C}_a^r[i]$.

Next, suppose ϕ, $\widetilde{\phi}$ are both repulsive. Then we have the $\mathbb{C}$-linear operators $B, \widetilde{B} \colon \mathcal{C}_a[i]^{b} \to \mathcal{C}_a^1[i]$ given, for $f \in \mathcal{C}_a[i]^{b}$ and $t \geqslant a$, by

$$Bf(t) := e^{\Phi(t)} \int_\infty^t e^{-\Phi(s)} f(s)\, ds, \qquad \widetilde{B}f(t) := e^{\widetilde{\Phi}(t)} \int_\infty^t e^{-\widetilde{\Phi}(s)} f(s)\, ds.$$

Now assume $\mathfrak{r} \in \mathcal{C}_a[i]^{b}$. Then we have the $\mathbb{C}$-linear operator

$$B_{\ltimes\mathfrak{r}} := \mathfrak{r}^{-1} \circ B \circ \mathfrak{r} \ : \ \mathcal{C}_a[i]^{b} \to \mathcal{C}_a^1[i].$$

A computation as above shows $\widetilde{B} = B_{\ltimes\mathfrak{r}}$; thus $B_{\ltimes\mathfrak{r}}$ maps $\mathcal{C}_a[i]^{b}$ into $\mathcal{C}_a[i]^{b} \cap \mathcal{C}_a^1[i]$, and the operator $B_{\ltimes\mathfrak{r}} \colon \mathcal{C}_a[i]^{b} \to \mathcal{C}_a[i]^{b}$ is continuous with $\|B_{\ltimes\mathfrak{r}}\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\widetilde{\phi}}\big\|_a$. If $\widetilde{\phi} \in \mathcal{C}_a^r[i]$, then $B_{\ltimes\mathfrak{r}}$ maps $\mathcal{C}_a[i]^{b} \cap \mathcal{C}_a^r[i]$ into $\mathcal{C}_a[i]^{b} \cap \mathcal{C}_a^{r+1}[i]$.

More on twists and right-inverses of linear operators over Hardy fields. In this subsection we adopt the assumptions in force for Lemma 6.1.5, which we repeat here. Thus H is a Hardy field, $K = H[i]$, $r \in \mathbb{N}^{\geqslant 1}$, and $f_1, \dots, f_r \in K$. We fix $a_0 \in \mathbb{R}$ and functions in $\mathcal{C}_{a_0}[i]$ representing the germs $f_1, \dots, f_r$, denoted by the same symbols. We let a range over $[a_0, \infty)$, and we denote the restriction of each $f \in \mathcal{C}_{a_0}[i]$ to $[a, \infty)$ also by f. For each a we then have the $\mathbb{C}$-linear map $A_a \colon \mathcal{C}_a^r[i] \to \mathcal{C}_a[i]$ given by

$$A_a(y) \ = \ y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y.$$

We are in addition given a splitting $(\phi_1, \dots, \phi_r)$ of the linear differential operator $A = \partial^r + f_1\partial^{r-1} + \cdots + f_r \in K[\partial]$ over K with $\operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \succcurlyeq 1$, as well as functions in $\mathcal{C}_{a_0}^{r-1}[i]$ representing $\phi_1, \dots, \phi_r$, denoted by the same symbols and satisfying $\operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \in (\mathcal{C}_{a_0})^{\times}$. This gives rise to the continuous $\mathbb{C}$-linear operators

$$B_j := B_{\phi_j} \ : \ \mathcal{C}_a[i]^{b} \to \mathcal{C}_a[i]^{b} \qquad (j = 1, \dots, r)$$

and the right-inverse

$$A_a^{-1} := B_r \circ \cdots \circ B_1 \ : \ \mathcal{C}_a[i]^{b} \to \mathcal{C}_a[i]^{b}$$

of A_a with the properties stated in Lemma 6.1.5.

Now let $\mathfrak{m} \in H^{\times}$ with $\mathfrak{m} \prec 1$, and set $\widetilde{A} := A_{\ltimes\mathfrak{m}} \in K[\partial]$. Let $\mathfrak{r} \in (\mathcal{C}_{a_0}^r)^{\times}$ be a representative of $\mathfrak{m}$. Then $\mathfrak{r} \in (\mathcal{C}_{a_0}^r)^{b}$ and $\widetilde{\phi}_j := \phi_j - \mathfrak{r}^{\dagger} \in \mathcal{C}_{a_0}^{r-1}[i]$ for $j = 1, \dots, r$.

We have the $\mathbb{C}$-linear maps

$$\widetilde{A}_j \ :=\ \partial - \widetilde{\phi}_j \ :\ \mathcal{C}^j_a[\mathrm{i}] \to \mathcal{C}^{j-1}_a[\mathrm{i}] \qquad (j = 1, \dots, r)$$

and for sufficiently large a a factorization

$$\widetilde{A}_a \ =\ \widetilde{A}_1 \circ \cdots \circ \widetilde{A}_r \ :\ \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}].$$

Below we assume this holds for all a, as can be arranged by increasing a_0. We call $f, g \in \mathcal{C}_a[\mathrm{i}]$ **alike** if f, g are both attractive or both repulsive. In the same way we define when germs $f, g \in \mathcal{C}[\mathrm{i}]$ are alike. Suppose that ϕ_j, $\widetilde{\phi}_j$ are alike for $j = 1, \dots, r$. Then we have continuous $\mathbb{C}$-linear operators

$$\widetilde{B}_j \ :=\ B_{\widetilde{\phi}_j} \ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \qquad (j = 1, \dots, r)$$

and the right-inverse

$$\widetilde{A}_a^{-1} \ :=\ \widetilde{B}_r \circ \cdots \circ \widetilde{B}_1 \ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

of $\widetilde{A}_a$, and the arguments in the previous subsection show that $\widetilde{B}_j = (B_j)_{\ltimes \mathfrak{r}} = \mathfrak{r}^{-1} \circ B_j \circ \mathfrak{r}$ for $j = 1, \dots, r$, and hence $\widetilde{A}_a^{-1} = \mathfrak{r}^{-1} \circ A_a^{-1} \circ \mathfrak{r}$. For $j = 0, \dots, r$ we set, in analogy with A_j° and B_j° from (6.1.2) and (6.1.3),

$$\widetilde{A}_j^\circ \ :=\ \widetilde{A}_1 \circ \cdots \circ \widetilde{A}_j \ :\ \mathcal{C}^j_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}], \quad \widetilde{B}_j^\circ \ :=\ \widetilde{B}_j \circ \cdots \circ \widetilde{B}_1 \ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}.$$

Then $\widetilde{B}_j$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}^j_a[\mathrm{i}]$, $\widetilde{A}_j^\circ \circ \widetilde{B}_j^\circ$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, and $\widetilde{B}_j^\circ = \mathfrak{r}^{-1} \circ B_j^\circ \circ \mathfrak{r}$ by the above.

A weighted version of Proposition 6.1.7. We adopt the setting of the subsection *Damping factors* of Section 6.1, and make the same assumptions as in the paragraph before Proposition 6.1.7. Thus H, K, A, $f_1, \dots, f_r$, $\phi_1, \dots, \phi_r$, a_0 are as in the previous subsection, $\mathfrak{v} \in \mathcal{C}^r_{a_0}$ satisfies $\mathfrak{v}(t) > 0$ for all $t \geqslant a_0$, and its germ $\mathfrak{v}$ is in H with $\mathfrak{v} \prec 1$. As part of those assumptions we also have $\phi_1, \dots, \phi_r \preccurlyeq_\Delta \mathfrak{v}^{-1}$ in the asymptotic field K, for the convex subgroup

$$\Delta \ :=\ \big\{\gamma \in v(H^\times) :\ \gamma = o(v\mathfrak{v})\big\}$$

of $v(H^\times) = v(K^\times)$. Also $\nu \in \mathbb{R}^>$ and $u := \mathfrak{v}^\nu|_{[a,\infty)} \in (\mathcal{C}^r_a)^\times$.

To state a weighted version of Proposition 6.1.7, let $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, and let $\mathfrak{m}$ also denote a representative in $(\mathcal{C}^r_{a_0})^\times$ of the germ $\mathfrak{m}$. Set $\mathfrak{r} := \mathfrak{m}|_{[a,\infty)}$, so we have $\mathfrak{r} \in (\mathcal{C}^r_a)^\times \cap (\mathcal{C}^r_a)^{\mathrm{b}}$ and thus $\mathcal{C}^r_a[\mathrm{i}]^{\mathfrak{r}} \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. (Note that $\mathfrak{r}$, like u, depends on a, but we do not indicate this dependence notationally.) With notations as in the previous subsection we assume that for all a we have the factorization

$$\widetilde{A}_a \ =\ \widetilde{A}_1 \circ \cdots \circ \widetilde{A}_r \ :\ \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}],$$

as can be arranged by increasing a_0 if necessary.

Proposition 6.5.7. *Assume H is real closed, $\nu \in \mathbb{Q}$, $\nu > r$, and the elements ϕ_j, $\phi_j - \mathfrak{m}^\dagger$ of $\mathcal{C}_{a_0}[\mathrm{i}]$ are alike for $j = 1, \dots, r$. Then:*

(i) *the $\mathbb{C}$-linear operator $uA_a^{-1} \colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$ into $\mathcal{C}^r_a[\mathrm{i}]^{\mathfrak{r}}$;*
(ii) *its restriction to a $\mathbb{C}$-linear map $\mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}} \to \mathcal{C}^r_a[\mathrm{i}]^{\mathfrak{r}}$ is continuous; and*
(iii) *denoting this restriction also by uA_a^{-1}, we have $\|uA_a^{-1}\|^{\mathfrak{r}}_{a;r} \to 0$ as $a \to \infty$.*

Proof. Let $f \in \mathcal{C}_a[i]^{\tau}$, so $g := \tau^{-1} f \in \mathcal{C}_a[i]^{\mathrm{b}}$. Let $i \in \{0,\dots,r\}$; then with $\widetilde{B}_j^{\circ}$ as in the previous subsection and $u_{i,j}$ as in Lemma 6.1.6, that lemma gives

$$\tau^{-1}\big(uA_a^{-1}(f)\big)^{(i)} = \sum_{j=r-i}^{r} u_{i,j}u \cdot \tau^{-1} B_j^{\circ}(\tau g) \ = \sum_{j=r-i}^{r} u_{i,j}u\widetilde{B}_j^{\circ}(g).$$

The proof of Proposition 6.1.7 shows $u_{i,j}u \in \mathcal{C}_a[i]^{\mathrm{b}}$ with $\|u_{i,j}u\|_a \to 0$ as $a \to \infty$. Set

$$\widetilde{c}_{i,a} \ := \ \sum_{j=r-i}^{r} \|u\, u_{i,j}\|_a \cdot \|\widetilde{B}_j\|_a \cdots \|\widetilde{B}_1\|_a \in [0,\infty) \qquad (i = 0,\dots,r).$$

Then $\big\|\tau^{-1}\big[uA_a^{-1}(f)\big]^{(i)}\big\|_a \leqslant \widetilde{c}_{i,a}\|g\|_a = \widetilde{c}_{i,a}\|f\|_a^{\tau}$ where $\widetilde{c}_{i,a} \to 0$ as $a \to \infty$. This yields (i)–(iii). □

Weighted variants of results in Section 6.2. In this subsection we adopt the hypotheses in force for Lemma 6.2.1. To summarize those, H, K, A, $f_1,\dots,f_r$, $\phi_1,\dots,\phi_r$, a_0, $\mathfrak{v}$, ν, u, Δ are as in the previous subsection, $d, r \in \mathbb{N}^{\geqslant 1}$, H is real closed, $R \in K\{Y\}$ has order $\leqslant d$ and weight $\leqslant w \in \mathbb{N}^{\geqslant r}$. Also $\nu \in \mathbb{Q}$, $\nu > w$, $R \prec_{\Delta} \mathfrak{v}^{\nu}$, $\nu\mathfrak{v}^{\dagger} \not\sim \operatorname{Re}\phi_j$ and $\operatorname{Re}\phi_j - \nu\mathfrak{v}^{\dagger} \in (\mathcal{C}_{a_0})^{\times}$ for $j = 1,\dots,r$. Finally, $\widetilde{A} := A_{\ltimes \mathfrak{v}^{\nu}} \in K[\partial]$ and $\widetilde{A}_a(y) = u^{-1}A_a(uy)$ for $y \in \mathcal{C}_a^r[i]$. Next, let $\mathfrak{m}$, τ be as in the previous subsection. As in Lemma 6.2.12 we consider the continuous operator

$$\Phi_a\colon \mathcal{C}_a^r[i]^{\mathrm{b}} \times \mathcal{C}_a^r[i]^{\mathrm{b}} \to \mathcal{C}_a^r[i]^{\mathrm{b}}$$

given by

$$\Phi_a(f,y) \ := \ \Xi_a(f+y) - \Xi_a(f) \ = \ u\widetilde{A}_a^{-1}\big(u^{-1}\big(R(f+y) - R(f)\big)\big)\,.$$

Here is our weighted version of Lemma 6.2.12:

Lemma 6.5.8. *Suppose the elements $\phi_j - \nu\mathfrak{v}^{\dagger}$, $\phi_j - \nu\mathfrak{v}^{\dagger} - \mathfrak{m}^{\dagger}$ of $\mathcal{C}_{a_0}[i]$ are alike, for $j = 1,\dots,r$, and let $f \in \mathcal{C}_a^r[i]^{\mathrm{b}}$. Then the operator $y \mapsto \Phi_a(f,y)$ maps $\mathcal{C}_a^r[i]^{\tau}$ into itself. Moreover, there are $E_a, E_a^{+} \in \mathbb{R}^{\geqslant}$ such that for all $g \in \mathcal{C}_a^r[i]^{\mathrm{b}}$ and $y \in \mathcal{C}_a^r[i]^{\tau}$,*

$$\|\Phi_a(f,y)\|_{a;r}^{\tau} \ \leqslant \ E_a \cdot \max\big\{1, \|f\|_{a;r}^{d}\big\} \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\tau},$$

$$\begin{aligned}&\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r}^{\tau} \ \leqslant \\ &\qquad E_a^{+} \cdot \max\big\{1, \|f\|_{a;r}^{d}\big\} \cdot \max\big\{1, \|g\|_{a;r}^{d}\big\} \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\tau}.\end{aligned}$$

We can take these E_a, E_a^{+} such that $E_a, E_a^{+} \to 0$ as $a \to \infty$, and do so below.

Proof. Let $y \in \mathcal{C}_a^r[i]^{\tau}$. By Taylor expansion we have

$$R(f+y) - R(f) \ = \ \sum_{|\boldsymbol{i}|>0} \frac{1}{\boldsymbol{i}!} R^{(\boldsymbol{i})}(f) y^{\boldsymbol{i}} \ = \ \sum_{|\boldsymbol{i}|>0} S_{\boldsymbol{i}}(f) y^{\boldsymbol{i}} \quad \text{where } S_{\boldsymbol{i}}(f) := \frac{1}{\boldsymbol{i}!} \sum_{\boldsymbol{j}} R_{\boldsymbol{j}}^{(\boldsymbol{i})} f^{\boldsymbol{j}},$$

and $u^{-1}S_{\boldsymbol{i}}(f) \in \mathcal{C}_a[i]^{\mathrm{b}}$. So $h := u^{-1}\big(R(f+y) - R(f)\big) \in \mathcal{C}_a[i]^{\tau}$, since $\mathcal{C}_a[i]^{\tau}$ is an ideal of $\mathcal{C}_a[i]^{\mathrm{b}}$. Applying Proposition 6.5.7(i) with $\phi_j - \nu\mathfrak{v}^{\dagger}$ in the role of ϕ_j yields $\Phi_a(f,y) = u\widetilde{A}_a^{-1}(h) \in \mathcal{C}_a^r[i]^{\tau}$, establishing the first claim. Next, let $g \in \mathcal{C}_a^r[i]^{\mathrm{b}}$. Then $\Phi_a(f,g+y) - \Phi_a(f,g) = \Phi_a(f+g,y)$ by (6.2.1). Therefore,

$$\Phi_a(f,g+y) - \Phi_a(f,g) \ = \ u\widetilde{A}_a^{-1}(h), \quad h \ := \ u^{-1}\big(R(f+g+y) - R(f+g)\big), \text{ so}$$

$$\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r}^{\tau} \ = \ \|u\widetilde{A}_a^{-1}(h)\|_{a;r}^{\tau} \ \leqslant \ \|u\widetilde{A}_a^{-1}\|_{a;r}^{\tau} \cdot \|h\|_a^{\tau}.$$

By Corollary 6.5.6 we have

$$\|h\|_a^{\mathfrak{r}} \leqslant D \cdot \max_{|\boldsymbol{i}|>0} \|u^{-1}S_{\boldsymbol{i}}(f+g)\|_a \cdot \big(1+\|y\|_{a;r}+\cdots+\|y\|_{a;r}^{d-1}\big)\cdot\|y\|_{a;r}^{\mathfrak{r}}$$

where $D = D(d,r) := \big(\binom{d+r+1}{r+1}-1\big)$. Let D_a be as in the proof of Lemma 6.2.2. Then $D_a \to 0$ as $a \to \infty$, and Lemma 6.2.11 gives for $|\boldsymbol{i}|>0$,

$$\begin{aligned}
\|u^{-1}S_{\boldsymbol{i}}(f)\|_a &\leqslant D_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \\
\|u^{-1}S_{\boldsymbol{i}}(f+g)\|_a &\leqslant D_a \cdot \max\big\{1, \|f+g\|_{a;r}^d\big\} \\
&\leqslant 2^d D_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\}\cdot \max\big\{1, \|g\|_{a;r}^d\big\}.
\end{aligned}$$

This gives the desired result for $E_a := \|u\widetilde{A}_a^{-1}\|_{a;r}^{\mathfrak{r}} \cdot D \cdot D_a$ and $E_a^+ := 2^d E_a$, using also Proposition 6.5.7(iii) with $\phi_j - \nu\mathfrak{v}^\dagger$ in the role of ϕ_j. $\square$

Lemma 6.5.8 allows us to refine Theorem 6.2.3 as follows:

Corollary 6.5.9. *Suppose the elements $\phi_j - \nu\mathfrak{v}^\dagger$, $\phi_j - \nu\mathfrak{v}^\dagger - \mathfrak{m}^\dagger$ of $\mathcal{C}_{a_0}[\mathrm{i}]$ are alike, for $j = 1, \dots, r$, and $R(0) \preccurlyeq \mathfrak{v}^\nu\mathfrak{m}$. Then for sufficiently large a the operator Ξ_a maps the closed ball $B_a := \big\{f \in \mathcal{C}_a^r[\mathrm{i}] : \|f\|_{a;r} \leqslant 1/2\big\}$ of the normed space $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ into itself, has a unique fixed point in B_a, and this fixed point lies in $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$.*

Proof. Take a such that $\|\mathfrak{r}\|_a \leqslant 1$. Then by (6.5.2), B_a contains the closed ball

$$B_a^{\mathfrak{r}} := \big\{f \in \mathcal{C}_a^r[\mathrm{i}] : \|f\|_{a;r}^{\mathfrak{r}} \leqslant 1/2\big\}$$

of the normed space $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$. Let $f, g \in B_a^{\mathfrak{r}}$. Then $\Xi_a(g) - \Xi_a(f) = \Phi_a(f, g-f)$ lies in $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ by Lemma 6.5.8, and with E_a as in that lemma,

$$\begin{aligned}
\|\Xi_a(f) - \Xi_a(g)\|_{a;r}^{\mathfrak{r}} &= \|\Phi_a(f, g-f)\|_{a;r}^{\mathfrak{r}} \\
&\leqslant E_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(1+\cdots+\|g-f\|_{a;r}^{d-1}\big)\cdot\|g-f\|_{a;r}^{\mathfrak{r}} \\
&\leqslant E_a \cdot d \cdot \|g-f\|_{a;r}^{\mathfrak{r}}.
\end{aligned}$$

Taking a so that moreover $E_a d \leqslant \frac{1}{2}$ we obtain

$$\|\Xi_a(f) - \Xi_a(g)\|_{a;r}^{\mathfrak{r}} \leqslant \tfrac{1}{2}\|f-g\|_{a;r}^{\mathfrak{r}} \quad \text{for all } f, g \in B_a^{\mathfrak{r}}. \tag{6.5.4}$$

Next we consider the case $g = 0$. Our hypothesis $R(0) \preccurlyeq \mathfrak{v}^\nu\mathfrak{m}$ gives $u^{-1}R(0) \in \mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$. Proposition 6.5.7(i),(ii) with $\phi_j - \nu\mathfrak{v}^\dagger$ in the role of ϕ_j gives $\Xi_a(0) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ and $\|\Xi_a(0)\|_{a;r}^{\mathfrak{r}} \leqslant \|u\widetilde{A}_a^{-1}\|_{a;r}^{\mathfrak{r}}\|u^{-1}R(0)\|_a^{\mathfrak{r}}$. Using Proposition 6.5.7(iii) we now take a so large that $\|\Xi_a(0)\|_{a;r}^{\mathfrak{r}} \leqslant \frac{1}{4}$. Then (6.5.4) for $g = 0$ gives $\Xi_a(B_a^{\mathfrak{r}}) \subseteq B_a^{\mathfrak{r}}$. By Lemma 6.5.2 the normed space $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ is complete, hence Ξ_a has a unique fixed point in $B_a^{\mathfrak{r}}$. $\square$

Now suppose in addition that $A \in H[\partial]$ and $R \in H\{Y\}$. Set

$$(\mathcal{C}_a^r)^{\mathfrak{r}} := \big\{f \in \mathcal{C}_a^r : \|f\|_{a;r}^{\mathfrak{r}} < \infty\big\} = \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}} \cap \mathcal{C}_a^r,$$

a real Banach space with respect to $\|\cdot\|_{a;r}^{\mathfrak{r}}$. Increase a_0 as at the beginning of the subsection *Preserving reality* of Section 6.2. Then we have the map

$$\operatorname{Re}\Phi_a\colon (\mathcal{C}_a^r)^{\mathrm{b}} \times (\mathcal{C}_a^r)^{\mathrm{b}} \to (\mathcal{C}_a^r)^{\mathrm{b}}, \qquad (f,y) \mapsto \operatorname{Re}\big(\Phi_a(f,y)\big).$$

Suppose the elements $\phi_j - \nu\mathfrak{v}^\dagger$, $\phi_j - \nu\mathfrak{v}^\dagger - \mathfrak{m}^\dagger$ are alike for $j = 1, \dots, r$, and let a and E_a, E_a^+ be as in Lemma 6.5.8. Then this lemma yields:

Lemma 6.5.10. *Let $f, g \in (\mathcal{C}_a^r)^{\mathrm{b}}$ and $y \in (\mathcal{C}_a^r)^{\mathfrak{r}}$. Then* $(\operatorname{Re}\Phi_a)(f,y) \in (\mathcal{C}_a^r)^{\mathfrak{r}}$ *and*

$$\|\operatorname{Re}(\Phi_a)(f,y)\|_{a;r}^{\mathfrak{r}} \leqslant E_a \cdot \max\{1, \|f\|_{a;r}^d\} \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\mathfrak{r}},$$

$$\|(\operatorname{Re}\Phi_a)(f,g+y) - (\operatorname{Re}\Phi_a)(f,g)\|_{a;r}^{\mathfrak{r}} \leqslant$$
$$E_a^+ \cdot \max\{1, \|f\|_{a;r}^d\} \cdot \max\{1, \|g\|_{a;r}^d\} \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\mathfrak{r}}.$$

The same way we derived Corollary 6.5.9 from Lemma 6.5.8, this leads to:

Corollary 6.5.11. *If $R(0) \preccurlyeq \mathfrak{v}^\nu \mathfrak{m}$, then for sufficiently large a the operator* $\operatorname{Re}\Xi_a$ *maps the closed ball $B_a := \{f \in \mathcal{C}_a^r : \|f\|_{a;r} \leqslant 1/2\}$ of the normed space $(\mathcal{C}_a^r)^{\mathrm{b}}$ into itself, has a unique fixed point in B_a, and this fixed point lies in $(\mathcal{C}_a^r)^{\mathfrak{r}}$.*

Revisiting Lemma 6.2.13. Here we adopt the setting of the previous subsection. As usual, a ranges over $[a_0, \infty)$. We continue our investigation of the differences $f - g$ between solutions f, g of the equation $(*)$ on $[a_0,\infty)$ from Section 6.2 which we began in Lemma 6.2.5, and so we take f, g, E, ε, h_a, θ_a as in that lemma. Recall that in the remarks preceding Lemma 6.2.13 we defined continuous operators $\Phi_a, \Psi_a \colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ by

$$\Phi_a(y) := \Phi_a(g,y) = \Xi_a(g+y) - \Xi_a(g), \quad \Psi_a(y) := \Phi_a(y) + h_a \qquad (y \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}).$$

As in those remarks, we set $\rho := \|f - g\|_{a_0;r}$ and

$$B_a := \{y \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} : \|y - h_a\|_{a;r} \leqslant 1/2\},$$

and take $a_1 \geqslant a_0$ so that $\theta_a \in B_a$ for all $a \geqslant a_1$. Then by (6.2.4) we have $\|y\|_{a;r} \leqslant 1+\rho$ for $a \geqslant a_1$ and $y \in B_a$. Next, take $a_2 \geqslant a_1$ as in Lemma 6.2.13; thus for $a \geqslant a_2$ and $y, z \in B_a$ we have $\Psi_a(y) \in B_a$ and $\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \leqslant \frac{1}{2}\|y - z\|_{a;r}$. As in the previous subsection, $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, $\mathfrak{m}$ denotes also a representative in $(\mathcal{C}_{a_0}^r)^\times$ of the germ $\mathfrak{m}$, and $\mathfrak{r} := \mathfrak{m}|_{[a,\infty)} \in (\mathcal{C}_a^r)^\times \cap (\mathcal{C}_a^r)^{\mathrm{b}}$, so $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}} \subseteq \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$.

In the rest of this subsection $\phi_1 - \nu\mathfrak{v}^\dagger, \dots, \phi_r - \nu\mathfrak{v}^\dagger \in K$ are γ-repulsive for $\gamma := v\mathfrak{m} \in v(H^\times)^>$, and $h_a \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ for all $a \geqslant a_2$. Then Corollary 4.5.5 gives $a_3 \geqslant a_2$ such that for all $a \geqslant a_3$ and $j = 1, \dots, r$, the functions $\phi_j - u^\dagger, \phi_j - (u\mathfrak{r})^\dagger \in \mathcal{C}_a[\mathrm{i}]$ are alike and hence $\Psi_a\big(\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}\big) \subseteq \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ by Lemma 6.5.8. Thus $\Psi_a^n(h_a) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ for all n and $a \geqslant a_3$.

For $a \geqslant a_2$ we have $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$ in $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ by Corollary 6.2.14; we now aim to strengthen this to "in $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$" (possibly for a larger a_2). Towards this:

Lemma 6.5.12. *There exists $a_4 \geqslant a_3$ such that $\|\Psi_a(y) - \Psi_a(z)\|_{a;r}^{\mathfrak{r}} \leqslant \frac{1}{2}\|y - z\|_{a;r}^{\mathfrak{r}}$ for all $a \geqslant a_4$ and $y, z \in B_a \cap \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$.*

Proof. For $a \geqslant a_3$ and $y, z \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$, and with E_a^+ as in Lemma 6.5.8 we have

$$\|\Psi_a(y) - \Psi_a(z)\|_{a;r}^{\mathfrak{r}} \leqslant$$
$$E_a^+ \cdot \max\{1, \|g\|_{a;r}^d\} \cdot \max\{1, \|z\|_{a;r}^d\} \cdot \big(1 + \|y-z\|_{a;r} + \cdots + \|y-z\|_{a;r}^{d-1}\big) \cdot \|y-z\|_{a;r}^{\mathfrak{r}}.$$

For each $a \geqslant a_1$ and $y, z \in B_a$ we then have

$$\max\{1, \|z\|_{a;r}^d\} \cdot \big(1 + \|y-z\|_{a;r} + \cdots + \|y-z\|_{a;r}^{d-1}\big) \leqslant (1+\rho)^d \cdot d,$$

so taking $a_4 \geqslant a_3$ with

$$E_a^+ \max\{1, \|g\|_{a_0;r}^d\}(1+\rho)^d d \leqslant 1/2 \quad \text{for all } a \geqslant a_4,$$

we have $\|\Psi_a(y) - \Psi_a(z)\|_{a;r}^{\mathfrak{r}} \leqslant \frac{1}{2}\|y - z\|_{a;r}^{\mathfrak{r}}$ for all $a \geqslant a_4$ and $y, z \in B_a \cap \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$. $\square$

Let a_4 be as in the previous lemma.

Corollary 6.5.13. *Suppose $a \geqslant a_4$. Then $\theta_a \in \mathcal{C}^r_a[\mathrm{i}]^{\tau}$ and $\lim_{n\to\infty} \Psi^n_a(h_a) = \theta_a$ in the normed space $\mathcal{C}^r_a[\mathrm{i}]^{\tau}$. In particular, $f-g, (f-g)', \dots, (f-g)^{(r)} \preccurlyeq \mathfrak{m}$.*

Proof. We have $\Phi_a(h_a) = \Psi_a(h_a) - h_a \in \mathcal{C}^r_a[\mathrm{i}]^{\tau}$, so $M := \|\Phi_a(h_a)\|^{\tau}_{a;r} < \infty$. Since $\Psi_a(B_a) \subseteq B_a$, induction on n using Lemma 6.5.12 gives

$$\|\Psi^{n+1}_a(h_a) - \Psi^n_a(h_a)\|^{\tau}_{a;r} \leqslant M/2^n \qquad \text{for all } n.$$

Thus $\big(\Psi^n_a(h_a)\big)$ is a cauchy sequence in the normed space $\mathcal{C}^r_a[\mathrm{i}]^{\tau}$, and so converges in $\mathcal{C}^r_a[\mathrm{i}]^{\tau}$ by Lemma 6.5.2. In the normed space $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ we have $\lim_{n\to\infty} \Psi^n_a(h_a) = \theta_a$, by Corollary 6.2.14. Thus $\lim_{n\to\infty} \Psi^n_a(h_a) = \theta_a$ in $\mathcal{C}^r_a[\mathrm{i}]^{\tau}$ by Lemma 6.5.1. $\square$

An application to slots in H. Here we adopt the setting of the subsection *An application to slots in H* in Section 5.10. Thus $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field, $K := H[\mathrm{i}]$, $\mathrm{I}(K) \subseteq K^{\dagger}$, and $(P, 1, \widehat{h})$ is a slot in H of order $r \geqslant 1$; we set $w := \operatorname{wt}(P)$, $d := \deg P$. *Assume also that K is 1-linearly surjective if $r \geqslant 3$.*

Proposition 6.5.14. *Suppose $(P, 1, \widehat{h})$ is special, ultimate, Z-minimal, deep, and strongly repulsive-normal. Let $f, g \in \mathcal{C}^r[\mathrm{i}]$ and $\mathfrak{m} \in H^{\times}$ be such that*

$$P(f) \ = \ P(g) \ = \ 0, \qquad f, g \ \prec \ 1, \qquad v\mathfrak{m} \in v(\widehat{h} - H).$$

Then $(f-g)^{(j)} \preccurlyeq \mathfrak{m}$ for $j = 0, \dots, r$.

Proof. We arrange $\mathfrak{m} \prec 1$. Let $\mathfrak{v} := |\mathfrak{v}(L_P)| \in H^{>}$, so $\mathfrak{v} \prec^{\flat} 1$, and set $\Delta := \Delta(\mathfrak{v})$. Take $Q, R \in H\{Y\}$ where Q is homogeneous of degree 1 and order r, $A := L_Q \in H[\partial]$ has a strong $\widehat{h}$-repulsive splitting over K, $P = Q - R$, and $R \prec_{\Delta} \mathfrak{v}^{w+1} P_1$, so $\mathfrak{v}(A) \sim \mathfrak{v}(L_P)$ by Lemma 3.1.1. Multiplying P, Q, R by some $b \in H^{\times}$ we arrange that A is monic, so $A = \partial^r + f_1 \partial^{r-1} + \dots + f_r$ with $f_1, \dots, f_r \in H$ and $R \prec_{\Delta} \mathfrak{v}^w$. Let $(\phi_1, \dots, \phi_r) \in K^r$ be a strong $\widehat{h}$-repulsive splitting of A over K, so $\phi_1, \dots, \phi_r$ are $\widehat{h}$-repulsive and

$$A \ = \ (\partial - \phi_1) \cdots (\partial - \phi_r), \qquad \operatorname{Re} \phi_1, \dots, \operatorname{Re} \phi_r \ \succcurlyeq \ \mathfrak{v}^{\dagger} \ \succcurlyeq \ 1.$$

By Corollary 3.1.6 we have $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$. Thus we can take $a_0 \in \mathbb{R}$ and functions on $[a_0, \infty)$ representing the germs $\phi_1, \dots, \phi_r$, $f_1, \dots, f_r$, f, g and the $R_{\boldsymbol{j}}$ with $\boldsymbol{j} \in \mathbb{N}^{1+r}$, $|\boldsymbol{j}| \leqslant d$, $\|\boldsymbol{j}\| \leqslant w$ (using the same symbols for the germs mentioned as for their chosen representatives) so as to be in the situation described in the beginning of Section 6.2, with f and g solutions on $[a_0, \infty)$ of the differential equation $(*)$ there. As there, we take $\nu \in \mathbb{Q}$ with $\nu > w$ so that $R \prec_{\Delta} \mathfrak{v}^{\nu}$ and $\nu \mathfrak{v}^{\dagger} \not\sim \operatorname{Re} \phi_j$ for $j = 1, \dots, r$, and then increase a_0 to satisfy all assumptions for Lemma 6.2.1. Corollary 3.3.15 gives $v(\mathfrak{v}^{\nu}) \in v(\widehat{h} - H)$, so $\phi_j - \nu \mathfrak{v}^{\dagger} = \phi_j - (\mathfrak{v}^{\nu})^{\dagger}$ $(j = 1, \dots, r)$ is $\widehat{h}$-repulsive by Lemma 4.5.13(iv), so γ-repulsive for $\gamma := v\mathfrak{m} > 0$. Now A splits over K, and K is 1-linearly surjective if $r \geqslant 3$, hence $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$ by Lemma 5.10.22. Thus by Corollary 5.10.16 we have $y, y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $y \in \mathcal{C}^r[\mathrm{i}]$ with $A(y) = 0$, $y \prec 1$. In particular, $\mathfrak{m}^{-1} h_a, \mathfrak{m}^{-1} h_a', \dots, \mathfrak{m}^{-1} h_a^{(r)} \prec 1$ for all $a \geqslant a_0$. Thus the assumptions on $\mathfrak{m}$ and the h_a made just before Lemma 6.5.12 are satisfied for a suitable choice of a_2, so we can appeal to Corollary 6.5.13. $\square$

The assumption that K is 1-linearly surjective for $r \geqslant 3$ was only used in the proof above to obtain $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$. So if A as in this proof satisfies $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$, then we can drop this assumption about K, also in the next corollary.

Corollary 6.5.15. *Suppose $(P, 1, \widehat{h})$, f, g, $\mathfrak{m}$ are as in Proposition 6.5.14. Then*

$$f - g \in \mathfrak{m}\, \mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}.$$

Proof. If $\mathfrak{m} \succcurlyeq 1$, then Lemma 6.4.1(ii) applied with $y = (f-g)/\mathfrak{m}$ and $1/\mathfrak{m}$ in place of $\mathfrak{m}$ gives what we want. Now assume $\mathfrak{m} \prec 1$. Since $\widehat{h}$ is special over H, Proposition 6.5.14 applies to $\mathfrak{m}^{r+1}$ in place of $\mathfrak{m}$, so $(f-g)^{(j)} \preccurlyeq \mathfrak{m}^{r+1}$ for $j = 0, \dots, r$. Now apply Lemma 6.5.3 to suitable representatives of $f - g$ and $\mathfrak{m}$. $\square$

Later in this section we use Proposition 6.5.14 and its Corollary 6.5.15 to strengthen some results from Section 6.4. In Section 7.7 we give further refinements of that proposition for the case of firm and flabby slots, but these are not needed for the proof of our main result, given in Section 6.7.

Weighted refinements of results in Section 6.4. We now adopt the setting of the subsection *Reformulations* of Section 6.4. Thus $H \supseteq \mathbb{R}$ is a real closed Hardy field with asymptotic integration, and $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ is its algebraic closure, with value group $\Gamma := v(H^\times) = v(K^\times)$. The next lemma and its corollary refine Lemma 6.4.5. Let P, Q, R, L_Q, w be as introduced before that lemma, set $\mathfrak{v} := |\mathfrak{v}(L_Q)| \in H^>$, and, in case $\mathfrak{v} \prec 1$, $\Delta := \Delta(\mathfrak{v})$.

Lemma 6.5.16. *Let $f \in K^\times$ and $\phi_1, \dots, \phi_r \in K$ be such that*

$$L_Q \;=\; f(\partial - \phi_1)\cdots(\partial - \phi_r), \qquad \operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \;\succcurlyeq\; 1.$$

Assume $\mathfrak{v} \prec 1$ and $R \prec_\Delta \mathfrak{v}^{w+1}Q$. Let $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, $P(0) \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}Q$. Suppose that for $j = 1, \dots, r$ and all $\nu \in \mathbb{Q}$ with $w < \nu < w+1$, $\phi_j - (\mathfrak{v}^\nu)^\dagger$ and $\phi_j - (\mathfrak{v}^\nu\mathfrak{m})^\dagger$ are alike. Then $P(y) = 0$ and $y, y', \dots, y^{(r)} \preccurlyeq \mathfrak{m}$ for some $y \prec \mathfrak{v}^w$ in $\mathcal{C}^{<\infty}[\mathrm{i}]$. If $P, Q \in H\{Y\}$, then there is such y in $\mathcal{C}^{<\infty}$.

Proof. Note that $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$ by Corollary 3.1.6 and that $R \prec_\Delta \mathfrak{v}^{w+1}Q$ gives $f^{-1}R \prec_\Delta \mathfrak{v}^w$. Take $\nu \in \mathbb{Q}$ such that $w < \nu < w+1$, $f^{-1}R \prec_\Delta \mathfrak{v}^\nu$ and $\nu\mathfrak{v}^\dagger \not\sim \operatorname{Re}\phi_j$ for $j = 1, \dots, r$. Set $A := f^{-1}L_Q$. From $\nu < w+1$ and

$$R(0) \;=\; -P(0) \;\prec_\Delta\; \mathfrak{v}^{w+2}\mathfrak{m}Q$$

we obtain $f^{-1}R(0) \prec_\Delta \mathfrak{v}^\nu\mathfrak{m}$. Thus we can apply successively Corollary 6.5.9, Lemma 6.2.1, and Corollary 6.3.5 to the equation $A(y) = f^{-1}R(y)$, $y \prec 1$ in the role of $(*)$ in Section 6.2 to obtain the first part. For the real variant, use instead Corollary 6.5.11 and Lemma 6.2.6. $\square$

Lemma 6.5.16 with $\mathfrak{m}^{r+1}$ for $\mathfrak{m}$ has the following consequence, using Lemma 6.5.3:

Corollary 6.5.17. *Let $f \in K^\times$ and $\phi_1, \dots, \phi_r \in K$ be such that*

$$L_Q \;=\; f(\partial - \phi_1)\cdots(\partial - \phi_r), \qquad \operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \;\succcurlyeq\; 1.$$

Assume $\mathfrak{v} \prec 1$ and $R \prec_\Delta \mathfrak{v}^{w+1}Q$. Let $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, $P(0) \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}^{r+1}Q$. Suppose that for $j = 1, \dots, r$ and all $\nu \in \mathbb{Q}$ with $w < \nu < w+1$, $\phi_j - (\mathfrak{v}^\nu)^\dagger$ and $\phi_j - (\mathfrak{v}^\nu\mathfrak{m}^{r+1})^\dagger$ are alike. Then for some $y \prec \mathfrak{v}^w$ in $\mathcal{C}^{<\infty}[\mathrm{i}]$ we have $P(y) = 0$ and $y \in \mathfrak{m}\,\mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}$. If $P, Q \in H\{Y\}$, then there is such y in $\mathcal{C}^{<\infty}$.

Remark. If H is a $\mathcal{C}^\infty$-Hardy field, then Lemma 6.5.16 and Corollary 6.5.17 go through with $\mathcal{C}^{<\infty}[\mathrm{i}]$, $\mathcal{C}^{<\infty}$ replaced by $\mathcal{C}^\infty[\mathrm{i}]$, $\mathcal{C}^\infty$, respectively. Likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. (Use Corollary 6.3.5.)

Next a variant of Lemma 6.4.6. *In the rest of this subsection* $(P,\mathfrak{n},\widehat{h})$ *is a deep, strongly repulsive-normal, Z-minimal slot in H of order $r \geqslant 1$ and weight $w := \operatorname{wt}(P)$. We assume also that $(P,\mathfrak{n},\widehat{h})$ is special (as will be the case if H is r-linearly newtonian, and $\upomega$-free if $r>1$, by Lemma 3.2.36).*

Lemma 6.5.18. *Let $\mathfrak{m} \in H^{\times}$ be such that $v\mathfrak{m} \in v(\widehat{h}-H)$, $\mathfrak{m} \prec \mathfrak{n}$, and $P(0) \preccurlyeq \mathfrak{v}(L_{P_{\times\mathfrak{n}}})^{w+2}\,(\mathfrak{m}/\mathfrak{n})^{r+1}\,P_{\times\mathfrak{n}}$. Then for some $y \in \mathcal{C}^{<\infty}$,*

$$P(y) \;=\; 0, \quad y \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^{\infty}$, then there is such y in $\mathcal{C}^{\infty}$; likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.

Proof. Replace $(P,\mathfrak{n},\widehat{h})$, $\mathfrak{m}$ by $(P_{\times\mathfrak{n}},1,\widehat{h}/\mathfrak{n})$, $\mathfrak{m}/\mathfrak{n}$ to arrange $\mathfrak{n}=1$. Then L_P has order r, $\mathfrak{v}(L_P) \prec^{\flat} 1$, and $P = Q - R$ where $Q,R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, $L_Q \in H[\partial]$ has a strong $\widehat{h}$-repulsive splitting $(\phi_1,\dots,\phi_r) \in K^r$ over $K = H[i]$, and $R \prec_{\Delta^*} \mathfrak{v}(L_P)^{w+1}P_1$ with $\Delta^* := \Delta\big(\mathfrak{v}(L_P)\big)$. By Lemma 3.1.1(ii) we have $\mathfrak{v}(L_P) \sim \mathfrak{v}(L_Q) \asymp \mathfrak{v}$, so $\operatorname{Re}\phi_j \succcurlyeq \mathfrak{v}^{\dagger} \succcurlyeq 1$ for $j=1,\dots,r$, and $\Delta = \Delta^*$. Moreover, $P(0) \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}^{r+1}Q$. Let $\nu \in \mathbb{Q}$, $\nu > w$, and $j \in \{1,\dots,r\}$. Then $0 < v(\mathfrak{v}^{\nu}) \in v(\widehat{h}-H)$ by Corollary 3.3.15, so ϕ_j is γ-repulsive for $\gamma = v(\mathfrak{v}^{\nu})$, hence ϕ_j and $\phi_j - (\mathfrak{v}^{\nu})^{\dagger}$ are alike by Corollary 4.5.5. Likewise, $0 < v(\mathfrak{v}^{\nu}\mathfrak{m}^{r+1}) \in v(\widehat{h}-H)$ since $\widehat{h}$ is special over H, so ϕ_j and $\phi_j - (\mathfrak{v}^{\nu}\mathfrak{m}^{r+1})^{\dagger}$ are alike. Thus $\phi_j - (\mathfrak{v}^{\nu})^{\dagger}$ and $\phi_j - (\mathfrak{v}^{\nu}\mathfrak{m}^{r+1})^{\dagger}$ are alike as well. Hence Corollary 6.5.17 gives $y \prec \mathfrak{v}^{w}$ in $\mathcal{C}^{<\infty}$ with $P(y)=0$ and $y \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$. For the rest use the remark following that corollary. $\square$

Corollary 6.5.19. *Suppose $\mathfrak{n}=1$, and let $\mathfrak{m} \in H^{\times}$ be such that $v\mathfrak{m} \in v(\widehat{h}-H)$. Then there are $h \in H$ and $y \in \mathcal{C}^{<\infty}$ such that:*

$$\widehat{h}-h \;\preccurlyeq\; \mathfrak{m}, \qquad P(y) \;=\; 0, \qquad y \;\prec\; 1,\ y \in (\mathcal{C}^r)^{\preccurlyeq}, \qquad y-h \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^{\infty}$, then we have such $y \in \mathcal{C}^{\infty}$; likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.

Proof. Suppose first that $\mathfrak{m} \succcurlyeq 1$, and let $h := 0$ and y be as in Lemma 6.4.6 for $\phi = \mathfrak{n} = 1$. Then $y \prec 1$, $y \in (\mathcal{C}^r)^{\preccurlyeq}$, so $y\mathfrak{m} \prec 1$, $y/\mathfrak{m} \prec 1$, $y/\mathfrak{m} \in (\mathcal{C}^r)^{\preccurlyeq}$ by the Product Rule. Next assume $\mathfrak{m} \prec 1$ and set $\mathfrak{v} := |\mathfrak{v}(L_P)| \in H^{>}$. By Corollary 3.3.15 we can take $h \in H$ such that $\widehat{h}-h \prec (\mathfrak{v}\mathfrak{m})^{(w+3)(r+1)}$, and then by Lemma 3.2.37 we have

$$P_{+h}(0) \;=\; P(h) \;\prec\; (\mathfrak{v}\mathfrak{m})^{w+3}P \;\preccurlyeq\; \mathfrak{v}^{w+3}\mathfrak{m}^{r+1}P_{+h}.$$

By Lemma 4.5.35, $(P_{+h},1,\widehat{h}-h)$ is strongly repulsive-normal, and by Corollary 3.3.8 it is deep with $\mathfrak{v}(L_{P_{+h}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Hence Lemma 6.5.18 applies to the slot $(P_{+h},1,\widehat{h}-h)$ in place of $(P,1,\widehat{h})$ to yield a $z \in \mathcal{C}^{<\infty}$ with $P_{+h}(z)=0$ and $(z/\mathfrak{m})^{(j)} \preccurlyeq 1$ for $j=0,\dots,r$. Lemma 6.4.1 gives $z^{(j)} \prec 1$ for $j=0,\dots,r$. Set $y := h+z$; then $P(y)=0$, $y^{(j)} \prec 1$ and $\big((y-h)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for $j=0,\dots,r$. $\square$

We now use the results above to approximate zeros of P in $\mathcal{C}^{<\infty}$ by elements of H:

Corollary 6.5.20. *Suppose H is Liouville closed, $\operatorname{I}(K) \subseteq K^{\dagger}$, $\mathfrak{n}=1$, and our slot $(P,1,\widehat{h})$ in H is ultimate. Assume also that K is 1-linearly surjective if $r \geqslant 3$. Let $y \in \mathcal{C}^{<\infty}$ and $h \in H$, $\mathfrak{m} \in H^{\times}$ be such that $P(y)=0$, $y \prec 1$, and $\widehat{h}-h \preccurlyeq \mathfrak{m}$. Then*

$$y-h \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}.$$

Proof. Corollary 6.5.19 gives $h_1 \in H$, $z \in \mathcal{C}^{<\infty}$ with $\widehat{h} - h_1 \preccurlyeq \mathfrak{m}$, $P(z) = 0$, $z \prec 1$, and $\big((z - h_1)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r$. Now

$$\frac{y-h}{\mathfrak{m}} \;=\; \frac{y-z}{\mathfrak{m}} + \frac{z-h_1}{\mathfrak{m}} + \frac{h_1-h}{\mathfrak{m}}$$

with $\big((y - z)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r$ by Corollary 6.5.15. Also $(h_1 - h)/\mathfrak{m} \in H$ and $(h_1 - h)/\mathfrak{m} \preccurlyeq 1$, so $\big((h_1 - h)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for all $j \in \mathbb{N}$. □

The above corollary is the only part of this section used towards establishing our main result, Theorem 6.7.22. But this use, in proving Theorem 6.7.13, is essential, and obtaining Corollary 6.5.20 required much of the above section.

6.6. Asymptotic Similarity

Let H be a Hausdorff field and $\widehat{H}$ an immediate valued field extension of H. Equip $\widehat{H}$ with the unique field ordering making it an ordered field extension of H such that $\mathcal{O}_{\widehat{H}}$ is convex [ADH, 3.5.12]. Let $f \in \mathcal{C}$ and $\widehat{f} \in \widehat{H}$ be given.

Definition 6.6.1. Call f **asymptotically similar** to $\widehat{f}$ over H (notation: $f \sim_H \widehat{f}$) if $f \sim \phi$ in $\mathcal{C}$ and $\phi \sim \widehat{f}$ in $\widehat{H}$ for some $\phi \in H$. (Note that then $f \in \mathcal{C}^\times$ and $\widehat{f} \neq 0$.)

Recall that the binary relations $\sim$ on $\mathcal{C}^\times$ and $\sim$ on $\widehat{H}^\times$ are equivalence relations which restrict to the same equivalence relation on $H^\times$. As a consequence, if $f \sim_H \widehat{f}$, then $f \sim \phi$ in $\mathcal{C}$ for any $\phi \in H$ with $\phi \sim \widehat{f}$ in $\widehat{H}$, and $\phi \sim \widehat{f}$ in $\widehat{H}$ for any $\phi \in H$ with $f \sim \phi$ in $\mathcal{C}$. Moreover, if $f \in H$, then $f \sim_H \widehat{f} \Leftrightarrow f \sim \widehat{f}$ in $\widehat{H}$, and if $\widehat{f} \in H$, then $f \sim_H \widehat{f} \Leftrightarrow f \sim \widehat{f}$ in $\mathcal{C}$.

Lemma 6.6.2. *Let $f_1 \in \mathcal{C}$, $f_1 \sim f$, let $\widehat{f_1} \in \widehat{H}_1$ for an immediate valued field extension $\widehat{H}_1$ of H, and suppose $\widehat{f} \sim \theta$ in $\widehat{H}$ and $\widehat{f_1} \sim \theta$ in $\widehat{H}_1$ for some $\theta \in H$. Then: $f \sim_H \widehat{f} \Leftrightarrow f_1 \sim_H \widehat{f_1}$.*

For $\mathfrak{n} \in H^\times$ we have $f \sim_H \widehat{f} \Leftrightarrow \mathfrak{n}f \sim_H \mathfrak{n}\widehat{f}$. Moreover, by Lemma 5.1.1:

Lemma 6.6.3. *Let $g \in \mathcal{C}$, $\widehat{g} \in \widehat{H}$, and suppose $f \sim_H \widehat{f}$ and $g \sim_H \widehat{g}$. Then $1/f \sim_H 1/\widehat{f}$ and $fg \sim_H \widehat{f}\,\widehat{g}$. Moreover,*

$$f \preccurlyeq g \text{ in } \mathcal{C} \quad\Longleftrightarrow\quad \widehat{f} \preccurlyeq \widehat{g} \text{ in } \widehat{H},$$

and likewise with $\prec$, $\asymp$, or $\sim$ in place of $\preccurlyeq$.

Lemma 6.6.3 readily yields:

Corollary 6.6.4. *Suppose $\widehat{f}$ is transcendental over H and $Q(f) \sim_H Q(\widehat{f})$ for all $Q \in H[Y]^{\neq}$. Then we have:*

- (i) *a subfield $H(f) \supseteq H$ of $\mathcal{C}$ generated by f over H;*
- (ii) *a field isomorphism $\iota \colon H(f) \to H(\widehat{f})$ over H with $\iota(f) = \widehat{f}$;*
- (iii) *with $H(f)$ and ι as in* (i) *and* (ii) *we have $g \sim_H \iota(g)$ for all $g \in H(f)^\times$, hence for all $g_1, g_2 \in H(f)$: $g_1 \preccurlyeq g_2$ in $\mathcal{C} \Leftrightarrow \iota(g_1) \preccurlyeq \iota(g_2)$ in $\widehat{H}$.*

Also, ι in (ii) *is unique and is an ordered field isomorphism, where the ordering on $H(f)$ is its ordering as a Hausdorff field.*

Proof. To see that ι is order preserving, use that ι is a valued field isomorphism by (iii), and apply [ADH, 3.5.12]. □

Here is the analogue when $\widehat{f}$ algebraic over H:

Corollary 6.6.5. *Suppose $\widehat{f}$ is algebraic over H with minimum polynomial P over H of degree d, and $P(f) = 0$, $Q(f) \sim_H Q(\widehat{f})$ for all $Q \in H[Y]^{\neq}$ of degree $< d$. Then we have:*

(i) *a subfield $H[f] \supseteq H$ of $\mathcal{C}$ generated by f over H;*
(ii) *a field isomorphism $\iota\colon H[f] \to H[\widehat{f}]$ over H with $\iota(f) = \widehat{f}$;*
(iii) *with $H[f]$ and ι as in* (i) *and* (ii) *we have $g \sim_H \iota(g)$ for all $g \in H[f]^{\times}$, hence for all $g_1, g_2 \in H[f]$: $g_1 \preccurlyeq g_2$ in $\mathcal{C} \Leftrightarrow \iota(g_1) \preccurlyeq \iota(g_2)$ in $\widehat{H}$.*

Also, $H[f]$ and ι in (i) *and* (ii) *are unique and ι is an ordered field isomorphism, where the ordering on $H(f)$ is its ordering as a Hausdorff field.*

If $\widehat{f} \notin H$, then to show that $f - \phi \sim_H \widehat{f} - \phi$ for all $\phi \in H$ it is enough to do this for ϕ arbitrarily close to $\widehat{f}$:

Lemma 6.6.6. *Let $\phi_0 \in H$ be such that $f - \phi_0 \sim_H \widehat{f} - \phi_0$. Then $f - \phi \sim_H \widehat{f} - \phi$ for all $\phi \in H$ with $\widehat{f} - \phi_0 \prec \widehat{f} - \phi$.*

Proof. Let $\phi \in H$ with $\widehat{f} - \phi_0 \prec \widehat{f} - \phi$. Then $\phi_0 - \phi \succ \widehat{f} - \phi_0$, so $\widehat{f} - \phi = (\widehat{f} - \phi_0) + (\phi_0 - \phi) \sim \phi_0 - \phi$. By Lemma 6.6.3 we also have $\phi_0 - \phi \succ f - \phi_0$, and hence likewise $f - \phi \sim \phi_0 - \phi$. $\square$

We define: $f \approx_H \widehat{f} :\Leftrightarrow f - \phi \sim_H \widehat{f} - \phi$ for all $\phi \in H$. If $f \approx_H \widehat{f}$, then $f \sim_H \widehat{f}$ as well as $f, \widehat{f} \notin H$, and $\mathfrak{n}f \approx_H \mathfrak{n}\widehat{f}$ for all $\mathfrak{n} \in H^{\times}$. Hence $f \approx_H \widehat{f}$ iff $f, \widehat{f} \notin H$ and the isomorphism $\iota\colon H + Hf \to H + H\widehat{f}$ of H-linear spaces that is the identity on H and sends f to $\widehat{f}$ satisfies $g \sim_H \iota(g)$ for all nonzero $g \in H + Hf$.

Here is an easy consequence of Lemma 6.6.6:

Corollary 6.6.7. *Suppose $\widehat{f} \notin H$ and $f - \phi_0 \sim_H \widehat{f} - \phi_0$ for all $\phi_0 \in H$ such that $\phi_0 \sim \widehat{f}$. Then $f \approx_H \widehat{f}$.*

Proof. Take $\phi_0 \in H$ with $\phi_0 \sim \widehat{f}$, and let $\phi \in H$ be given. If $\widehat{f} - \phi \prec \widehat{f}$, then $f - \phi \sim_H \widehat{f} - \phi$ by hypothesis; otherwise we have $\widehat{f} - \phi \succcurlyeq \widehat{f} \succ \widehat{f} - \phi_0$, and then $f - \phi \sim_H \widehat{f} - \phi$ by Lemma 6.6.6. $\square$

Lemma 6.6.2 yields an analogue for $\approx_H$:

Lemma 6.6.8. *Let $f_1 \in \mathcal{C}$ be such that $f_1 - \phi \sim f - \phi$ for all $\phi \in H$, and let $\widehat{f_1}$ be an element of an immediate valued field extension of H such that $v(\widehat{f} - \phi) = v(\widehat{f_1} - \phi)$ for all $\phi \in H$. Then $f \approx_H \widehat{f}$ iff $f_1 \approx_H \widehat{f_1}$.*

Let $g \in \mathcal{C}$ be eventually strictly increasing with $g(t) \to +\infty$ as $t \to +\infty$; we then have the Hausdorff field $H \circ g = \{h \circ g : h \in H\}$, with ordered valued field isomorphism $h \mapsto h \circ g\colon H \to H \circ g$. (See Section 5.1.) Suppose

$$\widehat{h} \mapsto \widehat{h} \circ g \ : \ \widehat{H} \to \widehat{H} \circ g$$

extends this isomorphism to a valued field isomorphism, where $\widehat{H} \circ g$ is an immediate valued field extension of the Hausdorff field $H \circ g$. Then

$$f \sim_H \widehat{f} \iff f \circ g \sim_{H \circ g} \widehat{f} \circ g, \qquad f \approx_H \widehat{f} \iff f \circ g \approx_{H \circ g} \widehat{f} \circ g.$$

The complex version. We now assume in addition that H is real closed, with algebraic closure $K := H[i] \subseteq \mathcal{C}[i]$. We take i with $i^2 = -1$ also as an element of a field $\widehat{K} := \widehat{H}[i]$ extending both $\widehat{H}$ and K, and equip $\widehat{K}$ with the unique valuation ring of $\widehat{K}$ lying over $\mathcal{O}_{\widehat{H}}$; see the remarks following Lemma 4.1.2. Then $\widehat{K}$ is an immediate valued field extension of K. Let $f \in \mathcal{C}[i]$ and $\widehat{f} \in \widehat{K}$ below.

Call f **asymptotically similar** to $\widehat{f}$ over K (notation: $f \sim_K \widehat{f}$) if for some $\phi \in K$ we have $f \sim \phi$ in $\mathcal{C}[i]$ and $\phi \sim \widehat{f}$ in $\widehat{K}$. Then $f \in \mathcal{C}[i]^\times$ and $\widehat{f} \neq 0$. As before, if $f \sim_K \widehat{f}$, then $f \sim \phi$ in $\mathcal{C}[i]$ for any $\phi \in K$ for which $\phi \sim \widehat{f}$ in $\widehat{K}$, and $\phi \sim \widehat{f}$ in $\widehat{K}$ for any $\phi \in K$ for which $f \sim \phi$ in $\mathcal{C}[i]$. Moreover, if $f \in K$, then $f \sim_K \widehat{f}$ reduces to $f \sim \widehat{f}$ in $\widehat{K}^\times$. Likewise, if $\widehat{f} \in K$, then $f \sim_K \widehat{f}$ reduces to $f \sim \widehat{f}$ in $\mathcal{C}[i]^\times$.

Lemma 6.6.9. *Let $f_1 \in \mathcal{C}[i]$ with $f_1 \sim f$. Let $\widehat{H}_1$ be an immediate valued field extension of H, let $\widehat{K}_1 := \widehat{H}_1[i]$ be the corresponding immediate valued field extension of K obtained from $\widehat{H}_1$ as $\widehat{K}$ was obtained from $\widehat{H}$. Let $\widehat{f}_1 \in \widehat{K}_1$, and $\theta \in K$ be such that $\widehat{f} \sim \theta$ in $\widehat{K}$ and $\widehat{f}_1 \sim \theta$ in $\widehat{K}_1$. Then $f \sim_K \widehat{f}$ iff $f_1 \sim_K \widehat{f}_1$.*

For $\mathfrak{n} \in K^\times$ we have $f \sim_K \widehat{f} \Leftrightarrow \mathfrak{n}f \sim_K \mathfrak{n}\widehat{f}$, and $f \sim_K \widehat{f} \Leftrightarrow \overline{f} \sim_K \overline{\widehat{f}}$ (complex conjugation). Here is a useful observation relating $\sim_K$ and $\sim_H$:

Lemma 6.6.10. *Suppose $f \sim_K \widehat{f}$ and $\operatorname{Re}\widehat{f} \succcurlyeq \operatorname{Im}\widehat{f}$; then*

$$\operatorname{Re} f \ \succcurlyeq\ \operatorname{Im} f, \qquad \operatorname{Re} f \ \sim_H\ \operatorname{Re}\widehat{f}.$$

Proof. Let $\phi \in K$ be such that $f \sim \phi$ in $\mathcal{C}[i]$ and $\phi \sim \widehat{f}$ in $\widehat{K}$. The latter yields $\operatorname{Re}\phi \succcurlyeq \operatorname{Im}\phi$ in H and $\operatorname{Re}\phi \sim \operatorname{Re}\widehat{f}$ in $\widehat{H}$. Using that $f = (1+\varepsilon)\phi$ with $\varepsilon \prec 1$ in $\mathcal{C}[i]$ it follows easily that $\operatorname{Re} f \succcurlyeq \operatorname{Im} f$ and $\operatorname{Re} f \sim \operatorname{Re}\phi$ in $\mathcal{C}$. $\square$

Corollary 6.6.11. *Suppose $f \in \mathcal{C}$ and $\widehat{f} \in \widehat{H}$. Then $f \sim_H \widehat{f}$ iff $f \sim_K \widehat{f}$.*

Lemmas 6.6.3 and 6.6.6 go through with $\mathcal{C}[i]$, K, $\widehat{K}$, and $\sim_K$ in place of $\mathcal{C}$, H, $\widehat{H}$, and $\sim_H$. We define: $f \approx_K \widehat{f} :\Leftrightarrow f - \phi \sim_K \widehat{f} - \phi$ for all $\phi \in K$. Now Corollary 6.6.7 goes through with K, $\sim_K$, $\approx_K$ in place of H, $\sim_H$, $\approx_H$.

Lemma 6.6.12. *Suppose $f \in \mathcal{C}$, $\widehat{f} \in \widehat{H}$, and $f \sim_H \widehat{f}$. Then $f + gi \sim_K \widehat{f} + gi$ for all $g \in H$.*

Proof. Let $g \in H$, and take $\phi \in H$ with $f \sim \phi$ in $\mathcal{C}$ and $\phi \sim \widehat{f}$ in $\widehat{H}$. Suppose first that $g \prec \phi$. Then $gi \prec \phi$, and together with $f - \phi \prec \phi$ this yields $(f + gi) - \phi \prec \phi$, that is, $f + gi \sim \phi$ in $\mathcal{C}[i]$ (cf. the basic properties of the relation $\prec$ on $\mathcal{C}[i]$ stated before Lemma 5.1.1). Using likewise the analogous properties of $\prec$ on $\widehat{K}$ we obtain $\phi \sim \widehat{f} + gi$ in $\widehat{K}$. If $\phi \prec g$, then $f \preccurlyeq \phi \prec gi$ and thus $f + gi \sim gi$ in $\mathcal{C}[i]$, and likewise $\widehat{f} + gi \sim gi$ in $\widehat{K}$. Finally, suppose $g \asymp \phi$. Take $c \in \mathbb{R}^\times$ and $\varepsilon \in H$ with $g = c\phi(1+\varepsilon)$ and $\varepsilon \prec 1$. We have $f = \phi(1+\delta)$ where $\delta \in \mathcal{C}$, $\delta \prec 1$, so $f + gi = \phi(1+ci)(1+\rho)$ where $\rho = (1+ci)^{-1}(\delta + ci\varepsilon) \prec 1$ in $\mathcal{C}[i]$, so $f + gi \sim \phi(1+ci)$ in $\mathcal{C}[i]$. Likewise, $\widehat{f} + gi \sim \phi(1+ci)$ in $\widehat{K}$. $\square$

Corollary 6.6.13. *Suppose $f \in \mathcal{C}$ and $\widehat{f} \in \widehat{H}$. Then $f \approx_H \widehat{f}$ iff $f \approx_K \widehat{f}$.*

Proof. If $f \approx_K \widehat{f}$, then for all $\phi \in H$ we have $f - \phi \sim_K \widehat{f} - \phi$, so $f - \phi \sim_H \widehat{f} - \phi$ by Corollary 6.6.11, hence $f \approx_H \widehat{f}$. Conversely, suppose $f \approx_H \widehat{f}$. Then for all $\phi \in K$ we have $f - \operatorname{Re}\phi \sim_H \widehat{f} - \operatorname{Re}\phi$, so $f - \phi \sim_K \widehat{f} - \phi$ by Lemma 6.6.12. $\square$

Next we exploit that K is algebraically closed:

Lemma 6.6.14. *$f \approx_K \widehat{f} \implies Q(f) \sim_K Q(\widehat{f})$ for all $Q \in K[Y]^{\neq}$.*

Proof. Factor $Q \in K[Y]^{\neq}$ as

$$Q = a(Y - \phi_1) \cdots (Y - \phi_n), \qquad a \in K^{\times},\ \phi_1, \dots, \phi_n \in K$$

and use $f - \phi_j \sim_K \widehat{f} - \phi_j$ $(j = 1, \dots, n)$ and the complex version of Lemma 6.6.3. $\square$

This yields a more useful "complex" version of Corollary 6.6.4:

Corollary 6.6.15. *Suppose $f \approx_K \widehat{f}$. Then $\widehat{f}$ is transcendental over K, and:*

(i) *f generates over K a subfield $K(f)$ of $\mathcal{C}[\mathrm{i}]$;*
(ii) *we have a field isomorphism $\iota\colon K(f) \to K(\widehat{f})$ over K with $\iota(f) = \widehat{f}$;*
(iii) *$g \sim_K \iota(g)$ for all $g \in K(f)^{\times}$, hence for all $g_1, g_2 \in K(f)$:*

$$g_1 \preccurlyeq g_2 \text{ in } \mathcal{C}[\mathrm{i}] \iff \iota(g_1) \preccurlyeq \iota(g_2) \text{ in } \widehat{K}.$$

(*Thus the restriction of the binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ to $K(f)$ is a dominance relation on the field $K(f)$ in the sense of* [ADH, 3.1.1].)

In the next lemma $f = g + h\mathrm{i}$, $g, h \in \mathcal{C}$, and $\widehat{f} = \widehat{g} + \widehat{h}\mathrm{i}$, $\widehat{g}, \widehat{h} \in \widehat{H}$. Recall from Lemma 4.1.3 that if $\widehat{f} \notin K$, then $v(\widehat{g} - H) \subseteq v(\widehat{h} - H)$ or $v(\widehat{h} - H) \subseteq v(\widehat{g} - H)$.

Lemma 6.6.16. *Suppose $f \approx_K \widehat{f}$ and $v(\widehat{g} - H) \subseteq v(\widehat{h} - H)$. Then $g \approx_H \widehat{g}$.*

Proof. Let $\rho \in H$ be such that $\rho \sim \widehat{g}$; by Corollary 6.6.7 it is enough to show that then $g - \rho \sim_H \widehat{g} - \rho$. Take $\sigma \in H$ with $\widehat{g} - \rho \succcurlyeq \widehat{h} - \sigma$, and set $\phi := \rho + \sigma\mathrm{i} \in K$. Then

$$\mathrm{Re}(f - \phi) = g - \rho \quad \text{and} \quad \mathrm{Re}(\widehat{f} - \phi) = \widehat{g} - \rho \succcurlyeq \widehat{h} - \sigma = \mathrm{Im}(\widehat{f} - \phi),$$

and so by $f - \phi \sim_H \widehat{f} - \phi$ and Lemma 6.6.10 we have $g - \rho \sim_H \widehat{g} - \rho$. $\square$

Corollary 6.6.17. *If $f \approx_K \widehat{f}$, then $\mathrm{Re}\, f \approx_H \mathrm{Re}\, \widehat{f}$ or $\mathrm{Im}\, f \approx_H \mathrm{Im}\, \widehat{f}$.*

Let $g \in \mathcal{C}$ be eventually strictly increasing with $g(t) \to +\infty$ as $t \to +\infty$; we then have the subfield $K \circ g = (H \circ g)[\mathrm{i}]$ of $\mathcal{C}[\mathrm{i}]$. Suppose the valued field isomorphism

$$h \mapsto h \circ g\colon H \to H \circ g$$

is extended to a valued field isomorphism

$$\widehat{h} \mapsto \widehat{h} \circ g \ :\ \widehat{H} \to \widehat{H} \circ g,$$

where $\widehat{H} \circ g$ is an immediate valued field extension of the Hausdorff field $H \circ g$. In the same way we took a common valued field extension $\widehat{K} = \widehat{H}[\mathrm{i}]$ of $\widehat{H}$ and $K = H[\mathrm{i}]$ we now take a common valued field extension $\widehat{K} \circ g = (\widehat{H} \circ g)[\mathrm{i}]$ of $\widehat{H} \circ g$ and $K \circ g = (H \circ g)[\mathrm{i}]$. This makes $\widehat{K} \circ g$ an immediate extension of $K \circ g$, and we have a unique valued field isomorphism $y \mapsto y \circ g\colon \widehat{K} \to \widehat{K} \circ g$ extending the above map $\widehat{h} \mapsto \widehat{h} \circ g\colon \widehat{H} \to \widehat{H} \circ g$ and sending $\mathrm{i} \in \widehat{K}$ to $\mathrm{i} \in \widehat{K} \circ g$. This map $\widehat{K} \to \widehat{K} \circ g$ also extends $f \mapsto f \circ g\colon K \to K \circ g$ and is the identity on $\mathbb{C}$. See the commutative diagram below, where the labeled arrows are valued field isomorphisms and all unlabeled arrows are natural inclusions.

$$\begin{tikzcd}
 & \widehat{H}\circ g \arrow[rr] & & \widehat{K}\circ g \\
\widehat{H} \arrow[ur, "\widehat{h}\mapsto\widehat{h}\circ g"] \arrow[rr] & & \widehat{K} \arrow[ur, "y\mapsto y\circ g"] & \\
 & H\circ g \arrow[uu] \arrow[rr] & & K\circ g \arrow[uu] \\
H \arrow[uu] \arrow[ur, "h\mapsto h\circ g"'] \arrow[rr] & & K \arrow[uu] \arrow[ur, "f\mapsto f\circ g"'] &
\end{tikzcd}$$

Now we have

$$f \sim_K \widehat{f} \quad\Longleftrightarrow\quad f\circ g \sim_{K\circ g} \widehat{f}\circ g, \qquad f \approx_K \widehat{f} \quad\Longleftrightarrow\quad f\circ g \approx_{K\circ g} \widehat{f}\circ g.$$

At various places in the next section we use this for a Hardy field H and active $\phi > 0$ in H, with $g = \ell^{\text{inv}}$, $\ell \in \mathcal{C}^1$, $\ell' = \phi$. In that situation, $H^\circ := H \circ g$, $\widehat{H}^\circ := \widehat{H} \circ g$, and $h^\circ := h \circ g$, $\widehat{h}^\circ := \widehat{h} \circ g$ for $h \in H$ and $\widehat{h} \in \widehat{H}$, and likewise with K and $\widehat{K}$ and their elements instead of H and $\widehat{H}$.

6.7. Differentially Algebraic Hardy Field Extensions

In this section we are finally able to generate under reasonable conditions Hardy field extensions by solutions in $\mathcal{C}^{<\infty}$ of algebraic differential equations, culminating in the proof of our main theorem. We begin with a generality about enlarging differential fields within an ambient differential ring. Here, a *differential subfield* of a differential ring E is a differential subring of E whose underlying ring is a field.

Lemma 6.7.1. *Let K be a differential field with irreducible $P \in K\{Y\}^{\neq}$ of order $r \geqslant 1$, and E a differential ring extension of K with $y \in E$ such that $P(y) = 0$ and $Q(y) \in E^\times$ for all $Q \in K\{Y\}^{\neq}$ of order $< r$. Then y generates over K a differential subfield $K\langle y\rangle \supseteq K$ of E. Moreover, y has P as a minimal annihilator over K and $K\langle y\rangle$ equals*

$$\left\{ \frac{A(y)}{B(y)} :\ A, B \in K\{Y\},\ \text{order}\, A \leqslant r,\ \deg_{Y^{(r)}} A < \deg_{Y^{(r)}} P,\ B \neq 0,\ \text{order}\, B < r \right\}.$$

Proof. Let $p \in K[Y_0, \dots, Y_r]$ with distinct indeterminates $Y_0, \dots, Y_r$ be such that $P(Y) = p(Y, Y', \dots, Y^{(r)})$. The K-algebra morphism $K[Y_0, \dots, Y_r] \to E$ sending Y_i to $y^{(i)}$ for $i = 0, \dots, r$ extends to a K-algebra morphism $K(Y_0, \dots, Y_{r-1})[Y_r] \to E$ with p in its kernel, and so induces a K-algebra morphism

$$\iota\ :\ K(Y_0, \dots, Y_{r-1})[Y_r]/(p) \to E, \qquad (p)\ :=\ pK(Y_0, \dots, Y_{r-1})[Y_r].$$

Now p as an element of $K(Y_0, \dots, Y_{r-1})[Y_r]$ remains irreducible [122, Chapter IV, §2]. Thus $K(Y_0, \dots, Y_{r-1})[Y_r]/(p)$ is a field, so ι is injective, and it is routine to check that the image of ι is $K\langle y\rangle$ as described; see also [ADH, 4.1.6]. □

In passing we also note the obvious d-transcendental version of this lemma:

Lemma 6.7.2. *Let K be a differential field and E be a differential ring extension of K with $y \in E$ such that $Q(y) \in E^\times$ for all $Q \in K\{Y\}^{\neq}$. Then y generates* *over K a differential subfield $K\langle y\rangle$ of E. Moreover, y is* d*-transcendental over K and*

$$K\langle y\rangle \;=\; \left\{\frac{P(y)}{Q(y)}:\ P,Q\in K\{Y\},\ Q\neq 0\right\}.$$

We now apply the material above to generate Hardy field extensions.

Application to Hardy fields. *In the rest of this section H is a real closed Hardy field, $H\supseteq\mathbb{R}$, and $\widehat{H}$ is an immediate H-field extension of H.* Let $f\in\mathcal{C}^{<\infty}$ and $\widehat{f}\in\widehat{H}$. Note that if $Q\in H\{Y\}$ and $Q(f)\sim_H Q(\widehat{f})$, then $Q(f)\in\mathcal{C}^\times$. Hence by Lemma 6.7.1 with $E=\mathcal{C}^{<\infty}$, $K=H$, we have:

Lemma 6.7.3. *Suppose $\widehat{f}$ is* d*-algebraic over H with minimal annihilator P over H of order $r\geqslant 1$, and $P(f)=0$ and $Q(f)\sim_H Q(\widehat{f})$ for all $Q\in H\{Y\}\setminus H$ with* $\operatorname{order}Q<r$*. Then $f\notin H$ and:*

(i) *f is hardian over H;*

(ii) *we have a (necessarily unique) isomorphism $\iota\colon H\langle f\rangle\to H\langle\widehat{f}\rangle$ of differential fields over H such that $\iota(f)=\widehat{f}$.*

With an extra assumption ι in Lemma 6.7.3 is an isomorphism of H-fields:

Corollary 6.7.4. *Let $\widehat{f}$, f, P, r, ι be as in Lemma 6.7.3, and suppose also that $Q(f)\sim_H Q(\widehat{f})$ for all $Q\in H\{Y\}$ with* $\operatorname{order}Q=r$ *and* $\deg_{Y^{(r)}}Q<\deg_{Y^{(r)}}P$*. Then $g\sim_H\iota(g)$ for all $g\in H\langle f\rangle^\times$, hence for $g_1,g_2\in H\langle f\rangle$ we have*

$$g_1\preccurlyeq g_2 \text{ in } \mathcal{C} \iff \iota(g_1)\preccurlyeq\iota(g_2) \text{ in } \widehat{H}.$$

Moreover, ι is an isomorphism of H-fields.

Proof. Most of this follows from Lemmas 6.6.3 and 6.7.3 and the description of $H\langle f\rangle$ in Lemma 6.7.1. For the last statement, use [ADH, 10.5.8]. $\square$

Here is a d-transcendental version of Lemma 6.7.3:

Lemma 6.7.5. *Suppose $Q(f)\sim_H Q(\widehat{f})$ for all $Q\in H\{Y\}\setminus H$. Then:*

(i) *f is hardian over H;*

(ii) *we have a (necessarily unique) isomorphism $\iota\colon H\langle f\rangle\to H\langle\widehat{f}\rangle$ of differential fields over H with $\iota(f)=\widehat{f}$; and*

(iii) *$g\sim_H\iota(g)$ for all $g\in H\langle f\rangle^\times$, hence for all $g_1,g_2\in H\langle f\rangle$:*

$$g_1\preccurlyeq g_2 \text{ in } \mathcal{C} \iff \iota(g_1)\preccurlyeq\iota(g_2) \text{ in } \widehat{H}.$$

Moreover, ι is an isomorphism of H-fields.

This follows easily from Lemma 6.7.2.

Analogues for $K=H[\mathrm{i}]$. We have the d-valued extension $K:=H[\mathrm{i}]\subseteq\mathcal{C}^{<\infty}[\mathrm{i}]$ of H. As before we arrange that $\widehat{K}=\widehat{H}[\mathrm{i}]$ is a d-valued extension of $\widehat{H}$ as well as an an immediate extension of K. Let $f\in\mathcal{C}^{<\infty}[\mathrm{i}]$ and $\widehat{f}\in\widehat{K}$. We now have the obvious "complex" analogues of Lemma 6.7.3 and Corollary 6.7.4:

Lemma 6.7.6. *Suppose $\widehat{f}$ is* d*-algebraic over K with minimal annihilator P over K of order $r\geqslant 1$, and $P(f)=0$ and $Q(f)\sim_K Q(\widehat{f})$ for all $Q\in K\{Y\}\setminus K$ with* $\operatorname{order}Q<r$*. Then*

(i) *f generates over K a differential subfield $K\langle f\rangle$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$;*

(ii) *we have a (necessarily unique) isomorphism* $\iota\colon K\langle f\rangle \to K\langle \widehat{f}\rangle$ *of differential fields over* K *such that* $\iota(f) = \widehat{f}$.

Corollary 6.7.7. *Let* $\widehat{f}$, f, P, r, ι *be as in Lemma 6.7.6, and suppose also that* $Q(f) \sim_K Q(\widehat{f})$ *for all* $Q \in K\{Y\}$ *with* $\operatorname{order} Q = r$ *and* $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$. *Then* $g \sim_K \iota(g)$ *for all* $g \in K\langle f\rangle^\times$, *so for all* $g_1, g_2 \in K\langle f\rangle$ *we have:*

$$g_1 \preccurlyeq g_2 \text{ in } \mathcal{C}[\mathrm{i}] \iff \iota(g_1) \preccurlyeq \iota(g_2) \text{ in } \widehat{K}.$$

Thus the relation $\preccurlyeq$ *on* $\mathcal{C}[\mathrm{i}]$ *restricts to a dominance relation on the field* $K\langle f\rangle$.

From K being algebraically closed we obtain a useful variant of Corollary 6.7.7:

Corollary 6.7.8. *Suppose* $f \asymp_K \widehat{f}$, *and* $P \in K\{Y\}$ *is irreducible with*

$$\operatorname{order} P \ = \ \deg_{Y'} P \ = \ 1, \quad P(f) = 0 \ \text{ in } \ \mathcal{C}^{<\infty}[\mathrm{i}], \quad P(\widehat{f}) = 0 \ \text{ in } \ \widehat{K}.$$

Then P *is a minimal annihilator of* $\widehat{f}$ *over* K, f *generates over* K *a differential subfield* $K\langle f\rangle = K(f)$ *of* $\mathcal{C}^{<\infty}[\mathrm{i}]$, *and we have an isomorphism* $\iota\colon K\langle f\rangle \to K\langle\widehat{f}\rangle$ *of differential fields over* K *such that* $\iota(f) = \widehat{f}$ *and* $g \sim_K \iota(g)$ *for all* $g \in K\langle f\rangle^\times$. *Thus for all* $g_1, g_2 \in K\langle f\rangle$: $g_1 \preccurlyeq g_2$ *in* $\mathcal{C}[\mathrm{i}] \iff \iota(g_1) \preccurlyeq \iota(g_2)$ *in* $\widehat{K}$.

Proof. By Corollary 6.6.15, $\widehat{f}$ is transcendental over K, so P is a minimal annihilator of $\widehat{f}$ over K by [ADH, 4.1.6]. Now use Lemma 6.7.1 and Corollary 6.6.15. $\square$

This corollary leaves open whether $\operatorname{Re} f$ or $\operatorname{Im} f$ is hardian over H. This issue is critical for us and we treat a special case in Proposition 6.7.18 below. The example following Corollary 5.4.24 shows that that there is a differential subfield of $\mathcal{C}^{<\infty}[\mathrm{i}]$ such that the binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on it, but which is not contained in $F[\mathrm{i}]$ for any Hardy field F.

Sufficient conditions for asymptotic similarity. Let $\widehat{h}$ be an element of our immediate H-field extension $\widehat{H}$ of H. Note that in the next variant of [ADH, 11.4.3] we use ddeg instead of ndeg.

Lemma 6.7.9. *Let* $Q \in H\{Y\}^{\neq}$, $r := \operatorname{order} Q$, $h \in H$, *and* $\mathfrak{v} \in H^\times$ *be such that* $\widehat{h} - h \prec \mathfrak{v}$ *and* $\operatorname{ddeg}_{\prec\mathfrak{v}} Q_{+h} = 0$, *and assume* $y \in \mathcal{C}^{<\infty}$ *and* $\mathfrak{m} \in H^\times$ *satisfies*

$$y - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}, \qquad \left(\frac{y-h}{\mathfrak{m}}\right)', \dots, \left(\frac{y-h}{\mathfrak{m}}\right)^{(r)} \preccurlyeq 1.$$

Then $Q(y) \sim Q(h)$ *in* $\mathcal{C}^{<\infty}$ *and* $Q(h) \sim Q(\widehat{h})$ *in* $\widehat{H}$*; in particular,* $Q(y) \sim_H Q(\widehat{h})$.

Proof. We have $y = h + \mathfrak{m}u$ with $u = \frac{y-h}{\mathfrak{m}} \in \mathcal{C}^{<\infty}$ and $u, u', \dots, u^{(r)} \preccurlyeq 1$. Now

$$Q_{+h,\times\mathfrak{m}} \ = \ Q(h) + R \qquad \text{with } R \in H\{Y\},\ R(0) = 0,$$

which in view of $\operatorname{ddeg} Q_{+h,\times\mathfrak{m}} = 0$ gives $R \prec Q(h)$. Thus

$$Q(y) \ = \ Q_{+h,\times\mathfrak{m}}(u) \ = \ Q(h) + R(u), \qquad R(u) \ \preccurlyeq \ R \ \prec \ Q(h),$$

so $Q(y) \sim Q(h)$ in $\mathcal{C}^{<\infty}$. Increasing $|\mathfrak{m}|$ if necessary we arrange $\widehat{h} - h \preccurlyeq \mathfrak{m}$, and then a similar computation with $\widehat{h}$ instead of y gives $Q(h) \sim Q(\widehat{h})$ in $\widehat{H}$. $\square$

In the remainder of this subsection we assume that H *is ungrounded and* $H \neq \mathbb{R}$.

Corollary 6.7.10. *Suppose $\widehat{h}$ is* d*-algebraic over H with minimal annihilator P over H of order $r \geqslant 1$, and let $y \in \mathcal{C}^{<\infty}$ satisfy $P(y) = 0$. Suppose for all Q in $H\{Y\} \setminus H$ of order $< r$ there are $h \in H$, $\mathfrak{m}, \mathfrak{v} \in H^\times$, and an active $\phi > 0$ in H such that $\widehat{h} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$,* $\operatorname{ddeg}_{\prec\mathfrak{v}} Q^{\phi}_{+h} = 0$*, and*

$$\derive^j\left(\frac{y-h}{\mathfrak{m}}\right) \preccurlyeq 1 \qquad \textit{for } j = 0, \dots, r-1 \textit{ and } \derive := \phi^{-1}\partial.$$

Then $y \notin H$ and y is hardian over H.

Proof. Let $Q \in H\{Y\} \setminus H$ have order $< r$, and take h, $\mathfrak{m}$, $\mathfrak{v}$, ϕ as in the statement of the corollary. By Lemma 6.7.3 it is enough to show that then $Q(y) \sim_H Q(\widehat{h})$. We use $(\ \ )^\circ$ as explained at the beginning of Section 6.4. Thus we have the Hardy field H° and the H-field isomorphism $h \mapsto h^\circ \colon H^\phi \to H^\circ$, extended to an H-field isomorphism $\widehat{f} \mapsto \widehat{f}^\circ \colon \widehat{H}^\phi \to \widehat{H}^\circ$, for an immediate H-field extension $\widehat{H}^\circ$ of H°. Set $u := (y-h)/\mathfrak{m} \in \mathcal{C}^{<\infty}$. We have $\operatorname{ddeg}_{\prec\mathfrak{v}^\circ} Q^{\phi\circ}_{+h^\circ} = 0$ and $(u^\circ)^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r-1$; hence $Q^{\phi\circ}(y^\circ) \sim_{H^\circ} Q^{\phi\circ}(\widehat{h}^\circ)$ by Lemma 6.7.9. Now $Q^{\phi\circ}(y^\circ) = Q(y)^\circ$ in $\mathcal{C}^{<\infty}$ and $Q^{\phi\circ}(\widehat{h}^\circ) = Q(\widehat{h})^\circ$ in $\widehat{H}^\circ$, hence $Q(y) \sim_H Q(\widehat{h})$. $\square$

Using Corollary 6.7.4 instead of Lemma 6.7.3 we show likewise:

Corollary 6.7.11. *Suppose $\widehat{h}$ is* d*-algebraic over H with minimal annihilator P over H of order $r \geqslant 1$, and let $y \in \mathcal{C}^{<\infty}$ satisfy $P(y) = 0$. Suppose for all Q in $H\{Y\} \setminus H$ with* $\operatorname{order} Q \leqslant r$ *and* $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$ *there are $h \in H$, $\mathfrak{m}, \mathfrak{v} \in H^\times$, and an active $\phi > 0$ in H such that $\widehat{h} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$,* $\operatorname{ddeg}_{\prec\mathfrak{v}} Q^{\phi}_{+h} = 0$*, and*

$$\derive^j\left(\frac{y-h}{\mathfrak{m}}\right) \preccurlyeq 1 \qquad \textit{for } j = 0, \dots, r \textit{ and } \derive := \phi^{-1}\partial.$$

Then y is hardian over H and there is an isomorphism $H\langle y\rangle \to H\langle \widehat{h}\rangle$ of H-fields over H sending y to $\widehat{h}$.

In the next subsection we use Corollary 6.7.11 to fill in certain kinds of holes in Hardy fields. Recall from [ADH, remark after 11.4.3] that if $\widehat{h} \notin H$ and $Z(H, \widehat{h}) = \emptyset$, then $\widehat{h}$ is d-transcendental over H. The next result is a version of Corollary 6.7.11 for that situation. (This will not be used until Section 7.5 below.)

Corollary 6.7.12. *Suppose $\widehat{h} \notin H$ and $Z(H, \widehat{h}) = \emptyset$. Let $y \in \mathcal{C}^{<\infty}$ be such that for all $h \in H$, $\mathfrak{m} \in H^\times$ with $\widehat{h} - h \preccurlyeq \mathfrak{m}$ and all n there is an active ϕ_0 in H such that for all active $\phi > 0$ in H with $\phi \preccurlyeq \phi_0$ we have $\derive^n\big(\frac{y-h}{\mathfrak{m}}\big) \preccurlyeq 1$ for $\derive = \phi^{-1}\partial$. Then y is hardian over H, and there is an isomorphism $H\langle y\rangle \to H\langle\widehat{h}\rangle$ of H-fields over H sending y to $\widehat{h}$.*

Proof. Let $Q \in H\{Y\} \setminus H$; by Lemma 6.7.5 it is enough to show that $Q(y) \sim_H Q(\widehat{h})$. Since $Q \notin Z(H, \widehat{h})$, we obtain $h \in H$ and $\mathfrak{m}, \mathfrak{v} \in H^\times$ such that $\widehat{h} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+h} = 0$. Let $r := \operatorname{order} Q$ and choose an active $\phi > 0$ in H such that $\operatorname{ddeg}_{\prec\mathfrak{v}} Q^{\phi}_{+h} = 0$ and $\derive^n\big(\frac{y-h}{\mathfrak{m}}\big) \preccurlyeq 1$ for $\derive = \phi^{-1}\partial$ and $n = 0, \dots, r$. As in the proof of Corollary 6.7.10 this yields $Q(y) \sim_H Q(\widehat{h})$. $\square$

Generating immediate d-algebraic Hardy field extensions. *In this subsection H is Liouville closed, $(P,\mathfrak{n},\widehat{h})$ is a special Z-minimal slot in H of order $r\geqslant 1$, $K:=H[\mathrm{i}]\subseteq\mathcal{C}^{<\infty}[\mathrm{i}]$, $\mathrm{I}(K)\subseteq K^{\dagger}$, and K is 1-linearly surjective if $r\geqslant 3$.* We first treat the case where $(P,\mathfrak{n},\widehat{h})$ is a hole in H (not just a slot):

Theorem 6.7.13. *Assume $(P,\mathfrak{n},\widehat{h})$ is a deep, ultimate, and strongly repulsive-normal hole in H, and $y\in\mathcal{C}^{<\infty}$, $P(y)=0$, $y\prec\mathfrak{n}$. Then y is hardian over H, and there is an isomorphism $H\langle y\rangle\to H\langle\widehat{h}\rangle$ of H-fields over H sending y to $\widehat{h}$.*

Proof. Replacing $(P,\mathfrak{n},\widehat{h})$, y by $(P_{\times\mathfrak{n}},1,\widehat{h}/\mathfrak{n})$, $y/\mathfrak{n}$ we arrange $\mathfrak{n}=1$. Let Q in $H\{Y\}\setminus H$, $\operatorname{order}Q\leqslant r$, and $\deg_{Y^{(r)}}Q<\deg_{Y^{(r)}}P$. Then $Q\notin Z(H,\widehat{h})$, so we have $h\in H$ and $\mathfrak{v}\in H^{\times}$ such that $h-\widehat{h}\prec\mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}}Q_{+h}=0$. Take any $\mathfrak{m}\in H^{\times}$ with $\widehat{h}-h\preccurlyeq\mathfrak{m}\prec\mathfrak{v}$. Take $\mathfrak{w}\in H^{\times}$ with $\mathfrak{m}\prec\mathfrak{w}\prec\mathfrak{v}$. Then $\operatorname{ndeg}Q_{+h,\times\mathfrak{w}}=0$, so we have active ϕ in H, $0<\phi\prec 1$, with $\operatorname{ddeg}Q^{\phi}_{+h,\times\mathfrak{w}}=0$, and hence $\operatorname{ddeg}_{\prec\mathfrak{w}}Q^{\phi}_{+h}=0$. Thus renaming $\mathfrak{w}$ as $\mathfrak{v}$ we have arranged $\operatorname{ddeg}_{\prec\mathfrak{v}}Q^{\phi}_{+h}=0$.

Set $\delta:=\phi^{-1}\partial$; by Corollary 6.7.11 it is enough to show that $\delta^{j}\big(\frac{y-h}{\mathfrak{m}}\big)\preccurlyeq 1$ for $j=0,\dots,r$. Now using $(\ \ )^{\circ}$ as before, the hole $(P^{\phi\circ},1,\widehat{h}^{\circ})$ in H° is special, Z-minimal, deep, ultimate, and strongly repulsive-normal, by Lemmas 6.4.3 and 6.4.4. It remains to apply Corollary 6.5.20 to this hole in H° with h°, $\mathfrak{m}^{\circ}$, y° in place of h, $\mathfrak{m}$, y. $\square$

Corollary 6.7.14. *Let ϕ be active in H, $0<\phi\preccurlyeq 1$, and suppose the slot $(P^{\phi},\mathfrak{n},\widehat{h})$ in H^{ϕ} is deep, ultimate, and strongly split-normal. Then $P(y)=0$ and $y\prec\mathfrak{n}$ for some $y\in\mathcal{C}^{<\infty}$. If $(P^{\phi},\mathfrak{n},\widehat{h})$ is strongly repulsive-normal, then any such y is hardian over H with $y\notin H$.*

Proof. Lemma 6.4.6 gives $y\in\mathcal{C}^{<\infty}$ with $P(y)=0$, $y\prec\mathfrak{n}$. Now suppose $(P^{\phi},\mathfrak{n},\widehat{h})$ is strongly repulsive-normal, and $y\in\mathcal{C}^{<\infty}$, $P(y)=0$, $y\prec\mathfrak{n}$. Using Lemma 3.2.14 we arrange that $(P,\mathfrak{n},\widehat{h})$ is a hole in H. The hole $(P^{\phi\circ},\mathfrak{n}^{\circ},\widehat{h}^{\circ})$ in H° is special, Z-minimal, deep, ultimate, and strongly repulsive-normal. Then Theorem 6.7.13 with H°, $(P^{\phi\circ},\mathfrak{n}^{\circ},\widehat{h}^{\circ})$, y° in place of H, $(P,\mathfrak{n},\widehat{h})$, y shows that y° is hardian over H° with $y^{\circ}\notin H^{\circ}$. Hence y is hardian over H and $y\notin H$. $\square$

Achieving 1-linear newtonianity. For the proof of our main theorem we need to show first that for any d-maximal Hardy field H the corresponding $K=H[\mathrm{i}]$ is 1-linearly newtonian, the latter being a key hypothesis in Lemma 6.7.21 below. In this subsection we take this vital step: Corollary 6.7.20.

Lemma 6.7.15. *Every* d-*maximal Hardy field is* 1-*newtonian.*

Proof. Let H be a d-maximal Hardy field. Then H satisfies the conditions at the beginning of the previous subsection, by Corollary 5.5.19 and Theorem 5.6.2, for any special Z-minimal slot $(P,\mathfrak{n},\widehat{h})$ in H of order 1. By Corollary 1.8.29, H is 1-linearly newtonian. Towards a contradiction assume that H is not 1-newtonian. Then Lemma 3.2.1 yields a minimal hole $(P,\mathfrak{n},\widehat{h})$ in H of order $r=1$. Using Lemma 3.2.26 we replace $(P,\mathfrak{n},\widehat{h})$ by a refinement to arrange that $(P,\mathfrak{n},\widehat{h})$ is quasilinear. Then $(P,\mathfrak{n},\widehat{h})$ is special, by Lemma 3.2.36. Using Corollary 4.5.42 we further refine $(P,\mathfrak{n},\widehat{h})$ to arrange that $(P^{\phi},\mathfrak{n},\widehat{h})$ is eventually deep, ultimate, and strongly repulsive-normal. Now Corollary 6.7.14 gives a proper d-algebraic Hardy field extension of H, contradicting d-maximality of H. $\square$

In the rest of this subsection H has asymptotic integration. We have the d-valued extension $K := H[i] \subseteq \mathcal{C}^{<\infty}[i]$ of H and as before we arrange that $\widehat{K} = \widehat{H}[i]$ is a d-valued extension of $\widehat{H}$ as well as an immediate d-valued extension of K.

Lemma 6.7.16. *Suppose H is Liouville closed and $\mathrm{I}(K) \subseteq K^\dagger$. Let $(P, \mathfrak{n}, \widehat{f})$ be an ultimate linear minimal hole in K of order $r \geqslant 1$, where $\widehat{f} \in \widehat{K}$, such that $\dim_{\mathbb{C}} \ker_{\mathrm{U}} L_P = r$. Assume also that K is $\upomega$-free if $r \geqslant 2$. Let $f \in \mathcal{C}^{<\infty}[i]$ be such that $P(f) = 0$, $f \prec \mathfrak{n}$. Then $f \approx_K \widehat{f}$.*

Proof. Replacing $(P, \mathfrak{n}, \widehat{f})$, f by $(P_{\times\mathfrak{n}}, 1, \widehat{f}/\mathfrak{n})$, $f/\mathfrak{n}$ we arrange $\mathfrak{n} = 1$. Let $\theta \in K^\times$ be such that $\theta \sim \widehat{f}$; we claim that $f \sim \theta$ in $\mathcal{C}[i]$ (and so $f \sim_K \widehat{f}$). Applying Proposition 6.4.9 and Remark 6.4.11 to the linear minimal hole $(P_{+\theta}, \theta, \widehat{f} - \theta)$ in K gives $g \in \mathcal{C}^{<\infty}[i]$ such that $P_{+\theta}(g) = 0$ and $g \prec \theta$. Then $P(\theta + g) = 0$ and $\theta + g \prec 1$, thus $L_P(y) = 0$ and $y \prec 1$ for $y := f - (\theta + g) \in \mathcal{C}^{<\infty}[i]$. Hence $y \prec \theta$ by the version of Lemma 5.10.13 for slots in K; see the remark following Corollary 5.10.16. Therefore $f - \theta = y + g \prec \theta$ and so $f \sim \theta$, as claimed.

The refinement $(P_{+\theta}, 1, \widehat{f} - \theta)$ of $(P, 1, \widehat{f})$ is ultimate thanks to the K-version of Lemma 4.4.10, and $L_{P_{+\theta}} = L_P$, so we can apply the claim to $(P_{+\theta}, 1, \widehat{f} - \theta)$ instead of $(P, 1, \widehat{f})$ and $f - \theta$ instead of f to give $f - \theta \sim_K \widehat{f} - \theta$. Since this holds for all $\theta \in K$ with $\theta \sim \widehat{f}$, the K-version of Corollary 6.6.7 then yields $f \approx_K \widehat{f}$. $\square$

Corollary 6.7.17. *Let $(P, \mathfrak{n}, \widehat{f})$ be a linear hole of order 1 in K. (We do not assume here that $\widehat{f} \in \widehat{K}$.) Then there is an embedding $\iota\colon K\langle\widehat{f}\rangle \to \mathcal{C}^{<\infty}[i]$ of differential K-algebras such that $\iota(g) \sim_K g$ for all $g \in K\langle\widehat{f}\rangle^\times$.*

Proof. Note that $(P, \mathfrak{n}, \widehat{f})$ is minimal. We first show how to arrange that H is Liouville closed and $\upomega$-free with $\mathrm{I}(K) \subseteq K^\dagger$ and $\widehat{f} \in \widehat{K}$. Let H_1 be a maximal Hardy field extension of H. Then H_1 is Liouville closed and $\upomega$-free, with $\mathrm{I}(K_1) \subseteq K_1^\dagger$ for $K_1 := H_1[i] \subseteq \mathcal{C}^{<\infty}[i]$. Let $\widehat{H}_1$ be the newtonization of H_1; then $\widehat{K}_1 := \widehat{H}_1[i]$ is newtonian [ADH, 14.5.7]. Corollary 3.2.29 gives an embedding $K\langle\widehat{f}\rangle \to \widehat{K}_1$ of valued differential fields over K; let $\widehat{f}_1$ be the image of $\widehat{f}$ under this embedding. If $\widehat{f}_1 \in K_1 \subseteq \mathcal{C}^{<\infty}[i]$, then we are done, so assume $\widehat{f}_1 \notin K_1$. Then $(P, \mathfrak{n}, \widehat{f}_1)$ is a hole in K_1, and we replace H, K, $(P, \mathfrak{n}, \widehat{f})$ by H_1, K_1, $(P, \mathfrak{n}, \widehat{f}_1)$, and $\widehat{K}$ by $\widehat{K}_1$, to arrange that H is Liouville closed and $\upomega$-free with $\mathrm{I}(K) \subseteq K^\dagger$ and $\widehat{f} \in \widehat{K}$.

Replacing $(P, \mathfrak{n}, \widehat{f})$ by a refinement we also arrange that $(P, \mathfrak{n}, \widehat{f})$ is ultimate and $\mathfrak{n} \in H^\times$, by Proposition 4.4.18 and Remark 4.4.19. Then Proposition 6.4.9 yield an $f \in \mathcal{C}^{<\infty}[i]$ with $P(f) = 0$, $f \prec \mathfrak{n}$. Now Lemma 6.7.16 gives $f \approx_K \widehat{f}$, and it remains to appeal to Corollary 6.7.8. $\square$

Proposition 6.7.18. *Suppose H is $\upomega$-free and 1-newtonian. Let $(P, \mathfrak{n}, \widehat{f})$ be a linear hole in K of order 1 with $\widehat{f} \in \widehat{K}$, and $f \in \mathcal{C}^{<\infty}[i]$, $P(f) = 0$, and $f \approx_K \widehat{f}$. Then $\operatorname{Re} f$ or $\operatorname{Im} f$ generates a proper d-algebraic Hardy field extension of H.*

Proof. Let $\widehat{g} := \operatorname{Re} \widehat{f}$ and $\widehat{h} := \operatorname{Im} \widehat{f}$. By Lemma 4.1.3 we have $v(\widehat{g} - H) \subseteq v(\widehat{h} - H)$ or $v(\widehat{h} - H) \subseteq v(\widehat{g} - H)$. Below we assume $v(\widehat{g} - H) \subseteq v(\widehat{h} - H)$ (so $\widehat{g} \in \widehat{H} \setminus H$) and show that then $g := \operatorname{Re} f$ generates a proper d-algebraic Hardy field extension of H. (If $v(\widehat{h} - H) \subseteq v(\widehat{g} - H)$ one shows likewise that $\operatorname{Im} f$ generates a proper d-algebraic Hardy field extension of H.) The hole $(P, \mathfrak{n}, \widehat{f})$ in K is minimal, and by arranging $\mathfrak{n} \in H^\times$ we see that $\widehat{g}$ is d-algebraic over H, by a remark preceding Lemma 4.3.7.

Every element of $Z(H,\widehat{g})$ has order $\geqslant 2$, by Corollary 3.2.16 and 1-newtonianity of H. We arrange that the linear part A of P is monic, so $A = \partial - a$ with $a \in K$, $A(\widehat{f}) = -P(0)$ and $A(f) = -P(0)$. Then Example 1.1.7 and Remark 1.1.9 applied to $F = \mathcal{C}^{<\infty}$ yields $Q \in H\{Y\}$ with $1 \leqslant \operatorname{order} Q \leqslant 2$ and $\deg Q = 1$ such that $Q(\widehat{g}) = 0$ and $Q(g) = 0$. Hence $\operatorname{order} Q = 2$ and Q is a minimal annihilator of $\widehat{g}$ over H.

Towards applying Corollary 6.7.10 to Q, $\widehat{g}$, g in the role of P, $\widehat{h}$, y there, let R in $H\{Y\} \setminus H$ have order < 2. Then $R \notin Z(H,\widehat{g})$, so we have $h \in H$ and $\mathfrak{v} \in H^\times$ such that $\widehat{g} - h \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} R_{+h} = 0$. Take any $\mathfrak{m} \in H^\times$ with $\widehat{g} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$. By Lemma 6.6.16 we have $g \approx_H \widehat{g}$ and thus $g - h \preccurlyeq \mathfrak{m}$. After changing $\mathfrak{v}$ as in the proof of Theorem 6.7.13 we obtain an active ϕ in H, $0 < \phi \preccurlyeq 1$, such that $\operatorname{ddeg}_{\prec\mathfrak{v}} R^{\phi}_{+h} = 0$. Set $\delta := \phi^{-1}\partial$; by Corollary 6.7.10 it is now enough to show that $\delta\big((g-h)/\mathfrak{m}\big) \preccurlyeq 1$.

Towards this and using $(\ \ )^\circ$ as before, we have $f^\circ \approx_{K^\circ} \widehat{f}^\circ$, and $g^\circ \approx_{H^\circ} \widehat{g}^\circ$ by the facts about composition in Section 6.6. Moreover, $(g-h)^\circ \preccurlyeq \mathfrak{m}^\circ$, and H° is $\upomega$-free and 1-newtonian, hence closed under integration by [ADH, 14.2.2]. We now apply Corollary 6.7.8 with H°, K°, $P^{\phi\circ}$, f°, $\widehat{f}^\circ$ in the role of H, K, P, f, $\widehat{f}$ to give

$$\big(f^\circ/\mathfrak{m}^\circ\big)' \approx_{K^\circ} \big(\widehat{f}^\circ/\mathfrak{m}^\circ\big)',$$

hence $(g^\circ/\mathfrak{m}^\circ)' \approx_{H^\circ} (\widehat{g}^\circ/\mathfrak{m}^\circ)'$ by Lemmas 4.1.4 and 6.6.16. Therefore,

$$\big((g-h)^\circ/\mathfrak{m}^\circ\big)' = (g^\circ/\mathfrak{m}^\circ)' - (h^\circ/\mathfrak{m}^\circ)' \sim_H (\widehat{g}^\circ/\mathfrak{m}^\circ)' - (h^\circ/\mathfrak{m}^\circ)' = \big((\widehat{g}-h)^\circ/\mathfrak{m}^\circ\big)'.$$

Now $(\widehat{g}-h)^\circ/\mathfrak{m}^\circ \preccurlyeq 1$, so $\big((\widehat{g}-h)^\circ/\mathfrak{m}^\circ\big)' \prec 1$, hence $\big((g-h)^\circ/\mathfrak{m}^\circ\big)' \prec 1$ by the last display, and thus $\delta\big((g-h)/\mathfrak{m}\big) \prec 1$, which is more than enough. □

If K has a linear hole of order 1, then K has a proper d-algebraic differential field extension inside $\mathcal{C}^{<\infty}[i]$, by Corollary 6.7.17. We can now prove a Hardy analogue:

Lemma 6.7.19. *Suppose K has a linear hole of order 1. Then H has a proper* d*-algebraic Hardy field extension.*

Proof. If H is not d-maximal, then H has indeed a proper d-algebraic Hardy field extension, and if H is d-maximal, then H is Liouville closed, $\upomega$-free, 1-newtonian, and $\mathrm{I}(K) \subseteq K^\dagger$, by Proposition 5.3.2, Corollary 5.5.19, Theorem 5.6.2, and Lemma 6.7.15. So assume below that H is Liouville closed, $\upomega$-free, 1-newtonian, and $\mathrm{I}(K) \subseteq K^\dagger$, and that $(P,\mathfrak{n},\widehat{f})$ is a linear hole of order 1 in K. By Lemma 4.2.15 we arrange that $\widehat{f} \in \widehat{K} := \widehat{H}[i]$ where $\widehat{H}$ is an immediate $\upomega$-free newtonian H-field extension of H. Then $\widehat{K}$ is also newtonian by [ADH, 14.5.7]. By Remark 4.4.19 we can replace $(P,\mathfrak{n},\widehat{f})$ by a refinement to arrange that $(P,\mathfrak{n},\widehat{f})$ is ultimate and $\mathfrak{n} \in H^\times$. Proposition 6.4.9 now yields $f \in \mathcal{C}^{<\infty}[i]$ with $P(f) = 0$ and $f \prec \mathfrak{n}$. Then $f \approx_K \widehat{f}$ by Lemma 6.7.16, and so $\operatorname{Re} f$ or $\operatorname{Im} f$ generates a proper d-algebraic Hardy field extension of H, by Proposition 6.7.18. □

Corollary 6.7.20. *If H is* d*-maximal, then K is* 1*-linearly newtonian.*

Proof. Assume H is d-maximal. Then K is $\upomega$-free by Theorem 5.6.2 and [ADH, 11.7.23]. If K is not 1-linearly newtonian, then it has a linear hole of order 1, by Lemma 3.2.5, and so H has a proper d-algebraic Hardy field extension, by Lemma 6.7.19, contradicting d-maximality of H. □

Finishing the story. With one more lemma we will be done.

Lemma 6.7.21. *Suppose H is Liouville closed, ω-free, not newtonian, and $K := H[\mathrm{i}]$ is 1-linearly newtonian. Then H has a proper* d*-algebraic Hardy field extension.*

Proof. By Proposition 1.8.28, K is 1-linearly surjective and $\mathrm{I}(K) \subseteq K^{\dagger}$. Since H is not newtonian, neither is K, by [ADH, 14.5.6], and so by Lemma 3.2.1 we have a minimal hole $(P, \mathfrak{m}, \widehat{f})$ in K of order $r \geqslant 1$, with $\mathfrak{m} \in H^{\times}$. Then $\deg P > 1$ by Corollary 3.2.8. As in the proof of Lemma 6.7.19 we take for $\widehat{H}$ an immediate ω-free newtonian H-field extension of H and arrange $\widehat{f} \in \widehat{K} := \widehat{H}[\mathrm{i}]$. Now $\widehat{f} = \widehat{g} + \widehat{h}\mathrm{i}$ with $\widehat{g}, \widehat{h} \in \widehat{H}$. By Theorem 4.5.43, there are two cases:

1. $\widehat{g} \notin H$ and some Z-minimal slot $(Q, \mathfrak{m}, \widehat{g})$ in H has a special refinement $(Q_{+g}, \mathfrak{n}, \widehat{g} - g)$ such that $(Q^{\phi}_{+g}, \mathfrak{n}, \widehat{g} - g)$ is eventually deep, strongly repulsive-normal, and ultimate;
2. $\widehat{h} \notin H$ and some Z-minimal slot $(R, \mathfrak{m}, \widehat{h})$ in H has a special refinement $(R_{+h}, \mathfrak{n}, \widehat{h} - h)$ such that $(R^{\phi}_{+h}, \mathfrak{n}, \widehat{h} - h)$ is eventually deep, strongly repulsive-normal, and ultimate.

Suppose $\widehat{g} \notin H$ and $(Q, \mathfrak{m}, \widehat{g})$ is as in (1). Then $1 \leqslant \operatorname{order} Q \leqslant 2r$ by Lemma 4.3.7. *Claim*: $Q(y) = 0$ for some $y \in \mathcal{C}^{<\infty} \setminus H$ that is hardian over H. To prove this claim, take a special refinement $(Q_{+g}, \mathfrak{n}, \widehat{g} - g)$ of $(Q, \mathfrak{m}, \widehat{g})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the slot $(Q^{\phi}_{+g}, \mathfrak{n}, \widehat{g} - g)$ in H^{ϕ} is deep, strongly repulsive-normal, and ultimate. Corollary 6.7.14 applied to $(Q_{+g}, \mathfrak{n}, \widehat{g} - g)$ in place of $(P, \mathfrak{n}, \widehat{h})$ gives a $z \in \mathcal{C}^{<\infty} \setminus H$ that is hardian over H with $Q_{+g}(z) = 0$. Thus $y := g + z \in \mathcal{C}^{<\infty}$ is as in the Claim. Case (2) is handled likewise. □

Recall from the introduction that an *H-closed field* is an ω-free newtonian Liouville closed H-field. Recall also that Hardy fields containing $\mathbb{R}$ are H-fields. The main result of these notes can now be established in a few lines:

Theorem 6.7.22. *A Hardy field is* d*-maximal iff it contains $\mathbb{R}$ and is H-closed.*

Proof. The "if" part is a special case of [ADH, 16.0.3]. By Proposition 5.3.2 and Theorem 5.6.2, every d-maximal Hardy field contains $\mathbb{R}$ and is Liouville closed and ω-free. Suppose H is d-maximal. Then $K := H[\mathrm{i}]$ is 1-linearly newtonian by Corollary 6.7.20, so H is newtonian by Lemma 6.7.21. □

Theorem 6.7.22 and Corollary 6.3.9 yield Theorem B from the introduction in a refined form:

Corollary 6.7.23. *Any Hardy field F has a* d*-algebraic H-closed Hardy field extension. If F is a $\mathcal{C}^{\infty}$-Hardy field, then so is any such extension, and likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.*

Part 7. Applications

Here we apply the material in the previous parts. In Section 7.1 we show how to transfer first-order logical properties of the differential field $\mathbb{T}$ of transseries to maximal Hardy fields, proving in particular Theorem A and Corollaries 1–5 as well as the first part of Corollary 6 from the introduction. In Section 7.2 we obtain Corollary 7, elaborate on [ADH, Chapter 16], and relate Newton-Liouville closure to relative differential closure. In Section 7.3 we investigate embeddings of Hardy fields into $\mathbb{T}$, and finish the proof of Corollary 6. There we also determine the universal theory of Hardy fields. Section 7.4 contains applications of our main theorem to linear differential equations over Hardy fields, including proofs of Corollaries 8–11 from the introduction. The final Corollary 12 from the introduction is established in Section 7.5, where we focus on the structure of perfect and d-perfect Hardy fields.

7.1. Transfer Theorems

From [ADH, 16.3] we recall the notion of a *pre-$\Lambda\Omega$-field* $\boldsymbol{H} = (H, \mathrm{I}, \Lambda, \Omega)$: this is a pre-$H$-field H equipped with a $\Lambda\Omega$-cut $(\mathrm{I}, \Lambda, \Omega)$ of H. (See also Section 5.6.) A *$\Lambda\Omega$-field* is a pre-$\Lambda\Omega$-field $\boldsymbol{H} = (H; \dots)$ where H is an H-field. If $\boldsymbol{M} = (M; \dots)$ is a pre-$\Lambda\Omega$-field and H is a pre-H-subfield of M, then H has a unique expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{H}$ such that $\boldsymbol{H} \subseteq \boldsymbol{M}$. By [ADH, 16.3.19], a pre-H-field H has a unique expansion to a pre-$\Lambda\Omega$-field iff one of the following conditions holds:

(1) H is grounded;
(2) there exists $b \asymp 1$ in H such that $v(b')$ is a gap in H;
(3) H is ω-free.

In particular, each d-maximal Hardy field M (being ω-free) has a unique expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{M}$, namely $\boldsymbol{M} = \big(M; \mathrm{I}(M), \Lambda(M), \omega(M)\big)$, and then $\boldsymbol{M}$ is a $\Lambda\Omega$-field with constant field $\mathbb{R}$. Below we always view any d-maximal Hardy field as an $\Lambda\Omega$-field in this way.

Lemma 7.1.1. *Let H be a Hardy field. Then H has an expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{H}$ such that $\boldsymbol{H} \subseteq \boldsymbol{M}$ for every* d*-maximal Hardy field $M \supseteq H$.*

Proof. Since every d-maximal Hardy field containing H also contains $\mathrm{D}(H)$, it suffices to show this for $\mathrm{D}(H)$ in place of H. So we assume H is d-perfect, and thus a Liouville closed H-field. For each d-maximal Hardy field $M \supseteq H$ we now have $\mathrm{I}(H) = \mathrm{I}(M) \cap H$ by [ADH, 11.8.2], $\Lambda(H) = \Lambda(M) \cap H$ by [ADH, 11.8.6], and $\omega(H) = \overline{\omega}(H) = \overline{\omega}(M) \cap H = \omega(M) \cap H$ by Corollary 5.5.3, as required. $\square$

Given a Hardy field H, we call the unique expansion $\boldsymbol{H}$ of H to a pre-$\Lambda\Omega$-field with the property stated in the previous lemma the **canonical $\Lambda\Omega$-expansion** of H.

Corollary 7.1.2. *Let H, H^* be Hardy fields, with their canonical $\Lambda\Omega$-expansions $\boldsymbol{H}$, $\boldsymbol{H}^*$, respectively, such that $H \subseteq H^*$. Then $\boldsymbol{H} \subseteq \boldsymbol{H}^*$.*

Proof. Let M^* be any d-maximal Hardy field extension of H^*. Then $\boldsymbol{H} \subseteq \boldsymbol{M}^*$ as well as $\boldsymbol{H}^* \subseteq \boldsymbol{M}^*$, hence $\boldsymbol{H} \subseteq \boldsymbol{H}^*$. $\square$

In the rest of this section $\mathcal{L} = \{0, 1, -, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ is the language of ordered valued differential rings [ADH, p. 678]. We view each ordered valued differential field as an $\mathcal{L}$-structure in the natural way. Given an ordered valued differential field H and a subset A of H we let $\mathcal{L}_A$ be $\mathcal{L}$ augmented by names for the elements of A, and expand the $\mathcal{L}$-structure H to an $\mathcal{L}_A$-structure by interpreting the name of any $a \in A$ as the element a of H; cf. [ADH, B.3]. Let H be a Hardy field and σ be an $\mathcal{L}_H$-sentence. We now have our Hardy field analogue of the "Tarski principle" [ADH, B.12.14] in real algebraic geometry promised in the introduction:

Theorem 7.1.3. *The following are equivalent:*

(i) $M \models \sigma$ *for some* d*-maximal Hardy field* $M \supseteq H$*;*
(ii) $M \models \sigma$ *for every* d*-maximal Hardy field* $M \supseteq H$*;*
(iii) $M \models \sigma$ *for every maximal Hardy field* $M \supseteq H$*;*
(iv) $M \models \sigma$ *for some maximal Hardy field* $M \supseteq H$*.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are obvious, since "maximal $\Rightarrow$ d-maximal"; so it remains to show (i) $\Rightarrow$ (ii). Let M, M^* be d-maximal Hardy field extensions of H. By Lemma 7.1.1 and Corollary 7.1.2 expand M, M^*, H to pre-$\Lambda\Omega$-fields $\boldsymbol{M}$, $\boldsymbol{M}^*$, $\boldsymbol{H}$, respectively, such that $\boldsymbol{H} \subseteq \boldsymbol{M}$ and $\boldsymbol{H} \subseteq \boldsymbol{M}^*$. In [ADH, introduction to Chapter 16] we extended $\mathcal{L}$ to a language $\mathcal{L}^{\iota}_{\Lambda\Omega}$, and explained in [ADH, 16.5] how each pre-$\Lambda\Omega$-field $\boldsymbol{K}$ is construed as an $\mathcal{L}^{\iota}_{\Lambda\Omega}$-structure in such a way that every extension $\boldsymbol{K} \subseteq \boldsymbol{L}$ of pre-$\Lambda\Omega$-fields corresponds to an extension of the associated $\mathcal{L}^{\iota}_{\Lambda\Omega}$-structures. By [ADH, 16.0.1], the $\mathcal{L}^{\iota}_{\Lambda\Omega}$-theory $T^{\mathrm{nl},\iota}_{\Lambda\Omega}$ of H-closed $\Lambda\Omega$-fields eliminates quantifiers, and by Theorem 6.7.22, the canonical $\Lambda\Omega$-expansion of each d-maximal Hardy field is a model of $T^{\mathrm{nl},\iota}_{\Lambda\Omega}$. Hence $\boldsymbol{M} \equiv_H \boldsymbol{M}^*$ [ADH, B.11.6], so if $\boldsymbol{M} \models \sigma$, then $\boldsymbol{M}^* \models \sigma$. □

Corollaries 1 and 2 from the introduction are special cases of Theorem 7.1.3. By Corollary 6.3.8, $\mathcal{C}^{\infty}$-maximal and $\mathcal{C}^{\omega}$-maximal Hardy fields are d-maximal, so the theorem above also yields Corollary 5 from the introduction in the following stronger form:

Corollary 7.1.4. *If* $H \subseteq \mathcal{C}^{\infty}$ *and* $M \models \sigma$ *for some* d*-maximal Hardy field extension* M *of* H*, then* $M \models \sigma$ *for every* $\mathcal{C}^{\infty}$*-maximal Hardy field* $M \supseteq H$*. Likewise with* $\mathcal{C}^{\omega}$ *in place of* $\mathcal{C}^{\infty}$*.*

The structure induced on $\mathbb{R}$. In the next corollary H is a Hardy field and $\varphi(x)$ is an $\mathcal{L}_H$-formula where $x = (x_1, \dots, x_n)$ and $x_1, \dots, x_n$ are distinct variables. Also, $\mathcal{L}_{\mathrm{OR}} = \{0, 1, -, +, \cdot, \leqslant\}$ is the language of ordered rings, and the ordered field $\mathbb{R}$ of real numbers is interpreted as an $\mathcal{L}_{\mathrm{OR}}$-structure in the obvious way. By Theorem 6.7.22, d-maximal Hardy fields are H-closed fields, so from [ADH, 16.6.7, B.12.13] in combination with Theorem 7.1.3 we obtain:

Corollary 7.1.5. *There is a quantifier-free* $\mathcal{L}_{\mathrm{OR}}$*-formula* $\varphi_{\mathrm{OR}}(x)$ *such that for all* d*-maximal Hardy fields* $M \supseteq H$ *and* $c \in \mathbb{R}^n$ *we have*

$$M \models \varphi(c) \iff \mathbb{R} \models \varphi_{\mathrm{OR}}(c).$$

This yields Corollary 3 from the Introduction. We now justify what we claim about the examples after that corollary. The first of these examples is already covered by [ADH, 5.1.18, 11.8.25, 11.8.26], so we only deal with the second example here:

Proposition 7.1.6. *Let* $g_2, g_3 \in \mathbb{R}$*. Then the following are equivalent:*

(i) *there exists a hardian germ* $y \notin \mathbb{R}$ *such that* $(y')^2 = 4y^3 - g_2 y - g_3$*;*
(ii) $g_2^3 = 27 g_3^2$ *and* $g_3 \leqslant 0$*;*

For (i) $\Rightarrow$ (ii) we take a more general setting, and recycle arguments used in the proof of [ADH, 10.7.1]. Let K be a valued differential field such that $\partial\mathcal{O} \subseteq \sigma$

and $C \subseteq \mathcal{O}$. (This holds for any d-valued field with small derivation.) Consider a polynomial $P(Y) = 4Y^3 - g_2Y - g_3$ with $g_2, g_3 \in C$. Its discriminant is 16Δ where $\Delta := g_2^3 - 27g_3^2$. Take e_1, e_2, e_3 in an algebraic closure of C auch that

$$P(Y) = 4(Y - e_1)(Y - e_2)(Y - e_3).$$

Then

$$(7.1.1) \qquad e_1 + e_2 + e_3 \ = \ 0, \quad e_1e_2 + e_2e_3 + e_3e_1 \ = \ -\tfrac{1}{4}g_2, \quad e_1e_2e_3 \ = \ \tfrac{1}{4}g_3,$$

and $\Delta \neq 0$ iff e_1, e_2, e_3 are distinct. In the next two lemmas $y \in K$ and $(y')^2 = P(y)$. Then $y \preccurlyeq 1$: otherwise $(y')^2 \prec 4y^3 \sim P(y) = (y')^2$ by [ADH, 4.4.3], a contradiction. Hence $P(y) = (y')^2 \prec 1$. Moreover, if $y \asymp 1$, then $\Delta \neq 0$ or $g_3 \neq 0$.

Lemma 7.1.7. *Suppose $P'(y) \asymp 1$. Then $y \in \{e_1, e_2, e_3\}$ (so $y \in C$).*

Proof. The property $\partial\mathcal{O} \subseteq o$ means that the derivation of K is small with trivial induced derivation on its residue field. By [ADH, 6.2.1, 3.1.9] this property is inherited by any algebraic closure of K, and so is the property $C \subseteq \mathcal{O}$ by [ADH, 4.1.2]. Thus by passing to an algebraic closure we arrange that K is algebraically closed. Then C is also algebraically closed, so $e_1, e_2, e_3 \in C$ and thus $y - e_j \prec 1$ for some $j \in \{1, 2, 3\}$, say $y = e_1 + z$ where $z \prec 1$. Since $P'(y) \asymp 1$ we have $y - e_2 \asymp y - e_3 \asymp 1$ and thus

$$z \ \asymp \ 4z(e_1 - e_2 + z)(e_1 - e_3 + z) \ = \ P(y) \ = \ (y')^2 \ = \ (z')^2.$$

Now if $z \neq 0$, then $(z')^2 \prec z$, again by [ADH, 4.4.3], a contradiction. So $y = e_1$. ☐

In the next lemma K is in addition equipped with an ordering making K a valued ordered differential field whose valuation ring is convex. (Any H-field with small derivation satisfies the conditions we imposed.) Suppose $\Delta = 0$. Then e_1, e_2, e_3 lie in the real closure of C, and after arranging $e_1 \geqslant e_2 \geqslant e_3$, the first and the last of the equations (7.1.1) yield $e_1 = e_2 \Longleftrightarrow g_3 \leqslant 0$, and $e_2 = e_3 \Longleftrightarrow g_3 \geqslant 0$.

Lemma 7.1.8. *Suppose $\Delta = 0$ and $g_3 > 0$. Then $y \in C$.*

Proof. Passing to the real closure of K with convex valuation extending that of K, cf. [ADH, 3.5.18], we arrange that K, and hence C, is real closed. Arranging also $e_1 \geqslant e_2 \geqslant e_3$, we set $e := e_2 = e_3$. Then $e_1 = -2e > 0 > e$ and $P(Y) = 4(Y + 2e)(Y - e)^2$. We have $y \preccurlyeq 1$ and $P(y) \prec 1$. Suppose $y \notin C$. Then $P'(y) \prec 1$ by Lemma 7.1.7, so $y - e \prec 1$. Set $z := y - e$, so $0 \neq z \prec 1$ and hence

$$12ez^2 \ \sim \ 4(z + 3e)z^2 \ = \ P(y) \ = \ (y')^2 \ > \ 0,$$

contradicting $e < 0$. ☐

Proof of Proposition 7.1.6. Suppose $y \notin \mathbb{R}$ is a hardian germ such that $(y')^2 = P(y)$ with $P(Y) = 4Y^3 - g_2Y - g_3$ and $g_2, g_3 \in \mathbb{R}$. Then $y \preccurlyeq 1$ and $P(y) \prec 1$, but also $P'(y) \prec 1$ by Lemma 7.1.7, hence $\Delta \prec 1$. As $\Delta \in \mathbb{R}$ this gives $\Delta = 0$, so $g_3 \leqslant 0$ by Lemma 7.1.8. This proves (i) $\Rightarrow$ (ii). For the converse, let K be the Hardy field $\mathbb{R}$ in the considerations above and suppose $\Delta = 0$, so $e_1, e_2, e_3 \in \mathbb{R}$. Arrange $e_1 \geqslant e_2 \geqslant e_3$. If $g_3 = 0$, then $e_1 = e_2 = e_3 = 0$, and if $g_3 < 0$, then $e_1 = e_2 > 0$. In Corollaries 7.1.9 and 7.1.12 we deal exhaustively with these two cases. In particular, we show there that in each case there is a hardian $y \notin \mathbb{R}$ such that $(y')^2 = P(y)$, thus finishing the proof of Proposition 7.1.6.

Accordingly we assume below that $g_2, g_3 \in \mathbb{R}$ and $\Delta = 0$, so $e_1, e_2, e_3 \in \mathbb{R}$. Note that the $y \in \mathbb{R}$ such that $(y')^2 = P(y)$ are exactly e_1, e_2, e_3.

Corollary 7.1.9. *Suppose $e_1 = e_2 = e_3 = 0$ and $y \in \mathcal{C}^1$. Then*

$$y \in \mathcal{C}^{\times} \text{ and } (y')^2 = P(y) \quad \Longleftrightarrow \quad y = \frac{1}{(x-c)^2} \text{ for some } c \in \mathbb{R}.$$

Proof. We have $P(Y) = 4Y^3$. The direction $\Leftarrow$ is routine. For $\Rightarrow$, suppose $y \in \mathcal{C}^{\times}$ and $(y')^2 = 4y^3$. Then $y' \in \mathcal{C}^{\times}$, $y > 0$, $z := y^{-1/2} > 0$, and $z' = -\frac{1}{2}y^{-3/2}y'$. We have $y' < 0$: otherwise $0 < y' = 2y^{3/2}$ and so $z' = -1$, hence $z < 0$, a contradiction. Therefore $y' = -2y^{3/2}$, so $z' = 1$ and thus $y = \frac{1}{z^2} = \frac{1}{(x-c)^2}$ for some $c \in \mathbb{R}$. □

Lemma 7.1.12 below is an analogue of Lemma 7.1.9 for $e_1 = e_2 > 0$, but first we make some observations about hyperbolic functions. Recall that for $t \in \mathbb{R}$,

$$\sinh t \ := \ \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}), \qquad \cosh t \ := \ \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}), \text{ so}$$
$$\cosh^2 t - \sinh^2 t \ = \ 1, \qquad \frac{d}{dt}\sinh t \ = \ \cosh t, \qquad \frac{d}{dt}\cosh t \ = \ \sinh t.$$

We also set for $t \in \mathbb{R}$:

$$\operatorname{sech} t \ := \ \frac{1}{\cosh t} \text{ (hyperbolic secant)}, \qquad \tanh t \ := \ \frac{\sinh t}{\cosh t} \text{ (hyperbolic tangent)}$$

and for $t \neq 0$:

$$\operatorname{csch} t \ := \ \frac{1}{\sinh t} \text{ (hyperbolic cosecant)}, \quad \coth t \ := \ \frac{\cosh t}{\sinh t} \text{ (hyperbolic cotangent).}$$

Now $\sinh\colon \mathbb{R} \to \mathbb{R}$ is an increasing bijection, so $t \mapsto \operatorname{csch} t\colon \mathbb{R}^{>} \to \mathbb{R}^{>}$ is a decreasing bijection. We have $\operatorname{sech}^2 t = 1 - \tanh^2 t$, and for $t \neq 0$, $\operatorname{csch}^2 t = \coth^2 t - 1$. Moreover, $\frac{d}{dt}\operatorname{sech} t = -\tanh t \operatorname{sech} t$ and for $t \neq 0$: $\frac{d}{dt}\operatorname{csch} t = -\coth t \operatorname{csch} t$. Hence both $-\operatorname{sech}^2\colon \mathbb{R} \to \mathbb{R}$ and $\operatorname{csch}^2\colon \mathbb{R}^{\times} \to \mathbb{R}$ satisfy the differential equation $(u')^2 = 4u^2(u+1)$. We use these facts to prove:

Lemma 7.1.10. *Let $w \in \mathcal{C}^1$ and $e \in \mathbb{R}^{>}$. Then the following are equivalent:*

(i) *$w(t) > 0$, eventually, and $(w')^2 = ew^2(w+1)$*
(ii) *$w = \operatorname{csch}^2 \circ (c + \frac{\sqrt{e}}{2}x)$ for some $c \in \mathbb{R}$.*

Proof. Direct computation gives (ii) $\Leftarrow$ (i). Assume (i). Consider the decreasing bijection $t \mapsto u(t) := \operatorname{csch}^2(t)\colon \mathbb{R}^{>} \to \mathbb{R}^{>}$; let $u^{\mathrm{inv}} : \mathbb{R}^{>} \to \mathbb{R}^{>}$ be its (strictly decreasing) compositional inverse, so $u^{\mathrm{inv}} \in \mathcal{C}^1(\mathbb{R}^{>})$. Then for $v := u^{\mathrm{inv}} \circ w \in \mathcal{C}^1$ we have $v(t) > 0$, eventually, $v' = \frac{w'}{u' \circ v}$, and $u \circ v = w$. Thus

$$(v')^2 \ = \ \frac{(w')^2}{(u')^2 \circ v} \ = \ \frac{e}{4}.$$

Hence $v' = \frac{\sqrt{e}}{2}$, since $v' = -\frac{\sqrt{e}}{2}$ contradicts $v > 0$. Now use $w = u \circ v$. □

The increasing bijection $t \mapsto \cosh t\colon (0, +\infty) \to (1, +\infty)$ yields the increasing bijection $t \mapsto -\operatorname{sech}^2 t\colon (0, +\infty) \to (-1, 0)$. We use this to prove likewise:

Lemma 7.1.11. *Let $w \in \mathcal{C}^1$ and $e \in \mathbb{R}^{>}$. Then the following are equivalent:*

(i) *$-1 < w(t) < 0$, eventually, and $(w')^2 \ = \ ew^2(w+1)$;*
(ii) *$w = -\operatorname{sech}^2 \circ (c + \frac{\sqrt{e}}{2}x)$ for some $c \in \mathbb{R}$.*

Corollary 7.1.12. *Suppose $e_1 = e_2 > 0$. Then the hardian germs $y \notin \mathbb{R}$ such that $(y')^2 = P(y)$ are all in $\mathbb{R}(\mathrm{e}^{x\sqrt{3e_1}})$ and are given by*

$$y \;=\; e_1 + 3e_1 \cdot \operatorname{csch}^2 \circ (c + x\sqrt{3e_1}), \quad y \;=\; e_1 - 3e_1 \cdot \operatorname{sech}^2 \circ (c + x\sqrt{3e_1}),$$

where $c \in \mathbb{R}$.

Proof. We have $P(Y) = 4(Y - e_1)^2(Y + 2e_1)$. Let $y \in \mathcal{C}^1$ and $w := (y - e_1)/3e_1$. Then $(y')^2 = P(y)$ iff $(w')^2 = 12e_1 w^2(w + 1)$. There is no hardian $y < -2e_1$ with $(y')^2 = P(y)$, so we can use Lemmas 7.1.10 and 7.1.11 with $e := 12e_1$. $\square$

Uniform finiteness. We now let H be a Hardy field and $\varphi(x, y)$ and $\theta(x)$ be $\mathcal{L}_H$-formulas, where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$.

Lemma 7.1.13. *There is a $B = B(\varphi) \in \mathbb{N}$ such that for all $f \in H^m$: if for some* d*-maximal Hardy field extension M of H there are more than B tuples $g \in M^n$ with $M \models \varphi(f, g)$, then for every* d*-maximal Hardy field extension M of H there are infinitely many $g \in M^n$ with $M \models \varphi(f, g)$.*

Proof. Fix a d-maximal Hardy field extension M^* of H. By [10, Proposition 6.4] we have $B = B(\varphi) \in \mathbb{N}$ such that for all $f \in (M^*)^m$: if $M^* \models \varphi(f, g)$ for more than B many $g \in (M^*)^n$, then $M^* \models \varphi(f, g)$ for infinitely many $g \in (M^*)^n$. Now use Theorem 7.1.3. $\square$

In the proof of the next lemma we use that $\mathcal{C}$ has the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, hence $|H| = \mathfrak{c}$ if $H \supseteq \mathbb{R}$.

Lemma 7.1.14. *Suppose H is* d*-maximal and $S := \big\{f \in H^m : H \models \theta(f)\big\}$ is infinite. Then $|S| = \mathfrak{c}$.*

Proof. Let $d := \dim(S)$ be the dimension of the definable set $S \subseteq H^m$ as introduced in [10]. If $d = 0$, then $|S| = |\mathbb{R}| = \mathfrak{c}$ by remarks following [10, Proposition 6.4]. Suppose $d > 0$, and for $g = (g_1, \dots, g_m) \in H^m$ and $i \in \{1, \dots, m\}$, let $\pi_i(g) := g_i$. Then for some $i \in \{1, \dots, m\}$, the subset $\pi_i(S)$ of H has nonempty interior, by [10, Corollary 3.2], and hence $|S| = |H| = \mathfrak{c}$. $\square$

The two lemmas above together now yield Corollary 4 from the introduction.

Transfer between maximal Hardy fields and transseries. Let $\boldsymbol{T}$ be the unique expansion of $\mathbb{T}$ to a pre-$\Lambda\Omega$-field, so $\boldsymbol{T}$ is an H-closed $\Lambda\Omega$-field with small derivation and constant field $\mathbb{R}$.

Lemma 7.1.15. *Let H be a pre-H-subfield of $\mathbb{T}$ with $H \not\subseteq \mathbb{R}$. Then H has a unique expansion to a pre-$\Lambda\Omega$-field.*

Proof. If H is grounded, this follows from [ADH, 16.3.19]. Suppose H is not grounded. Then H has asymptotic integration by the proof of [ADH, 10.6.19] applied to $\Delta := v(H^\times)$. Starting with an $h_0 \succ 1$ in H with $h_0' \asymp 1$ we construct a logarithmic sequence (h_n) in H as in [ADH, 11.5], so $h_n \asymp \ell_n$ for all n. Hence $\Gamma^<$ is cofinal in $\Gamma_{\mathbb{T}}^<$, so H is $\boldsymbol{\omega}$-free by [ADH, remark before 11.7.20]. Now use [ADH, 16.3.19] again. $\square$

In the rest of this subsection H is a Hardy field with canonical $\Lambda\Omega$-expansion $\boldsymbol{H}$, and $\iota \colon H \to \mathbb{T}$ is an embedding of ordered differential fields, and thus of pre-H-fields.

Corollary 7.1.16. *The map ι is an embedding $\boldsymbol{H} \to \boldsymbol{T}$ of pre-$\Lambda\Omega$-fields.*

Proof. If $H \not\subseteq \mathbb{R}$, then this follows from Lemma 7.1.15. Suppose $H \subseteq \mathbb{R}$. Then ι is the identity on H, so extends to the embedding $\mathbb{R}(x) \to \mathbb{T}$ that is the identity on $\mathbb{R}$ and sends the germ x to $x \in \mathbb{T}$. Now use that $\mathbb{R}(x) \not\subseteq \mathbb{R}$ and Corollary 7.1.2. □

Recall from [ADH, B.4] that for any $\mathcal{L}_H$-sentence σ we obtain an $\mathcal{L}_{\mathbb{T}}$-sentence $\iota(\sigma)$ by replacing the name of each $h \in H$ occurring in σ with the name of $\iota(h)$.

Corollary 7.1.17. *Let σ be an $\mathcal{L}_H$-sentence. Then* (i)–(iv) *in Theorem 7.1.3 are also equivalent to:*

(v) $\mathbb{T} \models \iota(\sigma)$.

Proof. Let M be a d-maximal Hardy field extension of H; it suffices to show that $M \models \sigma$ iff $\mathbb{T} \models \iota(\sigma)$. For this, mimick the proof of (i) $\Rightarrow$ (ii) in Theorem 7.1.3, using Corollary 7.1.16. □

Corollary 7.1.17 yields the first part of Corollary 6 from the introduction, even in a stronger form. After an intermezzo on differential closure in Section 7.2 we prove the second part of that corollary in Section 7.3: Corollary 7.3.2. There we also use:

Lemma 7.1.18. *ι extends uniquely to an embedding $H(\mathbb{R}) \to \mathbb{T}$ of pre-H-fields.*

Proof. Let $\widehat{H}$ be the H-field hull of H in $H(\mathbb{R})$. Then ι extends uniquely to an H-field embedding $\widehat{\iota}\colon \widehat{H} \to \mathbb{T}$ by [ADH, 10.5.13]. By [ADH, remark before 4.6.21] and [ADH, 10.5.16] $\widehat{\iota}$ extends uniquely to an embedding $H(\mathbb{R}) \to \mathbb{T}$ of H-fields. □

We finish with indicating how Theorem A from the introduction (again, in strengthened form) follows from [103] and the results above:

Corollary 7.1.19. *If $P \in H\{Y\}$, $f < g$ in H, and $P(f) < 0 < P(g)$, then each* d*-maximal Hardy field extension of H contains a y with $f < y < g$ and $P(y) = 0$.*

Proof. By [103], the ordered differential field $\mathbb{T}_{\mathrm{g}}$ of grid-based transseries is H-closed with small derivation and has the differential intermediate value property (DIVP). Hence $\mathbb{T}$ also has DIVP, by completeness of T_H (see the introduction). Now use Corollary 7.1.17. □

Corollary 7.1.20. *Let $P \in H\{Y\}$ have odd degree. Then there is an H-hardian germ y with $P(y) = 0$.*

Proof. This follows from Theorem 6.7.23 and [ADH, 14.5.3]. Alternatively, we can use Corollary 7.1.19: Replace H by $\operatorname{Li}\big(H(\mathbb{R})\big)$ to arrange that $H \supseteq \mathbb{R}$ is Liouville closed, and appeal to the example following Corollary 1.3.9. □

Note that if $H \subseteq \mathcal{C}^{\infty}$, then in the previous two corollaries we have $H\langle y \rangle \subseteq \mathcal{C}^{\infty}$, by Corollary 6.3.9; likewise with $\mathcal{C}^{\omega}$ in place if $\mathcal{C}^{\infty}$.

7.2. Relative Differential Closure

Let $K \subseteq L$ be an extension of differential fields, and let r range over $\mathbb{N}$. We say that K is **r-differentially closed** in L for every $P \in K\{Y\}^{\neq}$ of order $\leqslant r$, each zero of P in L lies in K. We also say that K is **weakly r-differentially closed** in L if every $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ with a zero in L has a zero in K. We abbreviate "r-differentially closed" by "r-d-closed." Thus

$$K \text{ is } r\text{-d-closed in } L \quad \Longrightarrow \quad K \text{ is weakly } r\text{-d-closed in } L,$$

and

$$\begin{aligned} K \text{ is 0-d-closed in } L \quad &\Longleftrightarrow \quad K \text{ is weakly 0-d-closed in } L \\ &\Longleftrightarrow \quad K \text{ is algebraically closed in } L. \end{aligned}$$

Hence

$$K \text{ is weakly 0-d-closed in } L \quad \Longrightarrow \quad C \text{ is algebraically closed in } C_L. \tag{7.2.1}$$

Also, if K is weakly 0-d-closed in L and L is algebraically closed, then K is algebraically closed, and similarly with "real closed" in place of "algebraically closed". In [ADH, 5.8] we defined K to be *weakly r-d-closed* if every $P \in K\{Y\} \setminus K$ of order $\leqslant r$ has a zero in K. Thus

$$K \text{ is weakly } r\text{-d-closed} \iff \begin{cases} K \text{ is weakly } r\text{-d-closed in every differential field} \\ \text{extension of } K. \end{cases}$$

If K is weakly r-d-closed in L, then $P(K) = P(L) \cap K$ for all $P \in K\{Y\}$ of order $\leqslant r$; in particular,

$$K \text{ is weakly 1-d-closed in } L \quad \Longrightarrow \quad \partial K = \partial L \cap K. \tag{7.2.2}$$

Also,

$$K \text{ is 1-d-closed in } L \quad \Longrightarrow \quad C = C_L \text{ and } K^\dagger = L^\dagger \cap K. \tag{7.2.3}$$

Moreover:

Lemma 7.2.1. *Suppose K is weakly r-d-closed in L. If L is r-linearly surjective, then so is K, and if L is $(r+1)$-linearly closed, then so is K.*

Proof. The first statement is clear from the remarks preceding the lemma, and the second statement is shown similarly to [ADH, 5.8.9]. □

Sometimes we get more than we bargained for:

Lemma 7.2.2. *Suppose K is not algebraically closed, $C \neq K$, and K is weakly r-d-closed in L. Let $Q_1, \dots, Q_m \in K\{Y\}^{\neq}$ of order $\leqslant r$ have a common zero in L, $m \geqslant 1$. Then they have a common zero in K.*

Proof. Take a polynomial $\Phi \in K[X_1, \dots, X_m]$ whose only zero in K^m is the origin $(0, \dots, 0) \in K^m$. Then the differential polynomial $P := \Phi(Q_1, \dots, Q_m) \in K\{Y\}$ is nonzero (use [ADH, 4.2.1]) and has order $\leqslant r$. For $y \in L$ we have

$$Q_1(y) = \cdots = Q_m(y) = 0 \quad \Longrightarrow \quad P(y) = 0,$$

and for $y \in K$ the converse of this implication also holds. □

We say that K is **differentially closed** in L if K is r-d-closed in L for each r, and similarly we define when K is **weakly differentially closed** in L. We also use "d-closed" to abbreviate "differentially closed". If K, as a differential ring, is an elementary substructure of L, then K is weakly d-closed in L. The elements of L that are d-algebraic over K form the smallest differential subfield of L containing K which is d-closed in L; we call it the **differential closure** ("d-closure" for short) of K in L. Thus K is d-closed in L iff no d-subfield of L properly containing K is d-algebraic over K. This notion of being differentially closed does not seen prominent in the differential algebra literature, though the definition occurs (as "differentially algebraic closure") in [114, p. 102]. Here is a useful fact about it:

Lemma 7.2.3. *Let F be a differential field extension of L and E be a subfield of F containing K such that E is algebraic over K and $F = L(E)$.*

$$\begin{tikzcd} & F = L(E) \arrow[dl, dash] \arrow[dr, dash] & \\ E \arrow[dr, dash] & & L \arrow[dl, dash] \\ & K & \end{tikzcd}$$

Then K is d*-closed in L iff $E \cap L = K$ and E is* d*-closed in F.*

Proof. Suppose K is d-closed in L. Then K is algebraically closed in L, so L is linearly disjoint from E over K. (See [122, Chapter VIII, §4].) In particular $E \cap L = K$. Now let $y \in F$ be d-algebraic over E; we claim that $y \in E$. Note that y is d-algebraic over K. Take a field extension $E_0 \subseteq E$ of K with $[E_0 : K] < \infty$ (so E_0 is a d-subfield of E) such that $y \in L(E_0)$; replacing E, F by E_0, $L(E_0)$, respectively, we arrange that $n := [E : K] < \infty$. Let $b_1, \dots, b_n$ be a basis of the K-linear space E; then $b_1, \dots, b_n$ is also a basis of the L-linear space F. Let $\sigma_1, \dots, \sigma_n$ be the distinct field embeddings $F \to L^{\mathrm{a}}$ over L. Then the vectors

$$\big(\sigma_1(b_1), \dots, \sigma_1(b_n)\big), \dots, \big(\sigma_n(b_1), \dots, \sigma_n(b_n)\big) \in (L^{\mathrm{a}})^n$$

are L^{a}-linearly independent [122, Chapter VI, Theorem 4.1]. Let $a_1, \dots, a_n \in L$ be such that $y = a_1 b_1 + \cdots + a_n b_n$. Then

$$\sigma_j(y) = a_1 \sigma_j(b_1) + \cdots + a_n \sigma_j(b_n) \quad \text{for } j = 1, \dots, n,$$

hence by Cramer's Rule,

$$a_1, \dots, a_n \in K\big(\sigma_j(y), \sigma_j(b_i) : i, j = 1, \dots, n\big).$$

Therefore $a_1, \dots, a_n$ are d-algebraic over K, since $\sigma_j(y)$ and $\sigma_j(b_i)$ for $i, j = 1, \dots, n$ are. Hence $a_1, \dots, a_n \in K$ since K is d-closed in L, so $y \in E$ as claimed. This shows the forward implication. The backward direction is clear. $\square$

Corollary 7.2.4. *If -1 is not a square in L and i in a differential field extension of L satisfies $\mathrm{i}^2 = -1$, then: K is* d*-closed in L $\Leftrightarrow$ $K[\mathrm{i}]$ is* d*-closed in $L[\mathrm{i}]$.*

In the next lemma *extension* refers to an extension of valued differential fields.

Lemma 7.2.5. *Suppose K is an $\uplambda$-free H-asymptotic field and is r-*d*-closed in an r-newtonian ungrounded H-asymptotic extension L. Then K is also r-newtonian.*

Proof. Let $P \in K\{Y\}^{\neq}$ be quasilinear of order $\leqslant r$. Then P remains quasilinear when viewed as differential polynomial over L, by Lemma 1.8.9. Hence P has a zero $y \preccurlyeq 1$ in L, which lies in K since K is r-d-closed in L. $\square$

Relative differential closure in H-fields. We now return to the H-field setting. Let $\mathcal{L}_\partial = \{0, 1, -, +, \cdot, \partial\}$ be the language of differential rings, a sublanguage of the language $\mathcal{L} = \mathcal{L}_\partial \cup \{\leqslant, \preccurlyeq\}$ of ordered valued differential rings from Section 7.1.

Let now M be an H-closed field, and let H a pre-H-subfield of M whose valuation ring and constant field we denote by $\mathcal{O}$ and C. Construing H and M as $\mathcal{L}$-structures in the usual way, H is an $\mathcal{L}$-substructure of M. We also use the sublanguage $\mathcal{L}_\preccurlyeq := \mathcal{L}_\partial \cup \{\preccurlyeq\}$ of $\mathcal{L}$, so $\mathcal{L}_\preccurlyeq$ is the language of valued differential rings. We expand the $\mathcal{L}_\partial$-structure $H[i]$ to an $\mathcal{L}_\preccurlyeq$-structure by interpreting $\preccurlyeq$ as the dominance relation associated to the valuation ring $\mathcal{O}+\mathcal{O}i$ of $H[i]$; we expand likewise $M[i]$ to an $\mathcal{L}_\preccurlyeq$-structure by interpreting $\preccurlyeq$ as the dominance relation associated to the valuation ring $\mathcal{O}_{M[i]} = \mathcal{O}_M + \mathcal{O}_M i$ of $M[i]$. Then $H[i]$ is an $\mathcal{L}_\preccurlyeq$-substructure of $M[i]$. By $H \preccurlyeq_{\mathcal{L}} M$ we mean that H is an elementary $\mathcal{L}$-substructure of M, and we use expressions like "$H[i] \preccurlyeq_{\mathcal{L}_\preccurlyeq} M[i]$" in the same way; of course, the two uses of the symbol $\preccurlyeq$ in the latter are unrelated.

By Corollary 7.2.4, H is d-closed in M iff $H[i]$ is d-closed in $M[i]$.

Lemma 7.2.6. *Suppose M has small derivation. Then*

$$H \preccurlyeq_{\mathcal{L}_\partial} M \iff H[i] \preccurlyeq_{\mathcal{L}_\partial} M[i].$$

Also, if $H \preccurlyeq_{\mathcal{L}_\partial} M$, then $H \preccurlyeq_{\mathcal{L}} M$ and $H[i] \preccurlyeq_{\mathcal{L}_\preccurlyeq} M[i]$.

Proof. The forward direction in the equivalence is obvious. For the converse, let $H[i] \preccurlyeq_{\mathcal{L}_\partial} M[i]$. We have $M \equiv_{\mathcal{L}_\partial} \mathbb{T}$ by [ADH, 16.6.3]. Then [ADH, 10.7.10] yields an $\mathcal{L}_\partial$-formula defining M in $M[i]$, so the same formula defines $M \cap H[i] = H$ in $H[i]$, and thus $H \preccurlyeq_{\mathcal{L}_\partial} M$. For the "also" part, use that the squares of M are the nonnegative elements in its ordering, that $\mathcal{O}_M$ is then definable as the convex hull of C_M in M with respect to this ordering, and if $H \preccurlyeq_{\mathcal{L}_\partial} M$, then each $\mathcal{L}_\partial$-formula defining $\mathcal{O}_M$ in M also defines $\mathcal{O} = \mathcal{O}_M \cap H$ in H. $\square$

The next proposition complements [ADH, 16.0.3, 16.2.5]:

Proposition 7.2.7. *The following are equivalent:*

(i) *H is* d*-closed in M;*
(ii) *$C = C_M$ and $H \preccurlyeq_{\mathcal{L}} M$;*
(iii) *$C = C_M$ and H is H-closed.*

Proof. Assume (i). Then $C = C_M$ and H is a Liouville closed H-field, by (7.2.1), (7.2.2), and (7.2.3). We have $\omega(M) \cap H = \omega(H)$ since H is weakly 1-d-closed in M, and $\sigma\big(\Gamma(M)\big) \cap H = \sigma\big(\Gamma(M) \cap H\big) = \sigma\big(\Gamma(H)\big)$ since H is 2-d-closed in M and $\Gamma(M) \cap H = \Gamma(H)$ by [ADH, p. 520]. Now M is Schwarz closed [ADH, 14.2.20], so $M = \omega(M) \cup \sigma\big(\Gamma(M)\big)$, hence also $H = \omega(H) \cup \sigma\big(\Gamma(H)\big)$, thus H is also Schwarz closed [ADH, 11.8.33]; in particular, H is $\upomega$-free. By Lemma 7.2.5, H is newtonian. This shows (i) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is [ADH, 16.0.3], and (iii) $\Leftrightarrow$ (ii) follows from [ADH, 16.2.5]. $\square$

Next a consequence of [ADH, 16.2.1], but note first that $H(C_M)$ is an H-subfield of M and d-algebraic over H, and recall that each $\upomega$-free H-field has a Newton-Liouville closure, as defined in [ADH, p. 669].

$$\begin{tikzcd}[column sep=small] & M \arrow[d, dash] & \\ & H(C_M) \arrow[dl, dash] \arrow[dr, dash] & \\ C_M & & H \end{tikzcd}$$

Corollary 7.2.8. *If H is $\upomega$-free, then the differential closure of H in M is a Newton-Liouville closure of the $\upomega$-free H-subfield $H(C_M)$ of M.*

Let $\boldsymbol{M}$ be the expansion of M to a $\Lambda\Omega$-field, and let $\boldsymbol{H}$, $\boldsymbol{H}(C_M)$ be the expansions of H, $H(C_M)$, respectively, to pre-$\Lambda\Omega$-subfields of $\boldsymbol{M}$; then $\boldsymbol{H}(C_M)$ is a $\Lambda\Omega$-field. By Proposition 7.2.7, the d-closure H^{da} of H in M is H-closed and hence has a unique expansion $\boldsymbol{H}^{\mathrm{da}}$ to a $\Lambda\Omega$-field. Then $\boldsymbol{H} \subseteq \boldsymbol{H}(C_M) \subseteq \boldsymbol{H}^{\mathrm{da}} \subseteq \boldsymbol{M}$. For the Newton-Liouville closure of a pre-$\Lambda\Omega$-field, see [ADH, 16.4.8].

Corollary 7.2.9. *The $\Lambda\Omega$-field $\boldsymbol{H}^{\mathrm{da}}$ is a Newton-Liouville closure of $\boldsymbol{H}(C_M)$.*

Proof. Let $\boldsymbol{H}(C_M)^{\mathrm{nl}}$ be a Newton-Liouville closure of $\boldsymbol{H}(C_M)$. Since $\boldsymbol{H}^{\mathrm{da}}$ is H-closed and extends $\boldsymbol{H}(C_M)$, there is an embedding $\boldsymbol{H}(C_M)^{\mathrm{nl}} \to \boldsymbol{H}^{\mathrm{da}}$ over $\boldsymbol{H}(C_M)$, and any such embedding is an isomorphism, thanks to [ADH, 16.0.3]. $\square$

Relative differential closure in Hardy fields. Specializing to Hardy fields, assume below that H is a Hardy field and set $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$, an H-asymptotic extension of H. By definition, H is d-maximal iff H is d-closed in every Hardy field extension of H. The following contains Corollary 7 from the introduction:

Corollary 7.2.10. *Suppose H is* d-*maximal. Then K is weakly* d-*closed, hence linearly closed by* [ADH, 5.8.9]*, and linearly surjective. If E is a Hardy field extension of H, then K is* d-*closed in $E[\mathrm{i}]$.*

Proof. By our main Theorem 6.7.22, H is newtonian, hence K is weakly d-closed by [ADH, 14.5.7, 14.5.3], proving the first statement; the second statement follows from Corollary 7.2.4. $\square$

We now strengthen the second part of Corollary 7.2.10:

Corollary 7.2.11. *Suppose H is* d-*maximal and $L \supseteq K$ is a differential subfield of $\mathcal{C}^{<\infty}[\mathrm{i}]$ such that L is a* d-*valued H-asymptotic extension of K with respect to some dominance relation on L. Then K is* d-*closed in L.*

Proof. The d-valued field K is ω-free and newtonian by [ADH, 11.7.23, 14.5.7]. Also $L^\dagger \cap K = K^\dagger$ by Corollary 5.5.22. Now apply Theorem 2.6.6. $\square$

We do not require that the dominance relation on L in Corollary 7.2.11 is the restriction to L of the relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$.

Recall also that in Section 5.3 we defined the d-perfect hull $\mathrm{D}(H)$ of H as the intersection of all d-maximal Hardy field extensions of H. By the next result we only need to consider here d-algebraic Hardy field extensions of H:

Corollary 7.2.12. *If H is* d-*closed in some* d-*maximal Hardy field extension of H, then H is* d-*maximal. Hence*

$$\mathrm{D}(H) \;=\; \bigcap \big\{ M : M \text{ is a d-maximal d-algebraic Hardy field extension of } H \big\}.$$

Proof. The first part follows from Theorem 6.7.22 and (i) $\Rightarrow$ (iii) in Proposition 7.2.7. To prove the displayed equality we only need to show the inclusion "$\supseteq$". So let f be an element of every d-maximal d-algebraic Hardy field extension of H, and let M be any d-maximal Hardy field extension of H; we need to show $f \in M$. Let E be the d-closure of H in M. Then E is d-algebraic over H, and by the first part, E is d-maximal; thus $f \in E$, hence $f \in M$ as required. $\square$

We can now prove a variant of Lemma 5.3.1 for $\mathcal{C}^\infty$- and $\mathcal{C}^\omega$-Hardy fields:

Corollary 7.2.13. *Suppose H is a $\mathcal{C}^\infty$-Hardy field. Then*

$$\begin{aligned}\mathrm{D}(H) &= \bigcap \big\{M : M \supseteq H \text{ d-}\textit{maximal } \mathcal{C}^\infty\textit{-Hardy field}\big\}\\ &= \big\{f \in \mathrm{E}^\infty(H) : f \textit{ is } \text{d-}\textit{algebraic over } H\big\}.\end{aligned}$$

Likewise with ω in place of ∞.

Proof. With both equalities replaced by "$\subseteq$", this follows from the definitions and the remarks following Corollary 6.3.9. Let $f \in \mathrm{E}^\infty(H)$ be d-algebraic over H; we claim that $f \in \mathrm{D}(H)$. To prove this claim, let E be a d-maximal Hardy field extension E of H; it is enough to show that then $f \in E$. Now $F := E \cap \mathcal{C}^\infty$ is a $\mathcal{C}^\infty$-Hardy field extension of H which is d-closed in E, by Corollary 6.3.9, and hence d-maximal by the previous corollary. Thus we may replace E by F to arrange that $E \subseteq \mathcal{C}^\infty$, and then take a $\mathcal{C}^\infty$-maximal Hardy field extension M of E. Now $f \in \mathrm{E}^\infty(H)$ gives $f \in M$, and E being d-maximal and f being d-algebraic over E yields $f \in E$. The proof for ω in place of ∞ is similar. □

Combining Theorem 5.4.20 with Corollary 7.2.13 yields:

Corollary 7.2.14. *If $H \subseteq \mathcal{C}^\infty$ is bounded, then $\mathrm{D}(H) = \mathrm{E}(H) = \mathrm{E}^\infty(H)$. Likewise with ω in place of ∞.*

Question. Do the following implications hold for all H?

$$H \subseteq \mathcal{C}^\infty \implies \mathrm{E}(H) \subseteq \mathrm{E}^\infty(H), \qquad H \subseteq \mathcal{C}^\omega \implies \mathrm{E}(H) \subseteq \mathrm{E}^\infty(H) \subseteq \mathrm{E}^\omega(H).$$

Let $\mathrm{E} := \mathrm{E}(\mathbb{Q})$ be the perfect hull of the Hardy field $\mathbb{Q}$. Boshernitzan [33, (20.1)] showed that $\mathrm{E} \subseteq \mathrm{E}^\infty(\mathbb{Q}) \subseteq \mathrm{E}^\omega(\mathbb{Q})$. From Corollary 7.2.14 we obtain

$$\mathrm{E} \ = \ \mathrm{E}^\infty(\mathbb{Q}) \ = \ \mathrm{E}^\omega(\mathbb{Q}) \ = \ \mathrm{D}(\mathbb{Q}),$$

thus establishing [32, §10, Conjecture 1].

Note that E is 1-d-closed in all its Hardy field extensions, by Theorem 6.3.14. However, E is not 2-linearly surjective by [35, Proposition 3.7], so E is not weakly 2-d-closed in any d-maximal Hardy field extension of E (see Lemma 7.2.1) and E is not 2-linearly newtonian (see [ADH, 14.2.2]).

More generally, by Theorem 6.3.14 each d-perfect Hardy field is 1-d-closed in all its Hardy field extensions. Together with Lemma 6.7.15 and 7.2.5, this yields a generalization of Lemma 6.7.15:

Corollary 7.2.15. *Every* d-*perfect Hardy field is* 1-*newtonian.*

Let M be a d-maximal Hardy field extension of H and H^{da} the d-closure of H in M, so $H(\mathbb{R}) \subseteq H^{\mathrm{da}} \subseteq M$. From Corollary 7.2.8 we obtain a description of H^{da}:

Corollary 7.2.16. *If H is $\upomega$-free, then H^{da} is a Newton-Liouville closure of $H(\mathbb{R})$.*

Next, let $\boldsymbol{H}(\mathbb{R})$, $\boldsymbol{H}^{\mathrm{da}}$, $\boldsymbol{M}$ be the canonical $\Lambda\Omega$-expansions of the Hardy fields $H(\mathbb{R})$, H^{da}, M, respectively, so $\boldsymbol{H}(\mathbb{R}) \subseteq \boldsymbol{H}^{\mathrm{da}} \subseteq \boldsymbol{M}$. Corollary 7.2.9 yields:

Corollary 7.2.17. *$\boldsymbol{H}^{\mathrm{da}}$ is a Newton-Liouville closure of $\boldsymbol{H}(\mathbb{R})$.*

We begin with a direct consequence of facts about "Newton-Liouville closure" in [ADH, 14.5, 16.2]. Let $\boldsymbol{H}$ be a $\Lambda\Omega$-field with underlying H-field H. By [ADH, 14.5.10, 16.4.1, 16.4.8], the constant field of the Newton-Liouville closure of $\boldsymbol{H}$ is the real closure of $C := C_H$. Let $\boldsymbol{M}$ be an H-closed $\Lambda\Omega$-field extension of $\boldsymbol{H}$, with underlying H-field M, and let $\boldsymbol{H}^{\mathrm{da}}$ be the d-closure of $\boldsymbol{H}$ in $\boldsymbol{M}$.

Proposition 7.3.1. *Let $\boldsymbol{H}^*$ be a* d*-algebraic $\Lambda\Omega$-field extension of $\boldsymbol{H}$ such that the constant field of $\boldsymbol{H}^*$ is algebraic over C. Then there is an embedding $\boldsymbol{H}^* \to \boldsymbol{M}$ over $\boldsymbol{H}$, and the image of any such embedding is contained in $\boldsymbol{H}^{\mathrm{da}}$.*

Proof. The image of any embedding $\boldsymbol{H}^* \to \boldsymbol{M}$ over $\boldsymbol{H}$ is d-algebraic over H and thus contained in $\boldsymbol{H}^{\mathrm{da}}$. For existence, take a Newton-Liouville closure $\boldsymbol{M}^*$ of $\boldsymbol{H}^*$. Then $\boldsymbol{M}^*$ is also a Newton-Liouville closure of $\boldsymbol{H}$, by [ADH, 16.0.3], and thus embeds into $\boldsymbol{M}$ over $\boldsymbol{H}$. □

Let $\mathcal{L}$ be the language of ordered valued differential rings, as in Section 7.1. The second part of Corollary 6 in the introduction now follows from the next result:

Corollary 7.3.2. *Let H be a Hardy field, $\iota\colon H \to \mathbb{T}$ an ordered differential field embedding, and H^* a* d*-maximal* d*-algebraic Hardy field extension of H. Then ι extends to an ordered valued differential field embedding $H^* \to \mathbb{T}$, and so for any $\mathcal{L}_H$-sentence σ, $H^* \models \sigma$ iff $\mathbb{T} \models \iota(\sigma)$.*

Proof. We have $H(\mathbb{R}) \subseteq H^*$, and so by Lemma 7.1.18 we arrange that $H \supseteq \mathbb{R}$. Let $\boldsymbol{H}$, $\boldsymbol{H}^*$ be the canonical $\Lambda\Omega$-expansions of H, H^*, respectively, and let $\boldsymbol{T}$ be the expansion of $\mathbb{T}$ to a $\Lambda\Omega$-field. Then $\boldsymbol{H} \subseteq \boldsymbol{H}^*$, and by Lemma 7.1.16, ι is an embedding $\boldsymbol{H} \to \boldsymbol{T}$. By Proposition 7.3.1, ι extends to an embedding $\boldsymbol{H}^* \to \boldsymbol{T}$. □

At the end of Section 5.5 we introduced the Hardy field $H := \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$, and we now mimick this in $\mathbb{T}$ by setting $\ell_0 := x$ and $\ell_{n+1} := \log \ell_n$ in $\mathbb{T}$. This yields the unique ordered differential field embedding $H \to \mathbb{T}$ over $\mathbb{R}$ sending $\ell_n \in H$ to $\ell_n \in \mathbb{T}$ for all n. Its image is the H-subfield $\mathbb{R}(\ell_0, \ell_1, \dots)$ of $\mathbb{T}$. Since the sequence (ℓ_n) in $\mathbb{T}$ is coinitial in $\mathbb{T}^{>\mathbb{R}}$, each ordered differential subfield of $\mathbb{T}$ containing $\mathbb{R}(\ell_0, \ell_1, \dots)$ is an ω-free H-field, by the remark preceding [ADH, 11.7.20].

From Lemma 7.1.15 and Proposition 7.3.1 we obtain:

Corollary 7.3.3. *If $H \supseteq \mathbb{R}$ is an ω-free H-subfield of $\mathbb{T}$ and H^* is a* d*-algebraic H-field extension of H with constant field $\mathbb{R}$, then there exists an H-field embedding $H^* \to \mathbb{T}$ over H.*

Corollary 7.3.3 goes through with $\mathbb{T}$ replaced by its H-subfield

$$\mathbb{T}^{\mathrm{da}} := \big\{ f \in \mathbb{T} : f \text{ is d-algebraic (over } \mathbb{Q}) \big\},$$

a Newton-Liouville closure of $\mathbb{R}(\ell_0, \ell_1, \dots)$; see [ADH, 16.6] and Section 7.2 above. We now apply this observation to o-minimal structures. The *Pfaffian closure* of an expansion of the ordered field of real numbers is its smallest expansion that is closed under taking Rolle leaves of definable 1-forms of class $\mathcal{C}^1$. See Speissegger [192] for complete definitions, and the proof that the Pfaffian closure of an o-minimal expansion of the ordered field of reals remains o-minimal.

Corollary 7.3.4. *The Hardy field H of the Pfaffian closure of the ordered field of real numbers embeds as an H-field over $\mathbb{R}$ into $\mathbb{T}^{\mathrm{da}}$.*

Proof. Let $f\colon \mathbb{R}\to\mathbb{R}$ be definable in the Pfaffian closure of the ordered field of real numbers. The proof of [130, Theorem 3] gives $r\in\mathbb{N}$, semialgebraic $g\colon \mathbb{R}^{r+2}\to\mathbb{R}$, and $a\in\mathbb{R}$ such that $f|_{(a,\infty)}$ is $\mathcal{C}^{r+1}$ and $f^{(r+1)}(t) = g\big(t,f(t),\dots,f^{(r)}(t)\big)$ for all $t>a$. Take $P\in\mathbb{R}[Y_1,\dots,Y_{r+3}]^{\neq}$ vanishing identically on the graph of g; see [ADH, B.12.18]. Then $P\big(t,f(t),\dots,f^{(r+1)}(t)\big)=0$ for $t>a$. Hence the germ of f is d-algebraic over $\mathbb{R}$, and so H is d-algebraic over $\mathbb{R}$. As H contains the ω-free Hardy field $\mathbb{R}(\ell_0,\ell_1,\dots)$, we can use the remark following Corollary 7.3.3. $\square$

Question. Let H be the Hardy field of an o-minimal expansion of the ordered field of reals, and let $H^*\supseteq H$ be the Hardy field of the Pfaffian closure of this expansion. Does every embedding $H\to\mathbb{T}$ extend to an embedding $H^*\to\mathbb{T}$?

We mentioned in the introduction that an embedding $H\to\mathbb{T}$ as in Corollaries 7.3.2 and 7.3.4 can be viewed as an *expansion operator* for the Hardy field H and its inverse as a *summation operator.* The corollaries above concern the existence of expansion operators; this relied on the H-closedness of $\mathbb{T}$. Likewise, Theorem 6.7.22 and Proposition 7.3.1 also give rise to summation operators:

Corollary 7.3.5. *Let H be an ω-free H-field and let H^* be a* d*-algebraic H-field extension of H with C_{H^*} algebraic over C_H. Then any H-field embedding $H\to M$ into a* d*-maximal Hardy field extends to an H-field embedding $H^*\to M$.*

In particular, given any ordered differential subfield $H\supseteq\mathbb{R}(\ell_0,\ell_1,\dots)$ of $\mathbb{T}$ with d-closure H^* in $\mathbb{T}$, any $\mathcal{L}$-isomorphism between H and a Hardy field F extends to an $\mathcal{L}$-isomorphism between H^* and a Hardy field extension of F. For $H=\mathbb{R}(\ell_0,\ell_1,\dots)\subseteq\mathbb{T}$ (so $H^*=\mathbb{T}^{\mathrm{da}}$) we recover the main result of [104]:

Corollary 7.3.6. *The H-field $\mathbb{T}^{\mathrm{da}}$ is $\mathcal{L}$-isomorphic to a Hardy field $\supseteq\mathbb{R}(\ell_0,\ell_1,\dots)$.*

Any Hardy field that is $\mathcal{L}$-isomorphic to $\mathbb{T}^{\mathrm{da}}$ is d-maximal, so contains the Hardy field $\mathrm{E}=\mathrm{E}(\mathbb{Q})=\mathrm{D}(\mathbb{Q})$; see the remarks following Lemma 5.3.1. Thus we have an $\mathcal{L}$-embedding $e\colon \mathrm{E}\to\mathbb{T}^{\mathrm{da}}$, which we can view as an expansion operator for the Hardy field E. We suspect that $e(\mathrm{E})$ is independent of the choice of e.

In the remainder of this section we draw some consequences of Corollary 7.3.6 for the universal theory of Hardy fields.

The universal theory of Hardy fields. Recall from Section 7.1 that $\mathcal{L}=\{0,1,-,+,\cdot,\partial,\leqslant,\preccurlyeq\}$ is the language of ordered valued differential rings. Let $\mathcal{L}^\iota$ be $\mathcal{L}$ augmented by a new unary function symbol ι. We view each pre-H-field H as an $\mathcal{L}^\iota$-structure by interpreting the symbols from $\mathcal{L}$ in the natural way and ι by the function $\iota\colon H\to H$ given by $\iota(a):=a^{-1}$ for $a\in H^\times$ and $\iota(0):=0$.

Since every Hardy field extends to a maximal one, each universal $\mathcal{L}^\iota$-sentence which holds in every maximal Hardy field also holds in every Hardy field; likewise with "d-maximal", "perfect", or "d-perfect" in place of "maximal". We now use Corollary 7.3.6 to show:

Proposition 7.3.7. *Let Σ be the set of universal $\mathcal{L}^\iota$-sentences true in all Hardy fields. Then the models of Σ are the pre-H-fields with very small derivation.*

For this we need a refinement of [ADH, 14.5.11]:

Lemma 7.3.8. *Let H be a pre-H-field with very small derivation. Then H extends to an H-closed field with small derivation.*

Proof. By Corollary 1.1.25, replacing H by its H-field hull, we first arrange that H is an H-field. Let (Γ,ψ) be the asymptotic couple of H. Then $\Psi^{\geqslant 0}\neq\emptyset$ or (Γ,ψ) has gap 0. Suppose (Γ,ψ) has gap 0. Let $H(y)$ be the H-field extension from [ADH, 10.5.11] for $K:=H$, $s:=1$. Then $y\succ 1$ and $y^\dagger=1/y\prec 1$, so replacing $H(y)$ by H we can arrange that $\Psi^{\geqslant 0}\neq\emptyset$. Then every pre-$H$-field extension of H has small derivation, and so we are done by [ADH, 14.5.11]. $\square$

Proof of Proposition 7.3.7. The natural axioms for pre-H-fields with very small derivation formulated in $\mathcal{L}^\iota$ are universal, so all models of Σ are pre-H-fields with very small derivation. Conversely, given any pre-H-field H with very small derivation we show that H is a model of Σ: use Lemma 7.3.8 to extend H to an H-closed field with small derivation, and note that the $\mathcal{L}^\iota$-theory of H-closed fields with small derivation is complete by [ADH, 16.6.3] and has a Hardy field model by Corollary 7.3.6. $\square$

Similar arguments allow us to settle a conjecture from [6], in slightly strengthened form. For this, let $\mathcal{L}^\iota_x$ be $\mathcal{L}^\iota$ augmented by a constant symbol x. We view each Hardy field containing the germ of the identity function on $\mathbb{R}$ as an $\mathcal{L}^\iota_x$-structure by interpreting the symbols from $\mathcal{L}^\iota$ as described at the beginning of this subsection and the symbol x by the germ of the identity function on $\mathbb{R}$, which we also denote by x as usual. Each universal $\mathcal{L}^\iota_x$-sentence which holds in every maximal Hardy field also holds in every Hardy field containing x.

Proposition 7.3.9. *Let Σ_x be the set of universal $\mathcal{L}^\iota_x$-sentences true in all Hardy fields that contain x. Then the models of Σ_x are the pre-H-fields with distinguished element x satisfying $x'=1$ and $x\succ 1$.*

This follows from [ADH, 14.5.11] and the next lemma just like Proposition 7.3.7 followed from Lemma 7.3.8 and [ADH, 16.6.3].

Lemma 7.3.10. *The $\mathcal{L}^\iota_x$-theory of H-closed fields with distinguished element x satisfying $x'=1$ and $x\succ 1$ is complete.*

Proof. Let K_1, K_2 be models of this theory, and let x_1,x_2 be the interpretations of x in K_1, K_2. Then [ADH, 10.2.2, 10.5.11] gives an isomorphism $\mathbb{Q}(x_1)\to\mathbb{Q}(x_2)$ of valued ordered differential fields sending x_1 to x_2. To show that $K_1\equiv K_2$ as $\mathcal{L}^\iota_x$-structures we identify $\mathbb{Q}(x_1)$ with $\mathbb{Q}(x_2)$ via this isomorphism. View $\Lambda\Omega$-fields as $\mathcal{L}^\iota_{\Lambda\Omega}$-structures where $\mathcal{L}^\iota_{\Lambda\Omega}$ extends $\mathcal{L}^\iota$ as specified in [ADH, Chapter 16]. (See also the proof of Theorem 7.1.3.) By [ADH, 16.3.19] the $\boldsymbol{\omega}$-free H-fields K_1, K_2 uniquely expand to $\Lambda\Omega$-fields $\boldsymbol{K}_1$, $\boldsymbol{K}_2$. The H-subfield $\mathbb{Q}(x_1)$ of K_1 is grounded, so expands also uniquely to an $\Lambda\Omega$-field, and this $\Lambda\Omega$-field is a common substructure of both $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$. Hence $\boldsymbol{K}_1\equiv_{\mathbb{Q}(x_1)}\boldsymbol{K}_2$ by [ADH, 16.0.1, B.11.6]. This yields the claim. $\square$

From the completeness of the $\mathcal{L}^\iota$-theory of H-closed fields with small derivation and Lemma 7.3.10 in combination with Theorem 6.7.22 we obtain:

Corollary 7.3.11. *The set Σ of universal $\mathcal{L}^\iota$-sentences true in all Hardy fields is decidable, and so is the set Σ_x of universal $\mathcal{L}^\iota_x$-sentences true in all Hardy fields containing x.*

We finish with an example of a not-so-obvious property of asymptotic integrals, expressible by universal $\mathcal{L}^\iota$-sentences, which holds in all Hardy fields. For this, let $Y=(Y_0,\dots,Y_n)$ be a tuple of distinct indeterminates and $P,Q\in\mathbb{Z}[Y]^{\neq}$.

Example. For all hardian germs $\ell_0, \dots, \ell_{n+1}, y$, and $\vec{y} := (y, y', \dots, y^{(n)})$:

$$\begin{cases} \text{if } \ell_0' = 1,\ \ell_{j+1}'\ell_j = \ell_j' \text{ for } j = 0, \dots, n,\ P(\vec{y}) = 0, \text{ and } q := Q(\vec{y}) \neq 0, \text{ then} \\ (\ell_0 \cdots \ell_{n+1} q)' \neq 0,\ (\ell_0 \cdots \ell_{n+1} q)^\dagger \not\asymp q, \text{ and } \big(q/(\ell_0 \cdots \ell_{n+1} q)^\dagger\big)' \asymp q. \end{cases}$$

To see this, let $\ell_0, \dots, \ell_{n+1}, y$ be as hypothesized. Induction on j shows $\ell_j \in \operatorname{Li}(\mathbb{R})$ and $\ell_j \asymp \log_j x$ for $j = 0, \dots, n+1$. Put $H := \mathbb{R}(x)$ and $E := H\langle y \rangle$. Then $\operatorname{trdeg}(E|H) \leqslant n$, so Theorem 5.4.25 and Lemma 5.4.26 yield an $r \in \{0, \dots, n\}$ and $g \in E^{>}$ with $g \asymp \ell_r$ such that E is grounded with $\max \Psi_E = v(g^\dagger)$. Iterating Proposition 5.3.2 and [ADH, 10.2.3 and remark after it], starting with $g^\dagger$ and $\log g$ in the role of s and y in [ADH, 10.2.3], produces a grounded Hardy field F with $E \subseteq F \subseteq \operatorname{Li}(E)$ and $\max \Psi_F = v(f^\dagger)$ where $f \in F^\times$, $f \asymp \ell_{n+1}$. Then

$$\Psi_E \ < \ v(f^\dagger) \ < \ (\Gamma_E^{>})',$$

so $f^\dagger \not\asymp q := Q(\vec{y})$, and thus $(f^\dagger/q)^\dagger \asymp (\ell_0 \cdots \ell_{n+1} q)^\dagger$. Now the conclusion follows from [6, remarks after Lemma 2.7] applied to q, $f^\dagger$, F in the role of a, b_0, K.

7.4. Linear Differential Equations over Hardy Fields

In this section we draw some consequences of our main Theorem 6.7.22 for linear differential equations over Hardy fields. This also uses results from Section 5.10. Throughout this section H is a Hardy field and $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$. Recall from Corollary 7.2.10 that if H is d-maximal, then K is linearly surjective and linearly closed; we use this fact freely below. Let $A \in K[\partial]^{\neq}$ be monic and $r := \operatorname{order} A$.

Solutions in the complexification of a d-maximal Hardy field.

Theorem 7.4.1. *Suppose H is* d*-maximal. Then A splits over K and the $\mathbb{C}$-linear space of zeros of A in $\mathcal{C}^{<\infty}[\mathrm{i}]$ has a basis*

$$f_1\, \mathrm{e}^{\phi_1 \mathrm{i}},\ \dots,\ f_r\, \mathrm{e}^{\phi_r \mathrm{i}} \qquad \text{where } f_1, \dots, f_r \in K^\times,\ \phi_1, \dots, \phi_r \in H.$$

For any such basis, set $\alpha_j := \phi_j'\mathrm{i} + K^\dagger \in K/K^\dagger$ for $j = 1, \dots, r$. Then the spectrum of A is $\{\alpha_1, \dots, \alpha_r\}$, with

$$\operatorname{mult}_\alpha(A) \ = \ |\{j \in \{1, \dots, r\} : \alpha_j = \alpha\}| \quad \text{ for every } \alpha \in K/K^\dagger,$$

and for any $a_1, \dots, a_r \in K$ with $A \ = \ (\partial - a_r) \cdots (\partial - a_1)$ we have

$$\operatorname{mult}_\alpha(A) \ = \ |\{j \in \{1, \dots, r\} : a_j + K^\dagger = \alpha\}| \quad \text{ for every } \alpha \in K/K^\dagger.$$

(The spectrum of A is as defined in Section 2.3, and does not refer to eigenvalues of the $\mathbb{C}$-linear operator $y \mapsto A(y)$ on $\mathcal{C}^{<\infty}[\mathrm{i}]$.)

Proof. The first part follows from Corollaries 2.5.6, 5.10.20, and Lemma 5.10.22. For the rest, also use Lemma 5.10.19 and the proof of Corollary 5.10.20. □

Remarks. Suppose H is d-maximal; so $\mathrm{I}(K) \subseteq K^\dagger$ by Corollary 5.5.19. Hence by Corollary 5.10.20 we can choose the germs f_j, ϕ_j ($j = 1, \dots, r$) in Theorem 7.4.1 such that additionally $f_1\, \mathrm{e}^{\phi_1 \mathrm{i}}, \dots, f_r\, \mathrm{e}^{\phi_r \mathrm{i}}$ is a Hahn basis of $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$; then the f_j with $\phi_j = 0$ form a valuation basis of the valued $\mathbb{C}$-linear space $\ker_K A$. Fix such f_j, ϕ_j, and let $\langle\ ,\ \rangle$ be the "positive definite hermitian form" on the K-linear subspace $K[\mathrm{e}^{H\mathrm{i}}]$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$, as specified in the remarks after Corollary 5.10.32, with associated “norm” $\|\cdot\|$ on $K[\mathrm{e}^{H\mathrm{i}}]$ given by $\|f\| := \sqrt{\langle f, f\rangle} \in H^{\geqslant}$. Those remarks give

$$\langle f_j\,\mathrm{e}^{\phi_j\mathrm{i}}, f_k\,\mathrm{e}^{\phi_k\mathrm{i}}\rangle \;=\; \begin{cases} 0 & \text{if } \phi_j \neq \phi_k, \\ f_j\overline{f_k} & \text{if } \phi_j = \phi_k, \end{cases}$$

and so $\|f_j\,\mathrm{e}^{\phi_j\mathrm{i}}\| = |f_j|$.

Next, let $H_0 \supseteq \mathbb{R}$ be a Liouville closed Hardy subfield of H, set $K_0 := H_0[\mathrm{i}]$ and suppose $\mathrm{I}(K_0) \subseteq K_0^\dagger$, $A \in K_0[\partial]$, and A splits over K_0. Then we can can choose the ϕ_j, f_j in Theorem 7.4.1 such that $f_1\,\mathrm{e}^{\phi_1\mathrm{i}}, \dots, f_r\,\mathrm{e}^{\phi_r\mathrm{i}}$ is a Hahn basis of $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$, $\phi_1, \dots, \phi_r \in H_0$, and $vf_1, \dots, vf_r \in v(H_0^\times)$, by Corollaries 2.6.21 and 5.10.28.

For each $\phi \in H$, the $\mathbb{C}$-linear operator $y \mapsto A(y)$ on $\mathcal{C}^{<\infty}[\mathrm{i}]$ maps the $\mathbb{C}$-linear subspace $K\,\mathrm{e}^{\phi\mathrm{i}}$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ into itself (Lemma 5.5.25); more precisely, by Corollary 5.10.23:

Corollary 7.4.2. *Suppose H is* d*-maximal, and let $\phi \in H$. Then $A(K\,\mathrm{e}^{\phi\mathrm{i}}) = K\,\mathrm{e}^{\phi\mathrm{i}}$. Moreover, if $\phi'\mathrm{i}+K^\dagger \in K/K^\dagger$ is not an eigenvalue of A, then for each $b \in K$ there is a unique $y \in K$ with $A(y\,\mathrm{e}^{\phi\mathrm{i}}) = b\,\mathrm{e}^{\phi\mathrm{i}}$.*

Can the assumption “H is d-maximal” in Theorem 7.4.1 and Corollary 7.4.2 be weakened to “H is perfect”? The case $H = \mathrm{E} := \mathrm{E}(\mathbb{Q})$ is illuminating: $\mathrm{E} \supseteq \mathbb{R}$ is a Liouville closed H-field, so contains the germs x and e^{x^2}, but Boshernitzan [35, Proposition 3.7] showed that E is not 2-linearly surjective, as there is no $y \in \mathrm{E}$ with $y'' + y = \mathrm{e}^{x^2}$. In fact, the conclusion of Corollary 7.4.2 fails for $H = \mathrm{E}$:

Lemma 7.4.3. *Suppose $H = \mathrm{E}$. Then K is not* 1*-linearly surjective: there is no $y \in K$ with $y' - y\mathrm{i} = \mathrm{e}^{x^2}$.*

Proof. Suppose $y = a + b\mathrm{i}$ $(a, b \in H)$ satisfies $y' - y\mathrm{i} = \mathrm{e}^{x^2}$. Now

$$y' - y\mathrm{i} = (a' + b'\mathrm{i}) - (-b + a\mathrm{i}) = (a' + b) + (b' - a)\mathrm{i},$$

hence $a'+b = \mathrm{e}^{x^2}$ and $b' = a$, so $b''+b = \mathrm{e}^{x^2}$, contradicting [35, Proposition 3.7]. $\square$

It follows that the conclusion of Theorem 7.4.1 fails for $H = \mathrm{E}$:

Corollary 7.4.4. *Let $H = \mathrm{E}$ and $A = (\partial - 2x)(\partial - \mathrm{i})$. Then $\ker_{K[\mathrm{e}^{H\mathrm{i}}]} A = \mathbb{C}\,\mathrm{e}^{x\mathrm{i}}$.*

Proof. In Section 5.10 we identified the universal exponential extension of K with $K[\mathrm{e}^{H\mathrm{i}}]$. We have $\mathrm{e}^{x\mathrm{i}} \in \ker_{K[\mathrm{e}^{H\mathrm{i}}]} A$. Suppose $\dim_{\mathbb{C}} \ker_{K[\mathrm{e}^{H\mathrm{i}}]} A = 2$. Then by Corollary 2.5.6, the eigenvalues of A are $2x + K^\dagger$ and $\mathrm{i} + K^\dagger$. Now $2x \in K^\dagger$, which gives $f \in K^\times$ with $A(f) = 0$, so $\ker_{K[\mathrm{e}^{H\mathrm{i}}]} A$ has basis $f, \mathrm{e}^{x\mathrm{i}}$. Also $\mathrm{i} \notin K^\dagger$ by a remark preceding Lemma 1.2.16, so [ADH, 5.1.14(ii)] yields $(\partial - \mathrm{i})(cf) = \mathrm{e}^{x^2}$ for some $c \in \mathbb{C}^\times$, contradicting the lemma above. $\square$

A vestige of linear surjectivity is retained by d-perfect Hardy fields:

Corollary 7.4.5. *Suppose $H \supseteq \mathbb{R}$ is Liouville closed and $\mathrm{I}(K) \subseteq K^\dagger$, and A splits over K. Then there are $\mathfrak{m}, \mathfrak{n} \in H^\times$ such that for each Hardy field extension F of H and $b \in F[\mathrm{i}]$ with $b \prec \mathfrak{n}$, there exists $y \in \mathrm{D}(F)[\mathrm{i}]$ that is the unique $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $A(y) = b$ and $y \prec \mathfrak{m}$. (So if H is* d*-perfect, then for such $\mathfrak{m}$, $\mathfrak{n}$ and all $b \in K$ with $b \prec \mathfrak{n}$ there is a unique $y \in K$ with $A(y) = b$ and $y \prec \mathfrak{m}$.)*

Proof. Let E be a d-maximal Hardy field extension of H. Theorem 7.4.1 and the remark following it yields a Hahn basis

$$f_1 \operatorname{e}^{\phi_1 i},\ \dots,\ f_r \operatorname{e}^{\phi_r i} \qquad (f_1,\dots,f_r \in E[i]^\times,\ \phi_1,\dots,\phi_r \in E)$$

of $\ker_{\mathcal{C}^{<\infty}[i]} A$ with $\phi_1,\dots,\phi_r \in H$ and $vf_1,\dots,vf_r \in \Gamma = v(K^\times)$. It follows from Corollaries 2.6.21 and 5.10.28 that $\mathscr{E}^{\mathrm{e}}(A) = \mathscr{E}^{\mathrm{e}}_{E[i]}(A) = \{vf_j : j = 1,\dots,r,\ \phi_j = 0\}$. By Corollary 1.8.10 the quantity $v^{\mathrm{e}}_A(\gamma)$, for $\gamma \in \Gamma$, does not change when passing from K to any ungrounded H-asymptotic extension of K.

Take $\mathfrak{m}, \mathfrak{n} \in H^\times$ with $\mathfrak{m} \prec f_1,\dots,f_r$ and $v\mathfrak{n} = v^{\mathrm{e}}_A(v\mathfrak{m})$. Consider a Hardy field extension F of H. Let $b \in F[i]^\times$, $b \prec \mathfrak{n}$, and let M be a d-maximal Hardy field extension of F. Then linear newtonianity of $L := M[i]$ and Corollary 1.5.7 yields $y \in L$ with $A(y) = b$, $vy \notin \mathscr{E}^{\mathrm{e}}_L(A) = \mathscr{E}^{\mathrm{e}}(A)$, and $v^{\mathrm{e}}_A(vy) = vb$. Then $v^{\mathrm{e}}_A(vy) = vb > v\mathfrak{n} = v^{\mathrm{e}}_A(v\mathfrak{m})$. Since $v\mathfrak{m} > \mathscr{E}^{\mathrm{e}}_L(A)$, this yields $y \prec \mathfrak{m}$ by Lemma 1.5.6. Suppose $z \in \mathcal{C}^{<\infty}[i]$, $A(z) = b$, and $y \neq z \prec \mathfrak{m}$. Then $u := y - z \in \ker_{\mathcal{C}^{<\infty}[i]} A$ and $0 \neq u \prec \mathfrak{m}$, so $f_j \prec \mathfrak{m}$ for some j by Corollary 5.10.18 (applied to E, $E[i]$ in place of H, K), a contradiction. This last argument also takes care of the case $b = 0$: there is no nonzero $u \prec \mathfrak{m}$ in $\mathcal{C}^{<\infty}[i]$ such that $A(u) = 0$. □

In [15] we shall prove that if H is ω-free and d-perfect, then K is linearly closed. (This applies to $H = \mathrm{E}$.) In particular, if the d-perfect hull $\mathrm{D}(H)$ of H is ω-free, then A splits over the algebraic closure $\mathrm{D}(H)[i]$ of $\mathrm{D}(H)$. (In Section 7.5 below we characterize when $\mathrm{D}(H)$ is ω-free.) Now if A splits over $\mathrm{D}(H)[i]$, then there are $g, \phi \in \mathrm{D}(H)$, $g \neq 0$, such that $A(g \operatorname{e}^{\phi i}) = 0$. The next lemma helps to clarify when for $H \supseteq \mathbb{R}$ we may take here g, ϕ in the Hardy subfield $\mathrm{Li}(H)$ of $\mathrm{D}(H)$.

Lemma 7.4.6. *Suppose $H \supseteq \mathbb{R}$. The following are equivalent:*

- (i) *there exists $y \neq 0$ in a Liouville extension of the differential field K such that $A(y) = 0$;*
- (ii) *there exists $f \in \mathrm{Li}(H)[i]$ such that f' is algebraic over K and $A(\mathrm{e}^f) = 0$;*
- (iii) *there exists $f \in \mathrm{Li}(H)[i]$ such that $A(\mathrm{e}^f) = 0$.*

Proof. Suppose (i) holds. Then Corollary 1.1.30 gives $y \neq 0$ in a differential field extension L of K such that $A(y) = 0$ and $g := y^\dagger$ is algebraic over K. We arrange that L contains the algebraic closure $K^{\mathrm{a}} = H^{\mathrm{rc}}[i]$ of K, where $H^{\mathrm{rc}} \subseteq \mathcal{C}^{<\infty}$ is the real closure of the Hardy field H. Thus $g \in K^{\mathrm{a}}$, and hence $A = B(\partial - g)$ where $B \in K^{\mathrm{a}}[\partial]$ by [ADH, 5.1.21]. Take $f \in \mathrm{Li}(H)[i]$ with $f' = g$ and set $z := \mathrm{e}^f \in \mathcal{C}^{<\infty}[i]^\times$. Then $z^\dagger = g$ and thus $A(z) = 0$. This shows (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i), let f be as in (iii) and $y := \mathrm{e}^f \in \mathcal{C}^{<\infty}[i]^\times$. By [ADH, 10.6.6] the differential field

$$L := \mathrm{Li}(H)[i] \subseteq \mathcal{C}^{<\infty}[i]$$

is a Liouville extension of $K = H[i]$. Now $L[y] \subseteq \mathrm{U}_L := L\big[\operatorname{e}^{\mathrm{Li}(H)i}\big] \subseteq \mathcal{C}^{<\infty}[i]$ and $y^\dagger = f' \in L$, so the differential fraction field $L(y)$ of L is a Liouville extension of L, and hence of K. □

Corollary 7.4.7. *If $H \supseteq \mathbb{R}$ is real closed and $A(y) = 0$ for some $y \neq 0$ in a Liouville extension of K, then $A(\mathrm{e}^f) = 0$ for some $f \in \mathrm{Li}(H)[i]$ with $f' \in K$.*

We assume $H \supseteq \mathbb{R}$ in the next three results. Let E be a d-maximal Hardy field extension of H. By Theorem 7.4.1, A splits over $E[i]$. When does A already split over the d-subfield $\mathrm{Li}(H)[i]$ of $E[i]$? Here is a necessary condition:

Corollary 7.4.8. *If A splits over* $\operatorname{Li}(H)[i]$*, then it splits over* $H^{\mathrm{rc}}[i]$.

Proof. We arrange $H = H^{\mathrm{rc}}$ and proceed by induction on r. The case $r = 0$ being trivial, suppose $r \geqslant 1$ and A splits over $L := \operatorname{Li}(H)[i]$. Then [ADH, 5.1.21] yields $y \neq 0$ in a differential field extension of L with constant field $\mathbb{C}$ and $y^\dagger \in L$ such that $A(y) = 0$. Now $L\langle y\rangle$ is a Liouville extension of L and hence of K, so Lemma 7.4.6 gives $f \in L$ with $f' \in K$ and $A(\mathrm{e}^f) = 0$. Then $A = B(\partial - f')$ where $B \in K[\partial]$ by [ADH, 5.1.21], and B splits over $\operatorname{Li}(H)[i]$ by [ADH, 5.1.22]. We can assume inductively that B splits over K, and then A does too. $\square$

With a weaker hypothesis on A, we have:

Lemma 7.4.9. *Suppose $A(y) = 0$ for some $y \neq 0$ in a Liouville extension of K. Then there is a monic $B \in K[\partial]$ of order $n \geqslant 1$ such that $A \in K[\partial]B$ and the $\mathbb{C}$-linear space of zeros of B in $\mathcal{C}^{<\infty}[i]$ has a basis*

$$g_1\,\mathrm{e}^{\phi_1 i},\ \ldots,\ g_n\,\mathrm{e}^{\phi_n i} \qquad \text{where } g_1,\ldots,g_n \in \operatorname{Li}(H)^\times,\ \phi_1,\ldots,\phi_n \in \operatorname{Li}(H)$$

and $g_1^\dagger,\ldots,g_n^\dagger,\phi_1',\ldots,\phi_n' \in H^{\mathrm{rc}}$. Any such B splits over $H^{\mathrm{rc}}[i]$.

Proof. Put $L := \operatorname{Li}(H)[i]$ and identify U_L with the differential subring $L\big[\,\mathrm{e}^{\operatorname{Li}(H)i}\,\big]$ of $\mathcal{C}^{<\infty}[i]$ as explained at the beginning of Section 5.10. We consider also the differential subfield $K^{\mathrm{a}} := H^{\mathrm{rc}}[i]$ of L, and use Lemma 2.2.12 to identify $\mathrm{U} := \mathrm{U}_{K^{\mathrm{a}}}$ with a differential subring of U_L, so for all $u \in \mathcal{C}^{<\infty}[i]^\times$ with $u^\dagger \in K^{\mathrm{a}}$ we have $u \in \mathrm{U}^\times$. Corollary 7.4.7 yields $f \in L$ such that $f' \in K^{\mathrm{a}}$ and $A(\mathrm{e}^f) = 0$. Then $g := \mathrm{e}^{\operatorname{Re} f} \in \operatorname{Li}(H)^\times$ and $\phi := \operatorname{Im} f \in \operatorname{Li}(H)$ with $\mathrm{e}^f = g\,\mathrm{e}^{\phi i}$, so $g^\dagger = \operatorname{Re} f'$, $\phi' = \operatorname{Im} f'$, and thus $g^\dagger, \phi' \in H^{\mathrm{rc}}$. Now $(\mathrm{e}^f)^\dagger = f' \in K^{\mathrm{a}}$, so $y := \mathrm{e}^f \in \mathrm{U}^\times$. Let V be the $\mathbb{C}$-linear subspace of U spanned by the $\sigma(y)$ with $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$. Then $y \in V \subseteq \ker_{\mathrm{U}} A$ and so $n := \dim_{\mathbb{C}} V \in \{1,\ldots,r\}$. Corollary 2.5.9 yields a unique monic $B \in K^{\mathrm{a}}[\partial]$ of order n such that $V = \ker_{\mathrm{U}} B$. From $\sigma(V) = V$ for all $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$ we get $B \in K[\partial]$ by Corollary 2.2.16. Then $A \in K[\partial]B$ by [ADH, 5.1.15(i), 5.1.11]. To show V has a basis as described in the lemma, let $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$. Then $\sigma(y) \in \mathrm{U}^\times$, so $\sigma(y)^\dagger \in K^{\mathrm{a}} = H^{\mathrm{rc}} + H^{\mathrm{rc}}[i]$, hence $\sigma(y)^\dagger = g_\sigma^\dagger + \phi_\sigma' i$ with $g_\sigma, \phi_\sigma \in H^{\mathrm{rc}}$, $g_\sigma \neq 0$. Also $\big(g_\sigma\,\mathrm{e}^{\phi_\sigma i}\big)^\dagger = g_\sigma^\dagger + \phi_\sigma' i$, and thus $\sigma(y) = c_\sigma g_\sigma\,\mathrm{e}^{\phi_\sigma i}$ with $c_\sigma \in \mathbb{C}^\times$. This yields a basis of V as claimed. The final splitting claim follows from Corollary 2.5.9. $\square$

Lemma 7.4.9 yields the following corollary inspired by [187, Corollary 3].

Corollary 7.4.10. *If A is irreducible, then the following are equivalent:*

(i) *$A(y) = 0$ for some $y \neq 0$ in a Liouville extension of K;*

(ii) *the $\mathbb{C}$-linear space of zeros of A in $\mathcal{C}^{<\infty}[i]$ has a basis*

$$g_1\,\mathrm{e}^{\phi_1 i},\ \ldots,\ g_r\,\mathrm{e}^{\phi_r i} \qquad \text{where } g_1,\ldots,g_r \in \operatorname{Li}(H)^\times,\ \phi_1,\ldots,\phi_r \in \operatorname{Li}(H)$$

and $g_1^\dagger,\ldots,g_r^\dagger,\phi_1',\ldots,\phi_r'$ are algebraic over H.

Next we improve the bounds on the derivatives of solutions to linear differential equations from Corollary 5.7.2 when the coefficients of the equation are in K:

Corollary 7.4.11. *Let $\mathfrak{m} \in H$ with $0 < \mathfrak{m} \preccurlyeq 1$ and $y \in \mathcal{C}^r[i]$ be such that $A(y) = 0$ and $y \preccurlyeq \mathfrak{m}^n$. Then $y \in \mathcal{C}^{<\infty}[i]$ and*

$$y^{(j)} \ \preccurlyeq\ \mathfrak{m}^{n-j}\mathfrak{v}(A)^{-j} \qquad \text{for } j = 0,\ldots,n,$$

with $\prec$ in place of $\preccurlyeq$ if $y \prec \mathfrak{m}^n$.

Proof. First arrange that H is d-maximal. Choose a complement Λ_H of the $\mathbb{R}$-linear subspace $\mathrm{I}(H)$ of H, set $\Lambda := \Lambda_H \mathrm{i}$, and identify the universal exponential extension $\mathrm{U} = \mathrm{U}_K$ of K with the differential subring $K[\mathrm{e}^{H\mathrm{i}}]$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ as described at the beginning of Section 5.10. By Lemmas 5.10.19 and 5.10.22 we have $y \in \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A = \ker_{\mathrm{U}} A$ and

$$y \;=\; f_1\,\mathrm{e}^{\phi_1 \mathrm{i}} + \cdots + f_m\,\mathrm{e}^{\phi_m \mathrm{i}}, \quad f_1,\dots,f_m \in K, \quad \phi_1,\dots,\phi_m \in H,$$

where $\lambda_1 := \phi_1'\mathrm{i},\dots,\lambda_m := \phi_m'\mathrm{i} \in \Lambda$ are the distinct eigenvalues of A with respect to Λ and $f_1\,\mathrm{e}^{\phi_1\mathrm{i}},\dots,f_m\,\mathrm{e}^{\phi_m\mathrm{i}} \in \ker_{\mathrm{U}} A$. By Corollary 5.10.9 and Lemma 5.10.10, we have for $j=1,\dots,m$: $f_j\,\mathrm{e}^{\phi_j\mathrm{i}} \preccurlyeq \mathfrak{m}^n$, with $f_j\,\mathrm{e}^{\phi_j\mathrm{i}} \prec \mathfrak{m}^n$ if $y \prec \mathfrak{m}^n$. Hence we may arrange that $y = f\,\mathrm{e}^{\phi\mathrm{i}}$ where $f \in K$, $\phi \in H$, and $\lambda := \phi'\mathrm{i} \in \Lambda$ is an eigenvalue of A with respect to Λ, so $\lambda \preccurlyeq \mathfrak{v}^{-1}$ by Corollary 4.4.6, where $\mathfrak{v} := \mathfrak{v}(A) \preccurlyeq 1$.

Now for each $j \in \mathbb{N}$: $(\mathrm{e}^{\phi\mathrm{i}})^{(j)} = R_j(\lambda)\,\mathrm{e}^{\phi\mathrm{i}} \preccurlyeq \mathfrak{v}^{-j}$, using Lemma 1.1.20 if $\lambda \succcurlyeq 1$, and so by the Product Rule: if $g \in \mathcal{C}^j[\mathrm{i}]^{\preccurlyeq}$, then $(g\,\mathrm{e}^{\phi\mathrm{i}})^{(j)} \preccurlyeq \mathfrak{v}^{-j}$, and likewise with $\prec$ in place of $\preccurlyeq$. If $\mathfrak{m} \asymp 1$, then this observation with $g := f$ already yields the desired conclusion. Suppose $\mathfrak{m} \prec 1$. Then with $z := y\mathfrak{m}^{-n} = f\mathfrak{m}^{-n}\,\mathrm{e}^{\phi\mathrm{i}}$ this same observation with $g := f\mathfrak{m}^{-n}$ gives for $j = 0,\dots,n$: $z^{(j)} \preccurlyeq \mathfrak{v}^{-j}$, with $z^{(j)} \prec \mathfrak{v}^{-j}$ if $y \prec \mathfrak{m}^n$. Now $z \in \mathcal{C}^n[\mathrm{i}]$, so we can use Lemma 5.7.10 for $r = n$ and $\eta = |\mathfrak{v}|^{-1}$. $\square$

For $\mathfrak{m} = 1$ we obtain from Corollary 7.4.11:

Corollary 7.4.12. *Let $y \in \mathcal{C}^r[\mathrm{i}]$ be such that $A(y) = 0$ and $y \preccurlyeq 1$. Then $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ and $y^{(n)} \preccurlyeq \mathfrak{v}(A)^{-n}$ for all n, with $\prec$ in place of $\preccurlyeq$ if $y \prec 1$.*

Recall from (2.4.3) the *concomitant* $P_A \in K\{Y,Z\}$ of A. It yields a $\mathbb{C}$-bilinear map

$$(y,z) \mapsto [y,z]_A := P_A(y,z) \;:\; \mathcal{C}^{<\infty}[\mathrm{i}] \times \mathcal{C}^{<\infty}[\mathrm{i}] \to \mathcal{C}^{<\infty}[\mathrm{i}]$$

used in the next result, which is immediate from Corollaries 5.10.29 and 7.2.10.

Corollary 7.4.13. *Suppose H is* d*-maximal, and let f_j, ϕ_j be as in Theorem 7.4.1. Then the $\mathbb{C}$-linear space of zeros of the adjoint A^* of A in $\mathcal{C}^{<\infty}[\mathrm{i}]$ has a basis*

$$f_1^*\,\mathrm{e}^{-\phi_1\mathrm{i}},\ \dots,\ f_r^*\,\mathrm{e}^{-\phi_r\mathrm{i}} \qquad \text{where } f_j^* \in K^\times \ (j=1,\dots,r)$$

such that $\left[f_j\,\mathrm{e}^{\phi_j\mathrm{i}}, f_k^*\,\mathrm{e}^{-\phi_k\mathrm{i}}\right]_A = \delta_{jk}$ for $j,k = 1,\dots,r$.

Recall that A is said to be *self-adjoint* if $A^* = A$, and *skew-adjoint* if $A^* = -A$. (See Definition 2.4.12.) Self-adjoint operators play an important role in boundary value problems; see, e.g., [61, Chapter XIII]. The next result follows from Corollaries 2.4.30, 5.10.31 and 7.2.10 and applies to such operators:

Corollary 7.4.14. *Suppose H is* d*-maximal, and $A^* = (-1)^r A_{\ltimes a}$, $a \in K^\times$. Then there are $a_1,\dots,a_r \in K$ such that*

$$A \;=\; (\partial - a_r)\cdots(\partial - a_1) \quad \text{and} \quad a_j + a_{r-j+1} \;=\; a^\dagger \ \text{ for } j = 1,\dots,r.$$

For f_j, ϕ_j as in Theorem 7.4.1 we have: $\phi_1 + \cdots + \phi_r \preccurlyeq 1$, and for each $i \in \{1,\dots,r\}$ there is a $j \in \{1,\dots,r\}$ such that $\phi_i + \phi_j \preccurlyeq 1$.

Operators satisfying the hypothesis of Corollary 7.4.14 are *self-dual* in the sense of Section 2.4. For sources of such operators in physics, see [38]. The next result is immediate from Corollary 2.4.9 and Theorem 7.4.1 and gives a sufficient condition for such operators to have nontrivial zeros in complexified Hardy fields:

Corollary 7.4.15. *If A is self-dual (which is the case if A is skew-adjoint), r is odd, and L is a* d*-maximal Hardy field extension of H, then there are $y, z \in L$, not both zero, such that $A(y + z\mathrm{i}) = 0$.*

The space of zeros of a self-dual A has a special kind of basis, by Corollary 5.10.32:

Corollary 7.4.16. *Suppose A is self-dual and H is* d*-maximal. Then the $\mathbb{C}$-linear space of zeros of A in $\mathcal{C}^{<\infty}[\mathrm{i}]$ has a basis*

$$f_1 \operatorname{e}^{\phi_1 \mathrm{i}}, g_1 \operatorname{e}^{-\phi_1 \mathrm{i}}, \dots, f_m \operatorname{e}^{\phi_m \mathrm{i}}, g_m \operatorname{e}^{-\phi_m \mathrm{i}},\ h_1, \dots, h_n \qquad (2m + n = r)$$

where $f_1, \dots, f_m, g_1, \dots, g_m, h_1, \dots, h_n \in K^\times$, and $\phi_1, \dots, \phi_m \in H^{>\mathbb{R}}$ are apart.

Bounded operators. *In this subsection H is* d*-maximal.* (One can often reduce to this situation by extending a given Hardy field to a d-maximal Hardy field.) *We choose an $\mathbb{R}$-linear complement Λ_H of $\mathrm{I}(H)$ in H, set $\Lambda := \Lambda_H \mathrm{i}$, and identify $\mathrm{U} := \mathrm{U}_K$ with $K[\operatorname{e}^{H\mathrm{i}}]$ as explained at the beginning of Section 5.10. Also $A \in \mathcal{O}[\partial]$* (so $\mathfrak{v}(A) = 1$). Thus $\mathrm{U}^\times = K^\times \operatorname{e}^{H\mathrm{i}}$ and $V := \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A = \ker_{\mathrm{U}} A$. See Sections 5.2 and 5.10 for definitions of Lyapunov exponents and of $\mathcal{C}[\mathrm{i}]^{\preccurlyeq}$ and $\mathrm{U}^{\preccurlyeq}$.

Lemma 7.4.17. *$V \subseteq \mathrm{U}^{\preccurlyeq}$, and $\lambda(y) = \lambda(y, y', \dots, y^{(r-1)}) \in \mathbb{R}$ for all $y \in V^{\neq}$.*

Proof. Lemma 2.3.36 gives $\Sigma(A) \subseteq [\mathcal{O}]$, and Corollary 5.2.50 yields $V \subseteq \mathrm{U} \cap \mathcal{C}[\mathrm{i}]^{\preccurlyeq}$. Lemma 5.10.19 gives a basis $f_1 \operatorname{e}(h_1 \mathrm{i}), \dots, f_r \operatorname{e}(h_r \mathrm{i})$ of the $\mathbb{C}$-linear space V with $f_1, \dots, f_r \in K^\times$ and $h_1, \dots, h_r \in \Lambda_H$, and it says that then the eigenvalues of A with respect to Λ are $h_1 \mathrm{i}, \dots, h_r \mathrm{i}$. So for $j = 1, \dots, r$ we have $h_j \mathrm{i} - a \in K^\dagger = H + \mathrm{I}(H)\mathrm{i}$ with $a \in \mathcal{O}$. Then $h_j - \operatorname{Im} a \in \mathrm{I}(H) \subseteq \mathcal{O}_H$ and $\operatorname{Im} a \in \mathcal{O}_H$, so $h_j \in \Lambda_H \cap \mathcal{O}_H$. From $f_j \operatorname{e}(h_j \mathrm{i}) \in \mathcal{C}[\mathrm{i}]^{\preccurlyeq}$ we obtain $f_j \in K \cap \mathcal{C}[\mathrm{i}]^{\preccurlyeq} = \mathcal{O}_\Delta$, so $V \subseteq \mathrm{U}^{\preccurlyeq}$. The rest follows from Corollary 5.2.50 and Lemma 5.10.49. $\square$

For $y \in \mathcal{C}^1[\mathrm{i}]^\times$, in Section 5.10 we also defined

$$\mu(y) \;=\; \limsup_{t \to +\infty} \operatorname{Im} \frac{y'(t)}{y(t)} \in \mathbb{R}_{\pm\infty}.$$

The zeros of the characteristic polynomial $\chi_A \in \mathbb{C}[Y]$ of A (defined in Section 2.3) contain information about elements of $V \cap \mathrm{U}^\times$:

Lemma 7.4.18. *Let $f \in K^\times$ and $\phi \in H$ be such that $y = f \operatorname{e}^{\phi \mathrm{i}} \in V$. Then $y \in (\mathrm{U}^{\preccurlyeq})^\times$, $\lambda := \lambda(y) \in \mathbb{R}$, $\mu := \mu(y) \in \mathbb{R}$, $\phi - \mu x \prec x$, and with $\alpha := \phi' \mathrm{i} + K^\dagger$:*

$$\chi_A(-\lambda + \mu \mathrm{i}) = 0, \qquad \operatorname{mult}_\alpha(A) \;\leqslant \sum_{c \in \mathbb{C},\ \operatorname{Im} c = \mu} \operatorname{mult}_c(\chi_A).$$

Proof. Corollary 2.3.37 gives $y^\dagger \preccurlyeq 1$, so $y \in (\mathrm{U}^{\preccurlyeq})^\times$, $\lambda, \mu \in \mathbb{R}$ with $y^\dagger - (-\lambda + \mu \mathrm{i}) \prec 1$ and $\phi' \preccurlyeq 1$ by Lemma 5.10.48 and an observation following Corollary 5.10.50. Then $\phi' - \mu \prec 1$, so $\phi - \mu x \prec x$. The rest follows from Corollary 2.3.37 and Lemma 2.3.39. $\square$

A **Lyapunov basis** of V is a basis $y_1, \dots, y_r$ of the $\mathbb{C}$-linear space V such that for all $c_1, \dots, c_r \in \mathbb{C}$, not all zero, and $y = c_1 y_1 + \cdots + c_r y_r$ we have $\lambda(y) = \min\{\lambda(y_j) : c_j \neq 0\}$. There is a Lyapunov basis of V; indeed, by the remarks after Theorem 7.4.1 and Corollary 5.10.44:

Corollary 7.4.19. *The* $\mathbb{C}$*-linear space* V *has a Hahn basis*

$$f_1\,\mathrm{e}^{\phi_1 \mathrm{i}},\dots,f_r\,\mathrm{e}^{\phi_r \mathrm{i}} \qquad (f_1,\dots,f_r\in K^\times,\ \phi_1,\dots,\phi_r\in H),$$

and every such Hahn basis of V *is a Lyapunov basis of* V*.*

Question. By Perron [148, Satz 8] (see also [191, Satz VI]), V has a Lyapunov basis $y_1,\dots,y_r$ such that for each $\lambda\in\mathbb{R}$, the number of j with $\lambda(y_j)=\lambda$ is equal to $\sum_{\mu\in\mathbb{R}}\operatorname{mult}_{-\lambda+\mu\mathrm{i}}(\chi_A)$. Can we choose here $y_1,\dots,y_r$ to be a Hahn basis of V?

Corollary 7.4.20. *If* χ_A *has no real zeros, then* $K^\dagger$ *is not an eigenvalue of* A*, and so there is no* $y\in K^\times$ *such that* $A(y)=0$*.*

Proof. Take a Hahn basis of V as in Corollary 7.4.19. Then the eigenvalues of A are $\phi_1'\mathrm{i}+K^\dagger,\dots,\phi_r'\mathrm{i}+K^\dagger$, by Theorem 7.4.1. Suppose $K^\dagger$ is an eigenvalue of A. Then we have j with $\phi_j'\mathrm{i}\in K^\dagger=H+\mathrm{I}(H)\mathrm{i}$, so $\phi_j'\in I(H)$, hence $\phi_j\preccurlyeq 1$, and thus $\phi_j=0$. Then Lemma 7.4.18 yields a real zero of χ_A. For the rest, use that the f_j with $\phi_j=0$ form a basis of $\ker_K A$ by remarks after Theorem 7.4.1. $\square$

Example. The linear differential equation

$$y'''-\left(\mathrm{i}+\frac{1}{\mathrm{e}^x}\right)y''+\left(1-\frac{1}{\log x}\right)y'-\left(\mathrm{i}+\frac{1}{x^2}\right)y=0$$

has no nonzero solution in $F[\mathrm{i}]$ for any Hardy field F.

We can now prove a strong version of a theorem of Perron [147, Satz 5] in the setting of linear differential equations over complexified Hardy fields. (A precursor of Perron's theorem for $A\in\mathbb{C}(x)[\partial]$ is due to Poincaré [154].) Perron assumes additionally that the real parts of distinct zeros of χ_A are distinct.

Proposition 7.4.21. *Suppose all* (*complex*) *zeros of* χ_A *are simple. Let*

$$y_1=f_1\,\mathrm{e}^{\phi_1\mathrm{i}},\ \dots,\ y_r=f_r\,\mathrm{e}^{\phi_r\mathrm{i}} \qquad (f_1,\dots,f_r\in K^\times,\ \phi_1,\dots,\phi_r\in H)$$

be a Hahn basis of V*. Then the zeros of* χ_A *are*

$$c_1\ :=\ -\lambda(y_1)+\mu(y_1)\mathrm{i},\ \dots,\ c_r\ :=\ -\lambda(y_r)+\mu(y_r)\mathrm{i},$$

and $\big(y_j^{(n)}/y_j\big)-c_j^n\prec 1$ *for* $j=1,\dots,r$ *and all* n*.*

Proof. By Lemma 7.4.18 each c_j is a zero of χ_A, and we claim that there are no other. Let $c=-\lambda+\mu\mathrm{i}$ ($\lambda,\mu\in\mathbb{R}$) be a zero of χ_A. Then Corollary 1.8.47 and [ADH, 5.1.21, 5.8.7] yield $A\in K[\partial]\big(\partial-(p+q\mathrm{i})\big)$ with $p,q\in\mathcal{O}_H$, $p+\lambda, q-\mu\prec 1$. Taking $f\in H^\times$ and $\phi\in H$ with $f^\dagger=p$ and $\phi'=q$ we have $y:=f\,\mathrm{e}^{\phi\mathrm{i}}\in V^{\neq}$ and so $\lambda(y)=\lambda$ and $\mu(y)=\mu$.

Take $a_1,\dots,a_r\in\mathbb{C}$ such that $y=a_1f_1\,\mathrm{e}^{\phi_1\mathrm{i}}+\dots+a_rf_r\,\mathrm{e}^{\phi_r\mathrm{i}}$. As in the proof of Corollary 5.10.44 (but with r instead of m) we arrange, with $l\in\{1,\dots,r\}$, that $\phi_1,\dots,\phi_l$ are distinct and each ϕ_j with $l<j\leqslant r$ equals one of $\phi_1,\dots,\phi_l$. For $k=1,\dots,l$ we take (as in that proof, but with other notation) $h_k\in\Lambda_H$ such that $\phi_k-\phi(h_k\mathrm{i})\preccurlyeq 1$ and put $g_k:=\sum_{1\leqslant j\leqslant l,\,\phi_j=\phi_k}a_jf_j$ and $u_k:=\mathrm{e}^{(\phi_k-\phi(h_k\mathrm{i}))\mathrm{i}}$, and likewise $h\in\Lambda_H$ with $\phi-\phi(h\mathrm{i})\preccurlyeq 1$, and set $u:=\mathrm{e}^{(\phi-\phi(h\mathrm{i}))\mathrm{i}}$. Then

$$y\ =uf\,\mathrm{e}(h\mathrm{i})\ =\ g_1\,\mathrm{e}^{\phi_1\mathrm{i}}+\dots+g_l\,\mathrm{e}^{\phi_l\mathrm{i}}\ =\ u_1g_1\,\mathrm{e}(h_1\mathrm{i})+\dots+u_l\,\mathrm{g}_l\,\mathrm{e}(h_l\mathrm{i}).$$

Now $u,u_1,\dots,u_l\in K$, so $h=h_k$ for some $k\in\{1,\dots,l\}$, say $h=h_1$, hence $uf=u_1g_1$ and so $y=g_1\,\mathrm{e}^{\phi_1\mathrm{i}}$. Since the f_j with $\phi_j=\phi_1$ are valuation-independent and $f \asymp g_1$, this yields j with $\phi_j = \phi_1$ and $a_j \neq 0$ such that $f \asymp f_j$. Then $\lambda = \lambda(y) = \lambda(f) = \lambda(f_j) = \lambda(y_j)$. The proof of Lemma 7.4.18 gives

$$\mu \;=\; \mu(y) \;=\; \lim_{t\to\infty} \phi'(t), \qquad \mu(y_j) \;=\; \lim_{t\to\infty} \phi_j'(t) \;=\; \lim_{t\to\infty} \phi_1'(t).$$

But $\phi_1 - \phi(h_1 i) \preccurlyeq 1$, $\phi - \phi(hi) \preccurlyeq 1$, and $h = h_1$, so $\phi - \phi_1 \preccurlyeq 1$, hence $\phi' - \phi_1' \prec 1$, and thus $\mu = \mu(y_j)$. This yields $c = c_j$. For the last claim, let $j \in \{1,\dots,r\}$. Then for $z_j := y_j^\dagger$ we have $z_j - c_j \prec 1$ (see for example the proof of Lemma 7.4.18), and $y_j^{(n)}/y_j = R_n(z_j)$. Now use Lemma 1.1.20 if $c_j \neq 0$. If $c_j = 0$, then $z_j \prec 1$, so we can use that then $R_n(z_j) \prec 1$ for $n \geqslant 1$. $\square$

Corollary 7.4.22. *Suppose the real part of each complex zero of χ_A is negative. Then for all $y \in V$ and all n we have $y^{(n)} \prec 1$.*

Proof. By Corollary 7.4.19 it is enough to consider the case $y = f\,\mathrm{e}^{\phi \mathrm{i}} \in V$ where $f \in K^\times$, $\phi \in H$. Then $\lambda := \lambda(y) = \lambda(f) \in \mathbb{R}^>$ by Lemma 7.4.18, which for $0 < \varepsilon < \lambda$ gives $f \prec \mathrm{e}^{-(\lambda-\varepsilon)x} \prec 1$. Now use Corollary 7.4.12. $\square$

We use Corollary 7.4.22 to strengthen another theorem of Perron [149, 150] in the Hardy field context:

Corollary 7.4.23. *Suppose $a_0 := \chi_A(0) \neq 0$. Let $b \in K$, $b \preccurlyeq 1$. Then there exists $y \in K$ such that*

$$A(y) \;=\; b, \quad y - (b/a_0) \;\prec\; 1, \quad y^{(n)} \;\prec\; 1 \;\text{ for all } n \geqslant 1.$$

Moreover, if the real part of each complex zero of χ_A is negative, then all $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $A(y) = b$ satisfy $y - (b/a_0) \prec 1$ and $y^{(n)} \prec 1$ for all $n \geqslant 1$.

Proof. By Theorem 6.7.22 and [ADH, 14.5.7], K is r-linearly newtonian. As $\partial\mathcal{O} \subseteq \sigma$, for the first part it is enough to find $y \in K$ such that $A(y) = b$ and $y - (b/a_0) \prec 1$. Corollary 1.5.8 yields such y if $b \asymp 1$, so suppose $0 \neq b \prec 1$ (since for $b = 0$ we can take $y = 0$). From $A(1) \asymp 1$ and Proposition 1.5.2 we obtain $0 \notin v(\ker_K^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$. Corollary 1.5.7 then yields $y \in K^\times$ with $A(y) = b$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v_A^{\mathrm{e}}(vy) = vb$. Now $A(1) \asymp 1$ gives $v_A^{\mathrm{e}}(0) = 0$, so $y \prec 1$ by Lemma 1.5.6. The second statement now follows from Corollary 7.4.22. $\square$

Example. Each $y \in \mathcal{C}^2[\mathrm{i}]$ with

$$y'' + (2-\mathrm{i})(1 + x^{-1}\log x)y' + (1-\mathrm{i})y \;=\; 2 + \mathrm{e}^{-x^2}$$

satisfies $y \sim \mathrm{i}+1$ and $y^{(n)} \prec 1$ for each $n \geqslant 1$, and there is such a $y \in F[\mathrm{i}]$ for some Hardy field F.

Here is a version of Corollary 2.3.40 in the Hardy field setting:

Proposition 7.4.24. *Let H_0 be a* d*-perfect Hardy subfield of H such that $A \in K_0[\partial]$ for $K_0 := H_0[\mathrm{i}]$. Suppose $r := \operatorname{order}(A) \geqslant 1$ and χ_A has distinct zeros $c_1,\dots,c_r \in \mathbb{C}$ with $\operatorname{Re} c_1 \geqslant \cdots \geqslant \operatorname{Re} c_r$. Then there is a unique splitting $(a_1,\dots,a_r)$ of A over K such that $a_1 - c_1,\dots,a_r - c_r \prec 1$. If in addition $\operatorname{Re} c_1 > \cdots > \operatorname{Re} c_r$, then for this splitting of A over K we have $a_1,\dots,a_r \in \mathcal{O}_{K_0}$.*

Proof. The first claim holds by Corollary 2.3.40. Suppose $\operatorname{Re} c_1 > \cdots > \operatorname{Re} c_r$; it remains to show $a_1,\dots,a_r \in \mathcal{O}_{K_0}$. For this we proceed by induction on r. The case $r = 1$ being obvious, suppose $r > 1$. Let $y_1,\dots,y_r$ be a Hahn basis of V with $c_j = -\lambda(y_j) + \mu(y_j)\mathrm{i}$ for $j = 1,\dots,r$ as in Proposition 7.4.21. Lemma 5.5.21

yields $\theta_j \in H$ with $y_j = |y_j|\,\mathrm{e}^{\theta_j \mathrm{i}}$ and $|y_j| \in H^{>}$, for $j = 1, \dots, r$; these θ_j might be different from the ϕ_j of Proposition 7.4.21. Let now F be any d-maximal Hardy field extension of H_0; we claim that $|y_r|, \theta_r \in F$. To see this, use Lemma 5.5.21 and Proposition 7.4.21 applied to F in place of H to get $f \in F^{>}$, $\theta \in F$ such that $y := f\,\mathrm{e}^{\theta\mathrm{i}}$ satisfies $A(y) = 0$ and $c_r - y^\dagger \prec 1$. Then $y \in V \cap \mathcal{C}[\mathrm{i}]^\times$. Take $d_1, \dots, d_r \in \mathbb{C}$ such that $y = d_1 y_1 + \cdots + d_r y_r$. Lemma 5.10.51 applied to the $d_j y_j$ with $d_j \neq 0$ in place of $f_1, \dots, f_n$, and with c_r in the role of c, yields $i \in \{1, \dots, r\}$ such that $d_i \neq 0$, $c_i = c_r$, and $\operatorname{Re} c_j \leqslant \operatorname{Re} c_r$ for all j with $d_j \neq 0$. Hence $i = r$ is the only one such j and thus $y = d_r y_r$. This yields $|y_r| \in \mathbb{R}^{>} f \subseteq F^{>}$ and $\theta_r \in \theta + \mathbb{R} \subseteq F$ by the uniqueness part of Lemma 5.5.20, as claimed. This claim and d-perfectness of H_0 now give $|y_r|, \phi_r \in H_0$, hence $y_r^\dagger = |y_r|^\dagger + \theta_r'\mathrm{i} \in K_0$. By [ADH, 5.1.21] we get $A = B(\partial - y_r^\dagger)$ where $B \in K_0[\partial]$ is monic, and by [ADH, 5.6.3] we have $B \in \mathcal{O}_{K_0}[\partial]$ with $\chi_A = \chi_B \cdot (Y - c_r)$, hence the zeros of B are $c_1, \dots, c_{r-1}$. Now apply the inductive hypothesis to B. □

Example. The linear differential operator

$$\partial^3 - (1 - \mathrm{e}^{-\mathrm{e}^x})\mathrm{i}\partial^2 - (1 + \mathrm{i} + x^{-2}\log x^2)\partial + (\log\log x)^{-1/2} \in \mathcal{O}[\partial]$$

splits over $\mathrm{E}(\mathbb{Q})[\mathrm{i}]$. In fact, there is a unique splitting (a_1, a_2, a_3) of this linear differential operator over K with $a_1 - (1+\mathrm{i}) \prec 1$, $a_2 \prec 1$, and $a_3 + 1 \prec 1$, and we have $a_1, a_2, a_3 \in \mathrm{E}(\mathbb{Q})[\mathrm{i}]$.

Question. Can we drop the assumption $\operatorname{Re} c_1 > \cdots > \operatorname{Re} c_r$ in the last part of Proposition 7.4.24?

Next we derive consequences of Theorem 7.4.1 for matrix differential equations.

Matrix differential equations. *In this subsection H is* d*-maximal. We take an $\mathbb{R}$-linear complement Λ_H of $\mathrm{I}(H)$ in H, set $\Lambda := \Lambda_H \mathrm{i}$, and identify $\mathrm{U} = \mathrm{U}_K$ with $K[\mathrm{e}^{H\mathrm{i}}]$ as usual. Let N be an $n \times n$ matrix over K, where $n \geqslant 1$.* Recall from [ADH, 5.5] the definition of *fundamental matrix* for the matrix differential equation $y' = Ny$ over any differential ring extension of K.

Corollary 7.4.25. *There are $M \in \mathrm{GL}_n(K)$ and $\phi_1, \dots, \phi_n \in H$ with $\phi_1, \dots, \phi_n$ apart such that for $D := \operatorname{diag}(\mathrm{e}^{\phi_1\mathrm{i}}, \dots, \mathrm{e}^{\phi_n\mathrm{i}})$, the $n \times n$ matrix MD over $K[\mathrm{e}^{H\mathrm{i}}]$ is a fundamental matrix for $y' = Ny$. Moreover, for any such M and $\phi_1, \dots \phi_n$, setting $\alpha_j := \phi_j'\mathrm{i} + K^\dagger$ for $j = 1, \dots, n$, the spectrum of $y' = Ny$ is $\{\alpha_1, \dots, \alpha_n\}$, and for all $\alpha \in K/K^\dagger$,*

$$\operatorname{mult}_\alpha(N) \;=\; |\{j \in \{1, \dots, n\} : \alpha_j = \alpha\}|.$$

Proof. The hypothesis of [ADH, 5.5.14] is satisfied for $R := \mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}]$. To see why, let $L \in K[\partial]$ be monic of order n. Then Theorem 7.4.1 and a subsequent remark provide $f_1, \dots, f_n \in K^\times$ and $\phi_1, \dots, \phi_n \in H$ such that $\phi_1, \dots, \phi_n$ are apart and $f_1\,\mathrm{e}^{\phi_1\mathrm{i}}, \dots, f_n\,\mathrm{e}^{\phi_n\mathrm{i}}$ is a basis of $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} L = \ker_\Omega L$, where $\Omega := \operatorname{Frac}\mathrm{U}$. Hence $W := \operatorname{Wr}\big(f_1\,\mathrm{e}^{\phi_1\mathrm{i}}, \dots, f_n\,\mathrm{e}^{\phi_n\mathrm{i}}\big) \in \mathrm{GL}_n(\Omega)$. Also $\det W \in \mathrm{e}^{\phi_1\mathrm{i} + \cdots + \phi_n\mathrm{i}}\,K^\times \subseteq \mathrm{U}^\times$, hence $W \in \mathrm{GL}_n(\mathrm{U})$. Thus by the remarks preceding [ADH, 5.5.14]: W is a fundamental matrix for $y' = A_L y$ with U as the ambient differential ring. Note that $W = QD$ where $D = \operatorname{diag}(\mathrm{e}^{\phi_1\mathrm{i}}, \dots, \mathrm{e}^{\phi_n\mathrm{i}})$ and $Q \in \mathrm{GL}_n(K)$.

We now follow the proof of [ADH, 5.5.14]: take monic $L \in K[\partial]$ of order n such that $y' = Ny$ is equivalent to $y' = A_L y$, and take $P \in \mathrm{GL}_n(K)$ such that $P\operatorname{sol}_R(A_L) = \operatorname{sol}_R(N)$. With W the above fundamental matrix for $y' = A_L y$,

$PW \in \mathrm{GL}_n(R)$ is then a fundamental matrix for $y' = Ny$. So $M := PQ \in \mathrm{GL}_n(K)$ gives $MD = PW$ as a fundamental matrix for $y' = Ny$.

Let now any $M \in \mathrm{GL}_n(K)$, $\phi_1, \dots, \phi_n \in H$, and $D := \operatorname{diag}(\mathrm{e}^{\phi_1 i}, \dots, \mathrm{e}^{\phi_n i})$ be given such that MD is a fundamental matrix for $y' = Ny$. Let $f_1, \dots, f_n$ be the successive columns of M. Then $\mathrm{e}^{\phi_1 i} f_1, \dots, \mathrm{e}^{\phi_n i} f_n$ is a basis of the $\mathbb{C}$-linear space $\mathrm{sol}_{\mathrm{U}}(N)$. The "moreover" part now follows from Lemma 5.10.24. □

Recall from Corollary 5.2.46 that for $f = (f_1, \dots, f_n) \in \mathcal{C}[i]^n$,

$$\lambda(f) \ = \ \min\bigl\{\lambda(f_1), \dots, \lambda(f_n)\bigr\} \ = \ \lambda\bigl(|f_1| + \cdots + |f_n|\bigr) \in \mathbb{R}_{\pm\infty}.$$

If $f \in \mathcal{C}^1[i]$ and $f \notin \mathcal{C}^1[i]^\times$, we set $\mu(f) := -\infty$. With this convention,

$$\mu(f) \ := \max\bigl\{\mu(f_1), \dots, \mu(f_n)\bigr\} \qquad \text{for } f = (f_1, \dots, f_n) \in \mathcal{C}^1[i]^n.$$

We also turn K^n into a valued $\mathbb{C}$-linear space with valuation $v\colon K^n \to \Gamma_\infty$ given by $v(f) := \min\bigl\{v(f_1), \dots, v(f_n)\bigr\}$ for $f = (f_1, \dots, f_n) \in K^n$.

Corollary 7.4.26. *We can choose M, D as in Corollary 7.4.25 such that the successive columns $f_1, \dots, f_n$ of M have the property that for $k = 1, \dots, n$ the f_j with $\phi_j = \phi_k$ are valuation-independent. For any such M, D, the matrix MD is a Lyapunov fundamental matrix for $y' = Ny$.*

Proof. Take M, $\phi_1, \dots, \phi_n$ as in Corollary 7.4.25 such that $\phi_1, \dots, \phi_m$ are distinct, $m \leqslant n$, and each ϕ_j with $m < j \leqslant n$ is equal to some ϕ_k with $1 \leqslant k \leqslant m$. For $V := \mathrm{sol}_{\mathrm{U}}(N)$ this yields an internal direct sum decomposition

$$V \ = \ \mathrm{e}^{\phi_1 i}\, V_1 \oplus \cdots \oplus \mathrm{e}^{\phi_m i}\, V_m$$

into $\mathbb{C}$-linear subspaces of V. Now [ADH, remark before 2.3.10] yields for $k = 1, \dots, m$ a valuation basis of V_k. Modifying M accordingly, this yields M, D with the desired property. The rest follows from Corollary 5.10.44. □

If the matrix N is bounded, then the solutions of $y' = Ny$ grow only moderately, by Lemma 5.2.47; their oscillation is also moderate:

Lemma 7.4.27. *Suppose N is bounded. Let $M \in \mathrm{GL}_n(K)$ and $\phi_1, \dots, \phi_n \in H$ be such that for $D := \operatorname{diag}(\mathrm{e}^{\phi_1 i}, \dots, \mathrm{e}^{\phi_n i})$ the $n \times n$ matrix MD over $K[\mathrm{e}^{Hi}]$ is a fundamental matrix for $y' = Ny$. Then $\phi_1, \dots, \phi_n \preccurlyeq x$.*

Proof. Corollary 7.4.25 yields $\Sigma(N) = \{\phi_1' i + K^\dagger, \dots, \phi_n' i + K^\dagger\}$. The differential module over K associated to N (cf. [ADH, p. 277]) is bounded, by Example 2.3.32(1), hence each $\alpha \in \Sigma(N)$ has the form $a + K^\dagger$ with $a \in \mathcal{O}$, by Corollary 2.3.48. Together with Lemma 2.3.38 this yields $\phi_1, \dots, \phi_n \preccurlyeq x$. □

Corollary 7.4.28. *Suppose N is bounded. Then* $\mathrm{sol}_{\mathrm{U}}(N) \subseteq (\mathrm{U}^{\preccurlyeq})^n$.

Proof. Take M and D as in Lemma 7.4.27. It suffices to show that the entries of MD are in $\mathrm{U}^{\preccurlyeq}$. Such an entry equals $g\,\mathrm{e}^{\phi i}$ where g is an entry of M and $\phi \in \{\phi_1, \dots, \phi_n\}$. Lemma 5.10.17 gives $h \in \Lambda_H$ such that $\phi - \phi(hi) \preccurlyeq 1$. Now $\phi \preccurlyeq x$ by Lemma 7.4.27, so $\phi(hi) \preccurlyeq x$, and thus $h = \phi(hi)' \preccurlyeq 1$. Also $\mathrm{e}^{\phi i} = u\,\mathrm{e}(hi)$ with $u = \mathrm{e}^{(h - \phi(hi))i} \in K^\times$, so $g\,\mathrm{e}^{\phi i} = gu\,\mathrm{e}(hi)$. Lemma 5.2.47 gives $\lambda(g\,\mathrm{e}^{\phi i}) > -\infty$. Now $\lambda(g\,\mathrm{e}^{\phi i}) = \lambda(g) = \lambda(gu)$, so $gu \in \mathcal{O}_\Delta$. Then $h \in \Lambda_H \cap \mathcal{O}_H$ gives $g\,\mathrm{e}^{\phi i} \in \mathrm{U}^{\preccurlyeq}$. □

The next result shows how Lemma 7.4.27 also yields information for unbounded N. For $a \in H$ we let "$N \preccurlyeq a$" stand for "$g \preccurlyeq a$ for every entry g of N".

Corollary 7.4.29. *Suppose $N \preccurlyeq \ell'$, $\ell \in H^{>\mathbb{R}}$. Let $M \in \mathrm{GL}_n(K)$, $\phi_1, \dots, \phi_n \in H$, and $D := \mathrm{diag}(\mathrm{e}^{\phi_1 \mathrm{i}}, \dots, \mathrm{e}^{\phi_n \mathrm{i}})$ be such that MD is a fundamental matrix for $y' = Ny$ over $K[\mathrm{e}^{H\mathrm{i}}]$. Then $\phi_1, \dots, \phi_n \preccurlyeq \ell$, and there exists $m \geqslant 1$ such that $f \preccurlyeq \mathrm{e}^{m\ell}$ and $f \not\preccurlyeq \mathrm{e}^{-m\ell}$, for each column f of M.*

Proof. Put $\phi := \ell'$ and use the superscript $\circ$ as in $(\partial, \circ, \delta)$, Section 6.4. Then for $R := \mathcal{C}^{<\infty}[\mathrm{i}]$ and any fundamental matrix $F \in \mathrm{GL}_n(R)$ for $y' = Ny$, the matrix $F^\circ \in \mathrm{GL}_n(R)$ is a fundamental matrix for $z' = (\phi^{-1}N)^\circ z$. As H° is d-maximal and $(\phi^{-1}N)^\circ$ is bounded, we can apply Lemmas 7.4.27 and 5.2.47 (and a remark following Corollary 5.2.45) to M° and D° in the role of M and D, and convert this back to information about M and D as claimed. $\square$

In the next lemma and its corollary we assume N is bounded and M, D are as in Lemma 7.4.27. Let $\mathrm{st}(N)$ (the *standard part* of N) be the $n \times n$ matrix over $\mathbb{C}$ such that $N - \mathrm{st}(N) \prec 1$. For $f \in K^n$, put $|f| := \max\big\{|f_1|, \dots, |f_n|\big\} \in H$, so $vf = v|f|$.

Lemma 7.4.30. *Let $y = \mathrm{e}^{\phi \mathrm{i}} f$ where $f = f_k$ is the kth column of M and $\phi = \phi_k$, $k \in \{1, \dots, n\}$. Set $s := |f|^{-1} f \in \mathcal{O}^n$. Then $\lambda := \lambda(y) \in \mathbb{R}$, $\mu := \mu(y) \in \mathbb{R}$, and $-\lambda + \mu\mathrm{i} \in \mathbb{C}$ is an eigenvalue of $\mathrm{st}(N)$ with eigenvector $\mathrm{st}(s) \in \mathbb{C}^n$.*

Proof. Note that y is the kth column of MD. From Lemma 5.2.47 we get $\lambda \in \mathbb{R}$. Let g be a nonzero entry of f. Then for the corresponding entry $g\,\mathrm{e}^{\phi\mathrm{i}}$ of y we have $(g\,\mathrm{e}^{\phi\mathrm{i}})^\dagger = g^\dagger + \phi'\mathrm{i}$, so $\mathrm{Im}\big((g\,\mathrm{e}^{\phi\mathrm{i}})^\dagger\big) = \mathrm{Im}(g^\dagger) + \phi'$ with $\mathrm{Im}(g^\dagger) \prec 1$ by a remark preceding Lemma 1.2.16, and $\phi' \preccurlyeq 1$ by Lemma 7.4.27. Hence $\mu(g\,\mathrm{e}^{\phi\mathrm{i}}) = \lim_{t\to+\infty} \phi'(t) \in \mathbb{R}$. This gives $\mu = \lim_{t\to+\infty} \phi'(t) \in \mathbb{R}$ and so $\mu - \phi' \prec 1$.

Next, $y' = Ny$ gives $\phi'\mathrm{i}f + f' = Nf$, and then using also Corollary 5.10.47,

$$Nf \;=\; (-\lambda + \mu\mathrm{i})f + (\phi'\mathrm{i} - \mu\mathrm{i})f + \lambda f + f' \;=\; (-\lambda + \mu\mathrm{i})f + r, \quad r \in K^n,\ r \prec f.$$

Dividing by $|f| \in H^\times$ then yields the claim about $-\lambda + \mu\mathrm{i}$ and s. $\square$

The proof of Lemma 7.4.30 also gives the next corollary, where I_n denotes the $n \times n$ identity matrix over K, and $\mathrm{mult}_c\big(\mathrm{st}(N)\big) := \dim_{\mathbb{C}} \ker_{\mathbb{C}^n}\big(\mathrm{st}(N) - cI_n\big)$ for $c \in \mathbb{C}$:

Corollary 7.4.31. *For $k = 1, \dots, n$, let f_k be the kth column of M, so $y_k := f_k\,\mathrm{e}^{\phi_k \mathrm{i}}$ is the kth column of MD, and put $c_k := -\lambda(y_k) + \mu(y_k)\mathrm{i}$. If for a certain k the f_j with $\mu_j = \mu_k$ are valuation-independent, then for this k we have*

$$\mathrm{mult}_{c_k}\big(\mathrm{st}(N)\big) \;\geqslant\; \big|\{j : (\lambda_j, \mu_j) = (\lambda_k, \mu_k)\}\big|.$$

Question. Suppose N is bounded and $\mathrm{st}(N)$ is the $n \times n$ matrix over $\mathbb{C}$ such that $N - \mathrm{st}(N) \prec 1$. By Perron [151, Satz 13] there is a Lyapunov fundamental matrix F for $y' = Ny$ such that for each $\lambda \in \mathbb{R}$, the number of columns f of F with $\lambda(f) = \lambda$ equals $\sum_{\mu \in \mathbb{R}} \mathrm{mult}_{-\lambda+\mu\mathrm{i}}\big(\mathrm{st}(N)\big)$. Can one take here F of the form $F = MD$ where $M \in \mathrm{GL}_n(K)$ and $D = \mathrm{diag}(\mathrm{e}^{\phi_1 \mathrm{i}}, \dots, \mathrm{e}^{\phi_n \mathrm{i}})$ with $\phi_1, \dots, \phi_n \in H$?

Recall: a column vector $(y_1, \dots, y_n)^{\mathrm{t}} \in \mathcal{C}[\mathrm{i}]^n$ is said to be *bounded* if $y_1, \dots, y_n \preccurlyeq 1$.

Lemma 7.4.32. *Suppose $y' = Ny$ where $y \in \mathcal{C}^1[\mathrm{i}]^n$ is bounded. Then*

$$y \;=\; \mathrm{e}^{\phi_1 \mathrm{i}}\, z_1 + \cdots + \mathrm{e}^{\phi_m}\, z_m$$

where $m \leqslant n$, $\phi_1, \dots, \phi_m \in H$ are distinct and apart, $z_1, \dots, z_m \in K^n$ are bounded, and $\mathrm{e}^{\phi_1 \mathrm{i}}\, z_1, \dots, \mathrm{e}^{\phi_m \mathrm{i}}\, z_m \in \mathrm{sol}_{\mathrm{U}}(N)$.

Proof. Let $M \in \mathrm{GL}_n(K)$ and $\phi_1, \dots, \phi_n \in H$ be as in Corollary 7.4.25, in particular, $\phi_1, \dots, \phi_n$ are apart. For $j = 1, \dots, n$, let $f_j \in K^n$ be the jth column of M. Take $c_1, \dots, c_n \in \mathbb{C}$ such that $y = c_1 \mathrm{e}^{\phi_1 \mathrm{i}} f_1 + \cdots + c_n \mathrm{e}^{\phi_n \mathrm{i}} f_n$. We arrange that $\phi_1, \dots, \phi_m$ are distinct, $m \leqslant n$, and each ϕ_j with $m < j \leqslant n$ is equal to one of the ϕ_k with $1 \leqslant k \leqslant m$. This gives $y = \mathrm{e}^{\phi_1 \mathrm{i}} z_1 + \cdots + \mathrm{e}^{\phi_m} z_m$ with $z_1, \dots, z_m \in K^n$ and $\mathrm{e}^{\phi_1 \mathrm{i}} z_1, \dots, \mathrm{e}^{\phi_m \mathrm{i}} z_m \in \mathrm{sol}_{\mathrm{U}}(N)$. The $\preccurlyeq$-version of Corollary 5.10.18 with $\mathfrak{m} = 1$ then shows that $z_1, \dots, z_m$ are bounded. $\square$

See [18, Chapter 2] and [45, Chapter II, §3] for classical conditions on a matrix differential equation to have only bounded solutions.

Despite Corollary 7.4.25, the differential fraction field of $K[\mathrm{e}^{H\mathrm{i}}]$ is not pv-closed, since it is not even algebraically closed; see [ADH, 5.1.31] and Lemma 2.1.2. Combining Corollary 7.2.10 and [ADH, 5.4.2] also yields:

Corollary 7.4.33. *For every column $b \in K^n$ the matrix differential equation $y' = Ny + b$ has a solution in K^n.*

We also have a version of Corollary 7.4.15 for matrix differential equations:

Corollary 7.4.34. *Suppose $y' = Ny$ is self-dual. If α is an eigenvalue of $y' = Ny$, then so is $-\alpha$, with the same multiplicity. If n is odd, then the matrix differential equation $y' = Ny$ has a solution $y \neq 0$ in K^n.*

This follows from Corollaries 2.4.36 and 7.4.25. Note that the hypothesis on N in Corollary 7.4.34 is satisfied if $y' = Ny$ is self-adjoint or hamiltonian.

If $y' = Ny$ is self-adjoint, and $M \in \mathrm{GL}_n(K)$ and $\phi_1, \dots, \phi_n \in H$ (as in Corollary 7.4.25) are such that MD is a fundamental matrix for $y' = Ny$ where $D := \operatorname{diag}(\mathrm{e}^{\phi_1 \mathrm{i}}, \dots, \mathrm{e}^{\phi_n \mathrm{i}})$, then there exists $U \in \mathrm{GL}_n(\mathbb{C})$ such that the fundamental matrix $MDU \in \mathrm{GL}_n\big(K[\mathrm{e}^{H\mathrm{i}}]\big)$ of $y' = Ny$ is orthogonal as an element of $\mathrm{GL}_n(\Omega)$, where Ω is the differential fraction field of $K[\mathrm{e}^{H\mathrm{i}}]$; likewise with "hamiltonian" and "symplectic" instead of "self-adjoint" and "orthogonal": Lemmas 2.4.39 and 2.4.40.

Example. Any matrix differential equation $y' = Ny$ with

$$N = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \qquad (a, b, c \in K = H[\mathrm{i}])$$

has a nonzero solution $y = (y_1, y_2, y_3)^{\mathrm{t}} \in K^3$.

In the self-dual case we can improve on Corollary 7.4.25:

Corollary 7.4.35. *Suppose $y' = Ny$ is self-dual. Then there are $M \in \mathrm{GL}_n(K)$ and $\phi_1, \dots, \phi_n \in H$ that are apart such that*

(i) *for each $j \in \{1, \dots, n\}$ there is a $k \in \{1, \dots, n\}$ with $\phi_j = -\phi_k$;*
(ii) *with $D := \operatorname{diag}(\mathrm{e}^{\phi_1 \mathrm{i}}, \dots, \mathrm{e}^{\phi_n \mathrm{i}})$, the $n \times n$ matrix MD over $K[\mathrm{e}^{H\mathrm{i}}]$ is a fundamental matrix for $y' = Ny$.*

Proof. Corollary 2.4.34 yields a matrix differential equation $y' = A_L y$ over K, equivalent to $y' = Ny$, where $L \in K[\partial]$ is monic self-dual of order n. Then we can use Corollary 7.4.16 instead of Theorem 7.4.1 in the proof of Corollary 7.4.25. $\square$

Let Ω be the differential fraction field of $K[\mathrm{e}^{H\mathrm{i}}]$ and $V := \mathrm{sol}_\Omega(N) \subseteq K[\mathrm{e}^{H\mathrm{i}}]^n$, a $\mathbb{C}$-linear subspace of Ω^n. Then $\dim_{\mathbb{C}} V = n$ and $V = \mathrm{sol}_{\mathcal{C}^{<\infty}[\mathrm{i}]}(N)$. In the corollary below we assume that $y' = Ny$ is self-adjoint, and we equip V with the symmetric bilinear form $\langle\ ,\ \rangle$ defined after Lemma 2.4.38 (with Ω instead of K).

Corollary 7.4.36. *There are $m \leqslant n$ and distinct $\theta_1, \dots, \theta_m$ in $H^{>\mathbb{R}}$ that are apart, subspaces $V_1, \dots, V_m, W$ of the $\mathbb{C}$-linear space V with $W \subseteq K^n$, and for $j = 1, \dots, m$, nonzero subspaces V_j^+, V_j^- of K^n, such that*

$$\begin{aligned} V_j \ &= \ V_j^+ \,\mathrm{e}^{\theta_j \mathrm{i}} \oplus V_j^- \,\mathrm{e}^{-\theta_j \mathrm{i}} \qquad &&(\textit{internal direct sum of subspaces of } V_j),\\ V \ &= \ V_1 \perp \cdots \perp V_m \perp W \qquad &&(\textit{orthogonal sum with respect to } \langle\ ,\ \rangle). \end{aligned}$$

For any such m and θ_j, V_j, V_j^+, V_j^-, W we have $\dim_{\mathbb{C}} V_j^+ = \dim_{\mathbb{C}} V_j^-$ and $\langle\ ,\ \rangle$ restricts to a null form on $V_j^+ \,\mathrm{e}^{\theta_j \mathrm{i}}$ and on $V_j^- \,\mathrm{e}^{-\theta_j \mathrm{i}}$.

Proof. Take M and $\phi_1, \dots, \phi_n$ as in Corollary 7.4.35, and set

$$D \ := \ \operatorname{diag}(\mathrm{e}^{\phi_1 \mathrm{i}}, \dots, \mathrm{e}^{\phi_n \mathrm{i}}).$$

Let $(f_1, \dots, f_n)^{\mathrm{t}}$ be the jth column of M and $(g_1, \dots, g_n)^{\mathrm{t}}$ be the kth column of M. The jth column of MD is $f = (f_1 \,\mathrm{e}^{\phi_j \mathrm{i}}, \dots, f_n \,\mathrm{e}^{\phi_j \mathrm{i}})^{\mathrm{t}}$, and the kth column of MD is $g = (g_1 \,\mathrm{e}^{\phi_k \mathrm{i}}, \dots, g_n \,\mathrm{e}^{\phi_k \mathrm{i}})^{\mathrm{t}}$, and $f, g \in \operatorname{sol}_\Omega(N)$ by Corollary 7.4.35(ii). Thus by Lemma 2.4.38 applied to Ω in place of K we have

$$\langle f, g \rangle \ = \ (f_1 g_1 + \cdots + f_n g_n)\,\mathrm{e}^{(\phi_j + \phi_k)\mathrm{i}} \in \mathbb{C}.$$

Corollary 7.4.35(i) gives $l \in \{1, \dots, n\}$ with $\phi_k = -\phi_l$; then $\phi_j + \phi_k = \phi_j - \phi_l$. Hence if $\phi_j \neq -\phi_k$, then $\phi_j + \phi_k \succ 1$ by ϕ_j, ϕ_l being apart, so $\mathrm{e}^{(\phi_j+\phi_k)\mathrm{i}} \notin K$ by Corollary 5.5.23 and thus $\langle f, g \rangle = 0$. Taking $\theta_1, \dots, \theta_m$ to be the distinct positive elements of $\{\phi_1, \dots, \phi_n\}$, this yields the existence statement. The rest follows from Corollary 2.4.36, Lemma 5.10.24, and again Corollary 5.5.23. □

Remark. Suppose $y' = Ny$ is hamiltonian. Then Corollary 7.4.36 remains true with $\langle\ ,\ \rangle$ replaced by the alternating bilinear form ω on V of Lemma 2.4.41. (Same proof, using Lemma 2.4.41 instead of Lemma 2.4.38.)

The complex conjugation automorphism of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$ restricts to an automorphism of the differential integral domain $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}]$, which in turn extends uniquely to an automorphism $g \mapsto \overline{g}$ of the differential field Ω, with $\overline{\overline{g}} = g$ for all $g \in \Omega$. Let Ω_{r} be the fixed field of this automorphism of Ω. Then Ω_{r} is a differential subfield of Ω, and $\Omega = \Omega_{\mathrm{r}}[\mathrm{i}]$. Set $\mathrm{U}_{\mathrm{r}} := \Omega_{\mathrm{r}} \cap \mathrm{U}$. Then

$$\Omega_{\mathrm{r}} \ = \ \operatorname{Frac}(\mathrm{U}_{\mathrm{r}}) \ \text{ inside } \Omega, \qquad \mathrm{U}_{\mathrm{r}} \ = \ \mathrm{U} \cap \mathcal{C}^{<\infty} \ \text{ inside } \mathcal{C}^{<\infty}[\mathrm{i}], \quad \mathrm{U} \ = \ \mathrm{U}_{\mathrm{r}}[\mathrm{i}].$$

Assume in the rest of this subsection that $y' = Ny$ is anti-self-adjoint, and equip the $\mathbb{C}$-linear space $V = \operatorname{sol}_\Omega(N)$ with the positive definite hermitian form $\langle\ ,\ \rangle$ introduced after Lemma 2.4.45, with Ω in the role of K. Then we have the following analogue of Corollary 7.4.36:

Corollary 7.4.37. *There are $m \in \{1, \dots, n\}$, distinct $\theta_1, \dots, \theta_m \in H$ that are apart, and nonzero $\mathbb{C}$-linear subspaces $V_1, \dots, V_m$ of K^n such that V is the following orthogonal sum with respect to $\langle\ ,\ \rangle$:*

$$V \ = \ V_1 \,\mathrm{e}^{\theta_1 \mathrm{i}} \perp \cdots \perp V_m \,\mathrm{e}^{\theta_m \mathrm{i}}.$$

Proof. Corollary 7.4.25 gives $M \in \mathrm{GL}_n(K)$ and $\phi_1, \dots, \phi_n \in H$ that are apart such that MD is a fundamental matrix for $y' = Ny$ where $D := \operatorname{diag}\big(\mathrm{e}^{\phi_1 \mathrm{i}}, \dots, \mathrm{e}^{\phi_n \mathrm{i}}\big)$. For $j, k = 1, \dots, n$, let $f = (f_1 \,\mathrm{e}^{\phi_j \mathrm{i}}, \dots, f_n \,\mathrm{e}^{\phi_j \mathrm{i}})^{\mathrm{t}}$ be the jth column of MD and $g =$

$(g_1\,\mathrm{e}^{\phi_k i},\dots,g_n\,\mathrm{e}^{\phi_k i})^{\mathrm{t}}$ the kth column of MD, where $f_1,\dots,f_n$ and $g_1,\dots,g_n$ are in K. Then by Lemma 2.4.45,

$$\langle f,g\rangle \;=\; (f_1\overline{g_1}+\cdots+f_n\overline{g_n})\,\mathrm{e}^{(\phi_j-\phi_k)i}\in\mathbb{C}$$

and hence $\langle f,g\rangle=0$ if $\phi_j\neq\phi_k$, by Corollary 5.5.23. Taking $\theta_1,\dots,\theta_m$ to be the distinct elements of $\{\phi_1,\dots,\phi_n\}$, this yields the desired result. □

Corollary 7.4.37 and [122, Chapter XV, Corollary 5.2] yield $M\in\mathrm{GL}_n(K)$ and $\phi_1,\dots,\phi_n\in H$ that are apart, such that MD with $D:=\operatorname{diag}\big(\mathrm{e}^{\phi_1 i},\dots,\mathrm{e}^{\phi_n i}\big)$ is not only a fundamental matrix for $y'=Ny$ but also unitary as an element of $\mathrm{GL}_n(\Omega)$. (Cf. Lemma 2.4.46.)

Example (Schrödinger equation for quantum systems with n states [199, §3.4]). This is the matrix differential equation $y'=-\mathrm{i}Sy$ where the $n\times n$ matrix S over K (the Hamiltonian of the system) is hermitian, i.e., $S^{\mathrm{t}}=\overline{S}$. Then $y'=-\mathrm{i}Sy$ is anti-self-adjoint, so we have the positive definite hermitian form

$$(y,z)\mapsto\langle y,z\rangle=y_1\overline{z_1}+\cdots+y_n\overline{z_n}\quad\big(y=(y_1,\dots,y_n)^{\mathrm{t}},\ z=(z_1,\dots,z_n)^{\mathrm{t}}\big)$$

on the $\mathbb{C}$-linear space of solutions W of $y'=-\mathrm{i}Sy$ in $\mathcal{C}^{<\infty}[\mathrm{i}]$. There are $\phi_1,\dots,\phi_n\in H$ that are apart and $f_1,\dots,f_n\in K^n$ such that

$$f_1\,\mathrm{e}^{\phi_1 i},\dots,f_n\,\mathrm{e}^{\phi_n i}\qquad(\text{“wave functions”})$$

is an orthonormal basis of W with respect to $\langle\ ,\ \rangle$. Note again the striking fact that $\langle y,y\rangle$ is a positive real constant, not just an element of $H^{>}$, for every $y\in W^{\neq}$.

Definability. Here we drop the d-maximality assumption from earlier subsections. We begin with consequences of our earlier boundedness results for matrix differential equations depending on (constant) parameters. For this, let $H\supseteq\mathbb{R}$, let $C=(C_1,\dots,C_m)$ be a tuple of distinct indeterminates over K and let $N(C)$ be an $n\times n$ matrix over the polynomial ring $K[C]$, $n\geqslant 1$. Then for $c\in\mathbb{R}^m$ we have the matrix differential equation $y'=N(c)y$ over K. Combining Corollary 7.1.5 with Lemmas 7.4.39 and 7.4.40 yields:

Corollary 7.4.38. *The set of $c\in\mathbb{R}^m$ such that all solutions of $y'=N(c)y$ in $\mathcal{C}^1[\mathrm{i}]^n$ are bounded is semialgebraic, and so is the set of $c\in\mathbb{R}^m$ such that $y'=N(c)y$ has no nonzero bounded solution in $\mathcal{C}^1[\mathrm{i}]^n$.*

For matrix differential equations depending analytically on a single parameter, see [204, Chapter VII]. In this connection we record that for $m=1$ it follows from Corollary 7.4.38: if $y'=N(c)y$ has for arbitrarily large $c\in\mathbb{R}$ only bounded solutions in $\mathcal{C}^1[\mathrm{i}]^n$, then this happens for all sufficiently large $c\in\mathbb{R}$; if $y'=N(c)y$ has for arbitrarily large $c\in\mathbb{R}$ no nonzero bounded solution in $\mathcal{C}^1[\mathrm{i}]^n$, then this happens for all sufficiently large $c\in\mathbb{R}$.

By the next lemma, the property of a matrix differential equation over a complexified Hardy field to have only bounded solutions is “uniformly definable” from the entries in the matrix. Here we view the canonical $\Lambda\Omega$-expansion $\boldsymbol{H}$ of a Hardy field H as a structure for the language $\mathcal{L}^{\iota}_{\Lambda\Omega}$ from [ADH, Chapter 16], and we let $u=(u_{ij})$, $v=(v_{ij})$ be disjoint multivariables of size $n\times n$ with $n\geqslant 1$.

Lemma 7.4.39. *There is a quantifier-free $\mathcal{L}^{\iota}_{\Lambda\Omega}$-formula $\beta(u,v)$ such that for every Hardy field H and $n\times n$ matrix N over $K=H[\mathrm{i}]$:*

$$\boldsymbol{H}\models\beta(\operatorname{Re}N,\operatorname{Im}N)\quad\Longleftrightarrow\quad\textit{all solutions of } y'=Ny \textit{ in } \mathcal{C}^1[\mathrm{i}]^n \textit{ are bounded.}$$

Proof. Let N be an $n \times n$ matrix over K, $\phi \in H$, $z \in K^n$. Then $\mathrm{e}^{\phi\mathrm{i}}\, z \in K[\mathrm{e}^{H\mathrm{i}}]^n$ is a solution of $y' = Ny$ iff $z' + \phi'\mathrm{i}z = Nz$. Moreover, for d-maximal H it follows from Corollary 7.4.25 that all solutions of $y' = Ny$ in $\mathcal{C}^1[\mathrm{i}]^n$ are bounded iff all solutions $\mathrm{e}^{\phi\mathrm{i}}\, z$ with $\phi \in H$ and $z \in K^n$ are bounded. Now use that d-maximal Hardy fields are H-closed and that the theory of H-closed H-fields admits quantifier elimination in the language $\mathcal{L}^{\iota}_{\Lambda\Omega}$. ☐

Using Lemma 7.4.32 we obtain in the same way:

Lemma 7.4.40. *There is a quantifier-free $\mathcal{L}^{\iota}_{\Lambda\Omega}$-formula $\gamma(u,v)$ such that for every Hardy field H and $n \times n$ matrix N over $K = H[\mathrm{i}]$:*

$\boldsymbol{H} \models \gamma(\operatorname{Re} N, \operatorname{Im} N) \iff$ *some nonzero solution of $y' = Ny$ in $\mathcal{C}^1[\mathrm{i}]^n$ is bounded.*

Example. Let $a, b \in H$, and take $g, \phi \in \operatorname{Li}\big(H(\mathbb{R})\big)$ with $g \neq 0$, $g^{\dagger} = a$, and $\phi' = b$. Then $\big\{y \in \mathcal{C}^1[\mathrm{i}] : y' = (a + b\mathrm{i})y\big\} = \mathbb{C}g\,\mathrm{e}^{\phi\mathrm{i}}$, and $g\,\mathrm{e}^{\phi\mathrm{i}} \asymp g$. Thus if H is Liouville closed, then by [ADH, 11.8.19]:

$$\begin{aligned} & \text{every } y \in \mathcal{C}^1[\mathrm{i}] \text{ with } y' = (a+b\mathrm{i})y \text{ is bounded} \\ \iff\ & \text{some } y \in \mathcal{C}^1[\mathrm{i}]^{\neq} \text{ with } y' = (a+b\mathrm{i})y \text{ is bounded} \\ \iff\ & a \notin \Gamma(H) \\ \iff\ & a \leqslant 0 \text{ or } a \in \mathrm{I}(H). \end{aligned}$$

The real case. *In this subsection we assume $A \in H[\partial]$.* Recall that if H is d-maximal, then K is linearly closed, so [ADH, 5.1.35] yields the following, which includes Corollary 9 from the introduction:

Corollary 7.4.41. *If H is* d*-maximal, then A is a product of irreducible operators in $H[\partial]$ which are monic of order* 1 *or monic of order* 2*.*

The next result follows from Corollaries 5.5.19 and 5.10.34, and is a version of Theorem 7.4.1 in the case of a real operator:

Corollary 7.4.42. *Let E be a* d*-maximal Hardy field extension of H. Then the $\mathbb{C}$-linear space $V := \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A$ of zeros of A in $\mathcal{C}^{<\infty}[\mathrm{i}]$ has a basis*

$$g_1\,\mathrm{e}^{\phi_1\mathrm{i}},\ g_1\,\mathrm{e}^{-\phi_1\mathrm{i}},\ \ldots,\ g_m\,\mathrm{e}^{\phi_m\mathrm{i}},\ g_m\,\mathrm{e}^{-\phi_m\mathrm{i}},\ h_1,\ \ldots,\ h_n \qquad (2m+n=r),$$

where $g_j, \phi_j \in E^{>}$ with $\phi_j \succ 1$ $(j = 1,\ldots,m)$ and $h_k \in E^{\times}$ $(k = 1,\ldots,n)$. For any such basis of V, the $\mathbb{R}$-linear space $V \cap \mathcal{C}^{<\infty}$ of zeros of A in $\mathcal{C}^{<\infty}$ has basis

$$g_1 \cos\phi_1,\ g_1 \sin\phi_1,\ \ldots,\ g_m \cos\phi_m,\ g_m \sin\phi_m,\ h_1,\ \ldots,\ h_n,$$

and the $\mathbb{R}$-linear space $V \cap E$ has basis $h_1, \ldots, h_n$.

Remarks. Let E be a d-maximal Hardy field extension of H. The quantity $n = \dim_{\mathbb{R}} \ker_E A$ in Corollary 7.4.42 (and hence also $m = (r-n)/2$) is independent of the choice of E, by Theorem 7.1.3. Likewise, the number of distinct eigenvalues of A with respect to $E[\mathrm{i}]$ does not depend on E. In more detail, the tuple $(d, \mu_1, \ldots, \mu_d)$ where d is the number of distinct eigenvalues of A and $\mu_1 \geqslant \cdots \geqslant \mu_d \geqslant 1$ are their multiplicities, with respect to $E[\mathrm{i}]$, does not depend on E.

Corollary 7.4.42 yields:

Corollary 7.4.43. *If r is odd, then $A(y) = 0$ for some H-hardian germ $y \neq 0$.*

From Corollary 5.10.36 we obtain:

Corollary 7.4.44. *Suppose H is* d*-maximal, and let $\phi > \mathbb{R}$ be an element of H such that $\phi'\mathrm{i} + K^{\dagger}$ is not an eigenvalue of A. Then for every $h \in H$ there are unique $f, g \in H$ such that $A(f\cos\phi + g\sin\phi) = h\cos\phi$.*

Taking $A = \partial$, with $K^{\dagger}$ as the only eigenvalue (Example 2.3.1), we recover the following result due to Shackell [186, Theorem 2]; his proof is based on [35].

Corollary 7.4.45. *Let $h, \phi \in H$ and $\phi > \mathbb{R}$. The germ $h\cos\phi \in \mathcal{C}^{<\infty}$ has an antiderivative $f\cos\phi + g\sin\phi \in \mathcal{C}^{<\infty}$ with f, g in a Hardy field extension of H, and any Hardy field extension of H contains at most one such pair (f, g).*

Besides Corollary 7.4.44 we use here that by a remark preceding Lemma 1.2.16 we have $\phi'\mathrm{i} \notin K^{\dagger}$ for $\phi \in H$ with $\phi > \mathbb{R}$.

We also record a real version of Corollary 7.4.25, which follows from Corollary 7.4.42 in the same way that Corollary 7.4.25 followed from Theorem 7.4.1. Let I_m denote the $m \times m$ identity matrix. Recall that $\mathrm{U}_{\mathrm{r}} = K[\mathrm{e}^{H\mathrm{i}}] \cap \mathcal{C}^{<\infty}$.

Corollary 7.4.46. *Suppose H is* d*-maximal and N is an $n \times n$ matrix over H, $n \geqslant 1$. Then there are $M \in \mathrm{GL}_n(H)$ as well as $k, l \in \mathbb{N}$ with $2k + l = n$ and*

$$D = \begin{pmatrix} \boxed{D_1} & & & \\ & \ddots & & \\ & & \boxed{D_k} & \\ & & & \boxed{I_l} \end{pmatrix} \qquad \text{where } D_j = \begin{pmatrix} \cos\phi_j & \sin\phi_j \\ -\sin\phi_j & \cos\phi_j \end{pmatrix},\ \phi_j \in H,\ \phi_j > \mathbb{R}$$

such that the $n \times n$ matrix MD is a fundamental matrix for $y' = Ny$ with respect to U_{r}. In particular, if n is odd, then $y' = Ny$ for some $0 \neq y \in H^n$.

Proof. Let R and Ω be as in the proof of Corollary 7.4.25, and let $L \in H[\partial]$ be monic of order n. Then Corollary 7.4.42 yields $g_1, \dots, g_k, h_1, \dots, h_l$ and $\phi_1, \dots, \phi_k > \mathbb{R}$ in H, where $2k + l = n$, such that the $\mathbb{R}$-linear space $\ker_{\mathcal{C}^{<\infty}} L$ has basis

$$g_1\cos\phi_1,\ g_1\sin\phi_1,\ \dots,\ g_k\cos\phi_k,\ g_k\sin\phi_k,\ h_1,\ \dots,\ h_l.$$

This is also a basis of the $\mathbb{C}$-linear space $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} L = \ker_{\Omega} L$. Thus

$$W := \mathrm{Wr}(g_1\cos\phi_1, g_1\sin\phi_1, \dots, g_k\cos\phi_k, g_k\sin\phi_k, h_1, \dots, h_l) \in \mathrm{GL}_n(\Omega).$$

Note that $\mathrm{U}_{\mathrm{r}} = R \cap \mathcal{C}^{<\infty}$. It is routine to verify that $W = QD$ where Q is an $n \times n$ matrix over H and D is the $n \times n$ matrix over U_{r} displayed above. We have $\det D = 1$, hence

$$\det W = \det Q \in H \cap \Omega^{\times} = H^{\times} \subseteq \mathrm{U}_{\mathrm{r}}^{\times}$$

and thus $Q \in \mathrm{GL}_n(H)$ and $W \in \mathrm{GL}_n(\mathrm{U}_{\mathrm{r}})$. So by the remarks before [ADH, 5.5.14], W is a fundamental matrix for $y' = A_L y$ with U_{r} as the ambient differential ring.

Now take monic $L \in H[\partial]$ such that $y' = Ny$ is equivalent to $y' = A_L y$, with respect to the differential field H. Then take $P \in \mathrm{GL}_n(H)$ such that $P\,\mathrm{sol}_{\mathrm{U}_{\mathrm{r}}}(A_L) = \mathrm{sol}_{\mathrm{U}_{\mathrm{r}}}(N)$, and let W be as above. Then $PW \in \mathrm{GL}_n(\mathrm{U}_{\mathrm{r}})$ is a fundamental matrix for $y' = Ny$. With D, Q as before such that $W = QD$, and $M := PQ \in \mathrm{GL}_n(H)$, we have $MD = PW$. $\square$

Example. Let T be an $n \times n$ matrix over H, $n \geqslant 1$, and suppose T is skew-symmetric, that is, $T^{\mathrm{t}} = -T$. Then the purely imaginary matrix $S := -iT$ is hermitian, giving the Schrödinger equation $y' = Ty$ $(= -iSy)$ as in the example after Corollary 7.4.37.

If n is odd, then this equation has a solution $y \in E^n$ with $y_1^2 + \cdots + y_n^2 = 1$ for some Hardy field extension E of H; such y exhibits no oscillatory behavior and hence is a "degenerate" wave.

We don't know whether in Corollary 7.4.46 for $n \geqslant 4$ we can choose $\phi_1, \dots, \phi_k$ to be apart. In the next corollary we let $\langle\ ,\ \rangle : \Omega_{\mathrm{r}}^n \times \Omega_{\mathrm{r}}^n \to \Omega_{\mathrm{r}}$ denote the usual symmetric bilinear form on Ω_{r}^n, $n \geqslant 1$, where $\Omega_{\mathrm{r}} = \mathrm{Frac}(\mathrm{U}_{\mathrm{r}})$.

Corollary 7.4.47. *Suppose H is* d*-maximal, $y' = Ny$ is self-adjoint, and M, k, l, and D are as in Corollary 7.4.46. Then $\langle f, f\rangle \in \mathbb{R}^{>}$ for each column f of MD. Let $f_1, g_1, \dots, f_k, g_k, h_1, \dots, h_l \in \mathrm{U}_{\mathrm{r}}^n$ be the* 1*st,* 2*nd,…, nth column of MD. Then for $i = 1, \dots, k$, $j = 1, \dots, l$ we have $\langle f_i, g_i\rangle = \langle f_i, h_j\rangle = \langle g_i, h_j\rangle = 0$.*

Proof. For any columns f, g of MD we have $\langle f, g\rangle \in \mathbb{R}$, by Lemma 2.4.38. This proves the first claim. Let $i \in \{1, \dots, k\}$ and set $\phi := \phi_i$. Then $f_i = f\cos\phi - g\sin\phi$, $g_i = f\sin\phi + g\cos\phi$ where $f, g \in H^n$. Hence

$$\langle f_i, g_i\rangle \ = \ (\langle f, f\rangle - \langle g, g\rangle)\cos\phi\sin\phi + \langle f, g\rangle(\cos^2\phi - \sin^2\phi) \in \mathbb{R}.$$

Lemma 5.10.17 gives $\mathrm{e}^{2\phi i} = \theta\,\mathrm{e}(\lambda)$, $\mathrm{e}^{-2\phi i} = \theta^{-1}\,\mathrm{e}(-\lambda)$ with $\theta \in K^\times$, $\lambda \in \Lambda^{\neq}$, so the elements 1, $\mathrm{e}^{2\phi i}$, $\mathrm{e}^{-2\phi i}$ of $K[\mathrm{e}^{Hi}]$ are K-linearly independent. In view of

$$\cos\phi\sin\phi \ = \ \tfrac{1}{4i}(\mathrm{e}^{2\phi i} - \mathrm{e}^{-2\phi i}), \qquad \cos^2\phi - \sin^2\phi \ = \ \tfrac{1}{2}(\mathrm{e}^{2\phi i} + \mathrm{e}^{-2\phi i}),$$

this yields $\langle f_i, g_i\rangle = 0$. For $j \in \{1, \dots, l\}$ we have

$$\langle f_i, h_j\rangle = \langle f, h_j\rangle\cos\phi - \langle g, h_j\rangle\sin\phi \in \mathbb{R}, \quad \langle g_i, h_j\rangle = \langle f, h_j\rangle\sin\phi + \langle g, h_j\rangle\cos\phi \in \mathbb{R},$$

and we obtain likewise $\langle f_i, h_j\rangle = \langle g_i, h_j\rangle = 0$. □

Corollary 7.4.42 holds for $r = 1$ with the assumption "E is d-maximal" weakened to "E is Liouville closed". The next section has more about the case $r = 2$. The next lemma shows that Corollary 7.4.42 fails for $r = 3$ with the hypothesis "E is d-maximal" replaced by "E is perfect".

Lemma 7.4.48. *Suppose $H = \mathrm{E}(\mathbb{Q})$ and $A = (\partial - 2x)(\partial^2 + 1)$. Then with $\mathrm{U} := K[\mathrm{e}^{Hi}]$ we have $\ker_{\mathrm{U}} A = \mathbb{C}\,\mathrm{e}^{-xi} \oplus \mathbb{C}\,\mathrm{e}^{xi}$.*

The proof is similar to that of Corollary 7.4.4, using $\partial^2 + 1 = (\partial - i)(\partial + i)$ in $K[\partial]$ and the fact that $y'' + y \neq \mathrm{e}^{x^2}$ for all $y \in K$.

Remark. Let H and A be as in the previous lemma. There is an H-hardian germ y with $y \neq 0$ and $A(y) = 0$, but by the lemma, no such y is in H. Thus Theorem 1 in [161] is false.

If H is d-maximal and A has exactly one eigenvalue, then this eigenvalue is 0 by Corollary 2.5.21. This situation will be investigated in the next subsection.

Non-oscillation and disconjugacy. *In this subsection we continue to assume that $A \in H[\partial]$.* In light of Corollary 7.4.42 one may ask whether every non-oscillating $y \in \ker_{\mathcal{C}^{<\infty}} A$ is H-hardian. The answer is "no" for some A: Suppose y in H satisfies $y'' + y = x$. (If H is d-maximal, then H is linearly surjective and such y exists.) Then $y \succ 1$, and $y + \sin x \in \ker_{\mathcal{C}^{<\infty}} A$ is non-oscillating, but not H-hardian.

Here is a better question: if $y \in \ker_{\mathcal{C}^{<\infty}} A$ and $y - h$ is non-oscillating for all $h \in H$, does it follow that y is H-hardian? The next two results shows that the answer may depend on H. The first is a consequence of Corollary 5.10.39.

Lemma 7.4.49. *Suppose H is* d*-maximal. Then every $y \in \ker_{\mathcal{C}^{<\infty}} A$ such that $y-h$ is non-oscillating for all $h \in H$ lies in H.*

Lemma 7.4.50. *Let $H := \mathrm{E}(\mathbb{Q})$. Then there is a monic $A \in H[\partial]$ of order 5 and a $y \in \ker_{\mathcal{C}^{<\infty}} A$ such that $y-h$ is non-oscillating for all $h \in H$, but y is not hardian.*

Proof. Recall that each d-maximal Hardy field contains an f with $f'' + f = \mathrm{e}^{x^2}$, by Theorem 6.7.22. Take an H-hardian $z \in \mathcal{C}^{<\infty}$ with $z'' + z = \mathrm{e}^{x^2}$. Then $z - h \succ x^n$ for all n and all $h \in H$, by [35, Proposition 3.7 and Theorem 3.9]. Set

$$B_1 \ := \ \partial^3 - 2x\partial^2 + \partial - 2x \in H[\partial], \quad B_2 \ := \ \partial^2 + 4 \in H[\partial].$$

Then $B_1(z) = B_2(\sin 2x) = 0$, hence $y := z + \sin 2x \in \mathcal{C}^{<\infty}$ satisfies $A(y) = 0$ for some monic $A \in H[\partial]$ of order 5, by [ADH, 5.1.39]. For all $h \in H$ we have $y - h = (z-h) + \sin 2x \sim z - h$, so $y - h$ is non-oscillating. Towards a contradiction, assume y is hardian. Then y is H-hardian. Take an $H\langle y\rangle$-hardian $u \in \mathcal{C}^{<\infty}$ such that $u'' + u = \mathrm{e}^{x^2}$. Then $(u-z)'' + (u-z) = 0$, so $u - z = c\cos(x+d)$, with $c, d \in \mathbb{R}$. But $u - y = c\cos(x+d) - \sin 2x$ is not hardian: this is clear if $c = 0$, and otherwise follows from $B_2(u-y) = 3c\cos(x+d)$. This is the desired contradiction. □

We can also ask: if *no* $y \in \ker_{\mathcal{C}^{<\infty}} A$ oscillates, does it follow that $\ker_{\mathcal{C}^{<\infty}} A$ is contained in some Hardy field extension of H? We now extend Corollary 5.5.9 to give a positive answer:

Theorem 7.4.51. *The following are equivalent:*

(i) *no $y \in \ker_{\mathcal{C}^{<\infty}} A$ oscillates;*
(ii) *$\ker_{\mathcal{C}^{<\infty}} A \subseteq \mathrm{D}(H)$;*
(iii) *A splits over $\mathrm{D}(H)$;*
(iv) *A splits over some Hardy field extension of H.*

Proof. Corollary 7.4.42 gives (i) ⇒ (ii). For (ii) ⇒ (iii) use that A splits over $\mathrm{D}(H)$ whenever $\dim_{\mathbb{R}} \ker_{\mathrm{D}(H)} A = r$, by Corollary 7.4.42, and Corollary 2.5.5 with the remark following it. The implication (iii) ⇒ (iv) is obvious. Suppose (iv) holds; to show (i), arrange that A splits over H and H is Liouville closed. Then $\ker_{\mathcal{C}^{<\infty}} A$ is contained in H by Lemma 2.5.30, so (i) holds. □

Remark. The implication (i) ⇒ (ii) in Theorem 7.4.51 is also claimed in [162, Theorem 1]; but the proof given there is defective: in the proof of the auxiliary [162, Lemma 1] it is assumed that if $y \in \mathcal{C}^{<\infty}$ is non-oscillating and $A(y) = 0$, $y \neq 0$, then $y^\dagger$ is also non-oscillating; but $A = \partial^3 + \partial$, $y = 2 + \sin x$ contradicts this.

We say that A **does not generate oscillations** if it satisfies one of the equivalent conditions in the theorem above. Thus if $r \leqslant 1$, then A does not generate oscillations, and by Corollary 5.5.7, the operator $\partial^2 + g\partial + h$ $(g, h \in H)$ generates oscillations iff the germ $-\frac{1}{2}g' - \frac{1}{4}g^2 + h$ generates oscillations in the sense of Section 5.2. The property of A to not generate oscillations is uniformly definable in the canonical $\Lambda\Omega$-expansion $\boldsymbol{H}$ of H viewed as a structure in the language $\mathcal{L}^{\iota}_{\Lambda\Omega}$ from [ADH, Chapter 16] (see also the proof of Theorem 7.1.3); more precisely:

Corollary 7.4.52. *There is a quantifier-free $\mathcal{L}^{\iota}_{\Lambda\Omega}$-formula $\omega_r(x_1, \dots, x_r)$ such that for every Hardy field H and all $(h_1, \dots, h_r) \in H^r$:*

$$\boldsymbol{H} \models \omega_r(h_1, \dots, h_r) \quad \Longleftrightarrow \quad \begin{cases} \partial^r + h_1\partial^{r-1} + \cdots + h_r \in H[\partial] \ \textit{does} \\ \textit{not generate oscillations.} \end{cases}$$

Proof. Note that A does not generate oscillations iff A splits over some d-maximal Hardy field extension of H. Now use that the $\mathcal{L}^{\iota}_{\Lambda\Omega}$-theory of canonical $\Lambda\Omega$-expansions of d-maximal Hardy fields eliminates quantifiers, by [ADH, 16.0.1] and Theorem 6.7.22. $\square$

Example. For $\omega_2(x_1, x_2)$ we may take the $\mathcal{L}^{\iota}_{\Lambda\Omega}$-formula $\Omega(-2x_1' - x_1^2 + 4x_2)$. Let $\alpha, \beta \in \mathbb{R}$, and let $\boldsymbol{\omega}_n$ be as in Corollary 5.5.38. Then for $H = \mathbb{R}$ in that corollary,

$$\begin{aligned} \partial^2 + \alpha\partial + \beta \text{ does not generate oscillations } &\iff -\alpha^2 + 4\beta < \boldsymbol{\omega}_n \text{ for some } n \\ &\iff \alpha^2 - 4\beta \geqslant 0, \end{aligned}$$

and applying the corollary to $H = \mathbb{R}(x)$ gives

$$\begin{aligned} \left.\begin{array}{l} \partial^2 + \alpha x^{-1}\partial + \beta x^{-2} \text{ does not generate} \\ \text{oscillations} \end{array}\right\} &\iff (2\alpha - \alpha^2 + 4\beta)x^{-2} < \boldsymbol{\omega}_n \text{ for some } n \\ &\iff (1-\alpha)^2 - 4\beta \geqslant 0, \end{aligned}$$

in accordance with Corollary 7.1.5. (By the way, $y'' + \alpha x^{-1}y + \beta x^{-2}y = 0$ is Euler's differential equation of order 2, cf. [111, §22.3], [203, §20, V].)

From Corollary 2.5.20 we obtain:

Corollary 7.4.53. *Suppose H is* d*-perfect. Then:*

$$A \textit{ does not generate oscillations} \iff \operatorname{mult}_0(A) = r.$$

If H is moreover d*-maximal, then:*

$$A \textit{ does not generate oscillations} \iff A \textit{ has no eigenvalue different from } 0.$$

We say that $B \in H[\partial]^{\neq}$ does not generate oscillations if bB does not generate oscillations, for $b \in H^{\times}$ such that bB is monic. Using [ADH, 5.1.22] we obtain:

Corollary 7.4.54. *Let $B_1, B_2 \in H[\partial]^{\neq}$. Then B_1 and B_2 do not generate oscillations iff B_1B_2 does not generate oscillations.*

Note also that if E is a Hardy field extension of H, then A generates oscillations iff A generates oscillations when viewed as element of $E[\partial]$. Moreover, A generates oscillations iff its adjoint A^* does.

In the next corollary ϕ ranges over elements of $\mathrm{D}(H)^{>}$ that are active in $\mathrm{D}(H)$.

Corollary 7.4.55. *Suppose A does not generate oscillations. Then the $\mathbb{R}$-linear space $\ker_{\mathcal{C}^{<\infty}} A$ has a basis $y_1, \dots, y_r$ with all $y_j \in \mathrm{D}(H)$ and $y_1 \prec \cdots \prec y_r$, and there is a unique splitting $(a_r, \dots, a_1)$ of A over $\mathrm{D}(H)$ such that eventually we have $a_j + \phi^{\dagger} < a_{j+1}$ for $j = 1, \dots, r-1$.*

Proof. By Theorem 7.4.51, A splits over $\mathrm{D}(H)$. Now use Lemma 2.5.30 and Corollary 2.5.38 applied to the Liouville closed H-field $\mathrm{D}(H)$ in place of H. $\square$

Theorem 7.4.51 and Corollary 7.4.55 yield Corollary 10 from the introduction. The next lemma complements this Corollary 10 by taking a look at splittings over the differential ring extension $R := \mathcal{C}^{<\infty}$ of H:

Lemma 7.4.56. *Suppose $H \supseteq \mathbb{R}$ is Liouville closed, A splits over H, $a_1, \dots, a_r$ lie in R, and $A = (\partial - a_r) \cdots (\partial - a_1)$ in $R[\partial]$. Then $a_1, \dots, a_r \in H$.*

Proof. By induction on r. The case $r=0$ being trivial, suppose $r\geqslant 1$. By Lemma 2.5.19 we have $\ker_R A\subseteq H$. Take $y\in R^\times$ with $y^\dagger=a_1$. Then $A(y)=0$, hence $y\in H^\times$, so $a_1=y^\dagger\in H$. Set $B:=(\partial-a_r)\cdots(\partial-a_2)\in R[\partial]$, so $A=B(\partial-a_1)$. Now $A_{\ltimes y}\in H[\partial]$ and $A_{\ltimes y}=B_{\ltimes y}\partial$, so $B_{\ltimes y}\in H[\partial]$. Thus $B\in H[\partial]$, and B splits over H by [ADH, 5.1.22]. Hence, inductively, $a_2,\dots,a_r\in H$. ☐

Corollary 7.4.57. *Suppose A does not generate oscillations, and let $b\in H$. Then all $y\in\mathcal{C}^{<\infty}$ with $A(y)=b$ are in $\operatorname{D}(H)$.*

Proof. This follows from Corollary 7.4.55 and variation of constants [ADH, 5.5.22], using that $\operatorname{D}(H)$ is closed under integration. ☐

As promised in the remarks following Corollary 11 from the Introduction we now strengthen the Trench normal form of disconjugate linear differential operators. (See Section 5.2, just before Lemma 5.2.40, for "disconjugate" in the present context.) Below we use for $h\in\mathcal{C}$ the suggestive notation $\int^\infty h=\infty$ to indicate that for some $a\in\mathbb{R}$ and representative $h\in\mathcal{C}_a$ (and thus for every $a\in\mathbb{R}$ and every representative $h\in\mathcal{C}_a$) we have $\int_a^t h(t)dt\to+\infty$ as $t\to+\infty$.

Corollary 7.4.58. *Let $r\geqslant 1$. Then A does not generate oscillations iff A is disconjugate. Suppose A is disconjugate. Then there are $g_1,\dots,g_r\in\operatorname{D}(H)^>$ with*

$$(7.4.1)\quad A\ =\ g_1\cdots g_r(\partial g_r^{-1})\cdots(\partial g_2^{-1})(\partial g_1^{-1}),\qquad g_j\in\Gamma\big(\operatorname{D}(H)\big)\ \textit{for } j=2,\dots,r.$$

Given any such $g_1,\dots,g_r$, if $h_1,\dots,h_r\in(\mathcal{C}^{<\infty})^\times$ satisfy

$$(7.4.2)\quad A\ =\ h_1\cdots h_r(\partial h_r^{-1})\cdots(\partial h_2^{-1})(\partial h_1^{-1}),\qquad \int^\infty h_j\ =\ \infty\ \textit{for } j=2,\dots,r,$$

then $g_j\in\mathbb{R}^>\cdot h_j$ for $j=1,\dots,r$.

Proof. If A does not generate oscillations, then A is disconjugate by Lemma 5.2.40 and Theorem 7.4.51. The converse is clear. Now suppose A is disconjugate. Then Proposition 2.5.39 yields $g_1,\dots,g_r\in\operatorname{D}(H)^>$ such that (7.4.1) holds. Let $h_1,\dots,h_r\in(\mathcal{C}^{<\infty})^\times$ be such that (7.4.2) holds, and set $a_j:=(h_1\cdots h_j)^\dagger\in\mathcal{C}^{<\infty}$ $(j=1,\dots,r)$. Then $A=(\partial-a_r)\cdots(\partial-a_1)$, so $a_1,\dots,a_r\in\operatorname{D}(H)$ by Lemma 7.4.56, hence $h_1,\dots,h_r\in\operatorname{D}(H)$ as well. Thus $h_1,\dots,h_r>0$ and $h_2,\dots,h_r\in\Gamma\big(\operatorname{D}(H)\big)$. The uniqueness part of Proposition 2.5.39 gives $g_j\in\mathbb{R}^>\cdot h_j$ for $j=1,\dots,r$. ☐

This yields in particular Corollary 11 from the Introduction.

Example (Trench [200, p. 321]). Suppose $H=\mathbb{R}$, $r=3$, and $A=\partial^3-\partial$. Then A splits as $(\partial-1)\partial(\partial+1)$ over H, so A does not generate oscillations. In $\operatorname{D}(H)[\partial]$,

$$A\ =\ \mathrm{e}^x\,\partial\,\mathrm{e}^{-2x}\,\partial\,\mathrm{e}^x\,\partial\ =\ \mathrm{e}^{-x}\,\partial\,\mathrm{e}^x\,\partial\,\mathrm{e}^x\,\partial\,\mathrm{e}^{-x}\ =\ \mathrm{e}^x\,\partial\,\mathrm{e}^{-x}\,\partial\,\mathrm{e}^{-x}\,\partial\,\mathrm{e}^x,$$

where only the last factorization is as in (7.4.1).

Generating oscillations is an invariant of the type of A:

Lemma 7.4.59. *Suppose A does not generate oscillations and $B\in H[\partial]$ has the same type as A. Then B also does not generate oscillations.*

Proof. By [ADH, 5.1.19]: $r=\operatorname{order}(B)$ and we have $R\in H[\partial]$ of order $<r$ such that $1\in H[\partial]R+R[\partial]A$ and $BR\in H[\partial]A$. Now $\ker_{\mathcal{C}^{<\infty}}A=\ker_{\operatorname{D}(H)}A$, and [ADH, 5.1.20] gives an isomorphism $y\mapsto R(y)\colon \ker_{\operatorname{D}(H)}A\to\ker_{\operatorname{D}(H)}B$ of $\mathbb{R}$-linear spaces. Hence $\ker_{\mathcal{C}^{<\infty}}B=\ker_{\operatorname{D}(H)}B$, so B does not generate oscillations. ☐

Next, let M be a differential module over H. Recall from Section 2.3 the notions of M splitting, and of M splitting over a given differential field extension of H.

Lemma 7.4.60. *The following are equivalent:*

(i) *M splits over some Hardy field extension of H;*
(ii) *M splits over* $\mathrm{D}(H)$*;*
(iii) *M splits over* $\mathrm{E}(H)$.

Proof. Let E be a Hardy field extension of H such that M splits over E. To show that M splits over $\mathrm{D}(H)$, replace E by $\mathrm{D}(E)$ to arrange $E = \mathrm{D}(E)$. Next, using $E \otimes_H M \cong E \otimes_{\mathrm{D}(H)} (\mathrm{D}(H) \otimes_H M)$ and replacing H, M by $\mathrm{D}(H)$, $\mathrm{D}(H) \otimes_H M$, respectively, also arrange $H = \mathrm{D}(H)$. In particular, $H \not\subseteq \mathbb{R}$, so $M \cong H[\partial]/H[\partial]B$ for some monic $B \in H[\partial]$, by [ADH, 5.5.3]. Then B splits over E by [ADH, 5.9.2] and hence over H (Theorem 7.4.51), so M splits. This shows (i) $\Rightarrow$ (ii). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious. $\square$

We define: M **does not generate oscillations** if it satisfies one of the equivalent conditions in the previous lemma. If $M = H[\partial]/H[\partial]A$, then M does not generate oscillations iff A does not generate oscillations.

Corollary 7.4.61. *Let E be a Hardy field extension of H. Then M does not generate oscillations iff its base change $E \otimes_H M$ to E does not generate oscillations.*

Proof. Use $\mathrm{D}(E) \otimes_H M \cong \mathrm{D}(E) \otimes_E \big(E \otimes_H M\big)$ as differential modules over $\mathrm{D}(E)$. $\square$

If N is a differential submodule of M, then M does not generate oscillations iff N and M/N do not generate oscillations. Hence:

Corollary 7.4.62. *Let $A_1, \dots, A_m \in H[\partial]^{\neq}$, $m \geqslant 1$. Then $A_1, \dots, A_m$ do not generate oscillations iff* $\operatorname{lclm}(A_1, \dots, A_m) \in H[\partial]$ *does not generate oscillations.*

Let now N be an $n \times n$ matrix over H, where $n \geqslant 1$. We also say that the matrix differential equation $y' = Ny$ over H **does not generate oscillations** if the differential module over H associated to N [ADH, p. 277] does not generate oscillations. If a matrix differential equation over H generates oscillations, then so does every equivalent matrix differential equation over H. Moreover, given a Hardy field extension E of H, the matrix differential equation $y' = Ny$ over H does not generate oscillations iff $y' = Ny$ viewed as matrix differential equation over E does not generate oscillations (by Corollary 7.4.61).

Lemma 7.4.63. *Suppose $B \in H[\partial]$ is monic and N is the companion matrix of B. Then $y' = Ny$ does not generate oscillations iff B does not generate oscillations. For each Hardy field extension E of $H(\mathbb{R})$ we have an isomorphism*

$$y \mapsto (y, y', \dots, y^{(n-1)})^{\mathrm{t}} \colon \ker_E(B) \to \mathrm{sol}_E(N)$$

of $\mathbb{R}$-linear spaces.

Proof. For the first claim, use that $M_N \cong H[\partial]/H[\partial]B^*$ by [ADH, 5.5.8], and B does not generate oscillations iff B^* does not generate oscillations (remark before Corollary 7.4.55). For the second claim, see [ADH, pp. 271–272]. $\square$

Corollary 7.4.64. *Suppose H is* d*-perfect. Then $y' = Ny$ does not generate oscillations iff* $\operatorname{mult}_0(N) = n$.

Proof. Using [ADH, 5.5.9], arrange that N is the companion matrix of the monic $B \in H[\partial]$. Then $\operatorname{mult}_0(B) = \operatorname{mult}_0(N)$ by Lemma 2.4.35. Now use Corollary 7.4.53 and Lemma 7.4.63. □

Proposition 7.4.65. *The following are equivalent:*

(i) $y' = Ny$ *does not generate oscillations;*
(ii) $\mathrm{D}(H)$ *contains a fundamental matrix of solutions for* $y' = Ny$*;*
(iii) $\mathrm{E}(H)$ *contains a fundamental matrix of solutions for* $y' = Ny$*;*
(iv) *every maximal Hardy field extension of* H *contains a fundamental matrix of solutions for* $y' = Ny$*;*
(v) *some Hardy field extension of* H *contains a fundamental matrix of solutions for* $y' = Ny$*.*

Proof. Suppose $y' = Ny$ does not generate oscillations. Then $y' = Ny$ viewed as matrix differential equation over $\mathrm{D}(H)$ does not generate oscillations. Hence to show (ii) we may arrange that $H = \mathrm{D}(H)$. Then $H \not\subseteq \mathbb{R}$, so by [ADH, 5.5.9] we arrange that N is the companion matrix of the monic $B \in H[\partial]$. Then B does not generate oscillations, so H contains a fundamental matrix of solutions for $y' = Ny$, by Theorem 7.4.51 and Lemma 7.4.63. This proves (ii). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear. Suppose (v) holds. To prove (i), we first arrange that H is d-perfect and contains a fundamental matrix of solutions for $y' = Ny$, and as in the proof of (i) $\Rightarrow$ (ii) we then arrange that N is the companion matrix of some monic operator in $H[\partial]$. Then $y' = Ny$ does not generate oscillations, by Theorem 7.4.51 and Lemma 7.4.63. □

In [33, Definition 16.14], Boshernitzan defines $y' = Ny$ to be H*-regular* if it satisfies condition (iii) in the proposition above. In [33, Theorem 16.16] he then notes the following version of Corollary 7.4.57, with $\mathrm{E}(H)$ in place of $\mathrm{D}(H)$:

Corollary 7.4.66. *Suppose the matrix differential equation* $y' = Ny$ *does not generate oscillations. Let* $b \in H^n$ *be a column vector. Then each solution* y *in* $(\mathcal{C}^{<\infty})^n$ *to the differential equation* $y' = Ny + b$ *lies in* $\mathrm{D}(H)^n$*.*

Proof. By Proposition 7.4.65, $\mathrm{D}(H)$ contains a fundamental matrix of solutions for $y' = Ny$. Now use [ADH, 5.5.21] and $\mathrm{D}(H)$ being closed under integration. □

Here is an application of the material above to the parametrization of curves in euclidean n-space, where for simplicity we only treat the case $n = 3$, denoting the usual euclidean norm on $\mathbb{R}^3$ by $|\cdot|$.

Example (Frenet-Serret formulas). Let $U \subseteq \mathbb{R}$ be a nonempty open interval and $\gamma\colon U \to \mathbb{R}^3$ be a $\mathcal{C}^\infty$-curve, parametrized by arc length, that is, $|\gamma'(t)| = 1$ for all $t \in U$. Let $T := \gamma'$ and $\kappa := |T'|$ (the *curvature* of γ). Suppose $\kappa(t) \neq 0$ for each $t \in U$, set $N := T'/|T'|$ and $B := T \times N$ (vector cross product). Then for $t \in U$ the vectors $T(t), N(t), B(t) \in \mathbb{R}^3$ are orthonormal and $y = (T, N, B)\colon U \to \mathbb{R}^9$ is a solution of the matrix differential equation $y' = Fy$ in $\mathcal{C}^\infty(U)$ where

$$F = \begin{pmatrix} & \kappa I & \\ -\kappa I & & \tau I \\ & -\tau I & \end{pmatrix} \qquad (I = \text{the } 3 \times 3 \text{ identity matrix}),$$

for some $\mathcal{C}^\infty$-function $\tau\colon U \to \mathbb{R}$ (the *torsion* of γ).

Conversely, let $\mathcal{C}^\infty$-functions $\kappa, \tau\colon U \to \mathbb{R}$ such that $\kappa(t) > 0$ for all $t \in U$ be given. Then there is a $\mathcal{C}^\infty$-curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)\colon U \to \mathbb{R}^3$, parametrized by arc length, with curvature κ and torsion τ. In fact, γ is unique up to proper euclidean motions in $\mathbb{R}^3$. (See [193, Chapter 1] for these facts.) Fix such γ and assume in addition that $U = (c, +\infty)$ with $c \in \mathbb{R} \cup \{-\infty\}$ and that H is a d-maximal $\mathcal{C}^\infty$-Hardy field and contains the germs of κ, τ, also denoted by κ, τ. Then for some $\alpha \in K/K^\dagger$, the matrix differential equation $y' = Fy$ over K has spectrum $\{\alpha, -\alpha, 0\}$. (Example 2.4.37.) If $\alpha = 0$, then $y' = Fy$ does not generate oscillations, hence the germs of γ_1, γ_2, γ_3 lie in H and so γ does not exhibit oscillating behavior. If $\alpha \neq 0$, then $\alpha = \phi'\mathrm{i} + K^\dagger$ where $\phi \in H$, $\phi > \mathbb{R}$, and then by Corollaries 7.4.45 and 7.4.46, the germs of γ_1, γ_2, γ_3 lie in

$$H\cos\phi + H\sin\phi + H \subseteq \mathcal{C}^\infty.$$

For example, if $\kappa \in \mathbb{R}^>$ and $\tau \in \mathbb{R}$ are constant, then γ is the helix given by

$$t \mapsto \big(-a\cos(t/D), a\sin(t/D), bt/D\big)$$

where $D = \sqrt{a^2+b^2}$, $\kappa = a/D^2$, $\tau = b/D^2$.

7.5. Revisiting Second-Order Linear Differential Equations

In this section we analyze the oscillating solutions of second-order linear differential equations over Hardy fields in more detail. In particular, we prove Corollary 12 from the introduction. This is connected to the ω-freeness of the perfect hull of a Hardy field, which is characterized in Theorem 7.5.32. *Throughout this section H is a Hardy field and $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$.*

Parametrizing the solution space. Let $a, b \in H$. We now continue the study of the linear differential equation

$$(\widetilde{\mathrm{L}}) \qquad Y'' + aY' + bY \;=\; 0$$

over H from Section 5.3 (with slightly changed notation), and focus on the oscillating case, viewed in the light of our main theorem. (Corollaries 5.5.7 and 5.5.9 already dealt with the non-oscillating case, which didn't need our main result.) Most of the following theorem was claimed without proof by Boshernitzan [35, Theorem 5.4]:

Theorem 7.5.1. *Suppose $(\widetilde{\mathrm{L}})$ has an oscillating solution (in $\mathcal{C}^{<\infty}$). Then there are H-hardian germs $g > 0$, $\phi > \mathbb{R}$ such that for all $y \in \mathcal{C}^{<\infty}$,*

$$y \text{ is a solution of } (\widetilde{\mathrm{L}}) \iff y = cg\cos(\phi + d) \text{ for some } c, d \in \mathbb{R}.$$

Any such H-hardian germs g, ϕ are d*-algebraic over H and lie in a common Hardy field extension of H. If $\mathrm{D}(H)$ is ω-free, then these properties force $g, \phi \in \mathrm{D}(H)$, and determine g uniquely up to multiplication by a positive real number and ϕ uniquely up to addition of a real number.*

Remarks. If H is ω-free, then $\mathrm{D}(H)$ is ω-free by Theorem 1.4.1. Also, if H is not λ-free or $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^\uparrow$, then $\mathrm{D}(H)$ is ω-free, by Lemma 5.5.37. (See Section 5.5 or [ADH, 5.2] for the definition of the function σ, and recall that $\overline{\omega}(H)$ is the set of all $f \in H$ such that $f/4$ does not generate oscillations. If H is ω-free, then $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^\uparrow$, by Corollary 5.5.36.) Recall also that λ-freeness includes having asymptotic integration. In the last sentence of Theorem 7.5.1 we cannot drop the hypothesis that $\mathrm{D}(H)$ is ω-free; see Remark 7.5.34.

Let V be an $\mathbb{R}$-linear subspace of $\mathcal{C}$. A pair (g,ϕ) is said to **parametrize** V if

$$g \in \mathcal{C}^\times,\ g>0, \quad \phi \in \mathcal{C},\ \phi > \mathbb{R}, \qquad V \ = \ \big\{cg\cos(\phi+d) : c,d \in \mathbb{R}\big\};$$

equivalently, $g \in \mathcal{C}^\times$, $g>0$, $\phi \in \mathcal{C}$, $\phi > \mathbb{R}$, and $V = \mathbb{R}g\cos\phi + \mathbb{R}g\sin\phi$, by Corollary 5.5.15. If (g,ϕ) parametrizes V, then so does $(cg, \phi+d)$ for any $c \in \mathbb{R}^>$, $d \in \mathbb{R}$.

Example. The example following Corollary 5.2.24 shows that for $f \in \mathbb{R}^>$ the pair $(1, \frac{\sqrt{f}}{2}x)$ parametrizes $\ker_{\mathcal{C}^{<\infty}}(4\partial^2+f)$.

Suppose $V = \ker_{\mathcal{C}^{<\infty}}(\partial^2 + a\partial + b)$, and let $g \in \mathcal{C}^\times$, $g>0$, and $\phi \in \mathcal{C}$, $\phi > \mathbb{R}$. Then (g,ϕ) parametrizes V iff $g\,\mathrm{e}^{\phi\mathrm{i}} \in \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]}(\partial^2+a\partial+b)$.

For later use we record the next lemma where V is an $\mathbb{R}$-linear subspace of $\mathcal{C}^1$ and $V' := \{y' : y \in V\}$ (an $\mathbb{R}$-linear subspace of $\mathcal{C}$).

Lemma 7.5.2. *Suppose $H \supseteq \mathbb{R}$ is real closed and closed under integration, and $(g,\phi) \in H \times H$ parametrizes V. Set $q := \sqrt{(g')^2+(g\phi')^2}$ and $u := \arccos(g'/q)$. Then $q,u \in H$ and $(q, \phi+u) \in H \times H$ parametrizes V'.*

Proof. Note that u is as in Corollary 5.5.16 with g', $-g\phi'$ in place of g, h. Let $y \in V$, so $y = cg\cos(\phi+d)$ where $c,d \in \mathbb{R}$. Then

$$y' \ = \ cg'\cos(\phi+d) - cg\phi'\sin(\phi+d) \ = \ cq\cos(\phi+u+d).$$

Conversely, for $c,d \in \mathbb{R}$ we have $cq\cos(\phi+u+d) = y'$ for $y = cg\cos(\phi+d) \in V$. $\square$

Lemma 7.5.3. *Set $f := -2a'-a^2+4b$. Let h be an H-hardian germ such that $h>0$ and $h^\dagger = -\frac{1}{2}a$. Let $g \in \mathcal{C}^\times$, $g>0$ and $\phi \in \mathcal{C}$, $\phi > \mathbb{R}$. Then:*

(i) *(g,ϕ) parametrizes $\ker_{\mathcal{C}^{<\infty}} 4\partial^2+f$ iff (gh,ϕ) parametrizes $\ker_{\mathcal{C}^{<\infty}} \partial^2+a\partial+b$.*

Assume also that ϕ is hardian (so ϕ' is hardian with $\phi'>0$). Then:

(ii) *$(1/\sqrt{\phi'}, \phi)$ parametrizes $\ker_{\mathcal{C}^{<\infty}} 4\partial^2 + \sigma(2\phi')$.*

Proof. The arguments leading up to Corollary 5.5.7 yield (i). As to (ii), the definition of σ in [ADH, p. 262] gives

$$\sigma(2\phi') \ = \ \omega\big({-(2\phi')^\dagger} + 2\phi'\mathrm{i}\big) \ = \ \omega\big({-\phi'^\dagger} + 2\phi'\mathrm{i}\big) \ = \ \omega(2y^\dagger)$$

where $y := (1/\sqrt{\phi'})\,\mathrm{e}^{\phi\mathrm{i}}$. Hence $A(y)=0$ for $A = 4\partial^2+\sigma(2\phi')$ by the computation in [ADH, p. 258], and thus $\big(1/\sqrt{\phi'}, \phi\big)$ parametrizes $\ker_{\mathcal{C}^{<\infty}} A$. $\square$

Item (i) in Lemma 7.5.3 reduces the proof of Theorem 7.5.1 to the case $a=0$, and (ii) is about that case. Next we isolate an argument in the proof of [ADH, 14.2.18]:

Lemma 7.5.4. *Let E be a 2-newtonian H-asymptotic field with asymptotic integration, $e \in E^\times$, $f \in E$, and γ be active in E such that $e^2 = f - \sigma(\gamma)$ and $e \succ \gamma$. Then $\sigma(y) = f$ and $y \sim e$ for some $y \in E^\times$.*

Proof. Note that e is active in E since $e \succ \gamma$. By [ADH, 11.7.6] we have

$$\sigma(e) - f \ = \ \sigma(e) - \sigma(\gamma) - e^2 \ = \ \omega(-e^\dagger) - \omega(-\gamma^\dagger) - \gamma^2 \ \prec \ e^2,$$

and so $\omega(-e^\dagger) - f \sim -e^2$. Eventually $\phi \prec e$, so $(\phi/e)^\dagger \prec e$ by [ADH, 9.2.11]. Hence with R, Q as defined before [ADH, 14.2.18], eventually we have $R^\phi \prec e^2$, and thus $Q^\phi \sim e^2Y^2(Y^2-1)$. Now [ADH, 14.2.12] yields $u \in E$ with $u \sim 1$ and $Q(u) = 0$, thus $\sigma(y) = f$ for $y := eu \sim e$. $\square$

Let $A = 4\partial^2 + f \in H[\partial]$ where $f/4 \in H$ generates oscillations, and set $V := \ker_{\mathcal{C}^{<\infty}} A$. If H is ω-free, then $f \in \sigma(\Gamma(H))^{\uparrow}$, by Corollary 5.5.36. Theorem 7.5.1 now follows from Lemmas 7.5.5, 7.5.6, and 7.5.7 below, which give more information.

Lemma 7.5.5. *There is a pair of H-hardian germs parametrizing V. For any such pair (g, ϕ) we have $\sigma(2\phi') = f$ and $g^2\phi' \in \mathbb{R}^{>}$, so g, ϕ are* d*-algebraic over $\mathbb{Q}\langle f\rangle$ and lie in a common Hardy field extension of H. If $f \in \mathcal{C}^\infty$, then each pair of H-hardian germs parametrizing V is in $(\mathcal{C}^\infty)^2$; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Proof. The first statement follows from Corollaries 5.5.19, 7.2.10, and 5.10.35. Next, let (g, ϕ) be a pair of H-hardian germs parametrizing V. Set $y := g\,\mathrm{e}^{\phi\mathrm{i}} \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times$; then we have $A(y) = 0$ and hence $\omega(2y^\dagger) = f$ where $y^\dagger = g^\dagger + \phi'\mathrm{i} \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Now for $p, q \in \mathcal{C}^1$ we have $\omega(p + q\mathrm{i}) = \omega(p) + q^2 - 2(pq + q')\mathrm{i}$, so

$$\omega(p + q\mathrm{i}) \in \mathcal{C} \iff pq + q' = 0.$$

Therefore $2g^\dagger = -(2\phi')^\dagger = -(\phi')^\dagger$ and so $g^2\phi' \in \mathbb{R}^{>}$, and $\sigma(2\phi') = f$ [ADH, p. 262]. If $f \in \mathcal{C}^\infty$, then $y \in \mathcal{C}^\infty[\mathrm{i}]$, so $g^2 = |y|^2 \in \mathcal{C}^\infty$ and hence also $\phi \in \mathcal{C}^\infty$ since $g^2\phi' \in \mathbb{R}^{>}$; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. □

Lemma 7.5.6. *Suppose that $H \supseteq \mathbb{R}$ is real closed with asymptotic integration, and that $f \in \sigma(\Gamma(H))^{\uparrow}$. Then there is an active $e > 0$ in H such that $\phi' \sim e$ for all pairs (g, ϕ) of H-hardian germs parametrizing V.*

Proof. Choose a logarithmic sequence (ℓ_ρ) for H and set $\upgamma_\rho := \ell_\rho^\dagger$ [ADH, 11.5]; then $(\upgamma_\rho)$ is strictly decreasing and coinitial in $\Gamma(H)$ [ADH, p. 528]. Take ρ such that $f > \sigma(\upgamma_\rho)$. As in the proof of [ADH, 14.2.18], we increase ρ so that $f - \sigma(\upgamma_\rho) \succ \upgamma_\rho^2$, and take $e \in H^{>}$ with $e^2 = f - \sigma(\upgamma_\rho)$. Then $e \succ \upgamma_\rho$ and so $e \in \Gamma(H)^{\uparrow}$. Let (g, ϕ) be a pair of elements in a Hardy field extension E of H parametrizing V. We claim that $\phi' \sim e/2$ (so $e/2$ in place of e has the property desired in the lemma). We arrange that E is d-maximal. Then E is Liouville closed and $\phi > \mathbb{R}$, so $e, 2\phi' \in \Gamma(E)$ by [ADH, 11.8.19]. Now E is newtonian by Theorem 6.7.22, so Lemma 7.5.4 yields $u \sim 1$ in E such that $\sigma(eu) = f$. Now the map $y \mapsto \sigma(y) \colon \Gamma(E) \to E$ is strictly increasing [ADH, 11.8.29], hence $2\phi' = eu$ by Lemma 7.5.5, and thus $\phi' \sim e/2$. □

Lemma 7.5.7. *Suppose* $\mathrm{D}(H)$ *is ω-free or $f \in \sigma(\Gamma(H))^{\uparrow}$. Let H_i be a Hardy field extension of H with $(g_i, \phi_i) \in H_i \times H_i$ parametrizing V, for $i = 1, 2$. Then*

$$g_1/g_2 \in \mathbb{R}^{>}, \qquad \phi_1 - \phi_2 \in \mathbb{R}.$$

Thus $g, \phi \in \mathrm{D}(H)$ for any pair (g, ϕ) of H-hardian germs parametrizing V.

Proof. We arrange that H_1, H_2 are d-maximal and thus contain $\mathrm{D}(H)$. Replacing H by $\mathrm{D}(H)$ we further arrange that H is d-perfect and $f \in \sigma(\Gamma(H))^{\uparrow}$. Then $\phi_1' \sim \phi_2'$ by Lemma 7.5.6, and for $i = 1, 2$ we have $c_i \in \mathbb{R}^{>}$ with $\phi_i' = c_i/g_i^2$, by Lemma 7.5.5. Replacing g_i by $g_i/\sqrt{c_i}$ we arrange $c_i = 1$ $(i = 1, 2)$, so $g_1 \sim g_2$. Consider now the elements $g_1 \cos\phi_1$, $g_1 \sin\phi_1$ of V; take $a, b, c, d \in \mathbb{R}$ such that

$$g_1 \cos\phi_1 \;=\; ag_2\cos(\phi_2 + b), \qquad g_1 \sin\phi_1 \;=\; cg_2\cos(\phi_2 + d).$$

Then

$$g_1^2 \;=\; g_1^2(\cos^2\phi_1 + \sin^2\phi_1) \;=\; g_2^2\big(a^2\cos^2(\phi_2 + b) + c^2\cos^2(\phi_2 + d)\big), \tag{7.5.1}$$

and hence

$$a^2\cos^2(\phi_2 + b) + c^2\cos^2(\phi_2 + d) \;\sim\; 1.$$

Thus the 2π-periodic function

$$t \mapsto F(t) \ := \ a^2 \cos^2(t+b) + c^2 \cos^2(t+d) \ : \ \mathbb{R} \to \mathbb{R}$$

satisfies $F(t) \to 1$ as $t \to +\infty$, hence $F(t) = 1$ for all t, so $g_1 = g_2$ by (7.5.1). It follows that $\phi_1' = \phi_2'$, so $\phi_1 - \phi_2 \in \mathbb{R}$.

For the final claim, let (g, ϕ) be a pair of H-hardian germs parametrizing V. Let M be any d-maximal extension of H. Then Lemma 7.5.5 gives a pair $(g_M, \phi_M) \in M^2$ that also parametrizes V. By the above, $g/g_M \in \mathbb{R}^>$ and $\phi - \phi_M \in \mathbb{R}$, hence $g, \phi \in M$. Since M is arbitrary, this gives $g, \phi \in \mathrm{D}(H)$. $\square$

This finishes the proof of Theorem 7.5.1 (and Corollary 12 from the introduction).

Corollary 7.5.8. *Suppose that H is* d*-perfect. Then $\omega(H) = \overline{\omega}(H)$ is downward closed and $\sigma\big(\Gamma(H)\big)$ is upward closed.*

Proof. By Corollary 5.5.3, $\omega(H) = \overline{\omega}(H)$ is downward closed.

Let $f \in \sigma\big(\Gamma(H)\big)^{\uparrow}$. The last part of Lemma 7.5.7 gives $g, \phi \in H$ such that (g, ϕ) parametrizes V. Then $\sigma(2\phi') = f$ by Lemma 7.5.5. Now $2\phi' \in \Gamma(H)$ by [ADH, 11.8.19], so f lies in $\sigma\big(\Gamma(H)\big)$. Thus $\sigma\big(\Gamma(H)\big)$ is upward closed. $\square$

Recall from [ADH, 11.8] that $H \supseteq \mathbb{R}$ is said to be *Schwarz closed* if H is Liouville closed and $H = \omega\big(\Lambda(H)\big) \cup \sigma\big(\Gamma(H)\big)$.

Corollary 7.5.9. *Suppose H is* d*-perfect. Then the following are equivalent:*

(i) *H is Schwarz closed;*
(ii) *H is $\boldsymbol{\omega}$-free;*
(iii) *for all $f \in H$ the operator $4\partial^2 + f \in H[\partial]$ splits over K;*
(iv) *for all $a, b \in H$ the operator $\partial^2 + a\partial + b \in H[\partial]$ splits over K.*

Proof. The equivalence (iii) $\Leftrightarrow$ (iv) holds by Corollary 5.5.10, and the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow from [ADH, 11.8.33] and Corollary 7.5.8. $\square$

In the rest of this subsection $A = \partial^2 + a\partial + b$ $(a, b \in H)$. We set $V := \ker_{\mathcal{C}^{<\infty}} A$ and $f := -2a' - a^2 + 4b$, and we take H-hardian $h > 0$ such that $h^\dagger = -\frac{1}{2}a$. Note the relevance of Lemma 7.5.3(i) in this situation.

Corollary 7.5.10. *Suppose $f > 0, f \succ 1/x^2$. Then $f \notin \overline{\omega}(H)$, and for some H-hardian germ ϕ with $\phi' \sim \frac{1}{2}\sqrt{f}$, and $g := 1/\sqrt{\phi'}$ we have: (gh, ϕ) parametrizes V.*

Proof. By Theorem 5.6.2 we arrange that $H \supseteq \mathbb{R}$ is Liouville closed and $\boldsymbol{\omega}$-free. With notation as at the beginning of Section 5.6 we have $\boldsymbol{\omega}_\rho \sim 1/x^2$ for all ρ; hence $f/4 > \boldsymbol{\omega}_\rho$ for all ρ, so $f/4$ generates oscillations by [ADH, 11.8.21] and Corollary 5.5.36, and $f \notin \overline{\omega}(H)$, $f \in \sigma\big(\Gamma(H)\big)^{\uparrow}$. Lemma 7.5.5 gives a pair (g, ϕ) parametrizing $\ker_{\mathcal{C}^{<\infty}}(4\partial^2 + f)$ with H-hardian ϕ and $g := 1/\sqrt{\phi'}$. Now $\boldsymbol{\gamma} := 1/x$ is active in H with $\sigma(\boldsymbol{\gamma}) = 2\boldsymbol{\gamma}^2$ and so $f > \sigma(\boldsymbol{\gamma})$ and $f - \sigma(\boldsymbol{\gamma}) \sim f$. Then $\phi' \sim \frac{1}{2}\sqrt{f}$ by the proof of Lemma 7.5.6, so ϕ has the property stated in Corollary 7.5.10. $\square$

Corollary 7.5.11. *Suppose $f \notin \overline{\omega}(H)$ and let (g, ϕ) be a pair of H-hardian germs parametrizing V. Then $\phi \prec x$ iff $f \prec 1$, and the same with $\preccurlyeq$ in place of $\prec$. Also, if $f \sim c \in \mathbb{R}^>$, then $\phi \sim \frac{\sqrt{c}}{2}x$ and $(f < c \Rightarrow \phi'' > 0)$, $(f > c \Rightarrow \phi'' < 0)$.*

Proof. We arrange $H \supseteq \mathbb{R}$ is $\boldsymbol{\omega}$-free, Liouville closed, and $g, \phi \in H$. Then $y := 2\phi' \in \Gamma(H)$ by [ADH, 11.8.19]. Lemma 7.5.5 gives $\sigma(y) = f$; also $\sigma(c) = c^2$ for all $c \in \mathbb{R}^>$. As the restriction of σ to $\Gamma(H)$ is strictly increasing [ADH, 11.8.29], this yields the first part. Now suppose $f \sim c \in \mathbb{R}^{>}$, and take $\lambda \in \mathbb{R}^{>}$ with $\phi \sim \lambda x$. Then $y \sim 2\lambda$, and with $z := -y^{\dagger} \prec 1$ we have $f = \sigma(y) = \omega(z) + y^2 \sim 4\lambda^2$. Hence $\lambda = \sqrt{c}/2$. Suppose $f < c$; then $f = \sigma(y) < c = \sigma(\sqrt{c})$ yields $y < \sqrt{c}$, so $\phi' < \lambda$. With $g := \phi - \lambda x$ we have $g \prec x$, $g' \prec 1$ and $g' = \phi' - \lambda < 0$, so $g'' = \phi'' > 0$. The case $f > c$ is similar. $\square$

Combining Corollaries 7.5.5 and 7.5.11 yields:

Corollary 7.5.12. *Suppose $f \notin \overline{\omega}(H)$. Then for every $y \in V^{\neq}$ we have:*

(i) *if $f \prec 1$, then $y \not\preccurlyeq h$ (so y is unbounded if in addition $a \leqslant 0$);*
(ii) *if $f \succ 1$, then $y \prec h$; and*
(iii) *if $f \asymp 1$, then $y \preccurlyeq h$.*

Remarks. See [18, Chapter 6] for related (though generally weaker) results in a more general setting. For example, if $g \in \mathcal{C}$ is eventually increasing with $g \succ 1$ or $g \in \mathcal{C}^1$ and $g \sim 1$ with $\int |g'| \preccurlyeq 1$, then every $y \in \mathcal{C}^2$ with $y'' + gy = 0$ satisfies $y \preccurlyeq 1$; cf. §§6, 18 in loc. cit.

Corollary 7.5.13. *Suppose $f \notin \overline{\omega}(H)$, $H \supseteq \mathbb{R}$, and H does not have asymptotic integration or H is $\upomega$-free. Then the following are equivalent:*

(i) *$A(y) = 0$ for some $y \neq 0$ in a Liouville extension of K;*
(ii) *some pair $(g, \phi) \in \operatorname{Li}(H)^2$ with $g^{\dagger}$, ϕ' algebraic over H parametrizes V;*
(iii) *some pair in $\operatorname{Li}(H)^2$ parametrizes V;*
(iv) *every pair of H-hardian germs parametrizing V lies in $\operatorname{Li}(H)^2$.*

Proof. Suppose (i) holds. Then Lemma 7.4.6 gives $g, \phi \in L := \operatorname{Li}(H)$, $g \neq 0$, such that $g^{\dagger}$, ϕ' are algebraic over H and $A(g\,\mathrm{e}^{\phi \mathrm{i}}) = 0$. Replacing g by $-g$ if necessary we arrange $g > 0$. We have $\phi \succ 1$: otherwise $g\,\mathrm{e}^{\phi \mathrm{i}} \in E[\mathrm{i}]^{\times}$ for some Hardy field extension E of L, by Corollary 5.5.24, hence $\operatorname{Re}(g\,\mathrm{e}^{\phi \mathrm{i}}) \in V^{\neq}$ does not oscillate, or $\operatorname{Im}(g\,\mathrm{e}^{\phi \mathrm{i}}) \in V^{\neq}$ does not oscillate, a contradiction. Replacing ϕ by $-\phi$ if necessary we arrange $\phi > \mathbb{R}$. Then (g, ϕ) parametrizes V. This yields (ii). The implication (ii) $\Rightarrow$ (iii) is trivial. By the assumptions on H, $\operatorname{Li}(H)$, and thus $\operatorname{D}(H)$, is $\upomega$-free, so (iii) $\Rightarrow$ (iv) follows from Theorem 7.5.1. For (iv) $\Rightarrow$ (i), note that the differential fraction field of $K[\mathrm{e}^{H\mathrm{i}}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ is a Liouville extension of K. $\square$

Corollary 7.5.14. *Suppose $H \supseteq \mathbb{R}$ is Liouville closed and $f \notin \overline{\omega}(H)$. Then the following are equivalent:*

(i) *$g, \phi \in H$ for every pair (g, ϕ) of H-hardian germs parametrizing V;*
(ii) *there is a pair of germs in H parametrizing V;*
(iii) *$f \in \sigma(H^{\times})$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from Lemma 7.5.5 and the remarks preceding Lemma 7.5.3. Suppose $f \in \sigma(H^{\times})$. Since $f \notin \overline{\omega}(H)$ and $\omega(H)^{\downarrow} \subseteq \overline{\omega}(H)$, we have $f \notin \omega(H)^{\downarrow}$, so $f \in \sigma\big(\Gamma(H)\big)$ by [ADH, 11.8.31]. Also, $4\partial^2 + f$ splits over K but not over H (cf. [ADH, pp. 259, 262]) and $f/4$ generates oscillations. Hence Corollary 5.10.35 and the remark following it yield a pair of germs in H parametrizing V. Now (i) follows from Lemma 7.5.7. $\square$

The case of Theorem 7.5.1 where a, b are d-algebraic over $\mathbb{Q}$ is used later. In that case the Ψ-set of the Hardy subfield $H_0 := \mathbb{Q}\langle a, b\rangle$ of H is finite by Lemma 5.4.26, so H_0 has no asymptotic integration. Thus the relevance of the next result:

Corollary 7.5.15. *Suppose $f \notin \overline{\omega}(H)$ and H has no asymptotic integration. Then there is a pair $(g,\phi) \in \mathrm{D}(H)^2$ parametrizing V such that every pair of H-hardian germs parametrizing V equals $(cg, \phi+d)$ for some $c \in \mathbb{R}^>$ and $d \in \mathbb{R}$.*

Proof. The assumption on H gives that $\mathrm{D}(H)$ is $\upomega$-free. Now use Theorem 7.5.1. □

In the rest of this subsection $f \notin \overline{\omega}(H)$, and (g,ϕ) is a pair of H-hardian germs parametrizing V. Then $\sigma(2\phi') = f$ (cf. Lemma 7.5.5) and thus

$$P(2\phi') \;=\; 0 \quad \text{where } P(Y) \;:=\; 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in H\{Y\}.$$

Hence Theorem 5.4.25 applied to $E := H\langle\phi\rangle = H(\phi,\phi',\phi'')$ gives for grounded H elements $h_0, h_1 \in H^>$ and m, n with $h_0, h_1 \succ 1$ and $m+n \leqslant 3$, such that

$$\log_{m+1} h_0 \;\prec\; \phi \;\preccurlyeq\; \exp_n h_1.$$

In the next two lemmas we improve on these bounds:

Lemma 7.5.16. *Suppose $\ell_0 \in H^>$, $\ell_0 \succ 1$, and $\max \Psi_H = v(\upgamma_0)$ for $\upgamma_0 := \ell_0^\dagger$. Then $f - \omega(-\upgamma_0^\dagger) \succ \upgamma_0^2$ and $\phi \succ \log \ell_0$, or $f - \omega(-\upgamma_0^\dagger) \asymp \upgamma_0^2$ and $\phi \asymp \log \ell_0$.*

Proof. By Lemma 5.5.34 we have $f \notin \overline{\omega}(H) = \omega(-\upgamma_0^\dagger) + \upgamma_0^2 \mathcal{O}_H^{\downarrow}$ and hence $f = \omega(-\upgamma_0^\dagger) + \upgamma_0^2 u$ where $u \succcurlyeq 1$, $u > 0$, so $f - \sigma(\upgamma_0) = \upgamma_0^2(u-1)$. Suppose $u \succ 1$; then $f - \sigma(\upgamma_0) \sim u\upgamma_0^2 \succ \upgamma_0^2$, and the proof of Lemma 7.5.6 shows that then $\phi' \sim e/2$ where $e^2 = f - \sigma(\upgamma_0)$, so $e \succ \upgamma_0$ and thus $\phi \succ \log \ell_0$. Now suppose $u \asymp 1$, and put $\ell_1 := \log \ell_0$, $\upgamma_1 := \ell_1^\dagger$. Then by [ADH, 11.7.6],

$$f - \sigma(\upgamma_1) \;=\; \omega\big(-\upgamma_0^\dagger\big) - \omega\big(-\upgamma_1^\dagger\big) + u\upgamma_0^2 - \upgamma_1^2 \;\sim\; u\upgamma_0^2 \;\succ\; \upgamma_1^2,$$

and arguing as in the proof of Lemma 7.5.6 as before gives $\phi \asymp \log \ell_0$. □

Lemma 7.5.17. *Suppose $f \in \sigma\big(\Gamma(H)\big)^\uparrow$ or H is not $\uplambda$-free, and $u \in H^>$ is such that $u \succ 1$ and $v(u^\dagger) = \min \Psi_H$. Then $\phi \leqslant u^n$ for some $n \geqslant 1$.*

Proof. We have H-hardian $\phi \succ 1$, but this is not enough to get $\theta \in H^\times$ with $\phi \asymp \theta$. That is why we consider first the case that $H \supseteq \mathbb{R}$ is real closed with asymptotic integration, and $f \in \sigma\big(\Gamma(H)\big)^\uparrow$. Then Lemma 7.5.6 gives $e \in H^>$ such that $\phi' \sim e$, and as H has asymptotic integration we obtain $\theta \in H^\times$ with $\phi \asymp \theta$. Hence $\phi^\dagger \asymp \theta^\dagger \preccurlyeq u^\dagger$, and thus $\phi \leqslant u^n$ for some $n \geqslant 1$, by [ADH, 9.1.11].

We now reduce the general case to this special case. Take a d-maximal Hardy field extension E of H with $g, \phi \in E$. Suppose H is $\uplambda$-free. Then $f \in \sigma\big(\Gamma(H)\big)^\uparrow$. Also, $H(\mathbb{R})$ is $\uplambda$-free with the same value group as H, by Proposition 1.4.3, so $L := H(\mathbb{R})^{\mathrm{rc}} \subseteq E$ has asymptotic integration, with $v(u^\dagger) = \min \Psi_L$. Thus $\phi \leqslant u^n$ for some $n \geqslant 1$ by the special case applied to L in the role of H.

For the rest of the proof we assume H is not $\uplambda$-free. Then $H(\mathbb{R})$ is not $\uplambda$-free by Lemmas 1.4.13 and 1.4.14, and so $L := H(\mathbb{R})^{\mathrm{rc}} \subseteq E$ is not $\uplambda$-free by [ADH, 11.6.8]. Using [ADH, 10.3.2] we also have $v(u^\dagger) = \min \Psi_L$. Hence replacing H by L we arrange that $H \supseteq \mathbb{R}$ and H is real closed in what follows.

Suppose H has no asymptotic integration. As in the proof of Lemma 1.4.18 this yields an $\upomega$-free Hardy subfield $L \supseteq H$ of E such that $\Gamma_H^>$ is cofinal in $\Gamma_L^>$, so $v(u^\dagger) = \min \Psi_L$. Moreover, $f \in L \setminus \overline{\omega}(L) = \sigma\big(\Gamma(L)\big)^\uparrow$ by Corollary 5.5.36. Hence replacing H by L^{rc} we have a reduction to the special case.

Suppose H has asymptotic integration. Since H is not $\uplambda$-free, [ADH, 11.6.1] gives $s \in H$ creating a gap over H. Take $y \in E^\times$ with $y^\dagger = s$. Then vy is a gap in $H(y)$ by the remark following [ADH, 11.5.14], and thus a gap in $L :=$

$H(y)^{\mathrm{rc}}$. Moreover, $\Gamma_H^>$ is cofinal in $\Gamma_{H(y)}^>$ by [ADH, 10.4.5](i), hence cofinal in $\Gamma_L^>$, so $v(u^\dagger) = \min \Psi_L$. Thus replacing H by L yields a reduction to the "no asymptotic integration" case. $\square$

Corollary 7.5.18. *Let $a, b \in H := \mathbb{R}(x)$. Then $g, \phi \in \mathrm{D}(\mathbb{Q}) \subseteq \mathcal{C}^\omega$, and $\phi \preccurlyeq x^n$ for some $n \geqslant 1$. Moreover, $f \succ 1/x^2$ and $\log x \prec \phi$, or $f \asymp 1/x^2$ and $\log x \asymp \phi$.*

Proof. Apply Lemma 7.5.16 with $\ell_0 := x$, and Lemma 7.5.17 with $u = x$, and note that $f \prec 1/x^2$ is excluded by the standing assumption $f \notin \overline{\omega}(H)$. $\square$

Remark. Suppose $a = 0$. Then $b = f/4$, and $g^2\phi' \in \mathbb{R}^>$ by Lemma 7.5.5; hence bounds on ϕ give bounds on g. Thus by Corollary 7.5.18, if $b \in H := \mathbb{R}(x)$, then $g \succcurlyeq x^{-n}$ for some $n \geqslant 1$, and either $f \succ 1/x^2$, $g \prec \sqrt{x}$, or $f \asymp 1/x^2$, $g \asymp \sqrt{x}$.

Examples. Let $H := \mathbb{R}(x)$. Then for $a = 0$ and $b = \frac{5}{4}x^{-2}$ the standing assumption $f \notin \overline{\omega}(H)$ holds, since $f = 5x^{-2}$. The germ $y = x^{1/2}\cos\log x \in \mathcal{C}^\omega$ solves the corresponding second-order linear differential equation $4Y'' + fY = 0$. Other example: let H contain x and x^r where $r \in \mathbb{R}$, $r > -1$. Then for $a = 0$ and $b := \frac{1}{4}\big(x^{2r} - r(r+2)x^{-2}\big) \in \mathcal{C}^\omega$ the standing assumption $f \notin \overline{\omega}(H)$ holds in view of $f = 4b \sim x^{2r} \succ 1/x^2$. Here $z = x^{-r/2}\cos\left(\frac{x^{r+1}}{2(r+1)}\right) \in \mathcal{C}^\omega$ satisfies $4z'' + fz = 0$.

We now set $B := \partial^3 + f\partial + (f'/2) \in H[\partial]$, and observe:

Lemma 7.5.19. $B(1/\phi') = 0$.

Proof. We arrange that $H \supseteq \mathbb{R}$ contains ϕ and is Liouville closed, and identify the universal exponential extension $\mathrm{U} = \mathrm{U}_K$ of $K = H[\mathrm{i}]$ with a differential subring of $\mathcal{C}^{<\infty}[\mathrm{i}]$ as explained at the beginning of Section 5.10. Then

$$(\phi')^{-1/2}\,\mathrm{e}^{\phi \mathrm{i}}, (\phi')^{-1/2}\,\mathrm{e}^{-\phi \mathrm{i}} \in \ker_{\mathrm{U}} 4\partial^2 + f.$$

Thus $B(1/\phi') = 0$ by Lemma 2.4.23 applied to $\mathrm{Frac}(\mathrm{U})$ in the role of K. $\square$

For the canonical $\Lambda\Omega$-expansion of a Hardy field, see Section 7.1.

Lemma 7.5.20. *Let $\boldsymbol{E}$ be a pre-$\Lambda\Omega$-field extension of the canonical $\Lambda\Omega$-expansion of $H\langle\phi'\rangle$. Then $\ker_E B = C_E(1/\phi')$.*

Proof. Using [ADH, 16.3.20, remark after 4.1.13] we arrange $\boldsymbol{E}$ to be Schwarz closed. Then $f \notin \overline{\omega}(H) = \omega(E) \cap H$, hence $f \in \sigma(E^\times)$, so $\dim_{C_E} \ker_E B = 1$ by Lemma 2.5.25. $\square$

We can now complement Corollary 7.5.13:

Corollary 7.5.21. *Suppose ϕ' is algebraic over H. Then $(\phi')^2 \in H$ and $g^\dagger \in H$.*

Proof. Let $E := H^{\mathrm{rc}} \subseteq \mathcal{C}^{<\infty}$. Then by Corollary 7.1.2, the canonical $\Lambda\Omega$-expansion of E extends that of $H\langle\phi'\rangle$. Set $L := E[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$, so L is an algebraic closure of the differential field H. Put $u := 2\phi' \in E$, and let $\tau \in \mathrm{Aut}(L|H)$. Then $B(\tau(1/u)) = 0$ by Lemma 7.5.19. So $\mathrm{Re}\,\tau(1/u)$ and $\mathrm{Im}\,\tau(1/u)$ in E are also zeros of B, hence Lemma 7.5.20 yields $c \in \mathbb{C}^\times$ with $\tau(1/u) = c/u$ and thus $\tau(u) = c^{-1}u$. Now with

$$P(Y) \ := \ 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in H\{Y\}$$

we have $P(u) = 0$, so $P(\tau(u)) = 0$, hence

$$0 \ = \ P(u) - c^2P(\tau(u)) \ = \ P(u) - c^2P(c^{-1}u) \ = \ (1 - c^{-2})u^4$$

and thus $c \in \{-1, 1\}$, so $\tau(u^2) = u^2$. This proves the first statement. The second statement follows from the first and $g^2\phi' \in \mathbb{R}^>$ by Lemma 7.5.5. $\square$

Distribution of zeros. Let $a, b \in H$ and consider again the differential equation

$$(\widetilde{\mathrm{L}}) \qquad Y'' + aY' + bY \;=\; 0.$$

Below we use Theorem 7.5.1 to show that for any oscillating solution $y \in \mathcal{C}^{<\infty}$ of $(\widetilde{\mathrm{L}})$ the sequence of successive zeros of y grows very regularly, with growth comparable to that of the sequence of successive relative maxima of y, and also to that of a function whose germ is hardian. (For the equation $Y'' + fY = 0$, where $f \in \mathrm{E}(\mathbb{Q})$ generates oscillations, this was suggested after [33, §20, Conjecture 4].)

To make this precise we first define a preordering $\leqslant$ on the set $\mathbb{R}^{\mathbb{N}}$ of sequences of real numbers by

$$(s_n) \leqslant (t_n) \quad :\Longleftrightarrow \quad s_n \leqslant t_n \text{ eventually} \quad :\Longleftrightarrow \quad \exists m\, \forall n \geqslant m\ s_n \leqslant t_n.$$

(A preordering on a set is a reflexive and transitive binary relation on that set.) We say that $(s_n), (t_n) \in \mathbb{R}^{\mathbb{N}}$ are *comparable* if $(s_n) \leqslant (t_n)$ or $(t_n) \leqslant (s_n)$. The induced equivalence relation $\sim_{\text{tail}}$ on $\mathbb{R}^{\mathbb{N}}$ is that of having the same tail:

$$(s_n) \sim_{\text{tail}} (t_n) \quad :\Longleftrightarrow \quad (s_n) \leqslant (t_n) \text{ and } (t_n) \leqslant (s_n) \iff s_n = t_n \text{ eventually.}$$

To any germ $f \in \mathcal{C}$ we take a representative in $\mathcal{C}_0$, denoted here also by f for convenience, and associate to this germ the tail of the sequence $\big(f(n)\big)$, noting that this tail is independent of the choice of representative.

For example, if the germs of $f, g \in \mathcal{C}_0$ are contained in a common Hardy field, then the sequences $\big(f(n)\big)$, $\big(g(n)\big)$ are comparable. Given an infinite set $S \subseteq \mathbb{R}$ with a lower bound in $\mathbb{R}$ and without a limit point, the **enumeration** of S is the strictly increasing sequence (s_n) with $S = \{s_0, s_1, \dots\}$ (so $s_n \to +\infty$ as $n \to +\infty$).

We take representatives of a, b in $\mathcal{C}^1_e$ with $e \in \mathbb{R}$, denoting these by a and b as well, and set $f := -2a' - a^2 + 4b \in \mathcal{C}_e$. Let $y \in \mathcal{C}^2_e$ be oscillating with

$$y'' + ay' + by \;=\; 0, \qquad \text{(so the germ of } f \text{ does not lie in } \overline{\omega}(H)),$$

and let (s_n) be the enumeration of $y^{-1}(0)$. (See Lemma 5.2.10.) Theorem 7.5.1 yields $e_0 \geqslant e$, $g \in \mathcal{C}^{\times}_{e_0}$, and strictly increasing $\phi \in \mathcal{C}_{e_0}$ such that $y|_{e_0} = g\cos\phi$, and g, ϕ lie in a common Hardy field extension of H with (g, ϕ) parametrizing $\ker_{\mathcal{C}^{<\infty}}(\partial^2 + a\partial + b)$ (where g, ϕ also denote their own germs).

Lemma 7.5.22. *There is a strictly increasing $\zeta \in \mathcal{C}_{n_0}$ ($n_0 \in \mathbb{N}$) such that $s_n = \zeta(n)$ for all $n \geqslant n_0$ and the germ of ζ is hardian with H-hardian compositional inverse.*

Proof. Take $n_0 \in \mathbb{N}$ such that $s_n \geqslant e_0$ for all $n \geqslant n_0$, and then $k_0 \in \frac{1}{2} + \mathbb{Z}$ such that $\phi(s_n) = (k_0 + n)\pi$ for all $n \geqslant n_0$. Thus $n_0 = \big(\phi(s_{n_0})/\pi\big) - k_0$. Let $\zeta \in \mathcal{C}_{n_0}$ be the compositional inverse of $(\phi/\pi) - k_0$ on $[s_{n_0}, +\infty)$. Then ζ has the desired properties: the germ of ζ is hardian by Lemma 5.3.5. $\square$

If $a, b \in \mathcal{C}^{\infty}$, then we can choose ζ in Lemma 7.5.22 such that its germ is in $\mathcal{C}^{\infty}$; likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$. We do not know whether we can always choose ζ in Lemma 7.5.22 to have H-hardian germ. For ϕ not growing too slowly we can describe the asymptotic behavior of ζ in terms of ϕ:

Corollary 7.5.23. *If $\phi \succcurlyeq x^{1/n}$ for some $n \geqslant 1$, then in Lemma 7.5.22 one can choose $\zeta \sim \phi^{\text{inv}} \circ \pi x$.*

Proof. Let n_0, k_0, ζ be as in the proof of Lemma 7.5.22. Then

$$\zeta^{\text{inv}} \ \sim \ (\phi/\pi) - k_0 \ \sim \ \phi/\pi.$$

Now assume $\phi \succcurlyeq x^{1/n}$, $n \geqslant 1$. Then $\zeta^{\text{inv}} \succcurlyeq x^{1/n}$, so $\zeta \preccurlyeq x^n$, and thus the condition stated just before Lemma 5.1.9 is satisfied for $h := \zeta$. We can therefore use Corollary 5.1.11 with ϕ/π, ζ in the role of g, h to give $\phi^{\text{inv}} \circ \pi x \sim \zeta$. $\square$

Combining Corollaries 7.5.11, 7.5.23, and 5.1.11 we obtain:

Corollary 7.5.24. *If $f \sim c$ ($c \in \mathbb{R}^>$), then $s_n \sim \frac{2}{\sqrt{c}}\pi n$ as $n \to \infty$.*

Combining Corollary 7.5.18 with the proof of Lemma 7.5.22 yields crude bounds on the growth of (s_n) when $H = \mathbb{R}(x)$:

Corollary 7.5.25. *Suppose $a, b \in \mathbb{R}(x)$. If $f \asymp 1/x^2$, then for some $r \in \mathbb{R}^>$ we have $\mathrm{e}^{n/r} \leqslant s_n \leqslant \mathrm{e}^{rn}$ eventually. If $f \succ 1/x^2$, then for some $m \geqslant 1$ and every $\varepsilon \in \mathbb{R}^>$ we have $n^{1/m} \leqslant s_n \leqslant \mathrm{e}^{\varepsilon n}$ eventually.*

The next lemma is a version of the Sturm Convexity Theorem [21, p. 318] concerning the differences between consecutive zeros of y:

Lemma 7.5.26. *If $f \prec 1$, then the sequence $(s_{n+1} - s_n)$ is eventually strictly increasing with $s_{n+1} - s_n \to +\infty$ as $n \to \infty$. If $f \succ 1$, then $(s_{n+1} - s_n)$ is eventually strictly decreasing with $s_{n+1} - s_n \to 0$ as $n \to \infty$. Now suppose $f \sim c$ ($c \in \mathbb{R}^>$). Then $s_{n+1} - s_n \to 2\pi/\sqrt{c}$ as $n \to \infty$, and if $f < c$, then $(s_{n+1} - s_n)$ is eventually strictly decreasing, if $f = c$, then $(s_{n+1} - s_n)$ is eventually constant, and if $f > c$, then $(s_{n+1} - s_n)$ is eventually strictly increasing.*

Proof. We arrange $\phi \in \mathcal{C}^2_{e_0}$ such that $\phi'(t) > 0$ for all $t \geqslant e_0$. Take ζ as in the proof of Lemma 7.5.22. Then $\zeta \in \mathcal{C}^2_{n_0}$ with

$$\zeta' \ = \ \pi \frac{1}{\phi' \circ \zeta}, \qquad \zeta'' \ = \ -\pi^2 \frac{\phi'' \circ \zeta}{(\phi' \circ \zeta)^3}.$$

The Mean Value Theorem gives for every $n \geqslant n_0$ a $t_n \in (n, n+1)$ such that

$$s_{n+1} - s_n \ = \ \zeta(n+1) - \zeta(n) \ = \ \zeta'(t_n).$$

If $f \prec 1$, then Corollary 7.5.11 gives $\phi \prec x$, so $\zeta \succ x$, hence $\zeta' \succ 1$; this proves the first claim of the lemma. The other claims follow likewise using Corollary 7.5.11 and the above remarks on ζ' and ζ''. $\square$

Corollary 7.5.27. *Let $h \in \mathcal{C}_0$ and suppose the germ of h is in $\mathrm{E}(\mathbb{Q})$. Then the sequences (s_n) and $\big(h(n)\big)$ are comparable.*

Proof. Let ζ be as in Lemma 7.5.22, and note that the germs of ζ and h lie in a common Hardy field. $\square$

Let also $\underline{a}, \underline{b} \in H$, and take representatives of $\underline{a}$, $\underline{b}$ in $\mathcal{C}^1_{\underline{e}}$ ($\underline{e} \in \mathbb{R}$), denoting these by $\underline{a}$ and $\underline{b}$ as well. Let $\underline{y} \in \mathcal{C}^2_{\underline{e}}$ be an oscillating solution of the differential equation

$$Y'' + \underline{a}Y' + \underline{b}Y \ = \ 0,$$

and let $(\underline{s}_n)$ be the enumeration of $\underline{y}^{-1}(0)$.

Lemma 7.5.28. *The sequences (s_n) and $(\underline{s}_n)$ are comparable.*

Proof. We arrange that H is maximal and take ζ as in Lemma 7.5.22. This lemma also provides a strictly increasing $\underline{\zeta} \in \mathcal{C}_{\underline{n}_0}$ ($\underline{n}_0 \in \mathbb{N}$) such that $\underline{s}_n = \underline{\zeta}(n)$ for all $n \geqslant \underline{n}_0$ and the germ of $\underline{\zeta}$ is hardian with H-hardian compositional inverse. With ζ and $\underline{\zeta}$ denoting also their germs this gives $\zeta^{\mathrm{inv}} \leqslant \underline{\zeta}^{\mathrm{inv}}$ or $\zeta^{\mathrm{inv}} \geqslant \underline{\zeta}^{\mathrm{inv}}$, hence $\zeta \geqslant \underline{\zeta}$ or $\zeta \leqslant \underline{\zeta}$. Thus (s_n) and $(\underline{s}_n)$ are comparable. $\square$

Now y' also oscillates, so by Corollary 5.5.17 there is for all sufficiently large n exactly one $t \in (s_n, s_{n+1})$ with $y'(t) = 0$. Also $b \neq 0$ in H, since $b = 0$ would mean that $z := y'$ satisfies $z' + az = 0$, so z would be H-hardian. This leads to the following: Let $m \geqslant 1$ and suppose $y \in \mathcal{C}_e^{m+2}$ (and $y'' + ay' + by = 0$ with oscillating y as before). Then the zero sets of $y, y', \dots, y^{(m)}$ are eventually parametrized by hardian germs as follows:

Lemma 7.5.29. *For $i = 0, \dots, m$ we have an $n_i \in \mathbb{N}$ and a strictly increasing function $\zeta_i \in \mathcal{C}_{n_i}$, such that:*

(i) *$\zeta_i(n_i) \geqslant e$ and $\zeta_i(t) \to +\infty$ as $t \to +\infty$;*
(ii) *the germ of ζ_i is hardian with H-hardian compositional inverse;*
(iii) *$\{\zeta_i(n) : n \geqslant n_i\} = \{t \geqslant \zeta_i(n_i) : y^{(i)}(t) = 0\}$;*
(iv) *$\zeta_i^{\mathrm{inv}} - \zeta_0^{\mathrm{inv}} \preccurlyeq 1$;*
(v) *if $i < m$, then $\zeta_i(n) < \zeta_{i+1}(n) < \zeta_i(n+1)$ for all $n \geqslant n_{i+1}$.*

Proof. We arrange that H is maximal. For simplicity we only do the case $m = 1$; the general case just involves more notation. For ζ_0 we take a function ζ as constructed in the proof of Lemma 7.5.22, and also take n_0 as in that proof, so clauses (i), (ii), (iii) are satisfied for $i = 0$. Set $A := \partial^2 + a\partial + b \in H[\partial]$. As $b \neq 0$, we have the monic operator $A^\partial \in H[\partial]$ of order 2 as defined before Lemma 2.5.13, with $A^\partial(y') = 0$. Take a pair (g_1, ϕ_1) of elements of H parametrizing $\ker_{\mathcal{C}^{<\infty}} A^\partial$. Then with A^∂, g_1, ϕ_1 instead of A, g, ϕ, and taking suitable representatives of the relevant germs, the proof of Lemma 7.5.22 provides likewise an $n_1 \in \mathbb{N}$, a $k_1 \in \frac{1}{2} + \mathbb{Z}$, and a strictly increasing function $\zeta_1 \in \mathcal{C}_{n_1}$ satisfying clauses (i), (ii), (iii) for $i = 1$ and with compositional inverse given by $(\phi_1/\pi) - k_1$.

Recall that $\mathrm{U} = \mathrm{U}_K \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ is a differential integral domain extending K, and that $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} B = \ker_{\mathrm{U}} B$ for all $B \in K[\partial]^{\neq}$, by Theorem 7.4.1. Therefore $\ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A^\partial = \{y' : y \in \ker_{\mathcal{C}^{<\infty}[\mathrm{i}]} A\}$ by Lemma 2.5.13, so in view of $A \in H[\partial]$,

$$\ker_{\mathcal{C}^{<\infty}} A^\partial \ = \ \{y' : y \in \ker_{\mathcal{C}^{<\infty}} A\}.$$

Now (iv) for $i = 1$ follows from Lemmas 7.5.2 and 7.5.7.

As to (v), the remark preceding the lemma gives $\ell \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that for all $n \geqslant n_1 + \ell$ we have: $n + p \geqslant n_0$ and $\zeta_1(n)$ is the unique zero of y' in the interval $\big(\zeta(n+p), \zeta(n+p+1)\big)$. Set $n_1^* := n_1 + \ell + |p|$, and modify ζ_1 to $\zeta_1^* \colon [n_1^*, +\infty) \to \mathbb{R}$ by setting $\zeta_1^*(t) = \zeta_1(t - p)$. Then $\zeta(n) < \zeta_1^*(n) < \zeta(n+1)$ for all $n \geqslant n_1^*$. The compositional inverse of ζ_1^* is given by $(\phi_1/\pi) - (k_1 - p)$. Thus replacing ζ_1, n_1, k_1 by ζ_1^*, n_1^*, $k_1 - p$, all clauses are satisfied. $\square$

Define $N \colon \mathbb{R}^{\geqslant e} \to \mathbb{N}$ by

$$N(t) \ := \ \big|[e, t] \cap y^{-1}(0)\big| \ = \ \min\{n : s_n > t\},$$

so for $n \geqslant 1$: $N(t) = n \Leftrightarrow s_{n-1} \leqslant t < s_n$. Thus $N(t) \to +\infty$ as $t \to +\infty$; in fact:

Lemma 7.5.30. *$N \sim \phi/\pi$.*

Proof. Take n_0, k as in the proof of Lemma 7.5.22, so $\phi(s_n) = (k+n)\pi$ for $n \geqslant n_0$. Let $t \geqslant e$ be such that $N(t) \geqslant n_0 + 1$; then $s_{N(t)-1} \leqslant t < s_{N(t)}$ and thus

$$N(t) + k - 1 = \phi(s_{N(t)-1})/\pi \leqslant \phi(t)/\pi < \phi(s_{N(t)})/\pi = N(t) + k.$$

This yields $N \sim \phi/\pi$. $\square$

The quantity $N(t)$ has been studied extensively in connection with second order linear differential equations; see [91, Chapter IX, §5, and the literature quoted on p. 401]. For example, the lemma below is a consequence of a result due to Wiman [210] that holds under more general assumptions (see [91, Chapter IX, Corollary 5.3]), but also follows easily using our Hardy field calculus. Here we assume $a = 0$, so $f = 4b \in \mathcal{C}_e$.

Lemma 7.5.31. *Suppose $f(t) > 0$ for all $t \geqslant e$, and $\big(1/\sqrt{f}\big)' \prec 1$. Then*

$$N(t) \ \sim\ \frac{1}{2\pi}\int_e^t \sqrt{f(s)}\, ds \quad \textit{as } t \to +\infty.$$

Proof. From $\big(1/\sqrt{f}\big)' \prec 1$ we get $f^\dagger \prec \sqrt{f}$. Now f is hardian, so $f \preccurlyeq 1/x^2$ would give $f^\dagger \succcurlyeq 1/x$, which together with $f \preccurlyeq 1/x^2$ contradicts $f^\dagger \prec \sqrt{f}$. Thus $f \succ 1/x^2$. For the rest of the argument we arrange H is maximal with $(g,\phi) \in H^2$. Corollary 7.5.10 yields a pair $(g_1, \phi_1) \in H^2$ parametrizing $\ker_{\mathcal{C}^{<\infty}}(\partial^2 + b)$ such that $\phi_1' \sim (1/2)\sqrt{f}$. Then $\phi - \phi_1 \in \mathbb{R}$ by Lemma 7.5.7, so $\phi' \sim (1/2)\sqrt{f}$. Let $\phi_2 \in \mathcal{C}_e^1$ be given by $\phi_2(t) = (1/2)\int_e^t \sqrt{f(s)}\, ds$. Then $\phi_2' = (1/2)\sqrt{f}$, so (the germ of) ϕ_2 lies in H and $\sqrt{f} \succ 1/x$, so $\phi_2 > \mathbb{R}$. Hence by [ADH, 9.1.4(ii)] we have $\phi \sim \phi_2$. Now apply Lemma 7.5.30. $\square$

In view of Lemma 5.2.10 one may ask to what extent the results in this subsection generalize to higher-order linear differential equations over Hardy fields.

When is the perfect hull ω-free? Here we use the lemmas that made up the proof of Theorem 7.5.1 to characterize ω-freeness of the (d-) perfect hull of H:

Theorem 7.5.32. *The following are equivalent:*

(i) *H is not λ-free or $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$;*
(ii) *$\mathrm{D}(H)$ is ω-free;*
(iii) *$\mathrm{E}(H)$ is ω-free.*

In connection with this theorem recall that by Corollary 7.5.9, a d-perfect Hardy field is Schwarz closed iff it is ω-free, so in (ii), (iii) we could have also written "Schwarz closed" instead of "ω-free". The implication (i) $\Rightarrow$ (ii) was shown already in Lemma 5.5.37. To show the contrapositive of (iii) $\Rightarrow$ (i) suppose H is λ-free and $\overline{\omega}(H) \neq H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$. Since $\overline{\omega}(H) \subseteq H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$ this yields $\omega \in H$ with $\overline{\omega}(H) < \omega < \sigma\big(\Gamma(H)\big)$, and so by Lemma 7.5.33 below, $\mathrm{E}(H)$ is not ω-free. The proof of this lemma relies on Corollary 7.5.8, but additionally draws on some results from Sections 5.3 and 5.6.

Lemma 7.5.33. *Suppose H is λ-free, and $\omega \in H$, $\overline{\omega}(H) < \omega < \sigma\big(\Gamma(H)\big)$. Then*

$$\omega(E) < \omega < \sigma\big(\Gamma(E)\big) \qquad \textit{for } E := \mathrm{E}(H),$$

hence E is not ω-free.

Proof. We may replace H by any λ-free Hardy subfield L of E containing H such that $\Gamma^{<}$ is cofinal in $\Gamma_L^{<}$, by [ADH, 11.8.14, 11.8.29]. Using this observation and Proposition 1.4.3 we replace H by $H(\mathbb{R})$ to arrange $H \supseteq \mathbb{R}$. Next we replace H by $\operatorname{Li}(H) \subseteq E$ to arrange that H is Liouville closed, using Proposition 1.4.15. Now $\omega(E) = \overline{\omega}(E)$ is downward closed and $\overline{\omega}(E) \cap H = \overline{\omega}(H)$, so $\omega(E) < \omega$. Towards a contradiction, assume $\omega \in \sigma\big(\Gamma(E)\big)^{\uparrow}$. Take $\gamma \in \Gamma(E)$ with $\sigma(\gamma) = \omega$. Corollaries 5.6.3 and 5.6.5 also yield a germ $\widetilde{\gamma} \in (\mathcal{C}^{<\infty})^{\times} \setminus \{\gamma\}$ with $\widetilde{\gamma} > 0$ and $\sigma(\widetilde{\gamma}) = \omega$, and a maximal Hardy field extension M of H containing $\widetilde{\gamma}$. Since M is ω-free (by Theorem 5.6.2) and $\omega \notin \overline{\omega}(M)$, we have $\omega \in \sigma\big(\Gamma(M)\big)^{\uparrow}$ by Corollary 5.5.36 and so $\widetilde{\gamma} \in \Gamma(M)$ by [ADH, 11.8.31]. Since $E \subseteq M$ we have $\Gamma(E) \subseteq \Gamma(M)$. Then from $\sigma(\gamma) = \omega = \sigma(\widetilde{\gamma})$ we obtain $\gamma = \widetilde{\gamma}$ by [ADH, 11.8.29], a contradiction. □

Remark 7.5.34. Suppose $H \supseteq \mathbb{R}$ is Liouville closed and $\omega \in H$ satisfies

$$\overline{\omega}(H) \ < \ \omega \ < \ \sigma\big(\Gamma(H)\big).$$

Then the uniqueness in Theorem 7.5.1 fails, by Corollaries 5.6.3 and 5.6.5: any $\gamma > 0$ in any d-maximal Hardy field extension M of H with $\sigma(\gamma) = \omega$ yields a pair (g, ϕ) parametrizing $V := \ker_{\mathcal{C}^{<\infty}}(4\partial^2 + \omega)$ where $g := 1/\sqrt{\gamma}$ and $\phi \in M$, $\phi' = \frac{1}{2}\gamma$.

To finish the proof of Theorem 7.5.32 it remains to show the implication (ii) ⇒ (iii), which we do in Lemma 7.5.39 below. (This implication holds trivially if H is bounded, by Theorem 5.4.20.) We precede this lemma with some observations.

If ϕ is active in H, then the pre-H-field H^{ϕ} has small derivation $\delta = \phi^{-1}\partial$; so if $h \in H$, $h \prec 1$, then $\delta^n(h) \prec 1$ for all n. The next lemma yields a variant of this when h is multiplied by a germ in $\mathcal{C}^{<\infty}$ with sufficiently small derivatives:

Lemma 7.5.35. *Let $y = hz$, $h \in H$, $h \prec 1$ and $z \in \mathcal{C}^{<\infty}$, $z \preccurlyeq 1$ and $z^{(j)} \preccurlyeq \mathrm{e}^{-x}$ for $j = 1, \dots, n$. Let also ϕ be active in H with $0 < \phi \preccurlyeq 1/x$, and $\delta = \phi^{-1}\partial$. Then $\delta^j(y) \prec 1$ for $j = 0, \dots, n$.*

Proof. Let j, k with $k \leqslant j$ range over $\{1, \dots, n\}$. By the Product Rule for the derivation δ and the remark preceding the lemma it is enough to show that $\delta^j(z) \preccurlyeq 1$. Let $R_k^j \in \mathbb{Q}\{X\}$ be as in Lemma 5.3.4. Now $\lambda := -\phi^{\dagger} \asymp 1/x$, hence $R_k^j(\lambda) \preccurlyeq 1$, and $(\phi^{-j})^{\dagger} = -j\phi^{\dagger} \asymp \lambda \prec 1 = (\mathrm{e}^x)^{\dagger}$, hence also $\phi^{-j} \prec \mathrm{e}^x$. This yields

$$\delta^j(z) \ = \ \phi^{-j}\big(R_j^j(\lambda)z^{(j)} + \cdots + R_1^j(\lambda)z'\big) \preccurlyeq \phi^{-j}\,\mathrm{e}^{-x} \prec 1,$$

which is more than enough. □

We have an ample supply of oscillating germs z as in Lemma 7.5.35:

Lemma 7.5.36. *Let $z := \mathrm{e}^{-x}\sin x \in \mathcal{C}^{\omega}$; then $|z^{(n)}| \leqslant 2^n\,\mathrm{e}^{-x}$ for all n.*

In the next lemma and its corollary our Hardy field H contains $\mathbb{R}$ and is real closed, and $\widehat{H}$ is an immediate Hardy field extension of H. We now have the following perturbation result:

Lemma 7.5.37. *Suppose H is ungrounded, $\Psi_H^{>0} := \Psi_H \cap \Gamma_H^{>} \neq \emptyset$. Let $\widehat{f} \in \widehat{H} \setminus H$ and $Z(H, \widehat{f}) = \emptyset$. Let $g \in H$, $vg > v(\widehat{f} - H)$ and $z \in \mathcal{C}^{<\infty}$, $z \preccurlyeq 1$, $z^{(n)} \preccurlyeq \mathrm{e}^{-x}$ for all $n \geqslant 1$. Then $f := \widehat{f} + gz \in \mathcal{C}^{<\infty}$ is hardian over H, and we have an isomorphism $H\langle f\rangle \to H\langle\widehat{f}\rangle$ of H-fields over H sending f to $\widehat{f}$.*

Proof. The hypothesis $\Psi_H^{>0} \neq \emptyset$ and [ADH, 9.2.15] yields active ϕ in H with $\phi^\dagger \asymp \phi$. But also $t^\dagger \asymp t$ for $t := x^{-1}$ in $H(x)$, so $\phi \asymp t$ by the uniqueness in [ADH, 9.2.15]. Below ϕ ranges over the active elements of H such that $0 < \phi \preccurlyeq t$, and $\delta := \phi^{-1}\partial$. Let $h \in H$, $\mathfrak{m} \in H^\times$ be such that $\widehat{f} - h \preccurlyeq \mathfrak{m}$; by Corollary 6.7.12 it is enough to show that then $\delta^n\big(\frac{f-h}{\mathfrak{m}}\big) \preccurlyeq 1$ for all n. Now $u := \frac{\widehat{f}-h}{\mathfrak{m}} \in \widehat{H}$, $u \preccurlyeq 1$, and the valued differential field $\widehat{H}^\phi$ has small derivation, so $\delta^n(u) \preccurlyeq 1$ for all n. Moreover, $g/\mathfrak{m} \in H$, $g/\mathfrak{m} \prec 1$, so $\delta^n\big(\frac{g}{\mathfrak{m}} z\big) \prec 1$ for all n, by Lemma 7.5.35. Thus $\delta^n\big(\frac{f-h}{\mathfrak{m}}\big) = \delta^n\big(\frac{\widehat{f}-h}{\mathfrak{m}}\big) + \delta^n\big(\frac{g}{\mathfrak{m}} z\big) \preccurlyeq 1$, for all n. $\square$

Using Lemmas 7.5.36 and 7.5.37, and results of [ADH, 11.4] we now obtain:

Corollary 7.5.38. *Suppose H is ungrounded with $\Psi_H^{>0} \neq \emptyset$. Let (f_ρ) be a divergent pc-sequence in H of* d*-transcendental type over H and with a pseudolimit in* $\mathrm{E}(H)$. *Then (f_ρ) is a c-sequence.*

Proof. Let $f_\rho \rightsquigarrow \widehat{f} \in \mathrm{E}(H)$. Then by [ADH, 11.4.7, 11.4.13] the Hardy field $H\langle \widehat{f} \rangle$ is an immediate extension of H, and $Z(H, \widehat{f}) = \emptyset$. Suppose (f_ρ) is not a c-sequence. Then we can take $g \in H^\times$ with $vg > v(H - \widehat{f})$. By Lemmas 7.5.36 and 7.5.37, the germ $f := \widehat{f} + g\,\mathrm{e}^{-x}\sin x$ generates a Hardy field $H\langle f \rangle$ over H; however, no maximal Hardy field extension of H contains both $\widehat{f}$ and f, contradicting $\widehat{f} \in \mathrm{E}(H)$. $\square$

We can now supply the proof of the still missing implication (ii) $\Rightarrow$ (iii) in Theorem 7.5.32:

Lemma 7.5.39. *Suppose H is ω-free. Then* $\mathrm{E}(H)$ *is also ω-free.*

Proof. Since $E := \mathrm{E}(H)$ is Liouville closed and contains $\mathbb{R}$ we may replace H by the Hardy subfield $\mathrm{Li}\big(H(\mathbb{R})\big)$ of E, which remains ω-free by Theorem 1.4.1, and arrange that $H \supseteq \mathbb{R}$ and H is Liouville closed (so Corollary 7.5.38 applies). Towards a contradiction, suppose $\omega \in E$, $\omega(E) < \omega < \sigma\big(\Gamma(E)\big)$; then $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$. Choose a logarithmic sequence (ℓ_ρ) for H and define $\omega_\rho := \omega(-\ell_\rho^{\dagger\dagger})$. Then (ω_ρ) is a divergent pc-sequence in H with $\omega_\rho \rightsquigarrow \omega$, by [ADH, 11.8.30]. By [ADH, 13.6.3], (ω_ρ) is of d-transcendental type over H. Its width is $\big\{\gamma \in (\Gamma_H)_\infty : \gamma > 2\Psi_H\big\}$ by [ADH, 11.7.2], which contains $v(1/x^4) = 2v\big((1/x)'\big)$, so (ω_ρ) is not a c-sequence, contradicting Corollary 7.5.38. $\square$

Next we describe for $j = 1, 2$ a λ-free Hardy field $H_{(j)} \supseteq \mathbb{R}$ and $\omega_{(j)} \in H_{(j)}$ such that $\omega\big(\Lambda(H_{(j)})\big) < \omega_{(j)} < \sigma\big(\Gamma(H_{(j)})\big)$ (so $H_{(j)}$ is not ω-free by [ADH, 11.8.30]), and

(1) $\omega_{(1)} \in \overline{\omega}(H_{(1)})$;
(2) $\omega_{(2)} \notin \overline{\omega}(H_{(2)})$.

It follows that $\overline{\omega}(H_{(1)}) = H_{(1)} \setminus \sigma\big(\Gamma(H_{(1)})\big)^\uparrow$ by Lemma 5.5.35, hence condition (i) in Theorem 7.5.32 is satisfied for $H = H_{(1)}$, but it is *not* satisfied for $H = H_{(2)}$; thus $\mathrm{E}(H_{(1)})$ is ω-free, whereas $\mathrm{E}(H_{(2)})$ is not.

To construct such $H_{(j)}$ and $\omega_{(j)} \in H_{(j)}$ we start with a hardian translogarithmic germ ℓ_ω (see the remarks before Proposition 5.6.6), and set

$$\gamma := \ell_\omega^\dagger, \quad \lambda := -\gamma^\dagger, \quad \omega_{(1)} := \omega(\lambda), \quad \omega_{(2)} := \sigma(\gamma) = \omega_{(1)} + \gamma^2.$$

Using [ADH, Sections 11.5, 11.7] we see that the Hardy field $E := \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$ is $\upomega$-free and that the elements $\upomega_{(1)}, \upomega_{(2)} \in M := E\langle \ell_\omega \rangle$ are pseudolimits of the pc-sequence $(\upomega_n)$ in E. For $j = 1, 2$, we consider the Hardy subfield

$$H_{(j)} \ := \ E\langle \upomega_{(j)} \rangle$$

of M, an immediate $\uplambda$-free extension of E by [ADH, 13.6.3, 13.6.4], and therefore $\omega\big(\Lambda(H_{(j)})\big) < \upomega_{(j)} < \sigma\big(\Gamma(H_{(j)})\big)$ by [ADH, 11.8.30]. Moreover

$$\overline{\omega}\big(H_{(j)}\big) \ = \ \overline{\omega}(M) \cap H_{(j)} \ \text{ for } j = 1, 2,$$

so (1) holds since $\upomega_1 \in \omega(M) \subseteq \overline{\omega}(M)$, whereas $\upomega_2 \in \sigma\big(\Gamma(M)\big) \subseteq M \setminus \overline{\omega}(M)$, hence (2) holds.

Example 7.5.40. Set $H := \mathrm{E}(H_{(2)})$. Then the Hardy field H is perfect, so $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field with $\mathrm{I}(K) \subseteq K^\dagger$, but H is not $\upomega$-free. This makes good on a promise made before Lemma 4.4.32.

Antiderivatives of rational functions as phase functions. In this subsection $H = \mathbb{R}(x)$, so $K = H[i] = \mathbb{C}(x)$. If $f \in H \setminus \overline{\omega}(H)$ and $(g, \phi) \in \mathrm{Li}(H)^2$ parametrizes $\ker_{\mathcal{C}^{<\infty}}(4\partial^2 + f)$, then $(\phi')^2 \in H$ by Corollaries 7.5.13, 7.5.15, and 7.5.21. In Corollary 7.5.51 below we give a condition on such f, g, ϕ that ensures $\phi' \in H$, to be used in Section 7.6. We precede this with remarks about ramification in quadratic extensions of K. So let L be a field extension of K with $[L : K] = 2$.

Lemma 7.5.41. *Up to multiplication by -1, there is a unique $y \in L$ such that $L = K(y)$ and $y^2 = p(x)$ where $p \in \mathbb{C}[X]$ is monic and separable.*

Proof. By [ADH, 1.3.11], $A := \mathbb{C}[x]$ is integrally closed, so [ADH, 1.3.12, 1.3.13] yield a $y \in L$ with minimum polynomial $P \in A[Y]$ over K such that $L = K(y)$. Take $a, b \in A$ with $P = Y^2 + aY + b$. Replacing a, b, y by 0, $b - (a/2)^2$, $y + (a/2)$, respectively, we arrange $a = 0$. Thus $y^2 = p(x)$ for $p \in \mathbb{C}[X]$ with $p(x) = -b$, and replacing y by cy for suitable $c \in \mathbb{C}^\times$ we arrange that p is monic. If $c \in \mathbb{C}$ and $p \in (X - c)^2\mathbb{C}[X]$, then we may also replace p, y by $p/(X - c)^2$, $y/(x - c)$, respectively. In this way we arrange that p is separable. Suppose $L = K(z)$ and $z^2 = q(x)$ where $q \in \mathbb{C}[X]$ is monic and separable. Take $r, s \in K$ with $z = r + sy$. Then $s \neq 0$, and $q(x) = z^2 = \big(r^2 + s^2 p(x)\big) + (2rs)y$, hence $r = 0$ and so $q(x) = s^2 p(x)$. Since p, q are monic and separable, this yields $s^2 = 1$ and thus $z = -y$ or $z = y$. □

In the following y, p are as in Lemma 7.5.41. For each $c \in \mathbb{C}$ we have the valuation $v_c \colon K^\times \to \mathbb{Z}$ that is trivial on $\mathbb{C}$ with $v_c(x - c) = 1$, and we also have the valuation $v_\infty \colon K^\times \to \mathbb{Z}$ that is trivial on $\mathbb{C}$ with $v_\infty(x^{-1}) = 1$ [ADH, 3.1.30]. Given $f \in K^\times$ there are only finitely many $c \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ such that $v_c(f) \neq 0$; moreover, $\sum_{c \in \mathbb{C}_\infty} v_c(f) = 0$, with $f \in \mathbb{C}^\times$ iff $v_c(f) = 0$ for all $c \in \mathbb{C}$. Let $c \in \mathbb{C}_\infty$, and equip K with the valuation ring $\mathcal{O}_c$ of v_c. By [ADH, 3.1.15, 3.1.21], either exactly one or exactly two valuation rings of L lie over $\mathcal{O}_c$. The residue morphism $\mathcal{O}_c \to \operatorname{res}(K)$ restricts to an isomorphism $\mathbb{C} \to \operatorname{res}(K)$, and equipping L with a valuation ring lying over $\mathcal{O}_c$, composition with the natural inclusion $\operatorname{res}(K) \to \operatorname{res}(L)$ yields an isomorphism $\mathbb{C} \to \operatorname{res}(L)$; thus the valued field extension $L \supseteq K$ is immediate iff L is unramified over K, that is, $\Gamma_L = \Gamma = \mathbb{Z}$.

Lemma 7.5.42. *Suppose $c \neq \infty$. If $p(c) = 0$, then only one valuation ring of L lies over $\mathcal{O}_c$, and equipping L with this valuation ring we have $[\Gamma_L : \Gamma] = 2$. If $p(c) \neq 0$,* *then there are exactly two valuation rings of L lying over $\mathcal{O}_c$, and equipped with any one of these valuation rings, L is unramified over K.*

Proof. If $p(c)=0$, then $v_c(p)=1$, so by [ADH, 3.1.28] there is a unique valuation ring $\mathcal{O}_L$ of L lying over $\mathcal{O}_c$, and equipping L with $\mathcal{O}_L$ we have $[\Gamma_L:\Gamma]=2$. Now suppose $p(c)\neq 0$. We identify K with its image under the embedding of K into the valued field $K^{\mathrm{c}} := \mathbb{C}(\!(t)\!)$ of Laurent series over $\mathbb{C}$ which is the identity on $\mathbb{C}$ and sends $x-c$ to t. Take $\alpha\in\mathbb{C}^\times$ with $\alpha^2=p(c)$. Hensel's Lemma [ADH, 3.3.5] yields $z\in K^{\mathrm{c}}$ with $z^2=p(x)$ and $z\sim\alpha$. Let $\mathcal{O}_+$, $\mathcal{O}_-$ be the preimages of the valuation ring of the valued subfield $K(z)$ of K^{c} under the field isomorphisms $L=K(y)\to K(z)$ over K with $y\mapsto z$ and $y\mapsto -z$, respectively. Then $\mathcal{O}_+\neq\mathcal{O}_-$ lie over $\mathcal{O}_c$, and each turns L into an immediate extension of K. $\square$

In the next lemma we set $d := \deg p$, so $d\geqslant 1$.

Lemma 7.5.43. *If d is odd, then only one valuation ring of L lies over $\mathcal{O}_\infty$, and equipping L with this valuation ring we have $[\Gamma_L:\Gamma]=2$. If d is even, then there are exactly two valuation rings of L lying over $\mathcal{O}_\infty$, and equipped with any one of these valuation rings, L is unramified over K.*

Proof. We have $v_\infty(p)=-d$. Hence if d is odd, then we can argue using [ADH, 3.1.28] as in the proof of Lemma 7.5.42. Suppose d is even, so with $e=d/2$ we have $(y/x^e)^2=p(x)/x^d\sim 1$. Identify K with its image under the embedding of K into the valued field $K^{\mathrm{c}} := \mathbb{C}(\!(t)\!)$ of Laurent series over $\mathbb{C}$ which is the identity on $\mathbb{C}$ and sends x^{-1} to t. Then [ADH, 3.3.5] yields $z\in K^{\mathrm{c}}$ with $z\sim 1$ and $z^2=p(x)/x^d$. Let $\mathcal{O}_+$, $\mathcal{O}_-$ be the preimages of the valuation ring of the valued subfield $K(z)$ of K^{c} under the field isomorphisms $L\to K(z)$ over K with $y\mapsto x^e z$ and $y\mapsto -x^e z$, respectively. Then $\mathcal{O}_+\neq\mathcal{O}_-$ are valuation rings of L lying over $\mathcal{O}_\infty$, each of which turns L into an immediate extension of K. $\square$

Corollary 7.5.44. *There are at least two $c\in\mathbb{C}_\infty$ such that some valuation ring of L lying over $\mathcal{O}_c$ makes L ramified over K.*

Next we let C be any field of characteristic zero and consider the d-valued Hahn field $C(\!(t^{\mathbb{Q}})\!)$ with its strongly additive C-linear derivation satisfying $t'=1$. We let q, r, s range over $\mathbb{Q}$, and $z=\sum_q z_q t^q\in C(\!(t^{\mathbb{Q}})\!)^\times$ with all $z_q\in C$. Put

$$q_0 \ := vz \ = \ \min\operatorname{supp} z\in\mathbb{Q},$$

so $z\sim z_{q_0}t^{q_0}$. If $z\notin C(\!(t)\!)$, then we also set $q_1 := \min\big((\operatorname{supp} z)\setminus\mathbb{Z}\big)\in\mathbb{Q}\setminus\mathbb{Z}$, so $q_1\geqslant q_0$. In Lemmas 7.5.46 and 7.5.47 below we give sufficient conditions for z to be in $C(\!(t)\!)$. Set

$$w \ := \ z^2 \ = \ \sum_q w_q t^q \quad\text{where}\quad w_q \ = \ \sum_{r+s=q} z_r z_s,$$

so $w\sim z_{q_0}^2 t^{2q_0}$, and observe:

Lemma 7.5.45. *If $w\notin C(\!(t)\!)$ (and so $z\notin C(\!(t)\!)$), then*

$$\min\big((\operatorname{supp} w)\setminus\mathbb{Z}\big)=q_0+q_1,\quad w_{q_0+q_1}=2z_{q_0}z_{q_1}.$$

Lemma 7.5.46. *Suppose $\omega(z)\in t^{-1}C[[t]]$. Then $z\in C(\!(t)\!)$.*

Proof. Put $u := z' = \sum_q u_q t^q$, $u_q = (q+1)z_{q+1}$. If $q_0 \neq 0$, then $u \sim\ q_0 z_{q_0} t^{q_0-1}$. Hence $q_0 \geqslant -1$: otherwise $-\omega(z) = 2u + w \sim z_{q_0}^2 t^{2q_0}$, contradicting $\omega(z) \preccurlyeq t^{-1} \preccurlyeq t^{-2}$. Moreover, if $q_0 = -1$, then $(2u+w) - (-2z_{-1} + z_{-1}^2)t^{-2} \prec t^{-2}$ and so $z_{-1} = 2$. Towards a contradiction, suppose $z \notin C(\!(t)\!)$. We have $u \notin C(\!(t)\!)$. Indeed

$$\min\big((\operatorname{supp} u) \setminus \mathbb{Z}\big) = q_1 - 1, \quad u_{q_1-1} = q_1 z_{q_1}. \tag{7.5.2}$$

Also $w = -\omega(z) - 2u \notin C(\!(t)\!)$, and by the previous lemma

$$\min\big((\operatorname{supp} w) \setminus \mathbb{Z}\big) = q_0 + q_1, \quad w_{q_0+q_1} = 2z_{q_0} z_{q_1}. \tag{7.5.3}$$

From (7.5.2), (7.5.3), and $2u + w \in C(\!(t)\!)$ we get $q_1 - 1 = q_0 + q_1$ and $2q_1 z_{q_1} = -2z_{q_0} z_{q_1}$, hence $q_0 = -1$, $q_1 = -z_{q_0}$. Thus $q_1 = -2 < -1 = q_0$, a contradiction. $\square$

In [ADH, p. 519] we defined $\omega^\phi : E \to E$ for a differential field E and $\phi \in E^\times$.

Lemma 7.5.47. *Suppose $\omega^{-1/t^2}(z) \in C[[t]]^\times$. Then $z \in C(\!(t)\!)$.*

Proof. Put $u := -t^2 z' = \sum_q u_q t^q$ where $u_q = -(q-1)z_{q-1}$. If $q_0 \neq 0$, then $u \sim -q_0 z_{q_0} t^{q_0+1}$. We must have $q_0 = 0$: otherwise, if $q_0 < 1$, then $2q_0 < q_0 + 1$ and so $-\omega^{-1/t^2}(z) = 2u + w \sim z_{q_0}^2 t^{2q_0}$, contradicting $\omega^{-1/t^2}(z) \asymp 1$, whereas if $q_0 \geqslant 1$ then $2u + w \preccurlyeq t^{q_0+1} \preccurlyeq t^2$, again contradicting $\omega^{-1/t^2}(z) \asymp 1$. Now suppose $z \notin C(\!(t)\!)$. Then

$$\min\big((\operatorname{supp} u) \setminus \mathbb{Z}\big) = q_1 + 1, \quad u_{q_1+1} = -q_1 z_{q_1} \tag{7.5.4}$$

and by Lemma 7.5.45:

$$\min\big((\operatorname{supp} w) \setminus \mathbb{Z}\big) = q_1, \quad w_{q_1} = 2z_0 z_{q_1}. \tag{7.5.5}$$

Together with $2u + w \in C(\!(t)\!)$ this yields a contradiction. $\square$

We now apply the above with $C = \mathbb{C}$ to show:

Corollary 7.5.48. *Let $z \in L$ be such that $\omega(z) = f \in K$. If $v_c(f) \geqslant -1$ for all $c \in \mathbb{C}$, or $v_c(f) \geqslant -1$ for all but one $c \in \mathbb{C}$ and $v_\infty(f) = 0$, then $z \in K$.*

Proof. Let $c \in \mathbb{C}_\infty$ and let L be equipped with a valuation ring lying over $\mathcal{O}_c$. If $c \in \mathbb{C}$, then we have a valued differential field embedding $L \to \mathbb{C}(\!(t^{\mathbb{Q}})\!)$ over $\mathbb{C}$ with $x - c \mapsto t$, and identifying L with its image under this embedding, if $v_c(f) \geqslant -1$, then $f \in t^{-1}\mathbb{C}[[t]]$, hence $z \in \mathbb{C}(\!(t)\!)$ by Lemma 7.5.46, so $K(z) \subseteq \mathbb{C}(\!(t)\!)$ is unramified over K. If $c = \infty$, then we have a valued differential field embedding $L \to \mathbb{C}(\!(t^{\mathbb{Q}})\!)^{-1/t^2}$ over $\mathbb{C}$ with $x^{-1} \mapsto t$, and again identifying L with its image under this embedding, if $v_\infty(f) = 0$, then $f \in \mathbb{C}[[t]]^\times$ by Lemma 7.5.47, so $K(z)$ is unramified over K. Now use Corollary 7.5.44. $\square$

In the next two lemmas we fix $c \in \mathbb{C}_\infty$ and equip $K = \mathbb{C}(x)$ with $v = v_c$. Then the valued differential field K is d-valued, and for all $z \in K^\times$ with $vz = k \neq 0$ we have $z^\dagger \sim k(x-c)^{-1}$ if $c \neq \infty$, and $z^\dagger \sim -kx^{-1}$ if $c = \infty$. In these two lemmas we let $z \in K^\times$, and set $k := vz$, $f := \omega(z)$.

Lemma 7.5.49. *Suppose $z \succ 1$. If $c = \infty$, then $f \sim -z^2$. If $c \neq \infty$ and $f \preccurlyeq 1$, then $z - 2(x-c)^{-1} \preccurlyeq 1$.*

Proof. If $c = \infty$, then $x \succ 1$ and so $z^\dagger \sim -kx^{-1} \prec 1 \prec z$, hence $f = \omega(z) = -z(2z^\dagger + z) \sim -z^2$. Now suppose $c \neq \infty$ and $f \preccurlyeq 1$. Applying the automorphism of the differential field K over $\mathbb{C}$ with $x \mapsto x + c$ we arrange $c = 0$. So $x \prec 1$ and $z^\dagger \sim kx^{-1}$. We have $-z(2z^\dagger + z) = \omega(z) = f \preccurlyeq 1$, so $2z^\dagger \sim -z$, and thus $z \sim 2x^{-1}$, that is, $z - 2x^{-1} \preccurlyeq 1$. ☐

Lemma 7.5.50. *Suppose $c \neq \infty$ and $d \in \mathbb{C}^\times$ is such that $f - d(x-c)^{-2} \preccurlyeq 1$. Then $z \succ 1$, and for some $b \in \mathbb{C}$ with $b(2-b) = d$ we have $z - b(x-c)^{-1} \preccurlyeq 1$.*

Proof. We arrange again $c = 0$, so $\omega(z) = f \sim dx^{-2}$. If $z \preccurlyeq 1$, then $z' \prec x^{-1}$ and thus $dx^{-2} \sim \omega(z) = -(2z' + z^2) \prec x^{-1}$, contradicting $x \prec 1$. Thus $z \succ 1$, hence $z^\dagger \sim kx^{-1}$ with $k < 0$. Together with $-z(2z^\dagger + z) = \omega(z) \sim dx^{-2}$ this yields $k = -1$ and $z \sim bx^{-1}$ with $b \in \mathbb{C}^\times$, $b(2-b) = d$, so $z - bx^{-1} \preccurlyeq 1$. ☐

Let $f \in H \setminus \overline{\omega}(H)$, and suppose $(g, \phi) \in \mathrm{Li}(H)^2$ parametrizes $\ker_{\mathcal{C}^{<\infty}}(4\partial^2 + f)$. Here is the promised sufficient condition for $\phi' \in H$:

Corollary 7.5.51. *Suppose $v_c(f) \geqslant -1$ for all $c \in \mathbb{C}$, or $v_c(f) \geqslant -1$ for all but one $c \in \mathbb{C}$ and $v_\infty(f) = 0$. Then $\phi' \in H$.*

Proof. Put $y := g\,\mathrm{e}^{\phi \mathrm{i}} \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times$. The proof of Lemma 7.5.5 gives $4y'' + fy = 0$ and $\omega(z) = f$ for $z := 2y^\dagger = -\phi'^\dagger + 2\phi'\mathrm{i}$. We have the differential field extension $L := K[z] = K[\phi'] \subseteq \mathrm{Li}(H)[\mathrm{i}]$ of K. If $\phi' \notin H$, then $[L : K] = 2$, and then Corollary 7.5.48 gives $z \in K$, a contradiction. Thus $\phi' \in H$. ☐

In the next section on the Bessel equation the relevant f satisfies an even stronger condition, and this gives more information about ϕ:

Corollary 7.5.52. *Suppose $v_c(f) \geqslant 0$ for all $c \in \mathbb{C}^\times$, $v_\infty(f) = 0$, and $d \in \mathbb{C}$, $v_0\big(f - dx^{-2}\big) \geqslant 0$. Then there are $a, b \in \mathbb{C}$ and distinct $c_1, \dots, c_n \in \mathbb{C}^\times$ such that*

$$-\phi'^\dagger + 2\phi'\mathrm{i} \;=\; a + bx^{-1} + 2\sum_{j=1}^{n}(x - c_j)^{-1} \quad \textit{and} \quad b(2-b) = d.$$

Proof. Corollary 7.5.51 and its proof gives $z := -\phi'^\dagger + 2\phi'\mathrm{i} \in K^\times$ and $\omega(z) = f$. Consider first the case $d \neq 0$. Then by Lemma 7.5.49 we have $v_\infty(z) \geqslant 0$ and

$$v_c\big(z - 2(x-c)^{-1}\big) \;\geqslant\; 0 \;\text{ whenever } c \in \mathbb{C}^\times \text{ and } v_c(z) < 0.$$

Lemma 7.5.50 gives $v_0(z - bx^{-1}) \geqslant 0$ with $b \in \mathbb{C}$ such that $b(2-b) = d$. Taking $c_1, \dots, c_n$ as the distinct poles of z in $\mathbb{C}^\times$, this yields the desired result by considering the partial fraction decomposition of z with respect to $\mathbb{C}[x]$. Next, suppose $d = 0$. Then $v_c(f) \geqslant 0$ for all $c \in \mathbb{C}_\infty$, hence $f \in \mathbb{C} \cap H = \mathbb{R}$. Also $f > 0$, since $0 \in \overline{\omega}(H)$ and $f \notin \overline{\omega}(H)$. The example preceding Lemma 7.5.2, together with Corollary 7.5.15, gives $\phi = \frac{\sqrt{f}}{2}x + r$ with $r \in \mathbb{R}$, so $z = \sqrt{f} \cdot \mathrm{i}$, and this gives the desired result with $a = \sqrt{f} \cdot \mathrm{i}$, $b = 0$, $n = 0$. ☐

7.6. The Example of the Bessel Equation

We are going to use the results from Section 7.5 to obtain information about the solutions of the Bessel equation

$$(\mathrm{B}_\nu) \qquad x^2Y'' + xY' + (x^2 - \nu^2)Y \;=\; 0$$

of order $\nu \in \mathbb{R}$. For solutions in $\mathcal{C}^2_e$ ($e \in \mathbb{R}^>$), this is equivalent to the equation ($\widetilde{\mathrm{L}}$) in Section 7.5 with $a = x^{-1}$, $b = 1 - \nu^2 x^{-2}$, so that $f_\nu := -2a' - a^2 + 4b$ gives

$$f_\nu \;=\; -2(x^{-1})' - (x^{-1})^2 + 4\big(1 - \nu^2 x^{-2}\big) \;=\; 4 + (1-4\nu^2)x^{-2} \;\sim\; 4.$$

Thus $f_\nu \notin \overline{\omega}\big(\mathbb{R}(x)\big)$, and we have the isomorphism $y \mapsto x^{1/2}y$ of the $\mathbb{R}$-linear space $V_\nu \subseteq \mathcal{C}^{<\infty}$ of solutions of (B_ν) onto the $\mathbb{R}$-linear space of solutions in $\mathcal{C}^{<\infty}$ of

$$(\mathrm{L}_\nu) \qquad 4Y'' + f_\nu Y \;=\; 0.$$

The nonzero solutions of (B_ν) in $\mathcal{C}^2(\mathbb{R}^>)$ are known as (real) *cylinder functions*; cf. [205, §15.22].

Proposition 7.6.1. *There is a unique hardian germ ϕ_ν such that*

$$\phi_\nu - x \;\preccurlyeq\; x^{-1} \text{ and } V_\nu \;=\; \left\{ \frac{c}{\sqrt{x\phi_\nu'}}\cos(\phi_\nu + d) : \; c,d \in \mathbb{R} \right\}.$$

This germ ϕ_ν lies in $\mathrm{D}(\mathbb{Q}) \subseteq \mathcal{C}^\omega$. (*Recall that* $\mathrm{D}(\mathbb{Q}) = \mathrm{E}(\mathbb{Q})$.)

If $\nu^2 = \frac{1}{4}$, then $V_\nu = \mathbb{R}x^{-1/2}\cos x + \mathbb{R}x^{-1/2}\sin x$, and Proposition 7.6.1 holds with $\phi_\nu = x$. So suppose $\nu^2 \neq \frac{1}{4}$. Then Corollary 7.5.10 gives a germ $\phi \sim x$ in $\mathrm{D}(\mathbb{Q})$ such that (g,ϕ) parametrizes V_ν, where $g := (x\phi')^{-1/2}$. Using this fact, Proposition 7.6.1 now follows from Corollary 7.5.15, Lemma 7.5.5, and the next lemma about any such pair (g,ϕ):

Lemma 7.6.2. *We have $\phi - x - r - \frac{1}{2}(\nu^2 - \frac{1}{4})x^{-1} \preccurlyeq x^{-3}$ for some $r \in \mathbb{R}$.*

Proof. Set $z := 2\phi'$, so $z = 2 + \varepsilon$, $\varepsilon \prec 1$. From $\sigma(z) = f_\nu$ and multiplication by z^2,

$$2zz'' - 3(z')^2 + z^2(z^2 - f_\nu) \;=\; 0$$

and thus with $\mu := 4\nu^2 - 1 \in \mathbb{R}^\times$, $u := -(2zz'' - 3(z')^2) = 3(\varepsilon')^2 - 2y\varepsilon''$:

$$u \;=\; z^2(z^2 - f_\nu) \sim 4(z^2 - f_\nu) = 4(4\varepsilon + \varepsilon^2 + \mu x^{-2}), \text{ and thus}$$

$$(7.6.1) \qquad u/4 \;\sim\; \varepsilon(4+\varepsilon) + \mu x^{-2}.$$

We claim that $u \prec x^{-2}$. If $\varepsilon \preccurlyeq x^{-2}$, then $\varepsilon' \preccurlyeq x^{-3}$, $\varepsilon'' \preccurlyeq x^{-4}$, and the claim is valid. If $\varepsilon \succ x^{-2}$, then $\varepsilon^\dagger \preccurlyeq (x^{-2})^\dagger = -2x^{-1}$, so $\varepsilon' \preccurlyeq x^{-1}\varepsilon \prec x^{-1} \prec 1$, hence $\varepsilon'' \prec (x^{-1})' = -x^{-2}$, which again yields $u \prec x^{-2}$. The claim and (7.6.1) give $\varepsilon \sim -\frac{\mu}{4}x^{-2}$ and hence $\delta := \varepsilon + \frac{\mu}{4}x^{-2} \prec x^{-2}$. Indeed, we have $\delta \preccurlyeq x^{-4}$. To see why, note that $\varepsilon' \sim \frac{1}{2}\mu x^{-3}$ and $\varepsilon'' \sim -\frac{3}{2}\mu x^{-4}$, so $u \sim 6\mu x^{-4}$, and

$$\tfrac{3}{2}\mu x^{-4} \;\sim\; g/4 \;\sim\; \varepsilon(4+\varepsilon) + \mu x^{-2} \;=\; 4\delta + \varepsilon^2, \qquad \varepsilon^2 \;\sim\; \tfrac{\mu^2}{16}x^{-4}.$$

Now the lemma follows by integration from

$$\phi' - 1 + \tfrac{1}{2}(\nu^2 - \tfrac{1}{4})x^{-2} \;=\; \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\mu x^{-2} \;=\; \delta/2 \;\preccurlyeq\; x^{-4}. \qquad \square$$

With ϕ and r as in Lemma 7.6.2, it is $\phi - r$ that is the germ ϕ_ν in Proposition 7.6.1, and till further notice we set $\phi := \phi_\nu$, $f := f_\nu$, and $V := V_\nu$. Thus $\sigma(2\phi') = f$ and $\phi_\nu = \phi_{-\nu}$. As mentioned before, we do not know if $\mathrm{E}(\mathbb{Q})^{>\mathbb{R}}$ is closed under compositional inversion. Nevertheless:

Lemma 7.6.3. $\phi^{\mathrm{inv}} \in \mathrm{E}(\mathbb{Q})$.

Proof. Set $\alpha := \frac{1}{2}(\nu^2 - \frac{1}{4})$. Then $\phi = x + \alpha x^{-1} + o(x^{-1})$, so

$$\phi^{\mathrm{inv}} \;=\; x - \alpha x^{-1} + o(x^{-1})$$

by Corollary 5.1.12, and ϕ^{inv} is hardian. Let $P \in \mathbb{R}(x)\{Y\}$ be as in the remarks before Lemma 7.5.16 with $H = \mathbb{R}(x)$, so $P(2\phi') = 0$. Corollary 5.3.12 then gives $\widetilde{P} \in \mathbb{R}(x)\{Z\}$ such that for all hardian $y > \mathbb{R}$,

$$P(2y') \;=\; 0 \quad\Longleftrightarrow\quad \widetilde{P}(y^{\mathrm{inv}}) \;=\; 0,$$

in particular, $\widetilde{P}(\phi^{\mathrm{inv}}) = 0$. Let now H be any maximal Hardy field. Theorem 7.1.3 then yields $z \in H$ such that $z = x - \alpha x^{-1} + o(x^{-1})$ and $\widetilde{P}(z) = 0$, so $y := z^{\mathrm{inv}}$ is hardian and $P(2y') = 0$. Then $\sigma(2y') = f$, so $\big((xy')^{-1/2}, y\big)$ parametrizes V by Lemma 7.5.3 and a remark preceding that lemma. Also $y = x + \alpha x^{-1} + o(x^{-1})$ by Corollary 5.1.12. Thus $\phi = y$ by Proposition 7.6.1 and so $\phi^{\mathrm{inv}} = z \in H$. $\square$

This quickly yields some facts on the distribution of zeros of solutions: Let $y \in \mathcal{C}^2_e$ ($e \in \mathbb{R}^>$) be a nonzero solution of (B_ν) and let (s_n) be the enumeration of its zero set. From Corollary 7.5.24 and Lemma 7.5.26 we obtain a well-known result, see for example [91, Chapter XI, Exercise 3.2(d)], [203, §27, XIII]:

Corollary 7.6.4. *We have $s_n \sim \pi n$ and $s_{n+1} - s_n \to \pi$ as $n \to \infty$.*

Lemma 7.6.5. *There is a strictly increasing $\zeta \in \mathcal{C}_{n_0}$ ($n_0 \in \mathbb{N}$) whose germ is in $\mathrm{E}(\mathbb{Q})$ such that $s_n = \zeta(n)$ for all $n \geqslant n_0$.*

Proof. Take $e_0 \geqslant e$, a representative of ϕ in $\mathcal{C}^1_{e_0}$ denoted also by ϕ, and $c, d \in \mathbb{R}$, such that $\phi'(t) > 0$ and $y(t) = \big(c/\sqrt{t\phi'(t)}\big) \cdot \cos\big(\phi(t) + d\big)$ for all $t \geqslant e_0$. So we are in the situation described before Lemma 7.5.22. Next, take n_0, k_0, ζ as in the proof of that lemma. Then ζ is strictly increasing with $s_n = \zeta(n)$ for all $n \geqslant n_0$, and the germ of ζ, denoted by the same symbol, satisfies $\zeta = \phi^{\mathrm{inv}} \circ \big(\pi \cdot (x + k_0)\big)$. Now use Lemma 7.6.3 and $\mathrm{E}(\mathbb{Q}) \circ \mathrm{E}(\mathbb{Q})^{>\mathbb{R}} \subseteq \mathrm{E}(\mathbb{Q})$ (see the remark after Lemma 5.3.7), to conclude $\zeta \in \mathrm{E}(\mathbb{Q})$. $\square$

Lemma 7.6.5 yields an improvement of Corollary 7.5.27 in our (Bessel) case:

Corollary 7.6.6. *For any $h \in \mathcal{C}_0$ with hardian germ the sequences (s_n) and $\big(h(n)\big)$ are comparable.*

Lemma 7.5.26 also has the following corollary, the first part of which was observed by Porter [157] (cf. also [205, §15.8, 15.82]).

Corollary 7.6.7. *If $\nu^2 > \frac{1}{4}$, then the sequence $(s_{n+1} - s_n)$ is eventually strictly decreasing, and if $\nu^2 < \frac{1}{4}$, then $(s_{n+1} - s_n)$ is eventually strictly increasing.*

Finally, if $\underline{\nu} \in \mathbb{R}$ and $\underline{y} \in \mathcal{C}^2_{\underline{e}}$ with $\underline{e} \in \mathbb{R}^>$ is a nonzero solution of the Bessel equation of order $\underline{\nu}$, then (s_n) and the enumeration $(\underline{s}_n)$ of the zero set of $\underline{y}$ are comparable, by Lemma 7.5.28. This is related to classical results on the "interlacing of zeros" of cylinder functions; cf. [205, §§15.22, 15.24].

In the next lemma, [21, Chapter 10, Theorem 8] has $-\frac{1}{2}(\nu^2 - \frac{1}{4})x^{-1}$ instead of our $\frac{1}{2}(\nu^2 - \frac{1}{4})x^{-1}$. This sign error originated in an integration on [21, p. 327].

Lemma 7.6.8. *Let $y \in V^{\neq}$. Then there is a pair $(c, d) \in \mathbb{R}^\times \times [0, \pi)$ such that $y = \frac{c}{\sqrt{x\phi'}} \cos(\phi + d)$, and for any such pair we have*

$$y - \frac{c}{\sqrt{x}} \cos\big(x + d + \tfrac{1}{2}(\nu^2 - \tfrac{1}{4})x^{-1}\big) \;\preccurlyeq\; x^{-5/2}. \tag{7.6.2}$$

Proof. Proposition 7.6.1 yields $(c,d) \in \mathbb{R} \times [0,\pi)$ such that $y = \frac{c}{\sqrt{x\phi'}}\cos(\phi + d)$. Then $c \neq 0$. From $\phi' - 1 \preccurlyeq x^{-2}$ we get $\frac{1}{\sqrt{\phi'}} - 1 \preccurlyeq x^{-2}$, and for every $u \in \mathcal{C}$ we have $\cos(x+u) - \cos(x) \preccurlyeq u$. Using also Lemma 7.6.2 this yields (7.6.2). $\square$

We complement this with some uniqueness properties:

Lemma 7.6.9. *Let $y \in V^{\neq}$. Then there is a unique $(c,d) \in \mathbb{R}^{\times} \times [0,\pi)$ such that*

$$y - \frac{c}{\sqrt{x}}\cos(x+d) \ \prec\ \frac{1}{\sqrt{x}}, \tag{7.6.3}$$

and this is also the unique $(c,d) \in \mathbb{R}^{\times} \times [0,\pi)$ such that $y = \frac{c}{\sqrt{x\phi'}}\cos(\phi + d)$.

Proof. For $(c,d) \in \mathbb{R}^{\times} \times [0,\pi)$ with $y = \frac{c}{\sqrt{x\phi'}}\cos(\phi+d)$ we have (7.6.2), so

$$y - \frac{c}{\sqrt{x}}\cos(x+d) \ \preccurlyeq\ x^{-3/2}$$

in view of $\cos(x+u) - \cos(x) \preccurlyeq u$ for $u \in \mathcal{C}$. This gives (7.6.3). Suppose towards a contradiction that (7.6.3) also holds for a pair $(c^*, d^*) \in \mathbb{R}^{\times} \times [0,\pi)$ instead of (c,d), with $(c^*,d^*) \neq (c,d)$. Then $d \neq d^*$, say $d < d^*$, so $0 < \theta := d^* - d < \pi$. Then $c\cos(x+d) - c^*\cos(x+d+\theta) \prec 1$, and hence $c\cos(x) - c^*\cos(x+\theta) \prec 1$, which by a trigonometric identity turns into

$$(c - c^*\cos\theta)\cos(x) + c^*\sin\theta\sin(x) \ =\ \sqrt{(c-c^*\cos\theta)^2 + (c^*\sin\theta)^2}\cdot\cos(x+s) \prec 1$$

with $s \in \mathbb{R}$ depending only on c, c^*, θ; see the remarks preceding Lemma 5.5.14. This forces $c^*\sin\theta = 0$, but $\sin\theta > 0$, so $c^* = 0$, contradicting $c^* \in \mathbb{R}^{\times}$. $\square$

Corollary 7.6.10. *For any $(c,d) \in \mathbb{R}^{\times} \times [0,\pi)$ there is a unique $y \in V^{\neq}$ such that* (7.6.3) *holds. This y is given by $y = \frac{c}{\sqrt{x\phi'}}\cos(\phi+d)$.*

Remark 7.6.11. Lemmas 7.6.8, 7.6.9, and Corollary 7.6.10 remain valid when we replace $\mathbb{R}^{\times} \times [0,\pi)$ everywhere by $\mathbb{R}^{>} \times [0,2\pi)$. (Use that $\cos(\theta+\pi) = -\cos(\theta)$ for $\theta \in \mathbb{R}$.)

Call a germ in $\mathcal{C}$ *eventually convex* if it has a convex representative in $\mathcal{C}_r$ for some $r \in \mathbb{R}$; likewise with "concave" in place of "convex". The two lemmas below comprise a slightly weaker version of [105, Theorem 2]. By Lemma 7.6.2 we have $\phi = x + \alpha x^{-1} + O(x^{-3})$ where $\alpha := \frac{1}{2}(\nu^2 - \frac{1}{4})$, so with ϕ being hardian we obtain

$$\phi' \ =\ 1 - \alpha x^{-2} + O(x^{-4}), \qquad \phi'' \ =\ 2\alpha x^{-3} + O(x^{-5}).$$

Hence $\phi'' > 0$ if $\nu^2 > \frac{1}{4}$ and $\phi'' < 0$ if $\nu^2 < \frac{1}{4}$, and thus:

Lemma 7.6.12. *ϕ is eventually convex if $\nu^2 > \frac{1}{4}$, and eventually concave if $\nu^2 < \frac{1}{4}$.*

Lemma 7.5.2 yields $(q,\theta) \in \mathrm{D}(\mathbb{Q})^2$ parametrizing $V' := \{y' : y \in V\}$.

Lemma 7.6.13. *We have $\theta - x - r - \frac{1}{2}(\nu^2 + \frac{3}{4})x^{-1} \preccurlyeq x^{-3}$ for some $r \in \mathbb{R}$. Hence θ is eventually convex.*

Proof. Set $g := (x\phi')^{-1/2}, q := \sqrt{(g')^2 + (g\phi')^2}$, so $g, q \in \mathrm{D}(\mathbb{Q})$. The proof of Lemma 7.5.29(iv) and Lemmas 7.5.2 and 7.5.7 give $\theta = \phi + d + u$ with $d \in \mathbb{R}$ and $u = \arccos(g'/q)$. Now $\phi'^{\dagger} = 2\alpha x^{-3} + O(x^{-5})$ and so $g^{\dagger} = -\frac{1}{2}x^{-1} + O(x^{-3})$. Since $(\phi')^2 = 1 + O(x^{-2})$, this yields $((g^{\dagger})^2 + (\phi')^2)^{-1/2} = 1 + O(x^{-2})$ and thus

$$g'/q \ =\ g^{\dagger}\big((g^{\dagger})^2 + (\phi')^2\big)^{-1/2} \ =\ -\tfrac{1}{2}x^{-1} + O(x^{-3}).$$

Hence

$$(g'/q)' \;=\; \tfrac{1}{2}x^{-2} + O(x^{-4}), \qquad \bigl(1-(g'/q)^2\bigr)^{-1/2} \;=\; 1 + O(x^{-2}).$$

We obtain

$$u' \;=\; -\frac{(g'/q)'}{\sqrt{1-(g'/q)^2}} \;=\; -\tfrac{1}{2}x^{-2} + O(x^{-4})$$

and thus $u = c + \frac{1}{2}x^{-1} + O(x^{-3})$ with $c \in \mathbb{R}$, and so $\theta - x - r - (\alpha + \frac{1}{2})x^{-1} \preccurlyeq x^{-3}$ for $r := c + d$, as claimed. $\square$

Asymptotic expansions for ϕ and ϕ^{inv}. The arguments in this subsection demonstrate the efficiency of our transfer theorems from Section 7.1. They allow us to produce hardian solutions of algebraic differential equations from transseries solutions of these equations. Such transseries solutions may be constructed by purely formal computations in $\mathbb{T}$ (without convergence considerations). Our first goal is to improve on the relation $\phi \sim x + \frac{\mu-1}{8}x^{-1}$ from Lemma 7.6.2, where $\mu := 4\nu^2$:

Theorem 7.6.14. *The germ $\phi = \phi_\nu$ has an asymptotic expansion*

$$\phi \;\sim\; x + \frac{\mu-1}{8}x^{-1} + \frac{\mu^2-26\mu+25}{384}x^{-3} + \frac{\mu^3-115\mu^2+1187\mu-1073}{5120}x^{-5} + \cdots$$

Here we use the sign $\sim$ not in the sense of comparing germs, but to indicate an *asymptotic expansion*: for a sequence (g_n) in $\mathcal{C}^{<\infty}[i]$ with $g_0 \succ g_1 \succ g_2 \succ \cdots$ we say that $g \in \mathcal{C}^{<\infty}[i]$ has the asymptotic expansion

$$g \;\sim\; c_0g_0 + c_1g_1 + c_2g_2 + \cdots \qquad (c_0, c_1, c_2, \dots \in \mathbb{C})$$

if $g - (c_0g_0 + \cdots + c_ng_n) \prec g_n$ for all n (and then the sequence $c_0, c_1, c_2, \dots$ of coefficients is uniquely determined by $g, g_0, g_1, g_2, \dots$).

In the course of the proof of Theorem 7.6.14 we also obtain an explicit formula for the coefficient of x^{-2n+1} in the asymptotic expansion of the theorem. Towards the proof, set

$$(\nu, n) \;:=\; \frac{(\mu-1^2)(\mu-3^2)\cdots(\mu-(2n-1)^2)}{n!\,2^{2n}} \qquad \text{(Hankel's symbol)},$$

so $(\nu,0)=1$, $(\nu,1)=\frac{\mu-1}{4}$, and $(\nu,n)=(-\nu,n)$. Also, if $(\nu,n)=0$ for some n, then $\nu \in \frac{1}{2}+\mathbb{Z}$: and if $\nu = \frac{1}{2}+m$, then $(\nu,n)=0$ for $n \geqslant m+1$. Moreover, if $\nu \notin \frac{1}{2}+\mathbb{Z}$, then in terms of Euler's Gamma function (cf. [123, XV, §2, Γ3, Γ5]),

$$\begin{aligned}
(\nu,n) \;&=\; \frac{(-1)^n}{\pi n!}\cos(\pi\nu)\,\Gamma(\tfrac{1}{2}+n-\nu)\,\Gamma(\tfrac{1}{2}+n+\nu), \text{ and so}\\
(m,n) \;&=\; \frac{(-1)^{m+n}}{\pi n!}\Gamma(\tfrac{1}{2}+n-m)\,\Gamma(\tfrac{1}{2}+n+m).
\end{aligned}$$

(To prove the first identity, use $\Gamma(z+1) = z\Gamma(z)$ n times to give

$$\begin{aligned}
\Gamma(\tfrac{1}{2}+n-\nu) \;&=\; \Gamma(\tfrac{1}{2}-\nu+n) \;=\; \Bigl(\textstyle\prod_{j=0}^{n-1}(\tfrac{1}{2}-\nu+j)\Bigr)\cdot\Gamma(\tfrac{1}{2}-\nu); \text{ likewise}\\
\Gamma(\tfrac{1}{2}+n+\nu) \;&=\; \Gamma(\tfrac{1}{2}+\nu+n) \;=\; \Bigl(\textstyle\prod_{j=0}^{n-1}(\tfrac{1}{2}+\nu+j)\Bigr)\cdot\Gamma(\tfrac{1}{2}+\nu),
\end{aligned}$$

and then use $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ to get $\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\nu) = \frac{\pi}{\cos(\pi\nu)}$.)

Below we consider the H-subfield $\mathbb{R}(\!(x^{-1})\!)$ of $\mathbb{T}$, and set

$$y \;:=\; \sum_{n=0}^{\infty} y_n x^{-2n} \in \mathbb{R}(\!(x^{-1})\!) \quad \text{where } y_n \;:=\; (2n-1)!!\frac{(\nu,n)}{2^n}.$$

Here $(2n-1)!! := 1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{2^n n!}$, so $(-1)!! = 1$. Thus

$$y \;=\; 1+\left(\frac{\mu-1}{8}\right)x^{-2}+\frac{3!!}{2!}\left(\frac{\mu-1}{8}\right)\left(\frac{\mu-9}{8}\right)x^{-4}+ \\ \frac{5!!}{3!}\left(\frac{\mu-1}{8}\right)\left(\frac{\mu-9}{8}\right)\left(\frac{\mu-25}{8}\right)x^{-6}+\cdots.$$

The definition of the y_n yields the recursion

$$(7.6.4)\qquad y_0 \;=\; 1 \qquad \text{and} \qquad y_{n+1} \;=\; \left(\frac{2n+1}{n+1}\right)\left(\frac{\mu-(2n+1)^2}{8}\right)y_n.$$

Using this recursion and $\Gamma(1/2)=\sqrt{\pi}$ for $n=0$, induction on n yields for $\nu\notin \frac{1}{2}+\mathbb{Z}$:

$$y_n \;=\; \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2}+n)}{n!\,\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2}-n)}.$$

We now verify that y satisfies the linear differential equation

$$Y'''+fY'+(f'/2)Y \;=\; 0.$$

Here $f = 4+(1-\mu)x^{-2}$, so $f'/2=(\mu-1)x^{-3}$. Thus

$$(f'/2)y \;=\; \sum_n(\mu-1)y_n x^{-2n-3} \;=\; (\mu-1)x^{-3}+\sum_{n\geqslant 1}(\mu-1)y_n x^{-2n-3}.$$

We also have $y'=\sum_{n\geqslant 1}-2ny_n x^{-2n-1}$ and so

$$\begin{aligned} fy' \;&=\; \big(4+(1-\mu)x^{-2}\big)\sum_{n\geqslant 1}-2ny_n x^{-2n-1} \\ &=\; \sum_{n\geqslant 1}2n(\mu-1)y_n x^{-2n-3}-\sum_{n\geqslant 1}8ny_n x^{-2n-1} \\ &=\; \sum_{n\geqslant 1}2(\mu-1)ny_n x^{-2n-3}-\sum_{m\geqslant 0}8(m+1)y_{m+1}x^{-2m-3} \\ &=\; -8y_1x^{-3}+\sum_{n\geqslant 1}\big(2(\mu-1)ny_n-8(n+1)y_{n+1}\big)x^{-2n-3} \end{aligned}$$

and hence, using $\mu-1=8y_1$:

$$fy'+(f'/2)y \;=\; \sum_{n\geqslant 1}\big((2n+1)(\mu-1)y_n-8(n+1)y_{n+1}\big)x^{-2n-3}.$$

Moreover

$$y'' \;=\; \sum_{n\geqslant 1}2n(2n+1)y_n x^{-2n-2}, \quad y''' \;=\; \sum_{n\geqslant 1}-4n(2n+1)(n+1)y_n x^{-2n-3}.$$

This yields the claim by (7.6.4). We now identify the Hardy field $\mathbb{R}(x)$ with an H-subfield of $\mathbb{T}$ in the obvious way. Then $\mathbb{R}(x)\subseteq\mathbb{R}(\!(x^{-1})\!)$, and the above yields:

Lemma 7.6.15. *Let H be an H-closed field extending the H-field $\mathbb{R}(x)$ and set $B := \partial^3+f\partial+(f'/2)\in\mathbb{R}(x)[\partial]$. Then $\dim_{C_H}\ker_H B = 1$. If H extends the H-field $\mathbb{R}\langle x,y\rangle\subseteq\mathbb{T}$, then $\ker_H B = C_H y$.*

Proof. We have $\sigma(1/x) = 2/x^2 \prec 4 \sim f$, so $f \in \sigma\big(\Gamma(H)\big)^{\uparrow}$, hence $f \in \sigma(H^\times)\setminus\omega(H)$. Thus $\dim_{C_H} \ker_H B = 1$ by Lemma 2.5.25. Hence if H extends the H-field $\mathbb{R}\langle x, y\rangle$, then $\ker_H B = C_H y$ since $B(y) = 0$ by the argument preceding the lemma. □

Proposition 7.6.16. *There is a unique hardian germ $\psi = \psi_\nu$ such that*

$$\psi \ \sim\ 1 \quad \textit{and} \quad \psi''' + f\psi' + (f'/2)\psi \ =\ 0. \tag{7.6.5}$$

This ψ satisfies $\psi = 1/\phi' \in \mathrm{D}(\mathbb{Q})$ and has the asymptotic expansion

$$\psi \ \sim\ 1 + \frac{\mu-1}{8}x^{-2} + \cdots + (2n-1)!!\frac{(\nu,n)}{2^n}x^{-2n} + \cdots. \tag{7.6.6}$$

Moreover $\psi_{-\nu} = \psi_\nu$, and if $\nu = \frac{1}{2} + m$, then

$$\psi \ =\ 1 + \frac{\mu-1}{8}x^{-2} + \cdots + (2m-1)!!\frac{(\nu,m)}{2^m}x^{-2m}.$$

Proof. For any H-closed field $H \supseteq \mathbb{R}(x)$, consider the statement

$$\begin{cases} \text{there is a unique } \psi \in H \text{ such that (7.6.5) holds, and this } \psi \\ \text{satisfies } \psi - \displaystyle\sum_{n=0}^{m} y_n x^{-2n} \prec x^{-2m-1} \text{ for all } m. \end{cases}$$

This holds for $H = \mathbb{T}$ by Lemma 7.6.15, and hence also for any d-maximal Hardy field H by Corollary 7.1.17 (applied with $\mathbb{R}(x)$ in the role of H). Thus every d-maximal Hardy field contains a unique germ ψ satisfying (7.6.5), and every such ψ has an asymptotic expansion (7.6.6).

Now let ψ be any hardian germ satisfying (7.6.5). Take a d-maximal Hardy field H containing ψ; then $\mathbb{R}\langle x, \phi\rangle \subseteq \mathrm{D}(\mathbb{Q}) \subseteq H$. Let B be as in Lemma 7.6.15. Then Lemma 7.5.19 gives $B(1/\phi') = 0$. Since $1/\phi' \sim 1$, this yields $\psi = 1/\phi' \in \mathrm{D}(\mathbb{Q})$. For the rest, use that $\phi_\nu = \phi_{-\nu}$ and that for $\nu = \frac{1}{2} + m$ we have $y_n = 0$ for $n \geqslant m+1$. □

Corollary 7.6.17. *Let $\psi = \psi_\nu$ and suppose $\mu \neq 1$. Then*

$$\psi^{(n)} \ \sim\ (-1)^n(n+1)!\left(\frac{\mu-1}{8}\right)x^{-n-2} \quad \textit{for } n \geqslant 1.$$

In particular, ψ is eventually strictly increasing if $\nu^2 < 1/2$ and eventually strictly decreasing if $\nu^2 > 1/2$.

Lemma 7.6.18. *Let H be a Hausdorff field extension of $\mathbb{R}(x)$, $e\colon H \to \mathbb{T}$ a valued field embedding over $\mathbb{R}(x)$, and $h \in H$, $e(h) \in \mathbb{R}(\!(x^{-1})\!)$, say*

$$e(h) \ =\ h_{k_0}x^{-k_0} + h_{k_0+1}x^{-k_0-1} + \cdots \qquad \textit{where } k_0 \in \mathbb{Z} \textit{ and } h_k \in \mathbb{R} \textit{ for } k \geqslant k_0.$$

Then h has an asymptotic expansion

$$h \ \sim\ h_{k_0}x^{-k_0} + h_{k_0+1}x^{-k_0-1} + \cdots.$$

Moreover, if $e(h) \in \mathbb{R}[x, x^{-1}]$, then $e(h) = h$.

Proof. For $k \geqslant k_0$ we set $h_{\leqslant k} := h_{k_0}x^{-k_0} + \cdots + h_k x^{-k} \in \mathbb{R}[x, x^{-1}]$. Then

$$e(h - h_{\leqslant k}) \ =\ e(h) - h_{\leqslant k} = h_{k+1}x^{-k-1} + \cdots \prec x^{-k} \ =\ e(x^{-k}),$$

so $h - h_{\leqslant k} \prec x^{-k}$, giving the asymptotic expansion. If $e(h) \in \mathbb{R}[x, x^{-1}]$, take $k \geqslant k_0$ so large that $e(h) = h_{\leqslant k}$, and then $e(h) = e(h_{\leqslant k})$, so $h = h_{\leqslant k} \in \mathbb{R}[x, x^{-1}]$. □

We now prove Theorem 7.6.14, using also results and notations from the Appendix to this section. Corollary 7.3.2 yields an H-field embedding $e\colon \mathrm{D}(\mathbb{Q}) \to \mathbb{T}$ over $\mathbb{R}(x)$. Let ψ be as in Proposition 7.6.16. Then $e(\psi) = y$ and so

$$e(\phi') \;=\; e(1/\psi) \;=\; y^{-1} \;=\; z_0 + z_1\frac{x^{-2}}{1!} + z_2\frac{x^{-4}}{2!} + \cdots + z_n\frac{x^{-2n}}{n!} + \cdots$$

where

$$z_n \;:=\; B_n(-y_1 1!, \dots, -y_n n!) \in \mathbb{Q}[y_1, \dots, y_n] \subseteq \mathbb{Q}[\mu]$$

by Lemma 7.6.53 at the end of this section; here the B_n are as defined in (7.6.23). Using Lemma 7.6.18 and $\phi \sim x$ we obtain the asymptotic expansion

$$\phi \;\sim\; u_0 x + u_1\frac{x^{-1}}{1!} + u_2\frac{x^{-3}}{2!} + \cdots + u_n\frac{x^{-2n+1}}{n!} + \cdots \quad \text{where } u_n := \frac{z_n}{-2n+1}.$$

The first few terms of the sequence (z_n) are

$$\begin{aligned}
z_0 &= 1,\\
z_1 &= -y_1 = \frac{-(\mu-1)}{8},\\
z_2 &= -2y_2 + 2y_1^2 = \frac{-3(\mu-1)(\mu-9) + 2(\mu-1)^2}{64},\\
z_3 &= -6y_3 + 12y_1y_2 - 6y_1^3\\
&= \frac{-15(\mu-1)(\mu-9)(\mu-25) + 18(\mu-1)^2(\mu-9) - 6(\mu-1)^3}{512}
\end{aligned}$$

and so

$$\begin{aligned}
u_0 &= 1,\\
u_1 &= \frac{\mu-1}{8},\\
u_2 &= \frac{3(\mu-1)(\mu-9) - 2(\mu-1)^2}{192} = \frac{\mu^2 - 26\mu + 25}{192}\\
u_3 &= \frac{15(\mu-1)(\mu-9)(\mu-25) - 18(\mu-1)^2(\mu-9) + 6(\mu-1)^3}{2560}\\
&= \frac{3(\mu^3 - 115\mu^2 + 1187\mu - 1073)}{2560}.
\end{aligned}$$

This finishes the proof of Theorem 7.6.14. □

We turn to the compositional inverse ϕ^{inv} of ϕ. Recall: $\phi^{\mathrm{inv}} \in \mathrm{D}(\mathbb{Q})$ by Lemma 7.6.3. To prove the next result we use Corollary 7.6.67 in the Appendix to this section.

Corollary 7.6.19. *We have an asymptotic expansion*

$$\phi^{\mathrm{inv}} \;\sim\; x - \frac{\mu-1}{8}x^{-1} - \frac{(\mu-1)(7\mu-31)}{192}\frac{x^{-3}}{2!} + \cdots$$

Proof. Let $e\colon \mathrm{D}(\mathbb{Q}) \to \mathbb{T}$ and (u_n) be as above. Set

$$u := \sum_n u_n \frac{x^{-2n+1}}{n!} = x + \frac{\mu-1}{8}x^{-1} + \frac{\mu^2 - 26\mu + 25}{384}x^{-3} + \cdots \in \mathbb{R}(\!(x^{-1})\!) \subseteq \mathbb{T},$$

so $e(\phi) = u$. Let $P \in \mathbb{R}(x)\{Y\}$ be as in the proof of Lemma 7.6.3, so $P(2u') = e(P(2\phi')) = 0$. Corollary 5.3.12 and the remark following it yield a $\widetilde{P} \in \mathbb{R}(x)\{Z\}$ such that for all hardian $y > \mathbb{R}$,

$$P(2y') \ = \ 0 \quad \Longleftrightarrow \quad \widetilde{P}(y^{\text{inv}}) \ = \ 0$$

and such that this equivalence also holds for $y \in \mathbb{T}^{>\mathbb{R}}$ and y^{inv} the compositional inverse of y in $\mathbb{T}$. Hence $\widetilde{P}(e(\phi^{\text{inv}})) = e(\widetilde{P}(\phi^{\text{inv}})) = 0$ and $\widetilde{P}(u^{\text{inv}}) = 0$. The proof of Lemma 7.6.3 shows that each maximal Hardy field H contains a unique zero z of $\widetilde{P}$ such that $z = x - \frac{1}{8}(\mu-1)x^{-1} + o(x^{-1})$. By Corollary 7.1.17 this remains true with $\mathbb{T}$ in place of H. Now $e(\phi^{\text{inv}}) = x - \frac{1}{8}(\mu-1)x^{-1} + o(x^{-1})$, and by the remarks following Corollary 7.6.67 we have $u^{\text{inv}} = u^{[-1]} = x - \frac{1}{8}(\mu-1)x^{-1} + o(x^{-1})$. Hence $e(\phi^{\text{inv}}) = u^{\text{inv}}$ and thus ϕ^{inv} has an asymptotic expansion as claimed. $\square$

Remark. Corollary 7.6.67 yields the more detailed asymptotic expansion

$$\phi^{\text{inv}} \ \sim \ x - \sum_{j=1}^{\infty} g_j \frac{x^{-2j+1}}{j!}, \quad \text{where } g_j = \sum_{i=1}^{j} \frac{(2(j-1))!}{(2j-1-i)!} B_{ij}(u_1, \dots, u_{j-i+1}).$$

Liouvillian phase functions. The next proposition adds to Corollary 7.5.13 for the differential equation (L_ν). This subsection is not used in the rest of the section.

Proposition 7.6.20. *With $\phi = \phi_\nu$, the following are equivalent:*

(i) $\nu \in \frac{1}{2} + \mathbb{Z}$;
(ii) $1/\phi' \in \mathbb{R}[x^{-1}]$;
(iii) $f \in \sigma\big(\mathbb{R}(x)^{>}\big)$; *recall:* $f = 4 + (1-\mu)x^{-2}$;
(iv) $\phi \in \operatorname{Li}\big(\mathbb{R}(x)\big)$;
(v) $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ *for some* $y \neq 0$ *in a Liouville extension of* $\mathbb{C}(x)$;
(vi) *there are* $a, b \in \mathbb{C}$ *and distinct* $c_1, \dots, c_n \in \mathbb{C}^\times$ *such that*

$$-\phi'^{\dagger} + 2\phi' \mathrm{i} = a + bx^{-1} + 2\sum_{i=1}^{n}(x - c_i)^{-1} \quad \textit{and} \quad b = 1 + 2\nu \textit{ or } b = 1 - 2\nu.$$

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 7.6.16, and (ii) $\Rightarrow$ (iii) from $f = \sigma(2\phi')$. If $f \in \sigma\big(\mathbb{R}(x)^\times\big)$, then $\phi \in \operatorname{Li}\big(\mathbb{R}(x)\big)$ by Corollary 7.5.14 with $H := \operatorname{Li}\big(\mathbb{R}(x)\big)$; thus (iii) $\Rightarrow$ (iv). For the rest of the proof, recall that by Lemma 7.5.3 the pair $\big(1/\sqrt{\phi'}, \phi\big)$ parametrizes $\ker_{\mathcal{C}^{<\infty}}(4\partial^2 + f)$, so $\big(1/\sqrt{x\phi'}, \phi\big)$ parametrizes V_ν. Thus (iv) $\Leftrightarrow$ (v) by Corollary 7.5.13. Moreover, if (iv) holds, then $\big(1/\sqrt{\phi'}, \phi\big) \in \operatorname{Li}\big(\mathbb{R}(x)\big)^2$, so (vi) then follows from Corollary 7.5.52.

Suppose $a, b, c_1, \dots, c_n$ are as in (vi) and set

$$y \ := \ (\phi')^{-1/2}\, \mathrm{e}^{\phi \mathrm{i}}, \qquad z \ := \ 2y^\dagger \ = \ -\phi'^\dagger + 2\phi'\mathrm{i} \in \mathbb{C}(x),$$

so $4y'' + fy = 0$, hence $\omega(z) = f$. Then, as germs at $+\infty$,

$$z \ = \ a + (b+2n)x^{-1} + O(x^{-2}), \text{ so } z' \ = \ O(x^{-2}), \quad z^2 \ = \ a^2 + 2a(b+2n)x^{-1} + O(x^{-2})$$

and hence

$$f \ = \ 4 + (1-\mu)x^{-2} \ = \ \omega(z) \ = \ -(2z' + z^2) \ = \ -a^2 - 2a(b+2n)x^{-1} + O(x^{-2}),$$

so $b + 2n = 0$, hence $\nu = -n - \frac{1}{2}$ or $\nu = n + \frac{1}{2}$, and thus $\nu \in \frac{1}{2} + \mathbb{Z}$. $\square$

Remark. In the setting of analytic functions, the above equivalence (i) $\Leftrightarrow$ (v) goes back to Liouville [132]. For more on this, see [113, appendix], [116, §4.2], [164, Chapter VI], and [205, §4.74].)

For the next result, note that $\arctan(g) \in \operatorname{Li}\big(\mathbb{R}(x)\big)$ for $g \in \mathbb{R}(x)$.

Corollary 7.6.21. *Suppose $\nu \in \frac{1}{2}+\mathbb{Z}$. Then there are distinct $(a_1, b_1), \dots, (a_m, b_m)$ in $\mathbb{R}^\times \times \mathbb{R}$ such that*

$$\phi \;=\; x + \sum_{i=1}^{m} \arctan\left(\frac{a_i}{x-b_i}\right).$$

Proof. Take imaginary parts in the equality of Proposition 7.6.20(vi), integrate, and appeal to the defining property of ϕ in Proposition 7.6.1 in combination with the fact that for $a, b \in \mathbb{R}$ we have $\arctan\left(\frac{a}{x-b}\right) \preccurlyeq x^{-1}$. Here we also use that the derivative of $\arctan\left(\frac{a}{x-b}\right)$ is $\frac{-a}{(x-b)^2+a^2} = \operatorname{Im}\left(\frac{1}{x-c}\right)$ for $a, b \in \mathbb{R}$, $c = b - ai$. $\square$

Is $\phi, \phi^{\text{inv}} \in \operatorname{Li}\big(\mathbb{R}(x)\big)$ possible? The answer is "no" except for $\phi = x$:

Corollary 7.6.22. *Suppose $\phi \in \operatorname{Li}\big(\mathbb{R}(x)\big)$, $\phi \neq x$, and $\nu \geqslant 0$, so $\nu = \frac{1}{2} + m$ where $m \geqslant 1$. Then $\theta := 1/\phi^{\text{inv}}$ satisfies*

$$\theta' = -\theta^2(1 + y_1\theta^2 + \cdots + y_m\theta^{2m}) \quad \textit{where } y_i = (2i-1)!!\frac{(\nu, i)}{2^i} \textit{ for } i = 1, \dots, m$$

and $\theta \notin \operatorname{Li}\big(\mathbb{R}(x)\big)$.

Proof. By the Chain Rule and Proposition 7.6.16 we have

$$\theta' = -\theta^2(\phi^{\text{inv}})' = -\theta^2(\psi \circ \phi^{\text{inv}}) \quad \text{where } \psi = 1 + y_1x^{-2} + \cdots + y_mx^{-2m},$$

and this yields the first claim. Towards a contradiction, assume $\theta \in \operatorname{Li}\big(\mathbb{R}(x)\big)$. Then by [ADH, 10.6.6], θ lies in the Liouville extension $\operatorname{Li}\big(\mathbb{R}(x)\big)[i]$ of $\mathbb{C}(x) = \mathbb{R}(x)[i]$. Hence by Corollary 1.1.35, θ is algebraic over $\mathbb{C}(x)$. Also $\theta \notin \mathbb{C}$, so Lemma 1.1.36 yields $Q \in \mathbb{C}(Y)$ with $Q' = 1/P$ where $P := -Y^2(1 + y_1Y^2 + \cdots + y_mY^{2m})$. Thus

$$\phi' \;=\; \frac{1}{\psi} \;=\; -\frac{x^{-2}}{P(x^{-1})} \;=\; -x^{-2}Q'(x^{-1}) \;=\; Q(x^{-1})'$$

and so $\phi \in \mathbb{R}(x)$. This is impossible by the lemma below. $\square$

Lemma 7.6.23. *$\mu = 1 \iff \phi = x \iff \phi \in \mathbb{R}(x)$.*

Proof. The first equivalence is clear from the remarks following Proposition 7.6.1. Assume $\phi \in \mathbb{R}(x)$. Then by Proposition 7.6.20,

$$-\phi'^{\dagger} + 2\phi' i \;=\; a + bx^{-1} + 2\sum_{i=1}^{n}(x - c_i)^{-1} \quad (a, b \in \mathbb{C},\ \text{distinct } c_1, \dots, c_n \in \mathbb{C}^\times),$$

so $2\phi' i - a \in \partial F \cap \mathbb{C}F^\dagger$ for $F := \mathbb{C}(x)$, hence $2\phi' i = a$ by Corollary 1.2.14. Thus $\phi \in \mathbb{R}x + \mathbb{R}$. Since $\phi \sim x$ and $\phi - x \preccurlyeq x^{-1}$, this gives $\phi = x$. $\square$

Question. Does there exist a $\nu \notin \frac{1}{2} + \mathbb{Z}$ for which $\phi^{\text{inv}} \in \operatorname{Li}\big(\mathbb{R}(x)\big)$?

The Bessel functions. We can now establish some classical facts about distinguished solutions to the Bessel differential equation (B_ν): Corollaries 7.6.40, 7.6.41, 7.6.42 below. Our proofs use less complex analysis than those in the literature: we need just one contour integration, for Proposition 7.6.29 below. We assume some basic facts about Euler's Γ-function and recall that $1/\Gamma$ is an entire function with $-\mathbb{N}$ as its set of zeros (all simple), so Γ is meromorphic on the complex plane without any zeros and has $-\mathbb{N}$ as its set of poles. Our main reference for these and other properties of Γ used below is [123]. Let also $z \mapsto \log z \colon \mathbb{C} \setminus \mathbb{R}^{\leqslant} \to \mathbb{C}$ be the holomorphic extension of the real logarithm function, and for $z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$, set $z^\nu := \exp(\nu \log z)$. Let $\nu \in \mathbb{C}$ until further notice, and note that $(\nu, z) \mapsto z^\nu$ is analytic on $\mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant})$, and, keeping ν fixed, has derivative $z \mapsto \nu z^{\nu-1}$ on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$. Moreover, for $z \in \mathbb{C} \setminus \mathbb{R}$, $\nu, \nu_1, \nu_2 \in \mathbb{C}$, $t \in \mathbb{R}^>$ we have

$$z^{\nu_1+\nu_2} \ = \ z^{\nu_1} z^{\nu_2}, \quad (tz)^\nu \ = \ t^\nu z^\nu, \quad |z^\nu| \ = \ |z|^{\operatorname{Re}\nu}, \quad \overline{z^\nu} \ = \ \overline{z}^{\overline{\nu}},$$

and for $z_1, z_2 \in \mathbb{C}^\times$ with $\operatorname{Re} z_1 \geqslant 0$, $\operatorname{Re} z_2 > 0$: $z_1 z_2 \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$, $(z_1 z_2)^\nu = z_1^\nu z_2^\nu$.

Lemma 7.6.24. *Let $A, B \subseteq \mathbb{C}$ be nonempty and compact. Then*

$$\sum_n \max_{(\nu,z)\in A\times B} \left| \frac{(-1)^n}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n} \right| < \infty, \text{ so the series}$$
$$\sum_n \frac{(-1)^n}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n}$$

converges absolutely and uniformly on $A \times B$.

Proof. Take $R \in \mathbb{R}^>$ such that $|z| \leqslant 2R$ for all $z \in B$. Set $M_n := \max\limits_{\nu \in A} \left| \frac{1}{\Gamma(\nu+n+1)} \right|$ and take $n_0 \in \mathbb{N}$ such that $|\nu+n+1| \geqslant 1$ for all $n \geqslant n_0$ and $\nu \in A$. Then $M_{n+1} \leqslant M_n$ for $n \geqslant n_0$, by the functional equation for Γ, so the sequence (M_n) is bounded. Hence $\sum\limits_n \max\limits_{(\nu,z)\in A\times B} \left| \frac{(-1)^n}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n} \right| \leqslant \sum\limits_n \frac{M_n R^n}{n!} < \infty$. $\square$

By Lemma 7.6.24 and [123, V, §1, Theorem 1.1] we obtain a holomorphic function

$$z \mapsto J_\nu(z) \ := \ \sum_n \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} : \mathbb{C} \setminus \mathbb{R}^{\leqslant} \to \mathbb{C}.$$

For example, for $z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$, we have

$$J_{\frac{1}{2}}(z) \ = \ \sum_n \frac{(-1)^n}{n!\,\Gamma(n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+\frac{1}{2}}.$$

Note also that

$$J_{-m}(z) \ = \ \sum_{n \geqslant m} \frac{(-1)^n}{n!\,(n-m)!} \left(\frac{z}{2}\right)^{2n-m} \ = \ (-1)^m J_m(z) \tag{7.6.7}$$

and thus

$$J_{-m}(z) \sim \frac{(-1)^m}{m!} \left(\frac{z}{2}\right)^m, \quad J_m(z) \sim \frac{1}{m!} \left(\frac{z}{2}\right)^m \qquad \text{as } z \to 0.$$

Termwise differentiation shows that J_ν satisfies the differential equation (B_ν) on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$. The function J_ν is known as the *Bessel function of the first kind* of order ν.

Note that (B_ν) doesn't change when replacing ν by $-\nu$, so $J_{-\nu}$ is also a solution of (B_ν). Lemma 7.6.24 shows that the function

$$(\nu, z) \mapsto J_\nu(z) \ : \ \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant}) \to \mathbb{C}$$

is analytic, and that for fixed ν the function $z \mapsto z^{-\nu} J_\nu(z)$ on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$ extends to an entire function.

Termwise differentiation gives $(z^\nu J_\nu)' = z^\nu J_{\nu-1}$ and $(z^{-\nu} J_\nu)' = -z^{-\nu} J_{\nu+1}$, so

$$J_{\nu-1} \ = \ \frac{\nu}{z} J_\nu + J_\nu', \qquad J_{\nu+1} \ = \ \frac{\nu}{z} J_\nu - J_\nu', \tag{7.6.8}$$

by the Product Rule, and thus

$$J_{\nu-1} + J_{\nu+1} \ = \ \frac{2\nu}{z} J_\nu \qquad J_{\nu-1} - J_{\nu+1} \ = \ 2J_\nu'. \tag{7.6.9}$$

Note: if $\nu + 1 \notin -\mathbb{N}$, then for $z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$ and $z \to 0$ we have

$$J_\nu(z) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu, \ \text{ and for } \nu \neq 0: \ J_\nu'(z) \sim \frac{\nu}{2\Gamma(\nu+1)} \left(\frac{z}{2}\right)^{\nu-1}. \tag{7.6.10}$$

If $\nu - 1 \notin \mathbb{N}$, then (7.6.10) holds with $-\nu$ in place of ν. It follows that for $\nu \notin \mathbb{Z}$ the solutions J_ν, $J_{-\nu}$ of (B_ν) are $\mathbb{C}$-linearly independent. Set

$$w \ := \ \mathrm{wr}(J_\nu, J_{-\nu}) \ = \ J_\nu J_{-\nu}' - J_\nu' J_{-\nu}.$$

Then $w'(z) = -w(z)/z$ on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$ (cf. remarks following Lemma 5.2.4). This gives $c \in \mathbb{C}$ such that $w(z) = c/z$ on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$. If $\nu \notin \mathbb{Z}$, then $c \neq 0$, and by (7.6.10) and the remark following it we obtain $c = -\frac{2\nu}{\Gamma(\nu+1)\Gamma(-\nu+1)} = -\frac{2}{\Gamma(\nu)\Gamma(-\nu+1)}$. Hence using $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\pi\nu)$:

$$\mathrm{wr}(J_\nu, J_{-\nu})(z) \ = \ -\frac{2\sin(\pi\nu)}{\pi z} \quad \text{for } \nu \notin \mathbb{Z} \text{ and } z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}. \tag{7.6.11}$$

Next we express $J_{1/2}$ and $J_{-1/2}$ in terms of $\sin z$ and $\cos z$:

Lemma 7.6.25. *On $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$ we have*

$$J_{1/2}(z) \ = \ \sqrt{\frac{2}{\pi z}} \ \sin z, \qquad J_{-1/2}(z) \ = \ \sqrt{\frac{2}{\pi z}} \ \cos z.$$

Proof. For $\nu = 1/2$, a fundamental system of solutions of (B_ν) on $\mathbb{R}^>$ is given by $x^{-1/2}\cos x$, $x^{-1/2}\sin x$. This yields $a, b \in \mathbb{R}$ such that on $\mathbb{R}^>$,

$$J_{1/2}(t) \ = \ at^{-1/2}\cos t + bt^{-1/2}\sin t.$$

As $t \to 0^+$ we have:

$$t^{-1/2}\cos t \ = \ t^{-1/2} + O(t^{3/2}), \quad t^{-1/2}\sin t \ = \ t^{1/2} + O(t^{5/2}), \text{ and}$$

$$J_{1/2}(t) \sim \frac{1}{\Gamma(3/2)} \left(\frac{t}{2}\right)^{1/2} \ = \ \sqrt{\frac{2}{\pi}} \ t^{1/2} \ \ \text{(using (7.6.10))},$$

so $a = 0$, $b = \sqrt{\frac{2}{\pi}}$, giving the identity claimed for $J_{1/2}$. For $\nu = -1/2$ one can use the left identity (7.6.8) for $\nu = 1/2$. $\square$

From Lemma 7.6.25 and (7.6.8) we obtain by induction on n:

Corollary 7.6.26. *For each n there are $P_n, Q_n \in \mathbb{Q}[Z]$ with $\deg P_n = \deg Q_n = n$, both with positive leading coefficient, such that, with $Q_{-1} := 0$, we have on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$:*

$$\begin{aligned} J_{n+\frac{1}{2}}(z) &= \sqrt{\frac{2}{\pi z}}\,\big(P_n(z^{-1})\sin z - Q_{n-1}(z^{-1})\cos z\big), \\ J_{-n-\frac{1}{2}}(z) &= (-1)^n\sqrt{\frac{2}{\pi z}}\,\big(P_n(z^{-1})\cos z + Q_{n-1}(z^{-1})\sin z\big). \end{aligned}$$

For example,

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right).$$

For $\nu \notin \mathbb{Z}$ we have the solution

$$Y_\nu := \frac{\cos(\pi\nu)J_\nu - J_{-\nu}}{\sin(\pi\nu)}$$

of (B_ν) on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$. For fixed $z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$, the entire function

$$\nu \mapsto \cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)$$

has a zero at each $\nu \in \mathbb{Z}$, by (7.6.7), so the holomorphic function

$$\nu \mapsto Y_\nu(z)\colon \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$$

has a removable singularity at each $\nu \in \mathbb{Z}$, and thus extends to an entire function whose value at $k \in \mathbb{Z}$ is given by

$$Y_k(z) := \lim_{\nu \in \mathbb{C}\setminus\mathbb{Z},\ \nu \to k} Y_\nu(z), \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}.$$

In this way we obtain a two-variable analytic function

$$(\nu, z) \mapsto Y_\nu(z) \ :\ \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant}) \to \mathbb{C},$$

and thus for each $\nu \in \mathbb{C}$ a solution Y_ν of (B_ν) on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$, called the *Bessel function of the second kind* of order ν. Using (7.6.11) we determine the Wronskian of J_ν, Y_ν (first for $\nu \notin \mathbb{Z}$, and then by continuity for all ν):

$$\text{(7.6.12)} \qquad \mathrm{wr}(J_\nu, Y_\nu)(z) = -\frac{\mathrm{wr}(J_\nu, J_{-\nu})(z)}{\sin(\pi\nu)} = \frac{2}{\pi z} \quad (z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}),$$

hence J_ν, Y_ν are $\mathbb{C}$-linearly independent. The recurrence formulas (7.6.9) yield analogous formulas for the Bessel functions of the second kind:

$$\text{(7.6.13)} \qquad Y_{\nu-1} + Y_{\nu+1} = \frac{2\nu}{z}Y_\nu \qquad Y_{\nu-1} - Y_{\nu+1} = 2Y_\nu'.$$

Adding and subtracting these identities gives the analogue of (7.6.8)

$$\text{(7.6.14)} \qquad Y_{\nu-1} = \frac{\nu}{z}Y_\nu + Y_\nu', \qquad Y_{\nu+1} = \frac{\nu}{z}Y_\nu - Y_\nu'.$$

For $\nu \in \mathbb{R}$ we have $J_\nu(\mathbb{R}^{>}), Y_\nu(\mathbb{R}^{>}) \subseteq \mathbb{R}$, and for such ν we let J_ν, Y_ν denote also the germs (at $+\infty$) of their restrictions to $\mathbb{R}^{>}$. We can now state the main result of the rest of this section. It gives rather detailed information about the behavior of $J_\nu(t)$ and $Y_\nu(t)$ for $\nu \in \mathbb{R}$ and large $t \in \mathbb{R}^{>}$:

Theorem 7.6.27. *Let $\nu \in \mathbb{R}$. Then for the germs J_ν and Y_ν we have*

$$J_\nu = \sqrt{\frac{2}{\pi x \phi_\nu'}}\ \cos\left(\phi_\nu - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \qquad Y_\nu = \sqrt{\frac{2}{\pi x \phi_\nu'}}\ \sin\left(\phi_\nu - \frac{\pi\nu}{2} - \frac{\pi}{4}\right).$$

The proof will take some effort, especially for Y_ν with $\nu \in \mathbb{Z}$.

Below J denotes the analytic function $(\nu, z) \mapsto J_\nu(z) \colon \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant}) \to \mathbb{C}$.

Lemma 7.6.28. *Let $k \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$. Then*

$$Y_k(z) \;=\; \frac{1}{\pi}\left(\left(\frac{\partial J}{\partial \nu}\right)(k, z) + (-1)^k \left(\frac{\partial J}{\partial \nu}\right)(-k, z)\right).$$

In particular, $Y_{-k} = (-1)^k Y_k$ and $Y_0(z) = \dfrac{2}{\pi}\left(\dfrac{\partial J}{\partial \nu}\right)(0, z)$.

Proof. By l'Hôpital's Rule for germs of holomorphic functions at k,

$$\begin{aligned}
&\lim_{\nu \to k} \frac{\cos(\pi\nu) J(\nu, z) - J(-\nu, z)}{\sin(\pi\nu)} \\
&= \lim_{\nu \to k} \frac{-\pi \sin(\pi\nu) J(\nu, z) + \cos(\pi\nu)(\partial J/\partial\nu)(\nu, z) + (\partial J/\partial\nu)(-\nu, z)}{\pi \cos(\pi\nu)} \\
&= \frac{1}{\pi} \lim_{\nu \to k} \left((\partial J/\partial\nu)(\nu, z) + \frac{(\partial J/\partial\nu)(-\nu, z)}{\cos(\pi\nu)}\right),
\end{aligned}$$

and this yields the claims. □

The following asymptotic relation is crucial for establishing Theorem 7.6.27. It is due to Hankel [82] with earlier special cases provided by Poisson [155] ($\nu = 0$), Hansen [83] ($\nu = 1$) and Jacobi [108] ($\nu \in \mathbb{Z}$).

Proposition 7.6.29 (Hankel)**.** *Let $\nu \in \mathbb{R}$. Then for the germ J_ν we have:*

$$J_\nu - \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \;\preccurlyeq\; x^{-3/2}.$$

Proposition 7.6.29 with Lemma 7.6.9 and Remark 7.6.11 yield the identity for the germ J_ν in Theorem 7.6.27. As to Y_ν, let us simplify notation by setting $S := \sqrt{\frac{2}{\pi x \phi_\nu'}}$, $\alpha := \frac{\pi\nu}{2}$, $\theta := \phi_\nu - \frac{\pi}{4}$. Using the identity for J_ν in Theorem 7.6.27, the numerator in the definition of Y_ν turns into

$$S \cdot \big[\cos(2\alpha)\cos(\theta - \alpha) - \cos(\theta + \alpha)\big]$$

and the denominator into $\sin(2\alpha)$. Trigonometric addition formulas yield

$$\cos(2\alpha)\cos(\theta - \alpha) - \cos(\theta + \alpha) \;=\; \sin(2\alpha)\sin(\theta - \alpha).$$

For $\nu \notin \mathbb{Z}$, we have $\sin(2\alpha) \neq 0$, so this gives the identity for the germ Y_ν in Theorem 7.6.27. The identity for Y_ν with $\nu \in \mathbb{Z}$ will be dealt with after the proof of Proposition 7.6.29. First a useful reduction step:

Lemma 7.6.30. *Let $\nu \in \mathbb{R}$, and let J_ν, $J_{\nu+1}$ denote also the germs (at $+\infty$) of their restrictions to $\mathbb{R}^>$. Then the following are equivalent:*

(i) $J_\nu - \sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{\pi\nu}{2} - \dfrac{\pi}{4}\right) \;\preccurlyeq\; x^{-3/2}$;

(ii) $J_{\nu+1} - \sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{\pi(\nu+1)}{2} - \dfrac{\pi}{4}\right) \;\preccurlyeq\; x^{-3/2}$.

Proof. Put $\alpha_\nu := \frac{\pi\nu}{2} + \frac{\pi}{4}$ and $g_\nu := \sqrt{\frac{2}{\pi x \phi'_\nu}} \in \mathrm{D}(\mathbb{Q})$. The proof of Lemma 7.6.8 gives $\frac{1}{\sqrt{\phi'_\nu}} - 1 \preccurlyeq x^{-2}$, so $g_\nu - \sqrt{\frac{2}{\pi x}} \preccurlyeq x^{-5/2}$, hence $g_\nu \preccurlyeq x^{-1/2}$ and thus $g'_\nu \preccurlyeq x^{-3/2}$. Assume now (i). By (7.6.8) we have

$$J_{\nu+1} \;=\; \frac{\nu}{x} J_\nu - J'_\nu \tag{7.6.15}$$

and by (i)

$$\frac{1}{x} J_\nu \;=\; \sqrt{\frac{2}{\pi}}\, x^{-3/2}\cos(x-\alpha_\nu) + O(x^{-5/2}) \;=\; O(x^{-3/2}). \tag{7.6.16}$$

We have $J_\nu \in V_\nu$, so by Lemma 7.6.9 and (ii),

$$J_\nu \;=\; g_\nu \cos(\phi_\nu - \alpha_\nu),$$

and $\alpha_{\nu+1} = \alpha_\nu + \frac{\pi}{2}$ gives $\sin(t-\alpha_\nu) = \cos(t - \alpha_{\nu+1})$. Thus

$$\begin{aligned} -J'_\nu \;&=\; -g'_\nu \cos(\phi_\nu - \alpha_\nu) + g_\nu \phi'_\nu \sin(\phi_\nu - \alpha_\nu) \\ &=\; g_\nu \phi'_\nu \cos(\phi_\nu - \alpha_{\nu+1}) + O(x^{-3/2}). \end{aligned}$$

Also $\cos(x+u) - \cos x \preccurlyeq u$ for $u \in \mathcal{C}$ and $\phi_\nu - x \preccurlyeq x^{-1}$, so $\cos(\phi_\nu - \alpha_{\nu+1}) = \cos(x - \alpha_{\nu+1}) + O(x^{-1})$. Using $\phi'_\nu - 1 \preccurlyeq x^{-2}$ and $g_\nu - \sqrt{\frac{2}{\pi x}} \preccurlyeq x^{-5/2}$ this yields

$$g_\nu \phi'_\nu \cos(\phi_\nu - \alpha_{\nu+1}) \;=\; \sqrt{\frac{2}{\pi x}} \cos(x - \alpha_{\nu+1}) + O(x^{-3/2})$$

and so

$$-J'_\nu \;=\; \sqrt{\frac{2}{\pi x}} \cos(x - \alpha_\nu) + O(x^{-3/2}). \tag{7.6.17}$$

Combining (7.6.15), (7.6.16), (7.6.17) yields (ii). Likewise one proves (ii) $\Rightarrow$ (i), using

$$J_\nu \;=\; \frac{\nu+1}{x} J_{\nu+1} - J'_{\nu+1}$$

instead of (7.6.15). $\square$

Remark. Using the identities (7.6.14) instead of (7.6.8) shows that Lemma 7.6.30 also holds with sin, Y_ν, $Y_{\nu+1}$ in place of cos, J_ν, $J_{\nu+1}$.

Lemma 7.6.30 gives a reduction of Proposition 7.6.29 to the case $\nu > -1/2$. (We could also reduce to the case $\nu > 1$, say, but the choice of $-1/2$ is useful later.)

Lemma 7.6.31 (Poisson representation)**.** *Let* $\operatorname{Re}\nu > -\frac{1}{2}$ *and* $z \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$. *Then*

$$J_\nu(z) \;=\; \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} \mathrm{e}^{tzi}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

Proof. For $p, q \in \mathbb{C}$ with $\operatorname{Re} p, \operatorname{Re} q > 0$ and $B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ we have

$$B(p,q) \;=\; \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

see for example [144, Chapter 2, §1.6]. From the definition of $B(p,q)$ we obtain

$$\int_{-1}^{1} t^{2n}(1-t^2)^{\nu-\frac{1}{2}}\,dt \;=\; \int_0^1 s^{n-\frac{1}{2}}(1-s)^{\nu-\frac{1}{2}}\,ds \;=\; B\big(n+\tfrac{1}{2}, \nu+\tfrac{1}{2}\big).$$

In the equalities below we use this for the fourth equality (and one can appeal to a Dominated Convergence Theorem for the second):

$$\begin{aligned}
\int_{-1}^{1} \mathrm{e}^{tzi}(1-t^2)^{\nu-\frac{1}{2}}\,dt &= \int_{-1}^{1} \sum_m (1-t^2)^{\nu-\frac{1}{2}} t^m \frac{(iz)^m}{m!}\,dt \\
&= \sum_m \left(\int_{-1}^{1} t^m (1-t^2)^{\nu-\frac{1}{2}}\,dt\right) \frac{(iz)^m}{m!} \\
&= \sum_n \left(\int_{-1}^{1} t^{2n} (1-t^2)^{\nu-\frac{1}{2}}\,dt\right) \frac{(-1)^n z^{2n}}{(2n)!} \\
&= \sum_n B\big(n+\tfrac{1}{2}, \nu+\tfrac{1}{2}\big) \frac{(-1)^n z^{2n}}{(2n)!} \\
&= \Gamma\big(\nu+\tfrac{1}{2}\big) \sum_n \frac{(-1)^n 2^{2n}\,\Gamma\big(n+\tfrac{1}{2}\big)}{\Gamma(n+\nu+1)(2n)!} \left(\frac{z}{2}\right)^{2n} \\
&= \Gamma\big(\nu+\tfrac{1}{2}\big)\sqrt{\pi} \sum_n \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n},
\end{aligned}$$

where for the last equality we used $\Gamma\big(n+\frac{1}{2}\big) = \sqrt{\pi}\ (2n)!/(n!2^{2n})$, a consequence of the Gauss-Legendre duplication formula for the Gamma function (see [123, XV, §2, Γ8]). $\square$

We also need the following estimate:

Lemma 7.6.32. *Let $\lambda, t \in \mathbb{R}$ with $\lambda > -1$ and $t \geqslant 1$. Then*

$$t^{-(\lambda+1)}\Gamma(\lambda+1) \;=\; \int_0^\infty \mathrm{e}^{-st}\, s^\lambda\, ds \;\leqslant\; \int_0^1 \mathrm{e}^{-st}\, s^\lambda\, ds + \Gamma(\lambda+1)\,\mathrm{e}^{1-t},$$

and thus $\displaystyle\int_1^\infty \mathrm{e}^{-st}\, s^\lambda\, ds \;\leqslant\; \Gamma(\lambda+1)\,\mathrm{e}^{1-t}$.

Proof. We have

$$\int_0^\infty \mathrm{e}^{-st}\, s^\lambda\, ds \;=\; \int_0^\infty \mathrm{e}^{-u} \left(\frac{u}{t}\right)^\lambda \frac{du}{t} \;=\; t^{-(\lambda+1)}\Gamma(\lambda+1)$$

and

$$\int_1^\infty \mathrm{e}^{-st}\, s^\lambda\, ds \;=\; \mathrm{e}^{-t} \int_1^\infty \mathrm{e}^{-t(s-1)}\, s^\lambda\, ds \;\leqslant\; \mathrm{e}^{-t} \int_1^\infty \mathrm{e}^{-(s-1)}\, s^\lambda\, ds.$$

Now use $\displaystyle\int_1^\infty \mathrm{e}^{-(s-1)}\, s^\lambda\, ds = \mathrm{e} \int_1^\infty \mathrm{e}^{-s}\, s^\lambda\, ds \leqslant \mathrm{e} \int_0^\infty \mathrm{e}^{-s}\, s^\lambda\, ds = \mathrm{e}\,\Gamma(\lambda+1).$ $\square$

Corollary 7.6.33. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -1$ and $t \in \mathbb{R}^{\geqslant 1}$. Then*

$$\left|\int_0^1 \mathrm{e}^{-st}\, s^\lambda\, ds \;-\; t^{-(\lambda+1)}\Gamma(\lambda+1)\right| \;\leqslant\; \Gamma\big(\operatorname{Re}(\lambda)+1\big)\,\mathrm{e}^{1-t}.$$

Proof. The identities in the beginning of the proof of Lemma 7.6.32 generalize to

$$\int_0^\infty \mathrm{e}^{-st}\, s^\lambda\, ds \;=\; t^{-(\lambda+1)}\Gamma(\lambda+1).$$

Hence

$$\left|\int_0^1 \mathrm{e}^{-st}\, s^\lambda\, ds \;-\; t^{-(\lambda+1)}\Gamma(\lambda+1)\right| \;=\; \left|\int_1^\infty \mathrm{e}^{-st}\, s^\lambda\, ds\right| \;\leqslant\; \int_1^\infty \mathrm{e}^{-st}\, s^{\operatorname{Re}\lambda}\, ds,$$

and now use Lemma 7.6.32. □

By Lemma 7.6.30 the next result is more than enough to give Proposition 7.6.29. The proof is classical and uses Laplace's method, cf. [144, Chapter 3, §7].

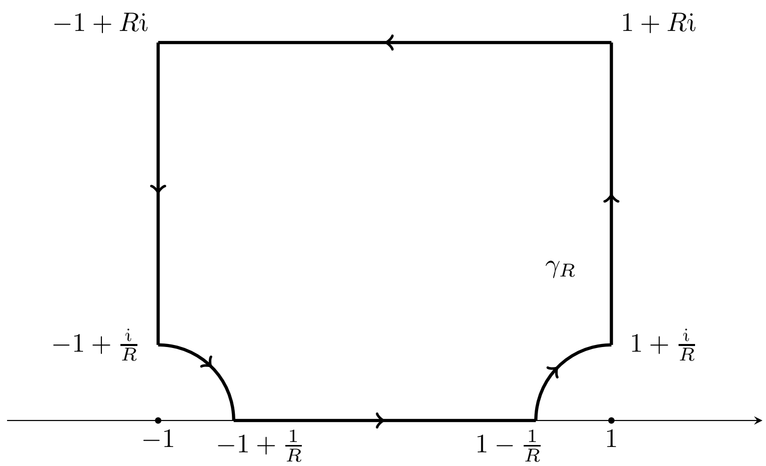
Lemma 7.6.34. *Suppose* $\operatorname{Re}\nu > -\frac{1}{2}$, *and let* t *range over* $\mathbb{R}^{\geqslant 1}$. *Then*

$$J_\nu(t) \;=\; \sqrt{\frac{2}{\pi t}}\,\cos\left(t - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O(t^{-\frac{3}{2}}) \quad \text{as } t \to +\infty.$$

Proof. We consider the holomorphic function

$$z \mapsto f_{\nu,t}(z) := \mathrm{e}^{tz\mathrm{i}}(1-z^2)^{\nu-\frac{1}{2}} \colon \mathbb{C}\setminus(\mathbb{R}^{\leqslant -1}\cup\mathbb{R}^{\geqslant 1}) \to \mathbb{C},$$

and set $I_\nu(t) := \displaystyle\int_{-1}^1 f_{\nu,t}(s)\, ds$. By Lemma 7.6.31 we have $J_\nu(t) = \dfrac{(\frac{t}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} I_\nu(t)$. To determine the asymptotic behavior of $I_\nu(t)$ as $t \to +\infty$ we integrate along the contour γ_R depicted below, where R is a real number > 1.



By Cauchy, $\int_{\gamma_R} f_{\nu,t}(z)\, dz = 0$ (see [123, III, §5]), and letting $R \to +\infty$ we obtain $I_\nu(t) = I_\nu^-(t) - I_\nu^+(t)$ where

$$I_\nu^-(t) \;:=\; \mathrm{i}\int_0^\infty f_{\nu,t}(-1+\mathrm{i}s)\, ds, \qquad I_\nu^+(t) \;:=\; \mathrm{i}\int_0^\infty f_{\nu,t}(1+\mathrm{i}s)\, ds.$$

Now

$$I_\nu^+(t) \;=\; \mathrm{i}\int_0^\infty \mathrm{e}^{t(1+s\mathrm{i})\mathrm{i}}\left(1-(1+s\mathrm{i})^2\right)^{\nu-\frac{1}{2}} ds \;=\; \mathrm{i}\,\mathrm{e}^{t\mathrm{i}}\int_0^\infty \mathrm{e}^{-st}(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}}\, ds.$$

The complex analytic function $(\kappa, z) \mapsto \left(1+\frac{\mathrm{i}}{2}z\right)^\kappa - 1$ on $\mathbb{C}\times\{z\in\mathbb{C} : |z|<2\}$ vanishes on the locus $z=0$, so we have a complex analytic function $(\kappa,z)\mapsto r_\kappa(z)$ on the same region such that $\left(1+\frac{\mathrm{i}}{2}z\right)^\kappa - 1 = z r_\kappa(z)$ for all (κ, z) in this region. For $\kappa = \nu - \frac{1}{2}$ this yields a continuous function $r\colon [0,1]\to\mathbb{C}$ such that $\left(1+\frac{\mathrm{i}}{2}s\right)^{\nu-\frac{1}{2}} =$

$1+r(s)s$ for all $s\in[0,1]$, and $r(0)=(\nu-\frac{1}{2})\frac{\mathrm{i}}{2}$. In view of an identity stated just before Lemma 7.6.24 this yields for $s\in(0,1]$:

$$(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}} \;=\; (-2s\mathrm{i})^{\nu-\frac{1}{2}}\big(1+\tfrac{\mathrm{i}}{2}s\big)^{\nu-\frac{1}{2}} \;=\; (-2s\mathrm{i})^{\nu-\frac{1}{2}}+r(s)s(-2s\mathrm{i})^{\nu-\frac{1}{2}},$$

so

$$\int_0^1 \mathrm{e}^{-st}(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds \;=\; \int_0^1 \mathrm{e}^{-st}(-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds+\int_0^1 \mathrm{e}^{-st}\,r(s)s(-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds.$$

By Corollary 7.6.33, as $t\to+\infty$,

$$\int_0^1 \mathrm{e}^{-st}(-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds \;=\; (-2\mathrm{i})^{\nu-\frac{1}{2}}t^{-(\nu+\frac{1}{2})}\Gamma\big(\nu+\tfrac{1}{2}\big)+O(\mathrm{e}^{-t}).$$

Take $C\in\mathbb{R}^{>}$ such that $|r(s)|\leqslant C$ for all $s\in[0,1]$, and set $\lambda:=\mathrm{Re}(\nu)-\frac{1}{2}$, so $\lambda>-1$. Then by Lemma 7.6.32,

$$\begin{aligned}\left|\int_0^1 \mathrm{e}^{-st}\,r(s)s(-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds\right| \;&\leqslant\; 2^\lambda C\int_0^1 \mathrm{e}^{-st}\,s^{\lambda+1}\,ds\\ &\leqslant\; 2^\lambda C\int_0^\infty \mathrm{e}^{-st}\,s^{\lambda+1}\,ds\\ &=\; 2^\lambda C\,\Gamma(\lambda+2)t^{-\lambda-2}.\end{aligned}$$

Hence, as $t\to+\infty$,

$$\int_0^1 \mathrm{e}^{-st}(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds \;=\; (-2\mathrm{i})^{\lambda}t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)+O(t^{-\lambda-2}). \tag{7.6.18}$$

Next, take $D\in\mathbb{R}^{>}$ such that $|(1-2\frac{\mathrm{i}}{s})^{\nu-\frac{1}{2}}|\leqslant D$ for all $s\in[1,\infty)$. For such s we have $|(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}}|\leqslant Ds^{2\lambda}\leqslant Ds^{2\,\mathrm{Re}\,\nu}$ and thus

$$\left|\int_1^\infty \mathrm{e}^{-st}(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds\right| \;\leqslant\; D\int_1^\infty \mathrm{e}^{-st}\,s^{2\,\mathrm{Re}\,\nu}\,ds,$$

hence by Lemma 7.6.32:

$$\int_1^\infty \mathrm{e}^{-st}(s^2-2s\mathrm{i})^{\nu-\frac{1}{2}}\,ds \;=\; O(\mathrm{e}^{-t})\quad\text{as } t\to+\infty. \tag{7.6.19}$$

Combining (7.6.18) and (7.6.19) yields

$$I_\nu^+(t) \;=\; \mathrm{i}(-2\mathrm{i})^{\nu-\frac{1}{2}}\,\mathrm{e}^{t\mathrm{i}}\,t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)+O(t^{-\lambda-2})\quad\text{as } t\to+\infty.$$

In the same way we obtain

$$I_\nu^-(t) \;=\; \mathrm{i}(2\mathrm{i})^{\nu-\frac{1}{2}}\,\mathrm{e}^{-t\mathrm{i}}\,t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)+O(t^{-\lambda-2})\quad\text{as } t\to+\infty.$$

Thus as $t\to+\infty$:

$$\begin{aligned}I_\nu(t) \;&=\; I_\nu^-(t)-I_\nu^+(t)\\ &=\; 2^{\nu-\frac{1}{2}}i\big[i^{\nu-\frac{1}{2}}\,\mathrm{e}^{-ti}-(-i)^{\nu-\frac{1}{2}}\,\mathrm{e}^{ti}\,\big]t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)+O(t^{-\lambda-2}).\end{aligned}$$

Using $\mathrm{i}^{\nu-\frac{1}{2}}=\mathrm{e}^{\frac{1}{2}(\nu-\frac{1}{2})\pi\mathrm{i}}$ and the like we have

$$\mathrm{i}\big[\mathrm{i}^{\nu-\frac{1}{2}}\,\mathrm{e}^{-t\mathrm{i}}-(-\mathrm{i})^{\nu-\frac{1}{2}}\,\mathrm{e}^{t\mathrm{i}}\,\big] \;=\; 2\cos\Big(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\Big),\ \text{and thus}$$

$$J_\nu(t) \;=\; \frac{(\frac{t}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\,I_\nu(t) \;=\; \sqrt{\frac{2}{\pi t}}\,\cos\Big(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\Big)+O(t^{-\frac{3}{2}})$$

as $t\to+\infty$. □

Here is a consequence of Hankel's result, cf. Lommel [133, p. 67]:

Corollary 7.6.35. *Let $\nu \in \mathbb{R}$. Then* $J_\nu^2 + J_{\nu+1}^2 = \dfrac{2}{\pi x} + O(x^{-2})$.

Proof. With $\alpha_\nu := \frac{\pi\nu}{2} + \frac{\pi}{4}$ we have by Proposition 7.6.29,

$$J_\nu \ = \ \sqrt{\frac{2}{\pi x}}\cos(x-\alpha_\nu) + O(x^{-3/2}), \qquad J_{\nu+1} \ = \ \sqrt{\frac{2}{\pi x}}\cos(x-\alpha_{\nu+1}) + O(x^{-3/2}).$$

Now use $\sin(x-\alpha_\nu) = \cos(x-\alpha_{\nu+1})$. $\square$

Since Proposition 7.6.29 is now established, so is Theorem 7.6.27, except for the Y_ν-identity when $\nu \in \mathbb{Z}$. To treat that case, and also for use in the next subsection, we now prove a uniform version of Lemma 7.6.34:

Lemma 7.6.36. *Let $\nu_0 \in \mathbb{C}$ and $\operatorname{Re}\nu_0 > -\frac{1}{2}$. Then there are reals $\varepsilon > 0$, $t_0 \geqslant 1$, and $C_0 > 0$, such that for all $\nu \in \mathbb{C}$ with $|\nu - \nu_0| < \varepsilon$ and all $t \geqslant t_0$:*

$$\operatorname{Re}\nu \ > \ -\frac{1}{2}, \qquad \left| J_\nu(t) - \sqrt{\frac{2}{\pi t}}\cos\Big(t - \frac{\pi\nu}{2} - \frac{\pi}{4}\Big)\right| \ \leqslant \ C_0 t^{-\frac{3}{2}}.$$

Proof. We follow the proof of Lemma 7.6.34, where in the beginning we introduced the complex analytic function $(\nu, z) \mapsto r_\nu(z)$ on $\mathbb{C} \times \big\{z \in \mathbb{C} : |z| < 2\big\}$. Take $\varepsilon \in \mathbb{R}^{>}$ and $C \in \mathbb{R}^{\geqslant 1}$ such that $0 < \varepsilon < \operatorname{Re}(\nu_0) + \frac{1}{2}$ and $|r_{\nu-\frac{1}{2}}(s)| \leqslant C$ for all $(\nu, s) \in B_0 \times [-1,1]$ where $B_0 := \{\nu \in \mathbb{C} : |\nu - \nu_0| < \varepsilon\}$. (To handle I_ν^+ we use this for $s \in [0,1]$, and to deal with I_ν^- we use $s \in [-1,0]$.) Take also have $D \in \mathbb{R}^{>}$ such that $|(1 - 2\frac{i}{s})^{\nu-\frac{1}{2}}| \leqslant D$ for all $\nu \in B_0$ and $s \geqslant 1$ (and thus also for $\nu \in B_0$ and $s \leqslant -1$). Next, set $\lambda_0 := \operatorname{Re}(\nu_0) - \frac{1}{2}$, and take $t_0 \geqslant 1$ such that $\mathrm{e}^{t-1} \geqslant t^{\lambda_0+\varepsilon+2}$ for all $t \geqslant t_0$. Below ν ranges over B_0 and t over $\mathbb{R}^{\geqslant t_0}$, and $\lambda := \operatorname{Re}(\nu) - \frac{1}{2}$, so $\lambda > -1$. Then, as in the proof of Lemma 7.6.34:

$$\left|\int_0^1 \mathrm{e}^{-st}\, r_{\nu-\frac{1}{2}}(s)s(-2si)^{\nu-\frac{1}{2}}\,ds\right| \ \leqslant \ 2^\lambda C\,\Gamma(\lambda+2)t^{-\lambda-2}.$$

Take $C_\Gamma \in \mathbb{R}^{>}$ such that $2^\lambda C\,\Gamma(\lambda+2) \leqslant C_\Gamma$ for all ν. Then

$$\left|\int_0^1 \mathrm{e}^{-st}\, r_{\nu-\frac{1}{2}}(s)s(-2si)^{\nu-\frac{1}{2}}\,ds\right| \ \leqslant \ C_\Gamma\, t^{-\lambda-2}.$$

By increasing C_Γ we arrange that $C_\Gamma \geqslant C$ and that for all ν,

$$2^\lambda\,\Gamma(\lambda+1),\, D\,\Gamma(2\lambda+2) \ \leqslant \ C_\Gamma.$$

We have $\mathrm{e}^{1-t} \leqslant t^{-\lambda-2}$, so by Corollary 7.6.33:

$$\left|\int_0^1 \mathrm{e}^{-st}(-2si)^{\nu-\frac{1}{2}}\,ds - (-2i)^{\nu-\frac{1}{2}}t^{-(\nu+\frac{1}{2})}\Gamma(\nu+\tfrac{1}{2})\right| \ \leqslant \ C_\Gamma\, t^{-\lambda-2}.$$

Combining this with an earlier displayed inequality yields:

$$\left|\int_0^1 \mathrm{e}^{-st}(s^2-2si)^{\nu-\frac{1}{2}}\,ds - (-2i)^{\nu-\frac{1}{2}}t^{-\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})\right| \ \leqslant \ 2C_\Gamma\, t^{-\lambda-2}. \tag{7.6.20}$$

As in the proof of Proposition 7.6.29 and using Lemma 7.6.32 we also have

$$\begin{aligned}\left|\int_1^\infty \mathrm{e}^{-st}(s^2-2si)^{\nu-\frac{1}{2}}\,ds\right| \ &\leqslant \ D\int_1^\infty \mathrm{e}^{-st}\,s^{2\operatorname{Re}(\nu)}\,ds \\ &\leqslant \ D\Gamma(2\lambda+2)\,\mathrm{e}^{1-t} \ \leqslant \ C_\Gamma\, t^{-\lambda-2}.\end{aligned}$$

Combining this with (7.6.20) we obtain:

$$\left|I_\nu^+(t) - \mathrm{i}(-2\mathrm{i})^{\nu-\frac{1}{2}}\,\mathrm{e}^{t\mathrm{i}}\,t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)\right| \;\leqslant\; 3C_\Gamma\,t^{-\lambda-2}.$$

In the same way,

$$\left|I_\nu^-(t) - \mathrm{i}(2\mathrm{i})^{\nu-\frac{1}{2}}\,\mathrm{e}^{-t\mathrm{i}}\,t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)\right| \;\leqslant\; 3C_\Gamma\,t^{-\lambda-2}$$

and so as in the proof of Proposition 7.6.29:

$$\left|I_\nu(t) - 2^{\nu+\frac{1}{2}}t^{-\nu-\frac{1}{2}}\Gamma\big(\nu+\tfrac{1}{2}\big)\cos\big(t-\tfrac{\pi\nu}{2}-\tfrac{\pi}{4}\big)\right| \;\leqslant\; 6C_\Gamma\,t^{-\lambda-2}.$$

Hence

$$\left|J_\nu(t) - \sqrt{\frac{2}{\pi t}}\,\cos\left(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\right| \;\leqslant\; \frac{6C_\Gamma}{\sqrt{\pi}\,2^{\mathrm{Re}(\nu)}|\Gamma(\nu+\frac{1}{2})|}t^{-\frac{3}{2}}.$$

Thus ε, t_0 as chosen, and a suitable C_0 have the required properties. □

To finish the proof of Theorem 7.6.27, it suffices by Lemma 7.6.9 and Remark 7.6.11 to show the following:

Lemma 7.6.37. *Let $k \in \mathbb{Z}$. Then for the germ Y_k we have:*

$$Y_k - \sqrt{\frac{2}{\pi x}}\,\sin\left(x-\frac{\pi k}{2}-\frac{\pi}{4}\right) \;\preccurlyeq\; x^{-\frac{3}{2}}.$$

Proof. By the remark after the proof of Lemma 7.6.30 it is enough to treat the case $k=0$. Lemma 7.6.36 with $\nu_0=0$ yields reals $t_0 \geqslant 1$, $C_0>0$, and ε with $0<\varepsilon<\frac{1}{2}$ such that for all $\nu\in\mathbb{C}$ with $|\nu|<\varepsilon$ and all $t\geqslant t_0$:

$$\left|J_\nu(t) - \sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\right| \;\leqslant\; C_0 t^{-\frac{3}{2}}.$$

Let $t\geqslant t_0$ be fixed and consider the entire function d given by

$$\begin{aligned} d(\nu) \;&:=\; J(\nu,t) - \sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\right), \text{ so} \\ d'(\nu) \;&=\; \left(\frac{\partial J}{\partial\nu}\right)(\nu,t) - \sqrt{\frac{\pi}{2t}}\sin\left(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\right) \end{aligned}$$

and hence by Lemma 7.6.28:

$$d'(0) \;=\; \left(\frac{\partial J}{\partial\nu}\right)(0,t) - \sqrt{\frac{\pi}{2t}}\sin\left(t-\frac{\pi}{4}\right) = \frac{\pi}{2}\left[Y_0(t) - \sqrt{\frac{2}{\pi t}}\sin\left(t-\frac{\pi}{4}\right)\right].$$

Also $|d'(0)| \leqslant \frac{1}{\varepsilon}\max_{|\nu|=\varepsilon}|d(\nu)|$, a Cauchy inequality, and thus

$$\left|Y_0(t) - \sqrt{\frac{2}{\pi t}}\sin\left(t-\frac{\pi}{4}\right)\right| \;=\; \frac{2}{\pi}\,|d'(0)| \;\leqslant\; \frac{2C_0}{\pi\varepsilon}t^{-\frac{3}{2}},$$

which gives the desired result for $k=0$ and thus for all $k\in\mathbb{Z}$. □

In the rest of this subsection we let ν range over $\mathbb{R}$ and derive some consequences of Theorem 7.6.27. Toward showing that the germ $\psi_\nu \in \mathrm{E}(\mathbb{Q})$ depends analytically on ν we introduce the real analytic function $\Psi\colon \mathbb{R}\times\mathbb{R}^> \to \mathbb{R}^>$ by

$$\Psi(\nu,t) \;:=\; \frac{\pi t}{2}\left[J(\nu,t)^2 + Y(\nu,t)^2\right],$$

and let $\Psi(\nu,-)$ be the function $t\mapsto\Psi(\nu,t)\colon \mathbb{R}^>\to\mathbb{R}^>$. Then by Theorem 7.6.27:

Corollary 7.6.38. *ψ_ν is the germ at $+\infty$ of $\Psi(\nu,-)$.*

For ϕ_ν we consider the real analytic function

$$\widetilde{\Phi}\ :\ \mathbb{R}\times\mathbb{R}^> \to \mathbb{R}, \quad (\nu,t)\mapsto \int_1^t \frac{1}{\Psi(\nu,s)}\,ds.$$

Let $\widetilde{\Phi}_\nu$ be the germ (at $+\infty$) of $t\mapsto \widetilde{\Phi}(\nu,t)$. Then $\widetilde{\Phi}_\nu' = \frac{1}{\psi_\nu} = \phi_\nu'$, so $\phi_\nu = \widetilde{\Phi}_\nu + c_\nu$ where c_ν is a real constant. To determine this constant we note that by Proposition 7.6.16 we have $1-\frac{1}{\psi_\nu}\preccurlyeq x^{-2}$, which gives the real number

$$\widetilde{c}_\nu\ :=\ \int_1^\infty\left(1-\frac{1}{\Psi(\nu,s)}\right)ds.$$

We also set $\widetilde{c}(\nu,t)=\int_1^t\left(1-\frac{1}{\Psi(\nu,s)}\right)ds$, so $\widetilde{c}(\nu,t)\to\widetilde{c}_\nu$ as $t\to+\infty$, and for $t>0$ we have $\widetilde{\Phi}(\nu,t)+\widetilde{c}(\nu,t)=t-1$. Taking germs we obtain $\widetilde{\Phi}_\nu+\widetilde{c}_\nu+1-x\prec 1$. Also $\phi_\nu - x\prec 1$, so $\widetilde{\Phi}_\nu + c_\nu - x\prec 1$, and thus $c_\nu=\widetilde{c}_\nu+1$. This suggests the function

$$\Phi:\mathbb{R}\times\mathbb{R}^>\to\mathbb{R},\quad (\nu,t)\mapsto\widetilde{\Phi}(\nu,t)+\widetilde{c}_\nu+1.$$

The above arguments yield:

Corollary 7.6.39. *For each ν the germ of $\Phi(\nu,-)$ is ϕ_ν.*

Thus as for ψ_ν the germ ϕ_ν has a unique real analytic representative on $\mathbb{R}^>$, namely $\Phi(\nu,-)$. Note that Φ is real analytic iff the function $\nu\mapsto\widetilde{c}_\nu\colon\mathbb{R}\to\mathbb{R}$ is real analytic, but we don't even know if this last function is continuous.

For the next result, cf. [205, §13.74]:

Corollary 7.6.40. *$J_\nu^2+Y_\nu^2$ is eventually strictly decreasing, $x(J_\nu^2+Y_\nu^2)$ is eventually strictly increasing if $|\nu|<1/2$ and eventually strictly decreasing if $|\nu|>1/2$.*

Proof. We have $\psi_\nu\sim 1+\frac{\mu-1}{8}x^{-2}$ by Proposition 7.6.16, and ψ_ν is hardian, thus $\psi_\nu'\preccurlyeq x^{-3}$, and so $(x^{-1}\psi_\nu)'=-x^{-2}\psi_\nu+x^{-1}\psi_\nu'\sim -x^{-2}$. This yields the claims. □

Corollary 7.6.41 (Schafheitlin [179, p. 86])**.** *If $\nu>1/2$, then, as elements of $\mathrm{D}(\mathbb{Q})$,*

$$\frac{2/\pi}{x}\ <\ J_\nu^2+Y_\nu^2\ <\ \frac{2/\pi}{(x^2-\nu^2)^{1/2}}.$$

Proposition 7.6.16 also yields (cf. [205, §13.75] or [144, Chapter 9, §9]):

Corollary 7.6.42. *The germ $J_\nu^2+Y_\nu^2$ has the asymptotic expansion*

$$J_\nu^2+Y_\nu^2\ \sim\ \frac{2}{\pi x}\sum_n(2n-1)!!\frac{(\nu,n)}{2^n}x^{-2n}.$$

Remark. Nicholson [142, 141] (see [205, §§13.73–13.75]) established Corollary 7.6.42, but "the analysis is difficult" [144, p. 340]. A simpler deduction of the integral representation of $J_\nu^2+Y_\nu^2$ used by Nicholson was given by Wilkins [208], see also [144, Chapter 9, §7.2]. For more on the history of this result, see [109, §1].

Theorem 7.6.27 and the recurrence relations (7.6.9) and (7.6.13) yield remarkable identities among the germs $\phi_\nu,\phi_{\nu-1},\phi_{\nu+1}\in\mathrm{D}(\mathbb{Q})$. For example:

Corollary 7.6.43. *Recalling that $\psi_\nu = 1/\phi'_\nu$, we have*

$$
\begin{aligned}
-\sqrt{\psi_{\nu-1}}\sin(\phi_{\nu-1}-\phi_\nu)+\sqrt{\psi_{\nu+1}}\sin(\phi_{\nu+1}-\phi_\nu) &= \frac{2\nu}{x}\sqrt{\psi_\nu},\\
\sqrt{\psi_{\nu-1}}\cos(\phi_{\nu-1}-\phi_\nu)-\sqrt{\psi_{\nu+1}}\cos(\phi_{\nu+1}-\phi_\nu) &= 0.
\end{aligned}
$$

Proof. Put $H_\nu := J_\nu + Y_\nu \mathrm{i} \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Then

$$
H_\nu \;=\; \sqrt{\psi_\nu}\cdot\sqrt{\frac{2}{\pi x}}\;\mathrm{e}^{(\phi_\nu-\frac{\pi\nu}{2}-\frac{\pi}{4})\mathrm{i}}, \qquad H_{\nu-1}+H_{\nu+1} \;=\; \frac{2\nu}{x}H_\nu,
$$

and dividing both sides of the equality on the right by $\sqrt{\frac{2}{\pi x}}\;\mathrm{e}^{(\phi_\nu-\frac{\pi\nu}{2}-\frac{\pi}{4})\mathrm{i}}$ gives

$$
\begin{aligned}
&\sqrt{\psi_{\nu-1}}\,\mathrm{e}^{(\phi_{\nu-1}-\phi_\nu+\pi/2)\mathrm{i}}+\sqrt{\psi_{\nu+1}}\,\mathrm{e}^{(\phi_{\nu+1}-\phi_\nu-\pi/2)\mathrm{i}}\\
&= \sqrt{\psi_{\nu-1}}\;\mathrm{i}\,\mathrm{e}^{(\phi_{\nu-1}-\phi_\nu)\mathrm{i}}-\sqrt{\psi_{\nu+1}}\;\mathrm{i}\,\mathrm{e}^{(\phi_{\nu+1}-\phi_\nu)\mathrm{i}} \;=\; \frac{2\nu}{x}\sqrt{\psi_\nu}.
\end{aligned}
$$

Now take real and imaginary parts in the last identity. $\square$

An asymptotic expansion for the zeros of Bessel functions. We are going to use Corollary 7.6.19 to strengthen a result of McMahon on parametrizing the zeros of Bessel functions: Corollary 7.6.51 and the remark following it. Lemma 7.6.46 below, due to Fourier [71] for $\nu = 0$ and to Lommel [133, p. 69] in general, is only included for completeness; its proof is based on the following useful identity also due to Lommel [134]:

Lemma 7.6.44. *Let $\alpha,\beta\in\mathbb{C}^\times$, $\mu,\nu\in\mathbb{C}$, and let $y_\mu, y_\nu\colon \mathbb{C}\setminus\mathbb{R}^{\leqslant}\to\mathbb{C}$ be holomorphic solutions of (B_μ) and (B_ν), respectively. Then on $\mathbb{C}\setminus\big(\alpha^{-1}\mathbb{R}^{\leqslant}\cup\beta^{-1}\mathbb{R}^{\leqslant}\big)$:*

$$
\begin{aligned}
&\frac{d}{dz}\left[z\left(\beta y_\mu(\alpha z)y'_\nu(\beta z)-\alpha y_\nu(\beta z)y'_\mu(\alpha z)\right)\right] \;=\\
&\left((\alpha^2-\beta^2)z-\frac{\mu^2-\nu^2}{z}\right)y_\mu(\alpha z)y_\nu(\beta z).
\end{aligned}
$$

Proof. Let $U\subseteq\mathbb{C}$ be open, let $g,\widetilde{g}\colon U\to\mathbb{C}$ be continuous, and let $y,\widetilde{y}\colon U\to\mathbb{C}$ be holomorphic such that $4y''+gy=4\widetilde{y}''+\widetilde{g}\widetilde{y}=0$. An easy computation gives

$$
\operatorname{wr}(y,\widetilde{y})' \;=\; \tfrac{1}{4}(g-\widetilde{g})y\widetilde{y}.
$$

Assume for now that $\alpha,\beta\in\mathbb{R}^>$ and apply this remark to $U:=\mathbb{C}\setminus\mathbb{R}^{\leqslant}$ and

$$
\begin{aligned}
y(z) &:= (\alpha z)^{1/2}y_\mu(\alpha z), &\quad g(z) &:= 4\alpha^2+(1-4\mu^2)z^{-2}\\
\widetilde{y}(z) &:= (\beta z)^{1/2}y_\nu(\beta z), &\quad \widetilde{g}(z) &:= 4\beta^2+(1-4\nu^2)z^{-2}.
\end{aligned}
$$

Then $4y''+gy=0$, since $z\mapsto z^{1/2}y_\mu(z)\colon\mathbb{C}\setminus\mathbb{R}^{\leqslant}\to\mathbb{C}$ satisfies (L_μ). Likewise, $4\widetilde{y}''+\widetilde{g}\widetilde{y}=0$. This yields the claimed identity for $\alpha,\beta\in\mathbb{R}^>$ by a straightforward computation using that $(\alpha z)^{\frac{1}{2}}=\alpha^{\frac{1}{2}}z^{\frac{1}{2}}$ for such α and for $z\in\mathbb{C}\setminus\mathbb{R}^{\leqslant}$.

Next, for general α, β, z we note that $U_3:=\big\{(\alpha,\beta,z)\in(\mathbb{C}^\times)^3:\ \alpha z,\beta z\notin\mathbb{R}^{\leqslant}\big\}$ is open in $\mathbb{C}^3$. Moreover, U_3 is connected. (Proof sketch: suppose $(\alpha,\beta,z)\in U_3$; then so is $(\mathrm{e}^{\theta\mathrm{i}}\,\alpha,\mathrm{e}^{\theta\mathrm{i}}\,\beta,\mathrm{e}^{-\theta\mathrm{i}}\,z)$ for $\theta\in\mathbb{R}$, so we can "connect to" $\alpha\in\mathbb{R}^>$; next keep α, z fixed, and rotate β to a point in $\mathbb{R}^>$ while preserving $\beta\notin z^{-1}\mathbb{R}^{\leqslant}$.) Both sides in the claimed identity define a complex analytic function on U_3. Now use analytic continuation as in [57, (9.4.4)]. $\square$

We now define certain improper complex integrals and state some basic facts about them. Let U be an open subset of $\mathbb{C}$ with $0 \in \partial U$ and let $f\colon U \to \mathbb{C}$ be holomorphic such that for some $\varepsilon \in \mathbb{R}^{>}$ we have $f(u) = O\big(|u|^{-1+\varepsilon}\big)$ as $u \to 0$. For $z \in U$ such that $(0, z] := \big\{tz :\ t \in (0,1]\big\} \subseteq U$, we set

$$\int_0^z f(u)\,du \ := \ \lim_{\delta \downarrow 0} \int_\delta^1 z f(tz)\,dt \quad \text{(the limit exists in } \mathbb{C}).$$

Suppose $(0, z] \subseteq U$ for all $z \in U$. Then the function $z \mapsto \int_0^z f(u)\,du$ on U is holomorphic with derivative f, and $\lim_{z\to 0} \int_0^z f(u)\,du = 0$; to see this, first show that for any $z_0 \in U$, open ball $B \subseteq U$ centered at z_0, and $z \in U$, we have $\int_0^z f(u)\,du = \int_0^{z_0} f(u)\,du + \int_{z_0}^z f(u)\,du$, where the last integral is by definition $\int_0^1 (z - z_0) f\big(z_0 + t(z - z_0)\big)\,dt$. Thus the integral below makes sense:

Corollary 7.6.45. *Let $\alpha, \beta, z \in \mathbb{C}^\times$ satisfy $\alpha z, \beta z \notin \mathbb{R}^{\leqslant}$, and let $\nu \in \mathbb{R}^{\geqslant -1}$. Then*

$$(7.6.21)\quad (\alpha^2 - \beta^2) \int_0^z u J_\nu(\alpha u) J_\nu(\beta u)\,du \ = \ z\big(\beta J_\nu(\alpha z) J_\nu'(\beta z) - \alpha J_\nu(\beta z) J_\nu'(\alpha z)\big).$$

Proof. Fixing $\alpha, \beta \in \mathbb{C}^\times$ and $\nu \in \mathbb{R}^{\geqslant -1}$, both sides in (7.6.21) are holomorphic functions of z on the open subset $\mathbb{C} \setminus \big(\alpha^{-1}\mathbb{R}^{\leqslant} \cup \beta^{-1}\mathbb{R}^{\leqslant}\big)$ of $\mathbb{C}$, with equal derivatives by Lemma 7.6.44. Moreover, both sides tend to 0 as $z \to 0$ in $\mathbb{C} \setminus \big(\alpha^{-1}\mathbb{R}^{\leqslant} \cup \beta^{-1}\mathbb{R}^{\leqslant}\big)$, using for the right hand side the first two terms in the power series for J_ν and J_ν'. $\square$

Lemma 7.6.46. *Let $\nu \in \mathbb{R}^{\geqslant -1}$. Then all zeros of J_ν are contained in $\mathbb{R}^{>}$.*

Proof. Let $\alpha \in \mathbb{C} \setminus \mathbb{R}^{\leqslant}$ be a zero of J_ν. From the power series for J_ν we see that then $\overline{\alpha}$ is also a zero of J_ν, and $\alpha \notin i\mathbb{R}$. Putting $\beta = \overline{\alpha}$ and $z = 1$ in (7.6.21) yields

$$(\alpha^2 - \overline{\alpha}^2) \int_0^1 t J_\nu(\alpha t) J_\nu(\overline{\alpha} t)\,dt \ = \ \overline{\alpha} J_\nu(\alpha) J_\nu'(\overline{\alpha}) - \alpha J_\nu(\overline{\alpha}) J_\nu'(\alpha) \ = \ 0.$$

If $\alpha \notin \mathbb{R}$, then this yields $\displaystyle\int_0^1 t J_\nu(\alpha t) J_\nu(\overline{\alpha} t)\,dt = 0$, but $J_\nu(\alpha t) J_\nu(\overline{\alpha} t) \in \mathbb{R}^{\geqslant}$ for all $t \in (0,1]$ and $J_\nu(\alpha t) \neq 0$ for some $t \in (0,1]$, a contradiction. Thus $\alpha \in \mathbb{R}$. $\square$

Taking $\alpha = \beta = 1$ in Lemma 7.6.44 yields (for all $\mu, \nu \in \mathbb{C}$):

$$(7.6.22)\quad \frac{d}{dz}\big[z\big(J_\mu'(z) J_\nu(z) - J_\mu(z) J_\nu'(z)\big)\big] \ = \ (\mu^2 - \nu^2) \frac{J_\mu(z) J_\nu(z)}{z} \quad \text{on } \mathbb{C} \setminus \mathbb{R}^{\leqslant}.$$

In the next result it is convenient to let J denote the analytic function $(\nu, z) \mapsto J_\nu(z)$ on $\mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant})$, so for $(\nu, z) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant})$ we have $J_\nu'(z) = \frac{\partial J}{\partial z}(\nu, z)$. In its proof we also use that for any complex analytic functions A, B on an open set $U \subseteq \mathbb{C}^2$ and (μ, z) and (ν, z) ranging over U:

$$\lim_{\mu \to \nu} \frac{\partial}{\partial z}\left(A(\mu, z) \frac{B(\mu, z) - B(\nu, z)}{\mu - \nu}\right) \ = \ \frac{\partial}{\partial z}\left(A(\nu, z) \frac{\partial B}{\partial \nu}(\nu, z)\right),$$

an easy consequence of $\frac{\partial^2 B}{\partial \nu \partial z} = \frac{\partial^2 B}{\partial z \partial \nu}$.

Corollary 7.6.47. *On $\mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^{\leqslant})$ we have*

$$\frac{d}{dz}\left[z\left(J_\nu(z) \cdot \frac{\partial^2 J}{\partial \nu \partial z}(\nu, z) - J_\nu'(z) \cdot \frac{\partial J}{\partial \nu}(\nu, z)\right)\right] \ = \ 2\nu \frac{J_\nu^2(z)}{z},$$

Proof. For $\mu, \nu \in \mathbb{C}$ with $\mu \neq \nu$ we obtain from (7.6.22) that on $\mathbb{C} \setminus \mathbb{R}^{\leqslant}$,

$$\frac{d}{dz}\left[z\left(J_\mu(z)\frac{J'_\mu(z)-J'_\nu(z)}{\mu-\nu} - J'_\mu(z)\frac{J_\mu(z)-J_\nu(z)}{\mu-\nu}\right)\right] \ = \ (\mu+\nu)\frac{J_\mu(z)J_\nu(z)}{z}.$$

Now let μ tend to ν and use the identity preceding the corollary. $\square$

Below $\nu \in \mathbb{R}$, so the set $Z_\nu := \mathbb{R}^{>} \cap J_\nu^{-1}(0)$ of positive real zeros of J_ν is infinite and has no limit point. Let $(j_{\nu,n})$ be the enumeration of Z_ν. Note that if $t \in Z_\nu$, then $J_{\nu+1}(t) = -J'_\nu(t)$ by (7.6.9), so $J_{\nu+1}(t) \neq 0$.

Proposition 7.6.48 (Schläfli [180])**.** *Let n be given. Then the function*

$$\nu \mapsto j(\nu) := j_{\nu,n} \colon \mathbb{R}^{>-1} \to \mathbb{R}^{>}$$

is analytic, and its derivative at $\nu > 0$ is given by

$$j'(\nu) \ = \ \frac{2\nu}{j(\nu)\, J^2_{\nu+1}\big(j(\nu)\big)} \int_0^{j(\nu)} J_\nu^2(s)\frac{ds}{s}.$$

In particular, the restriction of j to $\mathbb{R}^{>}$ is strictly increasing.

In the proof of Proposition 7.6.48 we use:

Lemma 7.6.49. *Let $\varepsilon \in \mathbb{R}^{>}$. Then there exists $\delta \in \mathbb{R}^{>}$ such that $J_\nu(t) \neq 0$ for all $\nu \geqslant -1+\varepsilon$ and $t \in (0,\delta]$.*

Proof. Take $\delta \in \mathbb{R}^{>}$ such that $\delta^2 < 4\log(1+\varepsilon)$. Then for $\nu \geqslant -1+\varepsilon$ and $0 < t \leqslant \delta$:

$$\left|\frac{\Gamma(\nu+1)J_\nu(t)}{(\frac{1}{2}t)^\nu} - 1\right| \ = \ \left|\sum_{n\geqslant 1}\frac{(-1)^n(\frac{1}{4}t^2)^n}{n!(\nu+n)\cdots(\nu+1)}\right| \ \leqslant \ \frac{\exp(\frac{1}{4}\delta^2)-1}{\varepsilon} \ < 1\ ,$$

hence $J_\nu(t) \neq 0$. $\square$

Proof of Proposition 7.6.48. Let $\nu_0 \in \mathbb{R}^{>-1}$. For each m we have $J_{\nu_0}(j_{\nu_0,m}) = 0$ and $J'_{\nu_0}(j_{\nu_0,m}) \neq 0$. So IFT (the Implicit Function Theorem [57, (10.2.2), (10.2.4)]) yields an interval $I = (\nu_0-\varepsilon, \nu_0+\varepsilon)$ with $\varepsilon \in \mathbb{R}^{>}$, $-1 < \nu_0-\varepsilon$, and for each $m \leqslant n$ an analytic function $j_m \colon I \to \mathbb{R}^{>}$ with $J_\nu\big(j_m(\nu)\big) = 0$ for $\nu \in I$ and $j_m(\nu_0) = j_{\nu_0,m}$. Shrinking ε if necessary we also arrange to have $\delta \in \mathbb{R}^{>}$ such that $j_{\nu_0,m} > \delta$ and $|j_m(\nu) - j_{\nu_0,m}| < \delta$ for all $\nu \in I$ and $m \leqslant n$, and $J'_\nu(t) \neq 0$ for all $\nu \in I$, $m \leqslant n$ and $t \in \mathbb{R}$ with $|t - j_{\nu_0,m}| < \delta$. Hence for $\nu \in I$ and $m \leqslant n$ (using IFT at all $\nu \in I$):

$$Z_\nu \cap (j_{\nu_0,m}-\delta, j_{\nu_0,m}+\delta) \ = \ \big\{j_m(\nu)\big\}, \qquad j_0(\nu) < j_1(\nu) < \cdots < j_n(\nu).$$

We claim that for $\nu \in I$ we have

$$\big\{t \in Z_\nu : \, t \leqslant j_n(\nu)\big\} = \big\{j_0(\nu), j_1(\nu), \dots, j_n(\nu)\big\}.$$

Suppose for example that $\nu_1 \in I$, $t_1 \in Z_{\nu_1}$, $t_1 < j_0(\nu_1)$; we shall derive a contradiction. (The assumption $\nu_1 \in I$, $t_1 \in Z_\nu$, $j_m(\nu_1) < t_1 < j_{m+1}(\nu_1)$, $m < n$, leads to a contradiction in the same way.) Using IFT again we see that

$$U := \big\{\nu \in I : \ \text{there is a } t \in Z_\nu \text{ with } t < j_0(\nu)\big\}$$

is open in I, and by an easy extra argument using Lemma 7.6.49 also closed in I. As $\nu_1 \in U$, this gives $U = I$, so $\nu_0 \in U$, a contradiction. The claim gives $j_m(\nu) = j_{\nu,m}$ for $\nu \in I$ and $m \leqslant n$. Taking $m = n$ it follows that j is analytic.

Next, let ν range over $\mathbb{R}^>$. Differentiating $J\big(\nu, j(\nu)\big) = 0$ yields

$$\frac{\partial J}{\partial \nu}\big(\nu, j(\nu)\big) + J_\nu'\big(j(\nu)\big) j'(\nu) \;=\; 0.$$

Using the primitive of $s \mapsto \frac{J_\nu^2(s)}{s}$ provided by Corollary 7.6.47 gives

$$\int_0^{j(\nu)} \frac{J_\nu^2(s)}{s}\, ds \;=\; -\frac{j(\nu)}{2\nu} J_\nu'\big(j(\nu)\big) \frac{\partial J}{\partial \nu}\big(\nu, j(\nu)\big).$$

Now combine the two displayed identities with $J_\nu'\big(j(\nu)\big) = -J_{\nu+1}\big(j(\nu)\big)$. □

We now bring in Lemma 7.6.36 to bound $j_{\nu,n}$ for $\nu > 0$ and sufficiently large n:

Proposition 7.6.50. *Let $\nu_0 \in \mathbb{R}^>$. Then there is an $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$:*

$$(n + \tfrac{1}{2}\nu_0 + \tfrac{1}{2})\pi \;\leqslant\; j_{\nu_0,n} \;\leqslant\; (n + \tfrac{1}{2}\nu_0 + 1)\pi.$$

Proof. Compactness and Lemma 7.6.36 yield $C_0, t_0 \in \mathbb{R}^>$ such that for all ν in the smallest closed interval I containing both ν_0 and $1/2$, and for all $t \geqslant t_0$:

$$\left| J_\nu(t) - \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \right| \;\leqslant\; C_0 t^{-3/2}.$$

We arrange that $t_0 \geqslant C_0\sqrt{\pi}$. Hence if $\nu \in I$ and $j_{\nu,n} \geqslant t_0$, then

$$\left|\cos\left(j_{\nu,n} - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right| \;\leqslant\; \frac{1}{\sqrt{2}},$$

and so we have a unique $k_{\nu,n} \in \mathbb{Z}$ with

$$\tfrac{1}{4}\pi \;\leqslant\; j_{\nu,n} - (\tfrac{1}{2}\nu + \tfrac{1}{4} + k_{\nu,n})\pi \;\leqslant\; \tfrac{3}{4}\pi.$$

Let ν_1 be the left endpoint of I (so $\nu_1 = 1/2$ or $\nu_1 = \nu_0$) and take $n_0 \in \mathbb{N}$ such that $j_{\nu_1,n_0} \geqslant t_0$. Then $j_{\nu,n} \geqslant t_0$ for $\nu \in I$ and $n \geqslant n_0$ by Proposition 7.6.48. Let $n \geqslant n_0$; we claim that $\nu \mapsto k_{\nu,n} \colon I \to \mathbb{Z}$ is constant. To see this, note that Proposition 7.6.48 yields $\delta \in (0, 1/4)$ such that for all $\nu, \widetilde{\nu} \in I$ with $|\nu - \widetilde{\nu}| < \delta$ we have $|j_{\nu,n} - j_{\widetilde{\nu},n}| < \pi/4$, which in view of

$$-\tfrac{1}{2}\pi \;\leqslant\; (j_{\nu,n} - j_{\widetilde{\nu},n}) - (\nu - \widetilde{\nu})\tfrac{\pi}{2} - (k_{\nu,n} - k_{\widetilde{\nu},n})\pi \;\leqslant\; \tfrac{1}{2}\pi,$$

gives $k_{\nu,n} = k_{\widetilde{\nu},n}$. Thus $\nu \mapsto k_{\nu,n} \colon I \to \mathbb{Z}$ is locally constant, and hence constant. Let k_n be the common value of $k_{\nu,n}$ for $\nu \in I$. Now $Z_{1/2} = \{m\pi : m \geqslant 1\}$ by Lemma 7.6.25, hence $j_{1/2,n} = (n+1)\pi$ and so

$$\tfrac{1}{4}\pi \;\leqslant\; (n+1)\pi - (\tfrac{1}{2} + k_n)\pi \;\leqslant\; \tfrac{3}{4}\pi.$$

This yields $k_n = n$. □

Corollary 7.6.51. *Suppose $\nu > 0$. There is a strictly increasing $\zeta \in \mathcal{C}_{n_0}$ ($n_0 \in \mathbb{N}$) whose germ is in $\mathrm{E}(\mathbb{Q})$ such that $j_{\nu,n} = \zeta(n)$ for all $n \geqslant n_0$ and which has for $s := \big(x + \frac{1}{2}\nu + \frac{3}{4}\big)\pi$ the asymptotic expansion*

$$\zeta \;\sim\; s - \left(\frac{\mu - 1}{8}\right) s^{-1} - \left(\frac{(\mu-1)(7\mu - 31)}{192}\right) \frac{s^{-3}}{2!} + \cdots\ .$$

Proof. Take n_0 as in Proposition 7.6.50 with $\nu_0 = \nu$. Theorem 7.6.27 yields $t_0 \in \mathbb{R}^>$ and a representative of $\phi = \phi_\nu$ in $\mathcal{C}^1_{t_0}$, also denoted by ϕ, such that for all $t \geqslant t_0$,

$$\phi'(t) > 0, \qquad J_\nu(t) = \sqrt{\frac{2}{\pi t \phi'(t)}}\, \cos\left(\phi(t) - \frac{\pi}{2}\nu - \frac{\pi}{4}\right).$$

Increasing n_0 if necessary we arrange that $j_{\nu,n_0} \geqslant t_0 + \frac{\pi}{2}$. Then for $n \geqslant n_0$,

$$\phi(j_{\nu,n}) - \left(\tfrac{1}{2}\nu + \tfrac{3}{4}\right)\pi \in \mathbb{Z}\pi.$$

Take $k \in \mathbb{Z}$ with $\phi(j_{\nu,n_0}) = \left(\frac{1}{2}\nu + \frac{3}{4} + k\right)\pi$. Then for $n \geqslant n_0$ we have $\phi(j_{\nu,n}) = \left(n - n_0 + \frac{1}{2}\nu + \frac{3}{4} + k\right)\pi$. By Proposition 7.6.50 we have for $n \geqslant n_0$,

$$(n + \tfrac{1}{2}\nu + \tfrac{1}{2})\pi \;\leqslant\; j_{\nu,n} \;\leqslant\; (n + \tfrac{1}{2}\nu + 1)\pi,$$

and thus for all $n \geqslant n_0$,

$$\phi\big((n + \tfrac{1}{2}\nu + \tfrac{1}{2})\pi\big) \;\leqslant\; \phi(j_{\nu,n}) \;=\; \big(n + \tfrac{1}{2}\nu + \tfrac{3}{4} + k - n_0\big)\pi \;\leqslant\; \phi\big((n + \tfrac{1}{2}\nu + 1)\pi\big).$$

Since $\phi - x \preccurlyeq x^{-1}$, this yields $k = n_0$, therefore $\phi(j_{\nu,n}) = \left(n + \frac{1}{2}\nu + \frac{3}{4}\right)\pi$ for $n \geqslant n_0$. Let $\phi^{\text{inv}} \in \mathcal{C}_{t_1}$ be the compositional inverse of ϕ, where $t_1 := \phi(t_0)$, and let $\zeta \in \mathcal{C}_{n_0}$ be given by $\zeta(t) := \phi^{\text{inv}}\left((t + \frac{1}{2}\nu + \frac{3}{4})\pi\right)$ for $t \geqslant n_0$. Then ζ is strictly increasing with $j_{\nu,n} = \zeta(n)$ for $n \geqslant n_0$. Taking ζ and ϕ^{inv} as germs we have $\zeta = \phi^{\text{inv}} \circ s$. Now $\phi^{\text{inv}} \in \mathrm{E}(\mathbb{Q})$ by Lemma 7.6.3, and $\mathrm{E}(\mathbb{Q}) \circ \mathrm{E}(\mathbb{Q})^{>\mathbb{R}} \subseteq \mathrm{E}(\mathbb{Q})$, so $\zeta \in \mathrm{E}(\mathbb{Q})$. The claimed asymptotic expansion for ζ follows from Corollary 7.6.19. $\square$

Remark. The asymptotic expansion for $j_{\nu,n}$ as $n \to \infty$ in Corollary 7.6.51 was obtained by McMahon [138]. (For $\nu = 1$, apparently Gauss was aware of it as early as 1797, cf. [205, p. 506].) What is new here is that we specified a function ζ with germ in $\mathrm{E}(\mathbb{Q})$ such that $j_{\nu,n} = \zeta(n)$ for all sufficiently large n.

In [144, p. 247], Olver states: "No explicit formula is available for the general term" of the asymptotic expansion for $j_{\nu,n}$ as $n \to \infty$ in Corollary 7.6.51. The remark after the proof of Corollary 7.6.19 yields the asymptotic expansion

$$\zeta \;\sim\; s - \sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} \frac{(2(j-1))!}{(2j-1-i)!} B_{ij}(u_1, \dots, u_{j-i+1}) \right) \frac{s^{-2j+1}}{j!},$$

which is perhaps as explicit as possible. The values of u_1, u_2, u_3 given before Corollary 7.6.19 yield the first few terms of this expansion:

$$\zeta \;\sim\; s \;-\; \frac{\mu - 1}{8} s^{-1} \;-\; \frac{(\mu-1)(7\mu - 31)}{192}\frac{s^{-3}}{2!} \;-\; \frac{(\mu-1)(83\mu^2 - 982\mu + 3779)}{2560}\frac{s^{-5}}{3!} \;-\; \cdots.$$

Appendix: inversion of formal power series. In this appendix we discuss multiplicative and compositional inversion of power series. We use [ADH, 12.5] and its notations. Thus $x, y_1, y_2, \dots, z$ are distinct indeterminates, and

$$R \;:=\; \mathbb{Q}[x, y_1, y_2, \dots], \qquad A \;:=\; R[[z]].$$

We also let K be a field of characteristic zero. Recall from [ADH, 12.5.1] the definition of the Bell polynomials $B_{ij} \in \mathbb{Q}[y_1, \dots, y_d]$, where $i \leqslant j$ and $d = j - i + 1$:

$$B_{ij} := \sum_{\substack{\boldsymbol{k}=(k_1,\dots,k_d)\in\mathbb{N}^d\\ |\boldsymbol{k}|=i,\ \|\boldsymbol{k}\|=j}} \frac{j!}{k_1!k_2!\cdots k_d!} \left(\frac{y_1}{1!}\right)^{k_1} \left(\frac{y_2}{2!}\right)^{k_2} \cdots \left(\frac{y_d}{d!}\right)^{k_d}.$$

(Also $B_{ij} := 0 \in \mathbb{Q}[y_1, y_2, \dots]$ for $i > j$.) Let

$$y \ := \ \sum_{n\geqslant 1} y_n \frac{z^n}{n!} \in zR[[z]].$$

By [ADH, (12.5.2)] we have in $R[[z]]$:

$$\frac{y^i}{i!} \ = \ \sum_{j\geqslant 0} B_{ij} \frac{z^j}{j!} \ = \ \sum_{j\geqslant i} B_{ij} \frac{z^j}{j!}.$$

Lemma 7.6.52. *Let $i \leqslant j$ and $d = j - i + 1$; then*

$$B_{ij}\left(\frac{y_2}{2}, \frac{y_3}{3}, \dots, \frac{y_{d+1}}{d+1}\right) \ = \ \frac{j!}{(i+j)!} B_{i,i+j}(0, y_2, y_3, \dots, y_{j+1}).$$

Proof. We have

$$y - y_1 z \ = \ z \sum_{n\geqslant 1} \left(\frac{y_{n+1}}{n+1}\right) \frac{z^n}{n!} \ = \ \sum_{n\geqslant 2} y_n \frac{z^n}{n!},$$

hence

$$\frac{(y - y_1 z)^i}{i!} \ = \ \sum_{j\geqslant 0} B_{ij}\left(\frac{y_2}{2}, \frac{y_3}{3}, \dots\right) \frac{z^{i+j}}{j!} \ = \ \sum_{k\geqslant 0} B_{ik}(0, y_2, y_3, \dots) \frac{z^k}{k!}. \qquad \square$$

For $j \in \mathbb{N}$ we set

$$B_j \ := \ \sum_{i=0}^{j} i!\, B_{ij} \in \mathbb{Q}[y_1, \dots, y_{j+1}]. \tag{7.6.23}$$

Note that

$$\frac{B_j}{j!} \ = \ \sum_{i=0}^{j} \sum_{\substack{\boldsymbol{k}=(k_1,\dots,k_d)\in\mathbb{N}^d\\ |\boldsymbol{k}|=i,\ \|\boldsymbol{k}\|=j,\ d=j-i+1}} \frac{i!}{k_1!\cdots k_d!} \left(\frac{y_1}{1!}\right)^{k_1} \left(\frac{y_2}{2!}\right)^{k_2} \cdots \left(\frac{y_d}{d!}\right)^{k_d}.$$

We have $B_0 = B_{00} = 1$ and $B_j = \sum_{i=1}^{j} i!\, B_{ij} \in \mathbb{Q}[y_1, \dots, y_j]$ for $j \geqslant 1$. Using the examples following [ADH, 12.5.4] we obtain

$$\begin{aligned}
B_1 \ &= \ y_1,\\
B_2 \ &= \ y_2 + 2y_1^2,\\
B_3 \ &= \ y_3 + 6y_1y_2 + 6y_1^3,\\
B_4 \ &= \ y_4 + 8y_1y_3 + 6y_2^2 + 36y_1^2y_2 + 24y_1^4,\\
B_5 \ &= \ y_5 + 10y_1y_4 + 20y_2y_3 + 60y_1^2y_3 + 90y_1y_2^2 + 240y_1^3y_2 + 120y_1^5.
\end{aligned}$$

We have $1-y \in 1+zR[[z]] \subseteq R[[z]]^\times$ with inverse $(1-y)^{-1} = \sum_{i\geqslant 0} y^i$, so for $m \geqslant 1$:

$$(1-y)^{-m} \ = \ \sum_{i\geqslant 0} \binom{i+m-1}{m-1} y^i = \sum_{j\geqslant 0}\left(\sum_{i=0}^{j} m^{\overline{i}} B_{ij}\right)\frac{z^j}{j!} \tag{7.6.24}$$

where $m^{\overline{i}} := m(m+1)\cdots(m+i-1)$ (so $m^{\overline{0}} = 1$, $1^{\overline{i}} = i!$). In particular,

$$(1-y)^{-1} \ = \ \sum_{j\geqslant 0} B_j \frac{z^j}{j!}.$$

Corollary 7.6.53. *Let* $f = \displaystyle\sum_{n\geqslant 1} f_n \frac{z^n}{n!} \in K[[z]]$ $(f_n \in K)$. *Then*

$$\begin{aligned}(1-f)^{-1} \ &= \ \sum_{j\geqslant 0} B_j(f_1,\dots,f_j)\frac{z^j}{j!} \\ &= \ 1 + f_1 z + (f_2 + 2f_1^2)\frac{z^2}{2!} + (f_3 + 6f_1 f_2 + 6f_1^3)\frac{z^3}{3!} + \cdots .\end{aligned}$$

Next we discuss the compositional inversion of formal power series. From [ADH, 12.5] recall that $zK^\times + z^2K[[z]]$ is a group under formal composition with z as its identity element. We denote the compositional inverse of any $f \in zK^\times + z^2K[[z]]$ by $f^{[-1]}$. We equip the field $K(\!(z)\!)$ of Laurent series with the strongly additive and K-linear derivation d/dz (so $z' = 1$).

Definition 7.6.54. Let $f = \sum_{k\in\mathbb{Z}} f_k z^k \in K(\!(z)\!)$ where $f_k \in K$ for $k \in \mathbb{Z}$. Then $\operatorname{res}(f) := f_{-1} \in K$ is the **residue** of f. (We also have the residue morphism $f \mapsto f(0)\colon K[[z]] \to K$ of the valuation ring $K[[z]]$ of $K(\!(z)\!)$.)

The map $f \mapsto \operatorname{res}(f)\colon K(\!(z)\!) \to K$ is strongly additive and K-linear.

Lemma 7.6.55. *Let* $f \in K(\!(z)\!)$. *Then* $\operatorname{res}(f') = 0$, *and if* $f \neq 0$, *then* $\operatorname{res}(f^\dagger) = vf$.

Proof. The first claim is clearly true. For the second, let $f = z^k g$ where $k = vf$, $g \in K[[z]]^\times$. Then $f^\dagger = kz^{-1} + g^\dagger \in kz^{-1} + K[[z]]$, so $\operatorname{res}(f^\dagger) = k = vf$. $\square$

Corollary 7.6.56. *Let* $f, g \in K(\!(z)\!)$. *Then* $\operatorname{res}(f'g) = -\operatorname{res}(fg')$ *and thus, if* $g \neq 0$ *and* $k \in \mathbb{Z}$, *then* $\operatorname{res}(f'g^k) = -k\operatorname{res}(fg^{k-1}g')$.

Proof. For the first claim, use the Product Rule and the first part of Lemma 7.6.55. The second claim follows from the first. $\square$

Corollary 7.6.57 (Jacobi [107]). *Let* $f \in zK[[z]]^{\neq}$ *and* $g \in K(\!(z)\!)$. *Then*

$$\operatorname{res}\big((g\circ f)f'\big) = vf\ \operatorname{res}(g).$$

Proof. By strong additivity and K-linearity it is enough to show this for $g = z^k$ $(k \in \mathbb{Z})$. If $k \neq -1$, then

$$\operatorname{res}\big((g\circ f)f'\big) = \operatorname{res}(f^k f') = \operatorname{res}\Big(\big(f^{k+1}/(k+1)\big)'\Big) = 0 = vf\ \operatorname{res}(g),$$

and if $k = -1$, then

$$\operatorname{res}\big((g\circ f)f'\big) = \operatorname{res}(f^\dagger) = vf = vf\ \operatorname{res}(g),$$

using Lemma 7.6.55. $\square$

We now obtain the Lagrange Inversion Formula, following [76]:

Theorem 7.6.58. *Let $f \in zK^\times + z^2K[[z]]$, $g \in K[[z]]$. Then*

$$g \circ f^{[-1]} \ = \ g(0) + \sum_{n \geqslant 1} \frac{1}{n} \operatorname{res}(g' f^{-n}) z^n.$$

Proof. Let $h := g \circ f^{[-1]} = \sum_n h_n z^n$ ($h_n \in K$). Then for $n \geqslant 1$ we have

$$\begin{aligned} \frac{1}{n} \operatorname{res}(g' f^{-n}) \ &= \ \operatorname{res}\big(g f^{-n-1} f'\big) \ = \ \operatorname{res}\big((h \circ f) f^{-n-1} f'\big) \\ &= \ \operatorname{res}\big(\big(hz^{-n-1} \circ f\big) \cdot f'\big) \ = \ \operatorname{res}\big(hz^{-n-1}\big) \ = \ h_n, \end{aligned}$$

using Corollary 7.6.56 for the first equality, and 7.6.57 for the next to last. □

Taking $g = z^m$ in the above yields:

Corollary 7.6.59. *If $f \in zK^\times + z^2K[[z]]$, then for $m \geqslant 1$:*

$$(f^{[-1]})^m \ = \ \sum_{n \geqslant m} \frac{m}{n} \operatorname{res}(z^{m-1} f^{-n}) z^n.$$

Remark. Theorem 7.6.58 stems from Lagrange [120] and Bürmann (cf. [98]). The identity in Corollary 7.6.59 is from Jabotinsky [106, Theorem II] and Schur [182].

We now use Corollary 7.6.59 to express the coefficients of $f^{[-1]}$ in terms of those of f; cf. [50, §3.8, Theorem E]. Here $f = z + \sum_{n \geqslant 2} f_n \frac{z^n}{n!}$ with $f_n \in K$ for $n \geqslant 2$. Then

$$g \ := \ f^{[-1]} \ = \ z + \sum_{n \geqslant 2} g_n \frac{z^n}{n!} \qquad (g_n \in K \text{ for } n \geqslant 2).$$

For $h \in zK[[z]]$, let $[\![h]\!]$ denote the (upper triangular) iteration matrix of h as in [ADH, 12.5], so $[\![h]\!]_{m,n} = 0$ for $m > n$, $[\![h]\!]_{0,0} = 1$, $[\![h]\!]_{0,n} = 0$ for $n \geqslant 1$.

Proposition 7.6.60. *For $1 \leqslant m \leqslant n$ we have*

$$[\![g]\!]_{m,n} \ = \ \sum_{i=0}^{n-m} \frac{(-1)^i (i+n-1)!}{(i+n-m)!(m-1)!} B_{i,i+n-m}(0, f_2, \dots, f_{n-m+1}).$$

Proof. By [ADH, remarks before 12.5.5] we have $g^m/m! = \sum_n [\![g]\!]_{m,n} z^n/n!$ and so by Corollary 7.6.59, $[\![g]\!]_{m,n} = (n-1)!/(m-1)! \operatorname{res}(z^{m-1} f^{-n})$ if $1 \leqslant m \leqslant n$. Set

$$h \ := \ 1 - \frac{f}{z} \ = \ \sum_{n \geqslant 1} h_n \frac{z^n}{n!} \qquad \text{where } h_n = -f_{n+1}/(n+1) \text{ for } n \geqslant 1.$$

Then $\operatorname{res}(z^{m-1} f^{-n})$ is the constant term of $z^m f^{-n}$ and so equals the coefficient of z^{n-m} in $z^{n-m} z^m f^{-n} = (z/f)^n = \big(\frac{1}{1-h}\big)^n$. Hence by (7.6.24) for $n \geqslant m \geqslant 1$:

$$\operatorname{res}(z^{m-1} f^{-n}) \ = \ \sum_{i=0}^{n-m} \frac{n^{\overline{i}}}{(n-m)!} B_{i,n-m}(h_1, \dots, h_{n-m-i+1}).$$

Lemma 7.6.52 gives for $n \geqslant m$ and $i \leqslant n - m$:

$$\begin{aligned} B_{i,n-m}\big(h_1, \dots, h_{n-m-i+1}\big) \ &= \ B_{i,n-m}\Big(\frac{-f_2}{2}, \dots, \frac{-f_{n-m-i+2}}{n-m-i+2}\Big) \\ &= \ \frac{(n-m)!}{(i+n-m)!} B_{i,i+n-m}\big(0, -f_2, \dots, -f_{n-m+1}\big) \\ &= \ \frac{(-1)^i (n-m)!}{(i+n-m)!} B_{i,i+n-m}\big(0, f_2, \dots, f_{n-m+1}\big). \end{aligned}$$

For $1 \leqslant m \leqslant n$ this yields the identity claimed for $[\![g]\!]_{m,n}$. □

Corollary 7.6.61 (Ostrowski [146]). *For $n \geqslant 2$ we have*

$$g_n = \sum_{\substack{\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbb{N}^n \\ |\boldsymbol{k}|=n-1,\ \|\boldsymbol{k}\|=2n-2}} (-1)^{n-k_1-1}\frac{(2n-k_1-2)!}{k_2!k_3!\cdots k_n!}\left(\frac{f_2}{2!}\right)^{k_2}\left(\frac{f_3}{3!}\right)^{k_3}\cdots\left(\frac{f_n}{n!}\right)^{k_n}.$$

Proof. Let $n \geqslant 2$. We have $g_n = [\![g]\!]_{1,n}$, so by Proposition 7.6.60:

$$g_n = \sum_{i=1}^{n-1}(-1)^i B_{i,i+n-1}(0, f_2, f_3, \dots, f_n) \qquad (n \geqslant 2).$$

Now use the definition of the Bell polynomials and reindex. □

One now easily determines the first few g_n:

$$\begin{aligned}
g_2 &= -f_2 \\
g_3 &= -f_3 + 3f_2^2 \\
g_4 &= -f_4 + 10f_3f_2 - 15f_2^3 \\
g_5 &= -f_5 + 15f_4f_2 + 10f_3^2 - 105f_3f_2^2 + 105f_2^4
\end{aligned}$$

We now establish analogues of some of these results for compositional inversion in the field $K(\!(x^{-1})\!)$ of Laurent series in x^{-1} over K. We have the usual valuation $v\colon K(\!(x^{-1})\!)^\times \to \mathbb{Z}$ on $K(\!(x^{-1})\!)$ with valuation ring $K[[x^{-1}]]$; so $v(x) = -1$. We also have the unique continuous derivation $f \mapsto f'$ on $K(\!(x^{-1})\!)$ such that $a' = 0$ for $a \in K$ and $x' = 1$. This valuation and derivation make $K(\!(x^{-1})\!)$ a d-valued field with small derivation.

As in $K(\!(z)\!)$, we have a well-behaved notion of composition in $K(\!(x^{-1})\!)$: for f, g in $K(\!(x^{-1})\!)$ with $f \succ 1$ and $g = \sum_k g_k x^k$ ($g_k \in K$ for $k \in \mathbb{Z}$), the family $(g_k f^k)$ is summable in $K(\!(x^{-1})\!)$, and we denote its sum by $g \circ f$. For $f \in K(\!(x^{-1})\!)$ with $f \succ 1$, the map $g \mapsto g \circ f$ is a strongly additive and K-linear field embedding, which is bijective if $f \asymp x$. This can be seen, for example, by relating composition in $K(\!(x^{-1})\!)$ to composition in $K(\!(z)\!)$: The strongly additive, K-linear map

$$\tau\colon K(\!(x^{-1})\!) \to K(\!(z)\!) \quad \text{with } \tau(x^{-k}) = z^k \text{ for all } k \in \mathbb{Z},$$

is an isomorphism of valued fields. Let $f \in K(\!(x^{-1})\!)$, $f \succ 1$. Then $\tau(1/f) \in zK[[z]]$, and we have a commutative diagram

$$\begin{tikzcd}
K(\!(x^{-1})\!) \arrow[r, "g \mapsto g\circ f"] \arrow[d, "\tau"] & K(\!(x^{-1})\!) \arrow[d, "\tau"] \\
K(\!(z)\!) \arrow[r, "h \mapsto h\circ\tau(1/f)"] & K(\!(z)\!)
\end{tikzcd}$$

of strongly additive, K-linear maps. Also $\tau(1/f) \in zK^\times + z^2K[[z]]$ if $f \asymp x$.

As in Definition 7.6.54, we define:

Definition 7.6.62. Let $f = \sum_{k\in\mathbb{Z}} f_k x^k \in K(\!(x^{-1})\!)$ where $f_k \in K$ for $k \in \mathbb{Z}$. Then $\operatorname{res}(f) := f_{-1} \in K$ is the **residue** of f.

The map $f \mapsto \operatorname{res}(f)\colon K(\!(x^{-1})\!) \to K$ is strongly additive and K-linear.

Lemma 7.6.63. *Let $f \in K(\!(x^{-1})\!)$. Then* $\operatorname{res}(f') = 0$*; if* $f \neq 0$*, then* $\operatorname{res}(f^\dagger) = -vf$.

Proof. The first claim is clearly true. For the second, let $f = x^k g$ where $k = -vf$, $g \in K(\!(x^{-1})\!)$, $g \asymp 1$. Then $f^\dagger = kx^{-1} + g^\dagger$ with $g^\dagger \preccurlyeq x^{-2}$, so $\operatorname{res}(f^\dagger) = k = -vf$. $\square$

Just like Lemma 7.6.55 led to Corollaries 7.6.56 and 7.6.57, Lemma 7.6.63 gives:

Corollary 7.6.64. *If $f, g \in K(\!(x^{-1})\!)$, then* $\operatorname{res}(f'g) = -\operatorname{res}(fg')$ *and thus, if also $g \neq 0$ and $k \in \mathbb{Z}$, then* $\operatorname{res}(f'g^k) = -k \operatorname{res}(fg^{k-1}g')$.

Corollary 7.6.65. *If $f, g \in K(\!(x^{-1})\!)$, $f \succ 1$, then* $\operatorname{res}\big((g \circ f)f'\big) = -vf \operatorname{res}(g)$.

For $f \in K(\!(x^{-1})\!)$, $f \asymp x$, let $f^{[-1]}$ be the compositional inverse of f. One verifies easily that if $f \in x + x^{-1}K[[x^{-1}]]$, then $f^{[-1]} \in x + x^{-1}K[[x^{-1}]]$. More generally:

Theorem 7.6.66. *Let $f, g \in x + x^{-1}K[[x^{-1}]]$. Then*

$$g \circ f^{[-1]} \;=\; x - \sum_{n \geqslant 1} \frac{1}{n} \operatorname{res}(g' f^n) x^{-n}.$$

Proof. Let $h := g \circ f^{[-1]} = \sum_k h_k x^k$ $(h_k \in K)$. Then for $k \in \mathbb{Z}^{\neq}$ we have

$$\begin{aligned}\frac{1}{k} \operatorname{res}(g' f^{-k}) \;&=\; \operatorname{res}\big(g f^{-k-1} f'\big) \;=\; \operatorname{res}\big((h \circ f) f^{-k-1} f'\big) \\ &=\; \operatorname{res}\big((h x^{-k-1} \circ f) f'\big) \;=\; \operatorname{res}\big(h x^{-k-1}\big) \;=\; h_k,\end{aligned}$$

using Corollary 7.6.64 for the first equality and 7.6.65 for the next to last one. This computation goes through for any $f, g \in K(\!(x^{-1})\!)$ with $f \asymp x$, but under the assumptions of the theorem gives the desired result. $\square$

Corollary 7.6.67. *Let $f = x + \sum_{n \geqslant 1} f_n \frac{x^{-2n+1}}{n!}$, all $f_n \in K$, and $g = f^{[-1]}$. Then*

$$g \;=\; x - \sum_{j \geqslant 1} g_j \frac{x^{-2j+1}}{j!} \quad \textit{where } g_j = \sum_{i=1}^{j} \frac{(2(j-1))!}{(2j-1-i)!} B_{ij}(f_1, \dots, f_{j-i+1}).$$

Proof. Put $F := \sum_{n \geqslant 1} f_n \frac{z^n}{n!} \in zK[[z]]$, $h := \sum_{n \geqslant 1} f_n \frac{x^{-2n}}{n!}$. Then $f = x(1+h)$, and $\operatorname{res}(f^n)$ is the coefficient of x^{-n-1} in $f^n/x^n = (1+h)^n$. Thus $\operatorname{res}(f^n) = 0$ if n is even. Now suppose n is odd, $n = 2j-1$ $(j \geqslant 1)$. Then the coefficient of $x^{-n-1} = x^{-2j}$ in $(1+h)^n$ equals the coefficient of z^j in the power series $(1+F)^n = \sum_{i=0}^{n} \frac{n!}{(n-i)!} \frac{F^i}{i!}$, and this coefficient in turn is given by

$$\frac{1}{j!} \sum_{i=1}^{j} \frac{n!}{(n-i)!} B_{ij}(f_1, \dots, f_{j-i+1}).$$

Now use Theorem 7.6.66 with x in the role of of g there. $\square$

Using the formulas for B_{ij} for small values of i, j given on [ADH, p. 554] we readily compute:

$$\begin{aligned}
g_1 &= f_1, \\
g_2 &= f_2 + 2f_1^2, \\
g_3 &= f_3 + 12 f_1 f_2 + 12 f_1^3, \\
g_4 &= f_4 + 24 f_1 f_3 + 18 f_2^2 + 180 f_1^2 f_2 + 120 f_1^4 \\
g_5 &= f_5 + 40 f_1 f_4 + 80 f_2 f_3 + 560 f_1^2 f_3 + 840 f_1 f_2^2 + 3660 f_1^3 f_2 + 1830 f_1^5
\end{aligned}$$

Remark. Suppose $K = \mathbb{R}$. Then $\mathbb{R}(\!(x^{-1})\!)$ is a subfield of $\mathbb{T}$, and the composition $(g, f) \mapsto g \circ f \colon \mathbb{T} \times \mathbb{T}^{>\mathbb{R}} \to \mathbb{T}$ in $\mathbb{T}$ (see the remarks after Corollary 5.3.12) extends the composition in $\mathbb{R}(\!(x^{-1})\!)$ defined above. All $f \in \mathbb{T}^{>\mathbb{R}}$ have a compositional inverse f^{inv} in $\mathbb{T}$, with $f^{\mathrm{inv}} = f^{[-1]}$ if $f \in \mathbb{R}(\!(x^{-1})\!)$. For $f > \mathbb{R}$ in the subfield $\mathbb{T}_{\mathrm{g}}$ of $\mathbb{T}$ consisting of the grid based series we have $f^{\mathrm{inv}} \in \mathbb{T}_{\mathrm{g}}$, and [103, Section 5.4.2] has a formula for the coefficients of f^{inv} in that case.

7.7. Holes and Slots in Perfect Hardy Fields

In this section $H \supseteq \mathbb{R}$ is a real closed Hardy field with asymptotic integration. We set $K := H[i] \subseteq \mathcal{C}^{<\infty}[i]$, an algebraically closed d-valued extension of H. Moreover, $\widehat{H}$ is an immediate H-field extension of H and $\widehat{K} := \widehat{H}[i]$ is the corresponding immediate d-valued extension of K as in Section 6.7. We also fix a d-maximal Hardy field extension H_* of H. The H-field H_* is newtonian, and the d-valued field extension $K_* := H_*[i] \subseteq \mathcal{C}^{<\infty}[i]$ of K is newtonian and linearly closed.

$$\begin{tikzcd}
K_* \arrow[dr, dash] \arrow[dd, dash] & & \widehat{K} \arrow[dd, dash] \\
& K \arrow[ur, dash] \arrow[dd, dash] & \\
H_* \arrow[dr, dash] & & \widehat{H} \\
& H \arrow[ur, dash] &
\end{tikzcd}$$

Recall that if $\mathrm{I}(K) \subseteq K^\dagger$ and $A \in K[\partial]^{\neq}$ splits over K, then A is terminal. In this section we show:

Theorem 7.7.1. *Suppose H is* d*-perfect and ω-free. Then every minimal hole in K of positive order is flabby. Moreover, H has no hole of order 1, every minimal hole in H of order 2 is flabby, and if all $A \in H[\partial]^{\neq}$ are terminal, then every minimal hole in H of positive order is flabby.*

In Corollary 7.7.50 below we also show that if H is d-perfect (but not necessarily ω-free), then every linear minimal hole $(P, \mathfrak{m}, \widehat{f})$ in K of order 1 with $\widehat{f} \in \widehat{K}$ is flabby. (See the discussion after the proof of Lemma 7.5.39 for an example of a d-perfect Hardy field that is not ω-free.)

The theorem above originated in an attempt to characterize ω-free d-perfect Hardy fields among Hardy fields containing $\mathbb{R}$ purely in terms of asymptotic differential algebra. We hope to return to this topic at a later occasion.

With the proof of Corollary 7.7.50 we also finish the proof of Theorem 7.7.1.

Asymptotic similarity and equivalence of slots. Let $(P, \mathfrak{m}, \widehat{f})$ be a slot in H of order $\geqslant 1$ where $\widehat{f} \in \widehat{H}$. If $f \in \mathcal{C}^{<\infty}$ is H-hardian and $(P, \mathfrak{m}, f)$ is a slot in H (we regard this as including the requirement that $f \notin H$ and the Hardy field $H\langle f \rangle$ is an immediate extension of H), then

$$f \approx_H \widehat{f} \iff (P, \mathfrak{m}, f) \text{ and } (P, \mathfrak{m}, \widehat{f}) \text{ are equivalent.}$$

Note that if $f \in \mathcal{C}$, $f \approx_H \widehat{f}$, and $g, h \in H$, $g \neq 0$, then $fg - h \approx_H \widehat{f}g - h$. From Corollary 3.2.29 and newtonianity of H_* we get a useful result about filling slots in H by elements of d-maximal Hardy field extensions of H:

Lemma 7.7.2. *If H is ω-free and $(P,\mathfrak{m},\widehat{f})$ is Z-minimal, then there exists $f\in H_*$ such that $(P,\mathfrak{m},f)$ is a hole in H equivalent to $(P,\mathfrak{m},\widehat{f})$, in particular, $P(f)=0$, $f\prec\mathfrak{m}$, and $f\approx_H\widehat{f}$.*

In Lemma 7.7.2 we cannot drop the assumption that H is ω-free. To see why, suppose H is d-perfect and not ω-free (such H exists by Example 7.5.40), and take $\omega\in H$ and $(P,\mathfrak{m},\lambda)$ as in Lemma 3.2.10 for H in the role of K there, so $P=2Y'+Y^2+\omega$ and $(P,\mathfrak{m},\lambda)$ is a minimal hole in H by Corollary 3.2.11. Since $\omega\notin\omega(H)$ and H is 1-d-closed in all its Hardy field extensions, no H-hardian germ f satisfies $P(f)=0$. Thus the conclusion of Lemma 7.7.2 fails for $\widehat{f}=\lambda$.

Corollary 3.2.30 yields a variant for P of order 1:

Lemma 7.7.3. *If H is λ-free and $(P,\mathfrak{m},\widehat{f})$ is Z-minimal of order 1 with a quasilinear refinement, then there exists $f\in H_*$ such that $H\langle f\rangle$ is an immediate extension of H and $(P,\mathfrak{m},f)$ is a hole in H equivalent to $(P,\mathfrak{m},\widehat{f})$.*

Here are complex versions of some of the above: Let $(P,\mathfrak{m},\widehat{f})$ be a slot in K of order $\geqslant 1$ where $\widehat{f}\in\widehat{K}$. If $f\in K_*$ and $(P,\mathfrak{m},f)$ is a slot in K (so $f\notin K$ and $K\langle f\rangle\subseteq K_*$ is an immediate extension of K), then

$$f\approx_K\widehat{f}\quad\Longleftrightarrow\quad(P,\mathfrak{m},f)\text{ and }(P,\mathfrak{m},\widehat{f})\text{ are equivalent.}$$

If $f\in\mathcal{C}[\mathrm{i}]$, $f\approx_K\widehat{f}$,and $g,h\in K$, $g\neq 0$, then $fg-h\approx_K\widehat{f}g-h$. Recall that H is ω-free iff K is, by [ADH, 11.7.23]. Again by Corollaries 3.2.29 and 3.2.30:

Lemma 7.7.4. *If H is ω-free and $(P,\mathfrak{m},\widehat{f})$ is Z-minimal as a slot in K, then there exists $f\in K_*$ such that $K\langle f\rangle$ is an immediate extension of K and $(P,\mathfrak{m},f)$ is a hole in K equivalent to $(P,\mathfrak{m},\widehat{f})$ (and thus $P(f)=0$, $f\prec\mathfrak{m}$, and $f\approx_K\widehat{f}$).*

Lemma 7.7.5. *If H is λ-free and, as a slot over K, $(P,\mathfrak{m},\widehat{f})$ is Z-minimal of order 1 with a quasilinear refinement, then there exists $f\in K_*$ such that $K\langle f\rangle$ is an immediate extension of K and $(P,\mathfrak{m},f)$ is a hole in K equivalent to $(P,\mathfrak{m},\widehat{f})$.*

In the rest of this section H is Liouville closed and $\mathrm{I}(K)\subseteq K^\dagger$. (These conditions are satisfied if H is d-perfect.) *We take an $\mathbb{R}$-linear complement Λ_H of $\mathrm{I}(H)$ in H, so $\Lambda:=\Lambda_H\mathrm{i}$ is a complement of $K^\dagger$ in K. Next we take an $\mathbb{R}$-linear complement Λ_{H_*} of $\mathrm{I}(H_*)$ in H_*, so $\Lambda_*:=\Lambda_{H_*}\mathrm{i}$ is a complement of $K_*^\dagger$ in K_*. Accordingly we identify in the usual way $\mathrm{U}:=\mathrm{U}_K:=K\big[\mathrm{e}(\Lambda)\big]$ with $K[\mathrm{e}^{H\mathrm{i}}]$ and likewise $\mathrm{U}_*:=\mathrm{U}_{K_*}:=K_*\big[\mathrm{e}(\Lambda_*)\big]$ with $K_*[\mathrm{e}^{H_*\mathrm{i}}]$.*

Zeros of linear differential operators close to the linear part of a slot. *In this subsection $(P,1,\widehat{h})$ with $\widehat{h}\in\widehat{H}$ is a normal or linear slot in H of order $r\geqslant 1$.* Then $\operatorname{order}(L_P)=r$, so $\dim_{\mathbb{C}}\ker_{\mathrm{U}_*}L_P=r$ by Theorem 7.4.1. Lemma 4.4.4(ii) then gives $\mathscr{E}^{\mathrm{u}}_{K_*}(L_P)=v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}_*}L_P)$. If L_P is terminal, then $\mathscr{E}^{\mathrm{u}}(L_P)=\mathscr{E}^{\mathrm{u}}_{K_*}(L_P)$, by Corollary 2.6.23. We use these remarks to deal with firm and flabby cases:

Lemma 7.7.6. *Suppose $(P,1,\widehat{h})$ is firm and ultimate and L_P is terminal. Then there is no $y\in\mathcal{C}^r[\mathrm{i}]^{\neq}$ such that $L_P(y)=0$ and $y\prec 1$.*

Proof. Suppose $y\in\mathcal{C}^r[\mathrm{i}]^{\neq}$, $L_P(y)=0$, and $y\prec 1$. Then $y\in\mathrm{U}_*$, so $y\prec_{\mathrm{g}}1$ by Lemma 5.10.8. The remarks above give $v_{\mathrm{g}}y\in\mathscr{E}^{\mathrm{u}}(L_P)$. Then $v_{\mathrm{g}}y\leqslant 0$ by Remark 4.4.29, contradicting $y\prec_{\mathrm{g}}1$. $\square$

Lemma 7.7.7. *Suppose $(P, 1, \widehat{h})$ is flabby. Then there exists $y \in \mathcal{C}^{<\infty}[\mathrm{i}]^{\neq}$ such that $L_P(y) = 0$ and $y \prec \mathfrak{m}$ for all $\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$. If in addition $(P, 1, \widehat{h})$ is Z-minimal, deep, and special, then $y', \dots, y^{(r)} \prec \mathfrak{m}$ for all such y and $\mathfrak{m}$.*

Proof. Flabbiness of $(P, 1, \widehat{h})$ and Lemmas 4.4.27 and 4.4.28 yield a $\gamma \in \mathscr{E}^{\mathrm{u}}(L_P)$ with $\gamma > v(\widehat{h} - H)$. Then $\gamma \in \mathscr{E}^{\mathrm{u}}_{K_*}(L_P)$ by Corollary 4.4.3, so a remark above gives $y \in \ker^{\neq}_{\mathrm{U}_*} L_P$ such that $v_{\mathrm{g}} y = \gamma$. Then $y \prec_{\mathrm{g}} \mathfrak{m}$ and thus $y \prec \mathfrak{m}$, for all $\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$. For the remainder, use Lemma 5.10.12. □

Next we consider a suitable perturbation A of L_P: *In the rest of this subsection we assume $L_P = A + B$ with $A, B \in K[\partial]$ satisfying*

$$\operatorname{order}(A) \ = \ r, \qquad \mathfrak{v} \ := \ \mathfrak{v}(A) \prec^\flat 1, \qquad B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1} A.$$

Then Lemma 3.1.1 gives $\mathfrak{v}(L_P) \sim \mathfrak{v}$. By Lemma 4.4.4(ii),(iii),

$$v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}_*} A) \ = \ \mathscr{E}^{\mathrm{u}}_{K_*}(A) \ = \ \mathscr{E}^{\mathrm{u}}_{K_*}(L_P), \qquad \mathscr{E}^{\mathrm{u}}(A) \ = \ \mathscr{E}^{\mathrm{u}}(L_P).$$

If A is terminal, then all five displayed sets are equal by Corollary 2.6.23. Recall also from Corollary 2.6.21 that if A splits over K, then A is terminal, and from Proposition 2.6.26 that if $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$, then A is terminal.

We can now generalize Proposition 5.10.15:

Proposition 7.7.8. *Suppose $(P, 1, \widehat{h})$ is ultimate, A is terminal, and $y \in \mathcal{C}^r[\mathrm{i}]$ satisfies $A(y) = 0$, $y \prec 1$. Then $y \prec \mathfrak{m}$ for all $\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$.*

Proof. We have $y \in \mathrm{U}_*$, so $y \prec_{\mathrm{g}} 1$ by Lemma 5.10.8. If $y = 0$, then we are done, so suppose $y \neq 0$. Then $0 < v_{\mathrm{g}} y \in \mathscr{E}^{\mathrm{u}}(L_P)$ by remarks before Proposition 7.7.8. Hence $v_{\mathrm{g}} y > v(\widehat{h} - H)$ by Lemma 4.4.12 if $(P, 1, \widehat{h})$ is normal, and by Lemma 4.4.13 if $(P, 1, \widehat{h})$ is linear, so $y \prec_{\mathrm{g}} \mathfrak{m}$ for all $\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$, and thus $y \prec \mathfrak{m}$ for all such $\mathfrak{m}$ by Corollary 5.10.9. □

Corollary 7.7.9. *Suppose A is terminal and $(P, 1, \widehat{h})$ is Z-minimal, deep, ultimate, and special. If $y \in \mathcal{C}^r[\mathrm{i}]$ satisfies $A(y) = 0$ and $y \prec 1$, then $y, y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$.*

Proof. First use Proposition 7.7.8 and then Lemma 5.10.12. □

Next we turn to firm and flabby cases.

Lemma 7.7.10. *Suppose A is terminal and $(P, 1, \widehat{h})$ is firm and ultimate. Then there is no $y \in \mathcal{C}^r[\mathrm{i}]^{\neq}$ such that $A(y) = 0$ and $y \prec 1$.*

Proof. Suppose $y \in \mathcal{C}^r[\mathrm{i}]^{\neq}$, $A(y) = 0$, and $y \prec 1$. Then $y \in \mathrm{U}_*$, so $y \prec_{\mathrm{g}} 1$ by Lemma 5.10.8. The remarks before Proposition 7.7.8 give $v_{\mathrm{g}} y \in \mathscr{E}^{\mathrm{u}}(L_P)$. Hence $v_{\mathrm{g}} y \leqslant 0$ by Remark 4.4.29, contradicting $y \prec_{\mathrm{g}} 1$. □

Lemma 7.7.11. *Suppose $(P, 1, \widehat{h})$ is flabby. Then there exists $y \in \mathcal{C}^{<\infty}[\mathrm{i}]^{\neq}$ such that $A(y) = 0$ and $y \prec \mathfrak{m}$ for all $\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$. If in addition $(P, 1, \widehat{h})$ is Z-minimal, deep, and special, then $y', \dots, y^{(r)} \prec \mathfrak{m}$ for all such y and $\mathfrak{m}$.*

Proof. Flabbiness of $(P, 1, \widehat{h})$ and Lemmas 4.4.27 and 4.4.28 yield a $\gamma \in \mathscr{E}^{\mathrm{u}}(L_P) = \mathscr{E}^{\mathrm{u}}(A)$ with $\gamma > v(\widehat{h} - H)$. The rest of the proof is the same as that of Lemma 7.7.7 with A instead of L_P. □

Remark. The material above in this subsection goes through if instead of $(P, 1, \widehat{h})$ with $\widehat{h} \in \widehat{H}$ being a normal or linear slot in H of order $r \geqslant 1$ we assume $(P, 1, \widehat{h})$ with $\widehat{h} \in \widehat{K}$ is a normal or linear slot in K of order $r \geqslant 1$, and "$\mathfrak{m} \in H^\times$ with $v\mathfrak{m} \in v(\widehat{h} - H)$" is replaced everywhere by "$\mathfrak{m} \in K^\times$ with $v\mathfrak{m} \in v(\widehat{h} - K)$".

To see this, use the K-versions of Lemmas 4.4.12, 4.4.13, 4.4.27, of Lemma 4.4.28 and Remark 4.4.29, and of Lemma 5.10.12; cf. the discussion at the end of the subsection *An application to slots in H* of Section 5.10.

Application to linear slots. In this subsection we apply the material in the last subsection to the study of linear slots (in H and in K). Until further notice $(P, \mathfrak{m}, \widehat{h})$ with $\widehat{h} \in \widehat{H}$ is a Z-minimal linear slot in H of order $r \geqslant 1$.

Lemma 7.7.12. *There exists $f \in H_*$ such that $P(f) = 0$ and $f \prec \mathfrak{m}$.*

Proof. We may replace $(P, \mathfrak{m}, \widehat{h})$ by a refinement whenever convenient. Hence by Remark 3.4.7 we may arrange that $(P, \mathfrak{m}, \widehat{h})$ is isolated. Then $P(0) \neq 0$, and $\gamma := v\widehat{h}$ is the unique element of $\Gamma \setminus \mathscr{E}^{\mathrm{e}}(L_P)$ such that $v^{\mathrm{e}}_{L_P}(\gamma) = v(P(0))$, by Lemmas 3.2.14 and 3.4.15. Now H_* is linearly newtonian, so Corollary 1.5.7 yields $f \in H_*^\times$ with $P(f) = 0$, $vf \notin \mathscr{E}^{\mathrm{e}}_{H^*}(L_P)$, and $v^{\mathrm{e}}_{L_P}(vf) = v(P(0))$. By Corollary 1.8.10, $v^{\mathrm{e}}_{L_P}(\gamma)$ does not change when passing from H to H_*, and $\gamma \notin \mathscr{E}^{\mathrm{e}}_{H_*}(L_P)$. Thus $vf = \gamma$ by Lemma 1.5.6; in particular, $f \prec \mathfrak{m}$. $\square$

Corollary 7.7.13. *Suppose $(P, \mathfrak{m}, \widehat{h})$ is flabby, and $f \in \mathcal{C}^r$ is such that $P(f) = 0$ and $f \prec \mathfrak{m}$. Then $f \in \mathcal{C}^{<\infty}$ and there exists $g \in \mathcal{C}^{<\infty}$ such that $P(g) = 0$, $g \prec \mathfrak{m}$, and $0 \neq f - g \prec \mathfrak{n}$ for all $\mathfrak{n} \in H^\times$ with $v\mathfrak{n} \in v(\widehat{h} - H)$. For any such g we have*

$$f \approx_H \widehat{h} \;\Rightarrow\; g \approx_H \widehat{h}, \quad H \subseteq \mathcal{C}^\infty \;\Rightarrow\; f, g \in \mathcal{C}^\infty, \quad H \subseteq \mathcal{C}^\omega \;\Rightarrow\; f, g \in \mathcal{C}^\omega.$$

If $(P, \mathfrak{m}, \widehat{h})$ is also deep and special, then $f - g \in \mathfrak{m}(\mathcal{C}^r)^\prec$ for any such g.

Proof. Lemma 6.3.4 gives $f \in \mathcal{C}^{<\infty}$. Replace $(P, \mathfrak{m}, \widehat{h})$, f, by $(P_{\times\mathfrak{m}}, 1, \widehat{h}/\mathfrak{m})$, $f/\mathfrak{m}$ to arrange $\mathfrak{m} = 1$. Lemma 7.7.7 then yields $y \in \mathcal{C}^{<\infty}[\mathrm{i}]^{\neq}$ such that $L_P(y) = 0$ and $y \prec \mathfrak{n}$ for all $\mathfrak{n} \in H^\times$ with $v\mathfrak{n} \in v(\widehat{h} - H)$. Replacing y by $\operatorname{Re} y$ or $\operatorname{Im} y$ we arrange $y \in \mathcal{C}^{<\infty}$. Then $g := f + y \in \mathcal{C}^{<\infty}$ satisfies $f \neq g$, $P(g) = 0$, and $g \prec 1$. The rest follows from remarks after Corollary 5.2.3 and from Lemma 7.7.7. $\square$

In the proof of the next corollary we use that if H is $\upomega$-free, then Lemma 7.7.2 yields an $f \in H_*$ such that $P(f) = 0$ and $f \approx_H \widehat{h}$.

Corollary 7.7.14. *Suppose $(P, \mathfrak{m}, \widehat{h})$ is ultimate and L_P is terminal. Then*

$$(P, \mathfrak{m}, \widehat{h}) \text{ is firm} \iff \text{there is a unique } f \in \mathcal{C}^r \text{ with } P(f) = 0 \text{ and } f \prec \mathfrak{m}.$$

If $(P, \mathfrak{m}, \widehat{h})$ is firm, $f \in \mathcal{C}^r$, $P(f) = 0$, $f \prec \mathfrak{m}$, then $f \in \mathrm{D}(H)$, there is no $g \neq f$ in $\mathcal{C}^r[\mathrm{i}]$ with $P(g) = 0$, $g \prec \mathfrak{m}$, and if in addition H is $\upomega$-free, then $f \approx_H \widehat{h}$.

Proof. We arrange $\mathfrak{m} = 1$ as before. Then Lemmas 7.7.6 and 7.7.12 yield "$\Rightarrow$". For "$\Leftarrow$" and the rest, use Corollary 7.7.13 and the remark after its proof, and observe that our d-maximal Hardy field extension H_* of H was arbitrary. $\square$

Corollary 7.7.15. *Suppose H is $\upomega$-free and* d*-perfect, and all $A \in H[\partial] \subseteq K[\partial]$ of order r are terminal. Then every Z-minimal linear slot in H of order r is flabby.*

Proof. Given a firm Z-minimal linear slot in H of order r we use Remark 4.4.15 and Lemma 4.4.25 to refine it to be ultimate. So we arrive at an ultimate firm Z-minimal linear slot in H of order r with terminal linear part. This contradicts H being d-perfect by Corollary 7.7.14. $\square$

Next the K-versions of Lemma 7.7.12 and its corollaries: Let $(P, \mathfrak{m}, \widehat{f})$ with $\widehat{f} \in \widehat{K}$ be a Z-minimal linear slot in K of order $r \geqslant 1$. Now K is λ-free [ADH, 11.6.8], so we can mimick the proof of Lemma 7.7.12 to obtain:

Lemma 7.7.16. *There exists $f \in K_*$ such that $P(f) = 0$ and $f \prec \mathfrak{m}$.*

The K-version of Lemma 7.7.7 leads to the K-version of Corollary 7.7.13:

Corollary 7.7.17. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is flabby, and $f \in \mathcal{C}^r[\mathrm{i}]$, $P(f) = 0$, and $f \prec \mathfrak{m}$. Then $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ and there exists $g \in \mathcal{C}^{<\infty}[\mathrm{i}]$ such that $P(g) = 0$, $g \prec \mathfrak{m}$, and $0 \neq f - g \prec \mathfrak{n}$ for all $\mathfrak{n} \in K^\times$ with $v\mathfrak{n} \in v(\widehat{f} - K)$. For any such g we have*

$$f \approx_K \widehat{f} \ \Rightarrow\ g \approx_K \widehat{f}, \quad H \subseteq \mathcal{C}^\infty \ \Rightarrow\ f, g \in \mathcal{C}^\infty[\mathrm{i}], \quad H \subseteq \mathcal{C}^\omega \ \Rightarrow\ f, g \in \mathcal{C}^\omega[\mathrm{i}].$$

If $(P, \mathfrak{m}, \widehat{f})$ is also deep and special, then $f - g \in \mathfrak{m}\mathcal{C}^r[\mathrm{i}]^\prec$ for any such g.

If H is ω-free, then Lemma 7.7.4 yields $f \in K_*$ with $P(f) = 0$ and $f \approx_K \widehat{f}$. This remark and K-versions of various results yield the K-version of Corollary 7.7.14:

Corollary 7.7.18. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is ultimate and L_P is terminal. Then:*

$$(P, \mathfrak{m}, \widehat{f}) \text{ is firm} \quad \Longleftrightarrow \quad \text{there is a unique } f \in \mathcal{C}^r[\mathrm{i}] \text{ with } P(f) = 0 \text{ and } f \prec \mathfrak{m}.$$

If $(P, \mathfrak{m}, \widehat{f})$ is firm, $f \in \mathcal{C}^r[\mathrm{i}]$, $P(f) = 0$, $f \prec \mathfrak{m}$, then $f \in \mathrm{D}(H)[\mathrm{i}]$, and also $f \approx_K \widehat{f}$ in case H is ω-free.

Using K-versions of various results (like Remark 4.4.19 instead of Remark 4.4.15), then yields the K-version of Corollary 7.7.15:

Corollary 7.7.19. *If H is ω-free and* d*-perfect, and all $A \in K[\partial]$ of order r are terminal, then every Z-minimal linear slot in K of order r is flabby.*

Linear slots in K of order 1 are Z-minimal, and we can say more in this case:

Corollary 7.7.20. *Suppose $r = 1$. If $(P, \mathfrak{m}, \widehat{f})$ is flabby, then it is ultimate. Moreover, L_P is terminal, so if $(P, \mathfrak{m}, \widehat{f})$ is firm and ultimate, then there is a unique $f \in \mathcal{C}^r[\mathrm{i}]$ with $P(f) = 0$ and $f \prec \mathfrak{m}$, and for this f we have: $f \in \mathrm{D}(H)[\mathrm{i}]$, with $f \approx_K \widehat{f}$ in case H is ω-free or $(P, \mathfrak{m}, \widehat{f})$ is a hole in K.*

Proof. Corollary 4.4.31(i) gives "flabby $\Rightarrow$ ultimate". Since L_P has order 1, it is terminal. Now use Corollary 7.7.18, and Lemma 7.7.3, noting in connection with that lemma that the linear slot $(P, \mathfrak{m}, \widehat{f})$ is quasilinear. $\square$

Our next goal is to establish refinements of Proposition 6.5.14 for the case of firm and flabby slots in H: Lemmas 7.7.33, 7.7.34, 7.7.36, and 7.7.42 below. Towards this goal, we introduce yet another useful concept of normality for slots.

Absolutely normal slots in H. *In this subsection $(P, \mathfrak{m}, \widehat{h})$ is a slot in H of order $r \geqslant 1$ with $\widehat{h} \in \widehat{H}$.* Given active $\phi > 0$ in H we take $\ell \in H$ with $\ell' = \phi$, and set $f^\circ := f \circ \ell^{\text{inv}}$ for $f \in \mathcal{C}[i]$, as usual; see Section 6.4. Recall from Section 5.3 that H° is Liouville closed with $K^\circ = H^\circ[i]$ and $\mathrm{I}(K^\circ) \subseteq (K^\circ)^\dagger$, and that H°_* is a d-maximal Hardy field extension of H°, with $K^\circ_* = H^\circ_*[i]$.

Since K_* is linearly closed, each linear differential operator $A \in K[\partial]^{\neq}$ splits over K_*. If $A \in K[\partial]^{\neq}$ splits strongly over K_*, then by Theorem 7.1.3 this remains true when K_* is replaced by $K_{**} := H_{**}[i] \subseteq \mathcal{C}^{<\infty}[i]$ for any d-maximal Hardy field extension H_{**} of H. We say that $(P, \mathfrak{m}, \widehat{h})$ is **absolutely normal** if it is strictly normal and its linear part splits strongly over K_*. If $(P, \mathfrak{m}, \widehat{h})$ is absolutely normal, then so is $(P_{\times\mathfrak{m}}, 1, \widehat{h}/\mathfrak{m})$. Moreover:

Lemma 7.7.21. *Suppose $(P, \mathfrak{m}, \widehat{h})$ is absolutely normal, and ϕ is active in H with $0 < \phi \preccurlyeq 1$. Then the slot $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{h}^\circ)$ in H° is absolutely normal.*

Proof. By Lemma 3.3.40, the slot $(P^\phi, \mathfrak{m}, \widehat{h})$ in H^ϕ is strictly normal, hence so is the slot $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{h}^\circ)$ in H°. By Lemma 4.2.12 the linear part $L_{P^\phi_{\times\mathfrak{m}}} = (L_{P_{\times\mathfrak{m}}})^\phi$ of $(P^\phi, \mathfrak{m}, \widehat{h})$ splits strongly over $K^\phi_* = H^\phi_*[i]$, hence the linear part of $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{h}^\circ)$ splits strongly over $K^\circ_* = H^\circ_*[i]$. □

Next we show how to achieve absolute normality:

Proposition 7.7.22. *Suppose $(P, \mathfrak{m}, \widehat{h})$ is Z-minimal, deep, and strictly normal, and $\widehat{h} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ with $\mathfrak{v} := \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$. Then for all sufficiently small $q \in \mathbb{Q}^>$, $(P, \mathfrak{n}, \widehat{h})$, for any $\mathfrak{n} \asymp \mathfrak{m}|\mathfrak{v}|^q$ in $H^\times$, is a deep, absolutely normal refinement of $(P, \mathfrak{m}, \widehat{h})$.*

Proof. Recall that $L_{P_{\times\mathfrak{m}}}$ splits over the H-asymptotic extension K_* of K. The argument in the proof of Corollary 4.2.14 shows that for all sufficiently small $q \in \mathbb{Q}^>$, $(P, \mathfrak{n}, \widehat{h})$, for any $\mathfrak{n} \asymp \mathfrak{m}|\mathfrak{v}|^q$ in $H^\times$, is a steep refinement of $(P, \mathfrak{m}, \widehat{h})$ whose linear part splits strongly over K_*. By Corollary 3.3.6 any such refinement $(P, \mathfrak{n}, \widehat{h})$ of $(P, \mathfrak{m}, \widehat{h})$ is deep, and for all sufficiently small $q \in \mathbb{Q}^>$, any such refinement of $(P, \mathfrak{m}, \widehat{h})$ also remains strictly normal, by Lemma 3.3.44 and Remark 3.3.45. □

Corollary 7.7.23. *If $(P, \mathfrak{m}, \widehat{h})$ is Z-minimal, deep, normal, and special, then $(P, \mathfrak{m}, \widehat{h})$ has a deep, absolutely normal refinement.*

This follows from Corollary 3.3.47 and Proposition 7.7.22.

Lemma 7.7.24. *Suppose H is $\upomega$-free and $(P, \mathfrak{m}, \widehat{h})$ is Z-minimal and special. Then there are a refinement $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ of $(P, \mathfrak{m}, \widehat{h})$ and an active $\phi > 0$ in H such that the slot $(P^{\phi\circ}_{+h^\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ - h^\circ)$ in H° is deep, absolutely normal, and ultimate.*

Proof. For any active $\phi > 0$ in H we may replace H, $(P, \mathfrak{m}, \widehat{h})$ by H°, $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{h}^\circ)$, and we may also replace $(P, \mathfrak{m}, \widehat{h})$ by any of its refinements. Since H is $\upomega$-free, Proposition 3.3.36 yields a refinement $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ of $(P, \mathfrak{m}, \widehat{h})$ and an active $\phi > 0$ in H such that the slot $(P^{\phi\circ}_{+h^\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ - h^\circ)$ in H° is normal. Replacing H, $(P, \mathfrak{m}, \widehat{h})$ by H°, $(P^{\phi\circ}_{+h^\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ - h^\circ)$ we arrange that $(P, \mathfrak{m}, \widehat{h})$ is normal. Proposition 4.4.14 now yields an ultimate refinement of $(P, \mathfrak{m}, \widehat{h})$. Applying Proposition 3.3.36 to this refinement and using Lemma 4.4.10, we obtain an ultimate refinement $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$

of $(P,\mathfrak{m},\widehat{h})$ and an active $\phi>0$ in H such that $(P^{\phi\circ}_{+h^\circ},\mathfrak{n}^\circ,\widehat{h}^\circ-h^\circ)$ is deep, normal, and ultimate. Again replacing H, $(P,\mathfrak{m},\widehat{h})$ by H°, $(P^{\phi\circ}_{+h^\circ},\mathfrak{n}^\circ,\widehat{h}^\circ-h^\circ)$, we arrange that $(P,\mathfrak{m},\widehat{h})$ is deep, normal, and ultimate. Now apply Corollary 7.7.23 to $(P,\mathfrak{m},\widehat{h})$ and use Lemma 4.4.10. $\square$

Corollary 7.7.25. *Suppose H is $\upomega$-free and r-linearly newtonian, and $(P,\mathfrak{m},\widehat{h})$ is Z-minimal. Then the conclusion of Lemma 7.7.24 holds.*

Proof. As in the beginning of the proof of Lemma 7.7.24, use Theorem 3.3.33 to arrange that $(P,\mathfrak{m},\widehat{h})$ is normal. Then $(P,\mathfrak{m},\widehat{h})$ is quasilinear by Corollary 3.3.21 and hence special by Lemma 3.2.36, so Lemma 7.7.24 applies to $(P,\mathfrak{m},\widehat{h})$. $\square$

Remark 7.7.26. By Corollary 3.2.6 the hypotheses of Corollary 7.7.25 are satisfied if H is $\upomega$-free and $(P,\mathfrak{m},\widehat{h})$ is a nonlinear minimal hole in H.

Absolutely normal slots in K. *Let $(P,\mathfrak{m},\widehat{f})$ be a slot in K of order $r\geqslant 1$, with $\widehat{f}\in\widehat{K}$.* Call $(P,\mathfrak{m},\widehat{f})$ **absolutely normal** if it is strictly normal and $L_{P_{\times\mathfrak{m}}}$ splits strongly over K_*. If $(P,\mathfrak{m},\widehat{f})$ is absolutely normal, then so is $(P_{\times\mathfrak{m}},1,\widehat{f}/\mathfrak{m})$. If $(Q,\mathfrak{n},\widehat{h})$ is a slot in H of order $\geqslant 1$ with $\widehat{h}\in\widehat{H}\subseteq\widehat{K}$, then it is a slot in K (Corollary 4.3.2), and $(Q,\mathfrak{n},\widehat{h})$ is absolutely normal as a slot in H iff it is absolutely normal as a slot in K.

Lemma 7.7.27. *Suppose $(P,\mathfrak{m},\widehat{f})$ is absolutely normal. Then there exists y in $\mathcal{C}^{<\infty}[i]\cap\mathfrak{m}\,\mathcal{C}^r[i]^{\preccurlyeq}$ such that $P(y)=0$ and $y\prec\mathfrak{m}$. If $H\subseteq\mathcal{C}^\infty$, then any such y lies in $\mathcal{C}^\infty[i]$, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Proof. Use Lemma 6.4.5 with $Q=(P_{\times\mathfrak{m}})_1$ and H_*, K_*, $P_{\times\mathfrak{m}}$ in place of H, K, P. For the last part, use Corollary 6.3.5 as in the proof of that lemma. $\square$

The K-versions of Lemma 7.7.21, Proposition 7.7.22, and Corollary 7.7.23 (with the same proofs) are as follows:

Lemma 7.7.28. *Suppose $(P,\mathfrak{m},\widehat{f})$ is absolutely normal, and ϕ is active in H with $0<\phi\preccurlyeq 1$. Then the slot $(P^{\phi\circ},\mathfrak{m}^\circ,\widehat{h}^\circ)$ in K° is absolutely normal.*

Proposition 7.7.29. *Suppose $(P,\mathfrak{m},\widehat{f})$ is Z-minimal, deep, and strictly normal, and $\widehat{f}\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$ with $\mathfrak{v}:=\mathfrak{v}(L_{P_{\times\mathfrak{m}}})$. Then for all sufficiently small $q\in\mathbb{Q}^>$, $(P,\mathfrak{n},\widehat{f})$, for any $\mathfrak{n}\asymp\mathfrak{m}|\mathfrak{v}|^q$ in $K^\times$, is a deep, absolutely normal refinement of $(P,\mathfrak{m},\widehat{f})$.*

Corollary 7.7.30. *If $(P,\mathfrak{m},\widehat{f})$ is Z-minimal, deep, normal, and special, then $(P,\mathfrak{m},\widehat{f})$ has a deep, absolutely normal refinement.*

The K-version of Lemma 7.7.24 is as follows (its proof uses Proposition 4.4.18 instead of Proposition 4.4.14, and the K-version of Lemma 4.4.10):

Lemma 7.7.31. *Suppose H is $\upomega$-free and $(P,\mathfrak{m},\widehat{f})$ is Z-minimal and special. Then there are a refinement $(P_{+f},\mathfrak{n},\widehat{f}-f)$ of $(P,\mathfrak{m},\widehat{f})$ and an active $\phi>0$ in H such that the slot $(P^{\phi\circ}_{+f^\circ},\mathfrak{n}^\circ,\widehat{f}^\circ-f^\circ)$ in K° is deep, absolutely normal, and ultimate.*

The K-version of Corollary 7.7.25 now follows in the same way:

Corollary 7.7.32. *Suppose H is $\upomega$-free, K is r-linearly newtonian, and $(P,\mathfrak{m},\widehat{f})$ is Z-minimal. Then the conclusion of Lemma 7.7.31 holds.*

Firm slots in H. *In this subsection $(P,\mathfrak{m},\widehat{h})$ is a slot in H of order $r\geqslant 1$, with $\widehat{h}\in\widehat{H}$.* We set $d:=\deg(P)$, $w:=\operatorname{wt}(P)$, and begin with a significant strengthening of Proposition 6.5.14 for firm slots in H:

Lemma 7.7.33. *Suppose $(P,\mathfrak{m},\widehat{h})$ is firm, ultimate, and strongly split-normal, and let $f,g\in\mathcal{C}^r[i]$ satisfy $P(f)=P(g)=0$ and $f,g\prec\mathfrak{m}$. Then $f=g$.*

Proof. The proof is similar to that of Proposition 6.5.14. We first replace $(P,\mathfrak{m},\widehat{h})$, f, g by $(P_{\times\mathfrak{m}},1,\widehat{h}/\mathfrak{m})$, $f/\mathfrak{m}$, $g/\mathfrak{m}$, to arrange $\mathfrak{m}=1$. We set $\mathfrak{v}:=|\mathfrak{v}(L_P)|\prec^\flat 1$ and $\Delta:=\Delta(\mathfrak{v})$, and take $Q,R\in H\{Y\}$ where Q is homogeneous of degree 1 and order r, $A:=L_Q\in H[\partial]$ splits strongly over K, $P=Q-R$, and $R\prec_\Delta\mathfrak{v}^{w+1}P_1$, so $\mathfrak{v}(A)\sim\mathfrak{v}(L_P)$. Multiplying P, Q, R by some $b\in H^\times$ we arrange that $A=\partial^r+f_1\partial^{r-1}+\cdots+f_r$ with $f_1,\dots,f_r\in H$ and $R\prec_\Delta\mathfrak{v}^w$. We have

$$(7.7.1)\quad A\ =\ (\partial-\phi_1)\cdots(\partial-\phi_r),\quad \phi_1,\dots,\phi_r\in K,\quad \operatorname{Re}\phi_1,\dots,\operatorname{Re}\phi_r\ \succcurlyeq\ \mathfrak{v}^\dagger\ \succcurlyeq\ 1.$$

Corollary 3.1.6 yields $\phi_1,\dots,\phi_r\preccurlyeq\mathfrak{v}^{-1}$. Take $a_0\in\mathbb{R}$ and functions on $[a_0,\infty)$ representing the germs $\phi_1,\dots,\phi_r$, $f_1,\dots,f_r$, f, g and the $R_{\boldsymbol{j}}$ with $\boldsymbol{j}\in\mathbb{N}^{1+r}$, $|\boldsymbol{j}|\leqslant d$, $\|\boldsymbol{j}\|\leqslant w$ (using the same symbols for the germs mentioned as for their chosen representatives) so as to be in the situation described in the beginning of Section 6.2, with f and g solutions on $[a_0,\infty)$ of the differential equation $(*)$ there. As there, we take $\nu\in\mathbb{Q}$ with $\nu>w$ so that $R\prec_\Delta\mathfrak{v}^\nu$ and $\nu\mathfrak{v}^\dagger\not\sim\operatorname{Re}\phi_j$ for $j=1,\dots,r$, and then increase a_0 to satisfy all assumptions for Lemma 6.2.1.

With $a\geqslant a_0$ and $h_a\in\mathcal{C}^r_a[i]$ as in Lemma 6.2.5 we have $A_a(h_a)=0$ and $h_a\prec 1$. Now A splits over K, so A is terminal as an element of $K[\partial]$, by Corollary 2.6.21. As $(P,1,\widehat{h})$ is firm and ultimate, this yields $h_a=0$ (for all $a\geqslant a_0$) by Lemma 7.7.10 and Corollary 5.2.2, and thus $f=g$ by Corollary 6.2.15. □

Next we prove variants of Lemma 7.7.33 by modifying the restrictive hypothesis of strong split-normality.

Lemma 7.7.34. *Suppose $(P,\mathfrak{m},\widehat{h})$ is firm, ultimate, and absolutely normal, and its linear part $L_{P_{\times\mathfrak{m}}}\in H[\partial]\subseteq K[\partial]$ is terminal. Then for all $f,g\in\mathcal{C}^r[i]$ such that $P(f)=P(g)=0$ and $f,g\prec\mathfrak{m}$ we have $f=g$.*

Proof. Replacing $(P,\mathfrak{m},\widehat{h})$ by $(P_{\times\mathfrak{m}},1,\widehat{h}/\mathfrak{m})$ we arrange $\mathfrak{m}=1$. Put $A:=L_P\in H[\partial]$ and $R:=P_1-P\in H\{Y\}$, so $R\prec_\Delta\mathfrak{v}^{w+1}P_1$ where $\Delta:=\Delta(\mathfrak{v})$, $\mathfrak{v}:=\mathfrak{v}(A)\prec^\flat 1$. Multiplying A, P, R on the left by some $b\in H^\times$ we arrange

$$A\ =\ \partial^r+f_1\partial^{r-1}+\cdots+f_r,\quad f_1,\dots,f_r\in H,\quad R\prec_\Delta\mathfrak{v}^w.$$

Then (7.7.1) holds with $\phi_1,\dots,\phi_r\in K_*$ instead of $\phi_1,\dots,\phi_r\in K$, and $\phi_1,\dots,\phi_r\preccurlyeq\mathfrak{v}^{-1}$ by Corollary 3.1.6. Now argue as at the end of the proof of Lemma 7.7.33 to get $f=g$, using that $A\in K[\partial]$ is terminal by assumption. □

From Corollary 2.6.21 and Lemma 7.7.34 we obtain:

Corollary 7.7.35. *If $(P,\mathfrak{m},\widehat{h})$ is firm, ultimate, and strictly normal, and its linear part splits strongly over K, then the conclusion of Lemma 7.7.34 holds.*

In the rest of this subsection we assume that all $A\in H[\partial]\subseteq K[\partial]$ of order r are terminal. Recall that by Lemmas 4.4.10 and 4.4.25, if $(P,\mathfrak{m},\widehat{h})$ is ultimate, then so is any refinement of it, and likewise with “firm” in place of “ultimate”.

Lemma 7.7.36. *Suppose $(P,\mathfrak{m},\widehat{h})$ is Z-minimal, deep, normal, special, ultimate, firm, and $f,g\in\mathcal{C}^r[\mathrm{i}]$, $P(f)=P(g)=0$, $f\approx_K\widehat{h}$, $g\approx_K\widehat{h}$. Then $f=g$.*

Proof. Corollary 7.7.23 gives a deep absolutely normal refinement $(P_{+h},\mathfrak{n},\widehat{h}-h)$ of $(P,\mathfrak{m},\widehat{h})$. Now apply Lemma 7.7.34 to $(P_{+h},\mathfrak{n},\widehat{h}-h)$, $f-h$, $g-h$ in the role of $(P,\mathfrak{m},\widehat{h})$, f, g. $\square$

Corollary 7.7.37. *Suppose H is $\upomega$-free and r-linearly newtonian, and $(P,\mathfrak{m},\widehat{h})$ is firm and Z-minimal. Then there is a unique $f\in\mathcal{C}^r[\mathrm{i}]$ with $P(f)=0$ and $f\approx_K\widehat{h}$. For this f we have $f\in\mathrm{D}(H)$ and $f\approx_H\widehat{h}$.*

Proof. Suppose $f,g\in\mathcal{C}^r[\mathrm{i}]$ satisfy $P(f)=P(g)=0$, $f\approx_K\widehat{h}$, and $g\approx_K\widehat{h}$; we claim that $f=g$. If $\phi>0$ is active in H, then by the remarks before Lemma 4.4.25, Lemma 6.4.3, and the remarks at the end of Section 6.6, and with the superscript $\circ$ having the usual meaning, we may replace H, K, $\widehat{H}$, $\widehat{K}$, $(P,\mathfrak{m},\widehat{h})$, f, g by H°, K°, $\widehat{H}^\circ$, $\widehat{K}^\circ$, $(P^{\phi\circ},\mathfrak{m}^\circ,\widehat{h}^\circ)$, f°, g°. Using this observation, Corollary 7.7.25 and the remarks before Lemma 7.7.36, we arrange that $(P,\mathfrak{m},\widehat{h})$ is ultimate and absolutely normal. The claim now follows from Lemma 7.7.34. From Lemma 7.7.2 we obtain $f\in H_*$ with $P(f)=0$ and $f\approx_H\widehat{h}$; then $f\approx_K\widehat{h}$ by Corollary 6.6.13. Our d-maximal Hardy field extension H_* of H was arbitrary, so the uniqueness statement just proved gives $f\in\mathrm{D}(H)$. $\square$

Corollary 7.7.38. *Suppose H is $\upomega$-free and $(P,\mathfrak{m},\widehat{h})$ is a firm minimal hole in H. Then the conclusion of Corollary 7.7.37 holds.*

Proof. If $(P,\mathfrak{m},\widehat{h})$ is nonlinear, then H is r-linearly newtonian by Corollary 3.2.6, so the hypotheses of Corollary 7.7.37 are satisfied, and so is its conclusion.

Suppose $(P,\mathfrak{m},\widehat{h})$ is linear. By Remark 4.4.15 we can refine $(P,\mathfrak{m},\widehat{h})$ to arrange it to be ultimate. By our standing assumption L_P is terminal, so we can appeal to Corollary 7.7.14. $\square$

Z-minimal slots in d-perfect Hardy fields. *In this subsection H is* d*-perfect.* By Corollary 7.2.15, H is 1-newtonian and so has no quasilinear Z-minimal slot of order 1, by Corollary 3.4.14. This allows us to add to the characterization of $\upomega$-freeness for d-perfect Hardy fields given in Corollary 7.5.9:

Corollary 7.7.39.

$$H \text{ is } \upomega\text{-free} \iff H \text{ has no hole of order } 1 \iff H \text{ has no slot of order } 1.$$

Proof. The first equivalence follows from Lemma 3.2.1 and Corollary 7.2.15. For the rest we observe that if H has a slot of order 1, then it also has a hole of order 1: Given a slot $(P,\mathfrak{m},\widehat{h})$ in H of order 1, take $Q\in Z(H,\widehat{h})$ of minimal complexity. Then $(Q,\mathfrak{m},\widehat{h})$ is a Z-minimal slot of order $\leqslant 1$ in H, hence is equivalent to a Z-minimal hole $(Q,\mathfrak{m},\widehat{b})$ in H, by Lemma 3.2.14, so $\operatorname{order} Q=1$ by a remark after Lemma 3.2.1. $\square$

Next, an immediate consequence of Corollaries 7.7.15 and 7.7.38:

Corollary 7.7.40. *Let $r\in\mathbb{N}^{\geqslant 1}$. If H is $\upomega$-free and all $A\in H[\partial]$ of order r are terminal, then every minimal hole in H of order r is flabby.*

From this we deduce:

Corollary 7.7.41. *Suppose H is $\upomega$-free. Then every minimal hole in H of order 2 and every linear slot in H of order 2 is flabby.*

Proof. By Corollaries 2.6.21 and 7.5.9, all $A \in H[\partial]$ of order 2 are terminal. Hence every minimal hole in H of order 2 is flabby, by Corollary 7.7.40. Every linear slot in H of order 2 is Z-minimal, by Corollary 7.7.39, and hence is flabby by Corollary 7.7.15. $\square$

In the next subsection we study flabby slots in H in more detail.

Flabby slots in H. *In this subsection $(P, \mathfrak{m}, \widehat{h})$ is a slot in H of order $r \geqslant 1$.* Note that if $(P, \mathfrak{m}, \widehat{f})$ is normal and $f \in \mathfrak{m}\mathcal{C}^r[i]^{\preccurlyeq}$, $P(f) = 0$, then by Corollary 6.3.6 we have $f \in \mathcal{C}^{<\infty}[i]$, and $f \in \mathcal{C}^{\infty}[i]$ if $H \subseteq \mathcal{C}^{\infty}$, $f \in \mathcal{C}^{\omega}[i]$ if $H \subseteq \mathcal{C}^{\omega}$.

Next some observations tacitly used in the proof of Lemma 7.7.42 below. For this, suppose $(P, \mathfrak{m}, \widehat{h})$ is flabby. Then $(P_{\times\mathfrak{m}}, 1, \widehat{h}/\mathfrak{m})$ is flabby by Lemma 4.4.26, and if $g \in (\mathcal{C}^r)^{\preccurlyeq}$ and $P_{\times\mathfrak{m}}(g) = 0$, $g \prec 1$, then $f := \mathfrak{m}g \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$ satisfies $P(f) = 0$ and $f \prec \mathfrak{m}$. Likewise, let $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ be a refinement of $(P, \mathfrak{m}, \widehat{h})$, and suppose $(P, \mathfrak{m}, \widehat{h})$ is also linear or normal. Then the slot $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ in H is flabby by Corollary 4.4.30, and if $g \in \mathfrak{n}\,(\mathcal{C}^r)^{\preccurlyeq}$ and $P_{+h}(g) = 0$, $g \prec \mathfrak{n}$, then $f := h + g \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$ satisfies $P(f) = 0$ and $f \prec \mathfrak{m}$. Finally, let ϕ be active in H, $0 < \phi \preccurlyeq 1$, and let the superscript $\circ$ have the usual meaning. Then the slot $(P^{\phi\circ}, \mathfrak{m}^{\circ}, \widehat{h}^{\circ})$ in H° is flabby, and if $g \in \mathfrak{m}^{\circ}\,(\mathcal{C}^r)^{\preccurlyeq}$ and $P^{\phi\circ}(g) = 0$, $g \prec \mathfrak{m}^{\circ}$, then taking $f \in \mathcal{C}^r$ with $f^{\circ} = g$ we have $f \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$, $P(f) = 0$, and $f \prec \mathfrak{m}$, using the remark after Lemma 6.4.2.

Lemma 7.7.42. *Suppose $(P, \mathfrak{m}, \widehat{h})$ is Z-minimal, deep, normal, special, and flabby. Then there are $f \neq g$ in $\mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$ such that $P(f) = P(g) = 0$, and $f, g \prec \mathfrak{m}$.*

Proof. Corollary 7.7.23 yields a deep absolutely normal refinement $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ of $(P, \mathfrak{m}, \widehat{h})$. Using the remarks preceding the lemma, we can replace $(P, \mathfrak{m}, \widehat{h})$ by $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ to arrange that $(P, \mathfrak{m}, \widehat{h})$ is absolutely normal, and then replacing $(P, \mathfrak{m}, \widehat{h})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{h}/\mathfrak{m})$ we also arrange $\mathfrak{m} = 1$. Set $d := \deg P$ and $w := \operatorname{wt}(P)$. Let $\mathfrak{v}$, Δ, A, R be as in the proof of Lemma 7.7.34. Multiplying A, P, R on the left by some $b \in H^{\times}$ we arrange $A = \partial^r + f_1\partial^{r-1} + \cdots + f_r$ with $f_1, \dots, f_r \in H$ and $R \prec_{\Delta} \mathfrak{v}^w$. Then (7.7.1) holds with $\phi_1, \dots, \phi_r \in K_*$ instead of $\phi_1, \dots, \phi_r \in K$, and $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$ by Corollary 3.1.6. Take $a_0 \in \mathbb{R}$ and functions on $[a_0, \infty)$ representing the germs $\phi_1, \dots, \phi_r$, $f_1, \dots, f_r$, and the $R_{\boldsymbol{j}}$ with $\boldsymbol{j} \in \mathbb{N}^{1+r}$, $|\boldsymbol{j}| \leqslant d$, $\|\boldsymbol{j}\| \leqslant w$ (using the same symbols for the germs mentioned as for their chosen representatives) so as to be in the situation described in the beginning of Section 6.2. Increasing a_0 if necessary and choosing ν as in the proof of Lemma 7.7.33 we arrange that $f_1, \dots, f_r$, and the $R_{\boldsymbol{j}}$ are in $\mathcal{C}^1_{a_0}$ and $\|R\|_{a_0} \leqslant 1/E$, with $E = E(d, r) \in \mathbb{N}^{\geqslant 1}$ as in Corollary 6.3.13, and the hypotheses of Lemma 6.2.1 are satisfied. Lemma 7.7.7 yields $h \in \mathcal{C}^{<\infty}[i]$ such that $A(h) = 0$, $h \neq 0$, and $h, h', \dots, h^{(r)} \prec 1$. Replacing h by $\operatorname{Re} h$ or $\operatorname{Im} h$ we arrange $h \in \mathcal{C}^{<\infty}$. Increasing a_0 again we arrange that h is represented by a function in $\mathcal{C}^r_{a_0}$, denoted by the same symbol, such that

$$A_{a_0}(h) \; = \; 0, \qquad \|h\|_{a_0;r} \; \leqslant 1/8,$$

and such that we are in the situation of Lemma 6.2.6, with a ranging over $[a_0, +\infty)$. Then Corollaries 6.2.8 and 6.2.9 yield for sufficiently large $a \geqslant a_0$ functions $f, g \in \mathcal{C}^r_a$

with $\|f\|_{a;r}, \|g\|_{a;r} \leqslant 1$ and $(\operatorname{Re}\Xi_a)(f) = f$, $(\operatorname{Re}\Xi_a)(g) = g + h$. Fix such a, f, g. Then $A_a(f) = R(f)$ and

$$A_a(g) \;=\; A_a(g+h) \;=\; A_a\big((\operatorname{Re}\Xi_a)(g)\big) \;=\; \operatorname{Re} A_a\big(\Xi_a(g)\big) \;=\; \operatorname{Re} R(g) \;=\; R(g),$$

and $f \prec 1$ and $g + h \prec 1$ by Lemma 6.2.6, so $g \prec 1$, hence f, g are solutions of $(*)$ on $[a, \infty)$. Denoting the germs of f, g also by f, g we have $P(f) = P(g) = 0$. Moreover, $f \neq g$ as germs: otherwise $f = g$ in $\mathcal{C}^r_a$ by the remark after the proof of Corollary 6.3.13, and thus $h = (\operatorname{Re}\Xi_a)(g) - g = (\operatorname{Re}\Xi_a)(f) - f = 0$ in $\mathcal{C}^r_a$, a contradiction. $\square$

Corollary 7.7.43. *Suppose H is $\upomega$-free and r-linearly newtonian, and $(P, \mathfrak{m}, \widehat{h})$ is Z-minimal and flabby. Assume also that $(P, \mathfrak{m}, \widehat{h})$ is linear or normal. Then the conclusion of Lemma 7.7.42 holds.*

Proof. Use Theorem 3.3.33 and the remarks preceding Lemma 7.7.42 to arrange that $(P, \mathfrak{m}, \widehat{h})$ is deep and normal. Then $(P, \mathfrak{m}, \widehat{h})$ is quasilinear by Corollary 3.3.21, and hence special by Lemma 3.2.36. Now Lemma 7.7.42 applies. $\square$

Note that the hypotheses of Corollary 7.7.43 hold if H is $\upomega$-free and $(P, \mathfrak{m}, \widehat{h})$ is a flabby normal nonlinear minimal hole in H, by Corollary 3.2.6.

Suppose H is $\upomega$-free, all $A \in H[\partial] \subseteq K[\partial]$ of order r are terminal, and $(P, \mathfrak{m}, \widehat{h})$ is a minimal hole in H. Then by Corollary 7.7.38 we have:

$$(P, \mathfrak{m}, \widehat{h}) \text{ is firm} \quad\Longrightarrow\quad \text{there is a unique } f \in \mathcal{C}^r \text{ with } P(f) = 0 \text{ and } f \approx_H \widehat{h}.$$

Thanks to Corollary 7.7.13, the converse of this implication also holds if $\deg P = 1$, but we do not know whether this is still the case when $\deg P > 1$. We now prove a partial generalization of Corollary 7.7.14:

Corollary 7.7.44. *Suppose H is $\upomega$-free, $(P, \mathfrak{m}, \widehat{h})$ is an ultimate minimal hole in H with terminal linear part, and $(P, \mathfrak{m}, \widehat{h})$ is linear or absolutely normal. Then*

$$(P, \mathfrak{m}, \widehat{h}) \textit{ is firm} \quad\Longleftrightarrow\quad \textit{there is a unique } f \in \mathcal{C}^r \textit{ with } P(f) = 0 \textit{ and } f \prec \mathfrak{m}.$$

If $(P, \mathfrak{m}, \widehat{h})$ is firm and $f \in \mathcal{C}^r$, $P(f) = 0$, and $f \prec \mathfrak{m}$, then $f \in \mathrm{D}(H)$ and $f \approx_H \widehat{h}$, and there is no $g \neq f$ in $\mathcal{C}^r[\mathrm{i}]$ with $P(g) = 0$ and $g \prec \mathfrak{m}$.

Proof. If $\deg P = 1$, then this follows from Corollary 7.7.14. Suppose $\deg P > 1$. Lemma 7.7.2 yields $f \in H_*$ with $P(f) = 0$ and $f \approx_H \widehat{h}$. This and Lemma 7.7.34 yield the forward direction of the displayed equivalence, as well as the rest in view of H_* being arbitrary. The converse holds by the remark after Corollary 7.7.43. $\square$

Remark. Suppose H is $\upomega$-free, all $A \in H[\partial] \subseteq K[\partial]$ of order r are terminal, and $(P, \mathfrak{m}, \widehat{h})$ is a minimal hole in H. Then Remarks 4.4.15 and 7.7.26 give a refinement $(P_{+h}, \mathfrak{n}, \widehat{h} - h)$ of $(P, \mathfrak{m}, \widehat{h})$ and an active $\phi > 0$ in H such that the minimal hole $(P^{\phi\circ}_{+h^\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ - h^\circ)$ in H° is ultimate, and linear or absolutely normal. Therefore the hypotheses of Corollary 7.7.44 are satisfied by H° and $(P^{\phi\circ}_{+h^\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ - h^\circ)$ in place of H and $(P, \mathfrak{m}, \widehat{h})$.

The next two subsections contain analogues of Lemmas 7.7.34, 7.7.36, and 7.7.42 for slots in K.

Firm slots in K. *In this subsection $(P, \mathfrak{m}, \widehat{f})$ is a slot in K of order $r \geqslant 1$, with $\widehat{f} \in \widehat{K}$.* Here are K-versions of Lemma 7.7.34 and its Corollary 7.7.35 with similar proofs:

Lemma 7.7.45. *If $(P, \mathfrak{m}, \widehat{f})$ is firm, ultimate, and absolutely normal, with terminal linear part, then for any $f, g \in \mathcal{C}^r[\mathrm{i}]$ with $P(f) = P(g) = 0$, $f, g \prec \mathfrak{m}$ we have $f = g$.*

Corollary 7.7.46. *If $(P, \mathfrak{m}, \widehat{f})$ is firm, ultimate, and strictly normal, and its linear part splits strongly over K, then the conclusion of Lemma 7.7.45 holds.*

In the rest of this subsection we assume that all $A \in K[\partial]$ of order r are terminal. (This holds if $r = 1$ or K is ω-free with a minimal hole of order r in K, because then K is r-linearly closed by Corollary 3.2.4.)

By the K-versions of Lemmas 4.4.10 and 4.4.25, if $(P, \mathfrak{m}, \widehat{f})$ is ultimate, then so is each of its refinements, and likewise with "firm" in place of "ultimate". The K-version of Corollary 7.7.23, and Lemma 7.7.45 in place of Lemma 7.7.34 then yields the K-version of Lemma 7.7.36:

Lemma 7.7.47. *If $(P, \mathfrak{m}, \widehat{f})$ is Z-minimal, deep, normal, special, ultimate, and firm, then there is at most one $f \in \mathcal{C}^r[\mathrm{i}]$ with $P(f) = 0$ and $f \approx_K \widehat{f}$.*

Using the K-version of Corollary 7.7.25, and Lemmas 7.7.4 and 7.7.45 instead of Lemmas 7.7.2 and 7.7.34 we obtain the K-version of Corollary 7.7.37:

Corollary 7.7.48. *If H is ω-free, K is r-linearly newtonian, and $(P, \mathfrak{m}, \widehat{f})$ is firm and Z-minimal, then there is a unique $f \in \mathcal{C}^r[\mathrm{i}]$ with $P(f) = 0$ and $f \approx_K \widehat{f}$, and this f is in $\mathrm{D}(H)[\mathrm{i}]$.*

Here is a K-analogue of Corollary 7.7.38:

Corollary 7.7.49. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is a firm minimal hole in K, and $r = \deg P = 1$ or H is ω-free. Then the conclusion of Corollary 7.7.48 holds.*

Proof. If $(P, \mathfrak{m}, \widehat{f})$ has complexity $(1, 1, 1)$, then by Remark 4.4.19 and the K-version of Lemma 4.4.25 we arrange that $(P, \mathfrak{m}, \widehat{f})$ is ultimate, so that the desired conclusion follows from Corollary 7.7.20. If K is ω-free and $(P, \mathfrak{m}, \widehat{f})$ has complexity $> (1, 1, 1)$, then $\deg P > 1$ by Corollary 3.2.8, so K is r-linearly newtonian by Corollary 3.2.6, and the desired conclusion follows from Corollary 7.7.48. $\square$

Corollary 7.7.50. *Suppose H is d-perfect. If $(P, \mathfrak{m}, \widehat{f})$ is a minimal hole in K of complexity $(1, 1, 1)$, then it is flabby. If H is ω-free, then every minimal hole in K of positive order is flabby.*

Proof. The first part is immediate from Corollary 7.7.49. For the second part, suppose H is ω-free and we are given a minimal hole in K of positive order. By Lemma 4.2.15 we can pass to an equivalent hole $(Q, \mathfrak{n}, \widetilde{a})$ in K with $\widetilde{a} \in \widetilde{H}[\mathrm{i}]$ for some immediate H-field extension $\widetilde{H}$ of H, so Corollary 7.7.49 applies to it. $\square$

Theorem 7.7.1 now follows from Corollaries 7.7.39, 7.7.40, 7.7.41, and 7.7.50.

Flabby slots in K. *Let $(P, \mathfrak{m}, \widehat{f})$ be a slot in K of order $r \geqslant 1$, $\widehat{f} \in \widehat{K}$.* Note that if $(P, \mathfrak{m}, \widehat{f})$ is normal and $f \in \mathfrak{m}\mathcal{C}^r[i]^{\preccurlyeq}$, $P(f) = 0$, then by Corollary 6.3.6 we have $f \in \mathcal{C}^{<\infty}[i]$, and $f \in \mathcal{C}^{\infty}[i]$ if $H \subseteq \mathcal{C}^{\infty}$, $f \in \mathcal{C}^{\omega}[i]$ if $H \subseteq \mathcal{C}^{\omega}$.

Suppose $(P, \mathfrak{m}, \widehat{f})$ is flabby. The remarks about multiplicative conjugates, refinements, and compositional conjugates preceding Lemma 7.7.42 then go through for the slot $(P, \mathfrak{m}, \widehat{f})$ in K instead of the slot $(P, \mathfrak{m}, \widehat{h})$ in H with $\mathcal{C}^r[i]$ replacing $\mathcal{C}^r$ and K° instead of H°; this uses the K-versions of Lemma 4.4.26 and Corollary 4.4.30. It helps in proving a complex version of Lemma 7.7.42:

Lemma 7.7.51. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is flabby, special, Z-minimal, deep, and strictly normal. Then there are $f \neq g$ in $\mathfrak{m}\,\mathcal{C}^r[i]^{\preccurlyeq}$ such that $P(f) = P(g) = 0$, $f, g \prec \mathfrak{m}$.*

Proof. Using Corollary 7.7.30 and the remarks preceding the lemma, we arrange that $\mathfrak{m} = 1$ and $(P, 1, \widehat{f})$ is absolutely normal. Now argue as in the proof of Lemma 7.7.42, using instead of Lemma 7.7.7 its K-version. We also appeal to Lemma 6.2.1, Theorem 6.2.3, and Lemma 6.2.4 instead of to Lemma 6.2.6 and Corollaries 6.2.8 and 6.2.9. Naturally, we don't need to take real or imaginary parts, and use Ξ_a instead of $\operatorname{Re} \Xi_a$. $\square$

Corollary 7.7.52. *Suppose H is ω-free, K is r-linearly newtonian, and $(P, \mathfrak{m}, \widehat{f})$ is Z-minimal and flabby. Assume also that $(P, \mathfrak{m}, \widehat{f})$ is linear or normal. Then the conclusion of Lemma 7.7.51 holds.*

Proof. Like that of Corollary 7.7.43, but using Corollary 3.3.48 instead of Theorem 3.3.33, and Lemma 7.7.51 instead of Lemma 7.7.42. $\square$

In particular, the conclusion of Lemma 7.7.51 holds if H is ω-free and $(P, \mathfrak{m}, \widehat{f})$ is a flabby normal nonlinear minimal hole in K.

Corollary 7.7.53. *Suppose that $(P, \mathfrak{m}, \widehat{f})$ is an ultimate minimal hole in K and, in case the complexity of $(P, \mathfrak{m}, \widehat{f})$ is $> (1, 1, 1)$, that H is ω-free and $(P, \mathfrak{m}, \widehat{f})$ is absolutely normal. Then*

$$(P, \mathfrak{m}, \widehat{f}) \text{ is firm} \iff \text{there is a unique } f \in \mathcal{C}^r[i] \text{ with } P(f) = 0 \text{ and } f \prec \mathfrak{m}.$$

If $(P, \mathfrak{m}, \widehat{f})$ is firm, $f \in \mathcal{C}^r[i]$, $P(f) = 0$, and $f \prec \mathfrak{m}$, then $f \in \mathrm{D}(H)[i]$ and $f \approx_K \widehat{f}$.

Proof. If $(P, \mathfrak{m}, \widehat{f})$ has complexity $(1, 1, 1)$, use Corollaries 7.7.18 and 7.7.20. Now suppose H is ω-free and $(P, \mathfrak{m}, \widehat{f})$ is absolutely normal of complexity $> (1, 1, 1)$. Then $\deg P > 1$ by Corollary 3.2.8, and $L_{P_{\times \mathfrak{m}}}$ is terminal by Corollaries 3.2.4 and 2.6.21. Thus the forward direction of the displayed equivalence follows from Lemmas 7.7.45 and 7.7.4, and the backward direction from the remark after Corollary 7.7.52. The rest follows by applying Lemma 7.7.4 to all choices of H_*. $\square$

Remark. Suppose $(P, \mathfrak{m}, \widehat{f})$ is a minimal hole in K. If $\deg P = 1$, then $(P, \mathfrak{m}, \widehat{f})$ refines to an ultimate hole in K by Remark 4.4.19. If H is ω-free and $\deg P > 1$, then Corollary 7.7.32 gives a refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ and an active $\phi > 0$ in H such that the minimal hole $(P^{\phi\circ}_{+f^{\circ}}, \mathfrak{n}^{\circ}, \widehat{f}^{\circ} - f^{\circ})$ in K° is ultimate and absolutely normal.

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List of Symbols

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The citation [ADH] refers to our book

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